Economic Interpretation of the Sensitivity Data in Optimization Solutions

Sensitivity Analysis

What is Sensitivity Analysis?

- Investigate how sensitive a solution is to changes in input parameters
- Deal, at the margin, with uncertainties in model inputs
- Assess the significance, robustness, applicability of a solution

In short, ascertain whether the model solution is a suitable basis for real-world decisions.

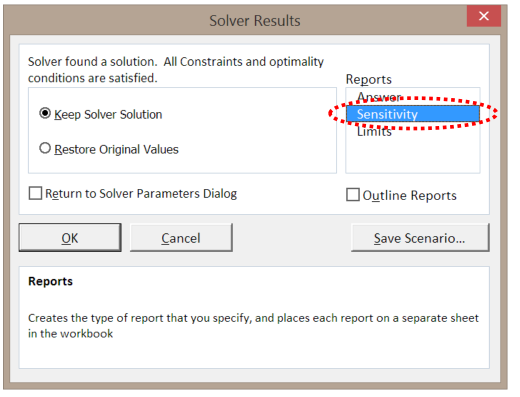
For optimization models, special sensitivity data is available

- facilitates/supplements the regular sensitivity analysis process
- automatically generated in a so-called "Sensitivity Report"

How to get a Sensitivity Report

In the *Solver Results* box:

- ❑ Select “Sensitivity” in the Reports field
- ❑ Click OK
- ❑ This will dump the sensitivity data in a new worksheet



Facts on the Sensitivity Report…

The Sensitivity Report…

- ✓ is computed as a by-product of solving the problem with the Simplex LP or GRG Nonlinear method without Integer constraints
- ✓ is not available if the model contains Integer (or Binary) constraints
- ✓ is a standard output, produced by any optimization package (not specific to Excel’s Solver)
- ✓ contains less information if the GRG Nonlinear method is used to obtain the solution than if the Simplex LP method is used, even if the model is linear
- ✓ may not be available if Solver was not able to reach an optimal solution (as may occur with the GRG Nonlinear method)

In what follows, we will study the Solver Sensitivity Report obtained for linear models solved with the Simplex LP method.

For linear models…

… the Sensitivity Report answers these questions:

- How much will a change in a constraint boundary affect the optimal objective function value?
- How much can an objective function coefficient change without affecting the optimal solution?
- How much will the introduction of a new decision variable affect the optimal objective function value?

Key contents of Sensitivity Report for a Linear Model

When the Simplex LP solving method is used, the sensitivity report will show:

- ❑ **Shadow Prices** for constraints
- ❑ **Allowable Changes** in Objective Function coefficients
- ❑ **Reduced Costs** for Decision Variables

This information enables the user to:

- diagnose the solution obtained, as well as possible alternatives
- identify potential improvements to the current optimized system
- react appropriately to changes in the environment

Example of Linear Model (from Exercise Set)

	A	B	C	D	E	F	G	H	I
1	*Transportation / Logistics*								
2									
3									
4	*Shipping cost per bottle*				Restaurant				
5		From / To:		A	B	C	D		
6		Malpas vineyard		7 F	8 F	13 F	9 F		
7		Peyrous vineyard		12 F	6 F	8 F	7 F		
8									
9									
10	*Profit per bottle*				Restaurant				
11		Price per bottle:		A	B	C	D		
12		Cost per bottle		69 F	67 F	70 F	66 F		
13		Malpas	23 F	39 F	36 F	34 F	34 F		
14		Peyrous	25 F	32 F	36 F	37 F	34 F		
15									
16									
17	*Delivery plan (Decision Variables and Constraints)*								
18				to A	to B	to C	to D	Produced	Capacity
19		From Malpas		1,800	1,700	0	0	3,500	3,500
20		From Peyrous		0	600	1,250	1,250	3,100	3,100
21		Total delivered		1,800	2,300	1,250	1,250		
22		Max Demand		1,800	2,300	1,250	1,750		
23									
24									
25	*Total Profit (Objective Fcn)*								
26									
27		241,750 F							
28									

Solver settings:

Maximize: B27 (Profit)
By Changing: D19:G20 (Shipments)
Subject To:
H19:H20 <= I19:I20 (Production <= Capacity)
D21:G21 <= D22:G22 (Deliveries <= Max_Demand)
D19:G20 >= 0 (Shipments non-negative)

Solving Method: Simplex LP

A so-called **Objective Coefficient** that is, the coefficient of a decision variable in a linear Objective Function

The so-called **Right-Hand Side (RHS)** of a constraint

Decision Variables are in the yellow cells

The **Objective Function** i.e., the function to be optimized

Example of Sensitivity Report for a Linear Model (from Exercise)

Microsoft Excel 14.0 Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D19	From Malpas to A	1800	0	39	1E+30	5
E19	From Malpas to B	1700	0	36	5	0
F19	From Malpas to C	0	-3	34	3	1E+30
G19	From Malpas to D	0	0	34	0	1E+30
D20	From Peyrous to A	0	-7	32	7	1E+30
E20	From Peyrous to B	600	0	36	0	2
F20	From Peyrous to C	1250	0	37	1E+30	3
G20	From Peyrous to D	1250	0	34	2	0

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D21	Total delivered to A	1800	5	1800	1250	500
E21	Total delivered to B	2300	2	2300	1250	500
F21	Total delivered to C	1250	3	1250	1250	500
G21	Total delivered to D	1250	0	1750	1E+30	500
H19	From Malpas Produced	3500	34	3500	500	1250
H20	From Peyrous Produced	3100	34	3100	500	1250

The amounts by which each **Objective Coefficient** can vary (all others remaining at their current values) without changing the optimal solution shown in the **Final Value** column.

A huge number: often a computer's way of saying "infinity", or "no limit".

For each constraint, these limits indicate the range of increase or decrease in the **Constraint Right Hand Side** over which the stated **Shadow Price** is valid.

The **decision variable** F19 is not used in the optimal solution, because its **Final Value** = 0.

Reduced Cost: if the **Objective Coefficient** of decision variable F19 improved by that amount (3 F/unit), then this variable would be included in the optimal solution, i.e. its **Final Value** would be > 0.

Shadow Price: measures by how many units the optimal objective value (here, total profit) would change if the **Constraint Right Hand Side** changed by one unit.

Shadow Prices: *Pricing changes in constraints*

Definition of Shadow Price

Shadow Price of a constraint: measures by how many units the objective function will change if the constraint bound changes by one unit, while everything else in the model remains unchanged.

Change in Obj. Fcn. = ***Change in Constraint*** × ***Shadow Price***

Example from "Production Planning" exercise:

- Objective Function: **Profit ($)** to be maximized
- Constraint: **Labor (hours)** available
- Constraint bound (also called RHS for "Right Hand Side") = **400,000 hrs.**
- Shadow Price = **50 ($/hr)**

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H11	Labor_Used	400000	50	400000	240000	60000

Meaning: **If 1 more hour of labor were available, profit would change by +$50**

Managerial Significance of Shadow Price

Shadow Prices can help us

- **Understand the benefits of changing constraints**

  Example: Production Capacity (a constraint) can be changed in future

  → *Compare benefit of changing a constraint with the cost of doing so*
- **Anticipate the impact of having to operate under tighter constraints**

  Example: tighter regulations, decreased access to resources
- **Identify decision opportunities of strategic importance**

The sign of the Shadow Price is meaningful !

- Relaxing the Constraint, i.e. making it *less* constraining

  ⇒ the Objective will improve (Max ↑ or Min ↓)

- Tightening the Constraint, i.e. making it *more* constraining

  ⇒ the Objective will deteriorate (Max ↓ or Min ↑)

Range of validity for Shadow Price

The Shadow Price applies as long as changes in the constraint Right Hand Side (*RHS*) remain within a certain range.

This range is defined by the "Allowable Increase" and "Allowable Decrease" of the Constraint R.H. Side.

Working through examples…

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Example of Shadow Price interpretation

Microsoft Excel 14.0 Sensitivity Report

Variable Cells

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D20	From Peyrous to A	0	-7	32	7	1E+30
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2300 bottles is Restaurant B's demand in our problem.

For each constraint, these limits indicate the range of increase or decrease in the **Constraint Right Hand Side** over which the stated **Shadow Price** is valid.

$1E+30 = 10^{30}$ is a computer's way of saying "infinity", or "no limit".

Shadow Price: measures by how many units the optimal objective value (here, profit) will change if the **Constraint Right Hand Side** changes by one unit.

Example: The optimal profit would increase by 2 if the demand from Restaurant B increased to 2301 bottles. The total profit would go down by 2x300 = 600 if Restaurant B demand decreased to 2000 bottles.

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Other facts about Shadow Prices

- ✓ A non-binding constraint has a Shadow Price = 0;
  indeed, changes in a non-binding constraint have no effect.

  Caution: Non-zero Shadow Price could be displayed as 0 due to rounding
  → **Display <u>decimals</u>** to check if Shadow Price is truly = 0

- ✓ Shadow Price indicates change in optimal objective value for unit change in the constraint, assuming <u>all other constraints remain unchanged</u>.

- ✓ Shadow prices indicate by how much the optimal <u>objective value</u> will change as a result of changes in constraints, but not how the <u>decision variables</u> will change.

- ✓ For certain constraints (e.g. "percentages must add up to 100%"), shadow prices do not have a meaningful interpretation.

Allowable Changes in Objective Coefficients:
Assessing the stability and uniqueness of the solution

Definition of Allowable Changes

"**Allowable Increase**" and "**Allowable Decrease**" in an Objective Function Coefficient indicate the limits within which this coefficient may vary <u>without altering</u> the optimal solution

Managerial significance of Allowable Changes

- Useful because the Objective Coefficients may be uncertain or subject to fluctuations (e.g. market prices)

→ The wider the allowable variation ranges for all coefficients, the more robust the optimal solution

Example of Allowable Change interpretation

Microsoft Excel 14.0 Sensitivity Report

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Example: The profit margin on "Peyrous to Restaurant C" could decrease by as much as 3 (i.e. go to 37 – 3 = 34) and the current optimal solution would not change, i.e. it would still be an optimal plan after the change.

Facts on allowable variations in O.F. coefficients

- ✓ If an Objective Function Coefficient changes within its allowable range, the optimal plan of action will not change
- ✓ If an Objective Function Coefficient changes beyond its allowable range, the optimal plan of action will change
- ✓ Wide allowable changes in O.F. coefficients → the solution is robust
  Narrow allowable changes → the solution is liable to change with small changes in O.F. coefficients
- ✓ The allowable changes in a coefficient are valid provided all other coefficients remain fixed at their current values
- ✓ **Detecting the existence of multiple solutions**: The presence of one or more '0' in the Allowable Increase and/or Decrease of Objective Coefficients indicates that alternative optimal solutions exist.

Reduced Costs:
Monitoring the optimality of decision variables

In the optimal solution, some decision variables may be **zero**, meaning that these variables are not used in the optimal plan (they are too "expensive")

Definition of Reduced Cost

Reduced Cost for a zero decision variable = the amount by which the Objective Coefficient of this variable would have to improve in order for the variable to be used (i.e., non zero) in the solution.

Managerial significance of Reduced Costs

- Define "trigger prices" at which the (currently zero) decision variables should be considered for use
- Indicate when the current optimal policy should be reconsidered
- Indicate how sensitive to price changes the current policy is

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Example of Reduced Cost interpretation

Microsoft Excel 14.0 Sensitivity Report

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Example: The profit margin on Malpas to Restaurant C has to improve to 34 – (–3) = 37 for it to become profitable to ship from Malpas to Restaurant C.

The **Decision Variable** F19 is not used in the optimal solution, because its **Final Value** = 0. In other words, it is not optimal to deliver C from Malpas

Reduced Cost: if the **Objective Coefficient** of decision variable F19 improved by that amount (3 F/unit), then this variable would be included in the optimal solution, i.e. its **Final Value** would be > 0.

The **Reduced Cost** tells you by how much the per unit profit of this variable would have to improve for it to become optimal to use that variable. Here it is 3F/unit.

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Final notes on Sensitivity Data

- Sensitivity Data can have great economic significance: may yield insights into critical trade-offs, vulnerabilities, and opportunities.
- Enhances your understanding of the solution: May draw your attention to issues that do not match your intuition. When in doubt, double check by re-running the model…
- Peculiar conditions may occur (e.g. "degeneracy" due to overlapping constraints) that render the interpretation of sensitivity information ambiguous. Always exercise judgment and caution in using sensitivity information.
- Not all questions of managerial interest can be answered from the Sensitivity Report.
- Traditional tweaking/what-if on the live model still has a role to play!