Optimization Models:

the Linear case

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Recall: an optimization model is entirely defined by:

- some Decision Variables $X_1, X_2, \ldots, X_n$
  whose values we want to decide
- one Objective Function $f(X_1, X_2, \ldots, X_n)$
  which describes the quantity to be optimized,
  i.e. the objective of the problem
- some Constraints $g_i(X_1, X_2, \ldots, X_n) \le b_i \quad i = 1, \ldots, m$
  which reflect the economic, legal, and technical
  realities under which we must operate

When the Objective Function f and the Constraints g_i's are all **linear functions**, the optimization model is said to be a "Linear Programming" (LP) problem.

In that case, the "Simplex LP" solving method should be used.

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Linear Optimization Model: Definition

Definition: An optimization model is **linear** if

- The **objective function** f is **linear**,

  $f(X_1, X_2, \ldots, X_n) = c_1X_1 + c_2X_2 + \ldots + c_nX_n \quad (+\, c_0 \;\; \textit{fixed cost?})$

  that is, it exhibits constant returns
- All the **constraints** are **linear**, that is, of the form:

  $a_{i1}\, X_1 + a_{i2}\, X_2 \; + \ldots + a_{in}\, X_n$ ≤ or ≥ b_i
- The **decision variables** $X_1, X_2, \ldots, X_n$ are **continuous**

Excel hints: if your model is linear,

- All model formulas can be expressed as a ***SUMPRODUCT*** of the decision variables with some constants
- Select *Simplex LP* as the *Solving Method*

(*Solver* will tell you if it finds your model to be non linear)

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Three Major Optimization Solving Methods

LP: Linear Models

... computationally ***easy*** *to solve*

The Objective Function and all Constraints are linear functions, and decision variables are continuous (not required to be integer)

→ Select Solving Method **Simplex LP**

ILP: Linear Models with Integer Variables

... computationally ***less easy*** *to solve*

The Objective Function and all Constraints are linear functions, and some decision variables are required to take integer values

→ Select Solving Method **Simplex LP** and use **Integer** or **Binary** constraints on those decision variables that require it

NLP: Non-Linear Models

... computationally ***difficult*** *to solve*

Something is not linear: the Objective Function, or some Constraint(s), or both

→ Select Solving Method **GRG Nonlinear**

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Example of Linear Model ("Diversifying Investments" Exercise)

❑ The **Decision Variables**

X_1 = *amount of money to invest in Municipal Bonds*
X_2 = *amount of money to invest in Certificates of Deposit*
X_3 = *amount of money to invest in Treasury Bills*
X_4 = *amount of money to invest in Growth Stocks*

❑ The **Objective Function**

Maximize: Return $= 0.04 \cdot X_1 + 0.025 \cdot X_2 + 0.03 \cdot X_3 + 0.07 \cdot X_4$

❑ The **Constraints**

$X_1 + X_2 + X_3 + X_4 \leq \$70{,}000$ (use $\leq$ rather than =)
$X_1 \leq 0.20 \cdot (X_1 + X_2 + X_3 + X_4)$
$X_4 \leq X_1 + X_2 + X_3$
$X_2 + X_3 \geq 0.30 \cdot (X_1 + X_2 + X_3 + X_4)$

Note: the Objective Function and all the Constraints involve linear operations of the Decision Variables.
→ The Simplex LP solving method should be used for solving

For Large-Scale Applications: Linear Optimization

Make the Objective Function and all Constraints to be linear functions of the Decision Variables

Powerful and reliable

- *Solution process is fast, even for very large models*
- *Solution process is guaranteed to find the global optimum*
- *Can solve very large models, with tens of thousands of variables and constraints!*
  (e.g., large portfolio models with linearized risk measures, airline operations models, transportation networks)

Applicable to a wide range of business problems

Most used in practice

Business Applications of Linear Optimization (non-exhaustive)

- ✓ Budget allocation, Capital budgeting, Financial planning **Max(*investment return*)**
- ✓ Diversification of portfolio **Max(*portfolio return*)** or **Min(*portfolio risk*)**
- ✓ Production planning **Max(*profit of mix*)** or **Min(*production cost*)**
- ✓ Design of 'blended' products (foods, chemicals, etc.) **Min(*production cost*)**
- ✓ Transportation, logistics **Min(*distribution cost*)**
- ✓ Vehicle routing **Min(*travel time* or *distance*)**
- ✓ Work scheduling **Min(*labor cost*)**
- ✓ Human resources planning **Min(*labor/training cost*)**
- ✓ Advertising **Max(*audience exposure*)**
- ✓ Assigning contracts to firms, jobs to people, … **Max(*performance*)** or **Min(*cost*)**

Often non-linear, but can sometimes be put in linear form:

- ✓ Inventory management **Min(*inventory cost*)**
- ✓ Project management **Min(*cost/duration of project "crashing"*)**
- ✓ Risk-Return portfolio selection with linearized risk measures **Min(*portfolio risk*)**

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"Assignment" Models

So-called "Assignment" problem = finding best matches

Assign {salespeople, facilities, contracts, ….} to {clients, locations, firms, ….}

according to their… {competencies, preferences, costs, ….}

so as to optimize overall… {performance, satisfaction, (cost) effectiveness, ….}

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Modeling Workers?

By building mathematical models of its own employees, IBM aims to improve productivity and automate management

2008: IBM embarks on research to harvest massive data on employees, and to build mathematical models of 50,000 of the company's consultants. The goal is to optimize them, so that they can be deployed with ever more efficiency.

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Assignment Models

- **Model formulation** (example from Exercise Set)
- **A special kind of transportation model**
  - With supply and demand = 1 unit at each origin and destination
  - If a particular assignment is not wanted or possible:
    put an arbitrary large cost to that assignment
- **Examples of large-scale applications**

  French public education system: matching teachers with open positions nationwide according to their wishes --thousands of individuals/positions each year!

  Residency assignment of medical students

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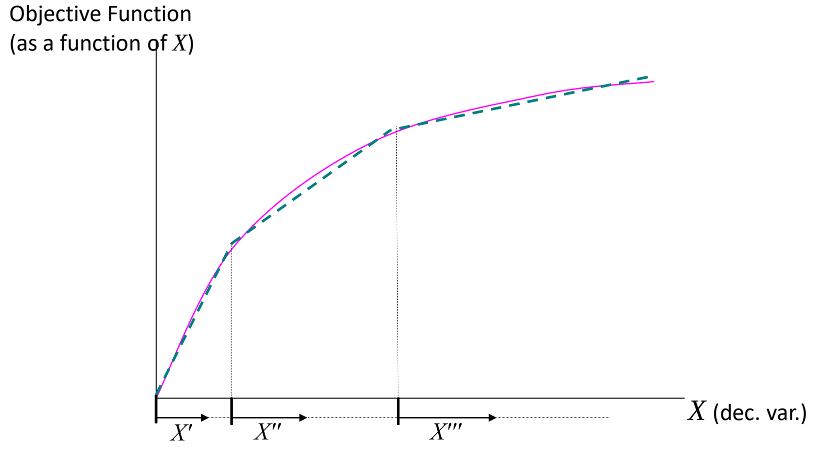
Linear modeling tricks: Dealing with non-constant returns

❑ Case of decreasing returns



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- Achieve a piecewise linear approximation of the decreasing returns
- This results in a set of incremental variables (X', X'',…)
- Replace X with X', X'',… in the objective function and constraints
- A worked-out example (from Exercise Set)

Q: *Why does it work?*

A: *The incremental variables will be used in the right order!*

❑ Case of increasing returns

Need another trick based on binary variables
(see Integer Linear Models, coming later)

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Tips for Designing Optimization Models

- First ask: can you pattern your model after one of the classic formulations?
- Proceed by trial and error to define decision variables, objective, and constraints
- Start small, then scale up: build a small, "toy" version of your model, with just a handful of variables and constraints to get a good handle on the design. Scaling up optimization models is usually straightforward.
- Remember that you can define decision variables as you please, and as many as you want; make it easier for you to express constraints and objective
- You may have to be creative with defining decision variables
- You may have to be creative with defining the objective

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Summary of Linear Optimization

Linear models play a very important role in Optimization

Only linear optimization can solve huge models reliably

There are many business applications of Linear Optimization

But not all applications can be formulated as a linear model…

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