Optimization I

Janne Kettunen

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Anomalies in LP Models

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Anomalies in LP Models

- Redundant constraints
- Alternate optimal solutions
- Unbounded solutions
- No feasible solution

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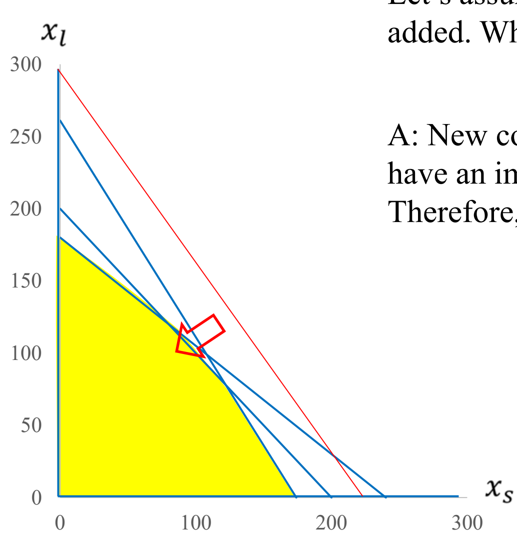
Redundant constraint



Let's assume a new constraint (on red) is added. What is its impact?

A: New constraint (on red) won't have an impact on the feasible region. Therefore, it is redundant.

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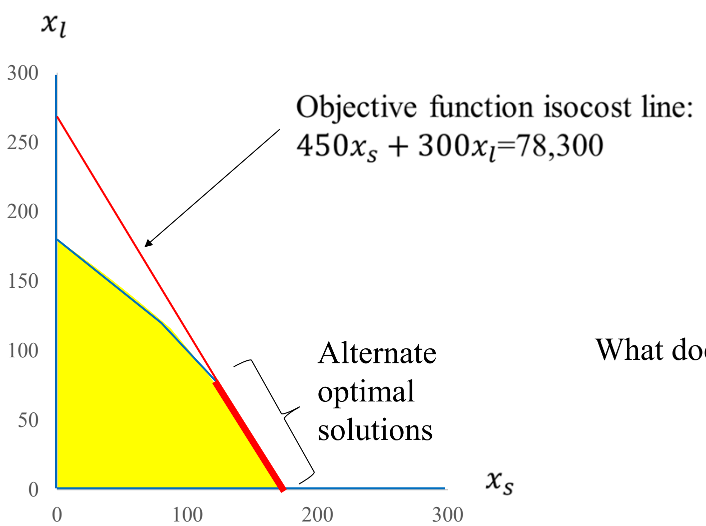
Alternate Optimal Solutions



What does this imply?

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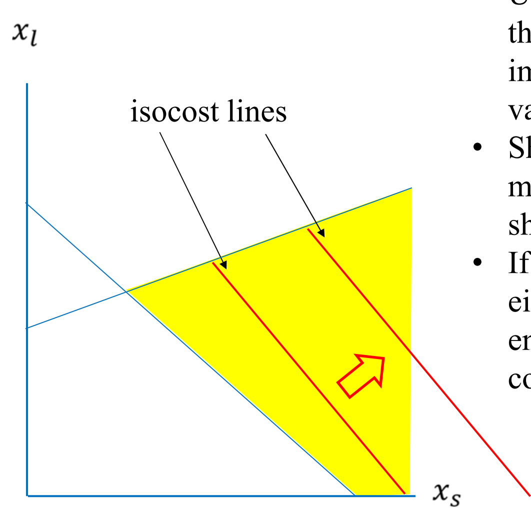
Unbounded Solutions



- Unboundedness occurs when the objective function can improve indefinitely by varying one or more variables
- Should not occur in realistic models, “real-life” constraints should prevent this
- If unboundedness occurs, either wrong input data was entered or a relevant constraint was omitted

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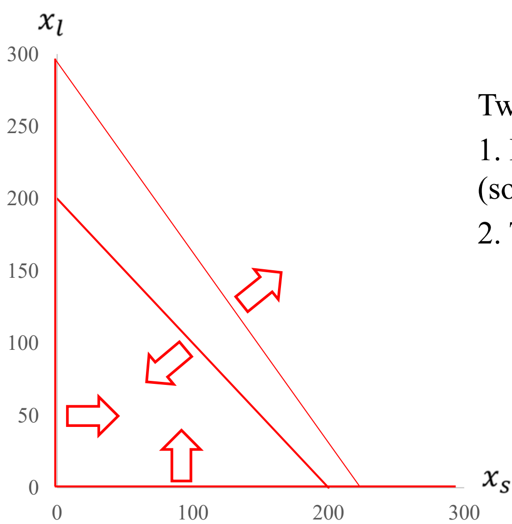
No Feasible Solution



Two reasons:

1. Mistake in the model
(some data is entered incorrectly)
2. The feasible region is empty

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Logic Behind Simplex Algorithm

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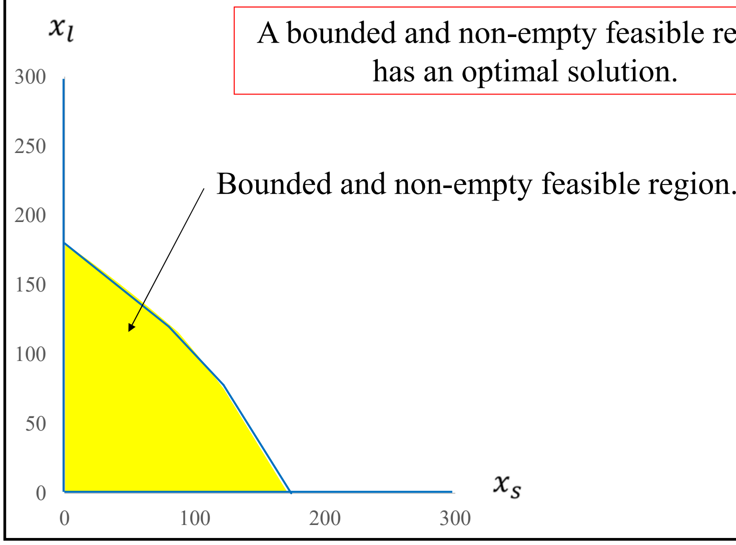
Existence of an Optimal Solution

A bounded and non-empty feasible region has an optimal solution.



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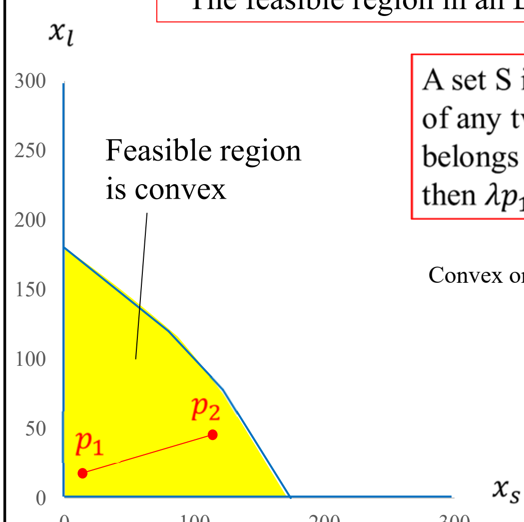
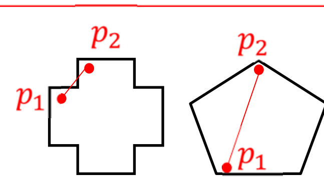
The Feasible Region in an LP

The feasible region in an LP is always convex.

A set S is convex when a line segment, of any two points belonging to the set, belongs also to the set, i.e., if $p_1, p_2 \in S$ then $\lambda p_1 + (1-\lambda) p_2 \in S$ for $\lambda \in [0,1]$.



Convex or not?



Non-convex Convex

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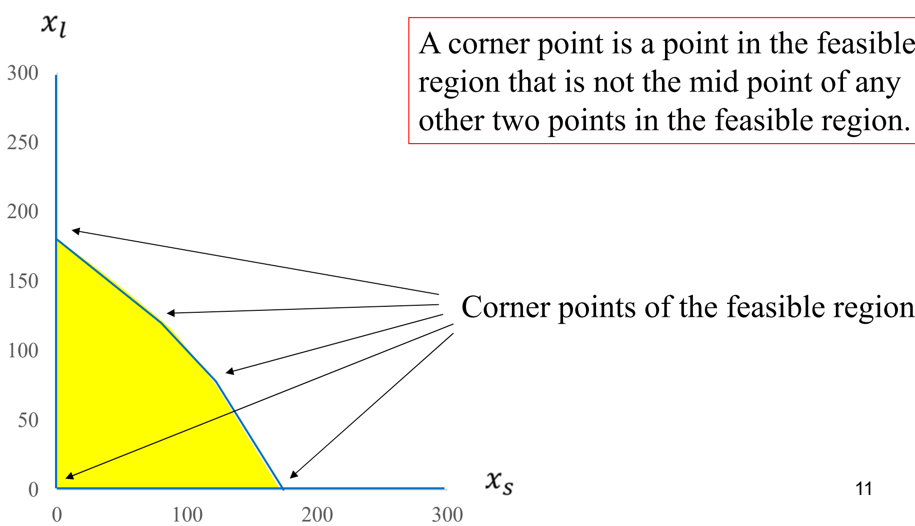
Corner Points Are only Defined for Convex Sets

A corner point is a point in the feasible region that is not the mid point of any other two points in the feasible region.



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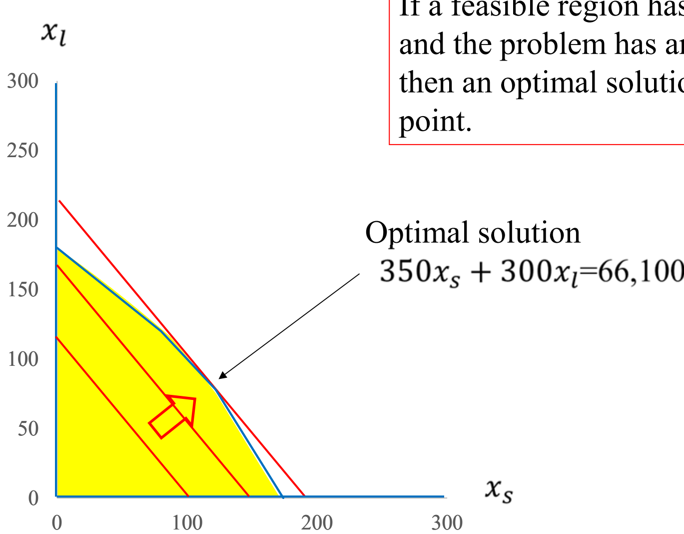
The Optimal Solution in a Corner 1/2

If a feasible region has a corner point and the problem has an optimal solution then an optimal solution is at a corner point.



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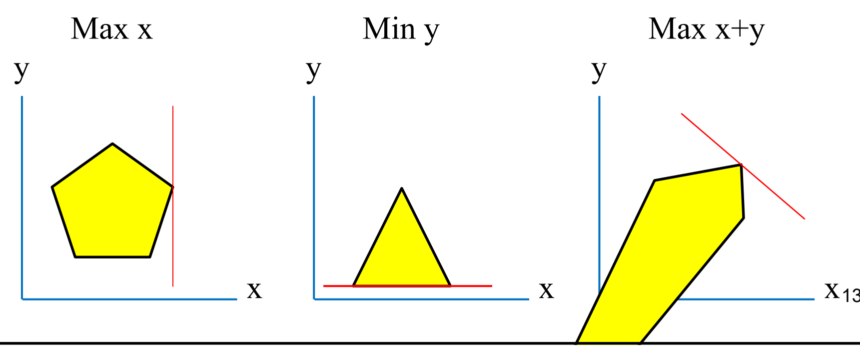
The Optimal Solution in a Corner 2/2

If a feasible region has a corner point and the problem has an optimal solution then an optimal solution is at a corner point.



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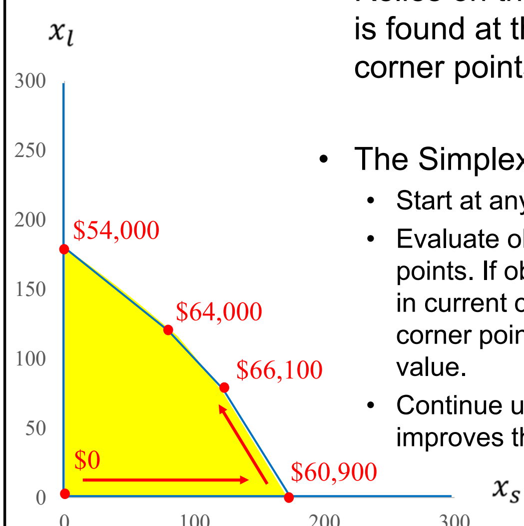
The Simplex Method



- Relies on the fact that the optimal solution is found at the corner point, (given that corner points exists in the feasible region)
- The Simplex method:
  - Start at any feasible corner point
  - Evaluate objective function in adjacent corner points. If objective function value is higher than in current corner point, then go to the adjacent corner point with the highest objective function value.
  - Continue until no adjacent corner point improves the objective function value.

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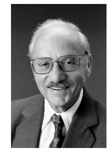
The Simplex Method



- Was developed in 1947 by George Dantzig
- It is still widely used
- Identifies if the LP is infeasible, finds the optimal solution if such exists, and checks for unboundedness
- Very fast for solving large practical problems

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LP Problem Manipulations

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Equivalence of Minimization and Maximization Problems

$$\begin{array}{c} \min 2x \\ ax \geq b \end{array} \quad \text{is equivalent to} \quad \begin{array}{c} \max -2x \\ ax \geq b \end{array}$$

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Linearizing Problems with Absolute Values

- Absolute value in a minimization (maximization) objective function with positive (negative) coefficients for decision variables:

$$\begin{array}{c} \min 2|x| \\ ax \geq b \end{array} \Longrightarrow \begin{array}{c} \min 2y \\ ax \geq b \\ y \geq x, y \geq -x \end{array}$$

Redundant if x>0

Redundant if x<0

- Absolute value in an inequality constraint:

$$\begin{array}{c} \min x \\ ax \geq b \\ |x| \leq 3 \end{array} \Longrightarrow \begin{array}{c} \min x \\ ax \geq b \\ x \leq 3, -x \leq 3 \end{array}$$

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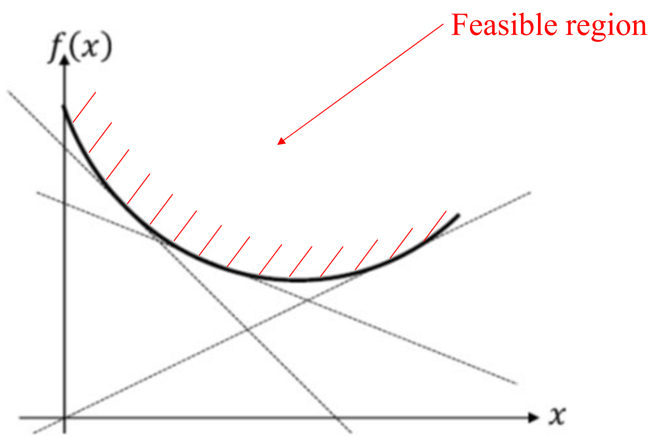
What If We Have A Minimization of Non-Linear Convex Problem?



A convex function approximated by a piecewise linear convex function

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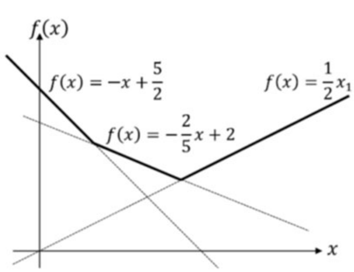
Linearized Version of the Problem

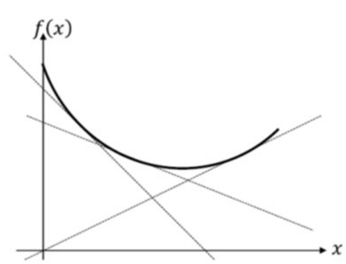
- The minimization of a max of piecewise linear convex function:

$$min\, max \left\{ -x + \frac{5}{2}, -\frac{2}{5}x + 2{,}1/2x \right\}$$

$$ax \geq b$$



A piecewise linear convex function



A convex function approximated by a piecewise linear convex function

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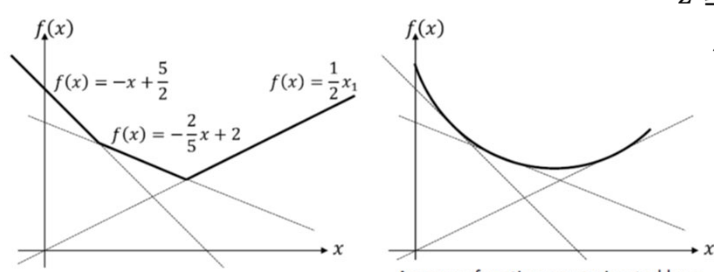
Linearizing Minmax Problems

- The minimization of a max of piecewise linear convex function:

$$\min \max \left\{-x+\frac{5}{2}, -\frac{2}{5}x+2, 1/2x\right\}$$
$$ax \geq b$$

$\Rightarrow$

$$\min z$$
$$ax \geq b$$
$$z \geq -x+\frac{5}{2},$$
$$z \geq -\frac{2}{5}x+2,$$
$$z \geq 1/2x$$



A piecewise linear convex function

A convex function approximated by a piecewise linear convex function

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Similarly Linearizing Maxmin Problems

- The maximization of a min of piecewise linear concave function:

$$\max \min \left\{x, 1, -\frac{1}{4}x+5\right\}$$
$$ax \geq b$$

$\Rightarrow$

$$\max z$$
$$ax \geq b$$
$$z \leq x,$$
$$z \leq 1,$$
$$z \leq -\frac{1}{4}x+5$$

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Linearizing Ratio Constraints

$$\min x_1 + x_2$$
$$\frac{x_1}{x_1 + x_2} \geq 5$$
$$ax \geq b$$

$\Rightarrow$

$$\min x_1 + x_2$$
$$x_1 \geq 5(x_1 + x_2)$$
$$ax \geq b$$

- Can be done if it is known that the denominator is positive for all possible values of the decision variables (for negative denominator the inequality sign would change)

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Duality in Linear Programming

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LP in Standard Form, Vector Notation

- LP standard form:

$$\min(or\ max)\ \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{subject to} \quad \boldsymbol{Ax} = \boldsymbol{b} \qquad \boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$$
$$\boldsymbol{x} \geq \boldsymbol{0} \qquad \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$$

Define:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$ $(n,1)$ vector of decision variables

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$ $(n,1)$ vector of objective function coefficients

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$ (m,n) matrix of coefficients for constraints

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$ (m,1) vector of coefficients for constraints' right hand side

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LP in Standard Form

- Any LP problem can be converted to the standard form:

$$\min(or\ max)\ \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{subject to} \quad \boldsymbol{Ax} = \boldsymbol{b} \qquad \boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$$
$$\boldsymbol{x} \geq \boldsymbol{0} \qquad \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$$

- The tricks to do the conversion are as follows:
  - 1. Convert inequality constraints to equality constraints by adding distinct, nonnegative, slack variables in every ≤ inequality and subtracting such slack variables in every ≥ inequality
  - 2. Eliminate nonpositive variables by substituting new variables equal to their negatives
  - 3. Eliminate unrestricted variables by substituting the difference of two new nonnegative variables

For example:

$$\begin{array}{c} min 2x_1 - 3x_2 \\ 3x_1 + 4x_2 \leq 4 \\ 2x_1 + x_2 \geq 1 \\ x_1 \geq 0,\ x_2 \leq 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} min 2x_1 + 3x_5 \\ 3x_1 - 4x_5 + x_3 = 4 \\ 2x_1 - x_5 - x_4 = 1 \\ x_1 \geq 0,\ x_5 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \end{array}$$

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Derivation of Dual Problem 1/3

- Let's start with the LP in the standard form and call this primal (P)

$$(P) \qquad z = min\boldsymbol{c}^T\boldsymbol{x}$$

$$\text{subject to} \quad \boldsymbol{Ax} = \boldsymbol{b} \qquad \mathbf{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$$

$$\mathbf{x} \geq \mathbf{0} \qquad \mathbf{x} \in \mathbb{R}^{\boldsymbol{n}}$$

- We define a relaxed problem $P(\boldsymbol{p})$ where the constraints can be violated at a fixed cost or price p_i

$$P(\boldsymbol{p}) \qquad z(\boldsymbol{p}) = min\boldsymbol{c}^T\boldsymbol{x} + \underbrace{\boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{Ax})}_{p_1(b_1 - \boldsymbol{a}_1\boldsymbol{x}) + \cdots + p_m(b_m - \boldsymbol{a}_m\boldsymbol{x})}$$

$$\text{subject to} \qquad \mathbf{x} \geq \mathbf{0} \quad \mathbf{x} \in \mathbb{R}^{\boldsymbol{n}}$$

Note that $z(\boldsymbol{p}) \leq z$ for any $\boldsymbol{p}$. This is because $\boldsymbol{b} - \boldsymbol{Ax}^* = \mathbf{0}$ for $\boldsymbol{x}^*$ that is optimal for P and otherwise $\boldsymbol{b} - \boldsymbol{Ax} < \mathbf{0}$.

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Derivation of Dual Problem 2/3

- Therefore, $z(\boldsymbol{p})$ is the lower bound on the optimal value of P
- A price vector $\boldsymbol{p}^*$ such that $z(\boldsymbol{p}^*)$=z is equivalent of solving the dual problem (D)

$$(D) \qquad \max_{\boldsymbol{p}\in\mathbb{R}^m}[z(\boldsymbol{p})] \quad \Longleftrightarrow \quad \max_{\boldsymbol{p}\in\mathbb{R}^m}\left[\min_{\boldsymbol{x}\geq\mathbf{0}} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{P}^T(\boldsymbol{b} - \boldsymbol{Ax})\right]$$

- The objective of the dual problem (D) is to find a vector $\boldsymbol{p}$ of prices that makes violating the constraints of P as inconvenient as possible
- The dual problem (D) is not linear but it can be reformulated as such

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Derivation of Dual Problem 3/3

- We can rewrite $\max_{\boldsymbol{p}\in\mathbb{R}^m}\left[\min_{\boldsymbol{x}\geq\boldsymbol{0}} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\right]$ as follows

$$\max_{\boldsymbol{p}\in\mathbb{R}^m}\left\{\boldsymbol{p}^T\boldsymbol{b} + \left[\min_{\mathbf{x}\geq\mathbf{0}}(\boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A})\boldsymbol{x}\right]\right\} \quad \Longleftrightarrow$$

$$\max_{\boldsymbol{p}\in\mathbb{R}^m}\left[\boldsymbol{p}^T\boldsymbol{b} + \begin{cases} 0 & \text{if } \boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A} \geq 0 \\ -\infty & \text{otherwise} \end{cases}\right]$$

- Since, the objective function is a maximization, the constraint $\boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A} \geq 0$ is optimal to impose, resulting in

$$\begin{array}{ll} & \max_{\boldsymbol{p}\in\mathbb{R}^m} \boldsymbol{p}^T\boldsymbol{b} \\ \text{subject to} & \boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A} \geq 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \text{(D)} & max\, \boldsymbol{p}^T\boldsymbol{b} \\ \text{subject to} & \boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A} \geq 0 \\ & \boldsymbol{p} \in \mathbb{R}^{\boldsymbol{m}} \end{array}$$

- The dual (D) is now a linear programming problem
- The components of $\boldsymbol{p}$ are dual variables

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The Importance of Dual Solution

- The optimal dual variables $\boldsymbol{p}^*$ (aka shadow prices) can provide intuitive interpretations, namely:
  - Penalty of breaking a constraint
  - Marginal cost
  - Fair prices

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Formulating and Solving LP Models

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To Start Consider Using Excel's Solver

- It is provided by Frontline Systems
- It is a good choice for a smaller problem as Excel is commonly available in any workplace
- However,
  - the problems that Solver can deal with are limited by their size up to 200 variables and 200 constraints
  - Also, Solver does not perform very well on challenging mathematical programming problems
- To improve the Solver's performance, it is possible to purchase a license to Frontline System's Solver (removes the size limitations and allows to invoke Gurobi solver, which is an industrial scale solver)

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Guidelines for Formulating Optimization Models in Excel

- 1. Organize the data for the model in the spreadsheet.
- 2. Reserve separate cells in the spreadsheet for each decision variable in the model.
- 3. Create a formula in a cell in the spreadsheet that corresponds to the objective function.
- 4. For each constraint, create a formula in a separate cell in the spreadsheet that corresponds to the left-hand side (LHS) of the constraint.

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How Solver Views the Model

- Target cell – The cell in the spreadsheet that represents the objective function
- Changing cells – The cells in the spreadsheet representing the decision variables
- Constraint cells – The cells in the spreadsheet representing the LHS formulae for the constraints

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Example Excel Formulation 1/2

- Decision variables:

$$x_s\text{=number of standard hot tubs to produce}$$
$$x_l\text{=number of luxury hot tubs to produce}$$

- Objective function:

$$Max\ 350x_s + 300x_l$$

- Constraints:

$$\begin{aligned} \text{pumps:}\quad & x_s+x_l \le 200 \\ \text{labor:}\quad & 9x_s+6x_l \le 1566 \\ \text{tubing:}\quad & 12x_s+16x_l \le 2880 \end{aligned}$$

$$\text{non-negativity:}\quad x_s \ge 0,\ x_l \ge 0$$

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Example Excel Formulation 2/2

Product Mix Model - Hot Tubs				
Resources Required (per Tub)				
Tub Type	**A**	**B**		
Pumps	1	1		
Labor (Hours)	9	6		
Tubing (Feet)	12	16		
Unit Profit	$350	$300		
Production Mix Decisions				
Hot Tub Type	**A**	**B**		
Number Produced	0	0		
Resource Constraints	**Used**		**Available**	
Pumps	0	<=	200	
Labor (Hours)	0	<=	1,566	
Tubing (Feet)	0	<=	2,880	
Financial Summary				
Hot Tub Type	**A**	**B**	**Total**	
Profit	$0.00	$0.00	$0.00	

NOTE: The <= text entries are not required. They are added to enhance readability.

"Product Mix – Hot Tubs.xlsx"

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Importance of Standalone Commercial Solvers

- Excel even with Gurobi solver is not capable to efficiently deal with large-scale optimization problems (faced in big data problems):
  - The size of the data can make it inefficient (or impossible) to import and manipulate the problem in Excel
  - Fast enough updating of data in Excel may not be possible for dynamic problems
  - Custom algorithms may be required to efficiently solve complex optimization models (e.g., multi-objective, challenging integer, non-linear, or stochastic programming models)
  - Efficient processing of data to generate the model and the output report might not be possible in Excel and requires the use of a programming language, e.g., Python or Java

- Therefore, after solving the first few examples with Solver, we will formulate and solve mathematical programming models using Python and Gurobi!

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Interpreting the Results of Excel's Solver

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Solver and Reports

Product Mix Model - Hot Tubs			
Resources Required (per Tub)			
Tub Type	**A**	**B**	
Pumps	1	1	
Labor (Hours)	9	6	
Tubing (Feet)	12	16	
Unit Profit	$350	$300	
Production Mix Decisions			
Hot Tub Type	**A**	**B**	
Number Produced	0	0	
Resource Constraints	**Used**		**Available**
Pumps	0	<=	200
Labor (Hours)	0	<=	1,566
Tubing (Feet)	0	<=	2,880
Financial Summary			
Hot Tub Type	**A**	**B**	**Total**
Profit	$0.00	$0.00	$0.00

NOTE: The <= text entries are not required. They are added to enhance readability.

"Product Mix – Hot Tubs.xlsx"

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Answer Report

Microsoft Excel 16.0 Answer Report
Worksheet: [Product Mix - Hot Tubs.xlsx]2 Tub Types
Report Created: 8/6/2020 1:49:16 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.031 Seconds.
Iterations: 2 Subproblems: 0
Solver Options
Max Time 100 sec, Iterations 100, Precision 0.000001
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 5%, So

Optimal objective function value

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$21	Profit Total	$0.00	$66,100.00

Optimal values for decision variables

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$12	Number Produced A	0	122	Contin
$C$12	Number Produced B	0	78	Contin

Constraints that are actively limiting (bounding) the problem

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$15	Pumps Used	200	$B$15<=$D$15	Binding	0
$B$16	Labor (Hours) Used	1566	$B$16<=$D$16	Binding	0
$B$17	Tubing (Feet) Used	2712	$B$17<=$D$17	Not Binding	168

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Sensitivity Report

The increase in the optimal objective function value per unit increase in the right hand side (RHS) of the constraints

The increase in the optimal objective function value per unit increase in the constraint on the decision variable (in this example nonnegativity constraints)

Increments and decrements allowed for objective coefficients (in this example unit profits) where the solved optimal values hold

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Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$12	Number Produced A	122	0	350	100	50
$C$12	Number Produced B	78	0	300	50	66.66666667

Increment and decrement for a constraint's RHS where the shadow price holds

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$15	Pumps Used	200	200	200	7	26
$B$16	Labor (Hours) Used	1566	16.66666667	1566	234	126
$B$17	Tubing (Feet) Used	2712	0	2880	1E+30	168

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Why Sensitivity Analysis of Results?

- The curse of optimization: "Being perfectly wrong or approximately correct"
- Model parameters are at best estimates of what might happen in reality (e.g., there is uncertainty in demand, resource requirements, prices, profit margins)
- Therefore, we want to understand how sensitive
  - (i) the value of the objective function is to the changes in input parameters and
  - (ii) the optimal decision is to the changes in input parameters
- Why can't we just solve the problem many times?
  - Takes too long time for large problems
  - Also, the sensitivity analysis (or sensitivity report) gives bounds for the parameters where the results hold rather than just solving for a single instance of the problem

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Limits Report

Microsoft Excel 16.0 Limits Report
Worksheet: [Product Mix - Hot Tubs.xlsx]2 Tub Types
Report Created: 8/6/2020 1:49:17 PM

Lower limit in the objective function value when a decision variable takes a value that results in the least contribution on the objective function

Cell	Objective Name	Value
$D$21	Profit Total	$66,100.00

Cell	Variable Name	Value	Lower Limit	Objective Result	Upper Limit	Objective Result
$B$12	Number Produced A	122	0	23400	122	66100
$C$12	Number Produced B	78	0	42700	78	66100

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Scaling Issues in Excel's Solver

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Scaling Issues 1/3

- For a model to be ***well-scaled***, the constant terms in the model should be of ***"similar magnitude"***:
  - Terms should not be different by *many* ***"orders of magnitude"***
  - E.g. if all the constant terms in the constraints are between 1 and 1,000, but the objective function coefficients are between 0.000001 and 0.000000001, then the model is ***ill-scaled***
- As the solution algorithm runs, intermediate calculations are made that make coefficients larger or smaller.
- If a problem is ill-scaled, memory storage accuracy may force the computer to use approximations of the actual numbers (round-off errors) that may be inaccurate.

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Scaling Issues 2/3

- Most problems can be formulated in a way to minimize scaling errors through appropriate definition of the decision variables. E.g.,
  - Let's say that the objective function coefficients for some model are in $'s and range between $25,000,000 and $100,000,000, RHS values reflect various resource availabilities and vary between 500 and 5000, and constraint coefficients represent unit resource consumptions and vary between 10 and 250.
  - Restating the objective function coefficients in $ millions makes their range (25 to 100) much closer in magnitude to the rest of the coefficients thus making the model "well-scaled."

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Scaling Issues 3/3

- Scaling & Excel's solver:

- Scaling problems sometimes prevents Excel's solver from being able to solve the problem accurately.
  - Solver may complain that the model is nonlinear.
  - This either means you have inadvertently added a nonlinearity, or the problem is ill-scaled.
- If necessary, you can also change the "Constraint Precision" level in Solver's Options dialog box (e.g., from 0.00001 to 0.0000001).

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Summary

- When solving an LP model, the outcome can be one of the following:
  - Unique optimal solution
  - Alternate optimal solutions
  - Unbounded solutions
  - There are no feasible solution

- Simplex is a powerful method to solve LP problems

- Many non-linear optimization problems can be linearized, e.g., when we have:
  - Absolute values
  - Minmax and Maxmin problems (that are created to approximate convex and concave non-linear functions)
  - Ratio constraints

- Dual solution allows to obtain shadow prices to conduct sensitivity analysis

- Use Excel and solver for small problems and Python and Gurobi for large problems

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