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Interpretation of the Risk Tolerance Coefficient in Terms of Maximum Acceptable Loss

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Users (or would-be users) of exponential expected utility often seek a concrete, intuitive meaning for the risk tolerance coefficient (RT) that they can grasp and explain to others easily. This paper shows an interpretation of RT as the maximum loss the decision maker is willing to be exposed to at a stated probability level, regardless of the upside potential. As an example, if you are facing a portfolio of projects having a 1 in 20 chance of total loss and you indicate that L is the maximum loss you would tolerate, then your risk tolerance is $L/3$. Other such examples, which may be better suited to other situations, are presented. In some contexts, this interpretation may be congruent with the way individuals naturally think about their risk-taking propensity, for example, as willingness to be exposed to losses or as in value-at-risk. This interpretation can also be helpful in determining whether assuming exponential utility is adequate for the situation being analyzed, and in eliciting the value of RT. The merits of this approach for thinking about RT are discussed.

Key words: risk tolerance; exponential utility; constant risk aversion; probable loss; downside risk; value-at-risk
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Decision makers can be seduced by the logic and rigor of the expected utility (EU) criterion for choice under risk. Still more attractive is the idea that risk-taking attitude may be quantified by a single parameter (at least, as an approximation) to make coherent evaluations of risks. The difficulties often begin when it comes to specifying an appropriate risk tolerance. Managers do not feel comfortable using a coefficient whose meaning remains fuzzy to them, or that they cannot easily explain to others. Teachers and practitioners of decision analysis are routinely confronted with demands for a concrete, nontheoretical definition of the risk tolerance coefficient. Because risk is often associated with probable loss, relating risk-taking attitude to acceptance of probable loss may be helpful. For example, managers in charge of an investment budget are often able to state the largest loss they could afford being exposed to, however enticing the projects' upside might be. Even in the presence of formidable potential rewards, companies remain keenly aware that there are realistic limits to the losses that could be sustained without compromising stable, continued operations.

The Risk Tolerance Coefficient of Exponential Utility

Various mathematical forms have been proposed to represent utility functions. The exponential form is a popular one: For a risk-averse decision maker, it is defined as $u(x) = -e^{-x/\rho}$, with the risk tolerance coefficient ρ a positive constant. For convenience, call the risk tolerance coefficient in the exponential utility function RT. Pratt (1964) defines the coefficient of risk aversion at wealth level w as $r(w) = -u''(w)/u'(w)$, and he shows that this function captures everything relevant about the utility function and dispenses with anything arbitrary. $r(w)$ essentially determines the risk premium for any small gamble as a function of current wealth w. The exponential utility presents a singular property: $r(w)$ is a constant, equal to $1/\rho$. The property of constant risk aversion turns out to be the great practical advantage of exponential utility because it makes EU analysis more tractable in several ways. Even if decreasing risk aversion may be deemed behaviorally more compelling, exponential utility is often used as an adequate approximation. See Kirkwood (2004) for a detailed analysis.

As another useful feature, the exponential utility function leads to a formula that is both attractive and practical for the certainty equivalent (CE) when gambles follow a normal distribution with mean μ and standard deviation σ: $\mathrm{CE} = \mu - \sigma^2/(2\rho)$ (Howard 1971, p. 538). This formula bypasses the possibly cumbersome calculation of expected utility and, with an exponential utility function, it gives a good approximation of CE if the gamble has a unimodal distribution and the skewness is not too pronounced. This greatly simplifies the estimation of CE in many cases.

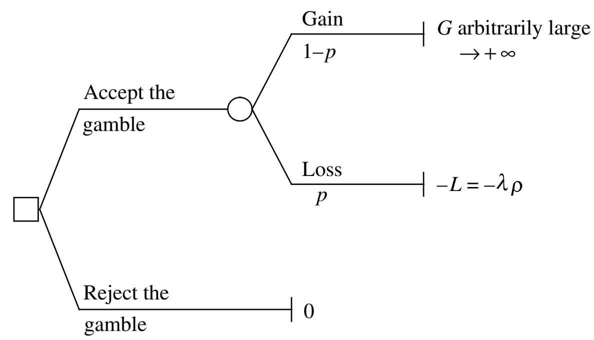
Because $r(w)$ is, by its definition, in the units of w^{-1}, ρ is of the same dimensionality as the payoff variable, that is, expressed in the same units. Thus, if payoffs are expressed in \$-million, ρ is stated as a value in \$-million. But what does it mean, concretely, to have a risk tolerance of, say, \$80,000?

A widely used interpretation is as follows: RT is approximately equal to the largest x for which the decision maker would accept the gamble $\{x, 0.5; -x/2, 0.5\}$ (Howard 1988). If we denote this value by x_{50}, that is, the (largest) value for which the decision maker is indifferent between taking the gamble or not, then the risk tolerance coefficient is[1] $\rho \approx 1.04x_{50}$. Another possibility is to consider the maximum amount, x_{75}, that the decision maker is willing to invest in a gamble having a 75% chance of doubling the ante (ending up with $2x - x$) and 25% chance of total loss of the investment, that is, the largest gamble of the type $\{x, 0.75; -x, 0.25\}$ that the decision maker would be willing to take on (McNamee and Celona 1990). This leads to $\rho \approx 1.10x_{75}$ (the exact value is $\rho = x_{75}/\ln 3$).

The Risk Tolerance Coefficient as Maximum Acceptable Loss

The above interpretations of RT turn out to be quite handy for giving practical meaning to ρ and probing for its value. In the same vein, let us consider a simple prospect that offers the potential of an arbitrarily large reward, and ask what is the most one is willing to risk in such a prospect. Figure 1 illustrates the decision, where the possible loss is expressed as a multiple of the risk tolerance, $\lambda\rho$. Because of the constant risk-aversion property of exponential utility, we can describe the situation in isolation of other (independent) background decisions, and just in terms of net gains and losses, independent of the decision maker's wealth.

Figure 1 **For What Loss Does the Gamble Become Unacceptable?**



The maximum loss for which the gamble is at the limit still acceptable is the value of $\lambda\rho$ such that:[2]

$$\begin{aligned}(1-p)u(+\infty) + pu(-\lambda\rho) &= u(0)\\ (1-p)\cdot 0 + p\cdot(-\mathrm{e}^{-(-\lambda\rho)/\rho}) &= -1\\ p\cdot(-\mathrm{e}^{\lambda}) &= -1\\ \mathrm{e}^{\lambda} &= \frac{1}{p}\\ \lambda &= -\ln p. \qquad (1)\end{aligned}$$

The gamble in Figure 1 is rather extreme: make a fortune or lose all of your stake. Real gambles may present a probability of losing all while offering tantalizingly huge rewards, but with some probability distributed between these extremes. The gamble in Figure 1 is preferred to all gambles that have a probability p (or larger) of losing $\lambda\rho$ (or more), due to first-order stochastic dominance. Thus, Equation (1) defines a frontier of acceptable gambles in the space of loss (as a multiple of risk tolerance) and probability of loss. Any gamble having a higher than p chance of losing more than $\lambda\rho$, where p and λ are related by (1), can never be acceptable to an exponential utility maximizer. Table 1 gives numerical values for Equation (1).

[1] The exact solution of the indifference equation is: $\rho = x_{50}/\ln((3+\sqrt{5})/2)$. $x = 0$ is always a trivial indifference point.

[2] If an infinite upside seems problematic, this assumption is not essential. A large enough positive payoff, of the order of 10 times ρ, will lead to almost the same numerical result.

Table 1 **Multiple of RT that an Individual is Willing to Lose at the Limit as a Function of the Probability of Loss**

Probability of loss p	Maximum acceptable loss (in multiples of RT)
0.50	0.693
0.37	1.000
0.35	1.050
0.33	1.099
0.25	1.39
0.10	2.30
0.05	3.00
0.01	4.61
1/1,000	6.91
1/10,000	9.21
1/100,000	11.51
1/1,000,000	13.82
1/10,000,000	16.12
1/100,000,000	18.42
1/1,000,000,000	20.72

Interpretations

Several kinds of statements relating to RT can be drawn from this. Let us give some examples.

According to Table 1, the absolute highest amount you would be willing to risk in any prospect that presents a 37% chance of loss is exactly your risk tolerance. An approximate but more mnemonic statement could be: "*Your Risk Tolerance is the maximum you would be willing to invest in a venture, no matter how promising it is, where the odds of losing your investment are approximately* 1 *to* 2."

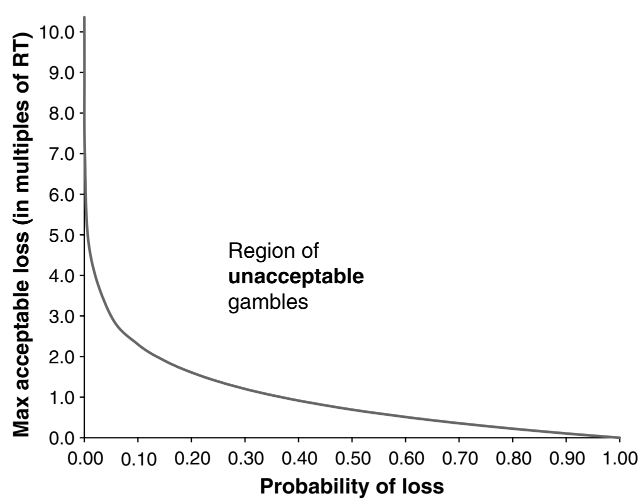
This particular example may be well suited to the case of start-ups in new technologies, such as electronics, biotech, nanotechnologies, clean energy, new materials, and the like. These innovations have staggering potential returns, but also sizeable probabilities of failure. The rewards in case of success are often orders of magnitude larger than the initial investments. However, around 1/3 may be a realistic probability of losing all the money invested in a start-up venture. The focal concern is: What is a reasonable limit on how much to invest, given such a prior chance of failure? The maximum a decision maker is willing to invest in such a project defines approximately his/her RT coefficient. This can help venture capital investors think about their risk tolerance level.

Value-at-risk has become a widespread measure of risk (Duffie and Pan 1997). It designates the maximum loss that can occur at a stated probability level, typically 5% or 1%. The value-at-risk is a fractile of the loss distribution reported as a monetary amount. Financial institutions (investment banks and insurance companies) use it as a benchmark to set the reserves they should provision to face future uncertainties. Suppose the maximum loss exposure acceptable to a company at the 0.05 level is equal to $\mathrm{VaR}_{0.05}$ (that is, it is willing and able to hold a portfolio of risks presenting a 5% chance of that loss), then from Table 1 the company's risk tolerance can be taken as $\rho = 3 \times \mathrm{VaR}_{0.05}$. Of course, the value-at-risk approach to managing risks is generally inconsistent with expected utility. However, it may be instructive to see what kind of RT it implies, if the decisions were to be examined in light of EU.

As a last example, consider very low probabilities of losses in Table 1. Suppose that a company with net total assets worth w_0 indicates that it would never risk a more than 1/1,000 chance of total ruination. Then its risk tolerance should be at most $w_0/6.9$. This is close to one of the rules of thumb proposed by Howard (1988): Set RT equal to about 1/6 the book-value of equity. As can be seen in Table 1, the maximum tolerable loss levels off, that is, it becomes rather insensitive to the probability of loss, as this probability becomes very small.

A graph of the maximum acceptable loss (measured in multiples of RT) as a function of probability of loss is shown in Figure 2. The curve delineates the region of unacceptable gambles.

Figure 2 **Maximum Acceptable Loss at a Given Probability of Loss**



Discussion

The above examples also highlight possible limitations of using exponential utility, and they suggest that caution should be exercised in using RT as in the preceding section. First and foremost, if an exponential utility function is intended only as a local approximation to another "true" utility function, then clearly the approximation will break down when considering very large gambles. Thus, the interpretation of RT proposed here should not entice the decision analyst to use exponential utility in situations where it is likely to be inaccurate. Rather, it should be viewed as a way of probing how large a decision maker's RT coefficient could be, perhaps appropriate only in certain contexts.

Next, the idea that a decision maker should not be willing to stand a 10% chance of losing more than 2.3 times his/her RT for a 90% chance at becoming immensely wealthy seems at first too conservative. On the other hand, this may just indicate that the decision maker needs to adopt a very, *very* large RT to make the statement true, perhaps much larger than what other forms of introspection may suggest. Thus, the present approach may help counterbalance overstatements of risk aversion (that is, understated RT) often observed when considering low to moderate risks as discussed by Smith (2004).

It could also be argued that these examples illustrate that it is unwise to push the thinking about risk-taking willingness to such extreme limit situations, as is done here. Nevertheless, it seems that EU analysis is particularly useful if the stakes are really high in terms of net changes in total wealth. Subjecting risk attitude assessment to "stress testing" by considering extreme cases may be a way to obtain a more robust measure of risk attitude, in the sense that the domain over which the measure was established will then certainly encompass its subsequent domain of application.

The interpretation of RT as maximum acceptable loss hinges on two key features of the exponential utility function: constant risk aversion and boundedness. Boundedness yields the present interpretation of RT, whereas constant risk aversion provides that RT is independent of wealth. Let us briefly examine these two properties.

Constant Risk Aversion

Having one (constant) value to characterize the decision maker's risk aversion over a very wide range of wealth levels may be descriptively inadequate. This is an inherent limitation of the exponential utility form, not necessarily a defect of any particular method of assessing the RT coefficient. Consider again the question "What is the most you are willing to invest in this 'sky-is-the-limit' venture that has a 1/3 chance of failure?" Would your answer to that question change if you were \$5 million wealthier? Maybe yes, but probably so would your answer to the $\{x, 0.5; -x/2, 0.5\}$ gamble in this situation. The former question makes it clear that RT may depend directly on disposable wealth, which may alert the analyst or decision maker to the limitations of using exponential utility. Hence, for practical purposes, a single measure of risk attitude such as RT may be best seen as a snapshot of the individual's present risk preferences, for use on current and foreseeable decisions, not for lifelong use. Risk aversion may need to be reassessed if the individual's or the firm's assets change significantly. Or, the assumption of constant risk aversion may have to be abandoned altogether if a sequence of decisions involves the possibility of massive wealth swings.

Boundedness of Utility

If firms and individuals are unwilling to trade off higher loss exposure against larger gains past a certain potential gain, then the idea that the utility of wealth does have a ceiling somewhere may make sense. Because EU theory does not prescribe a particular utility function, there is nothing irrational about having bounded utility, as long as the decision maker is prepared to act consistently with it.[3]

As shown above, under the exponential utility function, which is bounded above and has constant risk aversion, the maximum acceptable loss at a stated p level is: $L = -\rho \ln(p)$.

The hyperbolic utility function $u(x) = -1/x^{\alpha}$, $\alpha > 0$, for $x > 0$, is also bounded above but has decreasing risk aversion. For an individual with wealth w, the

[3] In fact, Arrow (1974, pp. 63–65), and other authors have shown that utility must be bounded above and below to be able to deal with all continuous distributions, and to avoid paradoxes of the St Petersburg type, in expected utility analysis. See Machina (1982, §2.2) for a brief review and additional references on the issue that unbounded utility may be incompatible with "reasonable" behavior.

maximum acceptable loss L under the hyperbolic utility function at a stated p level is limited to a *fraction* of wealth, specifically: $L = w(1 - p^{1/\alpha})$, regardless of how large the upside may be.

On the other hand, the log utility function $u(x) = \ln(x)$, $x > 0$, which is unbounded, implies that the decision maker should be willing to risk his/her entire wealth at the limit as the upside increases, even for a small probability of gain. For an individual with wealth w, the maximum acceptable loss under the log utility function is given by: $L = w - (w/(w+G)^{1-p})^{1/p}$, which goes to w as G goes to infinity. That is, for any given probability of loss, the acceptable loss can be arbitrarily close to total wealth by choosing G large enough. Some may find this unrealistic. The point is that each form of utility may have its own limitations for representing a decision maker's behavior over all possible choice situations.

Conclusion

The risk tolerance coefficient of an exponential utility function specifies the maximum acceptable loss at a given probability regardless of the potential gain. To the extent that this may be congruent with the way individuals and business managers think about their willingness to take risks, it can help associate a quick, meaningful interpretation to the value of RT. This can facilitate the elicitation, and promote a critical discussion, of risk tolerance and the adequacy of exponential utility as a model for prescriptive decision analysis.

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