CS 334 Fall 2021: Problem Set 9.

Problem 1. (15 points) Let $S = \{< M >: L(M) =< M >\}$. In words, S contains all and only TMs that accept only their own description. Show that neither S nor $\bar{S}$ (the complement of S) is TM-recognizable.
Hint: Assume otherwise, and use the recursion theorem to construct a program that uses the hypothetical enumerator.

Problem 2. (15 points) We have seen several examples of *undecidable* languages of the form

$$\mathcal{L} = \{\langle M\rangle: M\ is\ a\ TM\ whose\ language\ L(M)\ \ satisfies\ property\ P\}$$

Examples of P include: "$L(M) = \phi$", "$|L(M)|$ is regular."

Notice that in each example two conditions are satisfied:

a) If $L(M_1) = L(M_2)$ then $\langle M_1\rangle\ and\ \langle M_2\rangle$ are both in $\mathcal{L}$ or are both not in $\mathcal{L}, and$
b) There are TMs in $\mathcal{L}$ and TMs that are not in $\mathcal{L}$.

In this problem we will prove a powerful result: If P is any property that satisfies conditions a and b, then $\mathcal{L}$ is undecidable. In other words, every interesting property of TMs that depends on the language of the TM is undecidable. Fill in the blanks in the following outline to complete the proof of this result:

1. Let P be any property of the language of a TM that satisfies conditions a and b.
2. Assume that $\mathcal{L}$ is decidable and let TM $\mathcal{D}$ decide $\mathcal{L}$.
3. Since condition b is satisfied by P, let $\langle M\rangle \in \mathcal{L}$ and $\langle N\rangle \notin \mathcal{L}$
4. Now consider the following TM X:

   On input w:

   1. Compute own description $\langle X\rangle$
   2. If $\mathcal{D}\ accepts\ \langle X\rangle$ then ______________________
   3. If $\mathcal{D}\ rejects\ \langle X\rangle\ then$ ______________________

Since, in both cases $X\ contradicts\ \mathcal{D}$, we conclude that ________________________

Problem 3. (15 points) Recall that $A <_p B$ if there is a polynomial-time computable function f such that $w \in A \Leftrightarrow f(w) \in B$.

a. Show that the relation $<_p$ over languages is transitive.
b. Show that if $\forall A, B \in \boldsymbol{P}$, if $B \neq \phi$ and $B \neq \Sigma^*$ then $A <_p B$.
   (Hint: this is easier than it seems since $A \in \boldsymbol{P}$. Now use the definition.)
c. Show that if $\boldsymbol{P} = \boldsymbol{NP}$ then every language, other than ϕ and Σ^*, in $\boldsymbol{P}$ is $\boldsymbol{NP}$**-Complete.** (You can use the result of part (b) even if you didn't prove that.)

Problem 4. (15 points) A triangle in an undirected graph is a cycle of length 3. Show that the language $TRIANGLE = \{\langle G\rangle: graph\ G\ contains\ a\ triangle\}$ is in $\boldsymbol{P}$.