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Homework for

MA 346 Numerical Methods

Spring 2022 — Homework 4

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Exercise 1 (Contraction mapping theorem)

(a) In order to approximate $\sqrt{a}$, $0 < a < 2$, we set $a = 1 - b$, $|b| < 1$, $\sqrt{1-b} = 1 - x$, which yields the fixed-point problem

$$x = g(x) := \frac{1}{2}(x^2 + b).$$

We consider $D := [-|b|, |b|]$. Show that the function $g : D \to \mathbb{R}$ satisfies the assumptions in the contraction mapping theorem. State the contraction factor L.

(b) To approximate the solution of $x + \ln(x) = 0$ consider the following functions:

(i) $g_1(x) := e^{-x}$, and (ii) $g_2(x) := -\ln(x)$.

Investigate whether those functions are suitable to solve the above fixed-point problem by checking the assumptions in the contraction mapping theorem. Either derive a non-trivial interval $D \in \mathbb{R}$ in which the respective function has an unique fixed-point, give the corresponding contraction factor, and show that the respective function can be used, or argue why the function cannot be used. Hint: It can be helpful to first approximately locate where x and $-\ln(x)$ intersect.

Exercise 2
We search for solutions in $[1, 2]$ to the equation

$$x^3 - 3x^2 + 3 = 0.$$

(a) Compute the first iterates $x_0, \ldots, x_5$ of the secant method in $[1, 2]$.

(b) Compute the first iterates $x_0, \ldots, x_5$ using Newton's method with starting value $x_0 = 1.5$.

(c) Compute the first iterates using Newton's method with starting value $x_0 = 2.1$. Sketch the equation graph and try to explain the behavior.

Exercise 3

Consider the following ordinary differential equation (ODE):

$$\frac{du}{dt} = f(u).$$

To solve this numerically, you can use the backward Euler method, for some time step $\Delta t > 0$ (we will talk about this later in the semester):

$$\frac{u^{n+1} - u^n}{\Delta t} = f(u^{n+1}).$$

The numerical result from this process is the sequence $u^0, u^1, u^2, \ldots$, which can be interpreted as an approximation to the exact solution sampled at times $0, \Delta t, 2\,\Delta t, \ldots$.

(a) If $f(u) = au$ for some $a < 0$, derive a formula for u^{n+1} as a function of u^n.

(b) If $f(u)$ is a general nonlinear function, write down a formula for which u^{n+1} is a fixed point, i.e., determine g so that $u^{n+1} = g(u^{n+1})$.

(c) Derive conditions on Δt so that the fixed point iteration:

$$u^{n+1,k+1} = g(u^{n+1,k}), \quad k = 0, 1, 2, \ldots$$

converges. Notice there are two iterations here, one for n and one for k. This problem is asking about the iteration over k, for fixed n!

(d) If $f(u)$ is a general nonlinear function and is differentiable, write down an iteration which determines u^{n+1} from Newton's method.