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Homework for

MA 346 Numerical Methods

Spring 2022 — Homework 5

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Exercise 1 (Space of polynomials $\mathbb{P}_n$)
Let $\mathbb{P}_n$ be the space of functions defined on $[-1, 1]$ that can be described by polynomials of degree less or equal to n with coefficients in $\mathbb{R}$. $\mathbb{P}_n$ is a linear space in the sense of linear algebra, in particular, for $P, Q \in \mathbb{P}_n$ and $a \in \mathbb{R}$, also $P+Q$ and aP are in $\mathbb{P}_n$. Since the monomials $\{1, x, x^2, \dots, x^n\}$ are a basis for $\mathbb{P}_n$, the dimension of that space is $n + 1$.

(a) Show that for pairwise distinct points $x_0, x_1, \dots, x_n \in [-1, 1]$, the Lagrange polynomials $L_k(x)$ are in $\mathbb{P}_n$, and that they are linearly independent, that is, for a linear combination of the zero polynomial with Lagrange polynomials with coefficients α_k, i.e.,

$$\sum_{k=0}^{n} \alpha_k L_k(x) = 0 \text{ (the zero polynomial)}$$

necessarily follows that $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$. Note that this implies that the $(n + 1)$ Lagrange polynomials also form a basis of $\mathbb{P}_n$.

(b) Since both the monomials and the Lagrange polynomials are a basis of $\mathbb{P}_n$, each $P \in \mathbb{P}_n$ can be written as linear combination of monomials as well as Lagrange polynomials, i.e.,

$$P(x) = \sum_{k=0}^{n} \alpha_k L_k(x) = \sum_{k=0}^{n} \beta_k x^k, \tag{1}$$

with appropriate coefficients $\alpha_k, \beta_k \in \mathbb{R}$. As you know from basic matrix theory, there exists a basis transformation matrix that converts the coefficients $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_n)^T$ to the coefficients $\boldsymbol{\beta} = (\beta_0, \dots, \beta_n)^T$. Show that this basis transformation matrix is given by the Vandermonde matrix $V \in \mathbb{R}^{n+1 \times n+1}$ given by

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{pmatrix},$$

i.e., the relation between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in (1) is given by $\boldsymbol{\alpha} = V\boldsymbol{\beta}$. An easy way to see this is to choose appropriate x in (1).

Exercise 2 (Polynomial interpolation and error estimation)
Let us interpolate the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \exp(3x)$ using the nodes $x_i = i/2$, $i = 0, 1, 2$ by a quadratic polynomial $P_2 \in \mathbb{P}_2$.

(a) Use the monomial basis $1, x, x^2$ and compute (numerically) the coefficients $c_j \in \mathbb{R}$ such that $P_2(x) = \sum_{j=0}^{2} c_j x^j$. Plot P_2 and f in the same graph.

(b) Give an alternative form for P_2 using Lagrange interpolation polynomials $L_0(x)$, $L_1(x)$ and $L_2(x)$. Plot the three Lagrange basis polynomials in the same graph.

(c) Compare the exact interpolation error $E_f(x) := f(x) - P_2(x)$ at $x = 3/4$ with the estimate

$$|E_f(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|,$$

where $M_{n+1} = \max_{z\in[0,1]} |f^{(n+1)}(z)|$, $f^{(n+1)}$ is the $(n+1)$st derivative of f, and $\pi_{n+1}(x) = (x - x_0)(x - x_1)(x - x_2)$.

(d) Find a (Hermite) polynomial $P_3 \in \mathbb{P}_3$ that interpolates f and f' in x_0, x_1. Give the polynomial P_3 in the Hermite basis, plot f and P_3 in the same graph, and plot the four Hermite basis functions in another graph.

Exercise 3 (Runge's counter example)
Consider the function

$$f(x) = \frac{1}{1+x^2}, \quad -5 \leq x \leq 5.$$

We wish to approximate f by using Lagrange polynomials with equally spaces nodes. In other words, we choose $x_0, \ldots, x_n$ such that $x_j - x_{j-1}$ is the same for all $j = 1, \ldots, n+1$ and require $f(x_j) = P(x_j)$ for $j = 1, \ldots, n+1$.

a) Implement the Lagrange polynomials for growing n in Matlab. Then, using Lagrange polynomials as a basis, find the interpolant $P \in \mathbb{P}_n$ for $n = 5, 10, 15$ (still in Matlab).

b) Plot the function f and the interpolants from a) in the same graph.

c) Reflect on the result.