Joseph Kim MA 346 Homework 5 4/co1

I pledge my honor that I have abided by the Stevens Honor System - Joseph Kim

1) $x_0, x_1, \dots x_n \in [-1,1]$ $\sum_{k=0}^{n} \alpha_k L_k(x) = 0$ $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$

a)

- $L_k(x) \in P_n$
- Linearly independent $(\alpha_0 = \alpha_1 = \alpha_n = 0)$

If $L_0, \dots, L_k$ is basis for $P_n$, to find unique interpolant [redacted]

[redacted] [redacted]

We show that the only solution to

$$\sum_{k=0}^{n} \alpha_k L_k(x) = 0$$ is when $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$

<u>Notice</u>: when

$$x = x_0 \Rightarrow \sum_{k=0}^{n} \alpha_k L_k(x) = \alpha_0 = 0$$

Thus $\alpha_0$ must be 0.

Likewise

$x = x_1 \Rightarrow \alpha_1 = 0$

$x = x_2 \Rightarrow \alpha_2 = 0$

$\vdots$

$x = x_n \Rightarrow \alpha_n = 0$

That is if we evaluate our linear system of equations @ each $x_i$ $i \in [0, n] \wedge i \in \mathbb{Z}$,

$\alpha_i$ must be 0. Since our system must be true for any x, this suffices to show $\forall\ \alpha_i$ must $= 0$ for our system to be 0.

implies: $L_k(x_i)\alpha_k = f_k$ and $\alpha_0 \cdots \alpha_n = 0$

b) $P(x) = \sum_{k=0}^{n} \alpha_k L_k(x) = \sum_{k=0}^{n} \beta_k x^k \longrightarrow \alpha = V\beta$

$P_{n-1}(x) = \beta_0 x^0 + \beta_1 x^1 + \beta_2 x^2 + \dots \beta_{n-1} x^{n-1} = f_n$

$\hookrightarrow P_{n-1}(x_0) = \beta_0 + \beta_1 x_0 + \beta_2 x_0^2 + \dots \beta_{n-1} x_0^{n-1} = f_0$

$\vdots$

$P_{n-1}(x_n) = \beta_0 + \beta_1 x_n + \beta_2 x_n^2 + \dots \beta_{n-1} x_n^{n-1} = f_n$

equivalent to:
$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{n-1} \end{pmatrix}}_{\beta} = \underbrace{\begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix}}_{\alpha}$$

$P(x) = f(x) = \sum_{k=0}^{n} \alpha_k L_k(x)$

if $x = x_k$, $L_k(x_k) = \frac{x_k - x_i}{x_k - x_i} = 1$

so $\sum_{k=0}^{n} \alpha_k = \sum_{k=0}^{n} \beta_k x^k$

$V\beta = \alpha$

2) $f(x) = e^{3x}$ $\quad x_0 = 0 \quad f(0) = 1$

$[0,1]$ $\quad x_1 = \frac{1}{2} \quad f(\frac{1}{2}) = e^{1.5}$

$\quad x_2 = 1 \quad f(1) = e^{3}$

$$P_2(x) = \sum_{j=0}^{2} c_j x^j$$

$$= c_0 x^0 + c_1 x^1 + c_2 x^2$$

$$= c_0 + c_1 x + c_2 x^2$$

Divided differences.

$$\left.\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{e^{1.5} - 1}{\frac{1}{2} - 0} \\ f[x_1, x_2] &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{e^{3} - e^{1.5}}{1 - \frac{1}{2}} \end{aligned}\right\} \text{1st DD}$$

$$f[x_0, x_1, x_2] = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{2e^3 - 2e^{1.5} - 2e^{1.5} - 2}{1 - 0}$$

$$= 2(e^3 - 2e^{1.5} - 1) \quad \text{2nd DD}$$

$$P_2(x) = 1 + 2(e^{1.5} - 1)(x - \tfrac{1}{2}) + 2(e^3 - 2e^{1.5} - 1)(x - \tfrac{1}{2})(x - 1)$$

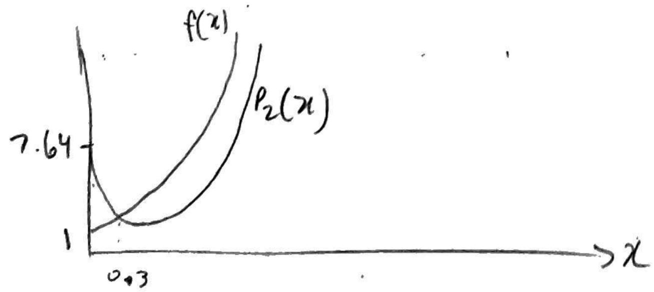
$$\approx 1 + 6.96338x - 3.48169 + 20.24432(x^2 - \tfrac{3}{2}x + \tfrac{1}{2})$$

$$\approx 7.6405 - 23.4031x + 2024432x^2$$

$\hookrightarrow c_0 = 7.6405$

$c_1 = -23.4031$

$c_2 = 20.24432$



b) $\tilde{P}_2(x) \rightarrow$ Lagrange interpolation polynomial

$x_0 = 0 \quad f(x_0) = 1$

$x_1 = \frac{1}{2} \quad f(x_1) = e^{1.5}$

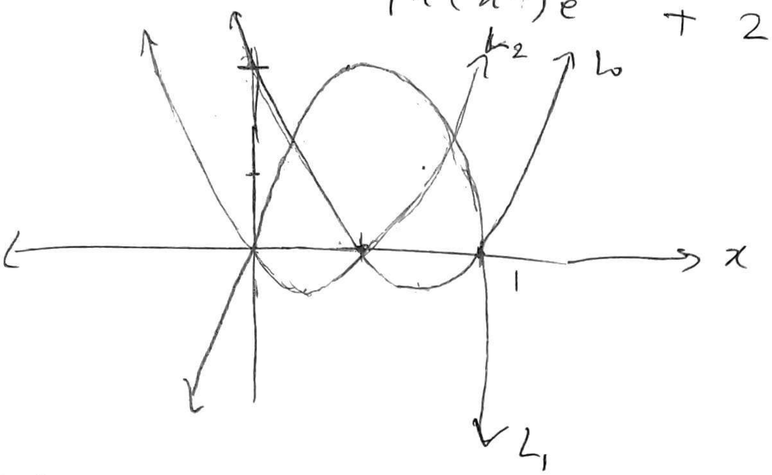
$x_2 = 1 \quad f(x_2) = e^3$

$$\hat{P}_2(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2$$

$$L_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \qquad L_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \qquad L_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_1)(x_2-x_0)}$$

$$\hat{P}_2(x) = \frac{(x-\frac{1}{2})(x-1)}{(-\frac{1}{2})(-1)}(1) + \frac{(x-0)(x-1)}{(\frac{1}{2}-0)(\frac{1}{2}-1)}e^{1.5} + \frac{(x-0)(x-\frac{1}{2})}{(1-\frac{1}{2})(1-0)}e^3$$

$$= 2(x-\tfrac{1}{2})(x-1) - 4x(x-1)e^{1.5} + 2x(x-\tfrac{1}{2})e^3$$




c) $E_f(x) = f(x) - P_2(x)$ @ $x = 0.75$ w/ $|E_f(x)| \leq \frac{M_{n+1}}{(n+1)!}|\pi_{n+1}(x)|$

$M_{n+1} = \max_{z \in [0,1]} |f^{(n+1)}(z)| \qquad \pi_{n+1}(x) = (x-x_0)(x-x_1)(x-x_2)$

$$E_f(0.75) = f(0.75) - P_2(0.75)$$

$$= e^{\frac{9}{4}} - [7.6405 - 23.4031(0.75) - 20.24432(0.75)^2]$$

$$\approx 8.012$$

$n = 3$

$f^{(4)}(x) = 3^4 e^{3x} = 81e^{3x}$

$\hookrightarrow M_{n+1} = 81e^3$

$$|E_f(0.75)| = \frac{81e^3}{4!}|(0.75-0)(0.75-0.5)(0.75-1)|$$

$$\approx 3.1776$$

d) $f(x) = e^{3x}$  Hermite in  $x_0 = 0$  $x_1 = \frac{1}{2}$  $M = 2$, data pts

$f'(x) = 3e^{3x}$

only need $M+1 = 3$ conditions

$n = 1$

$$L_0 = \frac{x - 0}{\frac{1}{2} - 0}$$

$$L_1 = \frac{x - \frac{1}{2}}{0 - \frac{1}{2}}$$

$-2(x - \frac{1}{2})$

$y_0 = 1$

$y_1 = e^{1.5}$

$$P_1(x) = L_0 y_0 + L_1 y_1$$
$$= 2x - 2(x - \tfrac{1}{2}) e^{1.5}$$

$L_0' = 2$

$L_1' = -2$

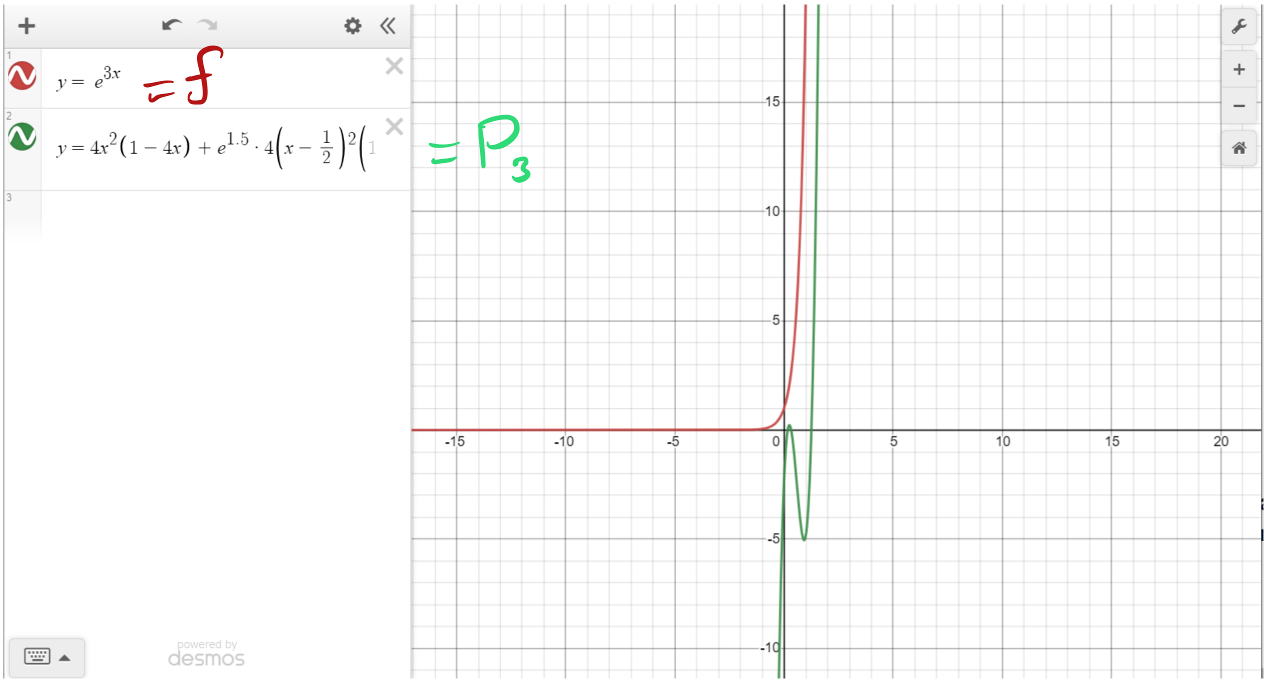
$$H_{1,0} = [1 - 2(x-0)(2)]\, 4x^2$$

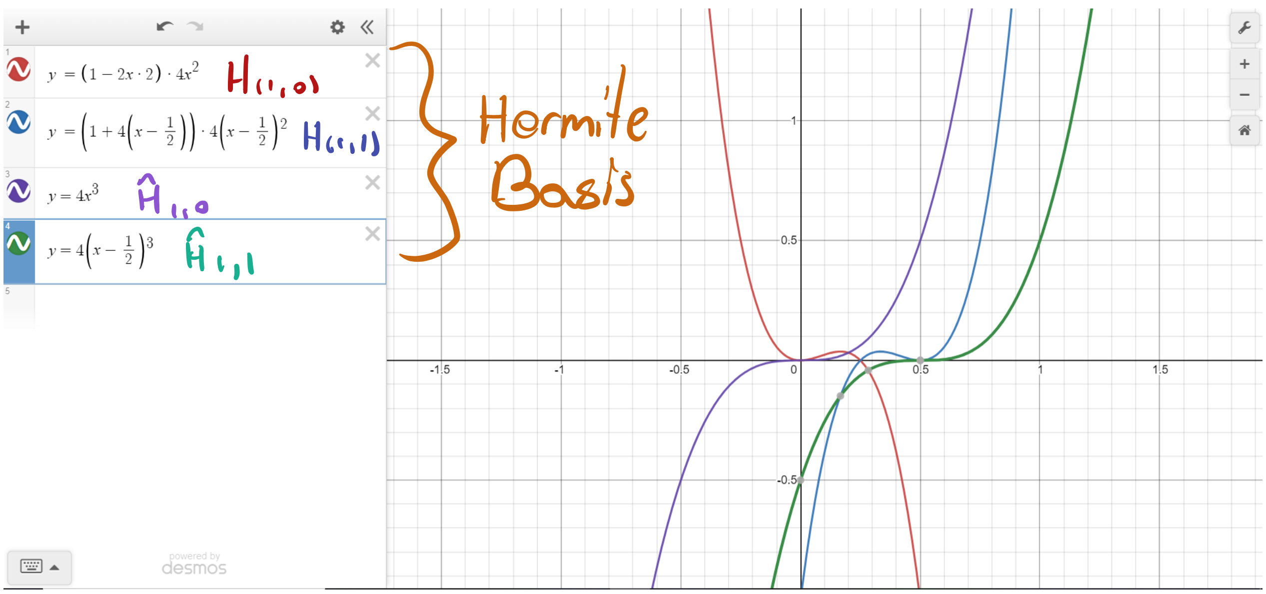
$$H_{1,1} = [1 - 2(x - \tfrac{1}{2})(-2)]\, 4(x - \tfrac{1}{2})^2$$

$$\hat{H}_{1,0} = (x - 0)(4x^2) = 4x^3$$

$$\hat{H}_{1,1} = (x - \tfrac{1}{2})(4)(x - \tfrac{1}{2})^2$$
$$= 4(x - \tfrac{1}{2})^3$$

$$H_3(x) = f(x_0) H_{1,0}(x) + f(x_1) H_{1,1}(x) - f(x_0)\hat{H}_{1,0} - f(x_1)\hat{H}_{1,1}(x)$$
$$= 4x^2[1 - 4x] + e^{1.5}\, 4(x - \tfrac{1}{2})^2 [1 + 4(x - \tfrac{1}{2})] - 4x^3 - e^{1.5}\, 4(x - \tfrac{1}{2})^3$$


$y = e^{3x}$
$= f$
$y = 4x^2(1-4x) + e^{1.5} \cdot 4\left(x - \frac{1}{2}\right)^2($
$= P_3$
powered by desmos


$y = (1 - 2x \cdot 2) \cdot 4x^2$
$H_{1,0}$
$y = \left(1 + 4\left(x - \frac{1}{2}\right)\right) \cdot 4\left(x - \frac{1}{2}\right)^2$
$H_{1,1}$
$y = 4x^3$
$\hat{H}_{1,0}$
$y = 4\left(x - \frac{1}{2}\right)^3$
$\hat{H}_{1,1}$
Hermite Basis
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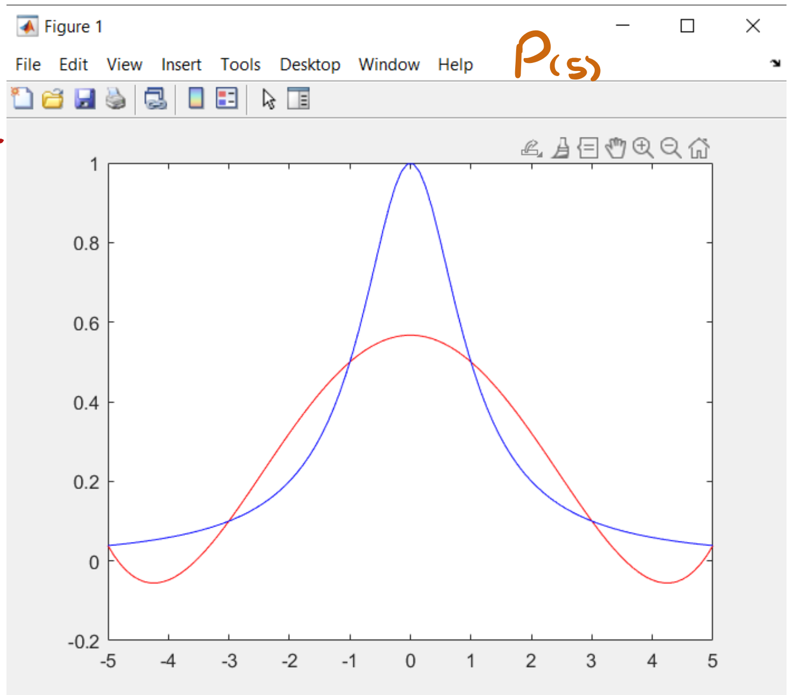
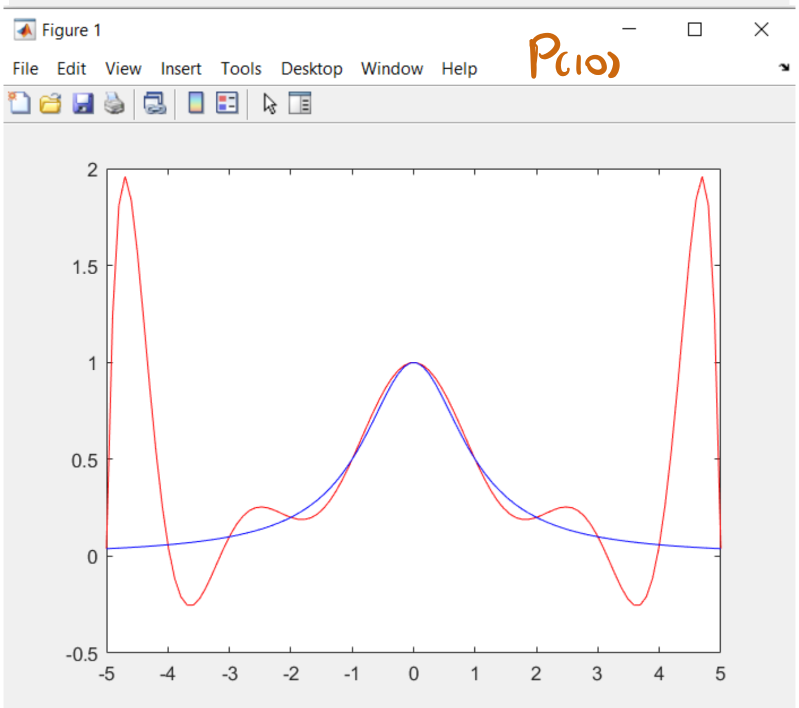
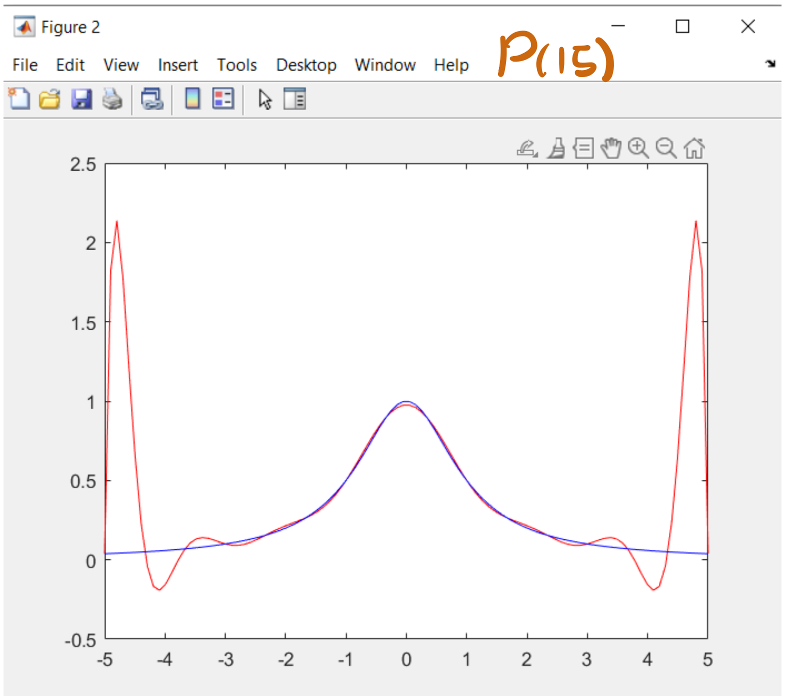
# #3. Part 1 provided below!

## 3.b)

- Interpolant
- Actual






3c) As you can see, as n increases we'd expect our Lagrange polynomial to better apx. our original polynomial $f(x)$.

For values close to $x=0$, we see this to be true. Notice how $P_{15}$ & $P_{10}$ almost perfectly fit our $f(x)$ from ~$[-2,2]$. However $P_5$ is a poor match.

That being said has n increases, the behavior of $P_n$ towards the endpoints becomes rapidly erratic. That is Lagrange fails to apx. $f(x)$, and gets worse at it as n increases, toward the endpoints of our interval

Welcome Guide eval_lagrange.m

Part 3a

```
%MA 346 Homework 5.3 Runge's Counter Example

function output = eval_lagrange(n)
%Original function f(x)

f= @(x) 1/(1+x.^2);

%Gives me n+1 points evenly spaced points between -5 and 5.
x=linspace(-5,5,n+1);

%Computes y_i
y = zeros(n, 1);

for i = 0:n
    y(i+1) = f(x(i+1));
end

%Computes L_i
L=zeros(n, n+1); %holds n coeff arrarys for each L(i)

% Loops through each Li
for i=1:n+1
    counter = 0;
    coef = 1;
    % Loops through each value other than i
    for j= 1:n+1
        if(i ~= j)
           % If first element, simply add in coef
           if (counter == 0)
                L(i, 2) = 1;
                L(i, 1) = -x(j);
           else
               % Multiply by cons
               temp = L(i,:).*(-x(j));
```

```
                temp = L(i,:).*(-x(j));
                % Multiply by x
                for d=n+1:-1:1
                    if(L(i, d) ~= 0)
                        L(i,d+1) = L(i, d);
                        L(i,d) = 0;
                    end
                end
                % Add x*P(n-1) + c*P(n-1)
                L(i,:) = L(i,:) + temp;
            end
            % keep track of Li denominator
            coef = coef * (x(i) - x(j));
            counter = counter + 1;
        end
    end
    % Divide polynomial by Li denominator + multiply by ai
    L(i, :) = (L(i,:) ./ coef) .* y(i);
end

% Get the total sum of each Li coef
total = zeros(n+1, 1);
for i=1:n+1
    for j=1:n+1
        total(i) = total(i) + L(j, i);
    end
end

% Graph result
t = linspace(-5,5);
f = 1./(1+t.^2);
plot(t, polyval(flip(total), t), "Red", t, f, "Blue");
```