Hw #6 MA 232 ~

I pledge my honor I have abided by the Stevens Honor System

#1)

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \end{array}\right] \rightarrow A\vec{x} = \vec{b} \rightarrow A^T \cdot A\vec{x} = A^T\vec{b}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc|c} 5 & 0 & 5 & 1 \\ 0 & 10 & 0 & 0 \\ 10 & 0 & 34 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 5 & 0 & 5 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 24 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1/5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/12 \end{array}\right] \rightarrow$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 17/60 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/12 \end{array}\right] \quad C = 17/60,\ D = 0,\ E = -1/12$$

Check:

$$\text{error} = \underset{\text{Act.}}{\vec{b}} - \underset{\text{Pred.}}{A\vec{x}}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -3/60 \\ 10/60 \\ 17/60 \\ 10/60 \\ -3/60 \end{bmatrix} = \begin{bmatrix} 3/60 \\ -10/60 \\ 43/60 \\ -12/60 \\ 3/60 \end{bmatrix} = \boxed{5/12}$$

$$\hat{q}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\hat{q}_2 = \frac{c_2 - \hat{q}_1(c_2 \cdot \hat{q}_1)}{\|c_2 - \hat{q}_1(c_2 \cdot \hat{q}_1)\|}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\left(\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}(-9)$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ +1 \\ 2 \end{bmatrix}$$

$$\hat{q}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\hat{q}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

#2) Step 1: For convenience, simplify A.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 6 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 + 2R_1 \end{array} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{array}{l} R_1 - R_2/-3 \\ R_2/-3 \\ R_3 + 2R_2 \end{array}$$

$$\rightarrow q_1 \times q_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{vmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix} = \hat{q}_3$$

$$\|\hat{q}_3\| = \sqrt{\frac{4+4+1}{9}} = 1 \checkmark$$

$$q_3 \cdot q_1 = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 0 \checkmark$$

$$q_3 \cdot q_2 = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 0 \checkmark$$

#1

#3) $U = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ Determinant gives <u>scaling</u> factor of Linear transformation.

$|U| = 1 \cdot 2 \cdot 3 = 6$ ← Scale

$|U^{-1}| = 1 \cdot 1/2 \cdot 1/3 = 1/6$ ← Inverse scale

$|U^2| = 1 \cdot 2^2 \cdot 3^2 = 36$ ← Scale^2

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ii) $U = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

$|U| = a \cdot b = ab$

$|U^{-1}| = 1/a \cdot 1/b = 1/ab$

$|U^2| = a^2 \cdot b^2 = a^2 b^2 = (ab)^2$

#4) Suppose A^{-1} DNE. (Assume A is square matrix, & B)

→ $\det(A) = 0$

Since $\det(A \cdot B) = \det(A) \cdot \det(B)$

→ $\forall B$, $\det(A \cdot B) = \det(A) \cdot \det(B) = 0 \cdot \det(B) = 0$

→ $(AB)^{-1}$ does not exist.

#5) $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$ $\det(A) = 5\left(\begin{vmatrix} 4 & 2 \\ -6 & -4 \end{vmatrix}\right) + 1 \begin{vmatrix} -6 & -6 \\ -6 & -4 \end{vmatrix} + 3 \begin{vmatrix} -6 & -6 \\ 4 & 2 \end{vmatrix}$

$= 5(-16+12) + (-12) + 3(12)$

$= 5(-4) - 12 + 36 = \boxed{4}$

Thus A^{-1} exists. → A has dimension 3.

#2

$$A = \begin{array}{c} \begin{bmatrix} -1 & 4 & 2 \\ 5 & -6 & -6 \\ 3 & -6 & -4 \end{bmatrix} \end{array}\begin{array}{l} R_2 \\ R_1 \\ R_3 \end{array} \rightarrow \begin{bmatrix} -1 & 4 & 2 \\ 0 & 14 & -2 \\ 0 & 6 & 2 \end{bmatrix}\begin{array}{l} R_1 \\ R_2 - 5R_1 \\ R_3 - 3R_2 \end{array} \rightarrow \begin{bmatrix} -1 & 4 & 2 \\ 0 & 14 & -2 \\ 0 & 0 & 40/14 \end{bmatrix}\begin{array}{l} R_1 \\ R_2 \\ R_3 - R_2 \cdot \frac{6}{14} \end{array}$$

Since every value in the values on the diagonal are unique, there will be 3 unique eigenvectors spanning the eigenbasis of A. → A is diagonalizable.

#6) $A = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}$

Find eigenvectors:

$$\det(A - \lambda I) = (-4-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 4-\lambda & 2 \end{vmatrix}$$

$$= (4-\lambda)((4-\lambda)^2 - 4) - 2(2(4-\lambda) - 4) + 2(4 - 2(4-\lambda))$$

$$= (4-\lambda)^3 - 4(4-\lambda) - 4(4-\lambda) + 8 + 8 - 4(4-\lambda)$$

$$= (4-\lambda)^3 - 12(4-\lambda) + 16$$

$$= -\lambda^3 + 3 \cdot 4\lambda^2 - 3 \cdot 4^2\lambda + 64 - 48 + 12\lambda + 16$$

$$= -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\begin{array}{r|rrrr} 8 & 1 & -12 & 36 & -32 \\ & & 8 & -32 & 32 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$$(\lambda - 8)(\lambda^2 - 4\lambda + 4) = 0$$

$$(\lambda - 8)(\lambda - 2)^2 = 0$$

$$\lambda = 8, 2$$

#6) $A = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \rightarrow \det(A - \lambda I) =$

$= (-4-\lambda)\begin{vmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & -4-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ -4-\lambda & 2 \end{vmatrix}$

$= (-4-\lambda)((-4-\lambda)^2-4) - 2(-8-2\lambda-4) + 2(4-2(-4-\lambda))$

$= -(4+\lambda)((\lambda^2+8\lambda+16)-4) + 16+4\lambda+8+8+16+4\lambda$

$= -(4+\lambda)(\lambda^2+8\lambda+12) + 48+8\lambda$

$= -((\lambda^3+8\lambda^2+12\lambda)+4\lambda^2+32\lambda+48) + 48+8\lambda$

$= -(\lambda^3+12\lambda^2+44\lambda+48) + 48+8\lambda$

$= -\lambda^3 - 12\lambda^2 - 36\lambda = \lambda(\lambda+6)^2 \rightarrow \lambda = 0, -6$

<u>$\lambda = -6$</u>

$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = -x_2 - x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{matrix} \rightarrow N(A_{\lambda=-6}) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

<u>$\lambda = 0$</u>

$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{matrix} R_3 \\ R_2 \\ R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 + 2R_1 \end{matrix}$

$\rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2/3 \\ R_3 - 3R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow N(A_{\lambda=0}) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

<u>Orthonormalize</u>

$1/\sqrt{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \hat{v}_1 \rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \text{proj}_{v_1}\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) \left(1/\sqrt{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$

$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 1/\sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$\hat{v}_3 = 1/\sqrt{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\hat{v}_2 = \sqrt{6}/2 \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

#3

$$Q^{-1}AQ = D = Q^TAQ$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & -\sqrt{6}/4 & 1/\sqrt{3} \\ -1/\sqrt{2} & -\sqrt{6}/4 & 1/\sqrt{3} \\ 0 & \sqrt{6}/2 & 1/\sqrt{3} \end{bmatrix}$$

$$\Rightarrow 2\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -\sqrt{6}/4 & -\sqrt{6}/4 & \sqrt{6}/2 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -\sqrt{6}/4 & 1/\sqrt{3} \\ -1/\sqrt{2} & -\sqrt{6}/4 & 1/\sqrt{3} \\ 0 & \sqrt{6}/2 & 1/\sqrt{3} \end{bmatrix}$$

$$= 2\begin{bmatrix} -3/\sqrt{2} & 3/\sqrt{2} & 0 \\ 3\sqrt{6}/4 & 3\sqrt{6}/4 & -\frac{6\sqrt{6}}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -\sqrt{6}/4 & 1/\sqrt{3} \\ -1/\sqrt{2} & -\sqrt{6}/4 & 1/\sqrt{3} \\ 0 & \sqrt{6}/2 & 1/\sqrt{3} \end{bmatrix}$$

$$= 2\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{D = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

#4

#7) QDQ^T is possible when A is symmetric.

$\rightarrow$ $b = 1$

ii) $A = SDS^{-1}$ is impossible if eigenbasis has smaller dimension than rank of A. The only way this can happen is if there exists a non-unique eigenvalue for a 2×2.

$$\rightarrow \begin{bmatrix} 2 & b \\ 1 & 0 \end{bmatrix} - I\lambda = (2-\lambda)(-\lambda) - b \qquad \text{Let } b = -1$$

$$= \lambda^2 - 2\lambda - b = 0 \quad \rightarrow$$

$$(\lambda - 1)^2 = 0 \quad \lambda = 1 \ (\times 2)$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \rightarrow \vec{x} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} < \dim(\lambda = 1)$$

this is 2

$\rightarrow$ $b = -1$ makes A nondiagnolizable.

iii) A^{-1} DNE if $\det(A) = 0 \rightarrow b = 0$

#8)*

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - R_2 \cdot 2 \end{matrix} \qquad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{matrix} A & & E_1 \end{matrix}$$

$$\rightarrow \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}}$$

$$\rightarrow A = E^{-1} D (E^{-1})^T \leftarrow \text{Since } A = A^T$$

$$E^{-1}: \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right]$$

$$\rightarrow A = \underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \underset{(E^{-1})^T}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$= (E^{-1}(\sqrt{D}))((\sqrt{D})(E^{-1})^T)$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \boxed{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = C\,C^T}$$

#6