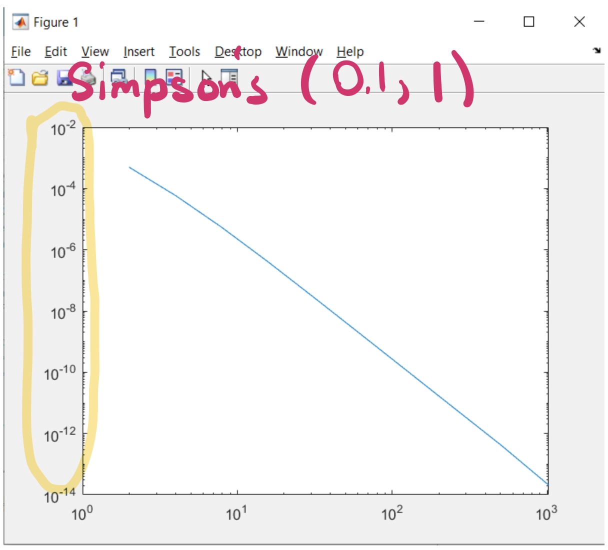
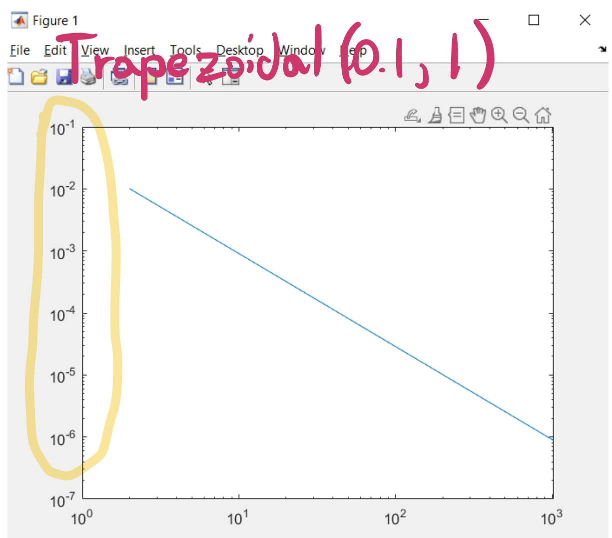
Pledge is on next page, needed to add in graphs.

1a. Code provided in zip.

Simpsons (0.1, 1)



Trapezoidal (0.1, 1)



As you can see by the loglog plots, Simpson's error is significantly smaller than the trapezoidal quadrature error (1 degree smaller as seen by 10^{-12} vs 10^{-6}).

Harris Spahic, Joseph Kim   MA 346 Homework 7   4/24/22

I pledge my honor that I have abided by the Stevens Honor System - Joseph Kim

1) b) $E_n = Cn^k$ need to find C, k

Take for ex, (arbitrarily)

$E_{10} = C\, 10^k \quad \cdots \quad (1)$ 2 eqns

$E_{80} = C\, 80^k \quad \cdots \quad (2)$ 2 unknowns

for Simpson's: $E_{10} \approx 2.291 \times 10^{-6}$

$E_{80} \approx 6.5531 \times 10^{-10}$

from loglog plot,

calculated as slope of loglog

Simpson: $k \approx -3.83$

trapezoid: $k \approx -1.976$

relatively close to calculated k values

$$C = \frac{E_{10}}{10^k}$$

$$= \frac{2.291 \times 10^{-6}}{10^k} \quad \cdots \quad (1)$$

$$C = \frac{E_{80}}{80^k} \quad \cdots \quad (2)$$

$$\frac{2.291 \times 10^{-6}}{10^k} = \frac{6.5531 \times 10^{-10}}{80^k}$$

$k \approx -3.92384$, $C \approx 0.019225$

for trapezoid: $E_{10} \approx 7.203 \times 10^{-4}$

$E_{80} \approx 1.14 \times 10^{-5}$

$$\frac{7.203 \times 10^{-4}}{10^k} = \frac{1.14 \times 10^{-5}}{80^k}$$

Calculated by hand

$k \approx -1.99383$, $C \approx 0.071014$

Actual error - Simpson: $E_n = Cn^k$

$= 0.019225\, n^{-3.92384}$

Trapezoid: $E_n = 0.071014\, n^{-1.99383}$

Theoretical estimates for error

Simpson: $E_n = \dfrac{(b-a)\, h^4}{180} f^{(4)}(\hat{x})$

Trapezoid: $E_n = \dfrac{(b-a)\, h^2}{12} f^{(2)}(\hat{x})$

$h = \dfrac{b-a}{n}$

Using theoretical estimates for Simpson:

$f(x) = x^{\frac{1}{2}}$

$f' = \frac{1}{2} x^{-\frac{1}{2}}$

$f^{(2)} = -\frac{1}{4} x^{-\frac{3}{2}}$

$f^{(3)} = \frac{3}{8} x^{-\frac{5}{2}}$

$f^{(4)} = -\frac{15}{16} x^{-\frac{7}{2}}$

$|f^{(2)}|$ and $|f^{(4)}|$ are decreasing functions so the max value on $(0.1, 1)$ is at $x = 0.1$

$|f^{(2)}(\tilde{x})| = |f^{(2)}(0.1)| \approx 7.9057$

$|f^{(4)}(\tilde{x})| = |f^{(4)}(0.1)| \approx 2964.6353$

$$E_n = \frac{-(b-a)\, h^4 f^{(4)}(\hat{x})}{180}$$

$$= \frac{(b-a)^5}{180 n^4} f^{(4)}(\hat{x})$$

$$E_{10} = \frac{(1-0.1)^5 (2964.6353)}{180 (10^4)}$$

$$\approx 9.7255 \times 10^{-4}$$ vs actual $E_{10} \approx 2.291 \times 10^{-6}$

$$E_{80} = \frac{(0.9)^5 (2964.6353)}{180 (80^4)}$$

$$\approx 2.3744 \times 10^{-7}$$ vs actual $E_{80} \approx 6.5531 \times 10^{-10}$

Using theoretical estimates for trapezoid:

$$E_n = \frac{-(b-a) h^2 f^{(2)}(\hat{x})}{12}$$

$$= \frac{(b-a)^3 f^{(2)}(\hat{x})}{12 n^2}$$

$$E_{10} = \frac{-(0.9)^3 (-7.9057)}{12 (10^2)}$$

$$\approx 0.0048$$ vs actual $E_{10} \approx 7.203 \times 10^{-4}$

$$E_{80} = \frac{-(0.9)^3 (-7.9057)}{12 (80^2)}$$

$$\approx 7.5042 \times 10^{-5}$$ vs actual $E_{80} \approx 1.14 \times 10^{-5}$

Evidently, the theoretical estimates give larger errors (and therefore more conservative values for the error) for theoretical simpson

$E_{10} \approx 9.7255 \times 10^{-4}$

$E_{80} \approx 2.3744 \times 10^{-7}$

$E_n = C n^x$

$$C = \frac{E_{10}}{10^k} \quad \cdots \quad (1)$$

$$= \frac{9.7255 \times 10^{-4}}{10^k}$$

$$C = \frac{E_{80}}{80^k}$$

$$= \frac{2.3744 \times 10^{-7}}{80^k} \quad \cdots \quad (2)$$

$$\frac{9.7255 \times 10^{-4}}{10^k} = \frac{2.3744 \times 10^{-7}}{80^k}$$

$k \approx -4.00000$ vs actual $k \approx -3.92384$,

for theoretical trapezoid

$E_{10} \approx 0.0048$

$E_{80} \approx 7.5042 \times 10^{-5}$

$$\frac{0.0048}{10^k} = \frac{7.5042 \times 10^{-5}}{80^k}$$

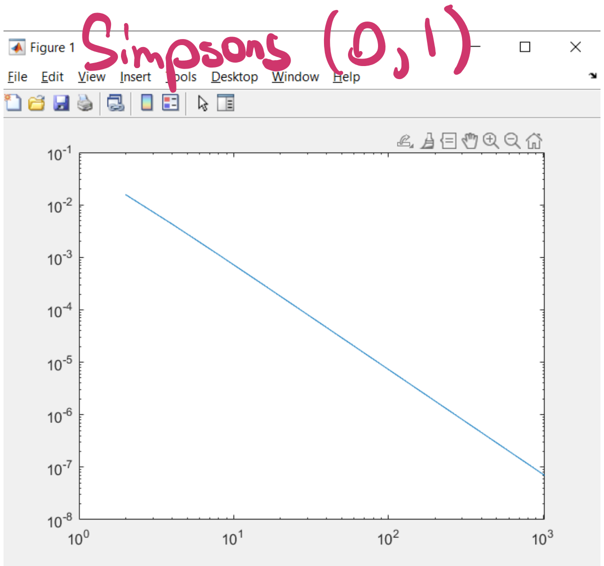
$k \approx -1.99973$ vs actual $k \approx -1.99383$

Clearly the values of x are similar between the theoretical and actual error.

c) The theoretical estimates cannot be computed because the max of $|f^{(2)}|$ and $|f^{(4)}|$ on $(0,1)$ are undefined.

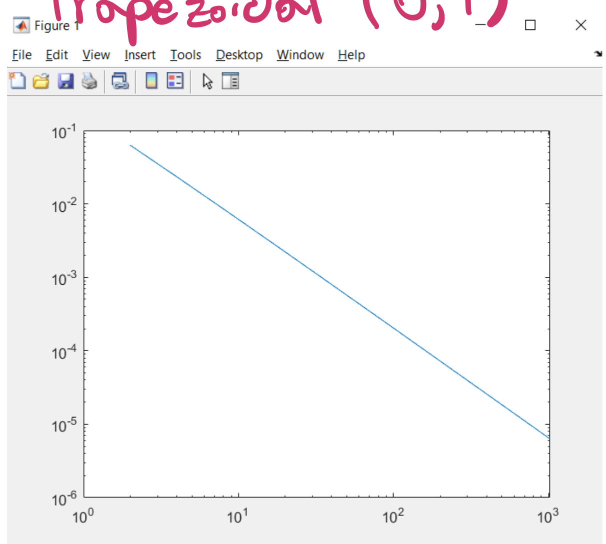
However we can apx a value k by plotting the loglog of our error.

Simpsons (0,1)



$K = -1.4999$

Trapezoidal (0,1)



$K = -1.4766$

Once again, simpson's is a smaller error than trapezoidal, meaning it better aproximates our integral.

2) a) It is given that the Gaussian quadrature is exact for polynomials of degree up to $2n-1$. $x^0, x^1, x^2 \ldots, x^{2n-1}$ are in the space of polynomials up to degree of $2n-1$ and are therefore exact.

b) 3 points $\rightarrow$ $n=3$ so the function is exact on P_5 $(2n-1=5)$

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} a_i f(x_i)$$

$$f(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

then $$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} a_5 x^5 dx + \int_{-1}^{1} a_4 x^4 dx + \int_{-1}^{1} a_3 x^3 dx + \int_{-1}^{1} a_2 x^2 dx + \int_{-1}^{1} a_1 x_1\, dx + \int_{-1}^{1} a_0\, dx$$

by linearity of integrals

$$\int_{-1}^{1} a_5 x^5 = \sum_{i=1}^{3} a_i f(x_i) = a_1 x_1^5 + a_2 x_2^5 + a_3 x_3^5$$ Gaussian quadrature

Applying Gaussian quadrature to the other integrals produces:

$$0 = a_1 x_1^5 + a_2 x_2^5 + a_3 x_3^5 \quad \ldots (1)$$
$$\frac{2}{5} = a_1 x_1^4 + a_2 x_2^4 + a_3 x_3^4 \quad \ldots (2)$$
$$0 = a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 \quad \ldots (3)$$
$$\frac{2}{3} = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \quad \ldots (4)$$
$$0 = a_1 x_1 + a_2 x_2 + a_3 x_3 \quad \ldots (5)$$
$$2 = a_1 + a_2 + a_3 \quad \ldots (6)$$

6 eqns

6 unknowns

Given: $x = [-x_1 \;\; 0 \;\; x_1]$

$w = a = [w_1 \;\; w_2 \;\; w_1]$

Clearly in eqns 1, 3, and 5 the solutions $[a_1 \; a_2 \; a_3] = [w_1 \; w_2 \; w_1]$ and $[x_1 \; x_2 \; x_3] = [-x_1 \; 0 \; x_1]$ are valid (1st and last term cancel, 2nd term is 0)

Now solving eqn 2, 4, 6

$$\frac{2}{5} = w_1(-x_1)^4 + w_2(0)^4 + w_1(x_1^4) \quad \ldots (2)$$
$$\frac{2}{5} = 2w_1 x_1^4$$
$$w_1 = \frac{1}{5x_1^4}$$

$$\frac{2}{3} = w_1(-x_1)^2 + w_2(0)^2 + w_1(x_1)^2 \quad \ldots (4)$$
$$\frac{2}{3} = 2w_1 x_1^2$$
$$\frac{2}{3} = \frac{2x_1^2}{5x_1^4}$$
$$x_1 = \pm\sqrt{\frac{6}{10}} = \pm\sqrt{\frac{3}{5}}$$

$\rightarrow$ Because the solution is $x = [-x_1 \;\; 0 \;\; x_1]$, it doesn't matter whether x_1 is taken as $+\sqrt{\frac{3}{5}}$ or $-\sqrt{\frac{3}{5}}$

$$w_1 = \frac{1}{5x_1^4}$$

$$= \frac{1}{5\left(\sqrt{\frac{3}{5}}\right)^4}$$

$$= \frac{5}{9}$$

$$2 = w_1 + w_2 + w_1 \quad \cdots \quad (6)$$

$$w_2 = 2 - 2w_1$$

$$= 2 - \frac{10}{9}$$

$$= \frac{8}{9}$$

$$[x_1 \quad w_1 \quad w_2] = \left[\sqrt{\frac{3}{5}} \quad \frac{5}{9} \quad \frac{8}{9}\right]$$