Linear Algebra - MA 232 → Hw #5

"I pledge my honor I have abided by the Stevens Honor System" - [illegible]

1. $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

$A\vec{x} = \lambda\vec{x}$ by def of eigen vector

$\rightarrow (A - \lambda I)\vec{x} = \vec{0} \rightarrow$ Find $N(A - \lambda I)$.

$\det\begin{bmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{bmatrix} = \vec{0}$ as causes space to "squish" to lower dim.

$\rightarrow +\lambda + \lambda^2 - 6 = 0 \rightarrow (\lambda+3)(\lambda-2) = 0$

$\boxed{\lambda = -3, +2}$

For $\lambda = -3$,

$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}\vec{x} = \vec{0} \rightarrow 2\vec{x}_1 = -3\vec{x}_2$

dependent ✓

i.e. $\vec{V}_1 = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

For $\lambda = 2$,

$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}\vec{x} = \vec{0} \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\vec{x} = \vec{0} \rightarrow -\vec{x}_1 = -\vec{x}_2$

i.e. $\vec{V}_2 = \begin{bmatrix} +1 \\ 1 \end{bmatrix}$

Note by the nature of eigenvalues, the transformation $\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ will not change its eigenvectors!!! Thus A^2 has eigen vectors $\begin{cases} V_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \\ V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$

Finding A's eigen values:

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$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

Eigen vectors $= \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$
(Ev)

$$\rightarrow \underbrace{\underset{Ev^{-1}}{\begin{bmatrix} 2/5 & 3/5 \\ -1/5 & 1/5 \end{bmatrix}} \underset{A^2}{\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}^2} \underset{Ev}{\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}}}_{\text{Used python numpy to calculate}} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\boxed{\begin{matrix} \lambda = 4 \\ \lambda = 9 \end{matrix}}$$

Notice λ of $A^2 = \lambda^2$ of A.

2. Suppose we have

$$A = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \\ x_{n1} & & \ddots \end{bmatrix} \rightarrow A^T = \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \\ x_{1n} & & \ddots \end{bmatrix}$$

Since $\forall A$, $\det(A) = \det(A^T)$

$\rightarrow \det(A - \lambda I) = \det(A^T - \lambda I)$ as

as $-\lambda I$ term only effects unchanged diagonal of A & A^T.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \rightarrow (1-\lambda)(2-\lambda) \Rightarrow \boxed{\lambda = 1, 2}$

$\lambda = 1$

$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{0} \rightarrow x_1 = -x_2$

$\vec{V}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 2$

$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \vec{x} = \vec{0} \rightarrow x_1 = 0$

$x_2 = x_2 \Rightarrow$ let $x_2 = 1$

$\Rightarrow \vec{V}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

A^T

$\lambda = 1$

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow \begin{matrix} x_1 = \overset{\text{Let } x_1 = 1}{x_1} \\ x_2 = 0 \end{matrix} \rightarrow \vec{V}_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 2$

$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0} \rightarrow x_1 = +x_2$

$\vec{V}_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{\text{Thus } \vec{V}_1 \neq \vec{V}_1' \text{ \& } \vec{V}_2 \neq \vec{V}_2'}$

3. i) $\det(A - I\lambda) = \vec{0}$

$\rightarrow (1-\lambda)(3-\lambda) = 0$

$\rightarrow \lambda = 1, 3$

$\lambda = 1$

$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 3$

$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\rightarrow S^{-1}AS = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

ii) $\det(B - I\lambda) = \vec{0}$

$\rightarrow (1-\lambda)(3-\lambda) - 3 = \vec{0} \rightarrow \cancel{3} - 4\lambda + \lambda^2 - \cancel{3} = \lambda(\lambda - 4) = 0$

$\rightarrow \lambda = 0, 4$

$\lambda = 0$

$\begin{bmatrix} -1 & 1 \\ 3 & 3 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 4$

$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$S = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$

$S^{-1} = \left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 0 & 4 & 1 & 1 \end{array}\right]$

$= \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & 1/4 & 1/4 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3/4 & 1/4 \\ 0 & 1 & 1/4 & 1/4 \end{array}\right]$

$B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

$S^{-1}AS = \begin{bmatrix} -3/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \rightarrow \det(A - I\lambda) = 0$

$\rightarrow \det \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix} = 0$

$= -\left(2 \det \begin{pmatrix} 2 & -\lambda \\ 2 & 0 \end{pmatrix} + (-\lambda) \det \begin{pmatrix} 2-\lambda & 2 \\ 2 & -\lambda \end{pmatrix}\right)$

$= -(2(2\lambda) - \lambda((2-\lambda)\cdot\lambda - 4))$

$= -(4\lambda - \lambda(-2\lambda + \lambda^2 - 4))$

$= -(8\lambda - \lambda^3 + 2\lambda^2) = +\lambda(\lambda^2 - 2\lambda - 8)$

$= +\lambda(\lambda - 4)(\lambda + 2)$

$\rightarrow \lambda = 0, +4, -2$

$\lambda = 0$

$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \vec{x} = \vec{0} \rightarrow \vec{V}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\rightarrow \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

$\lambda = 4$

$\begin{pmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{pmatrix} \vec{x} = \vec{0} \rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\rightarrow \begin{array}{l} x_1 = 2x_3 \\ x_2 = x_3 \end{array} \rightarrow \vec{V}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$\lambda = -2$

$\begin{pmatrix} 4 & 2 & 2 \\ 2 & +2 & 0 \\ 2 & 0 & +2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

$\rightarrow \vec{V}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

#5) i) $R^TR = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

$\det([2]) = 2$

$\det\left(\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\right) = 1$

$\det\begin{pmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{pmatrix} = 2\begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 \\ 4 & 5 \end{vmatrix} + 3\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix}$

$= 2(25-16) - 3(3) + 3(-3)$

$= 18 + 9 - 9 = 0$

$\rightarrow$ $0 \not> 0$ thus not positive definite

ii) $R^TR = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$\rightarrow \det(6) = 6$

$\det(R^TR) = 36 - 25 > 0$ ✓

Is positive definite

iii) $R^TR = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

$\rightarrow \det(1) = 1$ $\quad 1, 9 > 0$ ✓

$\det\begin{pmatrix} 1 & 2 \\ 2 & 13 \end{pmatrix} = 9$

Is positive definite