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Homework for

MA 346 Numerical Methods

Spring 2022 — Homework 6

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Exercise 1 (Numerical Integration)
Consider the following ordinary differential equation (ODE):

$$\frac{du(t)}{dt} = f(t, u(t)) \quad \text{on } [0, T] \text{ with } u(0) = u_0, \tag{1}$$

where $u_0 \in \mathbb{R}$ is given and f being continuously differentiable with respect to both arguments. To find an approximation to the function $u(t)$ that solves (1), one can proceed as follows: First one applies the Fundamental Theorem of Calculus (see first lecture) to obtain:

$$u(t) = u(0) + \int_0^t f(s, u(s))\, ds. \tag{2}$$

We may iterate this procedure as follows:

$$u(t_i) = u(t_{i-1}) + \int_{t_{i-1}}^{t_i} f(s, u(s))\, ds, \tag{3}$$

where $t_i = i\Delta t$, $i = 1, \ldots, n$, and $\Delta t = T/n$; we decomposed the interval $[0, T]$ in n small subintervals $[t_{i-1}, t_i]$ of length Δt.
As a second step one approximates the integral on the right-hand side of (3) with a quadrature formula.

a) Employ the following quadrature rule

$$\int_a^b g(x)\, dx \approx (b - a) g(a)$$

to approximate the integral on the right-hand side in (3) and therefore obtain a numerical method that approximates $u(t_i)$ for $t_i = i\Delta t$, $i = 1, \ldots, n$.

b) Employ the following quadrature rule

$$\int_a^b g(x)\, dx \approx (b - a) g(b)$$

to approximate the integral on the right-hand side in (3) and therefore obtain a numerical method that approximates $u(t_i)$ for $t_i = i\Delta t$, $i = 1, \ldots, n$.

c) What difference do you observe between the numerical methods you obtained in a) and b)?

d) Employ the trapezoidal rule

$$\int_a^b g(x)\,dx \approx (b-a)\frac{1}{2}(g(a)+g(b))$$

to approximate the integral on the right-hand side in (3) and therefore obtain a numerical method that approximates $u(t_i)$ for $t_i = i\Delta t$, $i = 1, \dots, n$.

e) Which method (a), b), or d)) do you think will converge the fastest to the true values of $u(t_i)$ for a decreasing Δt? Give a brief justification.

1a) $u(t_i) = u(t_{i-1}) + f(t_{i-1}, u(t_{i-1})) \cdot \Delta t$

1b) $u(t_i) = u(t_{i-1}) + f(t_i, u(t_{i-1})) \cdot \Delta t$

1c) The numerical method from 1a is equivalent to a (cumulative) left hand reimann-sum of our apx function f().

While 1b is the (cumulative) right hand reimann-sum of our apx. function f().

1d) $u(t_i) = u(t_{i-1}) + \Delta t\frac{1}{2} \cdot \left(f(t_{i-1}, u(t_{i-1})) + f(t_i, u(t_{i-1})) \right)$

1e) Notice 1a & 1b are constant aproximations of $u(t_i)$, whereas 1c is a linear aproximation of $u(t_i)$. Thus since linear systems have a higher order of convergence then constant systems, 1c will converge the fastest!