MA-346 Homework 3

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I pledge my honor that I have abided by the Stevens Honor System.

Exercise 1 (LU decomposition)

Consider the linear system of equations $Ax = b$ with

$$A = \begin{pmatrix} 6 & 4 & 4 \\ 3 & 4 & 6 \\ 6 & 6 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{1}$$

a)

Compute the LU decomposition of A defined in (1) <u>without</u> pivoting.

$$\begin{pmatrix} 6 & 4 & 4 \\ 3 & 4 & 6 \\ 6 & 6 & 4 \end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \begin{pmatrix} 6 & 4 & 4 \\ 0 & 2 & 4 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 6 & 4 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & -4 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & 4 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & -4 \end{pmatrix}$$

b)

Use the just determined LU decomposition of the matrix A defined in (1) to solve the linear system of equations $Ax = b$.

$$Ax = b \iff LUx = b$$

Let $y = Ux$.

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \implies y = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$Ux = y \implies \begin{pmatrix} 6 & 4 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & -4 \end{pmatrix} x = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \implies x = \begin{pmatrix} \frac{1}{4} \\ 0 \\ -\frac{1}{8} \end{pmatrix}$$

c)

Use the just determined LU decomposition of the matrix A defined in (1) to calculate the determinant of A.

Using $\det(AB) = \det A \det B$,

$$\det A = \det L \det U = (1 \cdot 1 \cdot 1)(6 \cdot 2 \cdot -4) = -48$$

Exercise 2 (LU decomposition)

We have given the following matrix A:

$$A := \begin{pmatrix} \alpha & -1 & 0 \\ -1 & \alpha & -1 \\ 0 & -1 & \alpha \end{pmatrix} \qquad \text{with } \alpha \in \mathbb{R}. \tag{2}$$

a)

Compute the LU decomposition of A defined in (2) without pivoting, where L has to be a unit lower triagonal matrix.

$$\begin{pmatrix} \alpha & -1 & 0 \\ -1 & \alpha & -1 \\ 0 & -1 & \alpha \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{1}{\alpha} R_1} \begin{pmatrix} \alpha & -1 & 0 \\ 0 & \frac{\alpha^2-1}{\alpha} & -1 \\ 0 & -1 & \alpha \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + \frac{\alpha}{\alpha^2-1} R_2} \begin{pmatrix} \alpha & -1 & 0 \\ 0 & \frac{\alpha^2-1}{\alpha} & -1 \\ 0 & 0 & \frac{\alpha^3-2\alpha}{\alpha^2-1} \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{\alpha} & 1 & 0 \\ 0 & -\frac{\alpha}{\alpha^2-1} & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{\alpha} & 1 & 0 \\ 0 & -\frac{\alpha}{\alpha^2-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha & -1 & 0 \\ 0 & \frac{\alpha^2-1}{\alpha} & -1 \\ 0 & 0 & \frac{\alpha^3-2\alpha}{\alpha^2-1} \end{pmatrix}$$

b)

Give conditions for α that are sufficient for the existence of a unit lower triagonal matrix L and an upper triagonal matrix U such that $A = LU$ for A defined in (2).

When $\alpha \in \left\{0, \pm 1, \pm\sqrt{2}\right\}$, the LU decomposition cannot exist because they will result in division by 0 or elements in U's diagonal becoming 0.

c)

Use the just determined LU decomposition of the matrix A defined in (2) to compute $\det(A)$.

Using $\det(AB) = \det A \det B$,

$$\det A = \det L \det U = (1 \cdot 1 \cdot 1)\left(\alpha \cdot \frac{\alpha^2-1}{\alpha} \cdot \frac{\alpha^3 - 2\alpha}{\alpha^2-1}\right) = \alpha^3 - 2\alpha, \alpha \notin \{0, \pm 1\}$$

Exercise 3 (LU decomposition with pivoting)

Consider the linear system of equations $Ax = b$ with

$$A = \begin{pmatrix} 0 & 1 & 5 \\ 6 & 1 & 3 \\ 5 & 1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$

a)

Employing partial pivoting (exchanging rows), compute the decomposition $PA = LU$ with a permutation matrix P, a unit lower triagonal matrix L and an upper triagonal matrix U.

Because $a_{11} = 0$, we must perform a pivot. We use

$$PA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 6 & 1 & 3 \\ 5 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 3 \\ 0 & 1 & 5 \\ 5 & 1 & 2 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - \frac{5}{6}R_1 - \frac{1}{6}R_2} \begin{pmatrix} 6 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{5}{6} & \frac{1}{6} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{5}{6} & \frac{1}{6} & 1 \end{pmatrix} \begin{pmatrix} 6 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix}$$

b)

Use the just determined decomposition to solve the linear system of equations $Ax = b$.

$$PAx = Pb$$

Using $PA = LU$, we have

$$LUx = Pb$$

Let $y = Ux$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{5}{6} & \frac{1}{6} & 1 \end{pmatrix} y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

$$y = \begin{pmatrix} 2 \\ 0 \\ \frac{4}{3} \end{pmatrix}$$

$$Ux = y \Longrightarrow \begin{pmatrix} 6 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix} x = \begin{pmatrix} 2 \\ 0 \\ \frac{4}{3} \end{pmatrix}$$

$$x = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}$$