MA 232 HW #3 "I pledge my honor I have abided by the Stevens Honor Code" - [illegible]

#1) $A\begin{bmatrix}2&3\\2&1\\1&0\\0&1\end{bmatrix} = \vec{0} \rightarrow A_{n\times 4}$, A has 2 pivots & for $\vec{c_i} \in A$

$\vec{C}_3 = -2\vec{c}_2 - 2\vec{c}_1$
$\vec{C}_4 = -\vec{c}_2 - 3\vec{c}_1$ } By def. of N(A)

$\rightarrow \boxed{A = \begin{bmatrix}1&0&-2&-3\\0&1&-2&-1\end{bmatrix}}$

Check: $\begin{bmatrix}1&0&-2&-3\\0&1&-2&-1\end{bmatrix}\begin{bmatrix}2&3\\2&1\\1&0\\0&1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \boxed{\vec{0}}$ ◼

#2) $\boxed{A = \begin{bmatrix}1&0&-1/2\\1&3&-2\\5&1&-3\end{bmatrix}}$ since $N(A) = \left(\begin{bmatrix}1\\1\\2\end{bmatrix}\right)$ & $\vec{x} = \begin{bmatrix}1\\1\\2\end{bmatrix} = 1\vec{c}_1 + 1\vec{c}_2 + 2\vec{c}_3 = \vec{0}$

$\rightarrow \begin{bmatrix}1\\1\\5\end{bmatrix} + \begin{bmatrix}0\\3\\1\end{bmatrix} = -2\vec{c}_3 = \begin{bmatrix}-1/2\\-2\\-3\end{bmatrix}$

#3) Notice $\vec{u}_1$ has 0's in its 2nd & 3rd rows,
that means $\vec{u}_1$ cannot be multiplied by any
constant to get $\vec{u}_2$ that has 1 in its second row.
→ Similarly, $\vec{u}_2$ can't make $\vec{u}_3$ b/c of its 3rd row.

<u>However</u>, $4\cdot\vec{u}_3 - \vec{u}_2 - \vec{u}_1 = 4\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix} = \vec{u}_4$

Thus $\vec{u}_1, \vec{u}_2$ & $\vec{u}_3$ are <u>independent</u>
but $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ are <u>dependent</u>.

#4) Rank(A) = 2 iff A has 2 pivots.

$\boxed{c=0,\ d=2}$ is the only way for that to happen, since if $c \neq 0 \rightarrow \vec{c}_3$ has a pivot & there is no way to eliminate a pivot in $\vec{c}_4$.

#5) $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \left[\vec{c}_1 \middle| \vec{c}_2 \middle| \vec{c}_3 \middle| \vec{c}_4 \middle| \vec{c}_5\right]$

$\boxed{C(A) = \left[\vec{c}_2\ \vec{c}_4\right]}$; $\underbrace{\vec{c}_1 = \vec{0},\ \vec{c}_3 = 2\vec{c}_2,\ \vec{c}_5 = 2\vec{c}_4 - 2\vec{c}_2}_{\text{Explanation}}$

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 \end{matrix} \Rightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} R_1 - 3R_2 \\ R_2 \\ R_3 - R_2 \end{matrix} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ -2x_3 + 2x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix}$$

$$\boxed{N(A) = \left(\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)} \text{ from } x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$R(A) = C(A^T) = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} \\ \vec{r}_1 | \vec{r}_2 | \vec{r}_3 \end{array} \Rightarrow \boxed{R(A) = \left[\vec{r}_1\ \vec{r}_2\right]} ;\ \vec{r}_3 = \vec{r}_2 - \vec{r}_1$$

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 - 2R_2 \\ R_4 - 3R_2 \\ R_5 - 4R_2 \end{matrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - R_4 \\ R_3 \\ R_4 \\ R_5 - 2R_4 \end{matrix} \Rightarrow \vec{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

$$\boxed{N(A^T) = \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)} \text{ from } x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

#6) $C(A) = \left(\begin{bmatrix}1\\2\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\3\\2\end{bmatrix} \right)$ as these 2 vectors span S, thus are independent.

We know,

$N(A^T) \Rightarrow A^T\vec{x} = \vec{0}$ implies orthoganality as it is equivalent to dot product.

$\rightarrow$ Let $A^T = \begin{bmatrix}1&2&2&3\\1&3&3&2\end{bmatrix} \rightarrow \begin{bmatrix}1&2&2&3\\0&1&1&-1\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0&5\\0&1&1&-1\end{bmatrix}$

$\rightarrow x_1 = -5x_4$

$x_2 = -x_3 + x_4$

$\rightarrow$ for $A\vec{x}$, $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-5x_4\\-x_3+x_4\\x_3\\x_4\end{bmatrix}$

$= x_3\begin{bmatrix}0\\-1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-5\\1\\0\\1\end{bmatrix}$

Free variables

$\rightarrow N(A^T) = \boxed{\left(\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\1\\0\\1\end{bmatrix} \right) = S^\perp}$

#7)

Since $x_1 + x_2 + x_3 + x_4 = \vec{0}$, $N(P^T) = \left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right)$

Since $S(P)$ is orthoganal to $N(P^T)$, the vector $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ is a basis for $P^\perp$.