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Homework for

MA 346 Numerical Methods

Spring 2022 — Homework 4

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Exercise 1 (Contraction mapping theorem)

(a) In order to approximate $\sqrt{a}$, $0 < a < 2$, we set $a = 1 - b$, $|b| < 1$, $\sqrt{1-b} = 1 - x$, which yields the fixed-point problem

$$x = g(x) := \frac{1}{2}(x^2 + b).$$

We consider $D := [-|b|, |b|]$. Show that the function $g : D \to \mathbb{R}$ satisfies the assumptions in the contraction mapping theorem. State the contraction factor L.

(b) To approximate the solution of $x + \ln(x) = 0$ consider the following functions:

(i) $g_1(x) := e^{-x}$, and (ii) $g_2(x) := -\ln(x)$.

Investigate whether those functions are suitable to solve the above fixed-point problem by checking the assumptions in the contraction mapping theorem. Either derive a non-trivial interval $D \in \mathbb{R}$ in which the respective function has an unique fixed-point, give the corresponding contraction factor, and show that the respective function can be used, or argue why the function cannot be used. Hint: It can be helpful to first approximately locate where x and $-\ln(x)$ intersect.

Exercise 2

We search for solutions in $[1, 2]$ to the equation

$$x^3 - 3x^2 + 3 = 0.$$

(a) Compute the first iterates $x_0, \ldots, x_5$ of the secant method in $[1, 2]$.

(b) Compute the first iterates $x_0, \ldots, x_5$ using Newton's method with starting value $x_0 = 1.5$.

(c) Compute the first iterates using Newton's method with starting value $x_0 = 2.1$. Sketch the equation graph and try to explain the behavior.

Exercise 3

Consider the following ordinary differential equation (ODE):

$$\frac{du}{dt} = f(u).$$

To solve this numerically, you can use the backward Euler method, for some time step $\Delta t > 0$ (we will talk about this later in the semester):

$$\frac{u^{n+1} - u^n}{\Delta t} = f(u^{n+1}).$$

The numerical result from this process is the sequence $u^0, u^1, u^2, \ldots$, which can be interpreted as an approximation to the exact solution sampled at times $0, \Delta t, 2\,\Delta t, \ldots$.

(a) If $f(u) = au$ for some $a < 0$, derive a formula for u^{n+1} as a function of u^n.

(b) If $f(u)$ is a general nonlinear function, write down a formula for which u^{n+1} is a fixed point, i.e., determine g so that $u^{n+1} = g(u^{n+1})$.

(c) Derive conditions on Δt so that the fixed point iteration:

$$u^{n+1,k+1} = g(u^{n+1,k}), \quad k = 0, 1, 2, \ldots$$

converges. Notice there are two iterations here, one for n and one for k. This problem is asking about the iteration over k, for fixed n!

(d) If $f(u)$ is a general nonlinear function and is differentiable, write down an iteration which determines u^{n+1} from Newton's method.

[illegible]

"I pledge my honor that I've abided by the Stevens Honor System"

1. <u>Contraction mapping thm</u>

Let $g \in C([a,b])$ be such that $g(x) \in [a,b] \ \forall x \in [a,b]$ and $g:[a,b] \to [a,b]$ a contraction which means that g is Lipschitz continous with Lipschitz constant $k<1$:

$$|g(x)-g(y)| \leq k|x-y| \quad \forall x,y \in [a,b]$$

$g(x)$ must satisfy ↑ on $D := [-|b|, |b|]$

1) $\frac{1}{2}(x^2+b)$ is the sum of two continous functions, thus it is continous.

2) <u>Endpoints</u> $(|b|<1)$

$$g(-|b|) = \frac{1}{2}((-|b|)^2+b)$$

show

$$\frac{1}{2}(b^2+b) \leq b$$

$\frac{1}{2}b^2 \leq \frac{1}{2}b$

$b^2 \leq b$

$b(b-1) \leq 0 \Rightarrow$ Only true

$0 < b < 1$ ✓

Same for $g(|b|)$ since x^2

check $g'(x)=0 \Rightarrow 2x$ @ 0

But $g(0)=b \in [-|b|, |b|]$

3) $|g(x)-g(y)| \leq k|x-y|$

$\Rightarrow \left| \frac{1}{2}(x^2+b) - \frac{1}{2}(y^2+b) \right| \leq k|x-y|$

$\Rightarrow \left| \frac{1}{2}(x^2-y^2) \right| \leq k|x-y|$

$\Rightarrow \left| \frac{1}{2}(x-y)(x+y) \right| \leq k|x-y|$

$\Rightarrow \frac{1}{2}|(x-y)||(x+y)| \leq k|x-y|$

Let $k := |b|$ for chosen $|b|$.

$\Rightarrow \frac{1}{2}|x-y||x+y| \leq |b||x-y|$

$\Rightarrow \frac{1}{2}|x+y| \leq |b|$

$\Rightarrow |x+y| \leq 2|b|$

But since $x, y \in [-|b|, |b|]$

$\max(|x+y|) = 2|b|$

$|x+y| \leq 2|b| \leq 2|b|$ ✓

Thus $k = |b| = L$ is a valid contraction factor.

1b.

Show $g_1(x) = e^{-x}$

Take $D := [0.5, 0.7]$

$\Rightarrow$ 1) $g_1(x) \in C(0.5, 0.7)$

$\Rightarrow$ 1) $g_1(x) \in C[0.5, \ldots$

2) $g_1(0.5) = e^{-1/2} = 0.60653$

$g_2(0.7) = e^{-0.7}$

+ No zero $g'(x) = -e^{-x}$
in interval

Thus, $e^{-1/2}$ is max.

3) Notice $d/dx \, e^{-x} = -e^{-x}$

$\Rightarrow |-e^{-x}| = |e^{-x}| < k$

For $x \in (0.5, 0.7)$, max $|e^{-x}|$
is $e^{-1/2} < k$.

Thus $0.8 = k$ $\blacksquare$

Show $g_2(x) = -\ln(x)$

By graphing we assume can't fix point iterate.

$$|-\ln(x)+\ln(y)| > k\,|x-y|$$

Let's take the case that $x > y$.

$\rightarrow xe^x > ye^y$

$\rightarrow x/y > \dfrac{e^y}{e^x} \rightarrow \ln\left(\dfrac{x}{y}\right) > y-x$

$$|\ln(y/x)| > k\,|x-y|$$

Well notice $|y-x| = |x-y|$

$\rightarrow |-\ln(x/y)| > k\,|y-x|$

$\rightarrow$ Since $x > y \rightarrow x/y > 1 \rightarrow \ln(x/y) > 0$

$\rightarrow \ln(x/y) > k\,|y-x|$

We can do same arguement for $y > x$ w.l.o.g. $\blacksquare$

#2) $x \in [1,2]$

s.t $x^3 - 3x^2 + 3 = 0$

a. Secant method

$$p_{n+1} = p_n - \frac{f(p_n)}{\left(\frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}\right)}$$

$\rightarrow$ Take $p_0 = 1$ $p_1 = 2$

$$\rightarrow P_3 = 2 - \frac{-1}{\left(\frac{-1 - 1}{2 - 1}\right)}$$

$$= 2 + \left(\frac{1}{-2/1}\right) = \boxed{3/2}$$

$$\rightarrow P_4 = 3/2 - \frac{-0.375}{\left(\frac{-0.375 + 1}{3/2 - 2}\right)}$$

$$= 3/2 + \frac{0.375}{\left(\frac{0.625}{-0.5}\right)}$$

$$= 1.2$$

$$\rightarrow P_5 = 1.2 - \frac{0.408}{\left(\frac{0.408 + 0.375}{1.2 - 1.5}\right)}$$

$$= 1.2 + \frac{0.408 \cdot 0.3}{0.408 + 0.375}$$

$$= 1.35632$$

b) $x_0 = 1.5$

$$f'(x) = 3x^2 - 6x$$

$$\rightarrow x_1 = 1.5$$

$$x_2 = 1.5 + \frac{0.375}{-2.25}$$

$$= 1.5 - 0.166$$

$$= 1.33\overline{3}$$

$$x_3 = 1.\overline{33} - \frac{0.037}{-2.\overline{66}}$$

$$= 1.33 + 0.0138$$

$$= 1.3472$$

$$x_4 = 1.347 - \frac{0.037037}{-2.6667}$$

$$= 1.347 + 0.000741$$

$$= 1.3472963$$

$$x_5 = 1.3472963 - \frac{5.72477 \times 10^{-9}}{-2.63155}$$

$$= 1.3472963 + 2.1699 \times 10^{-9}$$

$$= 1.3472963553$$

c) $x_0 = 2.1$

$$x_1 = 2.1 + \frac{0.968}{0.6299}$$

$$= 2.1 + 1.580$$

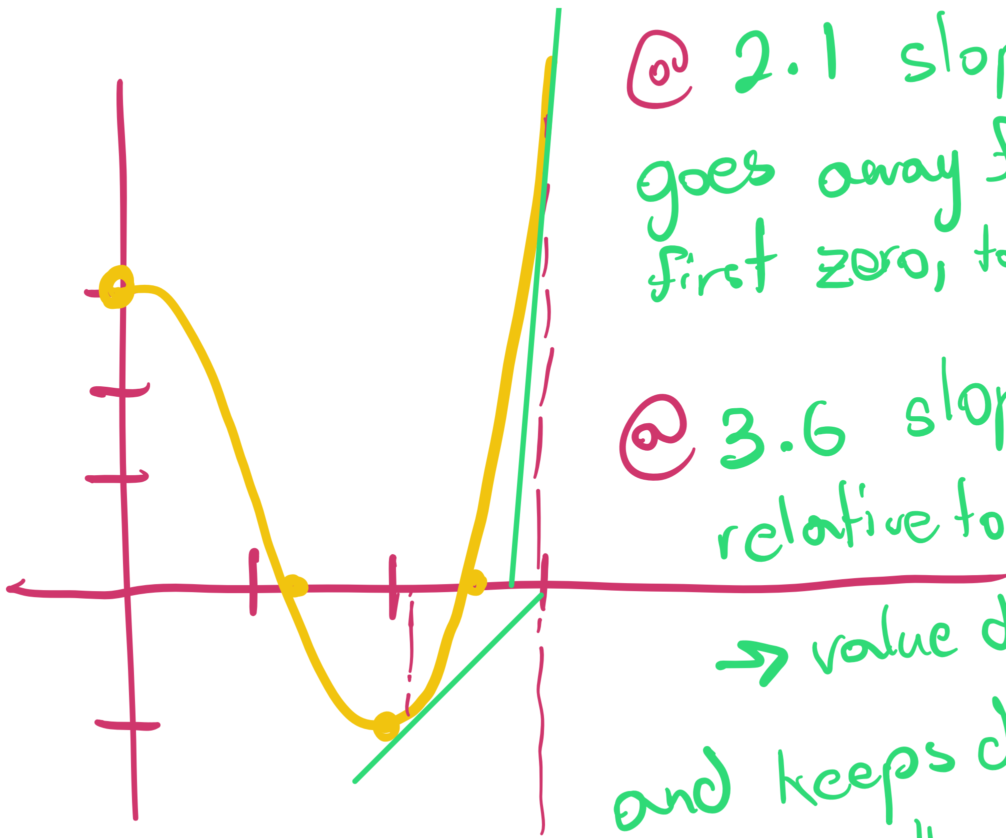
$$= 3.63809$$

$$x_2 = 3.63809 - \frac{11.445}{17.878}$$

$$= 3.63809 - 0.64018$$

$$= 2.9979$$

As this keeps going x_n converges to 2.532088....



@ 2.1 slope goes away from first zero, toward 3.6

@ 3.6 slope negative relative to x: 3.6

→ value decreases and keeps decreasing to zero thus forth

The key here is we don't know what zero we'll find with Newton's method.

#3) $$\frac{u^{n+1} - u^n}{\Delta t} = f(u^{n+1})$$

$$\rightarrow \frac{u^{n+1} - u^n}{\Delta t} = a u^{n+1}$$

$$\rightarrow u^{n+1} - u^n = a u^{n+1} \cdot \Delta t$$

$$\rightarrow u^{n+1} - a u^{n+1} \Delta t = u^n$$

$$\rightarrow u^{n+1}(1 - a\Delta t) = u^n$$

$$\rightarrow u^{n+1} = \frac{(1 - a\Delta t)}{u^n}$$

#3b) $\frac{u^{n+1}-u^n}{\Delta t} = f(u^{n+1})$

$\rightarrow u^{n+1} - u^n = \Delta t f(u^{n+1})$

$\rightarrow u^{n+1} = \Delta t f(u^{n+1}) + u^n$

Then since we replace $u^{n+1} = g(u^{n+1})$ to be a fixed point,

$\rightarrow g(x) = \Delta t f(x) + u^n$

#3c) ???

#3d) $u^{n+1} - \Delta t f(u^{n+1}) - u^n = 0$

$\rightarrow g(x) = \Delta t f(x) + u^n$

where $x = u^{n+1}$

Let $u_1^{n+1}, \ldots, u_k^{n+1}, \ldots$ represent iterations of Newton's method at value u^{n+1}, since n is const.

$$\rightarrow \quad u_{k+1}^{n+1} = u_k^{n+1} - \frac{g(u_k^{n+1})}{g'(u_k^{n+1})}$$