An Introduction to the equivariant Tamagawa Number Conjecture

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Introduction/Structure

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1. Determinant Modules
2. (A very small amount of) Algebraic K-Theory
3. BSD & the eTNC
4. Numerical Examples

References etc.

The first part of this talk and some of the general philosophy are based on notes of David Burns.

The latter part is all based on Werner Bley's paper 'Numerical evidence for the equivariant Birch and Swinnerton-Dyer conjecture, part I'.

Determinant modules

We give a brief introduction to *determinant modules*, restricting to the case of free R-modules M of finite rank r:

Definition

$$[M]_R = \bigwedge^r M \cong R \quad \text{and} \quad [M]_R^{-1} = \mathrm{Hom}_R([M]_R, R).$$

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This extends to finitely generated R-modules M for $R = \mathbb{Q}[G], \mathbb{C}[G]$, etc. for finite groups G by writing

$$R = \prod_i F_i \quad \text{and} \quad M = \bigoplus_i M_i$$

for F_i fields and M_i a free F_i-module, and taking

$$[M]_R = \prod_i [M_i]_{F_i}.$$

Determinant modules

This construction has the following properties:

1. Given $\mathcal{E} : 0 \to M \to N \to P \to 0$, we obtain canonical

$$\iota(\mathcal{E}) : [N]_R \xrightarrow{\sim} [M]_R \otimes_R [P]_R.$$

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3. Given $f : M \xrightarrow{\sim} N$, we obtain canonical isomorphism

$$t(f) : [M]_R \otimes_R [N]_R^{-1} \xrightarrow{[f]_R \otimes 1} [N]_R \otimes_R [N]_R^{-1} \xrightarrow{\mathrm{ev}_N} R,$$

   where $[f]_R$ is the map induced by f.

Determinant modules

To explicate the link to determinants, we note that for $f : M \xrightarrow{\sim} N$ the following commutes

$$\begin{tikzcd}
[M]_R[N]_R^{-1} \arrow[r, "t(f)"] \arrow[d, "\beta_M \otimes \beta_N"'] & R \\
R \otimes R \arrow[r, "\mathrm{id}"] & R \arrow[u, "\times \det(\Phi)"']
\end{tikzcd}$$

where the maps $\beta_\bullet$ are given by a choice of basis and Φ is the matrix of f with respect to the chosen bases.

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We can do pretty much the same thing for other rings (e.g. $R = \mathbb{Z}[G]$), although we lose the fact that $[M]_R$ is a free rank one R-module in general.

Some algebraic K-theory

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For a ring R, we denote by $K_0(R)$ the free abelian group on isomorphism classes $[M]$ of projective R-modules modulo the relations

$$[M] = [N] + [P]$$

whenever there exists

$$0 \to N \to M \to P \to 0.$$

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This is an abelian group whose elements can be represented by tuples $[P, \phi, Q]$ up to isomorphism, where P and Q are projective $\mathbb{Z}[G]$-modules and

$$\phi : P \otimes_{\mathbb{Z}[G]} F[G] \xrightarrow{\sim} Q \otimes_{\mathbb{Z}[G]} F[G].$$

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We have a canonical homomorphism

$$\delta : \zeta(F[G])^\times \to K_0(\mathbb{Z}[G], F).$$

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This works by mapping $[P, \phi, Q]$ to

$$((\prod_p [P_p]_{\mathbb{Z}_p[G]}[Q_p]^{-1}_{\mathbb{Z}_p[G]}, [P \otimes \mathbb{Q}[G]]_{\mathbb{Q}[G]}[Q \otimes \mathbb{Q}[G]]^{-1}_{\mathbb{Q}[G]}, \prod_p \theta_p), t(\phi)).$$

BSD & the eTNC

Now fix an elliptic curve $E/\mathbb{Q}$, a Galois extension $K/\mathbb{Q}$ with Galois group G and set $S = \{\text{bad primes}\} \cup \{\text{ramified primes}\}$. We state the eTNC for E/K.

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$$\Xi = [E(K) \otimes_{\mathbb{Z}} \mathbb{Q}]^{-1}_{\mathbb{Q}[G]}[(E(K) \otimes_{\mathbb{Z}} \mathbb{Q})^*]_{\mathbb{Q}[G]}[H_B^+]^{-1}_{\mathbb{Q}[G]}[H_{\mathrm{dR}}/H^0_{\mathrm{dR}}]_{\mathbb{Q}[G]}.$$

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We have an isomorphism $(E(K) \otimes_{\mathbb{Z}} \mathbb{Q})^* \xrightarrow{\sim} E(K) \otimes_{\mathbb{Z}} \mathbb{Q}$. Combining with the period isomorphism (from Deligne's conjecture), we obtain

$$\theta_\infty : \Xi \otimes_{\mathbb{Q}[G]} \mathbb{R}[G] \to \mathbb{R}[G].$$

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from which we obtain

$$R\Omega := ((\prod_\ell [R\Gamma_c(\mathbb{Z}_{S_\ell}, T_\ell)], \Xi, \prod_\ell \theta_\ell^{-1}), \theta_\infty) \in K_0(\mathbb{Z}[G], \mathbb{R}).$$

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Our hope is that this element will encode data about the leading terms at $s = 1$ of $L(E/\mathbb{Q}, \chi, s)$ for characters χ of G. In particular,

Conjecture (eTNC)

$$T\Omega := R\Omega + \delta((L^*(E/\mathbb{Q}, \chi, 1))_{\chi \in Irr(G)}) = 0.$$

Numerical evidence

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$$u := \frac{R}{\Omega Reg} \cdot (L^*(E/\mathbb{Q}, \chi, 1))_\chi \in \zeta(\mathbb{Q}[G])^\times.$$

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For ψ an abelian character and $\chi = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q})} \psi^\sigma$, we consider the $e_\chi\mathbb{Q}[G]$-vector space $e_\chi(E(K) \otimes_{\mathbb{Z}} \mathbb{Q})$.

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$$\mathrm{Reg}_\chi = (\det(\langle P_i, e_{\overline{\psi}^\sigma} P_j^* \rangle))_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})}.$$

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We now take the following hypotheses:

- The rationality conjecture holds;
- $Ш(E/K)$ is finite;
- All the ramified primes in K are primes of good reduction for E.

Numerical evidence

Assuming these, Bley shows:

Theorem

There exists a(n explicit) finite set of primes HP such that, for $\ell \notin HP$,

$$T\Omega_\ell = 0 \in K_0(\mathbb{Z}_\ell[G], \mathbb{C}_\ell) \iff \text{u has support in HP.}$$

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Here, $u = (u_1, \ldots, u_r) \in \zeta(\mathbb{Q}[G])^\times$ 'has support in HP' if $(u_i, p) = 1$ for all i and all $p \notin HP$.

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HP should contain the primes dividing $|Ш(E/K)|$, but we have no good way to compute this.

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HP should contain the primes dividing $|\text{Ш}(E/K)|$, but we have no good way to compute this. Hence we will have to assume BSD for our numerical computations.

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We use:

Theorem (*Longo, Tian–Zhang*)

Suppose E/K is a modular elliptic curve over a totally real number field. Then

$$\mathrm{ord}_{s=1} L(E/K, s) = 0 \implies rk(E/K) = 0.$$

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Moreover, from BSD we predict $|\text{Ш}(E/K)| = 5^3$. We take $HP = \{2, 3, 5, 11, 229\}$.

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Hence we have verified the eTNC for E/K away from $\ell = 2, 5$.

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- I lied to you! To properly do all this, we need a more nuanced definition of determinant modules (Picard categories, virtual objects, and more...).

Closing remarks

- This computational approach can also tell us about the Galois module structure of $\text{Ш}(E/K)$.
- I lied to you! To properly do all this, we need a more nuanced definition of determinant modules (Picard categories, virtual objects, and more...).
- To state the eTNC for general motives, we need to replace $E(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ by motivic cohomology.