

# Wild conductor exponents of curves

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# Wild conductor exponents

We can provably (and practically) compute  $n_{C,p,\text{wild}}$  in the following cases:

- $C$  an elliptic curve.
- $C$  hyperelliptic,  $p > 2$  or low genus.
- $C$  superelliptic,  $p$  prime to exponent.
- $C$  non-hyperelliptic of genus 3, 4 or 5,  $p > 2$ .
- $C$  plane quartic with a rational point.

## Theorem ( $M^2D^2$ )

Let  $C/\mathbb{Q} : y^2 = f(x)$  be a hyperelliptic curve,  $p > 2$ :

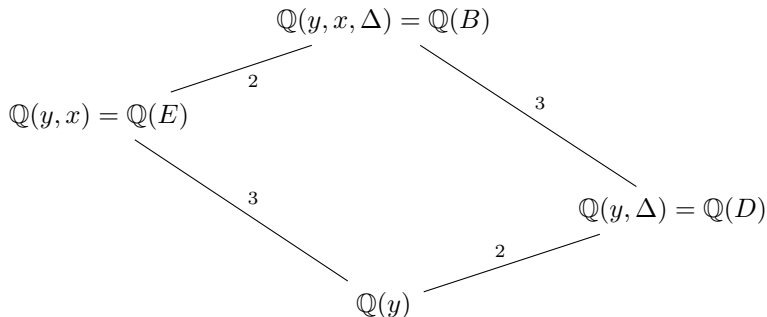
$$n_{C,p,\text{wild}} = \sum_{r \in R/G_{\mathbb{Q}_p}} v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p},$$

where  $R$  is the set of roots of  $f$  over  $\bar{\mathbb{Q}}_p$ .

Idea: generalise this formula to degree  $d$  covers of  $\mathbb{P}^1$ ,  $p > d$ .

# Toy example

Consider an elliptic curve  $E/\mathbb{Q} : y^2 = x^3 + ax + b$  with  $a \neq 0$ . We consider the degree 3 cover to  $\mathbb{P}^1$  and fill in the Galois diagram:



where  $\Delta^2 = \text{disc}_x(x^3 + ax + (b - y^2))$ . It turns out  $E[3] \cong \text{Jac}(D)[3]$  and  $E$  and  $D$  have the same wild conductor exponents at  $p \neq 3$ .

Replacing  $E \rightarrow \mathbb{P}^1$  by a degree 3 simply branched cover  $C \rightarrow \mathbb{P}^1$ ,  $\text{Jac}(D)[3] \cong \text{Jac}(C)[3] \oplus \text{stuff as wild inertia reps for } p \neq 3$ .

# Main theorem

## Theorem

Let  $C \rightarrow \mathbb{P}^1$  be a degree  $d$  simply branched cover. For  $p > d$ ,

$$n_{C,p,\text{wild}} = \sum_{r \in R/G_{\mathbb{Q}_p}} v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p},$$

where  $R$  is the set of  $\bar{\mathbb{Q}}_p$ -branch points.

## Idea of proof

Unramified cyclic covers + Galois theory + representation theory

# Perturbations

For  $g \in \mathbb{Q}_p[t]$ , write

$$w_p(g) = \sum_{r \in R/G_{\mathbb{Q}_p}} m(r) \cdot (v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p}),$$

where  $R$  is the set of  $\bar{\mathbb{Q}}_p$ -roots of  $g$  and  $r$  is a root with multiplicity  $m(r)$ .

## Lemma

For  $p^{\text{th}}$ -power free  $g$ , the quantity  $w_p(g)$  is locally constant.

Likewise, wild conductor exponents are locally constant. If we can perturb a degree  $d$  cover  $\pi : C \rightarrow \mathbb{P}^1$  to obtain  $\tilde{C} \rightarrow \mathbb{P}^1$  simply branched, then we can read off wild conductor exponents from the branch locus of  $\pi$ .

## Example

Suppose  $C : f(x, y) = 0$  is a smooth affine model for a curve and  $\deg_x f = d$ . Then, for  $p > d$ , we have  $n_{C,p,\text{wild}} = w_p(\text{disc}_x f)$ .