Motivic Pieces of Curves

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Abstract

For a curve C with automorphism group G over a number field, we associate to representations ρ of G certain L-functions $L(C, \rho, s)$ and study their behaviour at $s = 1$. In particular, we suggest an analogue of the conjectural BSD rank formula for Artin-twists and give some discussion on periods associated to these L-functions. Lastly, we explain how one might compute these L-functions numerically.

A curve, with Galois field of functions,
Over a number field, where we lay our scene,
From additive pieces guides to junctions,
Where covert factors make covert series unclean
...
From forth the loins of Tate's module sometimes,
A set of star-cross'd spaces split apart;
Who with dark and heinous, yet intricate, crimes,
Do in segregation sever the L-series at the heart.
...

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Contents

1 Introduction

Given an elliptic curve $E/\mathbb{Q}$, we associate to it a Hasse–Weil L-function defined via its Euler product

$$L(E,s) = \prod_{p \mid \Delta_{E/\mathbb{Q}}} (1 - a_p p^{-s})^{-1} \cdot \prod_{p \nmid \Delta_{E/\mathbb{Q}}} (1 - a_p p^{-s} + p^{1-2s})^{-1},$$

where $a_p = 1 + p - \#\widetilde{E}(\mathbb{F}_p)$. Conjectured by Birch and Swinnerton-Dyer in the 1960s, the 'BSD rank formula' predicts that that the order of vanishing of $L(E,s)$ at $s = 1$ should be equal to the Mordell–Weil rank of E, assuming meromorphic continuation.

Conjecture 1.1 (BSD rank formula)**.**

$$\operatorname{ord}_{s=1} L(E,s) = \operatorname{rk}(E/\mathbb{Q}).$$

Proving this conjecture is one of the biggest open problems in modern number theory, if not all mathematics; indeed, this is one of the Clay Mathematics Institute's 'Millennium Prize Problems', worth $\$10^6$ to any potential solver.

There is a natural generalisation of the BSD rank formula which replaces elliptic curves by more general abelian varieties, noting that the Euler factor at p in the above is precisely

$$\det(1 - p^{-s} \operatorname{Frob}_p^{-1} \mid V_\ell^{I_p})^{-1},$$

independent of choice of prime $\ell \neq p$, and where $V_\ell = T_\ell E \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is the $\mathbb{Q}_\ell$-vector space defined by the ℓ-adic Tate module of E. This leads us to define

$$L(A/K,s) = \prod_{\mathfrak{p}} \det(1 - N(\mathfrak{p})^{-s} \operatorname{Frob}_{\mathfrak{p}}^{-1} \mid V_\ell^{I_{\mathfrak{p}}})^{-1},$$

where this product is over proper prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$. Moreover, for each Artin representation of G_K, one may define an 'Artin-twisted' L-function

$$L(A/K,\rho,s) = \prod_{\mathfrak{p}} \det(1 - N(\mathfrak{p})^{-s} \operatorname{Frob}_{\mathfrak{p}}^{-1} \mid (\rho \otimes V_\ell)^{I_{\mathfrak{p}}})^{-1}.$$

These Artin-twists satisfy a so-called 'Artin formalism':

Lemma 1.2. (i) *For ρ_1 and ρ_2 factoring through* $\operatorname{Gal}(F/K)$ *we have*

$$L(A/K, \rho_1 \oplus \rho_2, s) = L(A/K,\rho_1,s) \cdot L(A/K,\rho_2,s).$$

(ii) *For F'/F a finite Galois extension and ρ factoring through* $\operatorname{Gal}(F'/F)$*, we have*

$$L(A/K, \operatorname{Ind}_{\operatorname{Gal}(F'/F)}^{\operatorname{Gal}(F'/K)} \rho, s) = L(A/F,\rho,s).$$

These properties, as well as general considerations of Deligne, led to a conjectural analogue of the BSD rank formula for Artin-twists of abelian varieties:

Conjecture 1.3 (Deligne–Gross)**.**

$$\operatorname{ord}_{s=1} L(A/K,\rho,s) = \langle A(F) \otimes_{\mathbb{Z}} \mathbb{C}, \rho \rangle,$$

for ρ an Artin representation factoring through $\operatorname{Gal}(F/K)$*, where this is the usual representation-theoretic inner product.*

1.1 Summary

We view Conjecture 1.3 as a prediction of a duality between the decompositions of the Mordell–Weil group and of the Tate module $V_\ell A$ into $\mathrm{Gal}(F/K)$-representations, and ask whether this duality holds when replacing $\mathrm{Gal}(F/K)$ by a finite group G whose action commutes with that of G_K. In particular, our aim is to state an analogue of this conjecture for Jacobians of curves with automorphisms.

Since the '60s, many of the concepts considered by Birch and Swinnerton-Dyer have been generalised. In particular, the insight of Grothendieck led to the conception of 'motives', offering a generalisation of abelian varieties and equipped with *motivic L-functions* which generalise the Hasse–Weil L-functions defined above. We will consider *pure motives*, which we view as a collection of cohomologies—called realisation data—equipped with comparison isomorphisms. Most relevant to us will be the ℓ-adic realisations; to a d-dimensional K-motive M, we associate a d-dimensional $\mathbb{Q}_\ell$-vector space $H_\ell(M)$ with a continuous representation

$$\rho_{M,\ell} : G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell}(H_\ell(M)).$$

We then define the L-function

$$L(M,s) = \prod_{\mathfrak{p}} \det(1 - N(\mathfrak{p})^{-s}\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \mid H_\ell(M)^{I_{\mathfrak{p}}})^{-1}$$

and take as a standing assumption that $L(M,s)$ has meromorphic continuation and satisfies a functional equation[1]. The general conjecture of Deligne which we will apply is the following (cf. [Del79, Conjecture 2.7]):

Conjecture 1.4 (Deligne)**.** *Let $F/\mathbb{Q}$ be Galois such that the Euler factors of $L(M,s)$ are defined over F. For each $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ we have*

$$\mathrm{ord}_{s=m} L(M,s) = \mathrm{ord}_{s=m} L^\sigma(M,s),$$

where this notation means applying σ on the level of local polynomials (with some assumptions on m, cf. Section 3.3).

To an abelian variety A of dimension g, we associate a motive $h^1(A)$ of dimension $2g$ such that $L(h^1(A),s) = L(A,s)$ (i.e. the 'ℓ-adic realisation' is the usual first étale cohomology, or the dual of $V_\ell A$). Suppose that A/K is an abelian variety with an action of the finite group G by automorphisms defined over K. We will seek to study the decomposition of $h^1(A)$ under the action of G, via the decompositions of the ℓ-adic realisations. Often an abelian variety with automorphisms will have *complex multiplication* (CM), recall:

Definition 1.5. Write $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. We say A has CM if $\mathrm{End}^0(A)$ contains a $2g$-dimensional commutative sub-algebra.

In the case that an elliptic curve E has CM, one finds that there is a factorisation

$$L(E,s) = L(\chi,s) \cdot L(\overline{\chi},s),$$

into complex conjugate Hecke L-functions. It is an observation of Shimura–Taniyama that this phenomenon is general. Indeed, they were able to show the following (slightly-imprecisely-stated) theorem away from all but finitely many primes, with the proof for all primes completed by Serre–Tate.

[1] This justifies our use of the terminology 'L-function' throughout; it is common in the literature to refer to the 'L-series' until analytic continuation is known.

Theorem 1.6 (Shimura–Taniyama, Serre–Tate)**.** *Suppose A/K is a simple abelian variety with CM. The L-function $L(A/K,s)$ factors as a product of conjugate Hecke L-functions.*

This observation translates in the motivic setting to a decomposition of $h^1(A)$ into 1-dimensional motives. It is a natural to wonder whether, under the condition that A have automorphisms, we may be able to draw similar conclusions about the structure of $h^1(A)$ as to those drawn by Shimura and Taniyama. In many cases these decompositions will align; elliptic curves with more automorphisms than just the inversion map on points automatically have CM, for example. In fact, all 1-dimensional submotives of abelian varieties correspond to Hecke motives (cf. [Sch88, Theorem 6.6.1]).

The special case to which we typically restrict ourselves is that of the Jacobian variety J_C associated to a curve C over a number field K. If C is a curve of genus g, then J_C is an abelian variety of dimension g, defined as the connected component of the identity in the *Picard group* of C. In particular, if E is an elliptic curve, then $E \cong J_E$ (this is a standard way to prove that the rational points of E form a group). Importantly, the Weil conjectures tell us that—similarly to the elliptic curves case above—the L-function $L(C/K,s) = L(J_C/K,s)$ can be computed just by counting points on the curve C over finite fields. Hence, we rarely have to worry about explicitly dealing with the Jacobian.

In the case of a curve C/K over a number field, J_C inherits automorphisms from C. Hence, if C is a curve on which the finite group G acts by automorphisms defined over K, then G acts on J_C; this is how we will construct examples. In particular, G acts on both the Mordell–Weil group of J_C over K and on $V_\ell J_C$—i.e. on the ℓ-adic realisation $H_\ell(h^1(A))$. Under certain circumstances we can view the latter as a complex representation of G, on which the action of G commutes with that of G_K. Hence we obtain an isotypic decomposition of V_ℓ under the action of G that consists of G_K-stable pieces. Naïvely, one might define L-functions via the action of G_K on these stable pieces of V_ℓ, but with such a definition one would not have an analogue of the Artin formalism when G is non-abelian. Instead we define new L-functions in the following way, which aligns with the naïve way in the case that G is abelian.

Definition 1.7 (=4.2)**.** Given a representation ρ of G, we define

$$L(C/K,\rho,s) = \prod_{\mathfrak{p}} \det(1 - N(\mathfrak{p})^{-s}\operatorname{Frob}_{\mathfrak{p}}^{-1} \mid \operatorname{Hom}_G(\rho, V_\ell)^{I_{\mathfrak{p}}})^{-1},$$

where this is the determinant of the map on $\operatorname{Hom}_G(\rho,V_\ell)^{I_{\mathfrak{p}}}$ defined by postcomposition.

Importantly, here we have an analogue of the Artin formalism because

$$\operatorname{Hom}_H(\rho, \operatorname{Res}^G_H V) \cong \operatorname{Hom}_G(\operatorname{Ind}^G_H \rho, V)$$

by Frobenius reciprocity. The analogue of the Artin formalism in this setting suggests that Definition 1.7 is the 'correct' definition.

As mentioned, these L-functions will often be related to Hecke L-functions. In the case of a curve C/K and F/K a finite extension of number fields, we often find that the L-functions $L(C/F,\rho,s)$ are related to the Hasse–Weil L-functions $L(C/K,s)$. Moreover, when C/F is isomorphic over a larger field to a curve D defined over K, we often find that the L-functions $L(C/F,\rho,s)$ correspond to Artin-twists of $L(D/K,s)$. One might note, then, that often

$$\operatorname{ord}_{s=1} L(C/K,\rho,s) = \langle J_C(K) \otimes_{\mathbb{Z}} \mathbb{C}, \rho\rangle,$$

as a consequence of the conjectures discussed above.

In addition to the above observations, then, because the actions of G_K and G on V_ℓ commute, it is natural to ask whether the duality predicted by the BSD rank formula for Artin-twists should also hold in this situation.

This leads us to conjecture:

Conjecture 1.8 (=4.20)**.**

$$\operatorname{ord}_{s=1} L(C/K, \rho, s) = \langle J_C(K) \otimes_{\mathbb{Z}} \mathbb{C}, \rho \rangle.$$

From a computational point of view, one can test this conjecture by numerically computing these L-functions by counting fixed points of certain endomorphisms on the reduced curves $\widetilde{C}(\mathbb{F}_{\mathfrak{p}})$ as $\mathfrak{p}$ varies, as a consequence of the Weil conjectures.

In general, we use standard tools of representation theory—in particular, Frobenius reciprocity and Artin induction—to emulate the deduction of Conjecture 1.3 from Deligne's conjecture and the usual BSD rank formula showing:

Theorem 1.9 (=4.19)**.** *Deligne's conjecture and the BSD rank formula imply Conjecture 1.8.*

The BSD rank formula is only one part of a stronger conjecture of Birch and Swinnerton–Dyer which predicts a formula for the leading term of the Hasse–Weil L-function $L(A/K, s)$ at $s = 1$ in terms of the arithmetic data of A. Potential analogous formulae for Artin-twists have been studied by V. Dokchitser, Evans and Wiersema in [DEW21]; hence it is natural to wonder whether we can replicate or extend their results in this setting. We give some discussion on the *periods* associated to the L-functions $L(C/K, \rho, s)$, which conjecturally give the 'irrational part' of the L-value at $s = 1$.

The full statement of BSD has been absorbed into a vast conjectural edifice—most notably, as a special case of the (equivariant) Tamagawa Number Conjecture of Bloch–Kato (and Burns–Flach). Through this lens, Burns and Macias Castillo considered refined BSD-type conjectures in [BC19], including of the type considered in [DEW21]. It is again natural to ask how one can fit analogous leading term conjectures on the L-functions of Definition 1.7 into this framework.

1.2 Structure

In Section 2 we discuss all the relevant background information, including abelian varieties, L-functions and complex multiplication.

In Section 3 we give an introduction to motives from two points of view, before introducing the full statement of Deligne's conjecture and discussing some example applications. In particular, we show how this conjecture can be used to deduce a rank formula for Artin-twisted L-functions, which we mimic in the sequel.

In Section 4 we define the L-functions with which we are concerned and deduce Conjecture 1.8 from standard conjectures. We also give some discussion on the relative scarcity of 'interesting examples' of curves C; we often dismiss the L-functions $L(C/K, \rho, s)$ which coincide with Hasse–Weil L-functions, Artin-twists, or Hecke L-functions as 'uninteresting', because Conjecture 1.8 tends to coincide with the BSD rank formula (for Artin-twists) and their periods are similarly well-understood. Lastly, we discuss examples in the case of elliptic and hyperelliptic curves.

In Section 5 we describe how one might compute the L-functions we define, and briefly discuss the first 'interesting' cases one might tackle computationally: the genus 3 family of Picard curves over $\mathbb{Q}(\zeta_3)$, equipped with an action of the cyclic group C_3.

Finally, in Section 6 we give some brief discussion on the next steps one might take in the study of these L-functions.

2 Background

We begin with a summary of the relevant background information: the main definitions and properties of abelian varieties and Jacobians of curves, followed by a summary of the background information on the relevant L-functions for our project.

2.1 Abelian varieties

Abelian varieties are higher dimensional generalisations of elliptic curves. Their study is a vast topic in number theory and algebraic geometry, so here we just recall some simple notions that will be useful later on. We recommend [Mil08] or [EvdGM] for a detailed treatment of the topic. More formally:

Definition 2.1. A group variety over a field K is a group object in the category of varieties over K; equivalently it's a variety A over K endowed with morphisms $m_A : A \times A \to A$, $i_A : A \to A$ and a point $0_A \in A(K)$ which endow $A(L)$ with the structure of group for every field extension L/K.

Definition 2.2. An abelian variety over K is a complete and connected group variety over K.

The group structure and additional properties have several deep implications; in particular, the group structure on an abelian variety is always abelian[2] and every abelian variety is projective. Therefore, for our purposes, it suffices to define abelian varieties to be projective and connected abelian group varieties.

Instead of studying abelian varieties up to isomorphism, it is often useful to relax the equivalence conditions and consider them up to *isogeny.*

Definition 2.3. A homomorphism of abelian varieties $f : A \to B$ is said to be an *isogeny* if satisfies one of the following equivalent conditions:

(i) f is surjective and $\dim(A) = \dim(B)$;

(ii) $\ker(f)$ is a finite group scheme and $\dim(A) = \dim(B)$;

(iii) f is a finite, flat and surjective morphism.

If such an f exists, we say A and B are *isogenous* and write $A \sim B$. Such an f induces an injection $f^* : \overline{K}(B) \hookrightarrow \overline{K}(A)$ on the level of function fields; we say $\deg f = [\overline{K}(A) : \overline{K}(B)]$ is the *degree* of f.

Example 2.4. For an abelian variety A and each $n \in \mathbb{Z}$, we have an isogeny $[n] : A \to A$ given by $P \mapsto nP$. The kernel of $[n]$ is precisely the n-torsion points of A, denoted $A[n]$. $\diamondsuit$

The property of being isogenous clearly satisfies reflexivity and transitivity. It is then an equivalence relation because, for every isogeny $f : A \to B$, there exists a *dual* isogeny $g : B \to A$ such that $g \circ f = [d]_A$ and $f \circ g = [d]_B$, where $d = \deg f$.

Definition 2.5. A non-trivial abelian variety A over a field K is said to be *simple* if it has no proper abelian subvarieties.

[2]Nevertheless, the name 'abelian variety' refers to a correspondence with abelian integrals studied by N. H. Abel.

Remark 2.6. It is important to underline that the simpleness property depends on the ground field K, since a base extension of a simple abelian variety might not be simple. Sometimes we write K-simple to make this dependence explicit. $\diamondsuit$

The definition of simple abelian variety is fundamental in light of the Poincaré Splitting theorem, which states that for every abelian subvariety $X \subset A$, there exists a *complementary* abelian subvariety Y, i.e. one such that $X \times Y \sim A$. Combining this with the fact that a homomorphism of simple abelian varieties is either an isogeny or the zero map, the following decomposition should now appear natural.

Proposition 2.7. *A non-trivial abelian variety over K is isogenous to a product of simple abelian varieties; moreover, the factors are unique up to isogeny.*

Even though we will always work over a number field K, it is important to characterise abelian varieties over the complex numbers. Indeed, as for elliptic curves, every abelian variety over $\mathbb{C}$ is isomorphic to a complex torus $\mathbb{C}^g/\Lambda$ for some $2g$-dimensional lattice Λ. The only difference between the general case and the case $g = 1$ is that not all tori can be obtained in this way; a complex torus arises from an abelian variety if it is *polarisable*, i.e. if it admits a Riemann form.

Remark 2.8. Using the lattice structure for an abelian variety of dimension g, we have that the group of n-torsion points $A(\overline{K})[n] = A(\mathbb{C})[n]$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$. $\diamondsuit$

This leads us to the following definition:

Definition 2.9 (Tate module)**.** Let ℓ be a rational prime. The *ℓ-adic Tate module $T_\ell A$* is the free $\mathbb{Z}_\ell$-module of rank $2g$ defined as the inverse limit of the system

$$A[\ell] \xleftarrow{[\ell]} A[\ell^2] \xleftarrow{[\ell]} \dots \xleftarrow{[\ell]} A[\ell^n] \xleftarrow{[\ell]} \dots .$$

Moreover, we define the $2g$-dimensional $\mathbb{Q}_\ell$-vector space $V_\ell = V_\ell A = T_\ell A \otimes \mathbb{Q}_\ell$.

Finally, we recall the Mordell–Weil theorem on the finite generation of rational points of abelian varieties.

Theorem 2.10 (Mordell–Weil)**.** *For an abelian variety A defined over a number field K, the K-rational points $A(K)$ form a finitely generated abelian group.*

2.2 Jacobians of curves

Let K be a number field with algebraic closure $\overline{K}$ and absolute Galois group G_K. For a curve C/K of genus $g > 1$ we consider the group of divisors $\mathrm{Div}_{\overline{K}}(C)$, i.e. the free abelian group generated by the points on $C(\overline{K})$, and its subgroup $\mathrm{Princ}_{\overline{K}}(C)$ of principal divisors. The *degree* of a divisor is the sum of its coefficients. We define the *Picard group* of C as the quotient

$$\mathrm{Pic}_{\overline{K}}(C) = \mathrm{Div}_{\overline{K}}(C)/\,\mathrm{Princ}_{\overline{K}}(C).$$

Because principal divisors have degree 0, we can talk about the degree of a divisor class in the Picard group. The classes of degree 0 form a subgroup which we denote by $\mathrm{Pic}^0_{\overline{K}}(C)$. For a non-singular curve C this subgroup can be given the structure of a projective abelian variety of dimension g defined over K. The resulting variety is called the Jacobian variety of C and denoted J_C. The K-rational points on J_C are the G_K-stable elements of $\mathrm{Pic}^0_{\overline{K}}(C)$:

$$J_C(K) = \mathrm{Pic}^0_{\overline{K}}(C)^{G_K}.$$

Similarly, one can define

- $\mathrm{Div}_K(C) := \mathrm{Div}_{\overline{K}}(C)^{G_K}$,
- $\mathrm{Princ}_K(C) := \mathrm{Princ}_{\overline{K}}(C)^{G_K}$,
- $\mathrm{Pic}_K(C) := \mathrm{Div}_K(C)/\,\mathrm{Princ}_K(C)$,
- $\mathrm{Pic}^0_K(C) := \{D \in \mathrm{Pic}_K(C) \mid \deg(D) = 0\}$.

Note that in general $J_C(K) \ncong \mathrm{Pic}^0_K(C)$, because a K-rational divisor class does not, in general, contain a K-rational divisor. However, when $C(K) \neq \emptyset$ they are isomorphic (e.g. [PS97, Proposition 3.2]). Moreover, under the same assumption, the variety J_C can be defined categorically because it represents a particular functor $P^0_C \in \mathrm{Fun}(\mathbf{Var}_K, \mathbf{Grp})$ (e.g. [Mil08, Theorem 1.2]). The construction of J_C is definitely non-trivial, but we only require some simple properties:

Remark 2.11. For every point $P_0 \in C(K)$, there is a regular map $\varphi_{P_0} : C \to J_C$ defined on points via $P \mapsto [P - P_0]$. Each pair of maps $\varphi_\bullet$ differs by a translation in J_C. ♢

Example 2.12. Fix an elliptic curve E with identity 0_E. For every divisor $D \in \mathrm{Div}^0_{\overline{K}}(E)$ there exists a unique point $P \in E(\overline{K})$ such that $D \sim (P) - (0_E)$ (e.g. [Sil09, Proposition III.3.4]). This implies that $\varphi_{0_E} : E \to J_E$ defines an isomorphism of elliptic curves. ♢

Remark 2.13. Every morphism of curves $f : C \to C'$ induces a homomorphism $f^* : \mathrm{Pic}^0(C') \to \mathrm{Pic}^0(C)$ functorially with respect to base extensions; thus f induces a morphism $J_{C'} \to J_C$. In particular, every automorphism of C induces an automorphism of J_C. We will be interested in the automorphisms of $V_\ell J_C$ induced by those of C. ♢

In general, every abelian variety can be realised as a quotient of the Jacobian of a certain curve. This means that sometimes it is sufficient to treat questions on abelian varieties only at the level of Jacobian varieties.

2.3 Weil conjectures

We now discuss some results on varieties over finite fields, as we will eventually need to study the reduction of abelian varieties over number fields modulo primes. These are the Weil conjectures, proved as the culmination of work by Dwork, Grothendieck, and Deligne, amongst others. All the relevant background can be found, for example, between [Har77, Appendix C] and the Stacks project [Sta18].

Fix a smooth proper variety[3] X over $\mathbb{F}_q$ of dimension d. We define the *zeta function* of X by

$$Z(X;T) = \exp\left(\sum_{n\geq 1} \#X(\mathbb{F}_{q^n})\frac{T^n}{n}\right).$$

We then have:

Theorem 2.14 (Weil conjectures)**.** *We have each of the following properties:*

1. *$Z(X;T)$ is a rational function*

$$Z(X;T) = \frac{P_1(T)\dots P_{2d-1}(T)}{P_0(T)\dots P_{2d}(T)}$$

such that for each $1 \leq i \leq 2d-1$ the polynomial $P_i(T) \in \mathbb{Z}[T]$ has complex roots with absolute value $q^{i/2}$ (i.e. $Z(X;T)$ satisfies the 'Riemann hypothesis').

[3]More generally, a scheme of finite type.

2. *$Z(X;T)$ satisfies a functional equation*

$$Z(X;q^{-(d-s)}) = \pm q^{\chi(\frac{d}{2}-s)} Z(X;q^{-s}),$$

where χ is the Euler characteristic of X.

3. *If X is the reduction modulo $\mathfrak{p}$ of a smooth proper variety Y over a number field, then $\deg P_i(T)$ is the i^{th} Betti number of $Y(\mathbb{C})$.*

Remark 2.15. Only the first two properties will be relevant to this project, so one needn't worry about the Betti numbers in property 3. ♢

The proof of the Weil conjectures relies on a so-called 'Weil cohomology theory' with coefficients in a characteristic 0 field K. This is a contravariant function $X \to \oplus_i H^i(X)$ from smooth projective varieties over $\overline{\mathbb{F}}_q$ to graded anti-commutative algebras over K which satisfies various properties (see [Sta18, 0FHA]). We will focus on the following properties, although there are many more:

1. $H^i(X)$ is a finite dimensional K-vector space which vanishes for $i \notin \{0,1,\dots,2\dim X\}$.
2. Every morphism $f : X \to Y$ of smooth projective varieties induces a pullback morphism of graded K-algebras $f^* : H^\bullet(Y) \to H^\bullet(X)$.
3. We have *Poincaré duality*, i.e. a perfect pairing

$$H^i(X) \times H^{2d-i}(X) \to H^{2d}(X).$$

Fact. For ℓ coprime to q, the étale cohomology $H^\bullet_{et}(X,\mathbb{Q}_\ell)$ is a Weil cohomology theory.

We now pivot from a general Weil cohomology theory to dealing with étale cohomology. The above properties give us:

Theorem 2.16 (Lefschetz trace formula)**.** *Let X be a smooth projective variety over $\mathbb{F}_q$ with base change $\overline{X}$ to $\overline{\mathbb{F}}_q$ with endomorphism f. Then*

$$\sum_{x\in \mathrm{Fix}(f)} i(f,x) = \sum_{k\geq 0} (-1)^k \operatorname{Tr}(f^* \mid H^k_c(\overline{X},\mathbb{Q}_\ell)),$$

where $i(f,x)$ is the index of the fixed point x, and $H^\bullet_c$ denotes étale cohomology with compact support.

Remark 2.17. We do not include a definition of the index $i(f,x)$ because in the cases with which we are concerned, each of the fixed points will have index 1. This gives us

$$\#\mathrm{Fix}(f) = \sum_{k\geq 0} (-1)^k \operatorname{Tr}(f^* \mid H^k_c(\overline{X},\mathbb{Q}_\ell)), \tag{2.1}$$

which we will discuss further in the setting of algebraic curves. ♢

Example 2.18. Suppose that X is an algebraic curve, so that we only get contributions in (2.1) for $k = 0,1,2$, and take $f = \Phi$ to be the q-power Frobenius. Writing μ_ℓ for the ℓ'th roots of unity in $\overline{\mathbb{F}}_q$, we have

$$H^k_c(\overline{X},\mu_\ell) = \begin{cases} \mu_\ell, & \text{if } k=0 \\ \mathrm{Pic}(\overline{X})[\ell], & \text{if } k=1 \\ \mathbb{Z}/\ell\mathbb{Z}, & \text{if } k=2 \\ 0 & \text{if } k\geq 3, \end{cases}$$

so that, after taking projective limits and tensoring with $\mathbb{Q}_\ell$, we find

$$\mathrm{rk}_{\mathbb{Q}_\ell} H_c^k(\overline{X}, \mathbb{Q}_\ell) = \begin{cases} 1, & \text{if } k = 0 \\ 2g, & \text{if } k = 1 \\ 1, & \text{if } k = 2 \\ 0, & \text{if } k \geq 3. \end{cases}$$

In particular, $H_c^1(\overline{X}, \mathbb{Q}_\ell)$ is the dual of $V_\ell J_X$, and hence the contribution from the term $k = 1$ is the sum of the eigenvalues of Φ on $V_\ell J_X$. Using Poincaré duality we find $\det(\Phi^* \mid H_c^2(\overline{X}, \mathbb{Q}_\ell)) \cdot \det(\Phi^* \mid H_c^0(\overline{X}, \mathbb{Q}_\ell)) = q$. Because the action on $H_c^2(\overline{X}, \mathbb{Q}_\ell)$ is trivial and these spaces are 1-dimensional, we find

$$\#\mathrm{Fix}(\Phi^n) = 1 + q^n - \sum_{i=1}^{2g} \gamma_i^n, \tag{2.2}$$

where the γ_i are the eigenvalues of Φ on V_ℓ. $\diamondsuit$

Equation (2.2) gives an example of the following special case of the Lefschetz trace formula, which will play an important role in our computation of L-functions in Section 5.

Theorem 2.19 ([Mil08, Proposition III.11.2]). *Let C be a smooth projective curve over $\mathbb{F}_q$ with base change $\overline{C}$ to $\overline{\mathbb{F}}_q$ with endomorphism α. Then*

$$\#\mathrm{Fix}(\alpha) = 1 + \deg(\alpha) - \sum_{i=1}^{2g} \gamma_i,$$

where the γ_i are the eigenvalues of the action on V_ℓ induced by α.

2.4 Representation theory

Here we briefly recall some tools from the representation theory of finite groups, which is sometimes overlooked by number theorists[4]. Throughout, ρ will denote a representation $\rho : G \to \mathrm{GL}(V)$ with G a finite group and V a complex vector space.

Remark 2.20. Equivalently, one may view a representation V as a vector space equipped with a linear action of the group G. As is standard, we will typically identify the homomorphism ρ with the vector space V under the specified action of G. $\diamondsuit$

Definition 2.21. For a field K we say that a representation ρ of G can be *realised over* K if for all $g \in G$ we can realise $\rho(g)$ as a matrix with coefficients in K. Representations that can be realised over $\mathbb{Q}$ are called *rational*.

Example 2.22. For an abelian variety A/K and a finite Galois extension F/K, the action of $\mathrm{Gal}(F/K)$ on $A(F) \otimes_{\mathbb{Z}} \mathbb{C}$ defines a rational representation. $\diamondsuit$

The following theorem—originally due to Brauer—describes a 'field of realisation' K_G for the irreducible representations of a finite group G.

Theorem 2.23 ([Isa76, Theorem 10.3]). *Let G be a finite group of exponent n. Every irreducible complex representation ρ of G can be realised over $K_G := \mathbb{Q}(\zeta_n)$.*

[4] At least, it was by some of the authors!

Note that K_G is Galois over $\mathbb{Q}$. Hence we obtain a Galois action on the irreducible representations ρ of G, where $\sigma \in \mathrm{Gal}(K_G/\mathbb{Q})$ acts on the level of coefficients in the matrix realisation of ρ. We denote the resulting representation by ρ^σ. Note that two irreducible representations are Galois conjugate if and only if their afforded characters are. In particular, we see that a representation is rational if and only if its character takes rational values. Moreover, we obtain the following lemma.

Lemma 2.24. *Let ρ be a rational representation of a finite group G with decomposition*

$$\rho = m_1\rho_1 + m_2\rho_2 + \ldots + m_k\rho_k$$

into irreducible representations of G. Then, for any $\sigma \in \mathrm{Gal}(K_G/\mathbb{Q})$, if $\rho_i = \rho_j^\sigma$ then $m_i = m_j$.

Let $\rho : G \to \mathrm{GL}(V)$ a finite dimensional complex representation of the finite group G and let $\chi = \mathrm{Tr}(\rho)$ be the associated character. If $H \leq G$, then $\mathrm{Res}_H^G \rho : H \hookrightarrow G \xrightarrow{\rho} \mathrm{GL}(V)$ is the *restriction* of ρ to H, affording the character $\mathrm{Res}_H^G \chi$.

Conversely, let χ a character of $H \leq G$, then the induced class function on G is defined by

$$\mathrm{Ind}_H^G \chi(g) = \frac{1}{|H|} \sum_{x \in G} \chi'(x^{-1}gx)$$

where $\chi'(x) = \chi(x)$ if $x \in H$ and 0 otherwise.

The class function $\mathrm{Ind}_H^G \chi$ is always a character and if the representation ρ of H affords the character χ then we define the induced representation $\mathrm{Ind}_H^G \rho$ as the representation of G affording the character $\mathrm{Ind}_H^G \chi$. The results on induction and restriction which will be relevant to us are *Frobenius reciprocity* and *Artin induction*:

Theorem 2.25 (Frobenius reciprocity)**.** *Let χ be a character of H and ξ a character of G. Then*

$$\langle \chi, \mathrm{Res}_H^G \xi \rangle_H = \langle \mathrm{Ind}_H^G \chi, \xi \rangle_G.$$

Here, $\langle \cdot, \cdot \rangle$ is the standard representation-theoretic inner product; in particular, for ξ an irreducible representation, $\langle \rho, \xi \rangle$ is the 'multiplicity of ξ in ρ'.

Remark 2.26. Frobenius reciprocity says that Ind_H^G and Res_H^G are adjoint functors. This gives

$$\mathrm{Hom}_H(\rho, \mathrm{Res}_H^G V) \cong \mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, V),$$

cf. Lemma 4.7. $\diamondsuit$

Theorem 2.27 (Artin induction)**.** *Let X be a family of subgroups of G such that $G = \bigcup_{g \in G, H \in X} g^{-1}Hg$. Then for every character χ of G there exists a positive integer d, a character χ_H and integers a_H for every $H \in X$ such that*

$$d\chi = \sum_{H \in X} a_H \mathrm{Ind}_H^G \chi_H.$$

In particular, the Artin induction formula holds for X the set of cyclic subgroups of G.

Artin induction is a weak version of the more-commonly-used *Brauer induction*, which allows one to take $d = 1$ by replacing cyclic groups by the set of subgroups containing direct products of cyclic groups and subgroups of prime-power order.

Theorem 2.28 (Brauer induction)**.** *Let X be the set of subgroups consisting of direct products of cyclic subgroups and p-groups for any prime p. Then for any character χ of G, there exists a character χ_H and integers a_H for each $H \in X$ such that*

$$\chi = \sum_{H \in X} a_H \operatorname{Ind}_H^G \chi_H.$$

Notation. From now on, we often adopt the standard convention of identifying a representation ρ with its afforded character $\operatorname{Tr}(\rho)$.

2.5 Hasse–Weil L-functions

All the relevant information for this section can be found between the standard reference [Sil09] and [DEW21], for example. We quickly recall the definition of Frobenius automorphisms: let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$, and fix a choice of decomposition group $G_\mathfrak{p} \subset G_K$ and inertia subgroup $I_\mathfrak{p} \triangleleft G_\mathfrak{p}$.

Definition 2.29 (Frobenius)**.** For $\mathfrak{p}$ a prime of $\mathcal{O}_K$, the Frobenius automorphism is $\operatorname{Frob}_\mathfrak{p} \in G_\mathfrak{p}/I_\mathfrak{p}$. We consider this an element of G_K, defined up to inertia and conjugation.

Remark 2.30. In finite abelian extensions L/K, the restriction of the Frobenius automorphism is characterised by the property

$$\operatorname{Frob}_\mathfrak{p} x - x^{N(\mathfrak{p})} \in \mathfrak{p}\mathcal{O}_\ell$$

for all $x \in \mathcal{O}_\ell$. ♢

Fix an abelian variety A of dimension g defined over a number field K. Note that G_K acts naturally on $A[\ell^n]$ for each n, from which $V_\ell A$ inherits an action, and which we use to define the Hasse–Weil L-function $L(A/K, s)$ via its Euler product as follows.

Definition 2.31 (Hasse–Weil L-function)**.**

$$L(A/K, s) = \prod_{\mathfrak{p} \triangleleft \mathcal{O}_K \text{ prime}} P_\mathfrak{p}(N(\mathfrak{p})^{-s})^{-1},$$

where $P_\mathfrak{p}(T) = \det(1 - T \operatorname{Frob}_\mathfrak{p}^{-1} \mid V_\ell^{I_\mathfrak{p}})$, independent of choice of $\mathfrak{p} \nmid l$.

Remark 2.32. This is well-defined because the Frobenius is well-defined up to conjugation and inertia. Moreover, for primes of good reduction $V_\ell^{I_\mathfrak{p}} = V_\ell$ and we need not worry about the inertia subgroup. ♢

Example 2.33. In the case of elliptic curves, we find that for primes of good reduction we have

$$P_\mathfrak{p}(T) = 1 - a_\mathfrak{p} T + N(\mathfrak{p}) T^2,$$

where $a_\mathfrak{p} = 1 + N(\mathfrak{p}) - |E(\mathbb{F}_\mathfrak{p})|$. For primes of bad reduction we have

$$P_\mathfrak{p}(T) = \begin{cases} 1, & \text{if } A \text{ has additive reduction at } \mathfrak{p} \\ 1 - T, & \text{if } A \text{ has split multiplicative reduction at } \mathfrak{p} \\ 1 + T, & \text{if } A \text{ has non-split multiplicative reduction at } \mathfrak{p} \end{cases},$$

which can again be summarised as $P_\mathfrak{p}(T) = 1 - a_\mathfrak{p} T$. ♢

Example 2.34. The example in which we will be primarily interested is that of Jacobians of curves. In this case we write $L(C/K,s)$ as shorthand for $L(J_C/K,s)$; this notation is appropriate because to understand $L(C/K,s)$, it suffices to understand the reduced curve modulo $\mathfrak{p}$ for primes $\mathfrak{p}$ of K. Indeed, for primes $\mathfrak{p}$ of good reduction, one finds that the zeta function

$$Z(\widetilde{C};T) = \frac{P(T)}{(1-T)(1-qT)}$$

of the reduced curve $\widetilde{C}/\mathbb{F}_\mathfrak{p}$ has numerator $P(T) = \det(1 - T\,\mathrm{Frob}_\mathfrak{p}^{-1} \mid (V_\ell J_C)^{I_\mathfrak{p}})$ (see [Mil08, Corollary 11.4]). $\diamondsuit$

The reason we are interested in these L-functions is that conjecturally they relate well-known arithmetic invariants of A to the value of $L(A/K,s)$ at $s=1$. This is made explicit by the famous conjecture of Birch and Swinnerton-Dyer (BSD), a weak version of which we will state shortly. To state BSD, we rely on the following conjecture:

Conjecture 2.35 (Hasse–Weil)**.** *$L(A/K,s)$ has meromorphic continuation to $\mathbb{C}$.*

Remark 2.36. This is known to hold for elliptic curves over $\mathbb{Q}$, as a consequence of the famous modularity theorem of Wiles, Breuil, Conrad, Diamond and Taylor, as well as for elliptic curves in some other special cases. It is also known for abelian varieties of CM-type by a result of Shimura and Taniyama (cf. Theorem 2.54). $\diamondsuit$

Remark 2.37. We can be much more precise about the conjectural continuation. We define the *completed* Hasse–Weil L-function

$$\Lambda(A/K,s) = \left(\frac{c(A/K)^{\frac{1}{2}}}{\pi^{d\cdot g}}\right)^s \left[\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\right]^{d\cdot g} L(A/K,s),$$

where $d = [K:\mathbb{Q}]$ and $c(A/K)$ is the *conductor* of A/K. Then conjecturally $\Lambda(A/K,s)$ satisfies a precise functional equation

$$\Lambda(A/K,s) = w(A/K)\Lambda(A/K,2-s), \tag{2.3}$$

where $w(A/K)$ is the *root number* and moreover $w(A/K) = \pm 1$ if A is *principally polarised*, e.g. if A is the Jacobian of a curve. $\diamondsuit$

We make the standing assumption that $L(A/K,s)$ has meromorphic continuation and satisfies its functional equation. Birch and Swinnerton-Dyer predict:

Conjecture 2.38 (BSD)**.** *For an abelian variety A over a number field K with dual A^*, the L-function $L(A/K,s)$ admits analytic continuation to a neighbourhood of $s=1$ and*

1. $\mathrm{rk}(A/K) = \mathrm{ord}_{s=1} L(A/K,s)$.
2. $|Ш_{A/K}|$ *is finite and*

$$\lim_{s\to 1}\frac{L(A/K,s)}{(s-1)^{\mathrm{rk}(A/K)}} = \frac{\Omega(A/K)\cdot R_{A/K}\cdot(\prod_\nu c_\nu)\cdot|Ш_{A/K}|}{|A(K)_{\mathrm{tors}}|\cdot|A^*(K)_{\mathrm{tors}}|} =: \mathrm{BSD}_{A/K}\,.$$

Here:

- $\Omega(A/K)$ is the *period* of A/K, c.f. Remark 3.32 and Section 3.3 in general;

- $R_{A/K}$ is the *regulator* of A/K, i.e. if $\{P_i\}_i$ and $\{Q_j\}_j$ generate $A(K)/A(K)_{\text{tors}}$ and $A^*(K)/A^*(K)_{\text{tors}}$ respectively, and $\langle\cdot,\cdot\rangle$ is the height pairing, then

$$R_{A/K} = (\det(\langle P_i, Q_j\rangle))_{i,j};$$

- For each place ν of K, c_ν is the *local Tamagawa number* $[A_\nu(K_\nu) : A\nu^0(K_\nu)]$, where $A_\nu^0(K_\nu)$ is the set of points with non-singular reduction;

- $Ш_{A/K}$ is the *Tate-Shafarevich group* of A/K, given cohomologically as

$$Ш_{A/K} = \bigcap_\nu \left(\ker\left(H^1(G_K, A) \to H^1(G_{K_\nu}, A_\nu)\right)\right).$$

Remark 2.39. Work of Kolyvagin, combined with the modularity theorem, shows that the rank part is known to hold for elliptic curves $E/\mathbb{Q}$ of rank 0 or 1. The standard minimalist conjecture says that this ought to be 100% of elliptic curves over the rationals.

Because it applies to essentially all of the elliptic curves we deal with in this project, we should also state a famous theorem of Coates–Wiles and Arthaud: suppose K is an imaginary quadratic field and $F = \mathbb{Q}$ or F/K is a finite abelian extension. If E/F is an elliptic curve with complex multiplication by K, then $L(E/F, 1) \neq 0 \implies E(F)$ is finite. $\diamondsuit$

The BSD rank formula naturally generalises to the setting of *Artin-twists*.

Definition 2.40 (Artin-twist)**.** Let ρ be a complex, continuous representation of G_K. The *Artin-twist* of $L(A/K, s)$ by ρ is the L-function given by

$$L(A/K, \rho, s) = \prod_{\mathfrak{p} \triangleleft \mathcal{O}_K \text{ prime}} P_{\mathfrak{p}}(\mathfrak{p}^{-s})^{-1},$$

where $P_{\mathfrak{p}}(T) = \det(1 - T\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \mid (\rho \otimes V_\ell)^{I_{\mathfrak{p}}})$, again independent of choice of $\mathfrak{p} \nmid l$.

Remark 2.41. We will often abuse terminology and talk about such a ρ factoring through $\mathrm{Gal}(F/K)$ as a representation of $\mathrm{Gal}(F/K)$. $\diamondsuit$

These twisted L-functions satisfy the following properties, which we refer to as the *Artin formalism.*

Lemma 2.42. (i) *For ρ_1 and ρ_2 factoring through* $\mathrm{Gal}(F/K)$ *we have*

$$L(A/K, \rho_1 \oplus \rho_2, s) = L(A/K, \rho_1, s) \cdot L(A/K, \rho_2, s).$$

(ii) *For F'/F a finite Galois extension and ρ a representation of* $\mathrm{Gal}(F'/F)$*, we have*

$$L(A/K, \mathrm{Ind}_{\mathrm{Gal}(F'/F)}^{\mathrm{Gal}(F'/K)}\, \rho, s) = L(A/F, \rho, s).$$

Example 2.43. In particular, taking $\rho = 1$ in property (ii) gives us a factorisation

$$L(A/F, s) = \prod_\chi L(A/K, \chi, s),$$

where this product is over the irreducible subrepresentations of the regular representation $\mathrm{Ind}_1^{\mathrm{Gal}(F/K)} 1$. When F/K is abelian, this product is just over the irreducible representations of $\mathrm{Gal}(F/K)$. $\diamondsuit$

The Hasse–Weil conjecture extends here to predict that $L(A/K, \rho, s)$ has meromorphic continuation to $\mathbb{C}$ for all ρ. Again, we take as a standing assumption that the Hasse–Weil conjecture holds and that $L(A/K, \rho, s)$ satisfies a functional equation. The analogue of the BSD rank formula is the Deligne–Gross conjecture, or 'BSD rank formula for Artin-twists'.

Conjecture 2.44 (Deligne–Gross). *For ρ factoring through $\mathrm{Gal}(F/K)$ we have*

$$\mathrm{ord}_{s=1} L(A/K, \rho, s) = \langle A(F)_{\mathbb{C}}, \rho \rangle,$$

where $A(F)_{\mathbb{C}}$ is the representation obtained from the Galois action on $A(F) \otimes_{\mathbb{Z}} \mathbb{C}$.

Remark 2.45. We recover the original BSD rank formula (Conjecture 2.38) as a special case of the Deligne–Gross conjecture by taking $F = K$. In this case $\mathrm{Gal}(F/K)$ is trivial, so

$$\langle A(F)_{\mathbb{C}}, \rho \rangle = \dim_{\mathbb{C}}(A(F) \otimes_{\mathbb{Z}} \mathbb{C}) = \mathrm{rk}(A/K),$$

where ρ is the unique representation of the trivial group $\mathrm{Gal}(F/K)$. ♢

2.6 Hecke characters and L-functions

In this section we will briefly go over the definition and basic properties of algebraic Hecke characters and their associated L-functions, mostly following [Sch88, Chapter 0]. These Hecke L-functions will come up later as examples of L-functions associated to submotives of abelian varieties.

There are several ways to define a Hecke character; in particular, one can do it either in terms of fractional ideals, or using the language of idèles. Although the latter strategy arguably leads to a more elegant definition, here we will stay closer to Hecke's original construction of what-he-called *Größencharaktere*, using fractional ideals. For further reading on these different definitions, the reader may consult [Neu99] or [Shu22]. Note that while most authors, ourselves included, use the terms 'Größencharakter' and 'Hecke character' to mean the same thing, this is not always the case. Further, there is a distinction between Hecke characters and algebraic Hecke characters; the former is a character taking values in $\mathbb{C}^\times$, while the latter maps to $F^\times$ for some number field F. We restrict our attention to algebraic Hecke characters.

We start by fixing some notation for the rest of this section. Let K and F be number fields, and fix an algebraic closure $\overline{F}$ of F (we usually take $\overline{F} \subset \mathbb{C}$). For a non-zero integral ideal $\mathfrak{a}$ of the ring of integers $\mathcal{O}_K$ we denote by $J^{\mathfrak{a}}$ the group of fractional ideals of K that are coprime to $\mathfrak{a}$ (two fractional ideals are called coprime if no prime $\mathfrak{p}$ appears in both their prime factorizations with non-zero exponent). Further, we assign to each embedding $\sigma : K \hookrightarrow \overline{F}$ an integer n_σ, and consider the tuple

$$T = (n_\sigma)_\sigma.$$

We call an element $a \in K$ *totally positive* if $\sigma(a) > 0$ for all real embeddings $\sigma : K \hookrightarrow \mathbb{R}$.

Definition 2.46 (Hecke character). An *algebraic Hecke character χ of K with values in F, of infinity type T and with conductor dividing* $\mathfrak{a}$ is a group homomorphism

$$\chi : J^{\mathfrak{a}} \to F^\times$$

such that, for any principal ideal $(a) \in J^{\mathfrak{a}}$ generated by a totally positive $a \in K^\times$ with

$a \equiv 1 \pmod{\mathfrak{a}}$, we have

$$\chi((a)) = a^T = \prod_{\sigma: K \hookrightarrow \overline{F}} \sigma(a)^{n_\sigma}.$$

Given an ideal $\mathfrak{b}$ such that $\mathfrak{a} \mid \mathfrak{b}$ we can restrict χ to a character with conductor dividing $\mathfrak{b}$ since $J^{\mathfrak{b}} \subset J^{\mathfrak{a}}$. The *conductor* of χ is the smallest divisor $\mathfrak{c}$ of $\mathfrak{a}$ such that χ is the restriction of a character with conductor dividing $\mathfrak{c}$. The infinity type T of a Hecke character is often identified with the homomorphism $\chi_\infty : K^\times \to F^\times$ sending $a \mapsto a^T$.

Example 2.47. Hecke characters are a generalisation of Dirichlet characters. To see this, let $\chi_D : (\mathbb{Z}/N\mathbb{Z})^\times \to S^1$ be a Dirichlet character. Further, let E denote the smallest cyclotomic field containing the image of χ_D. To realise χ_D as a Hecke character, we take $K = \mathbb{Q}$, $F = E$ and $\mathfrak{a} = (N)$. There is a one-to-one correspondence between ideals $\mathfrak{b} \in J^{\mathfrak{a}}$ and rational numbers $b \in \mathbb{Q}_{>0}$ coprime (both numerator and denominator) to N, given by taking their positive generator. Writing $b = s/t$ for coprime integers s and t and reducing modulo N we assign to each $\mathfrak{b}$ an element $\overline{b} \in (\mathbb{Z}/N\mathbb{Z})^\times$. We define the corresponding Hecke character $\chi_H : J^{\mathfrak{a}} \to F^\times$ by taking

$$\chi_H(\mathfrak{b}) = \chi_D(\overline{b}).$$

What is left to check is that there exists a compatible infinity type T for χ_H. Because $K = \mathbb{Q}$, there is only one trivial embedding σ. For $b \equiv 1 \pmod{N}$ we have $\chi_D(\overline{b}) = 1$, so the infinity type in this case must be $T = (n_\sigma) = (0)$. $\diamondsuit$

Hecke characters with trivial infinity type, such as Dirichlet characters, are said to be *of finite order*. These are exactly the Hecke characters whose image lies in the roots of unity in $F^\times$.

Note that the principal ideals $(a) \in J^{\mathfrak{a}}$ with a totally positive and $a \equiv 1 \pmod{\mathfrak{a}}$ form a subgroup of finite index in $J^{\mathfrak{a}}$. For every Hecke character χ there is an integer w—called the *weight* of χ—satisfying

$$w = n_\sigma + n_{\overline{\sigma}}$$

for every $\sigma : K \hookrightarrow \overline{F} \subset \mathbb{C}$, and independent of the chosen embedding $\overline{F} \hookrightarrow \mathbb{C}$. Two homomorphisms that agree on a finite index subgroup and map to a torsion-free group must be equal, so we find

$$\chi \cdot \overline{\chi} = N_{K/\mathbb{Q}}^w.$$

In fact, we have the following result (e.g. [Sch88, §0.3]).

Definition 2.48 (CM-field)**.** A number field F is a *CM-field* if there exists totally real $F_0 \subset F$ such that F/F_0 is a degree 2 totally imaginary extension.

Proposition 2.49. *Let K' be the maximal CM-subfield of K if it exists (cf.* [Mil20, Remark I.1.7]), or its maximal totally real subfield, otherwise.

(i) If K' is totally real, then every algebraic Hecke character of K is of the form

$$\chi = \mu \cdot N_{K/\mathbb{Q}}^{w/2},$$

where μ is a Hecke character of finite order and $w \in 2\mathbb{Z}$.

(ii) Otherwise, if K' is CM, then every algebraic Hecke character of K is of the form

$$\chi = \mu \cdot (\varphi \circ N_{K/K'}),$$

where μ is of finite order and φ is an algebraic Hecke character of K'.

Definition 2.50 (Hecke L-function)**.** For an algebraic Hecke character χ over K with values in F and conductor dividing $\mathfrak{a}$, we define a *Hecke L-function* for every complex embedding $\tau\colon F \hookrightarrow \mathbb{C}$ via

$$L(\chi_\tau, s) = \sum_{\mathfrak{b} \lhd \mathcal{O}_K} \frac{(\tau \circ \chi)(\mathfrak{b})}{N(\mathfrak{b})^s} = \prod_{\mathfrak{p} \lhd \mathcal{O}_K} \left(1 - \frac{(\tau \circ \chi)(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1},$$

where this product is over the finite primes of K, and we extend χ to all ideals of $\mathcal{O}_K$ by setting it to 0 on ideals not coprime to $\mathfrak{a}$.

Hecke's original motivation for studying Hecke characters was to find the largest possible class of L-functions that could be shown to satisfy a functional equation.

Theorem 2.51 (Hecke)**.** *Hecke L-functions have meromorphic continuation to $\mathbb{C}$ and satisfy a functional equation.*

2.7 Complex multiplication

We now give some background information on complex multiplication of abelian varieties. Recall that in the case of an elliptic curve E, generically we have $\mathrm{End}(E) \cong \mathbb{Z}$, although it is possible that $\dim_{\mathbb{Z}} \mathrm{End}(E) = 2$. In this case $\mathrm{End}(E)$ is isomorphic to an order in the ring of integers of an imaginary quadratic field F, and we say that E has *complex multiplication* by $\mathcal{O}_F$. We explain how this concept generalises to abelian varieties of dimension > 1, and discuss some important properties of the Hasse–Weil L-functions in these cases. We begin with a definition:

Definition 2.52. We say that an abelian variety A has *CM* (or is of *CM-type*, or is a *CM-abelian variety*)[5] if $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$ contains a commutative sub-algebra of dimension $2\dim(A)$ over $\mathbb{Q}$.

Recall that an abelian variety A has an isogeny decomposition

$$A \sim A_1 \times A_2 \times \cdots \times A_n, \tag{2.4}$$

into a product of (not-necessarily-distinct) simple abelian varieties. From the definition of CM we obtain the following criterion.

Lemma 2.53. *Let A be an abelian variety, and consider the decomposition* (2.4) *above. Then A has CM if and only if each simple factor A_i has CM.*

Proof. See e.g. [Mil20, Remark 3.5]. □

In the case that A is a simple abelian variety, A has CM if and only if there is a CM-field F of degree $2\dim(A)$ over $\mathbb{Q}$ contained in $\mathrm{End}^0(A)$.

The reason we will be concerned with CM-abelian varieties is that the examples we will come to deal with in this project will be Jacobians of curves with automorphisms. These Jacobians will have larger endomorphism rings than generic abelian varieties and hence often the examples we deal with have CM. In particular, we will appeal to the following theorem, due to Shimura–Taniyama and Serre–Tate.

[5]Typically these bracketed terms are used in the literature to distinguish CM-abelian varieties from those which do not have CM, but do have endomorphism ring strictly larger than $\mathbb{Z}$. We ignore this potential confusion with gay abandon.

Theorem 2.54. *Let A be a simple CM-abelian variety over a number field K, with all its endomorphisms defined over K. Then there exists an algebraic Hecke character χ taking values in a CM-field F of degree $2\dim(A)$ in $\mathrm{End}^0(A)$, such that*

$$L(A/K,s) = \prod_{\tau:F\hookrightarrow\mathbb{C}} L(\chi_\tau, s).$$

Remark 2.55. Together with Theorem 2.51 this implies that the Hasse–Weil conjecture holds for A/K. ♢

We can use this theorem to obtain factorisations of $L(A/K,s)$ even when not all the endomorphisms of A are defined over K. We borrow the following two examples of [Mil98, §13].

Example 2.56. Let A be a simple abelian variety over a number field K, and let K'/K be the smallest (Galois) extension such that all the elements of the centre of $\mathrm{End}^0(A)$ are defined over K'. Then

$$L(A/K',s) = L(A/K,s)^m,$$

where $m = [K' : K]$. ♢

Example 2.57. Let A/K be a simple CM-abelian variety such that K'/K is the smallest (Galois) extension such that the endomorphisms of A are defined over K'. Take $F \subset \mathrm{End}^0(A)$ a CM-field of degree $2\dim(A)$ over $\mathbb{Q}$. Let $\Sigma_F = \mathrm{Hom}(F,\mathbb{C})$. Then we have

$$L(A/K',s) = L(A/K,s)^m = \prod_{\tau\in\Sigma_F} L(\chi_\tau, s),$$

where $m = [K' : K]$. In this case we obtain

$$\prod_{\tau\in\Sigma_F} L(\chi_\tau, s) = \left(\prod_{\tau\in\Sigma_F/\,\mathrm{Gal}(K'/K)} L(\chi_\tau, s) \right)^m,$$

because $\mathrm{Gal}(K'/K)$ acts faithfully on F. Now taking m^{th} roots yields

$$L(A/K,s) = \prod_{\tau\in\Sigma_F/\,\mathrm{Gal}(K'/K)} L(\chi_\tau, s),$$

which we can do because, for example, we know the trailing coefficients of the local polynomials are 1. ♢

3 Motives

The Hecke L-functions, and Hasse–Weil L-functions and their Artin twists are all examples of a much more general class of L-function; these are the motivic L-functions. The aim of this project is to show how one can define another type of motivic L-function associated to curves with automorphisms, and to understand what the analogous conjectures might be for these L-functions. In particular, the formalism set out here will give us access to a vast conjectural edifice which might offer insight into properties of our L-functions. For some philisophical background and some of the technical points, we refer to [Mil12].

3.1 Motives, morally

Firstly we offer a somewhat imprecise definition of 'motives'; we give a naïve construction which will largely be sufficient for our purposes (cf. Remark 3.26). From this point of view, a 'motive' will be a pure motive over a number field K with coefficients in a number field F, which we view only as a tuple of 'realisation data' consisting of vector spaces with some additional structures and comparison isomorphisms. In particular, a K-motive M of dimension $d = \dim M$ and weight $w = w(M)$ with coefficients in F consists of:

1. A d-dimensional F-vector space $H_B(M)$ with Hodge filtration by free $F\otimes\mathbb{C}$-modules
$$H_B(M)\otimes\mathbb{C} = \bigoplus_{i+j=w} H^{i,j}(M),$$
equipped with an F-linear involution σ_∞, defined for each real place of K, which exchanges $H^{i,j}(M)$ and $H^{j,i}(M)$. We view this as an action of complex conjugation.

2. A d-dimensional F-vector space $H_{\mathrm{dR}}(M)$ with exhaustive decreasing filtration
$$\{\mathrm{Fil}^k H_{\mathrm{dR}}(M)\}_{k\in\mathbb{Z}}.$$

3. For each prime λ of F, a d-dimensional F_λ-vector space $H_\lambda(M)$ with a continuous homomorphism
$$\rho_\lambda : G_K \to \mathrm{GL}(H_\lambda(M)),$$
and, for each prime $\mathfrak{p}$ of K coprime to λ of F, a local polynomial
$$P_\mathfrak{p}(M,T;\lambda) = \det(1 - T\,\mathrm{Frob}_\mathfrak{p}^{-1} \mid H_\lambda(M)^{I_\mathfrak{p}}) \in F[T]$$
independent of λ. Moreover, we assert that there exists a finite *exceptional set* $S = S(M)$ of primes $\mathfrak{p}$ of K such that $\rho_\lambda(I_\mathfrak{p}) = 1$ for $\mathfrak{p} \notin S$, and that for $\mathfrak{p} \notin S$, fixing an embedding $F \hookrightarrow \mathbb{C}$, the complex roots of $P_\mathfrak{p}(M,T)$ have absolute value $N(\mathfrak{p})^{-w/2}$.

Writing $F_\ell = \prod_{\lambda|\ell} F_\lambda$ and $H_\ell = \oplus_{\lambda|\ell} H_\lambda$, the data of M also includes the following comparison isomorphisms:

1. $c_{B,\mathrm{dR}} : H_B(M)\otimes_\mathbb{Q}\mathbb{C} \xrightarrow{\sim} H_{\mathrm{dR}}(M)\otimes_\mathbb{Q}\mathbb{C}$ such that $c_{B,\mathrm{dR}}$ respects the action of complex conjugation and, for all k,
$$c_{B,\mathrm{dR}}\left(\bigoplus_{i\geq k} H^{i,j}(M)\right) = \mathrm{Fil}^k H_{\mathrm{dR}}(M).$$

2. $c_{\ell,B} : H_\ell(M) \xrightarrow{\sim} H_B(M) \otimes_F F_\ell$ for each ℓ, such that $c_{\ell,B}$ respects the action of complex conjugation.

3. $c_{\ell,\mathrm{dR}} : (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} H_\ell(M))^{G_{\mathbb{Q}_\ell}} \xrightarrow{\sim} H_{\mathrm{dR}}(M) \otimes_F F_\ell$, for each ℓ, where B_{dR} is Fontaine's *ℓ-adic de Rham period ring.*

Given motives M and N, we can obtain new motives $M \otimes N$, $M \oplus N$ and M^* by taking tensor products, direct sums and duals respectively, on the level of vector spaces in the realisation data. To abuse notation, we will denote isomorphic motives (i.e isomorphic on the level of realisation data with isomorphisms which respect the comparison isomorphisms) as equal.

Definition 3.1. We say a motive M is *indecomposable over K* if we cannot write $M = N_1 \oplus N_2$ for K-motives N_1 and N_2 of positive dimension.

To such an M and each embedding $\tau \in \Sigma_F$, we associate a *motivic L-function* $L(M, \tau, s)$ via the Euler product

$$L(M, \tau, s) = \prod_{\mathfrak{p}} \tau \cdot \det(1 - N(\mathfrak{p})^{-s} \operatorname{Frob}_{\mathfrak{p}}^{-1} \mid H_\lambda(M)^{I_{\mathfrak{p}}})^{-1}.$$

Remark 3.2. There is a natural notion of isomorphisms between motives, which induces isomorphisms on the level of ℓ-adic representations and thereby equality on the level of L-functions. We will often abuse notation and identify motives which are strictly-speaking isomorphic, i.e. we implicitly will work with motives *modulo isomorphism.* $\diamondsuit$

Instead of varying τ, if $F/\mathbb{Q}$ is Galois we will often find it convenient to fix τ and let automorphisms $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ act on $L(M, s)$ to the same effect. We write

$$L^\sigma(M, \tau, s) = L(M, \tau \circ \sigma, s).$$

Notation. We will sometimes write $L(M, s)$ for the tuple $(L(M, \tau, s))_{\tau \in \Sigma_F}$, but in certain cases we may write $L(M, s)$ for a choice of $L(M, \tau, s)$ when the specific embedding does not matter.

A priori, each L-function $L(M, \tau, s)$ converges only for $\Re(s) > 1 + w(M)/2$, but the following is predicted:

Conjecture 3.3. *$L(M, \tau, s)$ has meromorphic continuation and satisfies a functional equation.*

This conjectural functional equation relates the *completed* L-function $\Lambda(M, s) = L_\infty(M, s) \cdot L(M, s)$ with $\Lambda(M^*, 1 + w - s)$. The factor $L_\infty(M, s)$ is determined by a *Γ-factor* and *conductor.* The former is a product

$$\gamma(M) = \prod_{i=1}^{r} \Gamma\left(\frac{s + \lambda_i}{2}\right),$$

for some r and rational 'shifts' λ_i. The latter, $c(M)$, should be a positive integer. Then we have

$$L_\infty(M, s) = \left(\frac{c(M)}{\pi^r}\right)^{s/2} \cdot \gamma(M).$$

Remark 3.4. As in (2.3), the functional equation is of the form

$$\Lambda(M,s) = w(M)\Lambda(M^*, 1+w-s),$$

where $w(M) \in S^1$. However, in order to compute with L-functions in Magma [BCP97]—using their adaptation of T. Dokchitser's L-function calculator [Dok04]—one need only specify $\gamma(M)$ and $c(M)$ (cf. Section 5). ♢

We take this conjecture as a standing assumption. We also give some examples which will be relevant for this work:

Example 3.5. For an Artin representation (V,ρ) of G_K factoring through $\mathrm{Gal}(F/K)$, we obtain a motive $[\rho]$ defined over K with coefficients in F with $d([\rho]) = 1$ and $w([\rho]) = 0$ given by the following data:

- $H_B([\rho]) = V$, with $\sigma_\infty = \rho(c)$ where $c \in \mathrm{Gal}(\overline{K}/K)$ is complex conjugation and

$$H^{i,j}([\rho]) = \begin{cases} V \otimes \mathbb{C} & \text{if } i = j = 0 \\ 0 & \text{else.} \end{cases}$$

- $H_{\mathrm{dR}}([\rho]) = (V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^{G_K}$ and

$$\mathrm{Fil}^k H_{\mathrm{dR}}([\rho]) = \begin{cases} H_{\mathrm{dR}}([\rho]) & \text{if } k \leq 0 \\ 0 & \text{else.} \end{cases}$$

- $H_\lambda([\rho]) = V \otimes_F F_\lambda$, with G_K acting on the first factor.
- $c_{B,\mathrm{dR}}$ defined by the inclusion $H_{\mathrm{dR}}([\rho]) \subseteq V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$.
- $c_{\ell,B}$ given by the factor-wise identity.
- We omit an explicit description of $c_{\ell,\mathrm{dR}}$.

The L-functions $L(M,s)$ are the tuple $(L(\rho^\sigma, s))_{\sigma\in\mathrm{Gal}(F/K)}$. ♢

Example 3.6. To an abelian variety A/K, we obtain a motive $h^1(A)$ of dimension $\dim A$ and weight 2 with coefficients in $\mathbb{Q}$ and the following realisation data:

- $H_B(h^1(A)) = H^1(A(\mathbb{C}), \mathbb{Q})$ is the first singular cohomology, σ_∞ is induced by complex conjugation on $A(\mathbb{C})$ and

$$H^{i,j}(h^1(A)) = \begin{cases} H^0(A(\mathbb{C}), \Omega_h^1) & \text{if } (i,j) = (1,0) \\ H^1(A(\mathbb{C}), \Omega_h^0) & \text{if } (i,j) = (0,1) \\ 0 & \text{else,} \end{cases}$$

  where Ω_h^r is the sheaf of holomorphic r-forms on $A(\mathbb{C})$.

- $H_{\mathrm{dR}}([\rho]) = (V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^{G_K}$ and

$$\mathrm{Fil}^k H_{\mathrm{dR}}([\rho]) = \begin{cases} H_{\mathrm{dR}}([\rho]) & \text{if } k \leq 0 \\ 0 & \text{else.} \end{cases}$$

- $H_\ell(h^1(A)) = \mathrm{Hom}(T_\ell A, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, with G_K acting on the first factor.

- Comparison isomorphisms following from various classical results.

The L-function $L(h^1(A), s)$ is the usual Hasse–Weil L-function $L(A/K, s)$. $\diamondsuit$

Example 3.7. The inverse Tate motive $\mathbb{Q}(1)$ has dimension 1, weight -2 and realisation data:

- $H_B(\mathbb{Q}(1)) = \mathbb{Q}$, $\sigma_\infty = -1$,

$$H^{i,j}(\mathbb{Q}(1)) = \begin{cases} \mathbb{C} & \text{if } (i,j) = (-1,-1) \\ 0 & \text{else.} \end{cases}$$

- $H_{\mathrm{dR}}(\mathbb{Q}(1)) = \mathbb{Q}$ and

$$\mathrm{Fil}^k H_{\mathrm{dR}}(\mathbb{Q}(1)) = \begin{cases} \mathbb{Q} & \text{if } k \leq -1 \\ 0 & \text{else.} \end{cases}$$

- $H_\ell(\mathbb{Q}(1)) = T_l(\overline{\mathbb{Q}}^\times) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ with $G_\mathbb{Q}$ acting on the first factor by the ℓ-adic cyclotomic character.

- $c_{B,\mathrm{dR}}$ is multiplication-by-$2\pi i$.

The L-function $L(\mathbb{Q}(1), s)$ is the shifted Riemann zeta function $\zeta(s+1)$. $\diamondsuit$

Notation. The inverse Tate motive has an inverse $\mathbb{Q}(-1)$, *the Tate motive*. For integers n we write $\mathbb{Q}(n)$ for $\mathbb{Q}(\mathrm{sgn}(n))^{\otimes |n|}$, and for motives M we write $M(n)$ for $M \otimes \mathbb{Q}(n)$, so that $L(M(n), s) = L(M, s+n)$.

Let us introduce some notation before we discuss Hecke L-functions. We consider the category $\mathbf{M}_K^{\mathrm{av}}$ of 'motives coming from abelian varieties over K', defined as the subcategory in the category of K-motives generated as a Tannakian category by the Artin motives over K and the motives $h^1(A)$ of abelian varieties A/K (see [DM12, §6]). The subcategory obtained by further requiring that A has complex multiplication is denoted by $\mathbf{CM}_K$. For the subcategories of motives with coefficients in F we write $\mathbf{M}_K^{\mathrm{av}}(F)$ and $\mathbf{CM}_K(F)$, respectively.

Example 3.8. To a Hecke character χ of K of weight w taking values in F we can associate a motive $M(\chi) \in \mathbf{CM}_K(F)$ of weight w and dimension 1, such that for every embedding $\tau\colon F \hookrightarrow \mathbb{C}$ we have $L(M(\chi), \tau, s) = L(\chi_\tau, s)$. $\diamondsuit$

In fact, all 1-dimensional motives coming from abelian varieties correspond to Hecke characters in this way.

Theorem 3.9 ([Sch88, Theorem 2.6.6.1])**.** *For every motive $M \in \mathbf{M}_K^{\mathrm{av}}(F)$ of dimension 1, there exists an algebraic Hecke character χ of K with values in F such that M is isomorphic to $M(\chi)$.*

Remark 3.10. In this setting we may recast Theorem 2.54 as the statement that, if A has complex multiplication, then there exists a Hecke character χ taking values in a CM-field F with Galois closure F' and Galois group G, such that we have a decomposition

$$h^1(A) = \bigoplus_{\sigma \in G/H} M(\chi_\sigma)$$

on the level of motives, where $H = \mathrm{Gal}(F'/F)$ and $\chi_\sigma = \sigma \circ \chi$ is well-defined. $\diamondsuit$

3.2 Motives, technically

We now offer a more technical definition of 'Chow' motives.

Definition 3.11 (Algebraic cycle)**.** Let X be a smooth projective variety over k. A *prime algebraic cycle* Z of X is a closed irreducible subvariety. We write $C(X)$ for the free abelian group on prime algebraic cycles of X, the elements of which are *algebraic cycles.*

We have a decomposition

$$C(X) = \bigoplus_{j=0}^{\dim X} C^j(X),$$

where $C^j(X)$ is the free abelian group on prime algebraic cycles of codimension j.

Definition 3.12 (Rational equivalence)**.** Algebraic cycles Z and Z' of X are *rationally equivalent* if there exists an algebraic cycle $\hat{Z}$ on $X \times \mathbb{P}^1$ with fibre $Z - Z'$ over one point of $\mathbb{P}^1$ and 0 over another point.

By taking equivalence classes with respect to rational equivalence we obtain quotients $C^j_{\mathrm{rat}}(X)$, and well-defined maps

$$C^r_{\mathrm{rat}}(X) \times C^s_{\mathrm{rat}}(X) \to C^{r+s}_{\mathrm{rat}}(X)$$

given by the intersection product. Thereby the *Chow ring* $C_{\mathrm{rat}}(X)$ is a $\mathbb{Q}$-algebra and moreover there is a map

$$\mathrm{cl} : C_{\mathrm{rat}}(X) \to \bigoplus_{i=0}^{2\dim X} H^i(X),$$

where $H^i(X)$ is the i^{th} singular cohomology group, which doubles degrees and sends intersection products to cup products.

An element Z of $C^{\dim X}_{\mathrm{rat}}(X \times Y)$ can be thought of as a map on cohomology $H^\bullet(X) \to H^\bullet(Y)$ via the following commutative diagram

$$\begin{tikzcd}
 & H^\bullet(X \times Y) \arrow[r, "z \mapsto z \cup \mathrm{cl}(Z)"] & H^\bullet(X \times Y) \arrow[dr, "\pi_{Y,*}"] & \\
H^\bullet(X) \arrow[ur, "\pi_X^*"] \arrow[rrr, dashed, "x \mapsto \pi_{Y,*}(\pi_X^*(x) \cup \mathrm{cl}(Z)),"] & & & H^\bullet(Y)
\end{tikzcd}$$

where $\pi_\bullet$ is the projection onto $\bullet$. Singular cohomology is just one example of many cohomology theories we could use here[6]; the key point being that we can forgo making a choice of cohomology theory when defining morphisms in our category of motives.

Definition 3.13 (Category of effective motives)**.** The category $\mathcal{M}^{\mathrm{eff}}(k)$ of *effective motives* over k consists of objects $h(X, e)$ for each smooth projective variety X over k and idempotent $e \in \mathrm{End}(hX) := C^{\dim X}_{\mathrm{rat}}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We define

$$\mathrm{Hom}(h(X, e), h(Y, f)) := f \circ (C^{\dim X}_{\mathrm{rat}}(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}) \circ e.$$

We think of the object $h(X, e)$ as being the direct summand of some sort of 'universal cohomology' hX corresponding to projection onto the image of e. We have defined our morphisms in a clever way—as discussed above—to avoid having to say what this cohomology should be. Lastly, to allow us to take dual motives, we invert the Tate motive.

[6]Indeed, we could take any Weil cohomology theory.

Definition 3.14 (Tate motive)**.** Let $\Delta_X \in \mathrm{End}(hX)$ be projection onto the diagonal. We have

$$h(\mathbb{P}^1, \Delta_{\mathbb{P}^1}) = h(\mathbb{P}^1, e_0) \oplus h(\mathbb{P}^1, e_2),$$

where e_0 is projection onto $\mathbb{P}^1 \times \{0\}$ and e_2 is projection onto $\{0\} \times \mathbb{P}^1$. We call $h(\mathbb{P}^1, e_2)$ the *Tate motive.*

Remark 3.15. This notion of Tate motive aligns completely with the description of $\mathbb{Q}(-1)$ given earlier. ◇

Definition 3.16 (Category of Chow motives)**.** The category $\mathcal{M}(k)$ of *Chow motives* over k is obtained by inverting the Tate motive. Formally, we view its objects as data $h(X, e, n)$ where X is a smooth projective variety and $e \in \mathrm{End}(hX)$ is an idempotent as before. Now n is an integer and we define

$$\mathrm{Hom}(h(X, e, n), h(Y, f, m)) := f \circ (C^{\dim X + m - n}_{\mathrm{rat}}(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}) \circ g.$$

Remark 3.17. The moral picture here is that the motive $h(X, e, n)$ is a twist of the effective motive $h(X, e, 0) = h(X, e)$; i.e. '$h(X, e, n) = h(X, e)(n)$' in the earlier notation. ◇

We summarise some of the known properties of $\mathcal{M}(k)$:

Theorem 3.18. (i) *$\mathcal{M}(k)$ is an additive category, e.g.*

$$h(X, e, n) \oplus h(Y, f, n) = h(X \sqcup Y, e \oplus f, n).$$

(ii) *$\mathcal{M}(k)$ is a pseudo-abelian category, i.e. if $M = h(X, e, n)$ and $f \in \mathrm{End}(M)$ then*

$$M = h(X, e - efe, n) \oplus h(X, efe, n).$$

(iii) *There is a tensor product structure on $\mathcal{M}(k)$ via*

$$h(X, e, n) \otimes h(Y, f, m) = h(X \times Y, e \times f, n + m).$$

We would like to decompose hX according to the decomposition of $H^\bullet(X)$, but this is contingent in general on the following conjecture (see [Mur93, Conjectures A, B & C]):

Conjecture 3.19 (Murre)**.** *The diagonal element $\Delta_X \in C^{\dim X}_{rat}(X \times X)$ has a canonical decomposition into orthogonal idempotents*

$$\Delta_X = \pi_0 \oplus \cdots \oplus \pi_{2\dim X}$$

such that this decomposition induces the decomposition

$$H^\bullet(X) = H^0(X) \oplus \cdots \oplus H^{2\dim X}(X)$$

for any Weil cohomology theory H, under the analogous maps to cl. Moreover, this decomposition should satisfy some further nice properties.

Remark 3.20. This conjecture (excluding in some cases the 'further nice properties') is known to hold in several special cases; notably for curves, projective spaces, surfaces and abelian varieties. Therefore, bearing in mind our aims, little of our discussion is morally contingent on this conjecture. ◇

Assuming Conjecture 3.19, we obtain a decomposition

$$hX = h(X, \pi_0) \oplus \cdots \oplus h(X, \pi_{2\dim X}).$$

Definition 3.21. A *pure (Chow) motive* is a motive of the form $eh^i(X)(m) := h(X, e\pi_i e, m)$. To such motives we can assign a *weight* $w = w(M)$ such that $w(eh^i(X)(m)) = i - 2m$ and weights add under tensor products of motives.

Remark 3.22. We recover our naïve description of pure motives by assigning realisation data to an effective motive $M = eh^i(X)$ via

$$H_\bullet(M) = eH^i_\bullet(X),$$

and the additional data comes for free from general results about cohomology theories of varieties. This should help to elucidate our earlier notation. ♢

Definition 3.23. For a finite-dimensional semi-simple $\mathbb{Q}$-algebra A, the category $\mathcal{M}_A(k)$ of motives over k with an action of A is the sub-category of *motives with coefficients in* A.

Explicitly, the objects of $\mathcal{M}_A(k)$ are pairs (M, φ) where M is a motive over k and $\varphi : A \to \mathrm{End}(M)$ is a ring homomorphism. Upon decomposing $A_\mathbb{C} := A \otimes_\mathbb{Q} \mathbb{C}$ via

$$A_\mathbb{C} \cong \prod_{i=1}^r M_{n_i}(\mathbb{C})$$

and letting $\{e_i\}$ be the relevant projections, we obtain an equivalence of categories

$$\mathcal{M}_A(k) \cong \prod_{i=1}^r \mathcal{M}(k)$$

via $V \mapsto (e_i V)_i$ and $(V_i)_i \mapsto \prod_i V_i \otimes \mathbb{C}^{n_i}$—see, for example [BF01, §4]. To an object $(M, \varphi) \in \mathcal{M}_A(k)$ we can, under a suitable compatibility assumption (see [BF01, Conjecture 3 & Remark 7]), assign an L-function $L(M_\varphi, s)$ taking values in $\mathbb{C}^r$. We view this as a tuple of L-functions corresponding to the decomposition of $A_\mathbb{C}$.

3.3 Deligne's period conjecture

We are interested in applying the following conjecture of Deligne to the L-functions $L(C/K, \rho, s)$, for which we need the following definition:

Definition 3.24 (Critical value)**.** The integer n is a *critical value* for a motive M if neither $L_\infty(M, s)$ nor $L_\infty(M^*(-w), 1-s)$ have a pole at $s = n$. We say that a motive M is *critical* if 0 is a critical value for M, so that n is a critical value for M if and only if $M(n)$ is critical.

Example 3.25. For abelian varieties A, the motive $h^1(A)$ has $m = 1$ as a critical value. Moreover, because the poles of $\Gamma(s)$ are precisely at the non-positive integers, the form of the Γ-factor in the functional Equation (2.3) implies that any submotive of $h^1(A)$ also has $m = 1$ as a critical value. ♢

Remark 3.26. For many of our purposes, it is sufficient to use our loose definition of pure motives without worrying about the more technical categorical constructions. Indeed, in the article [Del79] in which Deligne offers the following conjecture, this loose definition is all he deems necessary (c.f. [loc. cit., (0.12)]). ♢

Conjecture 3.27 (Deligne's period conjecture). *Consider a motive M defined over $\mathbb{Q}$ with coefficients in F and with critical value m. Let $H_B(M)^+$ be the $+1$-eigenspace of σ_∞, and define*

$$\alpha_M^+ : H_B(M)^+ \otimes \mathbb{C} \hookrightarrow H_B(M) \otimes \mathbb{C} \xrightarrow{c_{B,\mathrm{dR}}} H_{dR}(M) \otimes \mathbb{C} \twoheadrightarrow H_{dR}^+(M) \otimes \mathbb{C},$$

where the last map is the natural quotient map. Denote by $c^+(M) \in (F \otimes_{\mathbb{Q}} \mathbb{C})^\times$ a representative of the class of $\det(\alpha_M^+)$ in $(F \otimes_{\mathbb{Q}} \mathbb{C})^\times / F^\times$, which we view as a tuple $(c^+(\sigma, M))_{\sigma \in \Sigma_F} \in (\mathbb{C}^\times / F^\times)^{\Sigma_F}$ indexed by the embeddings of F.

(i) $\mathrm{ord}_{s=m} L(M, \tau, s)$ *is independent of τ.*

(ii) *There exists $x \in F$ such that, for each $\sigma \in \Sigma_F$,*

$$L(M, \sigma, m) = \sigma(x) c^+(\sigma, M(m)).$$

Remark 3.28. Of course, Deligne's conjecture relies upon the validity of analytic continuation of $L(M, s)$. We take as a standing assumption throughout this document that for all motives M, the motivic L-function $L(M, s)$ has meromorphic continuation and satisfies its functional equation. ♢

Remark 3.29. Although the above conjecture is only stated over $\mathbb{Q}$, for each motive M over a number field K there exists a motive $\mathrm{Res}_{\mathbb{Q}}^K M$ over $\mathbb{Q}$ admitting the same L-functions as M. This is Weil's *restriction of scalars* functor; morally one replaces generators for $K/\mathbb{Q}$ by formal variables subject to the relevant relations. Thereby we may write $c^+(M)$ to mean $c^+(\mathrm{Res}_{\mathbb{Q}}^K M)$ is M is defined over a number field K. ♢

Example 3.30. The motive $\mathbb{Q}(n)$ is critical for even, positive n. Recalling Example 3.7, the map $c^+(\mathbb{Q}(n)) = (2\pi i)^n$. Thereby, Deligne's conjecture recovers the well-known result that

$$\zeta(2n) \sim_{\mathbb{Q}^\times} \pi^{2n},$$

for $n > 0$. ♢

Example 3.31. As all of the motives we wish to discuss arise as summands of motives associated to abelian varieties, we will be keenly interested in the interplay between Conjecture 3.27 and the BSD conjecture. The following discussion comes from [Del79, §4]: fix an abelian variety $A/\mathbb{Q}$ and bases $\{e_i\}$, $\{\omega_j\}$ for $H^1(A(\mathbb{C}), \mathbb{Z})^+ \subset H_B^+(h^1(A))$ and $H_{\mathrm{dR}}^+(h^1(A))$ respectively. The period $c^+(h^1(A)(1))$ is given (for these choices!) by

$$c^+(h^1(A)(1)) = \int_e \omega,$$

where e is the Pontryagin product of the e_i and ω is the exterior product of the ω_j. Recall that BSD predicts

$$L(h^1(A)(1), 0) \sim_{\mathbb{Q}} \int_{A(\mathbb{R})} |\omega| =: \Omega_+(A).$$

This is compatible with Deligne's conjecture because we find the relation

$$\Omega_+(A) = [A(\mathbb{R}) : A(\mathbb{R})^\circ] \cdot \left| \int_e \omega \right|,$$

where $A(\mathbb{R})^\circ$ is the connected component of the identity. ♢

Remark 3.32. This differs from the quantity $\Omega(A/\mathbb{Q})$ appearing in $\mathrm{BSD}_{A/\mathbb{Q}}$ by a rational factor; there exists a rational number a_ω such that

$$\omega a_\omega = \omega_p^0,$$

where ω_ν^0 is a Néron differential at the finite place ν. We then have

$$\frac{\Omega(A/\mathbb{Q})}{\Omega_+(A)} = \prod_p |a_\omega|_p \,.$$

One notes that the product formula guarantees that the product of $\Omega_+(A)$ and the above rational factor is independent of choice of ω. For abelian varieties over general number fields, we replace a_ω with a fractional ideal. ♢

Example 3.33. In [Eva21], Evans gives the following formula for the Deligne-period (associated to a fixed choice of complex embedding) of $h^1(A)\otimes[\chi]$ for an Artin representation χ:

$$\Omega(A,\rho) = \frac{\Omega_+(A)^{d_+(\rho)}\Omega_-(A)^{d_-(\rho)}w(\rho)^{\dim A}}{\sqrt{f_\rho^{\dim A}}}, \tag{3.1}$$

and shows its compatibility with the BSD-conjecture. Here:

- $\Omega_-(A) = \left|\int_{e^-}\omega\right|$, where e^- is the Pontyragin product of a choice of basis $\{e_i^-\}$ for $H_B^-(h^1(A))$.
- If V is a vector space over a number field realising ρ, then $d_\pm(\rho)$ is the dimension of the subspace on which complex conjugation acts by ±1.
- $w(\rho)$ is the root number appearing in the functional equation of $L(\rho, s)$.
- f_ρ is the conductor of ρ. ♢

3.4 An application to ranks

Here we show how the Deligne–Gross rank formula can be deduced from the usual BSD rank formula via Deligne's conjecture. We will mimic this proof later on to deduce a rank formula for a new class of L-functions.

Proposition 3.34. *Conjecture 3.27 (i) and the usual BSD rank formula (Conjecture 2.38) imply the Deligne–Gross rank formula (Conjecture 2.44).*

Proof. Fix an abelian variety A/K and a finite Galois extension F/K with $G = \mathrm{Gal}(F/K)$. Note that by Lemma 2.42 it suffices prove the result for irreducible representations ρ of G. Recall from Example 2.22 that the representation $A(F)_\mathbb{C}$ is rational, so Galois conjugate representations occur in $A(F)_\mathbb{C}$ with equal multiplicity by Lemma 2.24. Further note that $L(A/K, \rho^\sigma, s) = L^\sigma(A/K, \rho, s)$ for any $\sigma \in \mathrm{Gal}(K_G/\mathbb{Q})$. We first consider the case that $G \cong C_n$ is cyclic, inducting on the number of divisors of n. The case $n = 1$ is Remark 2.45.

For general n, consider a non-trivial subgroup $H \leq G$. By hypothesis, any representation of G factoring as

$$\rho : G \twoheadrightarrow G/H \xrightarrow{\widetilde{\rho}} \mathbb{C}^\times$$

satisfies $\mathrm{ord}_{s=1} L(A/K, \rho, s) = \langle A(F^H)_\mathbb{C}, \widetilde{\rho}\rangle$, since ρ and $\widetilde{\rho}$ define the same Artin representation. Further, the fact that ρ factors through $\mathrm{Gal}(F^H/K)$ implies that $\langle A(F)_\mathbb{C}, \rho\rangle =$

$\langle A(F^H)_{\mathbb{C}}, \widetilde{\rho}\rangle$, showing that ρ satisfies the Deligne–Gross rank formula. Two irreducible representations of G are Galois conjugate if and only if they factor faithfully through the same quotient G/H. Therefore, we have

$$\sum_{\rho \text{ irrep } G} \langle A(F)_{\mathbb{C}}, \rho\rangle = \sum_{d|n,\ d\neq n} \phi(d)\langle A(F)_{\mathbb{C}}, \rho_d\rangle + \phi(n)\langle A(F)_{\mathbb{C}}, \rho_n\rangle, \tag{3.2}$$

where ρ_d is irreducible with kernel $C_{n/d} \leq C_n$, and ϕ denotes Euler's totient function. On the other hand, from Deligne's conjecture and the induction hypothesis it follows that

$$\begin{aligned}\sum_{\rho \text{ irrep } G} \operatorname{ord}_{s=1} L(A/K,\rho,s) &= \sum_{d|n,\ d\neq n} \phi(d)\operatorname{ord}_{s=1} L(A/K,\rho_d,s) + \phi(n)\operatorname{ord}_{s=1} L(A/K,\rho_n,s)\\ &= \sum_{d|n,\ d\neq n} \phi(d)\langle A(F)_{\mathbb{C}}, \rho_d\rangle + \phi(n)\operatorname{ord}_{s=1} L(A/K,\rho_n,s). \qquad (3.3)\end{aligned}$$

The left-hand sides of (3.2) and (3.3) are equal by BSD and Example 2.43, so we find

$$\operatorname{ord}_{s=1} L(A/K,\rho_n,s) = \langle A(F)_{\mathbb{C}}, \rho_n\rangle.$$

For general G, we easily reduce to the cyclic case by Artin induction and Frobenius reciprocity; we spell out the proof here so we can skip over future applications. Artin induction gives

$$\rho^{\oplus d} = \bigoplus_{H<G \text{ cyclic}} a_H \operatorname{Ind}_H^G \rho_H$$

for ρ_H a representation of H, with d and each a_H integers. Hence

$$\begin{aligned}\operatorname{ord}_{s=1} L(A/K,\rho,s) &= \frac{1}{d}\sum_{H<G \text{ cyclic}} a_H \operatorname{ord}_{s=1} L(A/F,\operatorname{Ind}_H^G \rho_H,s)\\ &= \frac{1}{d}\sum_{H<G \text{ cyclic}} a_H \operatorname{ord}_{s=1} L(A/F^H,\rho_H,s)\\ &= \frac{1}{d}\sum_{H<G \text{ cyclic}} a_H \langle \operatorname{Res}_H^G A(F)_{\mathbb{C}}, \rho_H\rangle\\ &= \frac{1}{d}\sum_{H<G \text{ cyclic}} a_H \langle A(F)_{\mathbb{C}}, \operatorname{Ind}_H^G \rho_H\rangle\\ &= \langle A(F)_{\mathbb{C}}, \rho\rangle,\end{aligned}$$

by the Artin formalism and Frobenius reciprocity. □

4 Curves with automorphisms

4.1 Setting the scene

We consider a smooth projective curve C defined over a finite extension $K/\mathbb{Q}$, under the action of an automorphism group $G \leq \mathrm{Aut}_K(C)$. The L-function of C is given by

$$L(C/K,s) = \prod_{\mathfrak{p}} \det(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} N(\mathfrak{p})^{-s} \mid V_\ell^{I_\mathfrak{p}})^{-1},$$

where $V_\ell = T_\ell J_C \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, independent of choice of ℓ. Fixing an embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we note that V_ℓ admits an action of G, under which $V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{C}$ decomposes as a sum of irreducible representations of G. By choosing ℓ appropriately, we can view this as a decomposition of V_ℓ itself. Hence we obtain, for $\{\rho_i\}$ the irreducible representations of G,

$$V_\ell \cong \bigoplus_i \rho_i^{\oplus r_i}, \tag{4.1}$$

where $r_i = \dim \mathrm{Hom}_G(\rho_i, V_\ell)$. We make an immediate observation.

Lemma 4.1. *Suppose ρ is a representation of G, and that $\sigma \in \mathrm{Gal}(K_G/\mathbb{Q})$. Then ρ and ρ^σ occur with the same multiplicity in the decomposition* (4.1).

Proof. This follows from the fact that, for any $\alpha \in G$, the characteristic polynomial of α acting on V_ℓ is independent of ℓ (e.g. [EvdGM, Theorem 12.18 & Cor. 12.20]); indeed it lies in $\mathbb{Z}[T]$. Hence Galois conjugate eigenvalues must appear with the same multiplicity. If G were cyclic we would then be done. Else, suppose that ρ_i and ρ_j are Galois conjugate representations of G. From the above, we have an isomorphism on the level of restricted representations

$$\mathrm{Res}^G_{\langle\alpha\rangle} \rho_i^{\oplus r_i} \cong \mathrm{Res}^G_{\langle\alpha\rangle} \rho_j^{\oplus r_j},$$

given by the Galois action. This holds for all α, i.e. for all cyclic subgroups of G, so the characters of $\rho_i^{\oplus r_i}$ and $\rho_j^{\oplus r_j}$ are conjugate and $r_i = r_j$. $\square$

Note now that the action of G_K on V_ℓ commutes with that of G, and hence we obtain a factorisation

$$\det(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} N(\mathfrak{p})^{-s} \mid V_\ell^{I_\mathfrak{p}}) = \prod_i \det(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} N(\mathfrak{p})^{-s} \mid (\rho_i^{\oplus r_i})^{I_\mathfrak{p}}), \tag{4.2}$$

on the level of local polynomials. Taking this as inspiration, we offer the following definition which will give us an analogue of the Artin formalism.

Definition 4.2. For each representation ρ of G

$$L(C/K,\rho,s) = \prod_{\mathfrak{p}} \det(1 - N(\mathfrak{p})^{-s} \mathrm{Frob}_{\mathfrak{p}}^{-1} \mid \rho)^{-1},$$

where $\det(f \mid \rho)$ is the determinant of the map

$$\mathrm{Hom}_G(\rho, V_\ell)^{I_\mathfrak{p}} \to \mathrm{Hom}_G(\rho, V_\ell)^{I_\mathfrak{p}}$$

given by postcomposition with f.

Remark 4.3. We postpone the proof that these representations form a 'compatible system' of ℓ-adic representations to later on, see Proposition 4.16. $\diamondsuit$

Remark 4.4. There is an obvious notational clash with Artin-twists, but there is no ambiguity because it will always be clear from context whether ρ is a representation of the group G of automorphisms, or of G_K. We use this notation to stress the analogy, cf. Remark 4.17 ♢

First, we note that these L-functions are motivic.

Proposition 4.5. *The L-functions $L(C/K, \rho, s)$ are motivic.*

Remark 4.6. In light of Proposition 4.16, we can drop the dependence on [BF01, Conjecture 3] for our purposes. ♢

Proof. We may view $h^1(J_C)$ as a motive with coefficients in $\mathbb{Q}[G]$. If ρ is one-dimensional, then we obtain from the above a motive admitting L-function $L(C/K, \rho, s)$ via the projection onto the ρ-part of $\mathbb{Q}[G]$. For the general case, use Brauer induction. □

Notation. From now on we may denote the motive admitting (for an implicit fixed choice of embedding) the L-function $L(C/K, \rho, s)$ by $M^{\rho}_{C/K}$.

The construction of Definition 4.2 gives us an analogue of the Artin formalism (cf. Remark 2.26):

Lemma 4.7. (i) *For ρ_1 and ρ_2 representations of G,*

$$L(C/K, \rho_1 \oplus \rho_2, s) = L(C/K, \rho_1, s) \cdot L(C/K, \rho_2, s).$$

(ii) *For $H \leq G$, and ρ a representation of H we have*

$$L(C/K, \operatorname{Ind}_H^G \rho, s) = L(C/K, \rho, s).$$

Example 4.8. Taking H to be the trivial group we recover

$$L(C/K, s) = \prod_{\rho} L(C/K, \rho, s), \tag{4.3}$$

where this product is over irreducible ρ appearing in $\operatorname{Ind}_1^G 1$. Moreover, when G is abelian, the factorisation (4.3) is precisely the factorisation of L-functions induced by the factorisation (4.2). ♢

We often find that the L-functions arising in this way are themselves Hasse–Weil L-functions, their Artin-twists, or Hecke L-functions—or products thereof. These L-functions have all been well-studied, and so we will often be quick to dismiss these examples as ‘uninteresting’. There is an incredibly rich theory underlying these examples, so of course we mean only ‘have been seen before’.

Example 4.9. By Theorem 3.9, the 1-dimensional submotives of abelian varieties correspond to Hecke characters. Hence, if ρ and $\operatorname{Hom}_G(\rho, V_\ell)$ are 1-dimensional, then the L-function $L(C/K, \rho, s)$ is a Hecke L-function. ♢

Example 4.10. Consider the case of curves C/K such that J_C has CM. We saw in Theorem 2.54 that the L-function $L(C/K, s)$ decomposes into a product of Hecke characters. Indeed, recalling Remark 3.10, it is the case that $h^1(J_C)$ decomposes as a sum of 1-dimensional Hecke motives. Each $M^{\rho}_{C/K}$ then further decomposes as a sum of Hecke motives. In particular, each of the L-functions $L(C/K, \rho, s)$ is a product of Hecke L-functions. ♢

When dealing with an abelian automorphism group, we will typically restrict our interest to curves whose automorphisms are defined over the same field as the defining equation, because if the automorphisms are defined over a larger field then we are often in the situation of Example 2.56. In this case we often find that the L-functions $L(C/K, \rho, s)$ are just Hasse–Weil L-functions, and hence are uninteresting.

Example 4.11. Suppose $C/\mathbb{Q}$ admits $\mathbb{Q}$-simple Jacobian, such that the cyclic group $G \cong C_n$ acts on C by automorphisms with field of definition $K = \mathbb{Q}(\zeta_n)$. Suppose further that $\mathbb{Q}[G]$ is contained in the centre of $\mathrm{End}^0(J_C)$. It follows from Example 2.56 that $L(C/K, s) = L(C/\mathbb{Q}, s)^{\phi(n)}$. Lemma 4.1 says that the decomposition (4.3) is trivial or we have

$$L(C/K, s) = \prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} L(C/K, \rho^\sigma, s),$$

for ρ a faithful representation of G. In the latter case, because C is defined over $\mathbb{Q}$, we have that

$$\det(1 - T \operatorname{Frob}_{\mathfrak{p}^\sigma}^{-1} \mid \rho) = \det(1 - T \operatorname{Frob}_{\mathfrak{p}}^{-1} \mid \rho)^\sigma;$$

hence $\prod_{\mathfrak{p} | p} \det(1 - T \operatorname{Frob}_{\mathfrak{p}}^{-1} \mid \rho)$ is a rational polynomial. We also have

$$\det(1 - T \operatorname{Frob}_{\mathfrak{p}}^{-1} \mid \rho^\sigma) = \det(1 - T \operatorname{Frob}_{\mathfrak{p}}^{-1} \mid \rho)^\sigma,$$

and so it follows that the decomposition (4.3) is precisely the factorisation

$$L(C/K, s) = L(C/\mathbb{Q}, s)^{\phi(n)}$$

of Example 2.56. $\diamondsuit$

We note that the local polynomials of the L-functions from Definition 4.2 should be defined over a suitable cyclotomic field:

Proposition 4.12. *The local polynomials* $\det(1 - T \operatorname{Frob}_{\mathfrak{p}}^{-1} \mid \rho_i)$ *lie in* $K_G[T]$.

Proof. Write n for the exponent of G, such that $K_G = \mathbb{Q}(\zeta_n)$. Certainly the coefficients of each local polynomial lie in $\mathbb{Q}_\ell$ for each ℓ such that $K_G \subset \mathbb{Q}_\ell$, and moreover the local polynomials are independent of choice of ℓ. Hence they lie in the intersection

$$\bigcap_{\ell \in \mathcal{L}} (\mathbb{Q}_\ell \cap \overline{\mathbb{Q}}), \tag{4.4}$$

where $\mathcal{L} = \{\ell \text{ prime} \mid K_G \subset \mathbb{Q}_\ell\}$. It suffices to prove the claim that this intersection is K_G. Equivalently, we may take $\mathcal{L}$ to be the set

$$\{\ell \text{ prime} \mid l \equiv 1 \pmod{n}\},$$

which has density $1/[K_G : \mathbb{Q}] = 1/\phi(n)$ in the primes by Dirichlet's theorem on primes in arithmetic progressions. Suppose that F/K_G were a number field contained in the intersection (4.4), with $q(X) \in \mathbb{Z}[X]$ the absolute minimal polynomial of a primitive element for $F/\mathbb{Q}$. By Frobenius' density theorem (e.g. [SL96][7]), the proportion of primes ℓ for which $q(X)$ splits mod ℓ is then $1/|\operatorname{Gal}(q)|$, which is certainly smaller than $1/[K_G : \mathbb{Q}]$. In particular, there exists $\ell \in \mathcal{L}$ such that $F \not\subset \mathbb{Q}_\ell$. The claim follows. $\square$

Remark 4.13. Typically, we will consider affine patches of planar curves with automorphisms of the form $(x, y) \mapsto (\zeta_n x, \zeta_m y)$ and C will be defined over $K = K_G$, so that these local polynomials are defined over K. $\diamondsuit$

[7] This is also an excellent historical account of both Chebotarëv's and Frobenius' density theorems.

Proposition 4.14. *For all $\sigma \in \mathrm{Gal}(K_G/\mathbb{Q})$ we have*

$$L(C/K, \rho^\sigma, s) = L^\sigma(C/K, \rho, s).$$

We first give an example to demonstrate the strategy of proof:

Example 4.15. Consider the curve $C:\ y^3 = x^4 + x + \zeta_3$, with an action of C_3 given by $y \mapsto \zeta_3 y$. Under this action we obtain a decomposition

$$V_\ell \cong \rho^{\oplus 3} \oplus \overline{\rho}^{\oplus 3}$$

(cf. Section 5.2), where α acts on ρ by ζ_3.

Compute $\#\mathrm{Fix}(\alpha \circ \Phi^j)$ on $\widetilde{C}(\overline{\mathbb{F}}_\mathfrak{p})$, where Φ is the $N(\mathfrak{p})$-power Frobenius, for $j = 1, 2, 3$. This yields the traces $\sum_i \alpha_i \gamma_i^j$, where the α_i are the eigenvalues of α, the γ_i are those of $\mathrm{Frob}_\mathfrak{p}$, and the $\alpha_i \gamma_i^j$ are the eigenvalues of $\alpha \circ \mathrm{Frob}_\mathfrak{p}^j$. We obtain the trace of $\mathrm{Frob}_\mathfrak{p}^j$ on $\mathrm{Hom}_G(\rho, V_\ell)^{I_\mathfrak{p}} \cong \rho^{\oplus 3}$ as

$$\frac{\sum_i \alpha_i \gamma_i^j - \overline{\zeta}_3 \sum_i \gamma_i^j}{\zeta_3 - \overline{\zeta}_3},$$

so in this way we can obtain the characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ on $\mathrm{Hom}_G(\rho, V_\ell)^{I_\mathfrak{p}}$, e.g. via Newton's identities [Mea92]. Taking conjugates shows that

$$\frac{\sum_i \alpha_i \gamma_i^j - \zeta_3 \sum_i \gamma_i^j}{\overline{\zeta}_3 - \zeta_3} = \mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}^j \mid \overline{\rho}^{\oplus 3}) = \overline{\mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}^j \mid \rho^{\oplus 3})},$$

and it follows that the eigenvalues of $\mathrm{Frob}_\mathfrak{p}$ on $\mathrm{Hom}_G(\rho, V_\ell)^{I_\mathfrak{p}}$ are the conjugates of those on $\mathrm{Hom}_G(\overline{\rho}, V_\ell)^{I_\mathfrak{p}}$. ♢

Proof of Proposition 4.14. First we treat the case that $G = \langle \alpha \rangle \cong C_n$ is cyclic. Choose $\ell \equiv 1 \pmod n$ and let A be the diagonal matrix by which α acts on V_ℓ with entries $z_1, \ldots, z_d$. Also fix a prime $\mathfrak{p}$ and let Φ be the diagonal matrix of $\mathrm{Frob}_\mathfrak{p}$ on V_ℓ with respect to the same basis. Recall that since G is abelian, ρ is 1-dimensional and therefore $\mathrm{Hom}_G(\rho, V_\ell)^{I_\mathfrak{p}}$ is isomorphic to $V_\ell^\rho = \rho^{\oplus r_\rho}$ in the decomposition (4.1). We demonstrate how one can compute the L-function corresponding to ρ, where α acts on V_ℓ^ρ by eigenvalue $z = z_1$.

Let $p(X)$ of degree d be the minimal polynomial of A, and set $p_z(X) = p(X)/(X - z)$. Consider the matrix $\Phi p_z(A)$, which acts on $V_\ell^{\rho'}$ as the zero map for $\rho' \neq \rho$, and has trace $\mathrm{Tr}(\mathrm{Frob}_\mathfrak{p} \mid V_\ell^\rho) \prod_{i \neq 1}(z - z_i)$. Repeating for Φ^j for $j \in \{2, \ldots, g\}$, we can in this way obtain the minimal polynomial of $\mathrm{Frob}_\mathfrak{p}$ acting on V_ℓ^ρ using Newton's identities; hence the local factor of $L(C/K, \rho, s)$ at $\mathfrak{p}$.

We now ask what happens when we apply $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ to this construction. Note that expanding $p_z(A)$ gives

$$\mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}^j \mid V_\ell^\rho) = \frac{\sum_{k=0}^{d-1} e_k(z_2, \ldots, z_d) \, \mathrm{Tr}(\Phi^j A^{d-1-k})}{\prod_{i \neq 1}(z - z_i)},$$

where the e_k denote the elementary symmetric polynomials. By the Lefschetz trace formula (Theorem 2.19), $\mathrm{Tr}(\Phi^j A^{d-1-k}) \in \mathbb{Z}$ is integral and therefore fixed by σ. Hence we obtain $\mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}^j \mid V_\ell^\rho)^\sigma = \mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}^j \mid V_\ell^{\rho^\sigma})$. Therefore the local polynomial at $\mathfrak{p}$ satisfies

$$\det(1 - N(\mathfrak{p})^{-s} \, \mathrm{Frob}_\mathfrak{p}^{-1} \mid \rho^\sigma) = \det(1 - N(\mathfrak{p})^{-s} \, \mathrm{Frob}_\mathfrak{p}^{-1} \mid \rho)^\sigma.$$

This holds for all $\mathfrak{p}$. For the general case, use Artin induction and Lemma 4.7. □

Indeed, the proof of this proposition actually leads to an important compatibility result which says that the L-functions $L(C/K,\rho,s)$ are well-defined.

Proposition 4.16. *The local polynomials*

$$\det(1 - T\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \mid \mathrm{Hom}_G(\rho, V_\ell J_C)^{I_{\mathfrak{p}}})$$

are independent of choice of ℓ. In particular, the L-functions $L(C/K,\rho,s)$ are well-defined.

Proof. We obtain from the proof of Proposition 4.14, for one-dimensional ρ,

$$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}}^{j} \mid V_\ell^{\rho}) = \frac{\sum_{k=0}^{d-1} e_k(z_2,\ldots,z_d)\,\mathrm{Tr}(\Phi^j A^{d-1-k})}{\prod_{i\neq 1}(z-z_i)}.$$

Using the Lefschetz trace formula, the traces on the right-hand-side can be computed via point counts on curves over finite fields, which are independent of ℓ. We again conclude by Artin induction. □

4.2 An analogous rank formula

We now give some discussion on quotient curves and the analogy between the L-functions which we define and Artin-twists.

Remark 4.17. Consider, for example, the following S_3-diagram of number fields

$$\begin{tikzcd}
 & L=\mathbb{Q}(\zeta_3,\sqrt[3]{2}) \arrow[dl, dash] \arrow[dd, dash] \arrow[dr, dash] & \\
K=\mathbb{Q}(\sqrt[3]{2}) \arrow[dr, dash] & & F=\mathbb{Q}(\zeta_3) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

from which we obtain an identity

$$L(C/\mathbb{Q},s)^2 L(C/L,s) = L(C/F,s)L(C/K,s)^2$$

via the Artin-formalism for Artin-twists, for any curve $C/\mathbb{Q}$. By choosing such a curve and taking function fields over each of the above fields, we could view the above diagram as a Galois diagram of function fields.

Indeed, recall that a curve C/K corresponds to its function field $K(C)$, cf. [Sta18, 0BXX, Theorem 53.2.6]. If the curve admits an action by automorphisms defined over K of the finite group G, then the fixed field under $H \leq G$ is a subfield[8], which corresponds to a quotient curve C^H such that $h^1(J_{C^H}) \leq h^1(J_C)$.

Once we permit function field extensions which are not just extensions of the field of constants, then, we recover the L-functions of Definition 4.2 as a 'generalisation' of Artin-twists. ♢

[8]Indeed, $K(C)/K(C)^H$ is a Galois extension with Galois group H, validating the opening poem.

Given a curve C/K with an action by automorhpisms of K, we note that G also acts on $J_C(K)$, from which we obtain a representation $\chi_{C/K} = J_C(K) \otimes \mathbb{C}$. We are then interested in the following question, which is a potential analogue of the BSD rank formula for Artin-twists. Recalling our standing assumption that motivic L-functions have meromorphic continuation and satisfy their functional equations, we ask:

Question 4.18. *Do we have*

$$\operatorname{ord}_{s=1} L(C/K, \rho_i, s) = \langle \chi_{C/K}, \rho_i \rangle?$$

We recall from Example 3.25 that $m = 1$ is a critical value for the underlying motive $M^{\rho}_{C/K}$ which admits the L-function $L(C/K, \rho, s)$. Then it is natural to ask whether we can deduce Question 4.18 from Deligne's conjecture and the BSD rank formula, as in Proposition 3.34. We first give a brief remark on quotient curves and the analogy with Artin-twists, which we will use to replicate the proof of Proposition 3.34 in this setting.

Then we have:

Theorem 4.19. *Conjecture 3.27 (i) and the BSD rank formula imply an affirmative answer to Question 4.18.*

Proof. Mimic precisely the proof of Proposition 3.34: for G trivial we just apply the BSD rank formula to $J_C(K)$. For $G \cong C_n$, we consider the curve C^H for each subgroup H. In this case, for $\rho : G \twoheadrightarrow G/H \xrightarrow{\widetilde{\rho}} \mathbb{C}^\times$ we have

$$L(C^H/K, \widetilde{\rho}, s) = L(C/K, \rho, s),$$

and $\langle \chi_{C^H/K}, \widetilde{\rho} \rangle = \langle \chi_{C/K}, \rho \rangle$. The rest of the proof then follows as before, using Lemmas 4.1 and 4.7 and Proposition 4.14. $\square$

With this proposition under our belt, there is sufficient theoretic evidence to state:

Conjecture 4.20. *The answer to Question 4.18 is always in the affirmative.*

Remark 4.21. A conjecture of Deligne–Beilinson predicts that

$$\operatorname{ord}_{s=0} L(M, s) = \dim H^1_f(M^*(1)) - \dim H^0_f(M^*(1)),$$

for critical motives M, where here $H^\bullet_f(M)$ denotes a certain *motivic cohomology.* It is well-known that for an abelian variety A over a number field K we have

$$H^i_f(h^1(A)(1)) = \begin{cases} A(K) \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } i = 1 \\ 0, & \text{if } i = 0. \end{cases}$$

It is clear that the BSD rank formula follows from this conjecture, which is essentially the rank part of the eTNC ([BF01, Conjecture 4]); hence Conjecture 4.20 is an easy consequence of the eTNC. Of course, this is already known because BSD and Deligne's period conjecture are both implied by the eTNC; indeed Theorem 4.19 is significantly stronger than this observation. It may still be of use, however, to view the L-functions $L(C/K, \rho, s)$ through the lens of the eTNC. $\diamondsuit$

Hence we may view numerical evidence in favour of Conjecture 4.20 as evidence in favour of Deligne's conjecture and BSD.

Remark 4.22. We give a short discussion on the relative scarcity of 'interesting' examples in the non-abelian case. Firstly, consider the K-isogeny decomposition of J_C into simple abelian varieties over K:

$$\varphi : J_C \to A_1 \times \cdots \times A_n.$$

This corresponds to a decomposition

$$h^1(J_C) = h^1(A_1) \oplus \cdots \oplus h^1(A_n)$$

on the level of motives; hence the decomposition (4.3) induces a decomposition of each $h^1(A_i)$. This decomposition comes from the action of the subgroup $G_\varphi = \{\alpha \in G : \alpha(\ker\varphi) = \ker\varphi\}$, each element of which induces an automorphism of the product $A_1 \times \cdots \times A_n$. The action of G_φ may permute these isogeny factors and this permutation does not induce a further decomposition of the $h^1(A_i)$, so the 'interesting part' of the decomposition comes from the action of G_φ/H on $h^1(J_{C^H}) \leq h^1(J_C)$, where $H \lhd G_\varphi$ is the normal subgroup which acts by permutations. The quotient G_φ/H may well be abelian. To guarantee that this doesn't happen, we could restrict to looking at curves with simple Jacobian—but then we should require that $G = G_\varphi$ be simple: else for normal $H \lhd G$ we have $C^H \in \{\mathbb{P}^1_K, C\}$, by simplicity. If $C^H = \mathbb{P}^1_K$, then we may restrict to the action of G/H, which need not be non-abelian. Thereby we should have a large group of automorphisms; the smallest non-abelian simple group is A_5. This forces $\mathrm{End}^0(J_C)$ to contain many roots of unity, so J_C is highly likely to have CM by some abelian CM-field and hence be uninteresting.

Moreover, one can see on the LMFDB ([LMF23]) that the known curve of smallest genus with an action of A_5 by automorphisms which does not decompose as a product of elliptic curves has genus 6. This turns out to be too large for meaningful computation. $\diamondsuit$

4.3 Twisted pieces of curves

Recall that Conjecture 4.20 is an analogue of the BSD rank formula for Artin-twists (Conjecture 2.44). Here, we briefly discuss some special cases where Conjecture 4.20 aligns with Conjecture 2.44. In the notation of the previous, this occurs when our L-functions $L(C/K, \rho, s)$ happen to be Artin-twists of other curves. In the following, ρ will denote a representation of G, whilst χ denotes an Artin representation.

Firstly, for χ factoring through a finite Galois extension F/K, we obtain

$$L(C/F, s) = \prod_\chi L(C/K, \chi, s)$$

from the Artin formalism, where this product is over χ appearing in $\mathrm{Ind}_1^{\mathrm{Gal}(F/K)} 1$. On the level of F-motives this is an equality

$$h^1(J_C)_F = \bigoplus_\chi (h^1(J_C)_K \otimes [\chi]).$$

Combining with the decomposition coming from (4.3), we have

$$M^\rho_{C/F} = \bigoplus_\chi (M^\rho_{C/K} \otimes [\chi]) \tag{4.5}$$

for each representation ρ of G. This observation can help us to 'compute' $M^\rho_{C/K}$ in cases where we have an isomorphism from C to a better-understood curve C_0 over a larger field:

Example 4.23. Consider the elliptic curve $E:\ y^2 = x^3 + b$ over $K = \mathbb{Q}(\zeta_3)$. Then E has an action of C_3 via $\alpha : x \mapsto \zeta_3 x$; hence E has complex multiplication by $\mathbb{Z}[\zeta_3]$ and we have a factorisation into Hecke L-functions

$$L(E/K, s) = L(\xi, s) \cdot L(\overline{\xi}, s).$$

Moreover, we also have a factorisation coming from the decomposition of $V_\ell E$ under the action of C_3

$$L(E/K, s) = L(E/K, \rho, s) \cdot L(E/K, \overline{\rho}, s).$$

By Example 4.10, these L-function decompositions correspond. In the case that $b \in \mathbb{Q}$ we have $L(\xi, s) = L(\overline{\xi}, s) = L(E/\mathbb{Q}, s)$, as in Example 4.11. We can also explicitly compute these Hecke L-functions, for example, for $b \in \zeta_3 \cdot \mathbb{Z}$. Indeed, if $E:\ y^2 = x^3 + \zeta_3 c$ and $E_0:\ y^2 = x^3 + c$ for $c \in \mathbb{Z}$ we find

$$L(E/K, \rho, s) = L(E_0/\mathbb{Q}, \chi, s),$$

where $\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q}) \to \mathbb{C}^\times$ is an Artin representation. To see this, we rewrite $E:\ y^2 = \zeta_3^2 x^3 + c$, write $\overline{E}:\ y^2 = \zeta_3 x^3 + c$, and consider the decompositions under $\langle \alpha \rangle$

$$V_\ell E \cong \rho_E \oplus \overline{\rho}_E, \qquad V_\ell \overline{E} \cong \rho_{\overline{E}} \oplus \overline{\rho}_{\overline{E}}, \qquad V_\ell E_0 \cong \rho_0 \oplus \overline{\rho}_0$$

so that α acts on $\rho_E, \rho_{\overline{E}}$, and ρ_0 by the same eigenvalue. Over $\mathbb{Q}(\zeta_9)$ it is clear that these curves are isomorphic. Moreover we note that $\sigma\alpha\sigma = \overline{\alpha} : (x, y) \mapsto (\overline{\zeta}_3 \cdot x, y)$, where σ is complex conjugation, so

$$P \in \rho_E \implies \overline{P} \in \rho_{\overline{E}}.$$

It suffices to obtain the eigenvalues of $\mathrm{Frob}_\mathfrak{p}$ on ρ_E and $\rho_{\overline{E}}$ in terms of that on ρ_0. Let $f : E_0 \to E$ be induced by the isomorphism of curves $(x, y) \mapsto (\zeta_9^{-1} x, y)$. Then we have, for $P \in \rho_E$, on the level of ℓ^n-torsion,

$$\begin{aligned} \mathrm{Frob}_\mathfrak{p} f((x, y)) &= \mathrm{Frob}_\mathfrak{p}(\zeta_9^{-1} x, y) \\ &= (\zeta_9^{-N(\mathfrak{p})} \mathrm{Frob}_\mathfrak{p} x, \mathrm{Frob}_\mathfrak{p} y) \\ &= (\chi(\mathrm{Frob}_\mathfrak{p}) \cdot \zeta_9^{-1} \mathrm{Frob}_\mathfrak{p} x, \mathrm{Frob}_\mathfrak{p} y) \\ &= \chi(\mathrm{Frob}_\mathfrak{p}) \cdot f(\mathrm{Frob}_\mathfrak{p}(x, y)), \end{aligned}$$

so the eigenvalue of $\mathrm{Frob}_\mathfrak{p}$ on ρ_E is exactly $\chi(\mathrm{Frob}_\mathfrak{p}) \cdot \gamma$, where γ is the eigenvalue of $\mathrm{Frob}_\mathfrak{p}$ on ρ_0. This shows our claim. ♢

Remark 4.24. We can also re-run the previous example from a motivic point of view, which allows us to avoid using the explicit isomorphism from E to E_0. As $K = \mathbb{Q}(\zeta_3)$-motives, we have

$$h^1(E) = M^\rho_{E/K} \oplus M^{\overline{\rho}}_{E/K} \qquad \text{and} \qquad h^1(E_0) = M^\rho_0 \oplus M^{\overline{\rho}}_0,$$

for $M^\bullet_0 = M^\bullet_{E_0/K}$. Hence we obtain, via the Artin formalism and the F-isomorphism from E to E_0,

$$\bigoplus_\chi (M^\rho_{E/K} \otimes [\chi]) = \bigoplus_\chi (M^\rho_0 \otimes [\chi]),$$

where we have summand-wise equality because both sides correspond to the decomposition of $h^1(E)_F$ into Hecke motives. In particular, by Example 4.11, there exists χ such that $M^\rho_{E/K} = h^1(E_0)_\mathbb{Q} \otimes [\chi]$. This implies the desired equality on the level of L-functions. ♢

Following this same argument we can obtain the following:

Lemma 4.25. *Suppose C/K is a curve with an action of the abelian group G by automorphisms, and that D/K is another curve such that C is isogenous to D over a finite Galois extension F/K. Suppose further that the action of G on D/F is already defined on the level of D/K. For each representation ρ of G, if $M^{\rho}_{C/K}$ is indecomposable over K, then there exists an Artin representation χ_ρ factoring through* $\mathrm{Gal}(F/K)$ *such that*

$$M^{\rho}_{C/K} = M^{\rho}_{D/K} \otimes [\chi_\rho].$$

Proof. From Equation (4.5), we have

$$M^{\rho}_{D/F} = \bigoplus_{\chi} (M^{\rho}_{D/K} \otimes [\chi]) \quad \text{and} \quad M^{\rho}_{C/F} = \bigoplus_{\chi} (M^{\rho}_{C/K} \otimes [\chi]).$$

The conclusion follows by noting that $M^{\rho}_{C/F} = M^{\rho}_{D/F}$; indeed, we can take the identification

$$\bigoplus_{\chi} (M^{\rho}_{D/K} \otimes [\chi]) = \bigoplus_{\chi} (M^{\rho}_{C/K} \otimes [\chi])$$

to be summand-wise by comparing dimensions—and because of the assumption on indecomposability. □

In particular, we may apply Lemma 4.25 in cases such as Example 4.23:

Proposition 4.26. *Let $K = \mathbb{Q}(\zeta_n)$, and suppose that C/K is a curve with an action of the abelian group $G \cong C_n$ by automorphisms, and that $D/\mathbb{Q}$ is another curve admitting $\mathbb{Q}$-simple Jacobian such that C is isogenous to D over a finite Galois extension F/K. Suppose further that the action of G on D/F is already defined on the level of D/K, and that $\mathbb{Q}[G]$ is contained in the centre of* $\mathrm{End}^0(J_D)$*. For each representation ρ of G, if $M^{\rho}_{C/K}$ is indecomposable over K, then there exists an Artin representation χ_ρ factoring through* $\mathrm{Gal}(F/\mathbb{Q})$ *such that*

$$M^{\rho}_{C/K} = h^1(J_D)_{\mathbb{Q}} \otimes [\chi_\rho],$$

i.e.

$$L(C/K, \rho, s) = L(D/\mathbb{Q}, \chi_\rho, s).$$

Proof. By Example 4.11 we have $M^{\rho}_{D/K} = h^1(J_D)_{\mathbb{Q}}$, and Lemma 4.25 gives the result—noting that χ_ρ of the lemma can be viewed as a representation of $\mathrm{Gal}(F/\mathbb{Q})$. □

Remark 4.27. We do not expect that the above proposition is in its most general form, but state it this way for the sake of simplicity. ♢

4.4 Periods of pieces

Here we give some discussion on the problem of finding a recipe analagous to Example 3.33 for the L-functions $L(C/K, \rho, s)$. Firstly, we note that—although we were able to draw on the analogy with Artin-twists to understand ranks—this analogy falters when we try to understand leading terms; for example, we have no clear analogue of the root number $w(\rho)$ appearing in (3.1). We do already know explicit formulae for these periods in some special cases, however.

Example 4.28. Suppose $C/\mathbb{Q}$ is a curve such that the finite group $G \cong C_n$ acts by automorphisms defined over some larger field K, that the quotient curve C^G is a copy of

$\mathbb{P}^1$, and that $\mathbb{Q}[G]$ lies in the centre of $\mathrm{End}^0(J_C)$. Then the factorisation under the action of G is

$$L(C/K, s) = L(C/\mathbb{Q}, s)^{\varphi(n)},$$

i.e. if ρ is a primitive character of G, then

$$L(C/K, \rho, s) = L(C/\mathbb{Q}, s)$$

and we may take $c^+(M^{\rho}_{C/K}(1)) = \Omega_+(J_C)$. ♢

The following example, however, reminds us that it remains important to keep Example 3.33 in mind, c.f. Proposition 4.26.

Example 4.29. Let C/K be a curve over a number field with an action of a finite group G by automorphisms. Suppose $D/\mathbb{Q}$ is another curve with $\mathbb{Q}$-simple Jacobian such that C is isomorphic to D over some finite extension F/K which is Galois over $\mathbb{Q}$, that the action of G on D is already defined on the level of K, and that $\mathbb{Q}[G]$ lies in the centre of $\mathrm{End}^0(J_D)$. Then, for a representation ρ of G, there exists an Artin representation χ factoring through $\mathrm{Gal}(F/\mathbb{Q})$ such that

$$M^{\rho}_{C/K} = h^1(J_D)_{\mathbb{Q}} \otimes [\chi].$$

Thereby we may take, fixing a choice of embedding τ,

$$c^+(\tau, M^{\rho}_{C/K}(1)) = \Omega(J_D, \chi).$$ ♢

Remark 4.30. Coming up with an analogous (conjectural) explicit formula to (3.1) for the periods of the motives $M^{\rho}_{C/K}$ is an active area of thought. This naturally begs the question

"What do we mean by 'explicit' here?"

Of course, in some sense the formulation of Conjecture 3.27 gives an 'explicit' description of the period. We would like to have a description in terms of quantities which we can easily compute numerically, such as the periods associated to J_C and data associated to ρ. The former is not necessarily easy to compute, but Example 3.33 shows that we need at least to permit this. We lastly note that we do not necessarily conjecture that such a formula exists! ♢

4.5 Elliptic & hyperelliptic curves

First we recall the possible automorphisms of an elliptic curve.

Lemma 4.31. *Let E be an elliptic curve over a number field K. As a curve, the automorphism group of E is isomorphic to one of*

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}.$$

Proof. We can take E to be in short Weierstrass form, i.e.

$$E : \; y^2 = x^3 + ax + b.$$

The K-isomorphisms of curves in this form are given by $x \mapsto u^2 x$, $y \mapsto u^3 y$ for $u \in K^{\times}$ (e.g. [Sil09, Table 3.1]), with image

$$y^2 = x^3 + u^4 \cdot ax + u^6 \cdot b$$

so the possibilites are $u = \pm 1$, or $a = 0$ and $u^3 = 1$, or $b = 0$ and $u^4 = 1$. $\square$

Remark 4.32. A geometric point of view of this result is given by the complex structure of an elliptic curve. Indeed, automorphisms of a complex torus are induced by scalar multiplication that preserves the defining lattice. Assuming the lattice to be in the standard form $\mathbb{Z} \oplus \mathbb{Z}\tau$, it is easy to see that the only possible cases are multiplication by -1 and multiplication by i when the fundamental domain is a square, or by $e^{2\pi i/3}$ when the fundamental domain is composed of two equilateral triangles. $\diamondsuit$

The automorphism of order two in each of these possible cases corresponds to the inversion $P \mapsto -P$, which acts by -1 on the whole Tate module. Then the potentially interesting cases occur when we have an automorphism of order 3 or 4, i.e. when the elliptic curve E has complex multiplication by $\mathbb{Q}(\zeta_3)$ or $\mathbb{Q}(i)$. By Example 4.10, any non-trivial decomposition of $V_\ell E$ under the action of $\operatorname{Aut}_K C$ corresponds to the decomposition of $h^1(E)$ into 1-dimensional Hecke motives.

Elliptic curves are then uninteresting from our point of view, so we turn to the next natural case to consider: hyperelliptic curves. In [MP21], Müller and Pink classify all hyperelliptic curves with many[9] automorphisms (see [loc. cit., Table 1]). They find three infinite families, all of which have CM, and 15 more curves, 10 of which do not have CM. These 10 curves without CM (more precisely, their twists) are then potentially interesting to us. With the help of the LMFDB [LMF23], we make some notes on these curves, using the same labelling of curves as in the referenced table. We exclude the curves X_{16}, X_{17} and X_{18} of genera 20, 24 and 30 respectively—both because these genera are too large for meaningful computations with and because the LMFDB does not support curves of genus greater than 15 (of course, these reasons are not disjoint). Following the LMFDB, in the table below E stands in for an elliptic curve and A_n for an abelian variety of dimension n. We also note that these isogeny factors are not necessarily simple.

C	$g(C)$	$\operatorname{Aut}(C)$	Decomposition under isogeny
X_6	3	$C_2 \times S_4$	$E \times E \times E$
X_8	6	$\mathrm{GL}_2(\mathbb{F}_3)$	$E^2 \times E^4$
X_{10}	9	W_2	$E \times A_2 \times A_2^3$
X_{11}	12	W_3	$A_4 \times A_4^2$
X_{12}	5	$C_2 \times A_5$	E^5
X_{13}	9	$C_2 \times A_5$	$E^4 \times E^5$
X_{15}	15	$C_2 \times A_5$	$E^4 \times E^5 \times A_2^3$

Table 1: Non-CM hyperelliptic curves with many automorphisms.

We give some examples which show how one can make observations as in the above table.

Example 4.33. Consider the case of the genus 3 curve given by affine equation

$$X_6 : \; y^2 = x^8 + 14x^4 + 1.$$

We note that $y^2 = x^4 \pm 14x^2 + 1$ are affine equations of suitable quotients. The two elliptic curves represented by these equations therefore appear in the isogeny decomposition of X_6, but now counting dimensions shows that X_6 must have three simple isogeny factors—all of which are elliptic curves. Note that these isogenies are defined over $\mathbb{Q}(i)$, which is the

[9]This has a precise technical meaning, see [MP21, §2].

field of definition of the automorphisms of X_6, and so we consider this curve uninteresting as it boils down to studying elliptic curves. Likewise we will not be interested in twists of this curve. $\diamondsuit$

We are similarly uninterested in the curves X_8, X_{12} and X_{13}. We are however potentially interested in the curves X_{10}, X_{11} and X_{15}, or more precisely their twists with coefficients in the fields of definition of the relevant automorphism groups. Sadly, even genus 9 is currently beyond our computational ceiling (cf. Section 5).

We also show demonstrate a method to construct curves over $\mathbb{Q}$ with automorphisms defined over $\mathbb{Q}$, with an example which yields a hyperelliptic curve.

Example 4.34. Consider the map $\alpha : (x, y) \mapsto (x + xy, y + xy)$. We construct a genus 2 (hence hyperelliptic) curve by asserting that this defines an automorphism of order 3 on our curve. To do so we compute $\alpha^3(x, y) = (x + A(x, y), y + A(x, y))$, and take our curve to be defined by $C :\ A^*(x, y) = 0$. where $A^*(x, y)$ is the polynomial A/xy. We find

$$C :\ x^3y^3+2x^3y^2+2x^2y^3+x^3y+xy^3+6x^2y^2+5x^2y+5xy^2+x^2+y^2+7xy+3x+3y+3 = 0.$$

We can take a hyperelliptic model of C to have affine equation

$$y^2 + (x^3 + x + 1)y = -x^5 + 2x^4 - 3x^3 + x^2 - x,$$

on which the automorphism $\beta : (x : y : z) \mapsto (z : x^3 - 2x^2z + 3xz^2 + y - z^3 : x - z)$, acting on C via its action on the weighted projective space $\mathbb{P}^{1,3,1}_{\mathbb{Q}}$, has order 3. Under the action of $C_3 \cong \langle\beta\rangle$, we obtain a decomposition of the Tate module

$$V_\ell J_C \cong \rho^{\oplus 2} \oplus \overline{\rho}^{\oplus 2},$$

where ρ affords one of the two non-trivial irreducible characters of C_3. This curve turns out to be isogenous to the square of an elliptic curve over a cubic extension. We find that the local polynomials of $L(C/\mathbb{Q}, \rho, s)$ correspond to local polynomials of an Artin-twist of this elliptic curve, via Proposition 4.26. $\diamondsuit$

Remark 4.35. One can see on the LMFDB[10] that there are several known examples of genus 2 curves over $\mathbb{Q}$ with automorphism group C_3, but in each case we obtain a square of an elliptic curves over a cubic extension; hence these curves are uninteresting by Proposition 4.26. $\diamondsuit$

Lastly, we note that in Example 4.34 we relied on the fact that C was a genus 2 curve in order to find a suitable embedding in weighted projective space. In examples of higher genus, it is often far-from-clear how one might find a smooth embedding into projective space in this way. Notably, this can make the computations in the sequel more difficult.

[10] `https://www.lmfdb.org/HigherGenus/C/Aut/`

5 Computations

5.1 Computational strategies

Here we discuss how we may compute the Euler factors of the L-functions of Definition 4.2. We consider the scene as described in Section 4.1, using the same notation. Further, for any prime $\mathfrak{p}$ of good reduction we identify $\mathbb{F}_{\mathfrak{p}}$ with $\mathbb{F}_{N(\mathfrak{p})}$, and we consider the reduced curve $\widetilde{C}/\mathbb{F}_{\mathfrak{p}}$. Computing the local polynomial at $\mathfrak{p}$ of $L(C/K, \rho, s)$ boils down to computing the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}$ on $\mathrm{Hom}_G(\rho, V_\ell)^{I_{\mathfrak{p}}}$. For cyclic groups $G = \langle \alpha \rangle$, our strategy is summarised in the following algorithm. Throughout, all of our computations are implemented in Magma [BCP97].

Algorithm 5.1 (G cyclic).
Input: An irreducible representation ρ of G, and a prime $\mathfrak{p}$ of good reduction.
Output: The local polynomial of $L(C/K, \rho, s)$ at $\mathfrak{p}$.

1. Determine the eigenvalue δ of α on $\mathrm{Hom}_G(\rho, V_\ell)^{I_{\mathfrak{p}}}$. There is only one since G is cyclic, and it occurs with multiplicity $\dim \mathrm{Hom}_G(\rho, V_\ell)^{I_{\mathfrak{p}}}$.

2. Compute the eigenvalues $\{\gamma_i\}$ of $\mathrm{Frob}_{\mathfrak{p}}$ on $V_\ell^{I_{\mathfrak{p}}}$.

3. Using the output of the previous steps, compute the eigenvalues of $\mathrm{Frob}_{\mathfrak{p}}$ on $\mathrm{Hom}_G(\rho, V_\ell)^{I_{\mathfrak{p}}}$. This is spelled out in the proof of Proposition 4.14.

4. From these eigenvalues construct the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}^{-1}$ on $\mathrm{Hom}_G(\rho, V_\ell)^{I_{\mathfrak{p}}}$, giving the local polynomial at $\mathfrak{p}$ of $L(C/K, \rho, s)$.

Step 1. Although we mention the first step, in practice it is not necessary. Because G is cyclic, every eigenvalue of α corresponds uniquely to an irreducible representation ρ of G. We often choose ρ as correspondig to the piece on which α acts with a specified eigenvalue.

Step 2. The naïve approach would be to compute the eigenvalues of $\mathrm{Frob}_{\mathfrak{p}}$ on $V_\ell^{I_{\mathfrak{p}}}$ by counting points on $\widetilde{C}(\mathbb{F}_{N(\mathfrak{p})^j})$ for $j \in \{1, \ldots, g\}$, cf. Example 2.18 and Remark 5.2 below. However, much faster algorithms have been implemented in Magma to compute these: Kedlaya's algorithm has been implemented by Harrison ([Har12]) for hyperelliptic curves; another version of Kedlaya's algorithm ([Har07]) has been implemented by Minzlaff ([Min13]) for superelliptic curves; and an algorithm has been implemented by Tuitman ([Tui16], [Tui17]) for more general curves.

Step 3. Following the proof of Proposition 4.14, we may proceed as follows: set $f_{j,k} = \alpha^j \circ \mathrm{Frob}_{\mathfrak{p}}^k$ and compute $\#\mathrm{Fix}(f_{j,k})$ for $k \in \{1, \ldots, g\}$ and $j \in \{1, \ldots, d-1\}$ in the notation of the proposition (cf. Example 5.3), giving $\mathrm{Tr}(f_{i,j})$ via the Lefschetz trace formula. Thereby, we can compute the local polynomial directly as in the proposition. However, in practice, once we have computed $\#\mathrm{Fix}(f_{1,1})$ we run through the possible combinations of the eigenvalues $\{\delta_i\}$ of α and $\{\gamma_i\}$ of $\mathrm{Frob}_{\mathfrak{p}}$ to find all which give $\sum \delta_i \gamma_i = \#\mathrm{Fix}(f_{1,1})$. If there is more than one possibility, then we iterate over j and k until there is a unique possibility. If there is a unique arrangement, then the γ_i which pair with δ are the eigenvalues we are looking for. We note that, based on our experience, it seems rare to require even to compute $\#\mathrm{Fix}(f_{1,2})$.

Remark 5.2. A priori, to compute the eigenvalues of an endomorphism f on V_ℓ one should compute the traces $\mathrm{Tr}(f^i)$ for $i \in \{1, \dots, 2g\}$, and then apply Newton's identities. However, knowing the traces just for $i \in \{1, \dots, g\}$, we can compute the eigenvalues γ_i of f using the Weil conjectures. Using the Riemann hypothesis, we obtain the polynomial with roots $\{\gamma_i\}_i$ by setting $S = \mathbb{Q}[t, s_1, \dots, s_{2g}]$ and computing the image of the polynomial $\prod_i (t - s_i)$ in the quotient of S by the relations given by asserting that $\sum_i s_i^n = \sum_i \gamma_i^n$ and $s_{2i-1}s_{2i} = q$. The latter relations are valid because real roots occur with even multiplicity. We may then solve for the roots of this polynomial. ♢

We note that often counting the fixed points of endomorphisms f can be reduced to counting points on an auxilliary curve over $\mathbb{F}_\mathfrak{p}$:

Example 5.3. Consider the curve $C : \ y^3 = h(x)$ defined over $K = \mathbb{Q}(\zeta_3)$, with automorphism $\alpha : (x, y) \mapsto (x, \zeta_3 y)$. Fixing a prime $\mathfrak{p}$ of K, we consider the reduction $\widetilde{C}$ over $\mathbb{F}_\mathfrak{p} = \mathbb{F}_q$ of C and let Φ be the q-power Frobenius. Note that α descends to an automorphism of $\widetilde{C}$, and set $f = \alpha \circ \Phi$. Then the trace of $\alpha \circ \mathrm{Frob}_\mathfrak{p}$ on $V_\ell J_C$ is given by

$$1 + q - \#D(\mathbb{F}_q),$$

where $D : \ uy^3 = \widetilde{h}(x)$ is an auxilliary curve and u generates $\mathbb{F}_q^\times$, cf. Example 5.6. ♢

Remark 5.4. It may not always be possible to find such an auxilliary curve. Indeed, in the case of Example 4.34 we have an automorphism α of order 3 defined over the rationals by $x \mapsto x + xy$, $y \mapsto y + xy$. We have thus far been unable to find a suitable auxilliary curve, so instead to count fixed points we must resort to a Gröbner basis calculation in Magma. This quickly becomes computationally inefficient for even medium-sized primes. ♢

Remark 5.5. The computations outlined above are only for cyclic groups G, but Algorithm 5.1 can also be used to extract information in the non-cyclic case. For example, one could go through the steps for several cyclic subgroups of G and piece together that information to obtain the correct local polynomial. ♢

The computations described so far allow us to compute the local polynomials of $L(C/K, \rho, s)$. Our aim is to test whether there is an affirmative answer to Question 4.18 for C/K, so we must discuss how to compute $L(C/K, \rho, 1)$ once we have computed the local factors: in order to compute this value, we need to know the functional equation for $L(C/K, \rho, s)$. To do this, Magma requires the Γ-factor and conductor. The former we can easily guess because it divides the Γ-factor $(\Gamma(s/2)\Gamma((s+1)/2))^{dg}$ appearing in the functional Equation (2.3) for $L(C/K, s)$. For the conductor, we have the relation

$$\prod_\rho c(M^\rho_{C/K}) = c(J_C/K),$$

where this product is over ρ appearing in $\mathrm{Ind}_1^G 1$, from which it is often straightforward to deduce $c(M^\rho_{C/K})$, e.g. using Proposition 4.14. The major difficulty, then, is that it is not known that the conductor of C/K can be computed efficiently for curves of genus greater than 2. Often then, we must resort to guessing the functional equation by trying different values of the conductor and testing whether `CFENew` returns a small value when applied to our L-functions.

5.2 Picard Curves

We have discussed the cases of curves of genus 1 and 2, so the natural next case to consider is the genus 3 family of so-called *Picard curves*. These are smooth plane curves over a number field K given by equations of the form

$$y^3 = f(x),$$

where $f(x)$ is a quartic polynomial in x. If $\mathbb{Q}(\zeta_3) \subseteq K$, then these curves admit an automorphism of order 3 given by $\alpha : y \mapsto \zeta_3 \cdot y$. Indeed, these curves correspond to degree 3 Galois extensions $K(C)/K(x)$. We consider such a curve C defined over $K = \mathbb{Q}(\zeta_3)$ (and not over $\mathbb{Q}$, cf. Example 4.11), so for a choice of $\ell \equiv 1 \pmod 3$ we obtain a decomposition under the action of $C_3 = \langle \alpha \rangle$

$$V_\ell J_C \cong \rho^{\oplus 3} \oplus \overline{\rho}^{\oplus 3},$$

where again ρ affords one of the non-trivial irreducible characters of C_3. Note that the trivial character does not appear in this decomposition because the fixed field of α is $K(x)$, which corresponds to a copy of $\mathbb{P}^1_K$, which has genus 0. We obtain a corresponding factorisation of $L(C/K, s)$ into conjugate L-functions:

$$L(C, s) = L(C, \rho, s) \cdot L(C, \overline{\rho}, s).$$

We can see how Proposition 3.34 works very explicitly in this case: Deligne's conjecture says that

$$\operatorname{ord}_{s=1} L(C, \rho, s) = \operatorname{ord}_{s=1} \overline{L(C, \rho, s)}.$$

Because the K-rational points $J_C(K)$ cannot be fixed by complex conjugation, we have $J_C(K) \otimes \mathbb{C} \cong \rho^{\oplus r} \oplus \overline{\rho}^{\oplus r}$ as representations of $\operatorname{Aut}_K(C)$. BSD predicts that

$$\operatorname{rk}(J_C(K)) = \operatorname{ord}_{s=1} L(C/K, s) = 2r,$$

and $2 \operatorname{ord}_{s=1} L(C/K, \rho, s) = \operatorname{ord}_{s=1} L(C/K, s)$, i.e. that $\operatorname{ord}_{s=1} L(C/K, \rho, s) = \operatorname{rk}(J_C(K))/2 = r$. We aim to use Magma to verify this (at least as a proof of concept) by computing $\operatorname{ord}_{s=1} L(C/K, \rho, s)$, although, if one believes BSD, this is perhaps not so interesting because it is essentially equivalent to computing $\operatorname{ord}_{s=1} L(C/K, s)$. Therefore we are much more interested in the L-value $L(C/K, \rho, 1)$ when the order of vanishing is 0, in the hope of computing a Deligne-period.

To accurately compute with these L-functions in Magma, we would need to compute thousands of Euler factors, which would require computing power to which we do not presently have access[11].

Example 5.6. Consider the curve

$$C \colon y^3 = x^4 - x^3 - \zeta_3 x^2 - \zeta_3 x.$$

We compute the Euler factor at the prime $\mathfrak{p} = (5) \triangleleft \mathcal{O}_K$ of $L(C/K, \rho, s)$, where $\alpha : y \mapsto \zeta_3 y$ acts on $\operatorname{Hom}_G(\rho, V_\ell)$ by ζ_3: firstly, the local polynomial of $L(C/K, s)$ at $\mathfrak{p}$ is

$$15625T^6 + 3750T^5 - 450T^4 - 245T^3 - 18T^2 + 6T + 1.$$

Now we wish to count the fixed points of $\alpha \circ \Phi$, where Φ is the Frobenius $x \mapsto x^{25}$ on

[11] Access pending!

$\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_{25}$. The fixed points of this endomorphism on the reduction of C satisfy

$$(x, y) = (x^{25}, zy^{25}),$$

where z is the reduction of ζ_3 mod $\mathfrak{p}$. Hence, we find that such points correspond to points on the 'auxilliary' curve given by

$$D\colon uy^3 = x^4 - x^3 - zx^2 - zx$$

over $\mathbb{F}_{\mathfrak{p}}$, where u is the cube of a root of $X^{24} - z^{-1}$. We find that there are 26 points on this curve; hence the trace of $\alpha \circ \mathrm{Frob}_{\mathfrak{p}}$ on V_ℓ is $1 + 25 - 26 = 0$. We then find that there is a unique arrangement as in Step 3 of the algorithm and that

$$-125\zeta_3 T^3 - 10(\zeta_3 - 1)T^2 + 2(\zeta_3 - 1)T - 1$$

is the local polynomial we desire. $\diamondsuit$

6 Future steps

On the computational side of things, we have thus far—largely for bureaucratic and practical reasons—only been able to carry out a very limited number of numerical computations. As such, we would very much like to carry out more computations in 'interesting' settings. Hence, the problem of identifying interesting examples of curves with automorphisms and computing the relevant L-functions is certainly an avenue of potential future work. This is intimately linked to *explicit geometric class field theory* and its non-abelian analogues.

Recalling that Conjecture 4.20 was formulated as an analogue of the BSD rank formula for Artin-twists, we note that V. Dokchitser, Evans and Wiersema ([DEW21]) have studied potential analogues of BSD for Artin-twists. Hence, it is a natural next step to see if one can replicate their results in this new setting. In particular, one may seek to replicate the computations of this project in order to study the behaviour at $s = 1$ of the relevant L-functions beyond just the order of vanishing.

It is a future aim to come up with an explicit description of the periods associated to our L-functions, analagous to Example 3.33.

We also note that a potential integral refinement of Deligne's period conjecture for Jacobians of curves has been proposed in [ECW23], which is further relevant to the L-functions considered here.

BSD and other conjectures on special values of L-functions have been absorbed into a conjectural edifice largely governed by the equivariant Tamagawa Number Conjecture of Bloch–Kato and Burns–Flach. From this point of view, Burns and Macias Castillo ([BC19]) have studied the problems considered in [DEW21]. It is then natural to wonder whether one could mimic this work in our setting.

Lastly, we note that little of what we have done is specific to the case of Jacobians of curves, so one could easily extend many of the above results to the case of general abelian varieties.

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