MOTIVIC PIECES OF CURVES: L-FUNCTIONS AND PERIODS

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ABSTRACT. Given a curve C over a number field K equipped with the action of a finite group G by K-automorphisms, one obtains a factorisation of $L(C,s)$ into a product of L-functions of 'motivic pieces of curves' associated to irreducible K-representations of G. We describe an algorithm for explicitly computing these L-functions, demonstrating implementations in the cases of certain curves with actions by C_3, C_4 and D_{10}. We explain how this algorithm can be used to factor L-functions of curves with endomorphisms of Hecke type.

Towards applications, we explicitly formulate and numerically verify a version of Deligne's Period Conjecture for hitherto-uninvestigated L-functions arising from motivic pieces of superelliptic curves.

CONTENTS

1. INTRODUCTION

Given a curve C over a number field K, one associates to it an L-function

$$L(C,s) = \prod_{\mathfrak{p} \lhd \mathcal{O}_K} L_{\mathfrak{p}}(C, N(\mathfrak{p})^{-s})^{-1},$$

where $L_{\mathfrak{p}}(C,T) = \det(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} \cdot T \mid (T_\ell \mathrm{Jac}(C) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^{I_{\mathfrak{p}}})$ and these determinants are computed in practice by counting points on the special fibre of a regular model of C at each prime of $\mathcal{O}_K$. The Hasse–Weil Conjecture predicts that $L(C,s)$ admits analytic continuation to all of $\mathbb{C}$ and, famously, the Birch–Swinnerton-Dyer Conjecture (BSD) describes the order of vanishing and leading term of the Taylor expansion at $s = 1$ in terms of arithmetic invariants.

If, further, C admits an action by K-automorphisms of a finite group G, then $T_\ell \mathrm{Jac}(C) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ decomposes into irreducible G-representations. There is a corresponding factorisation on the level of L-functions:

$$L(C,s) = \prod_{\tau \text{ irred.}} L(C^\tau, s),$$

where $L(C^\tau, s)$ is the L-function associated to a 'motivic piece', as described in [8, §2] (cf. Definition 2.1). These will be the central objects of our study.

Formally, the motivic piece C^τ is a motive over K with coefficients in $\mathbb{Q}(\tau)$; it is the motive associated to the first cohomology of C and the τ-idempotent of $\mathbb{C}[G]$. The arithmetic of motivic L-functions is widely believed to be governed by a vast conjectural edifice which generalises BSD—for example, we are in the precise setting

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of the equivariant Tamagawa Number Conjecture of Burns and Flach [3, §4], after Bloch and Kato, for the pair $(h^1(C), \mathbb{C}[G])$.

We do not pursue such a broad viewpoint herein, but note that several standard conjectures on the lower levels of this edifice are interpreted explicitly for the L-functions $L(C^\tau, s)$ in [8] (see [*loc. cit.*, Conjectures 2.20 & 2.23, Theorem 2.24]). See §2 for explanations of the terms in (1.2).

Conjecture 1.1. *The L-functions $L(C^\tau, s)$ satisfy the following properties:*

Hasse–Weil: *$L(C^\tau, s)$ admits meromorphic continuation to the complex plane and the completed L-function $\Lambda(C^\tau, s)$ satisfies the functional equation*

$$(1.2) \qquad \Lambda(C^\tau, s) = w(C^\tau) \cdot \Lambda(C^{\tau^*}, 2 - s).$$

Deligne: *For $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sigma)/\mathbb{Q})$, we have $\mathrm{ord}_{s=1} L(C^\tau, s) = \mathrm{ord}_{s=1} L(C^{\sigma(\tau)}, s)$.*

Weak BSD: $\mathrm{ord}_{s=1} L(C^\tau, s) = \langle \tau, \mathrm{Jac}(C)(K) \otimes_\mathbb{Z} \mathbb{C} \rangle$.

Parity: *For τ self-dual, $w(C^\tau) = (-1)^{\langle \tau, \mathrm{Jac}(C)(K) \otimes_\mathbb{Z} \mathbb{C} \rangle}$.*

Further to the conjecture of Deligne appearing above, he makes a conjecture ([4, Conjecture 2.8], cf. Conjecture 5.3) linking the 'irrational parts' of motivic L-values to Deligne periods—that is, determinants of period matrices arising from the comparison isomorphism between the Betti and de Rham realisations.

The main achievement of the present work is the derivation of an algorithm in §3 for computing the Dirichlet series coefficients for $L(C^\tau, s)$, the primary novelty of which is the reduction of the computation of Euler factors to point counts on curves—just as in the case of $L(C, s)$. We implement this algorithm in several cases using the Magma [1] adaptation of T. Dokchitser's L-function calculator [5]. The motivations for this are myriad, but we focus primarily on numerical verification of Deligne's Period Conjecture in new settings. To do so, we work with superelliptic curves, a special case affording an explicit, testable formulation (Conjecture 5.18).

In particular, in §5.2 we discuss how the results of [22] and the associated Magma implementation [21] can be used to numerically compute the Deligne periods of motivic pieces of superelliptic curves. In §6, we combine this implementation with our algorithm to numerically verify Deligne's Period Conjecture for the motivic pieces of several superelliptic curves.

Theorem 1.3. *For the 14 genus 3 superelliptic curves with $L(C, 1) \neq 0$ considered in §6, there exist $a, b, c \in \mathbb{Z}$ with $|a|, |b| \leq 16$ and $|c| \leq 160$ such that*

$$(1.4) \qquad \left| \frac{L^{approx}(C^\tau, 1)}{\Omega_{C^\tau}^{approx}} - \frac{a + b \cdot \omega}{c} \right| < \frac{1}{c^4},$$

where τ is a non $\mathbb{Q}$-rational character of C_3 or C_4, $L^{approx}(C^\tau, 1)$ is our numerical value for $L(C^\tau, 1)$, $\Omega_{C^\tau}^{approx}$ is that for the Deligne period as in §5.2—both computed to precision at least 10—and $\omega = i$ or $\omega = \zeta_3$ depending on the exponent of C.

Remark 1.5. With Roth's Theorem in mind, we note that (1.4) is a vastly better approximation than one would expect to find for a non $\mathbb{Q}(\omega)$-rational. Theorem 1.3 therefore provides compelling evidence for the rationality of $L(C^\tau, 1)/\Omega_{C^\tau}$ and so for the validity of Deligne's Period Conjecture.

Example 1.6 (= Example 6.1). Consider the curve

$$C/\mathbb{Q}(\zeta_3): \quad y^3 = x^4 + x^3 + \zeta_3 x^2 + \zeta_3 - 2$$

and write τ for a non-trivial character of $C_3 \cong \langle y \mapsto \zeta_3 \cdot y \rangle$. With $100,000$ Dirichlet series coefficients, we compute $L(C^\tau, 1) \approx 1.8460900297\ldots - i \cdot 1.5447030118\ldots$ and $\Omega_{C^\tau} = -38.6473408677\ldots - i \cdot 105.727015294\ldots$. Together, these computations yield

$$\frac{L(C^\tau, 1)}{\Omega_{C^\tau}} \approx -0.0185185185185\ldots + i \cdot 0.0106916716517\ldots \approx \frac{\zeta_3 - 1}{81}.$$

Remark 1.7. Naturally, it would be of interest to go beyond the rationality predicted by Deligne's Period Conjecture to say something about the rational L-values it predicts to exist, *à la* Birch and Swinnerton-Dyer. Between [7] and [2], it is explained that L-values of Artin-twists of elliptic curves ought to depend on the Galois-module structure of the Tate–Shafarevich group. Utilising this philosophy, the recent work [19] in the same setting seeks to explicitly describe the ideal generated by the L-value of an Artin-twist in terms of this Galois-module structure. As this setting is a special case of motivic pieces of curves, one ought to believe that L-values of motivic pieces of curves depend (at least) on BSD-invariants and associated G-module structures.

It is also worth noting that, in the case of Artin-twisted L-functions of abelian varieties, integral refinements of Deligne's Period Conjecture are an active area of study (e.g. [12, 13]) and these may extend to our setting.

We also demonstrate how the algorithm discussed in §3 can be used to factorise L-functions of curves with extra endomorphisms of Hecke type (Definition 4.1), rather than extra automorphisms. In particular, by studying an action of the dihedral group of order 10 on a genus 6 cover, we recover the known factorisations of L-functions of a family of genus 2 curves over $\mathbb{Q}$ whose Jacobians are of GL_2-type into products of L-functions of classical newforms:

Proposition 1.8 (= Proposition 4.12). *Consider the one-parameter family of genus 2 curves*
$$C_t/\mathbb{Q}: \quad y^2 = x^6 - 2x^5 + x^4 + (6 - 4t)x^3 + 2x^2 + 4x + 1.$$
There is a degree 2 cover X_t of C_t admitting an action by the dihedral group of order 10 such that $C_t \cong X_t/C_2$ and so $L(C_t, s) = L(X_t^\rho, s) \cdot L(X_t^{\rho'}, s)$, where τ and τ' are conjugate irreducible two-dimensional representations. The factors $L(X_t^\rho, s)$ and $L(X_t^{\rho'}, s)$ are the L-functions of conjugate newforms with coefficients in $\mathbb{Q}(\sqrt{5})$.

In general, to factorise the L-function of a Jacobian $\mathrm{Jac}(C)$ of GL_2-type into L-functions of newforms, one must compute the conductor of C and thereby obtain the level of the associated newforms, reducing to a finite brute-force check. Proposition 1.8 demonstrates how—at least in principle—the results of this work allow, when C has endomorphisms of Hecke type, direct computation of Fourier coefficients without needing even full knowledge of the conductor of C.

Conventions. Unless otherwise stated, C/K denotes throughout a curve over a number field equipped with an action of the finite group G by K-automorphisms, with τ a representation of G. All of our computations are carried out in Magma and all code used is available at

https://github.com/hspen99/MotivicPiecesOfCurves [24].

2. MOTIVIC PIECES OF CURVES

The framework of motivic pieces of curves (or abelian varieties) was laid out by V. Dokchitser, Green, Konstantinou and Morgan in their work on the Parity Conjecture for Jacobians of curves [8, §2]. The set-up is as follows: fix a curve C over a number field K with the action of a finite group G by K-automorphisms, as well as embeddings $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell \subset \mathbb{C}$ for each prime ℓ such that $\overline{\mathbb{Q}} \subset \mathbb{C}$ is independent of ℓ. The G-action on C extends to $H^1_\ell(C) = (T_\ell \operatorname{Jac}(C) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell)^*$ for each prime ℓ.

Definition 2.1 (= [8, Definition 2.3]). For a G-representation τ, define
$$H^1_\ell(C^\tau) = \operatorname{Hom}_G(\tau, H^1_\ell(C)),$$
on which G_K acts by postcomposition. Implicitly this definition depends on ℓ, but in fact it yields a compatible system of ℓ-adic representations (cf. [8, Corollary 2.14]).

This system of ℓ-adic representations gives rise to an L-function in the usual way.

Definition 2.2. We write $L(C^\tau, s)$ for the L-function $L(H^1_\ell(C^\tau), s)$, given for $\Re(s) \geq 3/2$ by
$$L(C^\tau, s) := L(H^1_\ell(C^\tau), s) = \prod_{\mathfrak{p} \lhd \mathcal{O}_K} L_{\mathfrak{p}}(C^\tau, N(\mathfrak{p})^{-s})^{-1},$$
where $L_{\mathfrak{p}}(C^\tau, T) := \det(1 - \operatorname{Frob}_{\mathfrak{p}}^{-1} \cdot T \mid H^1_\ell(C^\tau)^{I_{\mathfrak{p}}})$ for any choice of $\mathfrak{p} \nmid \ell$. Here $I_{\mathfrak{p}}$ is the inertia group of the completion of K at $\mathfrak{p}$, and $\operatorname{Frob}_{\mathfrak{p}}$ is the arithmetic Frobenius.

We note that the τ-isotypic piece of $H^1_\ell(C)$ differs from the motivic piece corresponding to τ when $\dim \tau > 1$. This is because, for τ irreducible, we have a canonical isomorphism
$$\operatorname{Hom}_G(\tau, V)^{\oplus \dim \tau} \cong V^\tau,$$
where V is any G-representation and V^τ is the τ-isotypic piece thereof. This brings us to an important analogue of the 'Artin-formalism'.

Lemma 2.3 (= [8, Proposition 2.8]). *For representations τ_1 and τ_2 G, and ρ a representation of $H \leq G$:*
  (1) $H^1_\ell(C^{\tau_1 + \tau_2}) = H^1_\ell(C^{\tau_1}) + H^1_\ell(C^{\tau_2})$
  (2) $H^1_\ell(C^{\operatorname{Ind}^G_H \rho}) = H^1_\ell(C^\rho)$.

Proof. (1) follows from additivity of Hom. (2) follows from Frobenius reciprocity. □

The following is an important formula for the trace of Frobenius acting on the ℓ-adic representation of a motivic piece in terms of the Galois action on $H^1_\ell(C)$ as a whole.

Proposition 2.4 (= [8, Lemma 2.12]). *For $g \in G$ and $\alpha \in G_K$, we have*

$$(2.5) \qquad \operatorname{Tr}(\alpha \mid H^1_\ell(C^\tau)) = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(g^{-1} \mid \tau) \operatorname{Tr}(\alpha \cdot g \mid H^1_\ell(C)).$$

Immediately, this yields:

Lemma 2.6. *For $\sigma \in \operatorname{Gal}(\mathbb{Q}(\tau)/\mathbb{Q})$, we have*
$$\det(1 - \operatorname{Frob}_{\mathfrak{p}}^{-1} T \mid H^1_\ell(C^\tau))^\sigma = \det(1 - \operatorname{Frob}_{\mathfrak{p}}^{-1} T \mid H^1_\ell(C^{\sigma(\tau)})).$$

Another consequence of Proposition 2.4 is the following fact about conductors, which will prove helpful in our calculations. The conductor of an ℓ-adic representation is an ideal which—in some sense—measures bad reduction, and happens to be the fundamental invariant associated to L-functions. See, for example, [26] for the relevant background.

Lemma 2.7. *Let τ and τ' be conjugate representations of G. Then C^τ and $C^{\tau'}$ have equal conductor norms.*

Proof. The relevant conductors are determined by the action of inertia on the ℓ-adic representations $H^1_\ell(C^\tau)$ and $H^1_\ell(C^{\tau'})$. Inertia acts through a finite quotient, say I, on $H^1_\ell(C)$, and it suffices to show that $H^1_\ell(C^\tau)$ and $H^1_\ell(C^{\tau'})$ are conjugate as I-representations. This follows immediately from Proposition 2.4, which says that the characters of these I-representations are conjugate. $\square$

We write $\Lambda(C^\tau, s)$ for the completed L-function

$$\Lambda(C^\tau, s) := L_\infty(C^\tau, s) \cdot L(C^\tau, s),$$

where

$$(2.8) \qquad L_\infty(C^\tau, s) = \left(\frac{N}{\pi^{\langle \tau, H^1_\ell(C) \rangle}} \right)^{\frac{s}{2}} \cdot \left(\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \right)^{\frac{\langle \tau, H^1_\ell(C) \rangle \cdot [K:\mathbb{Q}]}{2}},$$

which one should think of as the contribution to the Euler product from infinite primes.

Remark 2.9. Here the conductor N appearing in (2.8) is obtained from the usual conductor $\mathcal{N}$ of $H^1_\ell(C)$ by $N = N_{K/\mathbb{Q}}(\mathcal{N}) \cdot |\Delta_K|^{\langle \tau, H^1_\ell(C) \rangle}$. This is really the conductor of $\mathrm{Res}_{K/\mathbb{Q}} C^\tau$ (cf. [20, p. 178]), which we use because Magma works with Dirichlet series over $\mathbb{Q}$.

With this in mind, the Hasse–Weil Conjecture for motivic pieces reads as follows:

Conjecture 2.10 (Hasse–Weil). *The L-function $L(C^\tau, s)$ admits meromorphic continuation to $\mathbb{C}$, and the completed L-function $\Lambda(C^\tau, s)$ satisfies the functional equation*

$$(2.11) \qquad \Lambda(C^\tau, s) = w(C^\tau) \cdot \Lambda(C^{\tau^*}, 2 - s),$$

where $w(C^\tau)$ is the root number of C^τ.

Remark 2.12. See [8, §2] for a discussion of the quantity $w(C^\tau)$, omitted from the present work because root numbers are not required in the Magma implementation for L-function computations.

Remark 2.13. A less explicit way of constructing the L-function $L(C^\tau, s)$ as in Definition 2.2 is as follows: construct the abelian variety $A_{\hat{\tau}}$ whose ℓ-adic cohomology is $\oplus_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\tau)/\mathbb{Q})} H^1_\ell(C^{\sigma(\tau)})$—this is a quotient of $\mathrm{Jac}(C)$ appearing with multiplicity $\dim \tau$ in the isogeny decomposition thereof. We have $\mathbb{Q}(\tau) \hookrightarrow \mathrm{End}(A_{\hat{\tau}}) \otimes \mathbb{Q}$ and the corresponding decomposition on the level of L-functions coming from the fact that $V_\ell(A)$ is a $\mathbb{Q}(\tau) \otimes \mathbb{Q}_\ell$-module is precisely that coming from that of $L(C, s)$ into L-functions of motivic pieces. This point of view will prove helpful in §4.3.

Remark 2.14. In this section we have held our focus on ℓ-adic cohomology, but we note that we can and will associate to C^τ other cohomology theories in much the same way.

3. The Algorithm

We now describe an algorithm for computing the Euler factors at good primes of $L(C^\tau, s)$. In light of (2.5), the important step is computing the traces $\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}} \cdot g \mid H^1_\ell(C^\tau)^{I_{\mathfrak{p}}})$ for $g \in G$; the novel aspect of our algorithm is the construction of auxiliary curves, point counts on which will yield such traces. The advantage of reducing the problem to point counts on curves—as opposed to relying, for example, on Gröbner basis techniques to count fixed points—is that there are many efficient algorithms implemented in Magma for these computations.

Algorithm 3.1.

Input: A (model of a) curve C with an action by automorphisms of G, a representation τ of G, and a prime $\mathfrak{p}$ of good reduction with $\mathfrak{p} \nmid |G|$.

Output: The local polynomial of $L(C^\tau, s)$ at $\mathfrak{p}$.

(1) For each $g \in G$ compute $\mathrm{Tr}((\mathrm{Frob}_\mathfrak{p} \cdot g)^i \mid H^1_\ell(C))$ for $i \in \{1, \ldots, g(C)\}$. We explain how to do this with $i = 1$, the rest is the same after base change:

   (a) Obtain a model for the quotient curve $C_g := C/\langle g \rangle$ over $k_\mathfrak{p}$.

   (b) Construct the following $C_d \times C_d$-diagram of function fields, where we identify $k_\mathfrak{p}$ with $\mathbb{F}_q$ and write d for the order of g:

$$
\begin{array}{ccccc}
 & & \mathbb{F}_{q^d}(C) & & \\
 & {\scriptstyle\langle\Phi\rangle}\diagup & \Big| & \diagdown{\scriptstyle\langle g\rangle} & \\
\mathbb{F}_q(C) & & {\scriptstyle\langle g\rangle\times\langle\Phi\rangle} & & \mathbb{F}_{q^d}(C_g) \\
 & {\scriptstyle\langle g\rangle}\diagdown & \Big| & \diagup{\scriptstyle\langle\Phi\rangle} & \\
 & & \mathbb{F}_q(C_g) & &
\end{array}
$$

Write $\widetilde{C}_g$ for the quotient curve $C_{\mathbb{F}_{q^d}}/\langle g \cdot \Phi \rangle$, which is defined over $\mathbb{F}_q$. By Lemma 3.4 below, we have

$$(\dagger) \qquad \mathrm{Tr}(\mathrm{Frob}_\mathfrak{p} \cdot g \mid H^1_\ell(C)) = 1 + q - \#(\widetilde{C}_g(\mathbb{F}_q)),$$

so it suffices to obtain equations for this curve; even if we find a singular curve birational to $\widetilde{C}_g$, we can adjust this point count for singularities. An equation can be found as follows:

   (i) Write r for the minimum degree of an extension of $\mathbb{F}_q$ which contains the d-th roots of unity and base change all curves to a degree r extension.

   (ii) Write $\mathbb{F}_{q^r}(C)/\mathbb{F}_{q^r}(C_g)$ as a Kummer extension with minimal polynomial $X^d - R$.

   (iii) Compute the function field $\mathbb{F}_{q^r}(C)[X]/(uX^d - R)$ of $(\widetilde{C}_g)_{\mathbb{F}_{q^r}}$, where u is as in Lemma 3.5 below.

   (iv) By taking the trace of a d-th root of R/u and computing its minimal polynomial, obtain an equation for $\widetilde{C}_g$ over $\mathbb{F}_q$.

(2) Compute $\mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}^i \mid H^1_\ell(C^\tau))$ for each i using (2.5).

(3) Output $\det(1 - \mathrm{Frob}_\mathfrak{p}^{-1} T \mid H^1_\ell(C^\tau))$, which can now be computed via Newton's identities.

Remark 3.2. The requirement that $\mathfrak{p}$ be of good reduction is necessary, although the method described above would compute the correct factor of the L-polynomial of the smooth normalisation of the special fibre of a regular model of $C_{K_\mathfrak{p}}$ at a bad prime $\mathfrak{p}$. To extract the full Euler factor, one must compute the action of G on the special fibre—in practice either the conductor exponent is small and this factor suffices, or it is easier to guess the Euler factor.

Remark 3.3. To extend Algorithm 3.1 to primes $\mathfrak{p} \mid |G|$, one may need to work with Artin–Schreier theory instead of Kummer theory. In principle this is not a problem, but it is harder to extract equations for the auxiliary curves as in step (1)(b)(iii). In practice, the order of G will have few divisors and again it will be easier to guess Euler factors at these primes.

It remains to justify $(\dagger)$ and step (1)(b)(iii), which we do with the following lemmata:

Lemma 3.4. *With the set-up as in Algorithm 3.1, equation (†) holds. That is,*

$$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}} \cdot g \mid H^1_\ell(C)) = 1 + q - \#\widetilde{C}_g(\mathbb{F}_q).$$

Proof. By Lefschetz's trace formula, it suffices to show that the number of fixed points of $g \cdot \Phi$ acting on $C_{\mathbb{F}_{q^d}}$ is equal to $\#\widetilde{C}_g(\mathbb{F}_q)$. This is clear because $\mathbb{F}_{q^d}(C)/\mathbb{F}_q(\widetilde{C}_g)$ is an extension of the constant field; $g \cdot \Phi$ is the Frobenius endomorphism of $\widetilde{C}_g$. $\quad\square$

Lemma 3.5. *Keep the set-up as in Algorithm 3.1, recalling in particular that $\mathfrak{p} \nmid |G|$ is coprime to d. Firstly, write $K(\zeta_d)(C)/K(\zeta_d)(C_g)$ as a Kummer extension with minimal polynomial $X_0^d - R_0$ so that $\mathbb{F}_{q^r}(C)/\mathbb{F}_{q^r}(C_g)$ is given by a Kummer extension with minimal polynomial $X^d - R$. Then $\mathbb{F}_{q^r}(\widetilde{C}_g)/\mathbb{F}_{q^r}(C_g)$ can be written as a Kummer extension with minimal polynomial*

$$uX^d - R, \quad where \quad u^{(q^r-1)/d} = z^{-1}.$$

Here z is a d-th root of unity constructed in the following way: choose a prime $\mathfrak{P}$ of $K(\zeta_d)$ above $\mathfrak{p}$, identifying the residue field of $\mathfrak{P}$ with $\mathbb{F}_{q^r}$, and set z to be the reduction modulo $\mathfrak{P}$ of $g(X_0)/X_0$.

Proof. Suppose without loss of generality C_g is given by a (possibly singular) affine equation $f(Y, Z) = 0$ over $\mathbb{F}_q$. The fixed points of $g \cdot \Phi$ acting on $C_{\mathbb{F}_{q^r}}$ are those with $(z \cdot X^{q^r}, Y^{q^r}, Z^{q^r}) = (X, Y, Z)$, i.e. $Y, Z \in \mathbb{F}_{q^r}$ and X satisfying $z \cdot X^{q^r} - X = 0$. The solutions to this equation in X lie in $\mathbb{F}_{q^{dr}}$; indeed if u_0 satisfies $X^{q^r-1} = z$ then the solutions are $u_0 \cdot \mathbb{F}_{q^r}$.

Substituting into the equations $\{X^d - R, f(Y, Z) = 0\}$ for $C_{\mathbb{F}_{q^r}}$, we obtain equations $\{uX^d - R, f(Y, Z) = 0\}$ for $(\widetilde{C}_g)_{\mathbb{F}_{q^r}}$, noting that $u = u_0^d \in \mathbb{F}_{q^r}$. $\quad\square$

Remark 3.6. Our implementations of Algorithm 3.1 are in Magma, within which one can work with general motivic L-functions using their implementation of a version of T. Dokchitser's L-function calculator. This works by estimating the error introduced by truncating the Dirichlet series for a given L-function at $N \in \mathbb{N}$. For a set precision, one can ask Magma for an approximation of the number of coefficients needed for accurate calculation using the command `LCfRequired`, and so in practice it suffices to work with truncated local polynomials rather than computing the whole thing as described above.

4. Endomorphisms of Jacobians

Here we recap some background on endomorphisms of Jacobians, with the point being that the framework of motivic pieces of curves—under certain conditions—can be used to describe factorisations of L-functions of curves whose Jacobians have extra *endomorphisms*, rather than automorphisms.

4.1. **Endomorphisms of Hecke type.** In [11], Ellenberg introduces the notion of endomorphism algebras of Hecke type in order to construct families of curves whose Jacobians have real multiplication, i.e. curves C of genus g for which there is a totally real field F of degree g and an injection $F \hookrightarrow \mathrm{End}(\mathrm{Jac}(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 4.1. Consider a curve C over k equipped with a cover $\pi : C \to B$ of another curve. Say the Galois closure of the field extension $k(C)/k(B)$ has Galois group G. This field corresponds to a curve D, which admits an action by G of automorphisms. Write $H \le G$ for the subgroup for which $C = D/H$. There is a map $\mathbb{Q}[H\backslash G/H] \hookrightarrow \mathrm{End}(\mathrm{Jac}(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$—say the image is an *endomorphism algebra of Hecke type* and that a curve C has extra endomorphisms of Hecke type if there exists a cover π such that this image is non-trivial.

Remark 4.2. Given a curve C whose Jacobian has extra endomorphisms of Hecke type, we obtain a factorisation of $L(C, s)$ corresponding to the factorisation of $L(D, s)$ under the action of G, in the notation of Definition 4.1.

Remark 4.3. Ellenberg notes that quotients of $\mathbb{Q}[H \backslash G / H]$ are central simple algebras over abelian number fields and so, in particular, not all non-trivial endomorphism algebras can arise in this way. Whether there are other obstructions to endomorphism algebras being of Hecke type remains an open question.

4.2. **GL_2-type and modularity.** We will also be interested in Jacobians of GL_2-type. Being of GL_2-type is a condition on the endomorphism algebra, which determines precisely when an L-function of an abelian variety decomposes as a product of L-functions of modular forms.

Definition 4.4. An abelian variety A of dimension g is of GL_2-type if $V_\ell(A)$ is a $G_\mathbb{Q}$-representation taking values in $GL(2, k \otimes \mathbb{Q}_\ell)$, where $k \hookrightarrow \text{End}(A) \otimes_\mathbb{Z} \mathbb{Q}$ is a number field of degree g.

Theorem 4.5. *Given a simple abelian variety A over $\mathbb{Q}$, there exists $N \in \mathbb{N}$ such that A is a quotient of $J_1(N)$ if and only if A is of GL_2-type.*

Proof. In [23] Ribet shows that this is a consequence of Serre's Conjecture, a proof of which is given in [17]. $\qquad\square$

Remark 4.6. A simple abelian variety A being a quotient of $J_1(N)$ means precisely that there exists a newform f of conductor N such that $L(A, s) = \prod_\tau L(f^\tau, s)$, where this product is over the Galois conjugates of f.

4.3. **An example.** It was pointed out to the author by T. Dokchitser that the two-parameter family

$$f_{t,y}(x) = x^5 + yx^4 + tx^3 + (-2t + y^2 + 3y + 5)x^2 + (t + y)x + (y + 3)$$
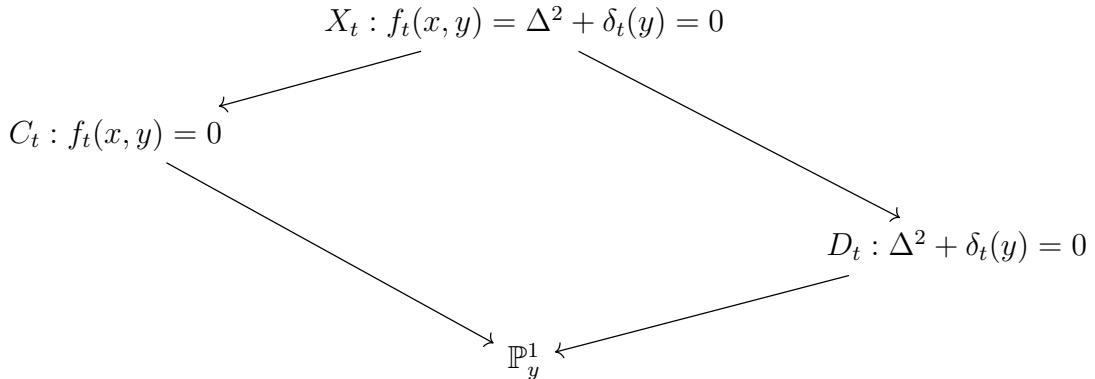
consists of polynomials with D_{10}-Galois group, where D_{10} is the dihedral group of order 10. The unique quadratic subfield of the Galois closure of $f_{t,y}(x)$ is generated by $\sqrt{-\delta_{t,y}}$, where

$$\begin{aligned} \delta_{t,y} = 4y^5 + 31y^4 \;\; &+ \;\; (118 - 26t)y^3 + (245 - 122t - t^2)y^2 \\ &+ \;\; (350 - 270t + 36t^2)y + (375 - 300t + 80t^2 + 4t^3). \end{aligned}$$

We write $f_t(x, y) = f_{t,y}(x)$ and consider this as a one-parameter family of genus 2 curves $C_t : f_t(x, y) = 0$, noting that a Weierstrass model is given by

(4.9) $C_t : \quad y^2 = x^6 - 2x^5 + x^4 + (6 - 4t)x^3 + 2x^2 + 4x + 1.$

We also write $\delta_t(y) = \delta_{t,y}$. We obtain the following $G = D_{10}$-diagram of curves:

$$X_t : f_t(x, y) = \Delta^2 + \delta_t(y) = 0$$

$$C_t : f_t(x, y) = 0$$

$$D_t : \Delta^2 + \delta_t(y) = 0$$

$$\mathbb{P}^1_y$$

Via [16, Theorem A] and the existence of the Brauer relation $\mathbf{1} - 2C_2 - C_5 + G$ in G, we obtain a factorisation $L(X_t, s) = L(C_t, s)^2 \cdot L(D_t, s)$. By comparing with

the decomposition under the action of G, writing ρ and ρ' for the conjugate two-dimensional irreducible representations, we find

$$(4.10) \qquad L(C_t, s) = L(X_t^\rho, s) \cdot L(X_t^{\rho'}, s).$$

Concomitantly, there is another decomposition of $L(C_t, s)$ afforded by Theorem 4.5:

Lemma 4.11. *The L-function $L(C_t, s)$ factors as a product of L-functions of classical modular forms with q-expansion coefficients in $\mathbb{Q}(\sqrt{5})$.*

Proof. It suffices to show that $\mathbb{Q}(\sqrt{5}) \hookrightarrow \operatorname{End}(\operatorname{Jac}(C_t)) \otimes_{\mathbb{Z}} \mathbb{Q}$, so that $\operatorname{Jac}(C_t)$ is of GL_2-type. To see this, note that $\pi_* \circ g_* \circ \pi^*$ acts as $(1 + \sqrt{5})/2$ on $\operatorname{Jac}(C_t)$, where $\pi : X_t \to C_t$ is the quotient map and g is an automorphisms of order 5 acting on X_t. $\qquad\square$

It turns out that the factorisation (4.10) and the factorisation described in Lemma 4.11 are the same, the salient point being that the endomorphism in the proof of Lemma 4.11 is of Hecke type.

Proposition 4.12. *$L(X_t^\rho, s)$ is the L-function of a classical newform.*

Proof. Firstly, the decomposition of $L(C_t, s)$ into L-functions of modular forms is precisely that coming from the action of $\mathbb{Q}(\sqrt{5}) \hookrightarrow \operatorname{End}(\operatorname{Jac}(C_t)) \otimes \mathbb{Q}$. By Remark 2.13, in the notation of which we have $A_{\hat{\rho}} = \operatorname{Jac}(C_t)$, this is the same decomposition coming from the action of D_{10} on X_t. $\qquad\square$

Example 4.13. The curve C_2 as in (4.9) has LMFDB label `529.a.529.1`, from which we see that its Jacobian is isogenous to that of the modular curve $X_0(23)$. We know that $L(X_0(23), s)$ factors as the product of the L-functions of the two newforms of level 23 and weight 2, whose L-functions can also be found on the LMFDB or via Magma's in-built L-series. We compute the Euler factors of $L(X_2^\tau, s)$ and $L(X_2^{\tau'}, s)$ for each prime up to $10,000$ and confirm that these match with those stored in Magma (see [24, `Dihedral.m`]). In light of Proposition 4.12, this provides a helpful sanity check for the validity of Algorithm 3.1.

Remark 4.14. Example 4.13 is the only instance discussed herein where we must implement an instance of explicit Kummer theory as in step 1(b)(ii) of Algorithm 3.1. To do this we used the results of [10] and an adaptation of the associated Mathematica [15] script available at `https://site.uvm.edu/ddummit/`.

In general, then, the algorithm we will describe in §3 gives a method to explicitly recover the factorisation of a Jacobian of GL_2-type whose endomorphisms are of Hecke type into a product of L-functions of newforms, provided that one can exhibit explicit covers as in Definition 4.1.

5. Periods

In this section we discuss the conjectural link between periods of motives and L-values, before explaining theoretically how to compute periods of motivic pieces in general and how to practically do so for pieces of superelliptic curves in particular.

5.1. **Deligne's Period Conjecture.** Firstly, note that in this section we think of number fields abstractly, rather than having in mind an embedding into $\mathbb{C}$.

Definition 5.1. Consider a $\mathbb{Q}$-motive M with coefficients in F. Let $H_B(M)^+$ be the $+1$-eigenspace of complex conjugation, and define

$$\alpha_M^+ : H_B(M)^+ \otimes \mathbb{C} \hookrightarrow H_B(M) \otimes \mathbb{C} \xrightarrow{c_{B,\mathrm{dR}}} H_{\mathrm{dR}}(M) \otimes \mathbb{C} \twoheadrightarrow H_{\mathrm{dR}}^+(M) \otimes \mathbb{C},$$

where the last map is the natural quotient map. Denote by $c^+(M, 0) \in (F \otimes_{\mathbb{Q}} \mathbb{C})^\times$ a representative of the class of $\det(\alpha_M^+)$ in $(F \otimes_{\mathbb{Q}} \mathbb{C})^\times / F^\times$, computed with respect to

F-bases of H_B and H_{dR}, which we view as a tuple $(c^+(\sigma, M, 0))_{\sigma \in \Sigma_F} \in (\mathbb{C}^\times / F^\times)^{\Sigma_F}$ indexed by the embeddings of F.

Moreover, we write $c^+(M, s_0)$ for $c^+(M(s_0), 0)$ and call this the *Deligne period* of M at s_0. If we have fixed an embedding $\iota : F \hookrightarrow \mathbb{C}$ and $F/\mathbb{Q}$ is Galois then, for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, we write $c^+(\iota \circ \sigma, M, s_0)$.

As mentioned above, Deligne conjectures that these periods describe the 'irrational parts' of L-values. To state this, we require one further definition.

Definition 5.2 (Critical value). Given a K-motive M with coefficients in F, say s_0 is a critical value for M if neither $L_\infty(M, s)$ nor $L_\infty(M^*(1), s)$ has a pole at s_0.

Conjecture 5.3 (= [4, Conjecture 2.8]). *Let M be a $\mathbb{Q}$-motive with coefficients in $F \subset \mathbb{C}$ and $s_0 \in \mathbb{Z}$ a critical value for M. There exists $\mathcal{L}(M, s_0) \in F$ such that, for $\sigma \in \Sigma_F$, we have*

$$(5.4) \qquad L(\sigma, M, s_0) = c^+(\sigma, M, s_0) \cdot \sigma(\mathcal{L}(M, s_0)).$$

Remark 5.5. Conjecture 5.3 depends implicitly on the conjectural analytic continuation of $L(M, s)$. We take this as a standing assumption, and in practice use Magma to numerically verify the functional equation before computing any L-values.

Conjecture 5.3 extends to motives over any number field via Weil restriction, under which L-functions are invariant.

Example 5.6. Fix an abelian variety over a number field, A/K. (Alternatively, a curve C/K seeing as C and $\mathrm{Jac}(C)$ have the same first cohomologies.) BSD then implicitly predicts

$$(5.7) \qquad \frac{L(A, 1)}{|\Delta_K|^{\dim A/2} \prod_{\nu | \infty} |\Omega_{A^\nu}|} \in \mathbb{Q}.$$

Here, this product is over the infinite places of K and Ω_{A^ν} is defined as follows: fix a basis $\mathcal{W}$ for the holomorphic differentials Ω_C^1 and, for a given ν, choose a generating set (henceforth basis) Γ_ν for the singular homology group $H_1(A^\nu(\mathbb{C}), \mathbb{Z})$ such that, if ν is a real place, then there exists some $\Gamma_\nu^+ \subset \Gamma_\nu$ which is a basis for $H_1(A^\nu(\mathbb{R}), \mathbb{Z})$.

$$(5.8) \qquad \Omega_{A^\nu} = \begin{cases} \det\left(\int_\gamma \omega \right)_{\omega \in \mathcal{W}, \gamma \in \Gamma_\nu^+} & \text{if } \nu \text{ is a real place} \\ \det\left(\int_\gamma \omega \right)_{\omega \in \mathcal{W} \cup \overline{\mathcal{W}}, \gamma \in \Gamma_\nu} & \text{if } \nu \text{ is a complex place.} \end{cases}$$

That this is consistent with Conjecture 5.3 can be seen from the formulation of BSD for abelian varieties over number fields in [14], applying restriction of scalars and referring to [4, §4].

In this context, to have coefficients in a number field F is to have an embedding $F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ of F into the endomorphism algebra. To compute the period $c^+(h^1(A), 1) \in (F \otimes_{\mathbb{Q}} \mathbb{C})^\times$ as in (5.4), one must then choose F-bases for the de Rham cohomology and singular homologies (of $\mathrm{Res}_{K/\mathbb{Q}} A$) and compute the determinants as in (5.8) with respect to these bases. The effect of forgetting this additional structure is to take the norm map $F \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C}$, in which case our periods and L-values are multiplied together and we recover the prediction (5.7) from Conjecture 5.3.

Remark 5.9. It is informative to see a brief explanation of the provenance of the discriminant factor in (5.7). We focus for simplicity on the case that K is totally real. Fix a basis $\{k_i\}$ for $K/\mathbb{Q}$ and assign labels $\{\nu_j\}$ to the embeddings of K. Fix also bases $\mathcal{W}$ for Ω_A^1 and Γ_j for $H_1(A^{\sigma_j}(\mathbb{R}))$ for each j. For $\omega \in \mathcal{W}$ the differentials $\{k_i \cdot \omega\}$ are linearly independent from the point of view of the restriction of scalars.

The period matrix of $\mathrm{Res}_{K/\mathbb{Q}} A$ is now made up of $\dim A \times \dim A$ blocks where the (i,j)-th block is given by

$$\left(\nu_j(k_i) \int_\gamma \omega \right)_{\omega \in \mathcal{W}, \gamma \in \Gamma_j} =: \nu_j(k_i) \cdot P_j$$

In particular, this matrix is equal to

$$\mathrm{diag}(P_j)_j \cdot \left((\nu_j(k_i))_{i,j} \otimes I_{g_C} \right),$$

where $\otimes$ denotes the Kronecker product, whose determinant—up to rationals—is as in (5.7).

We now return to the context of motivic pieces of curves. Given C/K, G as usual with τ now irreducible, write $\hat\tau$ for the rational representation $\bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\tau)/\mathbb{Q})} \sigma(\tau)$. We recall from Remark 2.13 that the piece $C^{\hat\tau}$ corresponds to the motive of an abelian variety, $A_{\hat\tau}$, a quotient of $\mathrm{Jac}(C)$. This abelian variety has coefficients in $\mathbb{Q}(\tau)$, with the corresponding L-values being associated to the pieces C^τ under different embeddings $\mathbb{Q}(\tau) \hookrightarrow \mathbb{C}$ or, equivalently, the Galois conjugate pieces under a fixed embedding. Applying Conjecture 5.3 to $A_{\hat\tau}$ then gives a prediction for the L-values $L(C^{\sigma(\tau)}, 1)$.

5.2. **Superelliptic curves.** Molin and Neurohr explicitly describe methods to compute period matrices associated to superelliptic curves over $\mathbb{C}$ to high precision in [22], with implementations of their work available at [21]. The methods they use do not give the full $2g \times 2g$-matrices as in the complex case of (5.8), but rather the $g \times 2g$-matrix

$$P_{\mathcal{W},\Gamma} = \left(\int_\gamma \omega \right)_{\omega \in \mathcal{W}, \gamma \in \Gamma}$$

for specific choices $\mathcal{W}$ and Γ of bases of Ω_C^1 and $H_1(C, \mathbb{Z})$, respectively. We explain how to extract periods of motivic pieces of superelliptic curves from their work.

Firstly, we fix a superelliptic curve $C/K : y^m = f(x)$, where f is separable, and an embedding $\sigma : K \hookrightarrow \mathbb{C}$. Write $n = \deg f$ and $\delta = \gcd(m, n)$. We consider the group of automorphisms $G \cong C_m$ generated by $\alpha : y \mapsto \zeta_m \cdot y$.

Proposition 5.10 (= [22, Proposition 3.8]). *The set*

$$\mathcal{W}_\sigma = \left\{ \omega_{i,j} := \frac{x^{i-1}\mathrm{d}x}{y^j} \mid -mi + jn - \delta \geq 0 \right\}$$

forms a basis for $\Omega_{C^\sigma}^1$.

The action of G on $\Omega_{C^\sigma}^1$ is thus clear; $\mathcal{W}_\sigma$ is already an eigenbasis for α. The story for singular homology is less straightforward. To construct a basis for $H_1(C^\sigma, \mathbb{Z})$, for each pair (a, b) of branch points of the rational map x, a branch $y_{a,b}$ of C is defined (see [22, p. 6] for details) and hence oriented paths

$$(5.11) \qquad \gamma_{[a,b]}^{(\ell)} = \{(x, \zeta_m^\ell \cdot y_{a,b}(x)) \mid x \in [a, b]\}, \quad \ell \in \mathbb{Z}/m\mathbb{Z}$$

are obtained. Here $[a, b]$ is the oriented line segment joining a and b in the complex plane. We obtain loops

$$\gamma_{a,b}^{(\ell)} = \gamma_{[a,b]}^{(\ell)} \cup \gamma_{[b,a]}^{(\ell)}$$

for each pair of branch points. Write X for the complete graph whose vertices are branch points of x, and E for the edges of any choice of spanning tree of X. Writing $\gamma_e^{(\ell)}$ for $\gamma_{a,b}^{(\ell)}$, where e is the edge between vertices $a, b \in X$, we are now able to describe generators for $H_1(C, \mathbb{Z})$.

Theorem 5.12 (= [22, Theorem 3.6]). *The set*

$$(5.13) \qquad \Gamma_\sigma = \{\gamma_e^{(\ell)} \mid 0 \le \ell < m - 1, e \in E\}$$

forms a generating set for $H_1(C^\sigma, \mathbb{Z})$.

Remark 5.14. The reason for the strict inequality in (5.13) is that $\prod_{\ell=0}^{m-1} \gamma_e^{(\ell)} = 1$ in $H_1(C, \mathbb{Z}) = \pi_1(C)^{\mathrm{ab}}$.

Molin and Neurohr then describe and implement a method for computing $\int_\gamma \omega$ for $\omega \in \mathcal{W}_\sigma$ and $\gamma \in \Gamma_\sigma$. The following lemma will allow us to circumvent the fact that this only applies to holomorphic differentials. Firstly, write $\mathcal{W}_{\sigma,\tau} = \{\omega \mid \omega \in \mathcal{W}_\sigma \cup \overline{\mathcal{W}}_\sigma, \alpha(\omega)/\omega = \sigma(\tau(\alpha))\}$.

Lemma 5.15. *Consider a curve* $C/\mathbb{Q}(\zeta_m) : y^m = f(x)$ *as above. There exists a matrix* T *with entries in* $\mathbb{Q}(\zeta_m)$ *such that, for a fixed embedding* $\sigma : \mathbb{Q}(\zeta_m) \hookrightarrow \mathbb{C}$,

$$\left(\int_\gamma \overline{\omega} \right)_{\omega \in \mathcal{W}_\sigma \cap \mathcal{W}_{\sigma,\tau}, \gamma \in \Gamma_\sigma} = \sigma(T) \cdot \left(\int_\gamma \omega \right)_{\omega \in \mathcal{W}_{\overline{\sigma}} \cap \mathcal{W}_{\overline{\sigma},\tau}, \gamma \in \Gamma_{\overline{\sigma}}} .$$

Proof. We identify $\mathcal{W}_{\overline{\sigma}}$ with $\overline{\mathcal{W}}_\sigma$. Then, for $\omega \in \mathcal{W}_\sigma$ and $\gamma \in H_1(C^\sigma(\mathbb{C}), \mathbb{Z})$, we have $\overline{\int_\gamma \overline{\omega}} = \int_{\overline{\gamma}} \omega$, where $\overline{\gamma}$ denotes the obvious loop in $H_1(C^{\overline{\sigma}}(\mathbb{C}), \mathbb{Z})$.

Explicitly, the comparison isomorphism $H_{\mathrm{dR}}^1(C) \otimes \mathbb{C} \to H_B^1(C) \otimes \mathbb{C}$ is given by the map

$$\omega \mapsto \left[\gamma \mapsto \int_\gamma \omega \right].$$

Restricting to the $\overline{\sigma}$-components gives an isomorphism $H_{\mathrm{dR}}^1(C^{\overline{\sigma}}) \otimes \mathbb{C} \to H_B^1(C^{\overline{\sigma}})$ and moreover the holomorphic differentials $\Omega_{C^{\overline{\sigma}}}^1$ are sent to a distinguished subspace in accordance with the Hodge decomposition. We note that $H^{(1,0)}(C^{\overline{\sigma}})$ and $H^{(0,1)}(C^\sigma)$ are exchanged under complex conjugation. The τ-part of the map

$$\Omega_{C^{\overline{\sigma}}}^1 \to H^{(1,0)}(C^{\overline{\sigma}}) \xrightarrow{\mathrm{c.c.}} H^{(0,1)}(C^\sigma),$$

is then given by the matrix $\left(\overline{\int_\gamma \omega} \right)_{\omega \in \mathcal{W}_{\overline{\sigma}} \cap \mathcal{W}_{\overline{\sigma},\tau}, \gamma \in \Gamma_{\overline{\sigma}}}$. On the other hand, the map

$$\Omega_{C^\sigma}^1 \xrightarrow{\mathrm{c.c.}} \Omega_{C^{\overline{\sigma}}}^1 \to H^{(0,1)}(C^\sigma)$$

is given by the matrix $\left(\int_\gamma \overline{\omega} \right)_{\omega \in \mathcal{W}_\sigma \cap \mathcal{W}_{\sigma,\tau}, \gamma \in \Gamma_{\sigma,E}}$. The result now follows because these correspond to the same comparison (possibly) with respect to different choices of bases. $\square$

Note that the graph X as above exists independently of any embedding $\mathbb{Q}(\zeta_m) \hookrightarrow \mathbb{C}$ and select a set E of edges of a spanning tree. For each such embedding $\sigma \in \Sigma_{\mathbb{Q}(\zeta_m)}$, we see from (5.11) that $\Gamma_{\sigma,E} := \{\gamma_e^{(0)} \mid e \in E\}$ is a $\mathbb{Q}(\zeta_m)$-basis for $H_1(C^\sigma, \mathbb{Z})$. Taking the restriction of scalars and writing $\Sigma_{\mathbb{Q}(\zeta_m)}^+$ for a set of coset representatives of $\Sigma_{\mathbb{Q}(\zeta_m)}$ modulo complex conjugation, we see that

$$\cup_{\sigma \in \Sigma_{\mathbb{Q}(\zeta_m)}^+} \{\gamma + \overline{\gamma} \mid \gamma \in \Gamma_{\sigma,E}\}$$

is a $\mathbb{Q}(\zeta_m)$-basis for $H_1(\mathrm{Res}_{\mathbb{Q}(\zeta_m)/\mathbb{Q}} C(\mathbb{R}), \mathbb{Z})$. Likewise, the +1-eigenspace of complex conjugation on $H_{\mathrm{dR}}^1(\mathrm{Res}_{\mathbb{Q}(\zeta_m)/\mathbb{Q}} C)$ has $\cup_{\sigma \in \Sigma_{\mathbb{Q}(\zeta_m)}^+} \{\omega + \overline{\omega} \mid \omega \in \mathcal{W}_\sigma\}$ as a $\mathbb{Q}(\zeta_m)$-basis.

There is a decomposition of these (co)homology spaces under the action of $\mathbb{Q}(\zeta_m)$. To take the determinant of the τ-part of the period map, we must compute the determinants of the matrices

$$\left(\int_\gamma \omega + \int_{\overline{\gamma}} \omega \right)_{\omega \in \mathcal{W}_{\sigma,\tau}, \gamma \in \Gamma_{\sigma,E}} .$$

In light of Lemma 5.15, however, it suffices to compute our Deligne period as

$$(5.16) \quad \Omega_{C^{\sigma(\tau)}} = \prod_{\sigma \in \Sigma^+_{\mathbb{Q}(\zeta_m)}} \det\left(\frac{P_{\sigma,\tau}}{P_{\bar{\sigma},\tau}}\right), \quad \text{where} \quad P_{\sigma,\tau} = \left(\int_\gamma \omega\right)_{\omega \in \mathcal{W}_\sigma \cap \mathcal{W}_{\sigma,\tau}, \gamma \in \Gamma_{\sigma,E}}.$$

Remark 5.17. There is no discriminant term appearing here as in Example 5.6 because we have already accounted for restriction of scalars above. However, one could state Conjecture 5.18 for superelliptic curves over finite extensions K of $\mathbb{Q}(\zeta_m)$, in which case an analogous term involving the relative discriminant $\Delta_{K/\mathbb{Q}(\zeta_m)}$ would appear.

Finally, Conjecture 5.3 then predicts:

Conjecture 5.18. *Let $C/\mathbb{Q}(\zeta_m) : y^m = f(x)$ be as above, choose τ a character of C_m. Fix an embedding $\mathbb{Q}(\zeta_m) \hookrightarrow \mathbb{C}$. There exists $\mathcal{L}(C^\tau) \in \mathbb{Q}(\zeta_m)$ such that, for $\sigma \in \mathrm{Gal}(\mathbb{Q}(\tau)/\mathbb{Q})$,*

$$L(C^{\sigma(\tau)}, 1) = \Omega_{C^{\sigma(\tau)}} \cdot \sigma(\mathcal{L}(C^\tau)).$$

In the sequel we combine [21] with our implementations of Algorithm 3.1 to numerically verify Conjecture 5.18 for the motivic pieces of several superelliptic curves over $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(i)$.

6. COMPUTATIONS

6.1. **Exponent three.** Here we present some computations of L-functions, L-values and periods for superelliptic curves $C/\mathbb{Q}(\zeta_3) : y^3 = f(x)$ with $\deg f = 4$ and f is separable. Such curves have genus 3 and are known as Picard curves.

Example 6.1. Consider the Picard curve

$$C/\mathbb{Q}(\zeta_3) : \quad y^3 = x^4 + x^3 + \zeta_3 x^2 + \zeta_3 - 2$$

and write τ for a non-trivial character of C_3. We use an implementation of Algorithm 3.1 (see [24, `Superelliptic_Exp3.m`]) to compute Dirichlet series coefficients $L(C^\tau, s)$, with the caveat being that we guess the correct Euler factor at the unique prime π above 3 and the conductor exponent at 3 from the conjectural functional equation. To guess the conductor, we assume that the bound

$$(6.2) \quad v_\pi(\mathcal{N}(C)) \le v_\pi(\Delta)$$

is satisfied, thereby reducing to a finite check. Note that (6.2) is not known to hold in this setting, but is generally expected to.

Taking $L_\pi(C^\tau, s) = 1$ and $v_\pi(N(C^\tau)) = 12$, we find that $L(C^\tau, s)$ seems to satisfy its functional equation—the output of Magma's `CFENew`, computed to precision 15, is less than 10^{-11} given $100,000$ Dirichlet series coefficients. Although we are primarily concerned with computing the L-functions of the pieces, we note that Magma estimates that a number of coefficients of the order 10^9 would be needed to naïvely verify the Hasse–Weil Conjecture for C to this same precision.

We compute $L(C^\tau, 1) \approx 1.84609002978320\ldots - i \cdot 1.54470301185825\ldots \ne 0$, making this a candidate for non-trivial numerical verification of Conjecture 5.18. We obtain the value $\Omega_{C^\tau} = -38.6473408677\ldots - i \cdot 105.727015294\ldots$ as in (5.16) via the method described in §5.2, and compute

$$(6.3) \quad \frac{L(C^\tau, 1)}{\Omega_{C^\tau}} \approx -0.0185185185185\ldots + i \cdot 0.0106916716517\ldots \approx \frac{\zeta_3 - 1}{81}.$$

The error in the latter approximation is approximately 10^{-13}—less than 81^{-6} and so vastly smaller than one would expect from an approximation with denominator of this size, and roughly to the same precision to which we computed the L-function.

Because Galois acts as complex multiplication on $\mathbb{Q}(\zeta_3)$, which respects the functional equation (2.11), we have both $L(C^{\overline{\tau}}, 1) = \overline{L(C^\tau, 1)}$ and $\Omega_{C^{\overline{\tau}}} = \overline{\Omega_{C^\tau}}$. Hence the Galois equivariance part of Conjecture 5.18 would automatically follow if (6.3) were a precise equality.

Remark 6.4. The algorithm described in [18] allows one to provably compute the conductor exponent at 3, so in principle one needn't guess the conductor here. In practice this is implemented only for curves defined over $\mathbb{Q}$, making guess-work the easier option, although there is no theoretical obstacle to extending the implementation to number fields.

Example 6.5. Now consider the curve

$$C/\mathbb{Q}(\zeta_3): \quad y^3 = x^4 - 3x^3 + (1 + \zeta_3)x.$$

Using the same methods and notation as in Example 6.1, we numerically verify the functional equation for $L(C^\tau, s)$ to precision 10^{-11} using approximately $360{,}000$ Dirichlet series coefficients, with $L_\pi(C^\tau, s) = 1$ and $v_\pi(N(C^\tau)) = 18$. As before, we compute $L(C^\tau, 1) \approx 2.29230558647\ldots + i \cdot 0.0927222735424\ldots$ and $\Omega_{C^\tau} = -99.3426636471\ldots - i \cdot 157.045532482\ldots$, giving

$$\frac{L(C^\tau, 1)}{\Omega_{C^\tau}} \approx -0.00617283950616\ldots + 0.0106916716517\ldots \approx \frac{\zeta_3}{81},$$

where the error in the latter approximation is again smaller than the precision to which our L-function was computed.

Remark 6.6. In both examples we have taken Picard curves $C : y^3 = f(x)$ where the coefficients of f generate $\mathbb{Q}(\zeta_3)$. If we were to take $C/\mathbb{Q}(\zeta_3)$ to be the base change of a curve defined over $\mathbb{Q}$, then we would simply find $L(C^\tau_{\mathbb{Q}(\zeta_3)}, s) = L(C, s)$.

6.2. Exponent four.

We consider the family of genus 3 curves of the form $C/\mathbb{Q}(i) : y^4 = f(x)$, where $\deg f = 3$. Here, the situation is somewhat kinder computationally than the case of Picard curves over $\mathbb{Q}(\zeta_3)$: fixing τ to be one of the two complex characters of C_4, we find that C^τ has dimension 2 because there is a 2-dimensional piece corresponding to the elliptic curve which arises as the quotient of C by the action of $C_2 \leq C_4$. Moreover, because we expect a large contribution to the conductor from primes dividing the exponent, we also typically deal with smaller conductors than in the Picard curve case. Altogether, this means we are able to generate several examples (see [24, `Superelliptic_Exp4.m`]).

In all of the examples presented in Table 1 we take $L_2(C^\tau, T) = 1$. The second column displays the conductor of $\mathrm{Res}_{K/\mathbb{Q}} C^\tau$, as in the functional equation (2.11). The final column displays a Gaussian integer whose which differs from our computed value of $160 \cdot L(C^\tau, 1)/\Omega_{C^\tau}$ by at most 10^{-10}, which is less than 160^{-4}—we ensure also that the output of Magma's `CFENew` is always less than this bound.

Remark 6.7. The following observation allows us to obtain provably correct values for the conductor norms in Table 1, assuming $L_2(C^\tau, T) = 1$: note that $D : \Delta^2 = \mathrm{disc}_x(y^4 - f(x))$ defines a genus 3 curve, the Jacobian of which decomposes under isogeny as a product of those of the genus 1 curve $\Delta^2 = \mathrm{disc}_x(y^2 - f(x))$ and the genus 2 curve $H : \Delta^2 = y \cdot \mathrm{disc}_x(y^2 - f(x))$. By the results discussed in [25, §5.1], in combination with Lemma 2.7, we deduce that the wild conductor exponent at 2 of C^τ is half that of H. In turn, the latter can be computed using the results of [6] alongside the associated implementation [9].

Remark 6.8. As before, the Galois equivariance part of Deligne's conjecture here would follow automatically from rationality. It would therefore be of great interest to carry out analogous computations in the case of exponent five superelliptic curves,

$f(x)$	$N(C^\tau)$	$160 \cdot L(C^\tau, 1)/\Omega_{C^\tau} \approx$
$x^3 + ix^2 - (1-i)x$	$2^{17} \cdot 5$	$3 - i$
$x^3 + (1+2i)x^2 - 2x$	$2^{14} \cdot 41$	$2 + 4i$
$x^3 + ix$	2^{20}	$1 - 2i$
$x^3 - x^2 + ix$	$2^{16} \cdot 17$	$2 + i$
$x^3 + (3+2i)x^2 + 2x$	$2^{14} \cdot 3^2 \cdot 17$	$-2 - 6i$
$x^3 - 3ix^2 - (3-i)x$	$2^{17} \cdot 5^2$	$-5i$
$x^3 - (1+i)x^2 - 4i$	$2^{16} \cdot 5 \cdot 13$	$10 + 10i$
$x^3 + (2+i)x^2 + 2$	$2^{16} \cdot 5 \cdot 17$	10
$x^3 - (3+i)x + (2+i)$	$2^{16} \cdot 97$	$-2 + i$
$x^3 + (4+2i)x^2 - 2x$	$2^{18} \cdot 41$	$4 + 8i$
$x^3 + (-3+7i)x^2 + (1-i)x$	$2^{19} \cdot 5 \cdot 13$	$-4 + 2i$
$x^3 - (1-i)x^2 - 4i$	$2^{16} \cdot 17 \cdot 37$	$8 - 16i$

TABLE 1. Numerical verification of Deligne's Period Conjecture for motivic pieces of superelliptic curves $C/\mathbb{Q}(i) : y^4 = f(x)$.

where there is a more interesting Galois action to verify on the L-values, but we are hindered by large conductor exponents at 5 and so such examples seem to lie narrowly beyond the reach of the computational capacity available to the author.

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