Hint to Homework 5 Problem 2

Consider a random walker on the two-dimensional lattice with the following step probabilities:

$$p_r = \frac{1}{4} + \delta, \quad p_l = \frac{1}{4} - \delta, \quad p_u = \frac{1}{4}, \quad p_d = \frac{1}{4},$$

where δ denotes the bias parameter.

Instead of directly computing the return probability P for a walker starting at the origin, one can study the more general function $u(i,j)$, defined as the probability that a walker beginning at (i,j) **ever** reaches $(0,0)$. Naturally, we have

$$u(0,0) = 1.$$

After leaving the origin, the walker moves to one of the neighboring lattice sites. Therefore, the return probability can be decomposed as

$$P = p_r\, u(1,0) + p_l\, u(-1,0) + p_u\, u(0,1) + p_d\, u(0,-1).$$

Similarly, for any point $(i,j) \neq (0,0)$, the function $u(i,j)$ must satisfy the recurrence

$$u(i,j) = p_r\, u(i+1,j) + p_l\, u(i-1,j) + p_u\, u(i,j+1) + p_d\, u(i,j-1),$$

with the condition $u(0,0) = 1$. This recurrence constitutes an infinite-dimensional linear system over the lattice $\mathbb{Z}^2 \setminus \{(0,0)\}$.

For practical computation, define the finite grid

$$\Lambda_N = \{(i,j) \in \mathbb{Z}^2 : |i| \leq N,\ |j| \leq N\},$$

and introduce the unknown function

$$u_N(i,j), \quad (i,j) \in \Lambda_N \setminus \{(0,0)\},$$

with the normalization condition

$$u_N(0,0) = 1.$$

The function $u_N(i,j)$ is assumed to satisfy the discrete relation

$$u_N(i,j) = p_r\, u_N(i+1,j) + p_l\, u_N(i-1,j) + p_u\, u_N(i,j+1) + p_d\, u_N(i,j-1),$$

for all $(i,j) \in \Lambda_N \setminus \{(0,0)\}$. And the boundary condition is

$$u_N(i,j) = 0 \quad \text{for} \quad (i,j) \notin \Lambda_N,$$

as it is reasonable to expect that

$$\lim_{N\to\infty} \sup_{N\notin\Lambda_N} u_N(i,j) = 0.$$

In matrix terms, the finite system is equivalent to solving a linear system

$$A_M\,\mathbf{v} = \mathbf{b},$$

where the number of unknowns is $M = (2N+1)^2 - 1$, $\mathbf{v}$ is the vector of values $u_N(i,j)$ for $(i,j) \in \Lambda_N \setminus \{(0,0)\}$.

Consequently, the overall probability of returning to the origin is given by

$$P = \lim_{N\to\infty} \Big[p_r\, u_N(1,0) + p_l\, u_N(-1,0) + p_u\, u_N(0,1) + p_d\, u_N(0,-1)\Big].$$

Your objective is to find the bias parameter δ so that $P = \frac{1}{2}$.