CS663: Digital Image Processing - Homework 3

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October 1, 2024

1 Homework 3 - Question 7

The given partial differential equation (PDE) is:

$$\frac{\partial I}{\partial t} = c\left(\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}\right)$$

This is the isotropic heat equation, where $I(x, y, t)$ represents an image (or a function) that evolves over time, and c is a non-negative constant.

Fourier Transform of the Heat Equation

To solve this problem, we apply the Fourier transform to both sides of the PDE. Using the Fourier differentiation theorem, which states that the Fourier transform of a derivative of a function becomes a multiplication by a polynomial in frequency, we transform the equation in the spatial domain into the frequency domain.
Let the Fourier transform of $I(x, y, t)$ be denoted as $\hat{I}(u, v, t)$, where u and v are the Fourier domain variables corresponding to the spatial variables x and y, respectively.

Applying Fourier Transform to Both Sides of the PDE

The heat equation in the spatial domain is:

$$\frac{\partial I(x, y, t)}{\partial t} = c\left(\frac{\partial^2 I(x, y, t)}{\partial x^2} + \frac{\partial^2 I(x, y, t)}{\partial y^2}\right)$$

Now, we take the 2D Fourier transform of both sides. The Fourier transform of the time derivative $\frac{\partial I(x,y,t)}{\partial t}$ is:

$$\mathcal{F}\left\{\frac{\partial I(x, y, t)}{\partial t}\right\} = \frac{\partial \hat{I}(u, v, t)}{\partial t}$$

Using the differentiation property of Fourier transforms, the Fourier transform of the second-order derivative $\frac{\partial^2 I(x,y,t)}{\partial x^2}$ is:

$$\mathcal{F}\left\{\frac{\partial^2 I(x,y,t)}{\partial x^2}\right\} = (j2\pi u)^2 \hat{I}(u,v,t)$$

Similarly, the Fourier transform of $\frac{\partial^2 I(x,y,t)}{\partial y^2}$ is:

$$\mathcal{F}\left\{\frac{\partial^2 I(x,y,t)}{\partial y^2}\right\} = (j2\pi v)^2 \hat{I}(u,v,t)$$

Substituting these into the Fourier-transformed PDE, we get:

$$\frac{\partial \hat{I}(u,v,t)}{\partial t} = -c\left((2\pi u)^2 + (2\pi v)^2\right)\hat{I}(u,v,t)$$

This is a first-order differential equation in time for $\hat{I}(u,v,t)$. We can solve it by separating variables:

$$\frac{1}{\hat{I}(u,v,t)}\frac{\partial \hat{I}(u,v,t)}{\partial t} = -c\left((2\pi u)^2 + (2\pi v)^2\right)$$

Integrating both sides with respect to t:

$$\ln(\hat{I}(u,v,t)) = -c\left((2\pi u)^2 + (2\pi v)^2\right)t + \ln(\hat{I}(u,v,0))$$

Exponentiation on both sides, we get:

$$\hat{I}(u,v,t) = \hat{I}(u,v,0)\cdot e^{-c\left((2\pi u)^2+(2\pi v)^2\right)t}$$

Here, $\hat{I}(u,v,0)$ is the initial Fourier transform of the image at $t=0$.
We recognize that the solution in the frequency domain has the form of the Fourier transform of a Gaussian. The exponential term $e^{-c\left((2\pi u)^2+(2\pi v)^2\right)t}$ is the Fourier transform of a Gaussian in the spatial domain.

We know that the fourier transform of a Gaussian is also a Gaussian, and the inverse fourier transform of the Gaussian in frequency domain will give a Gaussian in the spatial domain.

Thus, the solution in the spatial domain $I(x,y,t)$ is obtained by convolving the initial image $I(x,y,0)$ with a Gaussian. The standard deviation of this Gaussian depends on the parameters of the exponential term.
The Fourier transform of the Gaussian in the frequency domain is:

$$e^{-\frac{(2\pi\sigma)^2}{2}\left(u^2+v^2\right)}$$

Comparing this with our result $e^{-c\left((2\pi u)^2+(2\pi v)^2\right)t}$, we can identify the standard deviation σ by matching terms. Specifically:

$$\frac{\sigma^2}{2} = ct$$

Solving for σ, we get:

$$\sigma = \sqrt{2ct}$$

Conclusion

Thus, running the isotropic heat equation on an image is equivalent to convolving the image with a Gaussian of zero mean and standard deviation:

$$\sigma = \sqrt{2ct}$$

This shows that the heat equation diffuses the image over time, and the amount of diffusion (i.e., the standard deviation of the Gaussian) increases as t increases.