CS663: Digital Image Processing - Homework 2

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1 Homework 2 - Question 6

Rotational Invariant:

The Laplacian operator $\nabla^2 I$ in 2D Cartesian coordinates is defined as:

$$\nabla^2 I = I_{xx} + I_{yy}$$

where $I_{xx} = \frac{\partial^2 I}{\partial x^2}$ and $I_{yy} = \frac{\partial^2 I}{\partial y^2}$

A rotation of the coordinate system by an angle θ, transforming from (x, y) to (u, v), where:

$$u = x\cos\theta - y\sin\theta, \quad v = x\sin\theta + y\cos\theta$$

We aim to prove that the Laplacian operator is rotational invariant, i.e.,

$$I_{xx} + I_{yy} = I_{uu} + I_{vv}$$

Using the chain rule, we express the partial derivatives in the new coordinates (u, v):

First-order partial derivatives:

$$\frac{\partial I}{\partial u} = I_x \cos\theta + I_y \sin\theta$$

$$\frac{\partial I}{\partial v} = -I_x \sin\theta + I_y \cos\theta$$

Second-order partial derivatives:

$$\frac{\partial^2 I}{\partial u^2} = I_{xx}\cos^2\theta + 2I_{xy}\cos\theta\sin\theta + I_{yy}\sin^2\theta$$

$$\frac{\partial^2 I}{\partial v^2} = I_{xx}\sin^2\theta - 2I_{xy}\cos\theta\sin\theta + I_{yy}\cos^2\theta$$

So summing them up we evidently get, $I_{xx} + I_{yy} = I_{uu} + I_{vv}$ and hence Laplacian operator is rotational invariant.

Second Directional Derivative

The directional derivative of a function $I(x, y)$ in the direction of a unit vector $\mathbf{v}$ is given by:

$$d_v I(x, y) = \nabla I(x, y) \cdot v$$

where $\nabla I(x, y) = (I_x, I_y)$ is the gradient of the image.
The unit vector in the direction of the gradient is:

$$v = \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}}, \frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right)$$

The second directional derivative in the direction of $\mathbf{v}$ is given by:

$$I_v^2 = \frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2}$$

The unit vector perpendicular to the gradient direction is:

$$v_\perp = \left(\frac{-I_y}{\sqrt{I_x^2 + I_y^2}}, \frac{I_x}{\sqrt{I_x^2 + I_y^2}} \right)$$

The second directional derivative perpendicular to the gradient direction is given by:

$$d_{v_\perp}^2 I = \frac{I_x^2 I_{yy} - 2 I_x I_y I_{xy} + I_y^2 I_{xx}}{I_x^2 + I_y^2}$$