CS663: Digital Image Processing - Homework 2

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1 Homework 2 - Question 3

Part a:

A filter is separable if it can be expressed as the outer product of two 1D filters. In other words, a 2D filter $H(x, y)$ is separable if there exist two 1D filters $h_x(x)$ and $h_y(y)$ such that:

$$H(x, y) = h_x(x) \cdot h_y(y)$$

For example, a separable filter in 2D can be written as:

$$H(x, y) = h_x(x) \cdot h_y(y) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d & e & f \end{bmatrix} = \begin{bmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{bmatrix}$$

The Laplacian we are dealing with is:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -8 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

This structure cannot be achieved by the outer product of two 1D vectors, as there isn't any combination of h_x and h_y which could produce this pattern. So Laplacian with -8 in the center is not separable.

Part b:

The Laplacian operator is a second-order differential operator that captures the sum of the second derivatives in both the x and y-directions. The Laplacian is often written in discrete form for 2D images as:

$$\nabla^2 f(x, y) = [f(x+1, y) - 2f(x, y) + f(x-1, y)] + [f(x, y+1) - 2f(x, y) + f(x, y-1)]$$

Although the Laplacian filter is not separable, it can still be implemented using the sum of two 1D convolutions because the second derivative with respect to x and y can each be expressed as a 1D convolution. Therefore, the Laplacian can be computed as the sum of these two 1D convolutions:

$$\nabla^2 f(x, y) = ([1, -2, 1] * f(x, y))_x + ([1, -2, 1] * f(x, y))_y$$