CS663: Digital Image Processing - Homework 4

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Homework 4 - Question 1

Part a -

PCA matrix is defined as:

$$C = \frac{1}{N-1} X X^T$$

where X is the vector:

$$X = \begin{bmatrix} x_1 & x_2 \ldots x_{n-1} & x_n \end{bmatrix}$$

Symmetry

For a Matrix m to be symmetric it should satisfy the condition,

$$M = M^T$$

so trying that out wit C we get:

$$C^T = (\frac{1}{N-1} X X^T)^T$$

$$C^T = \frac{1}{N-1} (X^T)^T (X)^T$$

$$C^T = \frac{1}{N-1} X X^T$$

Finally,

$$C^T = C$$

Hence PCA matrix is symmetric.

Semi definite

For a matrix to to positive semi definite, it should satisfy the equation $v^T Mv >= 0$ for any vector v.

$$v^T Cv = \frac{1}{N-1} v^T X X^T v$$

So this can be re written as:

$$v^T Cv = \frac{1}{N-1} \|v^T X\|^2$$

And this quantity is surely greater than equal to 0 so the pcA matrix is positive semi definite.

Part b -

Let M be a symmetric matrix, and let v_1 and v_2 be two eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, respectively. We have the following eigenvalue equations:

$$Mv_1 = \lambda_1 v_1 \quad \text{and} \quad Mv_2 = \lambda_2 v_2$$

Taking the inner product of v_1^T with both sides of the second equation:

$$v_1^T M v_2 = \lambda_2 v_1^T v_2$$

Since M is symmetric, we know that:

$$v_1^T M v_2 = (Mv_1)^T v_2 = \lambda_1 v_1^T v_2$$

Thus, we have:

$$\lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$$

For $\lambda_1 \neq \lambda_2$, the only solution to this equation is:

$$v_1^T v_2 = 0$$

This shows that v_1 and v_2 are orthogonal.
To normalize the eigenvectors, we scale each eigenvector so that:

$$\|v_i\| = 1 \quad \text{for each eigenvector}\, v_i$$

Thus, the eigenvectors of the symmetric matrix C are orthonormal.

Part c -

We are approximating the difference between the original data points x_i and their truncated versions $\tilde{x}_i$ by truncating the eigen-coefficients for the smallest eigenvalues.
The difference is given by:

$$\|x_i - \tilde{x}_i\|^2 = \|V(\alpha_i - \tilde{\alpha}_i)\|^2$$

where α_i are the original eigen-coefficients and $\tilde{\alpha}i$ are the truncated eigen-coefficients. The truncation sets $\tilde{\alpha}il = 0$ for $l > k$, so:

$$\|x_i - \tilde{x}i\|^2 = \sum l = k + 1^d \alpha_{il}^2$$

Averaging over all data points gives:

$$\frac{1}{N}\sum_{i=1}^{N} \|x_i - \tilde{x}i\|^2 = \frac{1}{N}\sum i = 1^N \sum_{l=k+1}^{d} \alpha_{il}^2$$

The eigen-coefficients α_{il} have variance λ_l, the eigenvalue corresponding to the l-th eigenvector. Thus, the expected squared difference is approximately:

$$\sum_{l=k+1}^{d} \lambda_l$$

This is small when the eigenvalues $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_d$ are small, as they represent the least significant components of the data's variation.

Part -

The problem describes a situation where one variable (denoted as X_1) has a much larger variance than the other (denoted as X_2). This means that the covariance matrix of the data has one large eigenvalue (corresponding to the direction of X_1) and one small eigenvalue (corresponding to X_2).
The eigenvectors are:

$$\text{For the larger eigenvalue (100):} \quad (1; 0)$$

$$\text{For the smaller eigenvalue (1):} \quad (0; 1)$$

Thus, the covariance matrix in this case has the form:

$$\begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix}$$

where 100 and 1 are the eigenvalues corresponding to the directions of X_1 and X_2, respectively.