CS663: Digital Image Processing - Homework 5

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Homework 5 - Question 1

Translation

Let us assume an initial image $f_1(x, y)$ of size $N \times N$. Then, by translating the image by (x_0, y_0), we obtain a new image $f_2(x, y) = f_1(x - x_0, y - y_0)$. The Fourier transforms of these images would then be related as

$$F_2(\mu, \nu) = e^{-j2\pi(\mu x_0 + \nu y_0)} \times F_1(\mu, \nu)$$

The cross-power spectrum $C(\mu, \nu)$ of both images is then given by:

$$C(\mu, \nu) = \frac{F_2^*(\mu, \nu) F_1(\mu, \nu)}{|F_2(\mu, \nu)||F_1(\mu, \nu)|} = e^{j2\pi(\mu x_0 + \nu y_0)}$$

Taking the inverse Fourier transform of $C(\mu, \nu)$, we obtain a delta function that is centered at $(-x_0, -y_0)$:

$$F^{-1}\left(e^{j2\pi(\mu x_0 + \nu y_0)}\right) = \delta(x + x_0, y + y_0)$$

Through this, we are able to obtain the shift between the two images (i.e. (x_0, y_0)).

Time Compexity

The above method involves computing the Fourier transform of $N \times N$ images and the inverse Fourier transform of the cross-power spectrum, which have time complexity of $O(N^2 \log N)$ via FFT. The computation of the cross-power spectrum has a time complexity of $O(N^2)$.

The overall time complexity is thus $\mathbf{O}(\mathbf{N^2} \log \mathbf{N})$.

A pixel-wise approach would have a time complexity of $O(N^2W^2)$ (W being the window size) with the worst case being $O(N^4)$. This is significantly higher than the FFT-based approach.

Rotation

Let us again assume an initial image $f_1(x, y)$ of size $N \times N$. Then, by translating the image by (x_0, y_0) and rotated by an angle θ_0 , we obtain a new image $f_2(x, y) = f_1(x \cos\theta_0 + y \sin\theta_0 - x_0, -x \sin\theta_0 + y \cos\theta_0 - y_0)$.
The Fourier transforms of these images would then be related as

$$F_2(\mu, \nu) = e^{-j2\pi(\mu x_0 + \nu y_0)} \times F_1(\mu \cos\theta_0 + \nu \sin\theta_0, -\mu \sin\theta_0 + \nu \cos\theta_0)$$

With the same logic as before, the cross-power spectrum $C(\mu, \nu)$ of both images can be used to find the shift (x_0, y_0).

To find the rotation angle θ_0, we first take the magnitude of the Fourier transforms of both images, thereby obtaining:

$$M_2(\mu, \nu) = M_1(\mu \cos\theta_0 + \nu \sin\theta_0, -\mu \sin\theta_0 + \nu \cos\theta_0)$$

A rotation in the rectangular coordinates corresponds to a transation in the polar coordinates.

$$\rho = \sqrt{\mu^2 + \nu^2}$$

$$\theta_1 = \arctan\left(\frac{\nu}{\mu}\right)$$

$$\theta_2 = \arctan\left(\frac{\nu \cos\theta_0 - \mu \sin\theta_0}{\mu \cos\theta_0 + \nu \sin\theta_0}\right) = \arctan\left(\frac{\nu}{\mu}\right) - \theta_0$$

Thus, the rotation angle can be found by finding the shift in the polar coordinates, in the same way as the translation.

$$M_2(\rho, \theta) = M_1(\rho, \theta + \theta_0)$$