

# CS663: Digital Image Processing - Homework 5

Harsh | Pranav | Swayam

November 6, 2024

## Homework 5 - Question 1

### Translation

Let us assume an initial image  $f_1(x, y)$  of size  $N \times N$ . Then, by translating the image by  $(x_0, y_0)$ , we obtain a new image  $f_2(x, y) = f_1(x - x_0, y - y_0)$ . The Fourier transforms of these images would then be related as

$$F_2(\mu, \nu) = e^{-j2\pi(\mu x_0 + \nu y_0)} \times F_1(\mu, \nu)$$

The cross-power spectrum  $C(\mu, \nu)$  of both images is then given by:

$$C(\mu, \nu) = \frac{F_2^*(\mu, \nu)F_1(\mu, \nu)}{|F_2(\mu, \nu)||F_1(\mu, \nu)|} = e^{j2\pi(\mu x_0 + \nu y_0)}$$

Taking the inverse Fourier transform of  $C(\mu, \nu)$ , we obtain a delta function that is centered at  $(-x_0, -y_0)$ :

$$F^{-1}\left(e^{j2\pi(\mu x_0 + \nu y_0)}\right) = \delta(x + x_0, y + y_0)$$

Through this, we are able to obtain the shift between the two images (i.e.  $(x_0, y_0)$ ).

### Time Complexity

The above method involves computing the Fourier transform of  $N \times N$  images and the inverse Fourier transform of the cross-power spectrum, which have time complexity of  $O(N^2 \log N)$  via FFT. The computation of the cross-power spectrum has a time complexity of  $O(N^2)$ .

The overall time complexity is thus  $\mathbf{O(N^2 \log N)}$ .

A pixel-wise approach would have a time complexity of  $O(N^2 W^2)$  ( $W$  being the window size) with the worst case being  $O(N^4)$ . This is significantly higher than the FFT-based approach.

## Part b: Rotation

Let us again assume an initial image  $f_1(x, y)$  of size  $N \times N$ . Then, by translating the image by  $(x_0, y_0)$  and rotated by an angle  $\theta_0$ , we obtain a new image  $f_2(x, y) = f_1(x \cos \theta_0 + y \sin \theta_0 - x_0, -x \sin \theta_0 + y \cos \theta_0 - y_0)$ . The Fourier transforms of these images would then be related as

$$F_2(\mu, \nu) = e^{-j2\pi(\mu x_0 + \nu y_0)} \times F_1(\mu \cos \theta_0 + \nu \sin \theta_0, -\mu \sin \theta_0 + \nu \cos \theta_0)$$

With the same logic as before, the cross-power spectrum  $C(\mu, \nu)$  of both images can be used to find the shift  $(x_0, y_0)$ .

To find the rotation angle  $\theta_0$ , we first take the magnitude of the Fourier transforms of both images, thereby obtaining:

$$M_2(\mu, \nu) = M_1(\mu \cos \theta_0 + \nu \sin \theta_0, -\mu \sin \theta_0 + \nu \cos \theta_0)$$

A rotation in the rectangular coordinates corresponds to a translation in the polar coordinates.

$$\rho = \sqrt{\mu^2 + \nu^2}$$

$$\theta_1 = \arctan\left(\frac{\nu}{\mu}\right)$$

$$\theta_2 = \arctan\left(\frac{\nu \cos \theta_0 - \mu \sin \theta_0}{\mu \cos \theta_0 + \nu \sin \theta_0}\right) = \arctan\left(\frac{\nu}{\mu}\right) - \theta_0$$

Thus, the rotation angle can be found by finding the shift in the polar coordinates, in the same way as the translation.

$$M_2(\rho, \theta) = M_1(\rho, \theta + \theta_0)$$