# CS663: Digital Image Processing - Homework 5

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## Homework 5 - Question 4

The problem involves maximizing a quadratic form subject to constraints, using the method of Lagrange multipliers. The matrix $C$ is symmetric, and we aim to find the eigenvectors and eigenvalues of $C$. The solution proceeds in two parts: maximizing $J_1(f)$ and $J_2(g)$.

### Part 1: Maximizing $J_1(f)$

We aim to maximize:

$$J_1(f) = f^T Cf - \lambda_1(f^T f - 1) - \lambda_2(f^T e)$$

where: - $C$ is a symmetric matrix, - $f$ is the vector we are optimizing, - $\lambda_1$ and $\lambda_2$ are Lagrange multipliers, - The constraints are $f^T f = 1$ (normalization) and $f^T e = 0$ (orthogonality).

#### Taking the derivative w.r.t. $f$

To find the critical points, we take the derivative of $J_1(f)$ with respect to $f$:

$$\frac{\partial}{\partial f}(f^T Cf - \lambda_1(f^T f - 1) - \lambda_2(f^T e)) = 2Cf - 2\lambda_1 f - \lambda_2 e$$

Setting the total derivative to zero:

$$2Cf - 2\lambda_1 f - \lambda_2 e = 0$$

which simplifies to:

$$Cf = \lambda_1 f + \frac{\lambda_2}{2} e$$

#### Dot product with $e$

Next, we take the dot product of both sides with $e$:

$$e^T Cf = \lambda_1 e^T f + \frac{\lambda_2}{2} e^T e$$

Since $f^T e = 0$ (from the orthogonality constraint), this simplifies to:

$$e^T C f = \frac{\lambda_2}{2} e^T e$$

Solving for $\lambda_2$:

$$\lambda_2 = \frac{2e^T C f}{e^T e}$$

However, from the assumption that $f$ is orthogonal to $e$, we have $e^T C f = 0$, which implies:

$$\lambda_2 = 0$$

### Conclusion for $f$

With $\lambda_2 = 0$, the equation simplifies to:

$$Cf = \lambda_1 f$$

Thus, $f$ is an eigenvector of $C$ with eigenvalue $\lambda_1$. Since we assumed distinct eigenvalues, this eigenvalue corresponds to the second-largest eigenvalue.

## Part 2: Maximizing $J_2(g)$

Next, we maximize:

$$J_2(g) = g^T C g - \lambda_1(g^T g - 1) - \lambda_2(f^T g) - \lambda_3(e^T g)$$

where the constraints are $g^T g = 1$ (normalization), $f^T g = 0$ (orthogonality to $f$), and $e^T g = 0$ (orthogonality to $e$).

### Taking the derivative w.r.t. $g$

Taking the derivative of $J_2(g)$ with respect to $g$:

$$\frac{\partial}{\partial f}(g^T C g - \lambda_1(g^T g - 1) - \lambda_2(f^T g) - \lambda_3(e^T g)) = 2Cg - 2\lambda_1 g - \lambda_2 f - \lambda_3 e$$

Setting the total derivative to zero:

$$2Cg - 2\lambda_1 g - \lambda_2 f - \lambda_3 e = 0$$

which simplifies to:

$$Cg = \lambda_1 g + \frac{\lambda_2}{2} f + \frac{\lambda_3}{2} e$$

**Dot product with $e$**

Taking the dot product with $e$:

$$e^T Cg = \lambda_1 e^T g + \frac{\lambda_2}{2} e^T f + \frac{\lambda_3}{2} e^T e$$

Using the constraints $e^T g = 0$ and $e^T f = 0$, this simplifies to:

$$e^T Cg = \frac{\lambda_3}{2} e^T e$$

Solving for $\lambda_3$:

$$\lambda_3 = \frac{2e^T Cg}{e^T e}$$

Since $e^T Cg = 0$, we conclude:

$$\lambda_3 = 0$$

**Dot product with $f$**

Now, taking the dot product with $f$:

$$f^T Cg = \lambda_1 f^T g + \frac{\lambda_2}{2} f^T f + \frac{\lambda_3}{2} f^T e$$

Using the constraints $f^T g = 0$ and $f^T e = 0$, this simplifies to:

$$f^T Cg = \frac{\lambda_2}{2} f^T f = \frac{\lambda_2}{2}$$

Thus, $\lambda_2 = 0$.

**Conclusion for $g$**

With $\lambda_2 = 0$ and $\lambda_3 = 0$, the equation simplifies to:

$$Cg = \lambda_1 g$$

Thus, $g$ is an eigenvector of $C$, corresponding to the third-largest eigenvalue.