CS663: Digital Image Processing - Homework 5

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Homework 5 - Question 4

Part a - Reconstruction

Since we know the locations of the non-zero elements, we can formulate this reconstruction problem as follows:
The DFT of an $n \times n$ image $f(x, y)$ at a frequency (u, v) is given by:

$$F(u, v) = \sum_{x=0}^{n-1} \sum_{y=0}^{n-1} f(x, y) e^{-j2\pi\left(\frac{ux}{n} + \frac{vy}{n}\right)}.$$

Since $f(x, y)$ is sparse, we can simplify this sum to only include the non-zero locations:

$$F(u, v) = \sum_{(x_i, y_i) \in \Omega} f(x_i, y_i) e^{-j2\pi\left(\frac{ux_i}{n} + \frac{vy_i}{n}\right)},$$

where Ω is the set of known locations with non-zero values, and $|\Omega| = k$.
The non-zero values in $f(x, y)$ as a vector

$$f_\Omega = [f(x_1, y_1), f(x_2, y_2), \ldots, f(x_k, y_k)]^T$$

Let F_m represent the vector of m observed DFT coefficients at frequencies (u_i, v_i) for $i = 1, 2, \ldots, m$. We can set up the following linear system:

$$F_m = A f_\Omega,$$

where A is an $m \times k$ matrix with entries $A_{ij} = e^{-j2\pi\left(\frac{u_i x_j}{n} + \frac{v_i y_j}{n}\right)}$.
If $m \geq k$, we have enough equations to uniquely determine the k unknown values in f_Ω. If $m = k$ and A has full column rank (i.e., the rows of A are linearly independent), then A is invertible, and we can solve for f_Ω directly:

$$f_\Omega = A^{-1} F_m.$$

Alternatively, if $m > k$, we can use a least-squares approach to solve for f_Ω, which provides robustness against noise and dependencies between Fourier coefficients.

Part b - Minimum Value of m Needed for Reconstruction

- Uniqueness Condition:

  Since we have k unknowns in f_Ω, the minimum number of independent equations needed to uniquely determine these unknowns is k.

  Thus, **in an ideal, noise-free scenario**, if we collect exactly $m = k$ DFT coefficients and ensure they are independent, we can reconstruct f_Ω by solving a system of k equations in k unknowns.

- Mathematical Justification:

  With $m = k$, the system $F_m = Af_\Omega$ becomes exactly determined. If A has full rank, this system has a unique solution for f_Ω.

  Therefore, $m = k$ **is the minimum number of DFT coefficients required** for unique reconstruction of $f(x, y)$ in this idealized setting.

Part c - Unknown Locations of Non-Zero Elements

No, as multiple configurations of non-zero elements can yield identical DFT coefficients. For instance, if two different sets of pixels have the same Fourier representation due to their positions and values being interchanged.