CS663: Digital Image Processing - Homework 5

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November 6, 2024

Homework 5 - Question 5

Part a - Why the Standard Least Squares Solution Fails

The least squares solution would suggest solving for R by minimizing:

$$\|P_1 - RP_2\|_F^2,$$

which leads to:

$$R = P_1 P_2^T (P_2 P_2^T)^{-1}.$$

However, this solution does not guarantee that R will be orthonormal (i.e., $R^T R = I$). Without this constraint, R may not preserve lengths and angles, which is essential in many applications. Therefore, we need to modify our approach to ensure that R is orthonormal.

Part b - Setting Up the Cost Function and Justifying Steps

The objective function we aim to minimize is:

$$E(R) = \|P_1 - RP_2\|_F^2,$$

where R is an orthonormal matrix, meaning $R^T R = I$.
The Frobenius norm of a matrix A is defined as:

$$\|A\|_F^2 = \text{trace}(A^T A).$$

Therefore, we can rewrite $E(R)$ as:

$$E(R) = \text{trace}((P_1 - RP_2)^T (P_1 - RP_2)).$$

Expanding the term $(P_1 - RP_2)^T (P_1 - RP_2)$, we get:

$$E(R) = \text{trace}(P_1^T P_1 - P_1^T RP_2 - P_2^T R^T P_1 + P_2^T R^T RP_2).$$

Since R is an orthonormal matrix, we know $R^T R = I$. Thus, the term $P_2^T R^T RP_2$ simplifies to $P_2^T P_2$. Applying this condition, we obtain:

$$E(R) = \text{trace}(P_1^T P_1 + P_2^T P_2 - P_1^T RP_2 - P_2^T R^T P_1).$$

We know that $\text{trace}(A) = \text{trace}(A^T)$ for any square matrix A. This allows us to combine the last two terms as follows:

$$E(R) = \text{trace}(P_1^T P_1 + P_2^T P_2) - \text{trace}(P_1^T R P_2 + P_2^T R^T P_1).$$

Since $\text{trace}(P_1^T R P_2) = \text{trace}(P_2^T R^T P_1)$, we can rewrite this as:

$$E(R) = \text{trace}(P_1^T P_1 + P_2^T P_2) - 2\,\text{trace}(P_1^T R P_2).$$

Part c - Min $E(R)$ equivalent to Max $trace(P_1^T R P_2)$

The expression we derived,

$$E(R) = \text{trace}(P_1^T P_1 + P_2^T P_2) - 2\,\text{trace}(P_1^T R P_2),$$

has two parts:

- The term $\text{trace}(P_1^T P_1 + P_2^T P_2)$ is independent of R. It represents a constant based on the fixed matrices P_1 and P_2.
- The term $-2\,\text{trace}(P_1^T R P_2)$ depends on R. Minimizing $E(R)$ is therefore equivalent to maximizing $\text{trace}(P_1^T R P_2)$ with respect to R.

Thus, minimizing the objective function $E(R)$ is equivalent to solving the following problem:

$$\text{maximize } \text{trace}(P_1^T R P_2) \quad \text{subject to } R^T R = I.$$

Part d - Detailed Solution with Justification for Each Step

$\text{trace}(P_1^T R P_2) = \text{trace}(R P_2 P_1^T)$, this equality holds due to the cyclic property of trace. For any matrices A, B, and C of compatible dimensions, the trace of a product is invariant under cyclic permutations:

$$\text{trace}(ABC) = \text{trace}(BCA) = \text{trace}(CAB)$$

Applying this property, we have:

$$\text{trace}(P_1^T R P_2) = \text{trace}(R P_2 P_1^T)$$

This cyclic shift allows us to rewrite the expression in terms of $R P_2 P_1^T$, which becomes useful in subsequent steps when we apply the SVD.
We proceed by performing the SVD on the matrix $P_2 P_1^T$, which gives us:

$$P_2 P_1^T = U' S' V'^T$$

where U' and V' are orthogonal matrices, and S' is a diagonal matrix with non-negative real entries (the singular values of $P_2 P_1^T$).
Now, substituting this into our trace expression, we get:

$$\text{trace}(R P_2 P_1^T) = \text{trace}(R U' S' V'^T)$$

This substitution is valid since we are replacing $P_2 P_1^T$ with its decomposition, allowing us to simplify the trace further.
Using the cyclic property of trace again, we can rearrange terms inside the trace without changing the value:

$$\text{trace}(RU'S'V'^T) = \text{trace}(S'V'^T RU')$$

Defining $X = V'^T RU'$, we get:

$$\text{trace}(S'V'^T RU') = \text{trace}(S'X)$$

This expression isolates S', the singular values, which are instrumental in maximizing the trace expression in subsequent parts.

Part e - Finding the Matrix X that Maximizes trace$(S'X)$

We aim to maximize the expression trace$(S'X)$ with respect to matrix X, given that S' is a diagonal matrix with non-negative singular values. Here is how we approach this maximization problem:
Since $X = V'^T RU'$, and R is an orthonormal matrix, X inherits the constraints imposed by R. For the trace to be maximized, we must choose R (and hence X) in a way that aligns the entries of X with the entries of S'.
The trace trace$(S'X)$ is maximized when X matches the identity matrix in terms of the alignment of singular values. Thus, to maximize trace$(S'X)$, we need X to be a rotation matrix that aligns the directions of V' and U'.
The maximal alignment happens when $X = I$, as this configuration fully aligns the singular values in S' with those in X. Thus, the optimal choice for X is:

$$X = I$$

Part f - Determining R from the Optimal X

Since$X = I$ this leads to $V'^T RU' = I$, or equivalently:

$$R = V'U'^T$$

Part g - Additional Constraint if R is a Rotation Matrix

A rotation matrix in 2D or 3D has a determinant of +1, as opposed to reflection matrices, which have a determinant of -1. Thus:
Enforcing R as a Rotation Matrix: To ensure R is a rotation matrix, we need to impose the constraint:

$$\det(R) = +1$$

After computing $R = V'U'^T$, if we find that $\det(R) = -1$, we can adjust R to make it a rotation matrix by flipping the sign of one of the columns in U' or V'. This ensures that the determinant becomes +1 while maintaining the orthonormality of R.