chapter 2

equally likely outcome, the space where each element has the same probability of occuring.

if S is an equally likely space, then

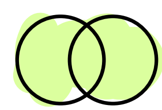
$$P(A) = \frac{\#\text{ of elements in } A}{\#\text{ of elements in } S}$$

eg throw pair of dice, $A = \{\text{probability } 10 \text{ or more}\}$
$= \{10, 11, 12\}$

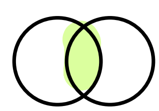
$$P(A) = \underset{\{10\}}{\frac{3}{36}} + \underset{\{11\}}{\frac{2}{36}} + \underset{\{12\}}{\frac{1}{36}} = \frac{1}{6}$$

set operations

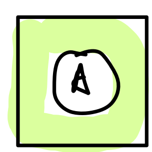
union $A \cup B = \{x : x \in A \text{ or } x \in B\}$



intersection $A \cap B = \{x : x \in A \text{ and } x \in B\}$

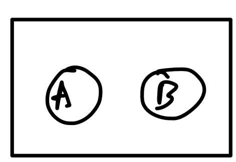


complement $A' = \{x : x \notin A\}$



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A & B are mutually exclusive if $A \cap B = \emptyset$



$P(A \cup B) = P(A) + P(B)$ if mutually exclusive

Multiplication principle

if a job has consecutive tasks,

$T_1 \to T_2 \to T_3 \to T_4$ and each task has n ways to do it

then total number of ways

$\downarrow$

$n_1 \cdot n_2 \cdot n_3 \cdots\cdots n_k$

n objects, pick k and permutate

$p(n,k) = n \cdot (n-1) \cdots\cdots (n-(k-1))$

$= P(n,k) = \frac{n!}{(n-k)!}$

Ex: 100 meter dash, 8 people, how many ways for medals

$= \frac{8!}{3!} = \underline{8} \cdot \underline{7} \cdot \underline{6} = 336$

how many ways for top 3 to advance

$= \frac{8 \cdot 7 \cdot 6}{3!} = C(8,3) =$ 8 choose 3

^ removes duplicates

$C(n,k) = \binom{n}{k} = \frac{p(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$

Ex $A_1, A_2 \cdots\cdots A_7$, 7 people, randomly pick 3,

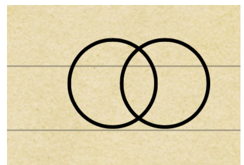
what's the prob that A is among the picked.

$S = \{$all possible ways to pick 3$\}$

$$P(E) = \frac{\#\text{ of elements in } E}{\#\text{ of elements in } S} = \frac{\binom{6}{2}}{\binom{7}{3}} = \frac{15}{35} = \frac{3}{7}$$

36 students, 4 freshman, randomly pick 7, what's the prob one freshmen

$$= \frac{\binom{4}{1}\binom{32}{6}}{\binom{36}{7}}$$



Ex, there are 10 balls, 5 red, 3 green. 2 blue.
Randomly select 3 without replacement, what's the prob

a. all red $\qquad \frac{\binom{5}{3}}{\binom{10}{3}} = \frac{1}{12}$

b none red

$$\frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} = \frac{1}{12}$$

c

$$\frac{\binom{5}{1}\binom{3}{1}\binom{2}{1}}{\binom{10}{3}} = \frac{30 \cdot 6}{10 \cdot 9 \cdot 8} = \frac{1}{4}$$

Ex. deck of cards

② P(full house)

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

$$p(3 \text{ spades}, 2 \text{ hearts}) = \frac{\binom{13}{3}\binom{13}{2}}{\binom{52}{5}}$$

$$P(555QQ) = \frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

$$P(55667) = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}$$

$$P(\text{2 pairs}) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$$

Additive rule

if A & B mutually exclusive

$$P(A \cup B) = P(A) + P(B)$$

but in general

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

10 balls, 6 red, 4 blue, randomly pick 4, the prob of one blue

$$1 - P(\text{no blue}) = 1 - \frac{\binom{6}{4}}{\binom{10}{4}}$$

Conditional probability, Independence & product

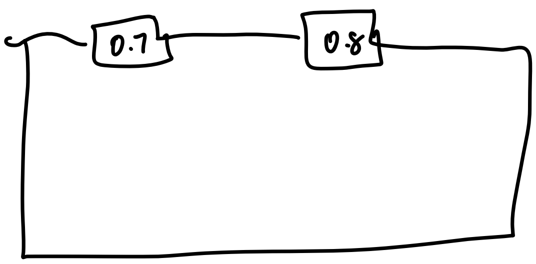
$$P(B|A) = \frac{P(A \cap B)}{P(A)} \rightarrow \text{probability of}$$

$$P(A \cap B) = P(A) \cdot P(B|A)$$

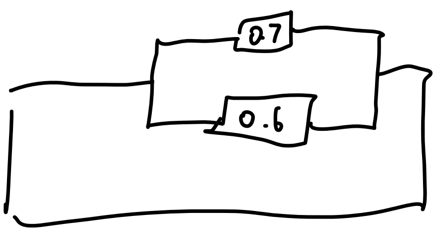
A & B independent if $P(A|B) = P(A)$

$$P(B|A) = P(B)$$

and if $P(A \cap B) = P(A)P(B)$



$P(\text{circuit working}) = 0.7 \cdot 0.8 = 0.56$



$P(\text{circuit working}) = 1 - (0.4 \cdot 0.3) = 0.88$

Bayes rule

A factory, 3 machines A, B, C. They produce 30%, 45%, and 25% products

A: 2% defective

B: 3% C 4%

$$P(\text{defective}) = P(\text{def} \cap A) + P(\text{def} \cap B) + P(\text{def} \cap C)$$

$$= 0.3 \cdot 0.02 + 0.45 \cdot 0.03 + 0.25 \cdot 0.04$$

$$= 0.006 + 0.0135 + 0.01 = 0.0295$$

P(def and made by A)

$$P(A \mid def) = \frac{P(A \cap def)}{P(def)}$$

$$P(B \mid def) = \frac{P(B \cap def)}{P(def)}$$

$$\boxed{\begin{array}{l} 5 \text{ fresh} \\ 2 \text{ soph} \end{array}} \xrightarrow{\text{step 1}} \boxed{\begin{array}{l} 2 \text{ fresh} \\ 5 \text{ soph} \end{array}} \xrightarrow[\text{step2}]{} \text{pick 1}$$

$$P(\text{step 1 Fresh} \mid \text{step 2 Fresh}) = P\left(\frac{\text{Step1 } F \cap \text{step2 } F}{P(\text{step 2. } F)}\right)$$

$$= \frac{\frac{5}{7}\cdot\frac{3}{8}}{\frac{5}{7}\cdot\frac{3}{8}+\frac{2}{7}\cdot\frac{2}{8}} = \frac{15}{19}$$

chapter 3 random variable

$$\bigcirc\!\!\begin{array}{l} 5 \text{ red} \\ 3 \text{ blue} \end{array}$$

random pick 3

X = # of blue picked

$S_X \{0, 1, 2, 3\}$

$$P(X=0) \quad P(X=1) \quad P(X=2)$$

$$\frac{\binom{5}{3}}{\binom{8}{3}} \quad \frac{\binom{3}{1}\binom{5}{2}}{\binom{8}{3}} \quad \frac{\binom{3}{2}\binom{5}{1}}{\binom{8}{3}} \quad \frac{\binom{3}{3}}{\binom{8}{3}}$$

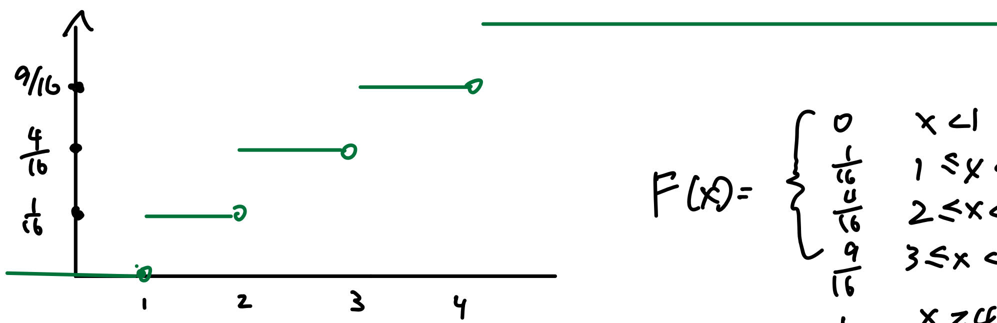
$$\frac{10}{56} + \frac{30}{56} + \frac{15}{56} + \frac{1}{56} = 1$$

for discrete random variable, f(x) is probability mass function

f(x) must be ≥ 0 at all times

$\sum_{x \in S_X} f(x) = 1$ $\Rightarrow$ sum of all x must be 1

cdf is the same thing but for continuous ones.



$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{16} & 1 \leq x < 2 \\ \frac{4}{16} & 2 \leq x < 3 \\ \frac{9}{16} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

for continuous random variable, f(x) is a probability density function (p.d.f)

① $f(x) \geq 0$ for all x

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

③ $p(a \leq x \leq b) = \int_a^b f(x)\,dx$

Ex $f(x) = c(x-2)$ $\quad 2 \leq x \leq 4$

$c = ?$ $\quad 1 = \int_2^4 c(x-2) = c\left.\left(\frac{x^2}{2} - 2x\right)\right|_2^4$

$= c(0+2)$

$1 = 2c$

$\frac{1}{2} = c$

② $(p>3)$ $\int_2^3 \frac{1}{2}(x-2)$

$= \frac{1}{2} \quad \frac{x^2}{2} - 2x \Big|_3^4$

$= \frac{1}{2}\left(\frac{9}{2}-6\right)$

$= \frac{1}{2}\cdot\left(-\frac{3}{2}\right)$

$= -\frac{3}{4}$

③ $F(x) = \int_2^x \frac{t-2}{2}\,dt$

$= \frac{1}{2}\left(\frac{x^2}{2} - 2x + 2\right)$

$= \frac{x^2}{4} - x + 1$

3.4 Joint probability

if $f(x,y)$ is joint probability

① $f(x,y) \geq 0$ ② Sum of $f(x,y)$ for x and y will be 1

Chapter 4 : Mathematical Expectation

Ex 10 balls, 5 labelled "1", 3 labelled "2", 2 labelled "3"

X = number on ball

$$f(x)\begin{cases} 0.5 & x=1 \\ 0.3 & x=2 \\ 0.2 & x=3 \end{cases}$$

$$E(X) = 1\cdot 0.5 + 2\cdot 0.3 + 3\cdot 0.2 = 1.7$$

$$E(X) = \begin{cases} \sum_x x f(x) & \text{discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx & \text{continuous} \end{cases}$$

Ex

3red
7 blue

X = num of red balls if pick 2

$$f(x) = \begin{cases} \frac{\binom{7}{2}}{\binom{10}{2}} = \frac{7}{15} & x=0 \\ \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{7}{15} & x=1 \\ \frac{\binom{3}{2}}{\binom{10}{2}} = \frac{1}{15} & x=2 \end{cases}$$

$$E(X) = 0\cdot\frac{7}{15} + 1\cdot\frac{7}{15} + 2\cdot\frac{1}{15} = \frac{3}{5}$$

$$X \sim f(x) = \frac{2(x+2)}{5}, \quad 0<x<1$$

find $E(x)$, $E(3X-1)$, $0<x<1$

Find $E(X)$, $E(3X-1)$, $E(X^2)$

$$E(X) = \int_0^1 x\cdot\frac{2(x+2)}{5} = \int_0^1 \frac{2(x^2+2x)}{5}\,dx = \frac{2x^3}{15} + \frac{4x^2}{10}\Big|_0^1 = \frac{2}{15} + \frac{4}{10} = \frac{8}{15}$$

$$E(3x-1) = \int_0^1 (3x-1)\cdot\frac{2(x+2)}{5} \quad \ldots$$

$$E(X^2) = \int_0^1 x^2\cdot\frac{2(x+2)}{5}$$

$\sigma^2 = Var(X) = E((X-\mu)^2)$

σ = standard deviation $(X) = \sqrt{Var(X)}$

For calculation $\sigma^2 = E(X^2) - \mu^2$

Covariance $= E(XY) - \mu_x \mu_y$

$E(aX + bY + c) = aE(X) + bE(Y) + c$

$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X, Y)$

chapter 5

- Bernoulli r.v. $f(x) = \begin{cases} p & x=1 \\ 1-p = q & x=0 \end{cases}$

(coin tossing) $E(X) = p$

$Var(X) = pq$

$X \sim (Ber(p))$

10 coin toss where $P(H) = 0.4$

$P(3H\ 7T) = 0.4^3\, 0.6^7 \cdot \binom{10}{3}$ ← amount of combinations

chance of 3 heads and 7 tails

Repeating Beroulli(p) n times

$f(x) = \binom{n}{x} p^x q^{n-x} \quad x = 0, 1, 2 \ldots n$

ex Driving school $p(\text{passing}) = 0.7$

20 students $p(\text{exactly } 15) = \binom{20}{15} 0.7^{15} 0.3^{5}$

② $P(\text{at least } 18 \text{ passed}) = \binom{20}{18} 0.7^{18} 0.3^{2} + \binom{20}{19} 0.7^{19} 0.3^{1} + \binom{20}{20} 0.7^{20}$

Multivariable Bernoulli

$$f(x_1, x_2 \dots x_k) = \binom{n}{x_1, x_2 \dots x_k} p_1^{x_1} \cdot p_2^{x_2} \cdot \dots \cdot p_k^{x_k}$$

$$\binom{n}{x_1, x_2 \dots x_k} = \frac{n!}{x_1! \, x_2! \dots x_k!}$$

Ex NYC → Boston

A: 0.20 B 0.3 C 0.5

$$P(5A, 8B, 12C) = \frac{25!}{5!\,8!\,12!} 0.2^5 0.3^8 0.5^{12}$$

5.4 Negative Binomial & Geometric distribution

$X \sim Geo(p)$

$f(x) = p \cdot q^{x-1}$ $x = 1, 2, 3 \dots$

$E(X) = \frac{1}{p}$ $Var(X) = \frac{1-p}{p^2}$

Ex throw dice seperately, the prob 5th throw is 1st throw for "5"

$= \left(\frac{5}{6}\right)^4 \cdot \left(\frac{1}{6}\right)$

Ex fair dice P(it takes < 10 times to get 1st "5")

or $= 1 - P(X \geq 10) = 1 - \dfrac{(\frac{5}{6})^9 \frac{1}{6}}{1 - \frac{5}{6}}$

or $= 1 - P(X \geq 10) = 1 - P$(the 1st 9 are all failures)
$= 1 - (\frac{5}{6})^9$

Poisson Distribution

counts of some rare event over the interval

$f(x) = e^{-\lambda} \dfrac{\lambda^x}{x!} \quad x = 0, 1, 2 \ldots \ldots$

Ex average radioactive particles pass though a container is 4

~ Poisson Process

① P(6 particles in 1 millisecond) $= e^{-4} \dfrac{4^6}{6!} = 0.1042$

② P(at least 5 in 1 millisecond)
$1 - (P(X = 0, 1, 2, 3, 4) =$

In application, when n is large, p is small & np is of the magnitude of 1, then we can use $Poi(\lambda = np)$ to approx $Bin(n, p)$

chapter 6