Question 5:

Exercise A:

(1) Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$ (leaving no remainder).

Hint: you may want to use the formula $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Solution:

Theorem:

For any positive integer n, 3 can divide $n^3 + 2n$ evenly (leaving no remainder).

Base Case:

P(n) = 1

$n^3 + 2n$

$= 1^3 + 2(1)$

$= 3$

Since $\frac{3}{3} = 1$, we have proven that $P(1)$ is true.

Inductive Step:

Since $P(1)$ is true we can assume that $P(k)$ is true for some positive integer k:

$$k^3 + 2k$$

Now we are going to prove that $P(k + 1)$ is true.

The following is our inductive hypothesis:

$$(k + 1)^3 + 2(k + 1)$$

$$k^3 + 3k^2 + 3k(1)^2 + 1^3 + 2k + 1$$

$$(k^3 + 2k) + (3k^2 + 3k + 3)$$

Since $(k^3 + 2k)$ is divisible by 3 and $(3k^2 + 3k + 3)$ is also divisible by 3 we can

conclude that $P(k + 1)$ is true. ■

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(2) Use strong induction to prove that any positive integer $n(n \geq 2)$ can be written as a product of primes.

Solution:

Theorem:

For any positive integer $n(n \geq 2)$ can be written as a product of primes.

Base Case:

P(n) = 2

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Since 2 is a prime number P(n) is true.

Inductive Step:

Question 6:

Exercise 7.4.1: (a-g):

Define P(n) to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

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a. Verify that P(3) is true.

Solution:

$$\sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 = \frac{3(3+1)(2(3)+1)}{6} = 14$$

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b. Express P(k).

Solution:

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

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c. Express P(k + 1).

Solution:

$$\sum_{j=1}^{k+1} j^2 = \frac{k+1(k+1)(2k+1)}{6}$$

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d. In an inductive proof that for every positive integer n.

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

What must be proven in the base case?

Solution:

The base case that must prove P(1) is true.

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e. In an inductive proof that for every positive integer n.

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

What must be proven in the inductive step?

Solution:

In the inductive step, $P(k + 1)$ must be proven true.

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f. What would be the inductive hypothesis in the inductive step from your previous answer?

Solution:

The inductive hypothesis would be the following:

$$P(k)\ is\ true\ which\ implies\ P(k + 1)$$

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g. Prove by induction that for any positive integer n

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

<u>Solution</u>:

<u>Theorem</u>:

For every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

<u>Proof</u>:

By induction on n.

<u>Base Case</u>:

$n = 1$

$$\sum_{j=1}^{n} j^2 = 1^2 = \frac{1(1+1)(2*1+1)}{6}$$

<u>Inductive Step</u>:

We will show that because P(1) is true, and by implication P(k+1) is true where $k \geq 1$.

$$\sum_{j=1}^{k+1} j^2 = \frac{k(k+1)(2k+1)}{6}$$

Starting with the left side of the equation to be proven:

$$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{k(k+1)(k+1)(2k+3)}{6}$$

Therefore, P(k+1) is true. ■

Exercise 7.4.3: **(c)**:

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c. Prove that for $n \geq 1$, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Solution:

Theorem:

For every positive integer n,

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Proof:

By induction on n.

Base Case:

$n = 1$

$$\sum_{j=1}^{1} \frac{1}{j^2} = \frac{1}{1^2} \leq 2 - \frac{1}{1}$$

$$= 1 \leq 1$$

Here we have proven that P(1) is true.

Inductive Step:

Since P(1) is true, by implication P(k) is true where for some positive integer k.

The following is the inductive hypothesis:

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$$

Now we are going to prove P(k+1).

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2}$$

$$\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)}$$

$$= 2 - \frac{(k+1)}{k(k+1)} + \frac{1}{k(k+1)}$$

$$= 2 - \frac{1}{k+1}$$

Based on the inductive step shown above P(k+1) is true. ■

Exercise 7.5.1: (a):

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a. Prove that for any positive integer n, 4 evenly divides $3^{2n-1} - 1$.

Solution:

Theorem:

For every positive integer n, 4 evenly divides $3^{2n} - 1$.

Proof:

By induction on n.

Base Case:

$n = 1$

$$3^{2(1)} - 1 = 8$$

Since 8 is evenly divisible by 4 the base case $P(1)$ is true.

Inductive Step:

The inductive hypothesis is, $P(k) = 3^{2k} - 1$, for some positive integer k.

Suppose that for a positive integer m, 4 evenly divides $3^{2k} - 1$. Then we will show $3^{2(k+1)} - 1 = 4m$.

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1$$
$$= 3^2 * (3^{2k} - 1)$$

Using the inductive hypothesis, $3^{2k} - 1$, we can say $3^2 * (3^{2k} - 1) = 4m$ or

$3^2 * (3^{2k} - 1)$ is divisible by 4. ■