Question 5:

Exercise 1.12.2:

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

b.

$p \rightarrow (q \land r)$ $\neg q$
$\therefore \neg p$

Result:

Line Number	Symbol(s)	Applied Law
1	$p \rightarrow (q \land r)$	Hypothesis
2	$p \rightarrow q$	Simplification, 1
3	$\neg q$	Hypothesis
4	$\neg p$	Modus Tollens, 2,3

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e.

$p \lor q$ $\neg p \lor r$ $\neg q$
$\therefore r$

Result:

Line Number	Symbol(s)	Applied Law
1	$p \lor q$	Hypothesis
2	$\neg p \lor r$	Hypothesis
3	$q \lor r$	Resolution, 1,2
4	$\neg q$	Hypothesis
5	r	Disjunctive Syllogism, 3,4

<u>Exercise 1.12.3</u>:

Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

c.

One of the rules of inference is Disjunctive syllogism:

$p \lor q$ $\neg p$
$\therefore q$

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

<u>Result</u>:

Line Number	Symbol(s)	Applied Law
1	$p \lor q$	Hypothesis
2	$\neg\neg p \lor q$	Double Negation, 1
3	$\neg(p \rightarrow q)$	Conditional, 2

4	$\neg p$	Hypothesis
5	$\neg\neg q$	Modus Ponens, 3,4
6	q	Double Negation, 5

Exercise 1.12.5:

Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

c.

I will buy a new car and a new house only if I get a job. I am not going to get a job.
∴ I will not buy a new car.

Result: Invalid argument.

p	q	r	$\neg p$	$\neg r$	$(p \land q) \to r$
T	T	T	F	F	T
T	T	F	F	T	F
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	T

F	F	F	T	T	T

Looking at the cells colored in red we can see that there is an instance in which both the hypotheses evaluate to True and the conclusion is False:

$(p \wedge q) \rightarrow r$ = T $\neg r$ = T
$\therefore \neg p$ = F

As a result, the argument is invalid.

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d.

I will buy a new car and a new house only if I get a job. I am not going to get a job. I will buy a new house.
∴ I will not buy a new car.

<u>Result</u>:

p = will buy a new car
q = will buy a new house
r = will get a job

$p \wedge (q \rightarrow r)$ $\neg r$ q
$\therefore \neg p$

Line Number	Symbol(s)	Applied Law
1	$p \wedge (q \rightarrow r)$	Hypothesis

2	$\neg r$	Hypothesis
3	$p \ \wedge \ \neg q$	Modens Tollens, 1,2
4	$\neg p$	Simplification, 3
5	q	Hypothesis
6	$\neg p$ ∧ q	Conjunction, 4,5
7	$\neg p$	Simplification, 6

Exercise 1.13.3:

b.

∃ x (P(x) ∨ Q(x)) ∃ x ¬Q(x)
∴ ∃ x P(x)

Result:

∃ x (P(x) ∨ Q(x)) ∃ x ¬Q(x)
∴ ∃ x P(x)

	P	Q
a	F	F
b	F	T

∃ x (P(a) ∨ Q(b)) = $F \vee T = T$ ∃ x ¬Q(a) $= \neg F = T$

∴ ∃ x P(b)	$= F$	$= F$

Exercise 1.13.5:

d.

Every student who missed class got a detention. Penelope is a student in the class. Penelope did not miss class.
Penelope did not get a detention.

Result: Invalid argument

x = domain of all students
M(x) = missed class
D(x) = received detention

$\forall x(M(x) \rightarrow D(x))$ $Penelope\ in\ x$ $\neg M(Penelope)$
$\therefore \neg D(Penelope)$

Providing the following values:

$M(x)$ = False
$D(x)$ = True

$\forall x(M(F) \rightarrow D(T))$ = True $Penelope\ in\ x$ = True $\neg M(Penelope)$ = True
$\therefore \neg D(Penelope)$ = False

As displayed above, the argument is invalid.

e.

Every student who missed class or got a detention did not get an A. Penelope is a student in the class. Penelope got an A.
Penelope did not get a detention.

Result:

x = domain of all students
P = Penelope
M(x) = missed class
D(x) = received detention
G(x) = received an "A"

$\forall x((M(x) \ \lor \ D(x)) \ \rightarrow \ \neg G(x))$ $P \ in \ x$ $G(P)$
$\therefore \neg D(P)$

Providing the following values:

$M(x)$ = True
$D(x)$ = True
G(x) = True

$\forall x((M(T) \ \lor \ D(T)) \ \rightarrow \ \neg G(T))$ = True $P \ in \ x$ = True $G(P)$ = True
$\therefore \neg D(P)$ = True

Line Number	Symbol(s)	Applied Law

1	$\forall x((M(x) \lor D(x)) \rightarrow \neg G(x))$	Hypothesis
2	$P \; in \; x$	Hypothesis
3	$M(P) \lor D(P)) \rightarrow \neg G(P)$	Universal Initialization, 1,2
4	$M(P) \lor (D(P) \rightarrow \neg G(P))$	Associative Law, 3
5	$M(P) \lor (\neg D(P) \lor \neg G(P))$	Conditional Law, 4
6	$M(P) \lor \neg (D(P) \land G(P))$	DeMorgan’s Law, 5
7	$M(P) \lor \neg (D(P))$	Simplification, 6
8	$\neg D(P) \lor M(P)$	Commutative Law, 7
9	$\neg D(P)$	Addition, 8
10	$G(P)$	Hypothesis
11	$\neg D(P) \land G(P)$	Conjunction
12	$\neg D(P)$	Simplification

<u>Question 6</u>:

<u>Exercise 2.4.1</u>:

Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as 2k+1, where k is an integer. An even integer is an integer that can be expressed as 2k, where k is an integer.

d. The product of two odd integers is an odd integer.

<u>Result</u>:

Theorem: If x and y are odd integers, the z is an odd integer.

Proof:

Let x, y and z be odd integers. We will prove that $xy = z$.

Since x,y and z are odd, x = 2k+1, y = 2k+1, z = 2k+1 for some integer k.

Plug x = 2k+1 and y = 2k+1 into $xy = z$ to get:

$xy =$
$(2k + 1) * (2k + 1) =$
$4k^2 + 2k + 1 =$
$2(2k^2 + k) + 1 =$

Since $z = 2m + 1$
m = $2k^2 + k$

Plug in m = $2k^2 + k$ to get:
z = $2(2k^2 + k) + 1$
is an odd integer, z is an odd integer ■

<u>Exercise 2.4.3</u>:

Prove the following statement using a direct proof.

b. If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Result:

Theorem: If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Proof:

Let x be a real number less than or equal to 3.

We will prove $12 - 7x + x^2 \geq 0$.

Since $x \leq 3 = x - 3 \leq 0$

And $12 - 7x + x^2 \geq 0 = (x - 3)(x - 4) \geq 0$

From the equation above, $(x - 3)(x - 4) \geq 0$, we can create either of the following equations:

$x - 3 \geq 0$
$x - 4 \geq 0$

Therefore we can say $x \leq 4$.

In conclusion, $12 - 7x + x^2 \geq 0$. ■

Question 7:

Exercise 2.5.1:

Prove each statement by contrapositive

d. For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Result:

Theorem: For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof:

Let n be an integer. We assume that n is even and $n^2 - 2n + 7$ is odd.

Since n is even, n = 2k where k is some integer.

Plug in n = 2k into $n^2 - 2n + 7$ to get:

$$(2k)^2 - 2(2k) + 7$$
$$4k^2 - 4k + 7$$
$$2(2k^2 - 2k) + 7$$

Since k is an integer, $2k^2 - 2k$ is also an integer.

Therefore, $n^2 - 2n + 7$ can be expressed as $2i + 1$, where $i = 2k^2 + 2k$ is an integer.

We conclude that $n^2 - 2n + 7$ is odd. ■

Exercise 2.5.4:

Prove each statement by contrapositive

a. For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$

Result:

Theorem: For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

Proof:

Let x and y be real numbers. We will prove that $x > y$.

$x^3 + xy^2 > x^2y + y^3 =$

$x(x^2 + y^2) > y(x^2 + y^2) =$

Thus, any values inserted into $(x^2 + y^2)$ would result in this quantity being a positive value.

In conclusion $x > y$ where one of the values must be a positive integer.

■

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b. For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$

Result:

Theorem: For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$

Proof:

Let x and y be real numbers. We shall prove that $x \leq 10$ and $y \leq 10$.

Since $x \leq 10$ and $y \leq 10$, $x + y \leq 20$.

We can conclude that for every pair of real numbers x and y the result will be ≤ 20.

■

Exercise 2.5.5:

c. For every pair of positive real numbers x and y, if $xy > 400$, then $x > 20$ or $y > 20$

Result:

Proof:

Let x and y be positive real numbers, where $x > 0$ and $y > 0$.

Since $0 < x \leq 20$ and $0 < y \leq 20$, we can conclude that $x * y \leq 400$.

■

Question 8:

Exercise 2.6.6:

Give a proof for each statement.

c. The average of three real numbers is greater than or equal to at least one of the numbers.

Result:

Proof:

Let x,y and z be real numbers where the average is less than all of the numbers.

Since the average of x, y and z is less than each of the numbers,
we can make the following equation:

$$\frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} < x + y + z$$

Therefore, any real numbers values supplied for x, y and z would contradict the inequality above.

■

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d. There is no smallest integer.

Result:

Proof:

Let x be an integer.

Suppose x - 1.

Since x - 1 < x, x cannot be the smallest integer. ■

Question 9:

Exercise 2.7.2:

b. If integers x and y have the same parity, then x+y is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Result:

Theorem: If integers x and y have the same parity, then x+y is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Proof:

Let x, y and z be real numbers.

Case 1:

x+y = z is even and each has the same parity.

Since x = 2k and y = 2k for some integer k.

Plug in the values for both x and y given you the following:

$2k + 2k = z$
$4k = z$

Regardless of the value that the integer k represents, z will always be a multiple of 4 and therefore even.

Case 2:

x+y = z is odd and each has the same parity.

Since x = 2k+1 and y = 2k + 1 for some integer k.

Plug in the values for both x and y given you the following:

$2k + 1 + 2k + 1 = z$
$4k + 2 = z$

Regardless of the value that the integer k represents, z will always be a multiple of 4 plus 2 and therefore cannot be even.

In conclusion $x+y$ is even. ■