Structured Nonlinear Feedback Design with Separable Control-Contraction Metrics

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***Abstract*—The abstract goes here.**

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I. Introduction

For linear systems, the problem of design controllers with a prescribed structure is known to be NP-hard **BlondelTsitsiklis1997** Consequently, for nonlinear systems, it is *at least* NP-hard.

a) Overview.: The motivation and background needed for this paper are stated in Section **??**. The problem under consideration is formalized in Section **??**. Section II presents the result of this paper. Illustrations of the proposed approach are provided in Section III. The proofs of the results are provided in Section IV. Section V collects final remarks.

b) Notation.: Let $N \in \mathbb{N}$ be a constant value. The notation $\mathbb{N}_{[1,N]}$ stands for the set $\{i \in \mathbb{N} : 1 \leq i \leq N\}$. Let $c \in \mathbb{R}$ be a constant value. The notation $\mathbb{R}_{[1,c]}$ (resp. $\mathbb{R}_{\diamond c}$) stands for the set $\{x \in \mathbb{R} : 1 \leq x \leq c\}$ (resp. $\{x \in \mathbb{R} : x \diamond c\}$, where $\diamond$ is a comparison operator, i.e., $\diamond \in \{<, \geq, =, \text{ etc}\}$). A matrix $M \in \mathbb{R}^{n\times n}$ with zero elements except (possibly) those $m_{ii}, \ldots, m_{nn}$ on the diagonal is denoted as $\mathtt{diag}(m_{ii}, \ldots, m_{nn})$. The notation $M \succ 0$ (resp. $M \succeq 0$) stands for M being positive (resp. semi) definite.

The notation $\mathcal{L}^{\infty}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ stands for the class of functions $u : \mathbb{R} \to \mathbb{R}^m$ that are locally essentially bounded. Given differentiable functions $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ the notation $\partial_f M$ stands for matrix with dimension $n \times n$ and with (i,j) element given by $\frac{\partial m_{ij}}{\partial x}(x)f(x)$.

A. Problem Formulation and Motivation

Class of systems. Consider the class of systems described by the differential equation

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t), \tag{1}$$

where, for positive times t, the *system state* $x(t)$ and the *system input variable* $u(t)$ evolve in the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R}^n \to \mathbb{R}^m$ are assumed to be smooth, i.e., infinitely differentiable and satisfy $f(0) = 0$ and $B(0) = 0$. From now on the dependence of on the time t will be omitted.

A function $u^* \in \mathcal{L}^{\infty}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ is said to be an *input signal or control for* (1). For such a control for (1), and for every *initial condition* x^*, there exists a unique solution to (1) ([1]) that is denoted by $X(t, x^*, u^*)$, when computed at time t. This solution is defined over an open interval $(\underline{t}, \overline{t})$, and it is said to be *forward complete* if $\overline{t} = +\infty$.

Stabilizability Notion. A forward complete solution $X^*(\cdot, x^*, u^*)$ to (1) is said to be *globally exponentially uniformly stabilizable* with *rate* $\lambda > 0$ if there exist a constant value $C > 0$ and a feedback law $k^* : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$, denoted as $k^*(\cdot, \cdot, X^*, u^*)$, such that the inequality

$$|X(t, x^*, u^*) - X(t, x, k^*)| \leq Ce^{-\lambda t}|x^* - x| \tag{2}$$

holds true, for every $t \geq 0$, and for every $x \in \mathbb{R}^n$ ([2]). Note that this not is a particular case of *incremental asymptotic stability*, the interested reader may address [3], [4] for further information on this stability concept.

Note that a stronger condition than the global exponential stabilizability of a particular solution is the requirement that every forward complete solution of the system is globally exponentially stabilizable. This concept is formalized in the following definition recalled from [2].

Definition 1. The system (1) is said to be *universally stabilizable* with rate λ if there exists a static feedback law k^* for system (2) that globally exponentially uniformly stabilizes any forward complete solution $X^*(\cdot, \cdot, u^*)$ to (2).

Note that Definition 1 reduces to the notion of stabilizability of equilibria, when $x^* = 0$ (for further reading on stabilizability, the reader may address [5]).

For each component $i \in \mathbb{N}_{[1,m]}$ of the feedback law $k^* = (k_1^*, \ldots, k_m^*)^\top$, denote the set of indexes

$$\mathscr{K}(i) = \left\{ j \in \mathbb{N}_{[1,n]} : k_i^* \text{ depends explicitely on } x_j \right\}.$$

The definition of the set $\mathscr{K}(\cdot)$ encompasses different structures. For instance, when $m = n$ full decentralization implies that, for each index $i \in \mathbb{N}_{[1,n]}$, the component k_i^* of the function k^* depends only on x_i. This is formalized by letting $\mathscr{K}(i) = \{i\}$. Moreover, at points where k_i^* is differentiable, the explicit dependence on x_i means that, for every index $j \in \mathbb{N}_{[1,n]}$ with $j \neq i$, $\partial k_i^* / \partial x_j \equiv 0$.

Define also the set of feedback laws

$$\Xi = \{k^* : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m : k_i^* \text{ has the property } \mathscr{K}(i)\}.$$

At this point, the problem under consideration in this paper can be stated as follows.

Problem 2.

1) Find a feedback law $k^* : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ for system (1) that globally exponentially uniformly stabilizes any forward complete solution $X(\cdot, x^*, u^*)$ to (1);

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2) The function $k^* = (k_1^*, \dots, k_m^*)$ belongs to the set Ξ.

The property described in item 2 of Problem 2 is particularly relevant for the design feedback laws with a prescribed structure (topology) for network systems rendering it universally stabilizable. Consider the network composed of systems described by the following equation.

$$\dot{x}_i = ?? \tag{3}$$

Section III shows how the approach proposed in this work is employed to design a decentralized controller for the network composed by interconnections of system (3).

B. Background

Riemannian metrics and differential formulation. A Riemannian metric is a positive-definite bilinear form that depends smoothly on $x \in \mathbb{R}^n$. In a particular coordinate system, for any pair of vectors δ_0, δ_1 of $\mathbb{R}^n$ the metric is defined as the inner product $\langle \delta_0, \delta_1 \rangle_x = \delta_0^\top M(x)\delta_1$, where $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a smooth function. Consequently, local notions of norm $|\delta_x|_x = \sqrt{\langle \delta_x, \delta_x \rangle_x}$ and orthogonality $\langle \delta_0, \delta_1 \rangle_x = 0$ can be defined. The metric is said to be *bounded* if there exists constant values $\underline{m} > 0$ and $\overline{m} > 0$ such that, for every $x \in \mathbb{R}^n$, $\underline{m} I_n \leq M(x) \leq \overline{m} I_n$, where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.

Let $\Gamma(x_0, x_1)$ be the set of piecewise-smooth curves $\gamma : [0,1] \to \mathbb{R}^n$ connecting $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. The *length* and *energy* of γ are, respectively, defined by the values

$$\ell(\gamma) = \int_0^1 |\gamma'(s)|_{\gamma(s)}\, ds \text{ and } e(\gamma) = \int_0^1 |\gamma'(s)|^2_{\gamma(s)}\, ds.$$

The Riemannian distance between x_0 and x_1, denoted as $\mathtt{dist}(x_0, x_1)$, is defined as the curve with the smallest length connecting them. This curve is said to be a *geodesic* and it is the solution to the optimization problem.

$$\mathtt{dist}(x_0, x_1) = \inf_{\gamma \in \Gamma(x_0, x_1)} \ell(\gamma). \tag{4}$$

A suitable framework to deal with exponential convergence of pair of solutions to (1) is provided by the *differential* (also known as variational or prolonged) dynamical system

$$\dot{\delta}_x = A(x,u)\delta_x + B(x)\delta_u, \tag{5}$$

where δ_x (resp. δ_u) is a vector of the Euclidean space $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). More precisely, it is the vector tangent to a piecewise smooth curve connecting a pair of points in $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). The matrix $A \in \mathbb{R}^{n\times n}$ has components given, for every $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$, by $A_{jk}(x,u) = \frac{\partial [f_j + b_j u_j]}{\partial x_k}(x,u)$ for indexes $j,k \in \mathbb{N}_{[1,n]}$.

The resulting system composed of Equations (1) and (5) is analyzed on the state space spanned by the vector $(x, \delta_x) \in \mathbb{R}^n \times \mathbb{R}^n$.

Similarly to (1), given a control δ_u for system (5), the solution to (5) computed at time $t \geq 0$ with $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ and issuing from the initial condition $\delta_x \in \mathbb{R}^n$ is denoted by $\Delta_x(t, x, \delta_x, u, \delta_u)$.

Lyapunov stability notions of solutions to (5) are similar to those of linear parameter-varying systems (LPVS) (see **Briat2015** for more information on LPVS).

The importance of the stability of (5) for system (1) can be understood as follows. Given fixed controls u and δu for systems (1) and (5), respectively. If every solution $|\Delta_x(t, x, \delta_x, u, \delta u)| \to 0$ exponentially as $t \to \infty$, then every pair of solutions to (1) converge to each other exponentially. The interested reader may address [6], [7] and references therein for further details.

A sufficient condition for the stability of (5) is provided by analyzing the derivative of a particular function along the solutions of systems (1) and (5). This function is recalled from [3] and [2].

Definition 3. A smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a *metric for system* (1) if there exist constant values $\underline{c} >$ and $\overline{c} >$ such that the inequality

$$\underline{c}|\delta_x|^2 \leq V(x, \delta_x) \leq \overline{c}|\delta_x|^2, \tag{6a}$$

holds, for every $(x, \delta_x) \in \mathbb{R}^n \times \mathbb{R}^n$. Given fixed controls u and δu for systems (1) and (5), respectively. A metric system (1) receives the adjective *contraction* if there exists a value $\lambda > 0$ such that the inequality

$$\frac{dV}{dt}(X(t,x,u), \Delta(t,x,\delta x, u, \delta u)) \leq -\lambda V(x, \delta_x) \tag{6b}$$

holds, for every pair $(x, \delta_x) \in \mathbb{R}^n \times \mathbb{R}^n$.

Note that a bounded Riemannian metric defined, for every $(x, \delta_x) \in \mathbb{R}^n \times \mathbb{R}^n$, as $V(x, \delta_x) = |\delta_x|_x^2$ and satisfying the set of inequalities (6) is a contraction metric for system (1). With an abuse of concept, from now the bounded Riemannian metric defined above will be called simply as *metric*.

The existence a contraction metric for system (1) with $u \equiv 0$ implies that every two solutions of this system converge to each other exponentially. The proof of this claim can be found in [8, Theorem 1], and [9, Theorems 5.7 and 5.33], and [10, Lemma 3.3].

For the class of systems considered in this paper, the following kind of metric is of interest, since it also allows the design a feedback law for system (5).

Definition 4 ([2])**.** A metric for system (1) is said to be a *control-contraction metric for system* (1) if there exists a constant value $\lambda > 0$ such that the condition

$$\delta_x^\top M(x) B(x) = 0 \tag{7a}$$

implies that the inequality

$$\delta_x^\top (\dot{M} + A^\top M + MA)\delta_x \leq \ -2\lambda \delta_x^\top M \delta_x \tag{7b}$$

holds, where $\dot{M} := \partial_{f+Bu} M$.

The set of equations (7) is an adaptation of Artstein-Sontag's condition for contraction. Given a control-contraction metric for system (1), Finsler's lemma (cf. [11, Lemma 11.1]) provides stabilizing a feedback law of the form $\delta_u = K\delta_x$ for system (5) defined for every $\delta_x \in \mathbb{R}^n$, where $K : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m}$.

The following result is recalled from [2] and provides a feedback law for system (1) given a feedback law for system (5).

Theorem 5. *If there exists a control-contraction metric for system* (1)*, then there exists a solution to item 1 of Problem 2.*

Another important result of Theorem 5 the feedback for system (1) obtained by integrating a feedback law designed for system (5).

As remarked in [2], the main advantage to look for control-contraction metric with respect to a control-Lyapunov function is that the former case can be formulate in terms of a convex optimization problem [12]. The steps to obtain a control to system (1) that solves item 1 of Problem 2 are shown below.

Step 1 (Offline LMI computation). Consider the change of variables $\eta = M\delta$ and define the matrix $W = M^{-1}$. The set of equations (7) is equivalent (cf. [11, Lemma 11.1]) to the existence of a bounded differentiable function $W : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $W = W^\top \succ 0$ and a function $Y : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times n}$ satisfying the following linear matrix inequality (LMI)

$$-\dot{W} + AW + WA^\top + BY + (BY)^\top + 2\lambda W \preceq 0, \qquad (8)$$

for every $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. Consequently, $M = W^{-1}$ is a control-contraction metric for system (1).

Step 2 (Online controller integration). The feedback law for system (1) can be obtained by integration as follows. Let $u^* : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ be a control for system (1) and $K = YW^{-1}$, from Hopf-Rinow theorem (cf. [13, Theorem 7.7]), for every x_0 and $x_1 \in \mathbb{R}^n$, there exist a smooth geodesic curve $\gamma : [0, 1] \to \mathbb{R}^n$ connecting them. This implies that the solution k^* to the integral equation

$$k^*(t, s) = u^*(t) + \int_0^s K(\gamma(\sigma), k^*(t, \sigma))\gamma'(\sigma)\, d\sigma, \qquad (9)$$

where $s \in [0, 1]$, is a feedback law for system (1).

Motivation. From the above steps, even if the problem of imposing a particular structure on Y to correspond to the constraints of Problem 2 was tractable, the integration of the controller would not necessarily satisfy these constraints. This is due to the fact that the solutions to the optimization problem (4) can not be distributely computed.

Contribution. In this paper, these limitations are addressed by imposing a block-diagonal structure over W. Moreover, when W is row-diagonal dominant, not only the solutions to the optimization problem (4) can be computed in parallel but also the LMI (9).

II. Results

Definition 6. A control-contraction metric V for system (1) receives the adjective *sum-separable* if M has a block-diagonal structure.

Definition 6 implies that there exist integers $N > 1$ and $n_i > 0$ such that $n_1 + \ldots + n_N = n$. Also, there exist smooth bounded functions $M_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i\times n_i}$ satisfying $M_i = M_i^\top \succ 0$, for every index $i \in \mathbb{N}_{[1,N]}$, and the equation

$$V(x, \delta_x) = \sum_{i=1}^{N} \delta_{x_i}^\top M_i(x_i)\delta_{x_i},$$

for every $(x_i, \delta x_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_i}$.

Although the requirement of a control-contraction metric M to have block-diagonal structure may be restrictive, for positive linear time-invariant systems the existence of a matrices $P = P^\top \succ 0$ with diagonal structure is not conservative [14]. Thus, the question of how restrictive is the requirement of M to be diagonal is open. The main result of the paper is stated below.

Compare SS-CCM with SS-LF from [15], [16]

To solve item 2 of Problem 2, the structure on the feedback defined in Equation (9) is obtained by imposing a a suitable constraint on the function Y to be satisfied together with the LMI (8).

For each index $i \in \mathbb{N}_{[1,N]}$, define the sets

$$\mathscr{K}_Y(i) = \{j \in \mathbb{N}_{[1,N]} \setminus \mathscr{K}(i) : Y_{ij} \equiv 0\}$$

and

$$\Xi_Y = \{Y : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times n} : \text{the } i\text{-th row } Y_i \text{ has the property } \mathscr{K}_Y(i)\}.$$

Note that the set $\mathscr{K}_Y(i)$ is the set index for which the corresponding columns of the row-vector Y_i are zero. By definition, this set is the complement of $\mathscr{K}(i)$. Consequently, given an index $j \in \mathscr{K}_Y$ and W is diagonal, the i-th line of the vector $YW^{-1}\delta_x$ does not depend on j-th component of the vector δ_x. As remarked in [14], the constraint "$Y \in \Xi_Y$" is linear.

Theorem 7. *If there exist a smooth functions $W : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $Y : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times n}$ such that, for every $(x, \delta x, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, $W = W^\top \succ 0$, W is block-diagonal $Y \in \Xi_Y$ and the pair (W, Y) is a solution to the LMI* (8)*, then there exists a solution to Problem 2.*

The detailed proof of Theorem 7 is provided in Section IV.

Although Theorem 7 provides a methodology to design distributed controllers, *a priori* the computation of the LMI (8) is cannot be done in parallel. The next result provides sufficient conditions to solve each component (8) independently, and according to the structure defined by the set Ξ. Before present it, the following concept of block-diagonally dominant matrix is recalled from [17].

Consider the smooth function $T : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$ with elements $i, j \in \mathbb{N}_{[1,N]}$ defined as

$$T_{ii} = -\dot{W}_i + A_{ii}W_i + W_iA_i^\top + B_iY_{ii} + (B_iY_{ii})^\top + 2\lambda_i W_i \qquad (11a)$$

and

$$T_{ij} = A_{ij}W_j + W_iA_{ji} + B_iY_{ij} + (B_jY_{ji})^\top. \qquad (11b)$$

Definition 8. The matrix T is said to be *block-diagonally dominant* if the inequality

$$|T_{ii}^{-1}|^{-1} \geq \sum_{\substack{j=1\\ j\neq i}}^{N} |T_{ij}|, \qquad (12)$$

holds, for every index $i \in \mathbb{N}_{[1,N]}$ and for every $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$.

Corollary 9. *If there exist a smooth functions $W : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $Y : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times n}$ such that, for every $(x, \delta x, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, $W = W^\top \succ 0$, W is block-diagonal $Y \in \Xi_Y$ and the pair (W, Y) is a solution to the LMI*

$$T_{ii} \preceq 0 \tag{13}$$

for every index $i \in \mathbb{N}_{[1,N]}$, and the matrix T is diagonally dominant, then pair of matrices W and Y is a solution to the LMI (8).

The detailed proof of Corollary 9 is provided in Section IV.

See the works of Ahmadi (Princeton) on scale-diag. dom.

III. Illustration

Let $N > 0$ be a constant integer, for each index $i \in \mathbb{N}_{[1,N]}$, consider the system given by

$$\begin{cases} \dot{x}_i &= -x_i - x_i^3 + y_i^2 + 0.01\left(x_{i-1}^3 - 2x_i^3 + x_{i+1}^3\right) \\ \dot{y}_i &= u_i, \end{cases} \tag{14}$$

where for convenience $x_0 = x_1$ and $x_N = x_{N+1}$.

For each index $i \in \mathbb{N}_{[1,N]}$, define the vectors $q_i = (x_i, y_i)$, $\breve{q}_i = (x_{i-1}, x_{i+1})$ and let $q = (q_1, \ldots, q_N)$. Denote also

$$f_i(q_i, \breve{q}_i) = \begin{bmatrix} -x_i - x_i^3 + y_i^2 + 0.01\left(x_{i-1}^3 - 2x_i^3 + x_{i+1}^3\right) \\ 0 \end{bmatrix}$$
$$B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that system (14) is not feedback linearizable in the sense of [18], because the vector fields

$$B = \mathtt{diag}\,(B_1, \ldots, B_N),$$
$$\frac{\partial f}{\partial q}B - \frac{\partial B}{\partial q}f = \mathtt{diag}\left(\begin{bmatrix} 2y_1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 2y_N \\ 0 \end{bmatrix}\right)$$

are not linearly independent when $y_1 = \cdots = y_N = 0$. Furthermore, due to the quadratic term on y, the only action of the controller on the x-subsystem is capable of is to move the x-component of solution to (14) towards the positive semi-axis. In other words, the controller cannot reduce the value of the x-component.

Employing the method provided in Theorem 7, the LMI (8) has been solved with optimization parser Yalmip [19], [20] and the solver Mosek for a network composed of four systems (i.e., $N = 4$) and for three different constraints on the matrix Y.

In all cases, the matrix W has a diagonal structure as follows

$$W = \mathtt{diag}\,(W_a, W_b, W_b, W_a)\,,$$

the definition of the matrices W_a and W_b differs, according to each case considered.

In the first one, no constraints were imposed on the matrix Y. The parser took 97 seconds to solve the LMI (8). The components of the matrix W and the matrix Y that form a solution to the (8) are

$$W_a = \mathtt{diag}\,(1.27, 1.53)\,, \quad W_b = \mathtt{diag}\,(1.08, 1.53)$$

and the matrix Y has the structure shown in Equation (15) with elements

$$\begin{aligned} \overline{Y}_{ij} &= -\,3y_i - 0.08x_iy_i \\ \underline{Y}_{ij} &= -\,2 + 0.009x_i - 1.2\sum_{k=1}^{4} y_k^2 + x_k^2. \end{aligned}$$

In the second case, the constraint imposed on the matrix Y was that each line of this matrix should depend on the system i and on its neighbors. The time taken by the parser to solve the LMI (8) was 98 seconds. The components of the matrix W and the matrix Y that form a solution to the (8) are

$$W_a = \mathtt{diag}\,(1.32, 1.59) \quad \text{and} \quad W_b = \mathtt{diag}\,(1.08, 1.61)\,.$$

and the matrix Y has the structure shown in Equation (16) with elements

$$\begin{aligned} Y_{ij} &= -\,3y_i - 0.03x_iy_i \qquad\qquad i,j \in \mathbb{N}_{[1,4]} \\ Y_{12} &= -\,4 - 0.01x_1 - y_1^2 - y_2^2 - 2x_1^2 - x_2^2 \\ Y_{24} &= -\,3 - 0.01x_2 - y_1^2 - y_2^2 - y_3^2 - x_1^2 - x_2^2 - x_3^2 \\ Y_{36} &= -\,3 - 0.01x_3 - y_2^2 - y_3^2 - y_4^2 - x_2^2 - x_3^2 - x_4^2 \\ Y_{48} &= -\,4 - 0.01x_4 - y_4^2 - y_3^2 - 2x_4^2 - x_3^2 \end{aligned}$$

In the last case, the fully decentralized scenario has been considered. The time taken by the parser to solve the LMI (8) was 101 seconds. In this case, $\mathscr{K}_Y(i) = \{j \in \mathbb{N}_{[1,N]} \setminus \mathscr{K}(i) : Y_{ij} \equiv 0\}$, where $\mathscr{K}(i) = \{i\}$. Imposing $Y \in \Xi_Y$ and solving the LMI (8) the following matrices were obtained

$$\begin{aligned} W &= \mathtt{diag}\,(W_a, W_b, W_b, W_a)\,, \\ Y &= \mathtt{diag}\,(Y_1, Y_2, Y_3, Y_4) \end{aligned}$$

where

$$W_a = \mathtt{diag}\,(1.43, 1.17)\,, \quad W_b = \mathtt{diag}\,(1.03, 2.17)$$

and, for each index $i \in \mathbb{N}_{[1,4]}$,

$$\begin{aligned} Y_i = [&-4.33y_i + 0.04x_iy_i, \\ &-7.78 - 0.01x_i - 2.73y_i^2 - 2.98x_i^2]. \end{aligned}$$

IV. Proof of the Results

Proof of Theorem 7. By assumption, the functions $W : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $Y : \mathbb{R}^n \times \mathbb{R}^m$ is a solution to the LMI (8). Apply the coordinate change $\eta = M\delta$ and define the matrices $W = M^{-1}$ and $K = YW^{-1}$. Since W is diagonal, the structure of Y is preserved and, consequently, $K \in \Xi_Y$.

The LMI (8) implies that the inequality

$$\delta_x^\top \left(\dot{M} + (A + BK)M + M(A + BK)^\top - 2\lambda M\right)\delta_x \leq 0$$

holds, for every $(x, \delta_x, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Consequently, the condition defined by the set of equations (7) holds. Thus, M is a sum-separable control-contraction metric for system (1).

It now remains to integrate $K\delta_x$ to obtain a feedback law for system (1) satisfying the constraints of Problem 2. Because M

$$Y = \begin{bmatrix} \overline{Y}_{11}, & \underline{Y}_{12}, & -0.004y_1, & 0, & -0.04y_1 + 0.003x_3y_1, & 0.005y_1y_3, & -0.007y_1 + 0.001x_4y_1, & 0 \\ -0.008y_2 - 0.002x_1y_2, & 0, & \overline{Y}_{23}, & \underline{Y}_{24}, & -0.007y_2 - 0.004x_3y_2, & 0, & -0.04y_2 - 0.004x_4y_2, & 0.005y_2y_4 \\ -0.04y_3 - 0.004x_1y_3, & 0, & -0.008y_3 + 0.004x_2y_3, & 0, & \overline{Y}_{35}, & \underline{Y}_{36}, & -0.008y_3 + 0.002x_4y_3, & 0 \\ -0.009y_4 + 0.001x_1y_4, & 0, & -0.04 - 0.003x_2y_4, & 0, & -0.004y_4 & 0, & \overline{Y}_{47}, & \underline{Y}_{48} \end{bmatrix} \tag{15}$$

$$Y = \begin{bmatrix} Y_{11}, & Y_{12}, & -0.01y_2 + 0.03x_2y_1, & 0, & 0, & 0, & 0, & 0 \\ -0.01y_2 - 0.002x_1y_2, & 0, & Y_{23}, & Y_{24}, & -0.01y_2 + 0.03x_3y_2, & 0, & 0, & 0 \\ 0, & 0, & -0.01y_3 + 0.03x_2y_3, & 0, & Y_{35}, & Y_{36}, & -0.01y_3 - 0.002x_4y_3, & 0 \\ 0, & 0, & 0, & 0, & -0.01y_4 + 0.03x_3y_4, & 0, & Y_{47}, & Y_{48} \end{bmatrix} \tag{16}$$

is a block-diagonal matrix, the length of any curve $\varphi : [0, 1] \to \mathbb{R}^n$ satisfies the following equation

$$\ell(\varphi) = \int_0^1 \sqrt{\sum_{i=1}^{N} \varphi_i'(s_i)^\top M_i(\varphi_i(s_i))\varphi_i'(s_i)\, ds}. \tag{17}$$

Since M is positive definite, the minimum of Equation (17) corresponds to a minimum of each component $i \in \mathbb{N}_{[1,N]}$. More precisely,

$$\gamma = \arg\inf \ell(\varphi) \Leftrightarrow \gamma_i = \arg\inf e(\varphi_i), \quad \forall i \in \mathbb{N}_{[1,N]}.$$

This implies that the minimization of Equation (17) can be computed separately.

From Hopf-Rinow theorem (cf. [13, Theorem 7.7]), for every index $i \in \mathbb{N}_{[1,N]}$, and for every x_i and $x_i^* \in \mathbb{R}^{n_i}$, there exists a solution to the optimization problem

$$\gamma_i = \arg \inf_{\varphi_i \in \Gamma(x_i^*, x_i)} e(\varphi_i).$$

This solution is a geodesic smooth curve $\gamma_i : [0, 1] \to \mathbb{R}^{n_i}$ connecting x_i to $x_i^* \in \mathbb{R}^{n_i}$.

From the definition of the tangent vectors δ_x and δ_u, given an input u^* for system (1), a feedback for system (1) that solves item 1 of Problem 2 is given by Equation (9).

From the fundamental theorem of calculus, Equation (9) is equivalent to the differential equation

$$\frac{dk^*}{ds}(t, s) = K(\gamma(s), k^*(t, s))\gamma'(s) =: \overline{K}(t, s, k^*). \tag{18}$$

which has a unique solution defined of a maximal interval of existence, for every initial condition, due to the smoothness of the functions K and γ. It remains to show that this interval of existence can be extended on $[0, 1]$. In other words, to show that there exist no $\bar{s} \in [0, 1]$ such that $k^*(t, s) \to \infty$, as $s \to \bar{s}$.

To show that a solution k^* is defined on the whole interval $[0, 1]$, the result [1, Theorem 2.12] is employed. This theorem states that, if the vector field has a linear growth with respect to the variable k^*, then the solutions to (18) are defined on the whole interval $[0, 1]$.

Since the vector field $\overline{K}$ is smooth, it also Lipschitz. This implies that, for every fixed $t \geq 0$, and for every $S > 0$, there exists a constants $c_1(S)$ and $c_2(S)$ such that the inequality

$$\left|\overline{K}(t, s, k^*)\right| \leq c_1(S) + c_2(S)|k^*|$$

holds, for every $(s, k^*) \in [-S, S] \times \mathbb{R}^m$. Applying [1, Theorem 2.12] pointwisely on $t \geq 0$ implies that a solution k^* to (18) is defined for every $s \in [0, 1]$.

Note that, due to the structure of the matrix K, the feedback law k^* defined in Equation (9) solves item 2 of Problem 2, i.e., $k^* \in \Xi$. Furthermore, the computation of a solution to Equation (18) can be partially distributed, because each line of the matrix K will depend only on specific components of the vector γ'.

This concludes the proof of Theorem 7. □

Proof of Corollary 9. By assumption, the pair of matrices W and Y is a solution to the LMI (13) and the matrix T with components defined by the set of equations (11) is block-diagonally dominant.

Equation (12) together with the fact that $|I_{n_i}| \leq |T_{ii}||T_{ii}^{-1}|$ implies that the inequality

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} |T_{ij}| \leq |T_{ii}| \tag{19}$$

holds, for every index $i \in \mathbb{N}_{[1,N]}$.

For every $\delta_\eta \in \mathbb{R}^n$, consider the product $\delta_\eta^\top T \delta_\eta$. Each line $i \in \mathbb{N}_{[1,N]}$ of this product is given by

$$\delta_{\eta_i}^\top \begin{bmatrix} T_{i1} & \cdots & T_{i(i-1)} & T_{ii} & T_{i(i+1)} & \cdots & T_{iN} \end{bmatrix} \delta_\eta.$$

Inequalities (19) and (13) imply that the inequalities

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{\eta_i}^\top T_{ij} \delta_{x_j} \leq \delta_{x_i}^\top T_{ii} \delta_{\eta_i} \leq 0$$

hold, for every $\delta_\eta \in \mathbb{R}^n$. Consequently, the inequality $\delta_\eta^\top T \delta_\eta \leq 0$ holds, for every $\delta_\eta \in \mathbb{R}^n$. This later inequality is equivalent to the LMI (8). This concludes the proof of Corollary 9. □

V. Conclusion

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