# Structured Nonlinear Feedback Design with Separable Control-Contraction Metrics

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***Abstract*—The abstract goes here.**

***Index Terms*—IEEE, IEEEtran, journal, LaTeX, paper, template.**

## I. Introduction

[1]

*a) Overview.:* The motivation and background needed for this paper are ~~precised~~ in Section **??**. The problem under consideration is formalized in Section **??**. Section II presents the result of this paper. Illustrations of the proposed approach are provided in Section III. The proofs of the results are provided in Section IV. Section V collects final remarks.

not a word

*b) Notation.:* Let $N \in \mathbb{N}$ be a constant value. The notation $\mathbb{N}_{[1,N]}$ stands for the set $\{i \in \mathbb{N} : 1 \leq i \leq N\}$. Let $c \in \mathbb{R}$ be a constant value. The notation $\mathbb{R}_{[1,c]}$ (resp. $\mathbb{R}_{\diamond c}$) stands for the set $\{x \in \mathbb{R} : 1 \leq x \leq c\}$ (resp. $\{i \in \mathbb{R} : i \diamond c\}$, where $\diamond$ is a comparison operator, i.e., $\diamond \in \{<, \geq, =, \text{etc}\}$). A matrix $M \in \mathbb{R}^{n\times n}$ with zero elements except (eventually) those $m_{ii}, \ldots, m_{nn}$ on the diagonal is denoted as $\mathtt{diag}(m_{ii}, \ldots, m_{nn})$. The notation $M \succ 0$ (resp. $M \succeq 0$) stands for $M$ is positive (resp. semi) definite.

x better real?

what?

The notation $\mathcal{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ stands for the class of functions $u : \mathbb{R} \to \mathbb{R}^m$ that are locally essentially bounded. Given differentiable functions $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ the notation $\dot{M}$ stands for matrix with dimension $n \times n$ and with $(i,j)$-element given by $\frac{\partial m_{ij}}{\partial x}(x) f(x)$.

where is this used?!

$\partial_f M$ ?

$\dot{M}$ = total derivative

### A. Problem Formulation and Motivation

*Class of systems.* Consider the class of systems described by the differential equation

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t), \tag{1}$$

where, for positive times $t$, the *system state* $x(t)$ and the *system input* $u(t)$ evolve in the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R}^n \to \mathbb{R}^m$ are assumed to be smooth, i.e., infinitely differentiable and satisfy $f(0) = 0$ and $B(0) = 0$. From now on the implicit dependence of the function $f$ and $B$ on the time $t$ will be omitted.

A function $u^* \in \mathcal{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ is said to be a *input signal or control for* (1). For such a control for (1), and for every *initial condition* $x^*$, there exists a solution to (1) ([2]). This solution is defined over an open interval $(\underline{t}, \overline{t})$, and is denoted, at time $t \in (\underline{t}, \overline{t})$, by $X(t, x^*, u^*)$. This solution is said to be *forward complete* if $\overline{t} = +\infty$.

*Stabilizability Notion.* A forward complete $X(t, x^*, u^*)$ to (1) is said to be *globally exponentially uniformly stabilizable* with *rate* $\lambda > 0$ if there exist a constant value $C > 0$ and a feedback law $k^* : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ such that the inequality

$$|X(t, x^*, u^*) - X(t, x, k^*)| \leq Ce^{-\lambda t}|x^* - x| \tag{2}$$

holds true, for every $t \geq 0$, and for every $x \in \mathbb{R}^n$ ([3]). Note that this is a particular case of *incremental asymptotic stability.* Although this later concept is not in the scope of this work, the interested reader may address [4], [5] for further information.

not true

Note that a stronger condition than the global exponential stabilizability of a particular solution is the requirement that every forward complete solution of the system is globally exponentially stabilizable. This concept is formalized in the following definition recalled from [3].

**Definition 1.** The system (1) is said to be *universally stabilizable* with rate $\lambda$ if there exists a feedback law $k^* : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \{x^*\} \times \{u^*\} \to \mathbb{R}^m$ for system (2) that globally exponentially uniformly stabilizes any forward complete solution $X(\cdot, x^*, u^*)$ to (2).

Note that Definition 1 reduces to the notion of stabilizability of equilibria, when $x^* = 0$ (for further reading on stabilizability, the reader may address [6]).

At this point, the problem under consideration in this paper can be stated as follows.

Define a such controller

**Problem 2.**

1) Find a feedback law $k^* : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \{x^*\} \times \{u^*\} \to \mathbb{R}^m$ for system (1) that globally exponentially uniformly stabilizes any forward complete solution $X(\cdot, x^*, u^*)$ to (1);
2) The function $k^* = (k_1^*, \ldots, k_m^*)$ belongs to the set

$$\begin{aligned}\Xi = \{k^* : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \{x^*\} \times \{u^*\} \to \mathbb{R}^m \mid \\ k_i^* \in \mathscr{K}(i), i \in \mathbb{N}_{[1,n]}\},\end{aligned} \tag{3a}$$

where

$$\mathscr{K}(i) = \{j \in \mathbb{N}_{[1,n]} : k_i^* \text{ depends explicitely on } x_j\}. \tag{3b}$$

Can this be clearer

The definition of the set $\mathscr{K}$ encompasses different structures. For instance, when $m = n$ full decentralization implies that, for each index $i \in \mathbb{N}_{[1,n]}$, the component $k_i^*$ of the function $k^*$ depends only on $x_i \in \mathbb{R}$. This is formalized by letting $\mathscr{K}(i) = \{i\}$. Moreover, at points where $k_i^*$ is differentiable, the explicit dependence on $x_i$ means that, for every index $j \in \mathbb{N}_{[1,n]}$ with $j \neq i$, $\partial k_i^*/\partial x_j \equiv 0$.

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$\partial_f M + \frac{\partial f}{\partial x} M + M \frac{\partial f}{\partial x}$

$k(x) = k(x)$ ?

$k_i : x_i \mapsto u_i$

$\dot{x} = f(x,t)$



The property described in item 2 of Problem 2 is particularly relevant for the design feedback laws with a prescribed structure (topology) for network systems rendering it universally stabilizable. Consider the network composed of systems described by the following equation.

$$\dot{x}_i = ?? \tag{4}$$

Section III shows how the approach proposed in this work is employed to design a decentralized controller for the network composed by interconnections of system (4).

### B. Background

*Riemannian metrics and differential formulation.* A Riemannian metric is a positive-definite bilinear form that depends smoothly on $x \in \mathbb{R}^n$. In a particular coordinate system, for any pair of vectors $\delta_0, \delta_1$ of $\mathbb{R}^n$ the metric is defined as the inner product $\langle \delta_0, \delta_1 \rangle_x = \delta_0^\top M(x)\delta_1$, where $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a smooth function. Consequently, local notions of norm $|\delta_x|_x = \sqrt{\langle \delta_x, \delta_x \rangle_x}$ and orthogonality $\langle \delta_0, \delta_1 \rangle_x = 0$ can be defined. The metric is said to be *bounded* if there exists constant values $\underline{M} > 0$ and $\overline{M} > 0$ such that, for every $x \in \mathbb{R}^n$, $\underline{M}I \leq M(x) \leq \overline{M}I$.

scalar?

Let $\Gamma(x_0, x_1)$ be the set of piecewise-smooth curves $\gamma : [0,1] \to \mathbb{R}^n$ connecting $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. The *length* and *energy* of $\gamma$ are, respectively, defined by the values

$$\ell(\gamma) = \int_0^1 |\gamma'(s)|_{\gamma(s)}\, ds \text{ and } e(\gamma) = \int_0^1 |\gamma'(s)|^2_{\gamma(s)}\, ds.$$

The Riemannian distance between $x_0$ and $x_1$, denoted as $\mathtt{dist}(x_0, x_1)$, is defined as the curve with the smallest length connecting them. This curve is said to be a *geodesic* and it is the solution to the optimization problem.

$$\mathtt{dist}(x_0, x_1) := \inf_{\gamma \in \Gamma(x_0, x_1)} \ell(\gamma). \tag{5}$$

A suitable framework to deal with exponential convergence of pair of solutions to (1) is provided by the *differential* (also known as variational or prolonged) dynamical system

$$\dot{\delta}_x = A(x,u)\delta_x + B(x)\delta_u, \tag{6}$$

where $\delta_x$ (resp. $\delta_u$) is a vector of the Euclidean space $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). More precisely, it is the vector tangent to a piecewise smooth curve connecting a pair of points in $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). The matrix $A \in \mathbb{R}^{n\times n}$ has components given, for every $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$, by

$$A_{jk}(x,u) = \frac{\partial [f_j + b_j u_j]}{\partial x_k}(x,u),$$

for indexes $j,k \in \mathbb{N}_{[1,n]}$. Similarly to (1), given a control $\delta_u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, the solution to (6) computed at time $t \geq 0$ with $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ and issuing ~~form~~ the initial condition $\delta_x \in \mathbb{R}^n$ is denoted by $\Delta_x(t, x, \delta_x, u, \delta_u)$.

from

The resulting system composed of Equations (1) and (6) is analyzed on the state space spanned by the vector $(x, \delta_x) \in \mathbb{R}^n \times \mathbb{R}^n$.

Lyapunov stability notions of solutions to (6) are similar to those of linear parameter-varying systems (LPVS) (see [7, Ch. 2 and 3] for more information on LPVS).

The importance of the stability of (6) for system (1) can be understood as follows. Let $\Delta_x(\cdot, x, \delta_x, 0, 0)$ be a solution to (6) and $X(\cdot, x, 0)$ be a solution to (1). If $|\Delta_x(t, x, \delta_x, 0, 0)| \to 0$ exponentially as $t \to \infty$, then these solutions satisfy inequality (2). The interested reader may address [8], [9] and references therein for further details.

for all

A sufficient condition for the stability of (6) is provided by analyzing the derivative of a particular function along the solutions of systems (1) and (6). This function is recalled from [4] and [3].

**Definition 3.** A smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a *metric for system* (1) if there exist constant values $\underline{c} >$ and $\overline{c} >$ such that the inequality

$$\underline{c}|\delta_x|^{1} \leq V(x, \delta_x) \leq \overline{c}|\delta_x|^{2},$$

holds, for every $(x, \delta_x) \in \mathbb{R}^n \times \mathbb{R}^n$. It is said to be a *contraction metric for system* (1) ~~with $u \equiv 0$~~ if there exists a value $\lambda > 0$ such that, for every $(x, \delta_x) \in \mathbb{R}^n \times \mathbb{R}^n$, the inequality

$$\cancel{L_{f+Bu}}V(x, \delta_x, 0, 0) \leq -\lambda V(x, \delta_x) \qquad \frac{\partial V}{\partial x} f$$

holds.

not true

Note that a bounded metric is a contraction metric for system (1). Also, if the bounded metric is a contraction metric for system (1) with $u \equiv 0$, then inequality (2) holds. The proof for this claim can be found in [10, Theorem 1], and [11, Theorems 5.7 and 5.33], and [12, Lemma 3.3].

When inputs to systems (1) and (6) are not identically zero, a suitable feedback law and a contraction metric for system (1) need to be found.

This problem is solved by

1) Finding a function $K : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ and a differentiable metric for system (1) such that the inequality
$$\delta_x^\top(\dot{M} + (A+BK)M + M(A+BK)^\top - 2\lambda M)\delta_x \leq 0,$$
holds, for every $(x, \delta_x, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$;
2) Obtain the function $k$ by integrating $K$ along a geodesic in $\mathbb{R}^n$.

not correct $\int K \frac{\partial \gamma}{\partial s} ds$

Note that item 1 is a co-design of a feedback law and a contraction metric. For this reason, the pair of functions $(k, V)$ satisfying items 1 and 2 is said to be a *contraction pair* for system (1). Item 2 is the construction of the a feedback law for system (1) by integration.

why?

The existence of a particular kind of metric satisfies items 1 and 2. Before presenting this result, the definition of this metric is recalled from [3].

**Definition 4.** A metric for system (1) is said to be a *control-contraction metric for system* (1) if there exists a constant value $c$ > 0 such that the condition

$\lambda$

$$\delta_x^\top M(x)B(x) = 0 \tag{7a}$$

implies that the inequality

$$\delta_x^\top(\dot{M} + A^\top M + MA)\delta_x \leq -2\lambda\delta_x^\top M\delta_x \tag{7b}$$

holds.

The set of equations (7) is an adaptation of Artstein-Sontag's condition for contraction.

integrability

The following result is recalled from [3] and provides a contraction pair for system (1).

~~**Theorem 5.** *If there exists a control-contraction metric for system (1), then there exists a contraction pair for system* (1).~~

As remarked in [3], the main advantage to look for control-contraction metric with respect to a control-Lyapunov function is that the former canse can be formulate in terms of a convex optimization problem [13]. The search for a contraction pair for system (1) can be done in terms of a convex optimization problem.

Step 1 (Offline LMI computation). Consider the change of variables $\eta = M\delta$ and define the matrix $W = M^{-1}$. The set of equations (7) is equivalent (cf. [14, Lemma 11.1]) to the existence of a bounded differentiable function $W : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $W = W^\top \succ 0$ and a function $Y : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ satisfying the following linear matrix inequality (LMI)

$$-\dot{W} + AW + WA^\top + BY + (BY)^\top + 2\lambda W \preceq 0, \qquad (8)$$

for every $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$.

Step 2 (Online controller integration). Once a solution to the LMI (8) has been found the feedback law for system (1) can be obtained by integration as follows. Let $u^* : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ be a control for system (1) and $K = YW^{-1}$, from Hopf-Rinow theorem (cf. [15, Theorem 7.7]), for every $x_0$ and $x_1 \in \mathbb{R}^n$, there exists a geodesic curve $\gamma : [0,1] \to \mathbb{R}^n$ connecting them. This implies that the functions

$$k(t) = u^*(t) + \int_0^1 K(\gamma(s))\gamma'(s)\, ds, \quad t \geq 0, \qquad (9)$$

and $M = W^{-1}$ are a contraction pair for system (1).

*Contribution.* At this point the two contributions of this paper becomes clear.

First, to impose a particular structure on the matrix $Y$ satisfying the LMI (8) without imposing any structure on $W$ is unfeasible, in general.

non convex intractable

Second, without imposing any structure on $W$, the solution to the optimization problem (5) cannot be distributed. In terms of execution, the feedback law (9) would need to compute a new geodesic at each new time step. This motivates approach proposed in this paper.

negative statements

## II. Results

**Definition 6.** A control-contraction metric $M$ for system (1) receives the adjective *sum-separable* if $M$ has a block-diagonal structure. More precisely, there exist an integer $N > 0$, differentiable bounded functions $M_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times n_i}$, where $i \in \mathbb{N}_{[1,N]}$, satisfying $M_i = M_i^\top \succ 0$ for some integers $n_i > 0$ such that $M = (M_1, \ldots, M_N)$ and $n = n_1 + \cdots + n_N$.

Although the requirement of a control-contraction metric $M$ to have block-diagonal structure may be restrictive, for positive linear time-invariant systems the existence of a symmetric positive definite matrices $P$ with diagonal structure used as Lyapunov functions is not conservative [16]. Thus, the question of how restrictive is the requirement of $M$ be diagonal is open. The main result of the paper is stated below.

and local dependence!

**Theorem 7.** *If there exist a differentiable function $W : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $W = W^\top \succ 0$, $W$ is block-diagonal, and a function $Y \in \Xi$ that is a solution to the LMI (8), then there exists a solution to Problem 2.*

The detailed proof of Theorem 7 is provided in Section IV. Theorem 7 implies that the a control-contraction metric $M$ for system (7) can be obtained by inverting $W$.

Write about the network case

A particular case of Theorem 7 allows the computation of the LMI (8) in parallel. This is formalized in the following result.

**Corollary 8.** *Consider the differentiable function $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with elements $i, j \in \mathbb{N}_{[1,N]}$ defined as*

$$T_{ii} = -\dot{W}_i + A_{ii}W_i + W_iA_i^\top + B_iY_{ii} + (B_iY_{ii})^\top + 2\lambda_i W_i$$

*and*

$$T_{ij} = A_{ij}W_j + W_iA_{ji} + B_iY_{ij} + (B_jY_{ji})^\top.$$

*If, for every index $i \in \mathbb{N}_{[1,N]}$ there exists a solution to the LMI*

$$T_{ii} \preceq 0 \qquad (10)$$

*such that the matrix $T$ is row-diagonal dominant, then there exists a solution to Problem 2.*

The detailed proof of Corollary 8 is provided in Section IV.

## III. Illustration

Here the an example of a network system will be used.

## IV. Proof of the Results

*Proof of Theorem 7.* By assumption, the function $W : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is differentiable, $W = W^\top \succ 0$, and it is also block-diagonal. Moreover, the functions $W$ and $Y \in \Xi$ is a solution to the LMI (8).

The LMI (8) implies that the inequality

$$\eta^\top(-\dot{W} + AW + WA^\top + BY + (BY)^\top)\eta \leq -2\lambda\eta^\top W\eta \qquad (11)$$

holds, for every $(x, \eta, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Apply the coordinate change $\eta = M\delta$ and define the matrices $W = M^{-1}$ and $K = YW^{-1}$, inequality (12) implies that the inequality

$$\delta_x^\top(\dot{M} + AM + MA^\top + MBK + (MBK)^\top)\delta_x \leq -2\lambda\delta_x^\top M\delta_x \qquad (12)$$

holds, for every $(x, \delta_x, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Consequently, the condition defined by the set of equations (7) holds. Thus, $M$ is a sum-separable control-contraction metric for system (1). □

*Proof of Corollary 8.* a □

$M_i(x_i)$

$M_i$

$\delta^\top M(x)\delta = \sum_{i=1}^{N} \delta_i^\top M_i(x_i)\delta_i$

## V. Conclusion

## References


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