# Relaxed and Hybridized Backstepping

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Humberto Stein Shiromoto‡, Vincent Andrieu*, Christophe Prieur‡

***Abstract*—In the present work, we consider nonlinear control systems for which there exist structural obstacles to the design of classical continuous backstepping feedback laws. We conceive feedback laws such that the origin of the closed-loop system is not globally asymptotically stable but a suitable attractor (strictly containing the origin) is practically asymptotically stable. A design method is suggested to build a hybrid feedback law combining a backstepping controller with a locally stabilizing controller. A constructive approach is also suggested employing a differential inclusion representation of the nonlinear dynamics. The results are illustrated for a nonlinear system which, due to its structure, does not have *a priori* any globally stabilizing backstepping controller.**

## I. Introduction

Over the years, research in control of nonlinear dynamical systems has led to many different tools for the design of (globally) asymptotically stabilizing feedbacks. These techniques require particular structures on the systems. Depending on the assumptions, the designer may use different approaches such as high-gain [1], backstepping [2] or forwarding [3]. However, in the presence of unstructured dynamics, these classical design methods may fail.

For systems where the classical backstepping technique can not be applied to render the origin globally asymptotically stable, the approach presented in this work may solve the problem by utilizing a combination of a backstepping feedback law that renders a suitable compact set globally attractive and a locally stabilizing controller. By assumption, this set is contained in the basin of attraction of the system in closed loop with the local controller. The main contribution of this work can be seen as a design of hybrid feedback laws for systems which *a priori* do not have a controller that globally stabilizes the origin. This methodology of hybrid stabilizers is by now well known [4] and it has been also applied for systems that do not satisfy the Brockett's condition ([5] and [6]). Hybrid feedback laws can have the advantage of rendering the equilibrium of the closed-loop system robustly asymptotically stable with respect to measurement noise and actuators' errors ([7] and [8]). We also present a technique to design a continuous local controller satisfying constraints on the basin of attraction of the closed-loop system by using a differential inclusion.

To our best knowledge, this is the first work suggesting a design method to adapt the backstepping technique to a given local controller in the context of hybrid feedback laws. Other works do exist in the context of continuous controllers ([9] and [10]). In contrast to these works, for the class of systems considered in this paper, *a priori* no continuous globally stabilizing controller does exist. Note that we address a different problem than [11], where a synergistic Lyapunov function and a feedback law are designed by backstepping.

In Section II, the stabilization problem under consideration is introduced, as well as the concepts on hybrid feedbacks and the set of assumptions. Based on this set of assumptions, a solution is given in Section III. In Section IV, sufficient conditions are developped to verify the assumptions by using linear matrix inequalities. An illustration is given in Section V. Proofs of the results are presented in Section VI. Concluding remarks are given in Section VII.

**Notation**. The boundary of a set $S \subset \mathbb{R}^n$ is denoted by $\partial S$, its convex hull is denoted by $\mathtt{co}\ (S)$ and its closure by $\bar{S}$. The identity matrix of order $n$ is denoted $I_n$. The null $m \times n$ matrix is denoted $0_{m\times n}$. For two symmetric matrices, $A$ and $B$, $A > B$ (resp. $A < B$) means $A - B > 0$ (resp. $A - B < 0$). Given a continuously differentiable function $f : \mathbb{R}^{n-1}\times\mathbb{R} \to \mathbb{R}^n$, $\partial_x f(\bar{x}, \bar{u})$ (resp. $\partial_u f(\bar{x}, \bar{u})$) stands for the partial derivative of $f$ with respect to $x$ (resp. to $u$) at $(\bar{x}, \bar{u}) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Let $V : \mathbb{R}^{n-1} \to \mathbb{R}$ be a continuously differentiable function, $L_f V(\bar{x}, \bar{u})$ stands for the Lie derivative of $V$ in the $f$-direction at $(\bar{x}, \bar{u}) \in \mathbb{R}^{n-1} \times \mathbb{R}$, i.e., $L_f V(\bar{x}, \bar{u}) := \partial_x V(\bar{x}) \cdot f(\bar{x}, \bar{u})$. Let $c_\ell \in \mathbb{R}_{>0}$ be a constant, we denote sets $\Omega_{c_\ell}(V) = \{x : V(x) < c_\ell\}$. By $r\mathbf{B}_1$ we denote an open ball with radius $r$ and centered at $x_0 = 0$.

## II. Problem statement

*A. Class of systems.* In this work, we consider the class nonlinear systems defined by

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2) + h_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2)u + h_2(x_1, x_2, u), \end{cases} \tag{1}$$

where $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is the state and $u \in \mathbb{R}$ is the input. Functions $f_1$, $f_2$, $h_1$ and $h_2$ are locally Lipschitz continuous satisfying $f_1(0, 0) = 0$, $h_1(0, 0, 0) = 0$, $h_2(0, 0, 0) = 0$ and, $\forall (x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $f_2(x_1, x_2) \neq 0$. In a more compact notation, the vector $(x_1, x_2) \in \mathbb{R}^{n-1}\times\mathbb{R}$ is denoted by $x$, the $i$-th component of $x_1 \in \mathbb{R}^{n-1}$ is denoted by $x_{1,i}$ and system (1) is denoted by $\dot{x} = f_h(x, u)$. When $h_1(x_1, x_2, u) \equiv 0$ and $h_2(x_1, x_2, u) \equiv 0$, system (1) is denoted by $\dot{x} = f(x, u)$.

Consider system $\dot{x} = f(x, u)$, assuming stabilizability of the $x_1$-subsystem and applying backstepping, one may design a feedback law $\varphi_b$ by solving an algebraic equation in the input variable $u$. The solution of this algebraic equation equation renders the origin globally asymptotically stable for $\dot{x} = f(x, \varphi_b(x))$ (e.g. [12]). However, because functions $h_1$ and $h_2$ depend on $u$, the design of a feedback law for (1) leads to an implicit equation in this variable (see (21) below for an example). Thus, backstepping may be difficult (if not impossible) to apply for (1).

This is the motivation to introduce the hybrid feedback law design problem ensuring global asymptotic stability of the origin for (1) in closed loop.

*B. Preliminaries.* In this section, we give a brief introduction on *hybrid feedback laws* $\mathbb{K}$ (see [7] for further

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details). It consists of a finite discrete set $Q$, closed sets $\mathscr{C}_q, \mathscr{D}_q \subset \mathbb{R}^n$ such that, $\forall q \in Q$, $\mathscr{C}_q \cup \mathscr{D}_q = \mathbb{R}^n$, continuous functions $\varphi_q : \mathbb{R}^n \to \mathbb{R}$ and outer semicontinuous set-valued functions $g_q : \mathbb{R}^n \rightrightarrows Q$. The system (1) in closed loop with $\mathbb{K}$ leads to a system with mixed dynamics[1]

$$\left\{ \begin{array}{rcll} \dot{x} & = & f_h(x, \varphi_q(x)), & x \in \mathscr{C}_q, \\ q^+ & \in & g_q(x), & x \in \mathscr{D}_q, \end{array} \right. \tag{2}$$

with state space given by $\mathbb{R}^n \times Q$.

In this work, we consider the framework provided in [13]. A set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is called a *compact hybrid time domain* ([13, Definition 2.3]) if $S = \cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. It is a hybrid time domain if, $\forall (T, J) \in S$, $S \cap ([0, T] \times \{0, \ldots, J\})$ is a compact hybrid time domain. A solution of (2) is a pair of functions $(x, q) : S \times S \to \mathbb{R}^n \times Q$ such that, during flows, $x$ evolves according to the differential equation $\dot{x} = f_h(x, \varphi_q(x))$, while $q$ remains constant, and the constraint $x \in \mathscr{C}_q$ is satisfied. During jumps, $q$ evolves according to the difference inclusion $q^+ \in g_q(x)$, while $x$ remains constant, and before a jump the constraint $x \in \mathscr{D}_q$ is satisfied.

Since, $\forall q \in Q$, each function $\varphi_q$ is continuous and each set-valued function $g_q$ is outer semicontinuous and locally bounded[2], system (2) satisfies the basic assumptions of [14].[3]

### C. Assumptions

**Assumption 1:** *(Local hybrid controller)*
*Let $\hat{Q} \subset \mathbb{N}$ be a finite discrete set, for each $\hat{q} \in \hat{Q}$,*

- *the sets $\mathscr{C}^\ell_{\hat{q}} \subset \mathbb{R}^n$ and $\mathscr{D}^\ell_{\hat{q}} \subset \mathbb{R}^n$ are closed, and $\mathscr{C}^\ell_{\hat{q}} \cup \mathscr{D}^\ell_{\hat{q}} = \mathbb{R}^n$;*
- *$\varphi^\ell_{\hat{q}} : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $g^\ell_{\hat{q}} : \mathbb{R}^n \rightrightarrows \hat{Q}$ is an outer semicontinuous set-valued function;*
- *$V^\ell_{\hat{q}} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a $\mathcal{C}^1$ function satisfying, $\forall x \in \mathbb{R}^n$, $\alpha_1(||x||) \leq V^\ell_{\hat{q}}(x) \leq \alpha_2(||x||)$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Moreover, let $c_\ell > 0$ be a constant, it also satisfies, $\forall \hat{q} \in \hat{Q}$,*

$$\forall x \in (\Omega_{c_\ell}(V^\ell_{\hat{q}}) \cap \mathscr{C}_{\hat{q}}) \setminus \{0\}, \quad L_{f_h} V^\ell_{\hat{q}}(x, \varphi^\ell_{\hat{q}}(x)) < 0, \tag{3}$$

$$\forall x \in (\Omega_{c_\ell}(V^\ell_{\hat{q}}) \cap \mathscr{D}_{\hat{q}}) \setminus \{0\}, \quad V^\ell_{\hat{q}^+}(x) - V^\ell_{\hat{q}}(x) < 0. \tag{4}$$

System (1) in closed loop with the local hybrid controller leads to the hybrid system

$$\left\{ \begin{array}{rcll} \dot{x} & = & f_h(x, \varphi^\ell_{\hat{q}}(x)), & x \in \mathscr{C}^\ell_{\hat{q}}, \\ \hat{q}^+ & \in & g^\ell_{\hat{q}}(x), & x \in \mathscr{D}^\ell_{\hat{q}}. \end{array} \right. \tag{5}$$

Due to [14, Theorem 20], Assumption 1 implies that the set $\{0\} \times \hat{Q}$ (which will be called origin) is locally asymptotically stable for (5). Whenever we are in a neighborhood of the origin, Equation (3) implies that the Lyapunov function $\mathbb{R}^n \times \hat{Q} \ni (x, \hat{q}) \mapsto V_{\hat{q}}(x) \in \mathbb{R}_{\geq 0}$ is strictly decreasing during a flow. Equation (4) implies that, during a transition from a controller $\hat{q}$ to a controller $\hat{q}^+$, the value $V_{\hat{q}}(x)$ strictly decreases to $V_{\hat{q}^+}(x)$.

The second assumption provides bounds for the terms that impeach the direct application of the backstepping method. It also concerns the global stabilizability of the origin for

$$\dot{x}_1 = f_1(x_1, x_2), \tag{6}$$

when $x_2$ is considered as an input.

**Assumption 2:** *(Bounds) There exist a $\mathcal{C}^1$ positive definite and proper function $V_1 : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq 0}$, a $\mathcal{C}^1$ function $\psi_1 : \mathbb{R}^{n-1} \to \mathbb{R}$, a $\mathcal{K}^\infty$ and locally Lipschitz function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

*a) (Stabilizing feedback to (6)): $\forall x_1 \in \mathbb{R}^{n-1}$, $L_{f_1} V_1(x_1, \psi_1(x_1)) \leq -\alpha(V_1(x_1))$;*
*In addition, there exist a continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ and two positive constants, $\varepsilon < 1$ and $M$, satisfying*

*b) (Bound on $h_1$): $\forall (x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, $||h_1(x_1, x_2, u)|| \leq \Psi(x_1, x_2)$ and $L_{h_1} V_1(x_1, \psi_1(x_1), u) \leq (1 - \varepsilon)\alpha(V_1(x_1)) + \varepsilon\alpha(M)$;*

*c) (Bound on $\partial_{x_2} h_1$): $\forall (x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, $||\partial_{x_2} h_1(x_1, x_2, u)|| \leq \Psi(x_1, x_2)$;*

*d) (Bound on $h_2$): $\forall (x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, $||h_2(x_1, x_2, u)|| \leq \Psi(x_1, x_2)$.*

Before introducing the last assumption, let the set $\mathbf{A} \subset \mathbb{R}^n$ be given by

$$\mathbf{A} = \{(x_1, x_2) \in \mathbb{R}^n : V_1(x_1) \leq M, x_2 = \psi_1(x_1)\}. \tag{7}$$

Since function $V_1$, given by Assumption 2, is proper, this set is compact. Moreover, we will prove in Proposition 1 that if Assumption 2 holds, then there exists a feedback law $\varphi_g$ rendering $\mathbf{A}$ globally practically stable for $\dot{x} = f_h(x, \varphi_g(x))$.

**Assumption 3:** *(Inclusion) For each $\hat{q} \in \hat{Q}$, each function $V^\ell_{\hat{q}}$ satisfies $\max_{x \in \mathbf{A}} V^\ell_{\hat{q}}(x) < c_\ell$.*

The first and third assumptions together ensure that, for each $\hat{q} \in \hat{Q}$, $\mathbf{A}$ is included in the basin of attraction of system (5). In the following section, it will be shown that the above assumptions are sufficient to solve the problem under consideration.

## III. Results

Before stating the first result, we recall the concept of global practical asymptotical stability. A compact set $\mathbf{S} \subset \mathbb{R}^n$ containing the origin is *globally practically asymptotically stabilizable* for (1) if, $\forall a \in \mathbb{R}_{>0}$, there exists a controller $\varphi_g$ such the set $\mathbf{S} + a\mathbf{B}_1$ contains a set that is globally asymptotically stable for $\dot{x} = f_h(x, \varphi_g(x))$ ([15]).

**Proposition 1:** *Under Assumption 2, $\mathbf{A}$ is globally practically stabilizable for (1).*

The proof of Proposition 1 is provided in Section VI-A. We can now state the main result.

**Theorem 1:** *Under Assumptions 1, 2 and 3, there exists a continuous controller $\varphi_g : \mathbb{R}^n \to \mathbb{R}$; a suitable choice of a constant value $\tilde{c}_\ell$ satisfying $0 < \tilde{c}_\ell < c_\ell$; a hybrid state feedback law $\mathbb{K}$ defined by $Q := \{1, 2\} \times \hat{Q}$ and, $\forall \hat{q} \in \hat{Q}$, subsets of $\mathbb{R}^n$*

$$\begin{gathered} \mathscr{C}_{1,\hat{q}} = \overline{\Omega_{c_\ell}(V^\ell_{\hat{q}})} \cap \mathscr{C}^\ell_{\hat{q}}, \quad \mathscr{C}_{2,\hat{q}} = \overline{\mathbb{R}^n \setminus \Omega_{\tilde{c}_\ell}(V^\ell_{\hat{q}})}, \\ \mathscr{D}_{2,\hat{q}} = \overline{\Omega_{\tilde{c}_\ell}(V^\ell_{\hat{q}})}, \\ \mathscr{D}_{1,\hat{q}} = (\overline{\Omega_{c_\ell}(V^\ell_{\hat{q}})} \cap \mathscr{D}^\ell_{\hat{q}}) \cup (\overline{\mathbb{R}^n \setminus \Omega_{c_\ell}(V^\ell_{\hat{q}})}) \end{gathered} \tag{8}$$

[1] Continuous and discrete dynamics.

[2] Because $\hat{Q}$ is finite.

[3] The interested reader might also consult [7] and [13] for the sufficient conditions for the existence of solutions and stability (of the origin or subsets of $\mathbb{R}^n$) concepts.

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*and functions*

$$
\begin{array}{c|c}
\begin{array}{rcl}\varphi_{1,\hat{q}}:\mathscr{C}_{1,\hat{q}} & \to & \mathbb{R}\\ x & \mapsto & \varphi^{\ell}_{\hat{q}}(x)\end{array} & \begin{array}{rcl}\varphi_{2,\hat{q}}:\mathscr{C}_{2,\hat{q}} & \to & \mathbb{R}\\ x & \mapsto & \varphi_{g}(x)\end{array}\\
\hline
\multicolumn{2}{c}{\begin{array}{rcl} g_{2,\hat{q}}:\mathscr{D}_{2,\hat{q}} & \rightrightarrows & Q\\ x & \mapsto & \{(1,\hat{q})\}\end{array}}\\
\hline
\end{array}
$$

$$
g_{1,\hat{q}}:\mathscr{D}_{1,\hat{q}} \rightrightarrows Q
$$

$$
x\mapsto\begin{cases}\{(1,g^{\ell}_{\hat{q}}(x))\}, & x\in(\Omega_{c_\ell}(V^{\ell}_{\hat{q}})\cap\mathscr{D}^{\ell}_{\hat{q}})\\ \{(2,\hat{q})\}, & x\in(\mathbb{R}^n\setminus\overline{\Omega_{c_\ell}(V^{\ell}_{\hat{q}})})\\ \{(1,g^{\ell}_{\hat{q}}(x)),(2,\hat{q})\}, & x\in(\partial\Omega_{c_\ell}(V^{\ell}_{\hat{q}})\cap\mathscr{D}^{\ell}_{\hat{q}})\end{cases} \quad (9)
$$

*such that the origin is globally asymptotically stable for system* (1) *in closed loop with* $\mathbb{K}$.

Theorem 1 is more than an existence result since its proof allows one to design a hybrid feedback law that globally stabilizes the origin. The complete proof of Theorem 1 is presented in Section VI-B.

*Remark 1: These results are useful for systems that do not satisfy the Brockett necessary condition for the existence of a continuous stabilizing controller and for which there exists a locally stabilizing hybrid feedback law (see e.g. [14, Example 38] and [16]).*

The problems concerning the design of a feedback law by backstepping to (1) that renders $\mathbf{A}$ globally practically stable and blends different feedback laws according to each basin of attraction is solved. In the next section, under Assumption 2, a local continuous feedback law satisfying Assumptions 1 and 3 is designed using an over-approximation of (1).

## IV. A method to design a local stabilizing controller

Based on the approach presented on [17], we formulate the nonlinear dynamics of (1) in terms of a Linear Differential Inclusion. Let $\hat{Q}$ be a singleton, we start by defining a neighborhood $\mathbf{V}_\mu$ of the origin such that: a) there exist a continuous feedback law $\varphi^{\ell}_{\hat{q}}$ and a Lyapunov function $V^{\ell}_{\hat{q}}$ satisfying, $\forall x\in\mathbf{V}_\mu\setminus\{0\}$, $L_{f_h}V^{\ell}_{\hat{q}}(x,\varphi^{\ell}_{\hat{q}}(x))<0$; b) it strictly contains an estimation of the basin of attraction of $\dot{x}=f_h(x,\varphi^{\ell}_{\hat{q}}(x))$ and a convex set containing $\mathbf{A}$. These two set inclusions follow from two over-approximations of $\mathbf{A}$.

Under Assumption 2, there exist[4] a finite set $\mathscr{P}\subset\mathbb{N}$ of indexes and $\{x_p\}_{p\in\mathscr{P}}$ vectors of $\mathbb{R}^n$ such that

$$
\mathbf{A}\subset \mathsf{co}\ (\{x_p\}_{p\in\mathscr{P}}). \quad (10)
$$

Let $\mu_u>0$ be a constant and $\mu=[\mu_1,\mu_2,\dots,\mu_n]\in\mathbb{R}^n$ be a vector of positive values such that $\mathsf{co}\ (\{x_p\}_{p\in\mathscr{P}})\subset \mathbf{V}_\mu=\{x:|x_i|\le\mu_i, i=1,2,\dots,n\}$.

Consider the function

$$
\tilde{f}_h(x,u)=f_h(x,u)-Fx-Gu, \quad (11)
$$

where $F$ and $G$ are the linearization of (1) around the origin:

$$
\dot{x}=Fx+Gu:=\partial_x f_h(0)x+\partial_u f_h(0)u. \quad (12)
$$

Since $f_h$ is $\mathcal{C}^1$, $\tilde{f}_h$ is also $\mathcal{C}^1$. In the following, an elementwise over-approximation is made of the matrices in (12). For each $l\in\mathscr{L}:=\{l\in\mathbb{N}:1\le l\le 2^{n^2}\}$, let $C_l\in\mathbb{R}^{n\times n}$ be a matrix with components given by either

$$
\begin{aligned} c^{+}_{ij} &= \max_{x\in\mathbf{V}_\mu,|u|\le\mu_u}\partial_{x_j}\tilde{f}_{h,i}(x,u) \text{ or}\\ c^{-}_{ij} &= \min_{x\in\mathbf{V}_\mu,|u|\le\mu_u}\partial_{x_j}\tilde{f}_{h,i}(x,u).\end{aligned} \quad (13)
$$

For each $m\in\mathscr{M}:=\{m\in\mathbb{N}:1\le m\le 2^n\}$, let $D_m\in\mathbb{R}^{n\times 1}$ be a vector with components given by either

$$
\begin{aligned} d^{+}_{i} &= \max_{x\in\mathbf{V}_\mu,|u|\le\mu_u}\partial_{u}\tilde{f}_{h,i}(x,u) \text{ or}\\ d^{-}_{i} &= \min_{x\in\mathbf{V}_\mu,|u|\le\mu_u}\partial_{u}\tilde{f}_{h,i}(x,u).\end{aligned} \quad (14)
$$

For each $i\in\mathscr{I}:=\{i\in\mathbb{N}:1\le i\le n\}$, the mean value theorem ensures, for all $x\in\mathbf{V}_\mu$ and $|u|\le\mu_u$, the existence of points $\overline{x}$ and $\overline{u}$ satisfying $\tilde{f}_{h,i}(x,u)=\partial_x\tilde{f}_{h,i}(\overline{x},\overline{u})x_i+\partial_u\tilde{f}_{h,i}(\overline{x},\overline{u})u$. This implies that, for all $x\in\mathbf{V}_\mu$, $|u|\le\mu_u$, $l\in\mathscr{L}$ and $m\in\mathscr{M}$, (1) may be over-approximated by

$$
\dot{x}\in\mathsf{co}\ \{(F+C_l)x+(G+D_m)u\}. \quad (15)
$$

*Remark 2: This linear differential inclusion allows us to go further than the linearization* (12) *because we take into account the gradient of the nonlinear terms. The precision of this over-approximation method depends basically on two aspects: the size of the neighborhood* $\mathbf{V}_\mu$ *considered and the rate of change of the nonlinear terms* $\tilde{f}_h$.

Let us consider the canonical basis in $\mathbb{R}^n$, i.e., the set of vectors $\{e_s\}^n_{s=1}$, where the components are all 0 except the $s$-th one which is equals to 1.

**Proposition 2:** *Assume that there exist a symmetric positive definite matrix* $W\in\mathbb{R}^{n\times n}$ *and a matrix* $H\in\mathbb{R}^{n\times 1}$ *satisfying, for all* $l\in\mathscr{L}$ *and* $m\in\mathscr{M}$,

$$
\begin{aligned} W(F+C_l)^T+H(G+D_m)^T+(F+C_l)W&\\ +(G+D_m)H^T<0,&\end{aligned} \quad (16)
$$

$$
\begin{bmatrix}\mu_s^2W & We_s\\ * & 1\end{bmatrix}\ge 0, s=1,2,\dots,n, \quad (17)
$$

$$
\begin{bmatrix}1 & x_p^T\\ * & W\end{bmatrix}\ge 0,\quad p\in\mathscr{P}, \quad (18)
$$

*and*

$$
\begin{bmatrix}\mu_u^2W & H\\ * & 1\end{bmatrix}\ge 0. \quad (19)
$$

*Then, by letting* $\hat{Q}=\{1\}$, $V^{\ell}_1(x)=x^TPx$, *where* $P=W^{-1}$, $c_\ell=1$, $\mathscr{C}^{\ell}_1=\overline{\Omega_1(V^{\ell}_1)}$, $\mathscr{D}^{\ell}_1=\overline{\mathbb{R}^n\setminus\mathscr{C}^{\ell}_1}$, $g^{\ell}_1(x)\equiv 1$ *and* $\varphi^{\ell}_1(x)=Kx$, *where* $K=H^TP$, *Assumptions 1 and 3 hold.*

The proof of Proposition 2 is provided in Section VI-C.

## V. Illustration

Let us consider a class of systems given by

$$
\begin{cases}\dot{x}_1 &=& x_1+x_2+\theta[x_1^2+(1+x_1)\sin(u)]\\ \dot{x}_2 &=& u,\end{cases} \quad (20)
$$

where $\theta\in\mathbb{R}_{>0}$ is a constant. We will show in the following that, due to the presence of the term $\theta(1+x_1)\sin(u)$ in the time-derivative of $x_1$, it is not possible to apply the backstepping technique to design a feedback law. Let $f_1(x_1,x_2)=x_1+x_2+\theta x_1^2$, $f_2(x_1,x_2)\equiv 1$, $h_1(x_1,x_2,u)=\theta(1+x_1)\sin(u)$ and $h_2(x_1,x_2,u)\equiv 0$. Before applying Proposition 1 and Theorem 1, we check their assumptions.

### A. Checking assumptions for (20)

*1) Assumption 2:* It is possible to check that item a) holds with $V_1(x_1)=x_1^2/2$, $\psi_1(x_1)=-(1+K_1)x_1-\theta x_1^2$,

[4]Because $\mathbf{A}$ is a compact set.

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and $\alpha(s) = 2K_1 s$, where $K_1 \in \mathbb{R}_{>0}$ is constant. Items b)-d) hold with $\Psi(x_1, x_2) = \theta(1 + |x_1|)$, $\varepsilon \leq 1 - 3\theta/(2K_1)$ and[5] $M \geq \theta/(4K_1\varepsilon)$.

Since Assumption 2a) holds, we could try to apply the backstepping technique directly in (20). Following this procedure, we intend to use $V(x_1, x_2) = V_1(x_1) + (x_2 - \psi_1(x_1))^2/2$ as a control Lyapunov function for (20). Taking its Lie derivative, algebraic computations yield, $\forall (x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} &L_{f_h} V(x_1, x_2, u) \leq -K_1 x_1^2 + x_1\theta(1 + x_1) \cdot \sin(u) \\ &+(x_2 - \psi_1(x_1))(u + x_1/2 + (1 + K_1 + 2\theta K_1 x_1) \\ &\cdot (x_1 + x_2 + \theta[x_1^2 + (1 + x_1) \cdot \sin(u)])). \end{aligned} \tag{21}$$

In order to have a term proportional to $(x_2 - \psi_1(x_1))^2$ in the right-hand side of (21), we should solve an implicit equation in the variable $u$ defined as $E(x_1, x_2, u) = -K_1 x_1^2 + L(x_2 - \psi_1(x_1))^2$, where $E$ is the right-hand side of (21) and $L > 0$ is a constant. Since this seems to be difficult (if not impossible), it motivates the design a hybrid feedback by applying Theorem 1.

*2) Assumptions 1 and 3:* From the previous definitions of $V_1$ and $\psi_1$, we get $\mathbf{A} = \{(x_1, x_2) : |x_1| < \sqrt{2M}, x_2 = -(1 + K_1)x_1 - \theta x_1^2\}$.

Let us first establish sets $\mathscr{P}$ and $\{x_p\}_{p\in\mathscr{P}}$ such that (10) holds. Define constants $a^+ = \max_{|x_1|<\sqrt{2M}} -(1 + K_1) - 2\theta x_1 = -(1+K_1)+2\theta\sqrt{2M}$ and $a^- = \min_{|x_1|<\sqrt{2M}} -(1 + K_1) - 2\theta x_1 = -(1 + K_1) - 2\theta\sqrt{2M}$ by computing the derivatives of $\psi_1$ and let $\mathscr{P} = \{1, 2, 3, 4\}$. From the mean value theorem we have, $\forall x \in \mathbf{A}$, $a^- \cdot x_1 \leq x_2 \leq a^+ \cdot x_1$. This implies that

$$\begin{aligned} \mathbf{A} \subseteq \mathsf{co}\ ((\{\sqrt{2M}\} \times \{x_2^{+,<0}, x_2^{-,<0}\}) \\ \cup(\{-\sqrt{2M}\} \times \{x_2^{+,>0}, x_2^{-,>0}\})), \end{aligned} \tag{22}$$

where

$$\begin{aligned} x_2^{+,>0} &= -a^+\sqrt{2M}, \quad x_2^{+,<0} = a^+\sqrt{2M}, \\ x_2^{-,>0} &= -a^-\sqrt{2M}, \quad x_2^{-,<0} = a^-\sqrt{2M}. \end{aligned} \tag{23}$$

Figure 1 shows this inclusion.

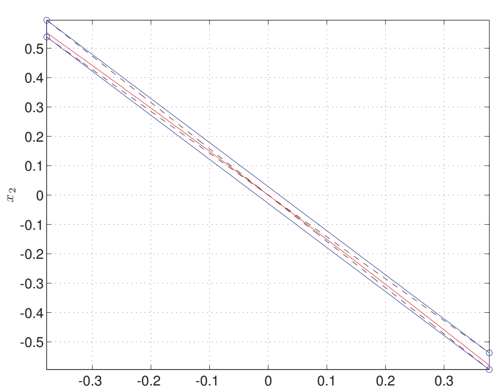

FIGURE 1: The sets $\mathbf{A}$ (in red) and the convex set defined in (22) (in blue) are presented in solid line. The circles are the vertexes of this set. The dashed straight lines which bound $\mathbf{A}$ are given by functions $x_1 \mapsto a^+ x_1$ and $x_1 \mapsto a^- x_1$.

[5] With $\varepsilon \leq 1 - 3\theta/(2K_1)$ and the condition $\varepsilon > 0$, we get the lower bound for $K_1 > \frac{3\theta}{2}$.

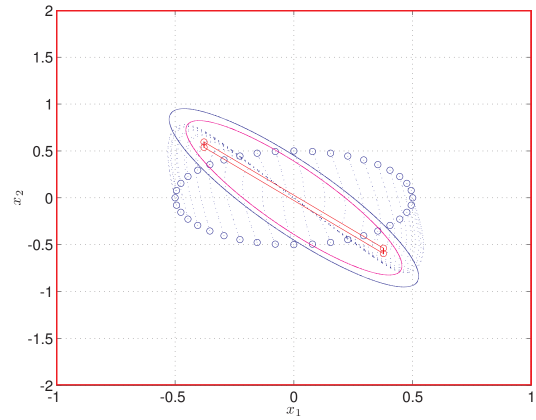

FIGURE 2: The sets $\mathbf{V}_\mu$ (in red), $\partial\Omega_1(x^T P x)$ (in blue), and the inclusion (22) (in red) at the center. Initial conditions are points given in a ball of radius 0.5 and centered at the origin.

A necessary condition for feasibility of the Linear Matrix Inequalites of Proposition 2 is $\mathbf{A} \subset \mathbf{V}_\mu$. This follows from the inequalities $\sqrt{2M} < \mu_1$ and $|a^\pm\sqrt{2M}| < \mu_2$. These inequalities imply that $K_1$ must satisfy

$$\frac{\theta}{2}\left(\frac{1}{\mu_1^2} + 3\right) < K_1 < \frac{\mu_2}{\mu_1} - 2\theta\mu_1 - 1. \tag{24}$$

*Remark 3: Equation* (24) *imposes a limitation on the speed of response, since $K_1$ is lower and upper bounded.*

Let $\theta = 0.1$, applying the technique presented in Section IV we define $\mu = [1, 2]$, $\mathbf{V}_\mu = \{(x_1, x_2) : |x_1| < 1, |x_2| < 2\}$ and $|u| < 2\pi$. Moreover, letting $K_1 = 0.5$ we get that (24) holds. From Assumption 2 and with this choice for $K_1$, we let $M = 0.0714$, and $\varepsilon = 0.7$.

The matrices $F$ and $G$ defined in (12) are given by $F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$ while the matrices $\{C_l\}_{l=1}^2$ and $\{D_m\}_{m=1}^2$ have elements defined by (13) and (14). The matrices that are not null are given by $C_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} -0.3 & 0 \\ 0 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$, and $D_2 = \begin{bmatrix} -0.3 \\ 1 \end{bmatrix}$. Applying Proposition 2 and using SeDuMi 1.3, we get $P = \begin{bmatrix} 16.1210 & 7.8345 \\ 7.8345 & 4.9138 \end{bmatrix}$ and $K = \begin{bmatrix} -11.2361 & -6.6087 \end{bmatrix}$.

Figure 2 shows some solutions of system (20) in closed loop with the feedback law $\varphi_\ell$ , the inclusions $\mathbf{A} \subset \overline{\Omega_1(V_1^\ell)}$ and $\overline{\Omega_1(V_1^\ell)} \subset \mathbf{V}_\mu$. From Proposition 2, Assumptions 1 and 3 hold with $c_\ell = 1$.

*B. Construction of the hybrid feedback law*

Since Assumption 2 holds, we get from the proof of Proposition 1 that the feedback law given by $\varphi_g(x_1, x_2) = -(1 + K_1 + 2\theta x_1)(x_1 + \theta x_1^2 + x_2) - x_1/(2K_V) - (x_1 - \psi_1(x_1))[c + c\Delta(x_1, x_2)^2/4]/K_V$, where $\Delta(x_1, x_2) = |x_1|\theta(1 + |x_1|) + K_V\theta(1 + |x_1|)(1 + |1 + K_1 + 2\theta x_1|)$, $a = 0.01$, $c = 10$ and $K_V = 1.6286 \times 10^3$, the set $\mathbf{A} + a\mathbf{B}_1$ contains a set that is globally asymptotically stable for $\dot{x} = f_h(x, \varphi_g(x))$. Now applying Theorem 1, we may design a hybrid feedback law $\mathbb{K}$. Let $\tilde{c}_\ell = 0.75$ the hybrid controller $\mathbb{K}$ defined by (8)-(9) in Theorem 1 is such that the origin is globally asymptotically stable for (20) in closed loop.

Consider a simulation with initial condition $(x_1, x_2, q) = (2, 0, 1)$. Figure 3 shows the time evolution of the $x_1$,

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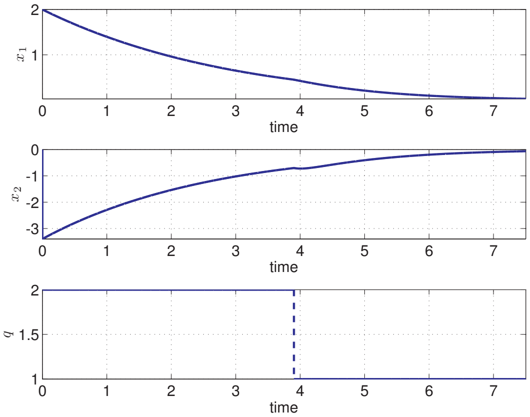


FIGURE 3

$x_2$ and $q$ components[6] of the solution of (20) in closed loop with $\mathbb{K}$. Firstly, (20) is in closed loop with $\varphi_g$ (for $t \in [0, 3.9]$), then (20) is in closed loop with $\varphi_\ell$, and the solution converges to the origin.

## VI. Proofs

### A. Proof of Proposition 1

In order to prove Proposition 1, the following lemma is required:

**Lemma 3:** *There exist positive constants $a'$ and $K_V$ and a function*

$$\begin{array}{rcl} V : \mathbb{R}^{n-1} \times \mathbb{R} & \to & \mathbb{R} \\ (x_1, x_2) & \mapsto & V_1(x_1) + K_V(x_2 - \varphi_1(x_1))^2 \end{array} \tag{25}$$

*such that the set $\Omega_{a'}(V)$ satisfies the inclusion*

$$\Omega_{a'}(V) \subset \mathbf{A} + a\mathbf{B}_1. \tag{26}$$

*Proof:* Consider the sequence of functions $V_k(x_1, x_2) = V_1(x_1) + k(x_1 - \varphi_1(x_1))^2$ and of values $a'_k = 1/k$. To prove this lemma by contradiction, assume that, $\forall k > 0$, inclusion (26) does not hold. In this case, for each $k > 0$, there exists a sequence $(x_{1,k}, x_{2,k})$ such that $V_k(x_{1,k}, x_{2,k}) \leq a'_k$ and $(x_{1,k}, x_{2,k}) \notin \mathbf{A} + a\mathbf{B}_1$. Note that, we have

$$\left\{ \begin{array}{rcccl} V_1(x_{1,k}) & < & M + \frac{1}{k} & < & 2M \\ (x_{2,k} - \varphi_1(x_{1,k}))^2 & < & \frac{M}{k} + \frac{1}{k^2} & < & 2M. \end{array} \right. \tag{27}$$

The function $V_1$, being proper, yields the sequence $(x_{1,k}, x_{2,k})_{k\in\mathbb{N}}$ belongs to a compact subset. Hence, we can extract a converging subsequence $(x_{1,l}, x_{2,l})$ with $\lim_{l\to\infty}(x_{1,l}, x_{2,l}) = (x_1^*, x_2^*)$. We have, with (27), $V_1(x_1^*) \leq M$ and $x_2^* - \varphi_1(x_1^*) = 0$. Hence $(x_1^*, x_2^*) \in \mathbf{A}$. This contradicts the fact that $(x_{1,l}, x_{2,l}) \notin \mathbf{A} + a\mathbf{B}_1$. Consequently, $\exists a'$ and $K_V$ such that (26) holds. ■

We are now able to prove Proposition 1.

*Proof:* Let $a \in \mathbb{R}_{>0}$ be a constant. We will show that there exists a continuous controller $\varphi_g$ rendering $\mathbf{A} + a\mathbf{B}_1$ globally asymptotically stable for $\dot{x} = f_h(x, \varphi_g(x))$.

Define $r_1(x_1, x_2, u) := f_1(x_1, x_2) + h_1(x_1, x_2, u)$. From items a) and b) of Assumption 2 we get, $\forall (x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$\begin{array}{l} L_{r_1}V_1(x_1, x_2, u) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] \\ \qquad\qquad + L_{r_1}V_1(x_1, x_2, u) - L_{r_1}V_1(x_1, \psi_1(x_1), u). \end{array} \tag{28}$$

Defining, $\forall s \in [0, 1]$, $\eta_{x_1,x_2}(s) = sx_2 + (1 - s)\psi_1(x_1)$, we have $\partial_s r_1(x_1, \eta_{x_1,x_2}(s), u) = \partial_{x_2} r_1(x_1, \eta_{x_1,x_2}(s), u)(x_2 - \psi_1(x_1))$. This implies that $r_1(x_1, x_2, u) - r_1(x_1, \psi_1(x_1), u) = (x_2 - \psi_1(x_1)) \cdot \int_0^1 \partial_{x_2} r_1(x_1, \eta_{x_1,x_2}(s), u)\, ds$. Hence, Equation (28) becomes $L_{r_1}V_1(x_1, x_2, u) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + \partial_{x_1}V_1(x_1)(x_2 - \psi_1(x_1)) \cdot \int_0^1 \partial_{x_2} r_1(x_1, \eta_{x_1,x_2}(s), u)\, ds$.

Let $\tilde{\psi}$ be the feedback law defined, $\forall (x_1, x_2, \bar{u}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, by

$$\begin{array}{l} \tilde{\psi}(x_1, x_2, \bar{u}) = \frac{1}{f_2(x_1,x_2)} \left[ \frac{\bar{u}}{K_V} + L_{f_1}\psi_1(x_1, x_2) \right. \\ \qquad \left. - \frac{1}{K_V} \partial_{x_1}V_1(x_1) \cdot \int_0^1 \partial_{x_2} f_1(x_1, \eta_{x_1,x_2}(s))\, ds \right], \end{array} \tag{29}$$

where $K_V$ is given by Lemma 3. Letting $u = \tilde{\psi}(x_1, x_2, \bar{u})$, Equation (25) implies that $L_{f_h}V(x_1, x_2, \tilde{\psi}(x_1, x_2, \bar{u})) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + (x_2 - \psi_1(x_1))[\bar{u} + \Upsilon(x_1, x_2, \tilde{\psi}(x_1, x_2, \bar{u}))]$, with $\Upsilon(x_1, x_2, \tilde{\psi}(x_1, x_2, \bar{u})) = \partial_{x_1}V_1(x_1) \cdot \int_0^1 \partial_{x_2} h_1(x_1, \eta_{x_1,x_2}(s), \tilde{\psi}(x_1, x_2, \bar{u}))\, ds + K_V h_2(x_1, x_2, \tilde{\psi}(x_1, x_2, \bar{u})) - K_V L_{h_1}\psi_1(x_1, x_2, \tilde{\psi}(x_1, x_2, \bar{u}))$. From Items b)-d) of Assumption 2, $\Upsilon$ satisfies $|\Upsilon(x_1, x_2, \tilde{\psi}(x_1, x_2, \bar{u}))| \leq \Delta(x_1, x_2)$, where

$$\begin{array}{l} \Delta(x_1, x_2) = ||\partial_{x_1}V_1(x_1)|| \int_0^1 \Psi(x_1, \eta_{x_1,x_2}(s))\, ds \\ \qquad\qquad + K_V \Psi(x_1, x_2)(1 + ||\partial_{x_1}\psi_1(x_1)||). \end{array} \tag{30}$$

From Cauchy-Schwartz inequality we get, for each positive constant $c$ and $\forall (x_1, x_2, \bar{u}) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, $(x_2 - \psi_1(x_1))\Upsilon(x_1, x_2, \bar{u}) \leq \frac{1}{c} + \frac{c}{4}(x_2 - \psi_1(x_1))^2\Delta(x_1, x_2)^2$. Taking

$$\bar{u} = \tilde{u} := -(x_2 - \psi_1(x_1)) \left[ c + \tfrac{c}{4}\Delta(x_1, x_2)^2 \right] \tag{31}$$

it yields, for all $(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $c \geq 1$,

$$\begin{array}{l} L_{f_h}V(x_1, x_2, \tilde{\psi}(x_1, x_2)) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] \\ \qquad\qquad\qquad + \frac{1}{c} - c(x_2 - \psi_1(x_1))^2, \end{array} \tag{32}$$

where, in order to simplify the presentation, we denoted $\tilde{\psi}(x_1, x_2, \tilde{u})$ by $\tilde{\psi}(x_1, x_2)$.

Since $V_1$ is proper, the set $\mathbf{A}_{\geq 0} = \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}^n : \varepsilon\alpha(V_1(x_1)) + (x_2 - \psi_1(x_1))^2 \leq \varepsilon\alpha(M) + 1\}$, is compact. Let $\zeta = \max_{(x_1,x_2)\in\mathbf{A}_{\geq 0}}\{V(x_1, x_2)\}$, for all $c > 1$ and $(x_1, x_2) \in \mathbb{R}^n \setminus \overline{\Omega_\zeta(V)}$, we get $L_{f_h}V(x_1, x_2, \psi(x_1, x_2)) < 0$. In other words, $\overline{\Omega_\zeta(V)}$ is globally asymptotically stable for $\dot{x} = f_h(x, \psi(x))$.

Let $K_\alpha > 0$ be the Lipschitz constant of $\alpha$ in the compact set $[0, \zeta]$. Hence, $\forall (x_1, x_2) \in \overline{\Omega_\zeta(V)}$, $|\alpha(V_1(x_1)) - \alpha(V(x_1, x_2))| \leq \frac{K_V K_\alpha}{2}(x_2 - \psi_1(x_1))^2$.

From (32) we get, for all $c > 1$ and $(x_1, x_2) \in \overline{\Omega_\zeta(V)}$, $L_{f_h}V(x_1, x_2, \tilde{\psi}(x_1, x_2)) \leq \varepsilon[\alpha(M) - \alpha(V(x_1, x_2))] + \frac{1}{c} - \left(c - \varepsilon\frac{K_V K_\alpha}{2}\right)(x_2 - \psi_1(x_1))^2$.

Consider $a'$ given by Lemma 3 and let $c_g = \max\left\{\frac{1}{\varepsilon[\alpha(a') - \alpha(M)]}, \varepsilon\frac{K_V K_\alpha}{2}, 1\right\}$ it gives, for all $c > c_g$ and $(x_1, x_2) \in \overline{\Omega_\zeta(V)}$, $L_{f_h}V(x_1, x_2, \tilde{\psi}(x_1, x_2)) \leq \varepsilon\left[\alpha(a') - \alpha(V(x_1, x_2))\right]$. Thus, with $c > c_g$, $\forall (x_1, x_2) \in \mathbb{R}^n \setminus \overline{\Omega_{a'}(V)}$, we get $L_{f_h}V(x_1, x_2, \tilde{\psi}(x_1, x_2)) < 0$. Therefore, the set $\Omega_{a'}(V)$ is an attractor for $\dot{x} = f_h(x, \tilde{\psi}(x))$. Together with (26), we conclude that $\mathbf{A} + a\mathbf{B}_1$ contains $\Omega_{a'}(V)$ which is globally asymptotically stable with the control $\varphi_g(x) = \tilde{\psi}(x)$ and $c > c_g$ given by (29) and (31), that is $\varphi_g(x_1, x_2) = \frac{1}{K_V f_2(x_1,x_2)}[K_V L_{f_1}\psi_1(x_1, x_2) - \partial_{x_1}V_1(x_1) \cdot \int_0^1 \partial_{x_2} f_1(x_1, \eta_{x_1,x_2}(s))\, ds - (x_2 - \psi_1(x_1)) \cdot (c + \frac{c}{4}\Delta^2(x_1, x_2))]$.This concludes the proof of Proposition 1. ■

[6]Regarding $q$, here it is shown only its first component, because the second one does not change.

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### B. Proof of Theorem 1

*Proof:* Under Assumption 2, Proposition 1 can be applied. This allows one to consider a constant $a$ and also choose two constant values[7] $0 < \tilde{c}_\ell < c_\ell$ such that, for each $\hat{q} \in \hat{Q}$,

$$\max_{x \in \mathbf{A}+a\mathbf{B}_1} V^\ell_{\hat{q}}(x) < \tilde{c}_\ell \ . \tag{33}$$

Let us consider the controller $\varphi_g$ given by the proof of Proposition 1 and use it to design a hybrid feedback law $\mathbb{K}$ building an hysteresis of local controllers $\varphi^\ell_{\hat{q}}$ and $\varphi_g$ on appropriate domains (see also [14, Page 51] or [4] for similar concepts applied to different control problems). Define $Q = \{1, 2\} \times \hat{Q}$. Consider the subsets (8) and the functions defined in (9). The state of system (1) in closed loop with $\mathbb{K}$ is $(x, q) \in \mathbb{R}^n \times Q$.

Case 1. Assume that $q = (2, \hat{q})$.

i. If $x \in \mathscr{C}_{2,\hat{q}}$. Then from (9), we have $\varphi_{2,\hat{q}}(x) = \varphi_g(x)$. From Assumptions 2 and Proposition 1, $\mathbf{A}$ is globally practically asymptotically stable for $\dot{x} = f_h(x, \varphi_g(x))$ and $\mathbf{A} \subset \mathscr{D}_{2,\hat{q}}$. Moreover, the solution will not jump until the $x$ component enters in $\mathscr{D}_{2,\hat{q}}$;

ii. If $x \in \mathscr{D}_{2,\hat{q}}$. Then from (9), we have $g_{2,\hat{q}}(x) = \{(1, \hat{q})\}$ and, after the jump, the local hybrid controller is selected. Since the value of $x$ does not change during a jump and $\mathscr{D}_{2,\hat{q}} \subset \Omega_{c_\ell}(V^\ell_{\hat{q}})$, it follows from Assumption 1, that the origin is locally asymptotically stable for (5).

Case 2. Assume that $q = (1, \hat{q})$.

i. If $x \in \mathscr{D}_{1,\hat{q}}$. Then from (8) and (9), we have either

i.a. $q^+ = (2, \hat{q})$ and, after the jump, the global controller $\varphi_g$ is selected. Since before this jump, the $x$ component must be inside $\overline{\mathbb{R}^n \setminus \Omega_{c_\ell}(V^\ell_{\hat{q}})}$, $\overline{\mathbb{R}^n \setminus \Omega_{c_\ell}(V^\ell_{\hat{q}})} \subset \mathscr{C}_{2,\hat{q}}$ and the $x$ component does change after the jump, the solution follows the behavior prescribed by Case 1.i., after the jump; or i.b. $q^+ = (1, g^\ell_{\hat{q}}(x))$ and, after the jump, a local controller is selected. Since before this jump, the $x$ component must be inside $x \in \overline{\Omega_{c_\ell}(V^\ell_{\hat{q}})} \cap \mathscr{D}^\ell_{\hat{q}}$ and the $x$ component does change after the jump, it follows from Assumption 1 that the solution converges to the origin;

ii. If $x \in \mathscr{C}_{1,\hat{q}}$. Then from (9), we have $\varphi_{1,\hat{q}}(x) = \varphi^\ell_{\hat{q}}(x)$. From Assumption 1, the origin is locally asymptotically stable for (5). Thus, the origin is locally stable and globally attractive. This concludes the proof. ■

### C. Proof of Proposition 2

*Proof:* Equation (16) may be rewritten in terms of a Linear Matrix Inequality in the matrix variables (2) and $W$ given by $W(F + C_l)^T + H(G + D_m)^T + (F + C_l)W + (G + D_m)H^T < 0$. Multiplying this equation at left by and right by a symmetric positive definite matrix $P$ we get, $\forall l \in \mathscr{L}$, $\forall m \in \mathscr{M}$, and $\forall x \in \mathbf{V}_\mu$, $x \neq 0$, $x^T(F + C_l + (G + D_m)K)^T Px + x^T P(F + C_l + (G + D_m)K)x < 0$.

Equation (17) is equivalent to $\mu_s^2 W - W e_s e_s^T W^T > 0$, $\forall s = 1, 2, \ldots, n$. For each $s = 1, 2, \ldots, n$, this inequality implies that $x^T e_s e_s^T x < \mu_s^2 x^T Px \leq \mu_s^2, \forall x \in \overline{\Omega_1(V^\ell_{\hat{q}})}$. Since $e_s \cdot x = x_s$, we get $x_s^2 < \mu_s^2$, $\forall x \in \overline{\Omega_1(V^\ell_{\hat{q}})}$, $s = 1, 2, \ldots, n$. In other words, $\overline{\Omega_1(V^\ell_{\hat{q}})} \subset \mathbf{V}_\mu$. Equation (18) implies that $x_p^T Px_p \leq 1, \forall p \in \mathscr{P}$ and thus $\mathrm{co}\,(\{x_p\}_{p \in \mathscr{P}}) \subset \overline{\Omega_1(x^T Px)}$. By Schur complement, (19) is equivalent to $\mu_u^2 W - HH^T \geq 0$. This implies $\mu_u^2 W^{-T} \geq K^T K$. Then, $\forall x \in \overline{\Omega_1(V^\ell_{\hat{q}})}$, $x^T K^T Kx \leq \mu_u^2 x^T Px \leq \mu_u^2$. It concludes the proof of Proposition 2. ■

## VII. Conclusion

A design of hybrid feedback laws method has been presented in this paper to combine a backstepping controller with a local feedback law. This allows us to define a stabilizing control law for nonlinear control systems for which the backstepping design procedure can not be applied to globally stabilize the origin. We have also developed conditions to check the assumptions needed for the presented results. In a future work, the authors intend to use these techniques for other classes of nonlinear systems (e.g., cascade systems or in forwarding form).

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[7]Such values exist since Assumptions 2 and 3 hold, and since $V^\ell_{\hat{q}}$ is a proper function.