Hull-White One-Factor Model

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架構

利率模型介紹

Hull-White
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利率模型介紹

利率模型簡介

One-Factor

Multi-Factor

risk

無風險利率
(short rate)

除無風險利率外之因子，
利用PCA挑選影響最大的前
幾個因子放入模型

ex: level, slope, curvature changes

動態利率模型分類

均衡模型 (Equilibrium Model)	無套利模型 (No-Arbitrage Model)
基於流動性偏好理論建立	基於預期理論建立
Vasicek (1977)	Hull-White (1990)
CIR (1985)	HJM (1990, 1992)

" In an equilibrium model, today's term structure of interest rates is an output.
In a no-arbitrage model, today's term structure of interest rates is an input."
-- John C. Hull, "Options, Futures, and Other Derivatives", 8^{th} ed, p.690

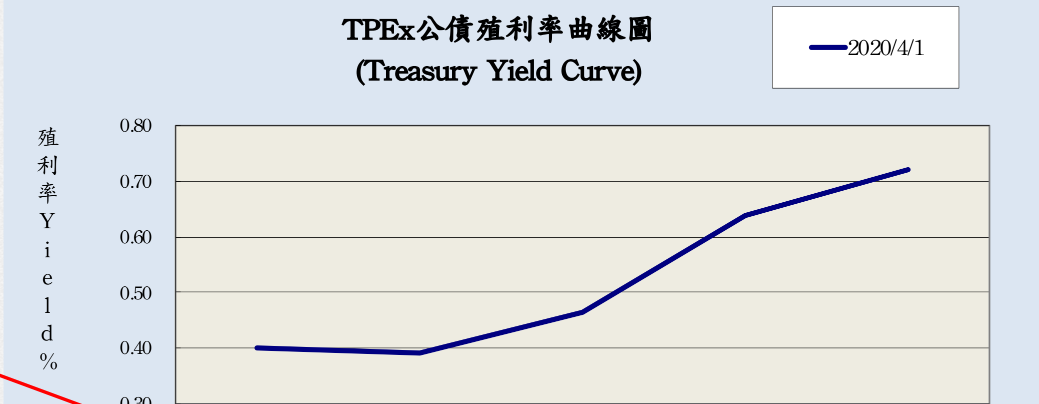
02

Hull-White One-Factor Model

背景

- 假設現在在時間 $t = 0$
- 即期利率 Spot Rate：$R(t,T)$

R(0,2)

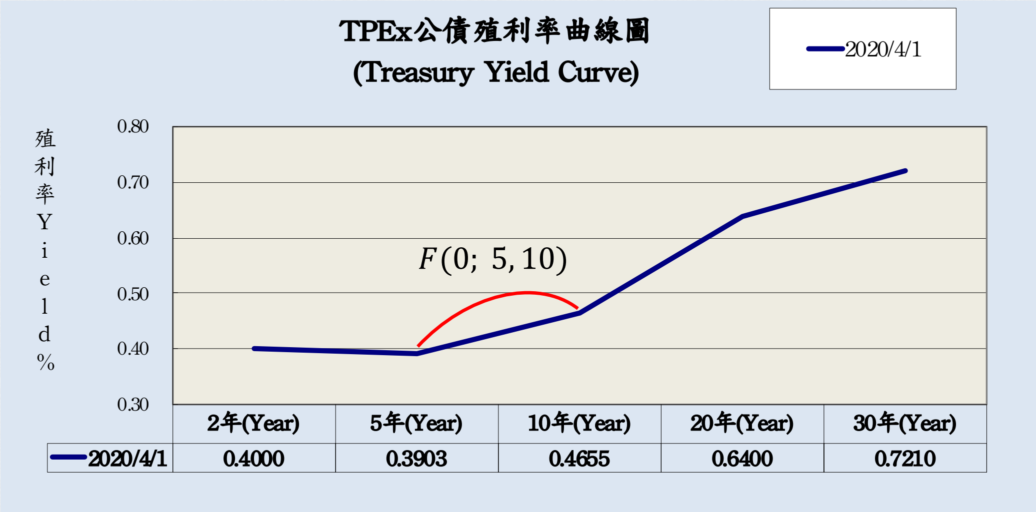


	2年(Year)	5年(Year)	10年(Year)	20年(Year)	30年(Year)
2020/4/1	0.4000	0.3903	0.4655	0.6400	0.7210

資料來源：台灣證券櫃買中心

背景

- 遠期利率 Forward Rate：$F(t; S, T)$

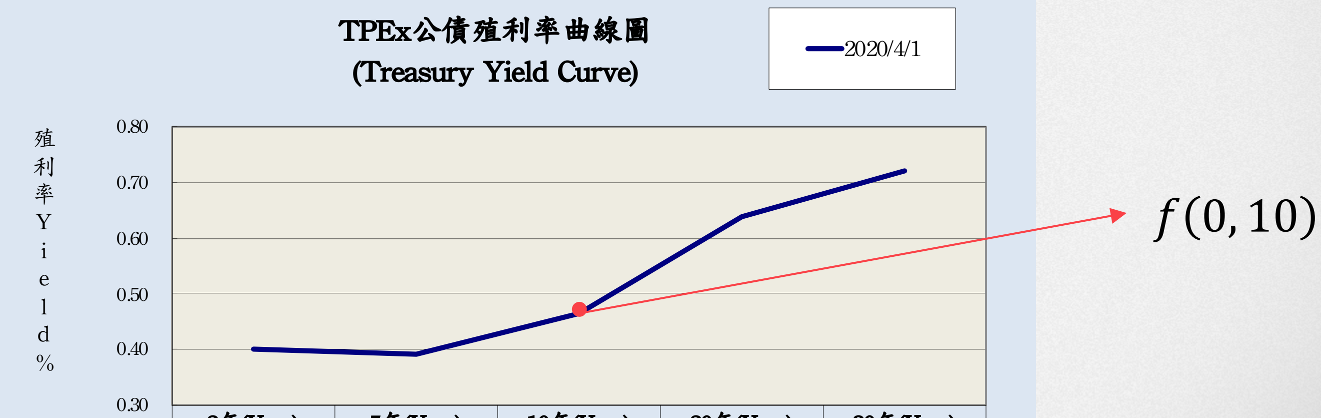


資料來源：台灣證券櫃買中心

背景

- 瞬間遠期利率 Instantaneous Forward Rate

$$f(t,T) := \lim_{S \to T^+} F(t;S,T)$$



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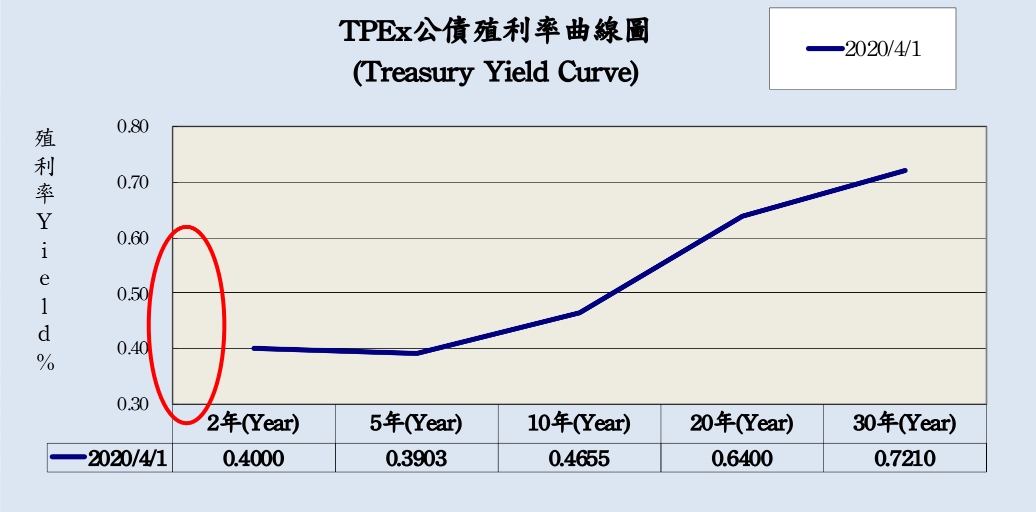
資料來源：台灣證券櫃買中心

背景

- 瞬間即期利率 (instantaneous spot rate) ，又稱作短利 (short rate)

$$r(t) = f(t,t)$$



資料來源：台灣證券櫃買中心

背景

- Hull-White Model，又稱作 Extended Vasicek Model，short rate model的一種

Vasicek Model

$$dr = a(b - r)dt + \sigma dz$$

Hull-White Model

$$dr = a\left(\frac{\theta(t)}{a} - r\right)dt + \sigma dz$$

模型介紹

- 假設$r(t)$服從$normal$分配

long-term level of short rate at time t

$$dr(t) = a\left(\boxed{\frac{\theta(t)}{a}} - r(t)\right)dt + \sigma d\boxed{z}$$

mean-reversion rate

Weiner process

$$\theta(t) = f_t(0,t) + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

待估參數：a、σ

背景

利率隨機過程：$dr = \mu_r dt + \sigma_r dz$

$$dr = (\theta(t) - ar)dt + \sigma dz$$

根據CIR論文：

設一contingent claim在時間t的價值$V(r,t)$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\mu_r + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma_r^2 = rV + \lambda(t)\frac{\partial V}{\partial r}\sigma_r$$

背景

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial r}\mu_r+\frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma_r^2=rV+\lambda(t)\frac{\partial V}{\partial r}\sigma_r$$

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial r}(\emptyset(t)-ar)+\frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma_r^2-rV=0$$

其中，$\emptyset(t)=\theta(t)-\lambda(t)\sigma$

Hull White模型下的零息債券價格

時間點t，到期日T的零息債券價格：

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

其中，

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$lnA(t,T) = ln\frac{P(0,T)}{P(0,t)} + B(t,T)f(0,t) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2aT} - 1)$$

Hull White模型下的零息債券價格

$$\frac{\partial P}{\partial r} = -B(t,T)P, \frac{\partial^2 P}{\partial r^2} = B(t,T)^2 P$$

$$\frac{\partial P}{\partial t} = A(t,T)\left[e^{-B(t,T)t} \cdot r \cdot \frac{-\partial B}{\partial t}\right] + \frac{\partial A}{\partial t} e^{-B(t,T)r}$$

$$= P(\frac{1}{A}\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t})$$

⇩

$$\boxed{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}(\emptyset(t) - ar) + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma_r^2 - rV = 0}$$

Hull White模型下的零息債券價格

$$P\left(\frac{1}{A}\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right) - (\emptyset(t) - ar)BP + \frac{1}{2}\sigma_r^2 B^2 P - rP = 0$$

$$\Downarrow$$

$$-Pr\left(\underbrace{r\frac{\partial B}{\partial t} - aB + 1}_{=0}\right) + \frac{P}{A}\left(\underbrace{\frac{\partial A}{\partial t} - \emptyset(t)AB + \frac{1}{2}\sigma_r^2 AB^2}_{=0}\right) = 0$$

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模型參數估計

估計σ

$$dr(t) = a\left(\frac{\theta(t)}{a} - r(t)\right)dt + \sigma dz$$

例子：

Date	Treasury Bill	dr
2020/3/2	1.135	-0.217
2020/3/3	0.918	-0.243
2020/3/4	0.675	-0.08
2020/3/5	0.595	-0.18
2020/3/6	0.415	–

單位%

資料來源：
https://finance.yahoo.com/quote/%5EIRX/history/

$\sigma = 1.135\%$

實務上，
估dr的資料期間通常和商品合約期限相同

估計a

目標式：

$$SE = min\sum_{i}(R_i^{Market} - R_i^{Hull-White})^2$$

$Hull - White$ 零息債券價格：

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

一般零息債券連續複利下的價格：

$$P(t,T) = e^{-R(t,T)(T-t)}$$

兩價格應相等，故

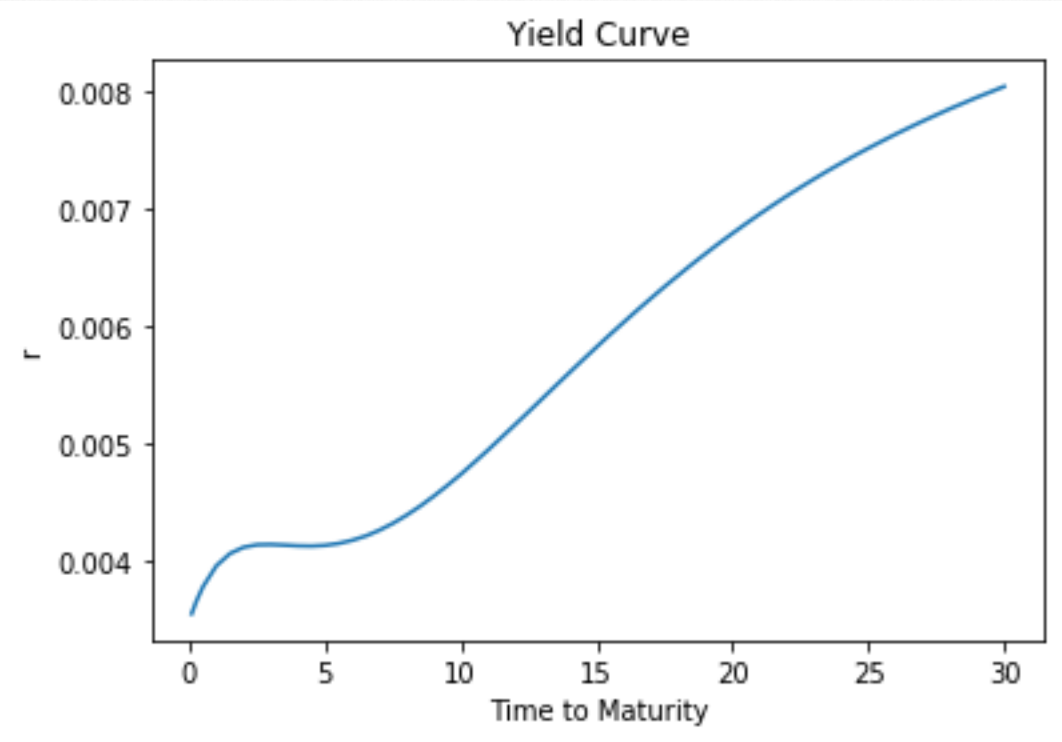
$$A(t,T)e^{-B(t,T)r(t)} = e^{-R(t,T)(T-t)}$$

$$R(t,T) = \frac{-lnA(t,T) + B(t,T)r(t)}{T-t}$$

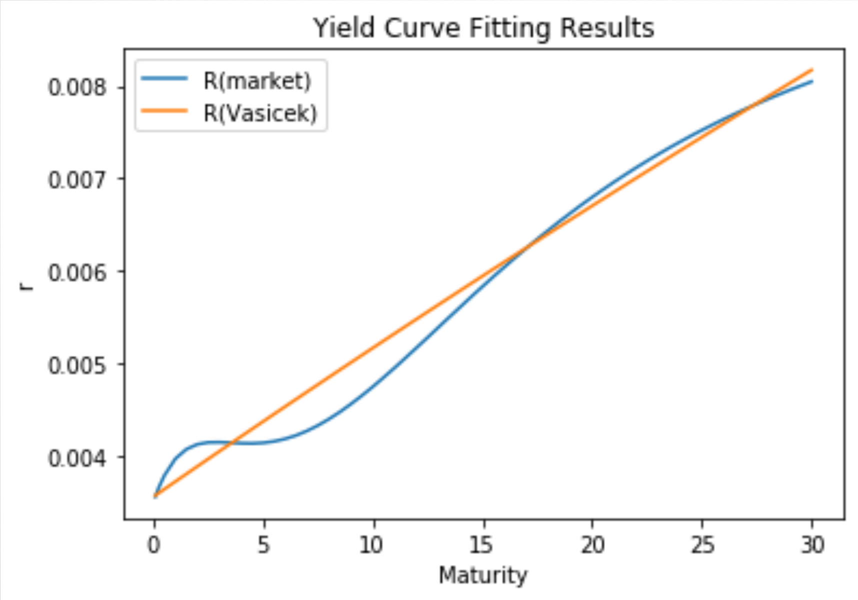
04

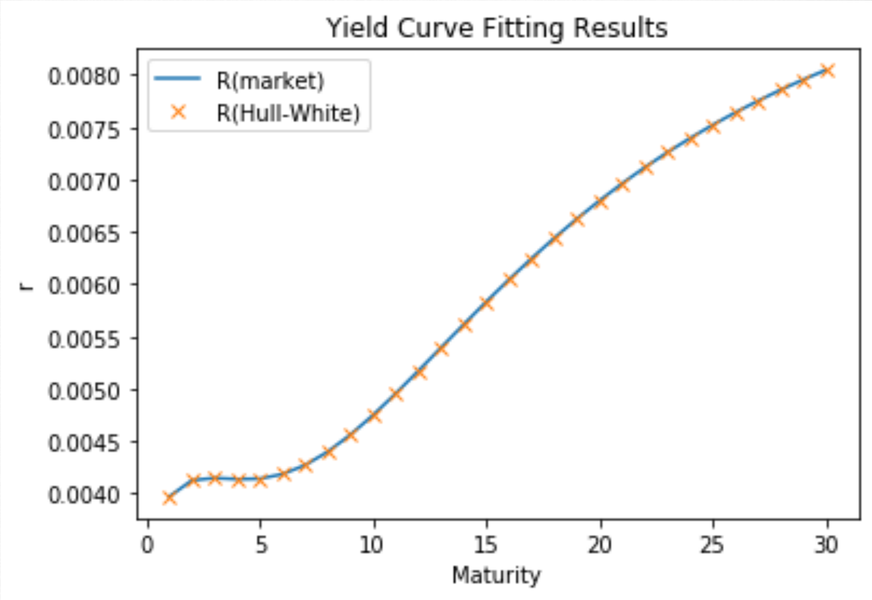
實作結果

Fitting Example



Vasicek vs Hull-White





估計σ

設現在時間 t 發行30年期債券，發行日為2016/9/27

$$r_{t-30}, r_{t-29}, \dots, r_{t-1}$$

$$\Downarrow$$

$$dr_{t-30}, dr_{t-29}, \dots, dr_{t-1}$$

$$\Downarrow$$

$$\sigma = 0.008693$$

估計a

將歷史波動度丟入Hull-White模型中，
且我們在時間$t = 0$時估計，故Hull-White即期利率如下：

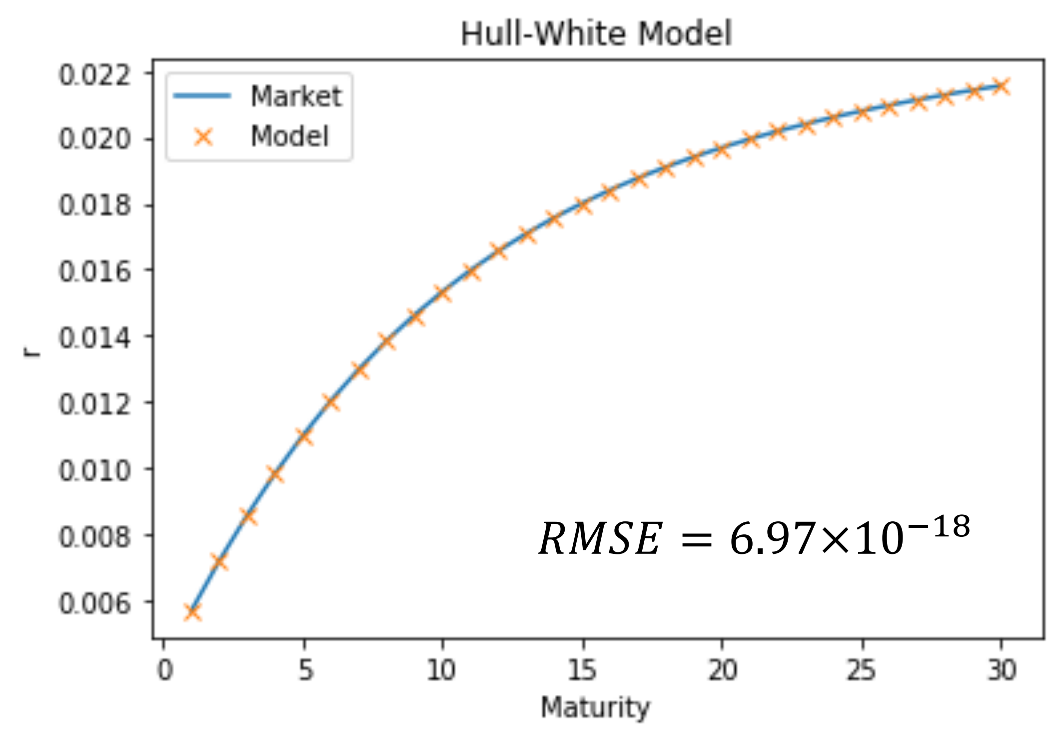
$$R_{Hull-White}(0,T) = \frac{-lnA(0,T) + B(0,T)r(0)}{T}$$

接著透過最佳化

$$min \sum_{t \in T} (R_{market}(0,t) - R_{Hull-White}(0,t))^2$$

得到 mean-reversion speed $\boldsymbol{a = 0.030295}$

結果


Hull-White Model
Market
Model
0.022
0.020
0.018
0.016
0.014
0.012
0.010
0.008
0.006
r
0
5
10
15
20
25
30
Maturity
$RMSE = 6.97 \times 10^{-18}$

THANK YOU