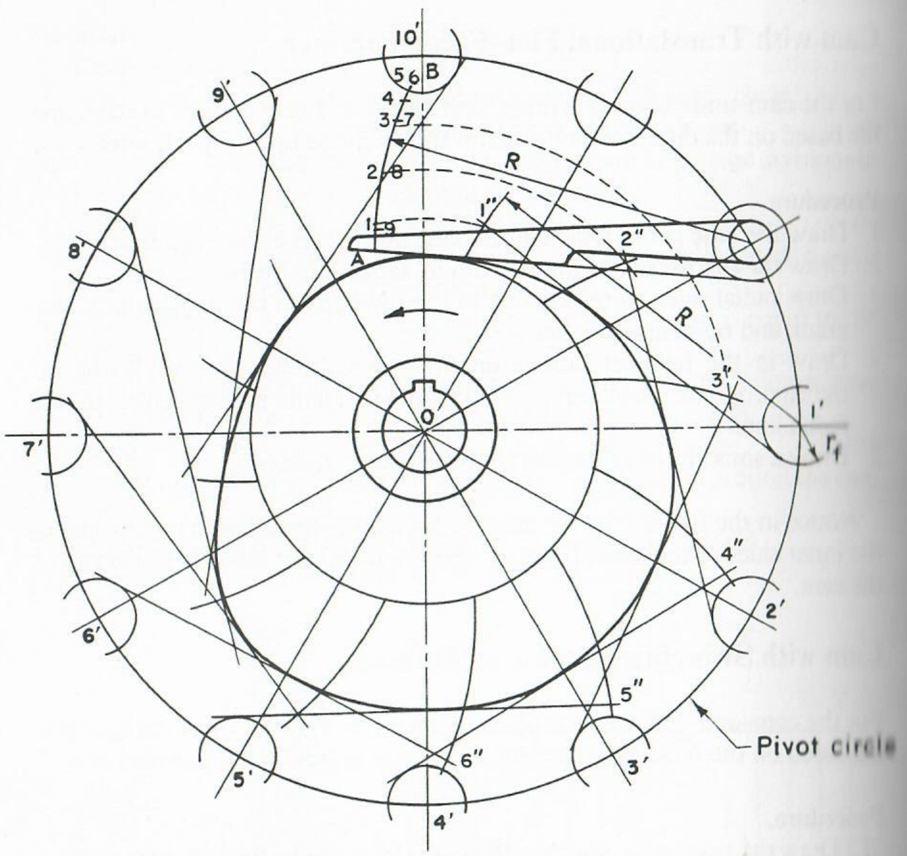

Fig. 9-5 Cam with swinging flat-faced follower.

11. Draw a smooth curve tangent to the family of straight lines representing the follower face.

9-2. THE ANALYTICAL METHOD OF DETERMINING CAM PROFILES

For high speed machines, the method by which points on the pitch curve are determined by graphical layout is not adequate, especially when the cam is manufactured by an incremental cutting process or by a numerically controlled milling machine. To develop equations for cam profile coordinates, the relative geometry of the cam and the follower must be described at both the lowest follower position and at some displaced follower position consistent with the rotational direction of the cam. Because of the voluminous calculations required in conjunction with its complicated geometry, the analytical method of determining cam profiles and cutter coordinates was subordinated to graphical techniques. With the widespread use of digital computers and programmable calculators, these calculations need no longer to be a deterrent.

The following coordinate systems have been used for deriving cam profiles:

- Polar coordinates.
- Rectangular coordinates (relative to the cam shaft center).
- Rectangular coordinates in which the ordinate or the Y-axis is represented by a line drawn from the cam shaft center to the center of the cam roll at its outer-most position, and the abscissa, or the X-axis, is normal to the Y-axis.

The third coordinate system, used by Cram (52), is not as commonly utilized as the first two types of coordinates.

Even if the choice of coordinates is the same, the form of the derived equation may be different, but their end results should agree. Variants of the equations representing the profile coordinates depend upon the approach used in derivation. Churchill and Hanson (49) employed the theory of envelopes, Davidson (55) suggested a method using instant centers, and others merely used brute force. We will introduce the method using the theory of envelopes to determine the cam profile and cutter-coordinate equations for the six major types of cam-follower arrangements. These major types are

Radial cams with roller followers
Translating cams with offset roller followers
Swinging cams with roller followers
Radial cams with flat-faced followers
Cams with swinging centric flat-faced followers
Cams with swinging eccentric flat-faced followers.

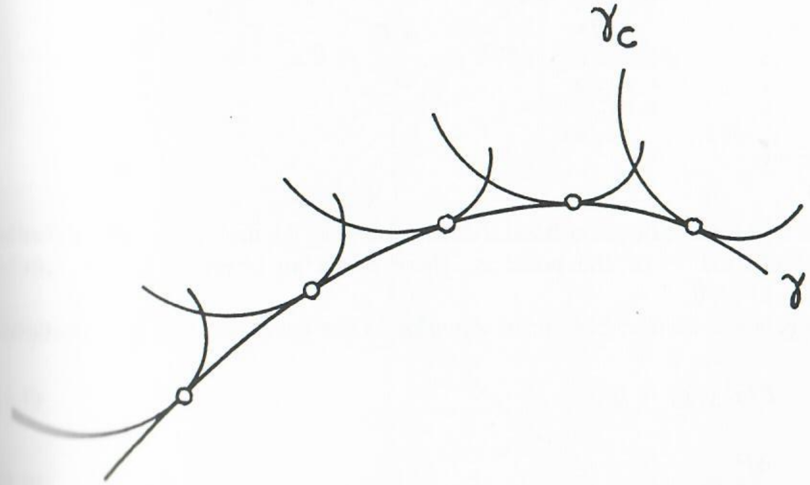

Fig. 9-6. Envelope of a family of curves depending on a parameter.

Envelope of a One-Parameter Family of Curves

Suppose $S\{\gamma_c\}$ is a family of smooth curves on a surface, depending on a parameter c. A smooth curve γ is called an envelope of the family S if

(1) for every point of the curve γ it is possible to give a curve γ_c of the family that is tangent to the curve γ at this point,
(2) for every curve γ_c of the family it is possible to give a point on the curve γ at which the curve γ_c is tangent to γ, and
(3) no curve of the family has a segment in common with the curve γ.

See Fig. 9-6 for this definition.

Now let us introduce a theorem.

Theorem: Suppose the curves γ_c of a family S are given by the equation $F(x, y, c) = 0$, where F is continuous and continuously differentiable for all its arguments in a neighborhood of the point (x_0, y_0, c_0). At the point (x_0, y_0, c_0) let the following relations hold:

$$F(x_0, y_0, c_0) = 0 \tag{9-1}$$

$$\frac{\partial F}{\partial c}(x_0, y_0, c_0) = 0 \tag{9-2}$$

$$\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial^2 F}{\partial c \partial x} & \dfrac{\partial^2 F}{\partial c \partial y} \end{vmatrix} \neq 0 , \quad \frac{\partial^2 F}{\partial c^2} = 0 . \tag{9-3}$$

Then in a certain neighborhood u of the point (x_0, y_0), and for c from a definite neighborhood V of the point c_0, there exists an envelope of the family $F(x, y, c) = 0$.

In general, we may obtain the equation of the envelope from the equations

$$F(x, y, c) = 0 \tag{9-4}$$

$$\frac{\partial F}{\partial c}(x, y, c) = 0 \tag{9-5}$$

by expressing x and y as a function of the parameter c or by expressing c as a function of the variables x, y and substituting for c into Eq. (9-4)

$$F(x, y, c(x, y)) = 0 ,$$

i.e., by elimination of the parameter c from Eq. (9-4).

Example 9-1

Consider the equation $F(x, y, c) = 0$ as

$$y - cx - \frac{4}{c} = 0 .$$

Then $\dfrac{\partial F}{\partial c} = -x + \dfrac{4}{c^2} = 0 .$

Eliminating c from the above two equations yields

$$y^2 = 16x ,$$

which is the envelope (Fig. 9-7).

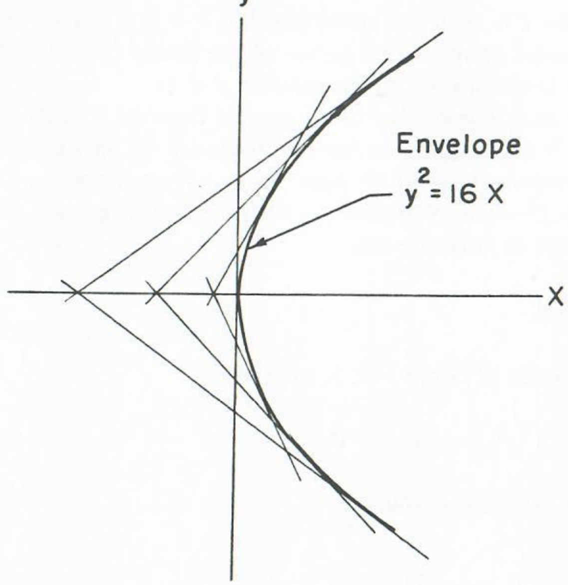

Fig. 9-7. Parabolic envelope.

Example 9-2

Consider the equation $F(x, y, c) = \theta$ as

$$[x^2 + (y - c)^2]^2 - b^2[x^2 - (y - c)^2] = 0 .$$

This represents a lemniscate of width $2b$ with its center located at $x = 0$, $y = c$. As c is varied, a series of lemniscates translated along the y-axis is determined.

Eq. (9-5), $\dfrac{\partial F}{\partial c} = 0$, requires

$$(y - c)[2x^2 + 2(y - c)^2 - b^2] = 0 .$$

A solution of this equation is $y = c$.

Substituting this solution into the equation $F(x, y, c) = 0$ yields

$$x^2(x^2 - b^2) = 0$$

or

$$x = 0, +b, -b .$$

The two lines $x = \pm b$ are envelopes, but $x = 0$ is not an envelope. $x = 0$ consists of nodal points of the curves of the family (Fig. 9-8). Note also that nothing new is obtained from the solution $y \neq c$.

It is important to note that in the general theorem stated above, the conditions in Eq. (9-3) are sufficient but not necessary for the existence of the envelope in the neighborhood of the point (x_0, y_0). If conditions in Eq. (9-3) are not fulfilled, Eqs. (9-4) and (9-5) need not define the envelope. An additional example is used here to illustrate this.

Example 9-3

Consider a family of curve $F(x, y, c) = 0$ as

$$(y - c)^2 - (x - c)^3 = 0 .$$

From Eq. (9-5) it follows that

$$\frac{\partial F}{\partial c} = 0 = -2(y - c) + 3(x - c)^2$$

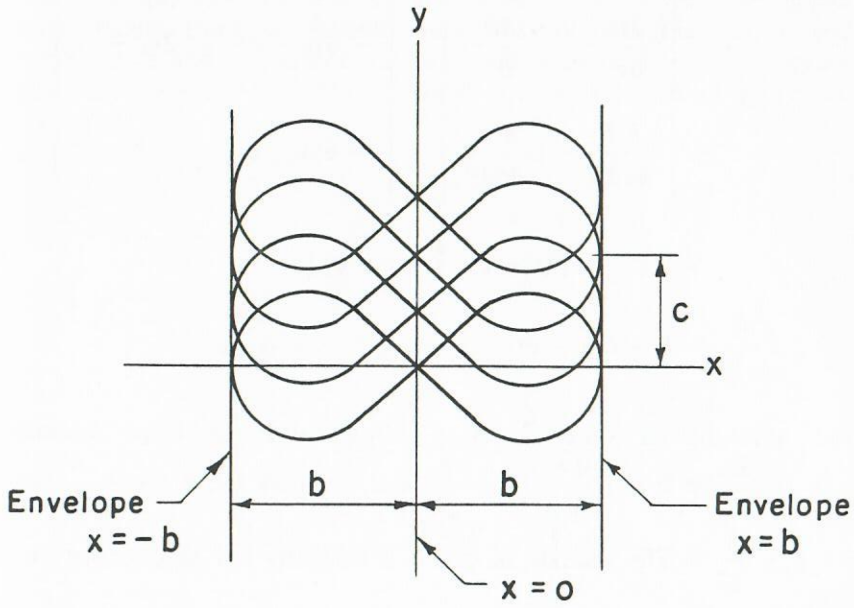

Fig. 9-8. Linearly translating lemniscates.

or

$$y - c = \frac{3}{2}(x - c)^2.$$

Substituting this into $F(x, y, c) = 0$ gives

$$\frac{9}{4}(x - c)^3\left(x - c - \frac{4}{9}\right) = 0.$$

Hence, the solutions are

case 1 $\quad x = c, \qquad y = c$ (or $y = x$)

case 2 $\quad x = c + \dfrac{4}{9}, \qquad y = c + \dfrac{8}{27}$ $\left(\text{or } y = x - \dfrac{4}{27}\right)$.

Conditions (9-3) in the general theorem are

$$\Delta = \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial^2 F}{\partial c \partial x} & \dfrac{\partial^2 F}{\partial c \partial y} \end{vmatrix} = \begin{vmatrix} -3(x-c)^2 & 2(y-c) \\ = 6(x-c) & -2 \end{vmatrix}$$

$$= 6(x-c)\,[\,(x-c) - 2(y-c)\,]$$

$$\frac{\partial^2 F}{\partial c^2} = 2 - 6(x-c)\,.$$

The solution of case 2 is a straight-line envelope because $\Delta = -\dfrac{32}{81} \neq 0$ and $\dfrac{\partial^2 F}{\partial c^2} = -\dfrac{2}{3} \neq 0$ when we use $x = c + \dfrac{4}{9}$ and $y = c + \dfrac{8}{27}$. The solution of case 1 is evidently not an envelope since $\Delta = 0$ and $\dfrac{\partial^2 F}{\partial c^2} = 2 \neq 0$ when we use $x = c,\ y = c$. It is merely a straight-line locus of cuspidal points. Fig. 9-9 shows the situation.

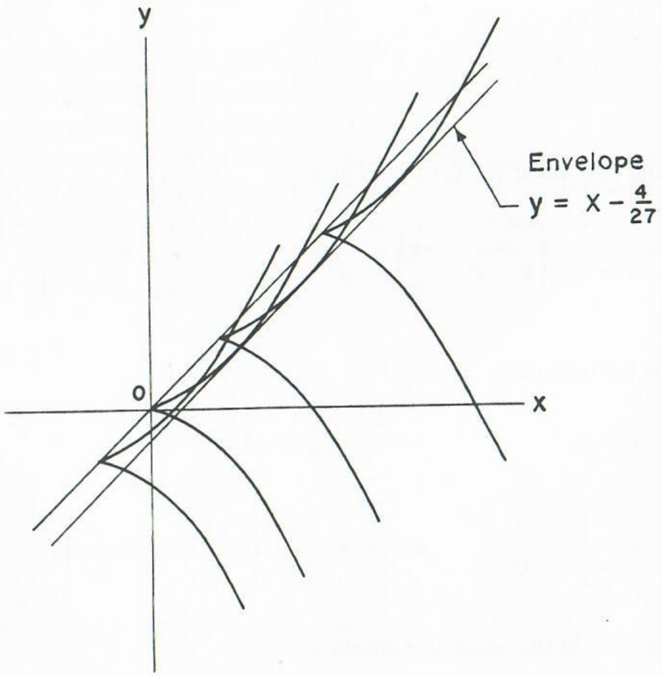

Fig. 9-9.

The envelope theory will now be applied to determine the cam working profile coordinates. For roller–follower cams, two envelopes (Fig. 9-10) — one inner envelope and one outer envelope — are obtained. Both envelopes are applicable for grooved cams. Only the inner envelope is used with plate cams.

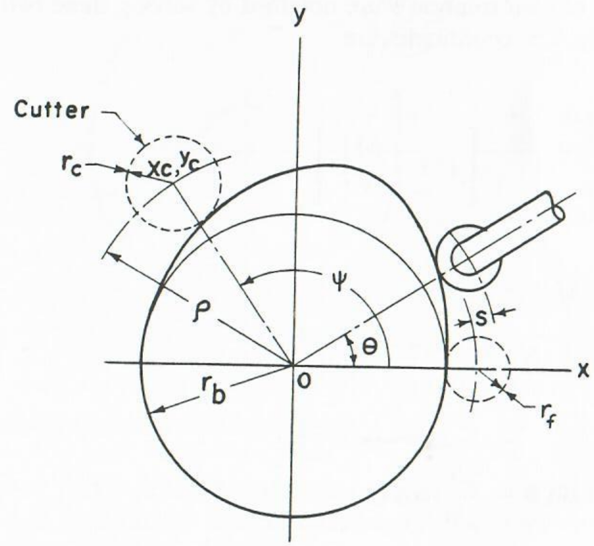

Fig. 9-10. Radial cam with roller follower.

Radial Cams with Roller Followers. Referring to Fig. 9-10, the radial distance r to the center of the follower is

$$r = r_b + r_f + s\,,$$

where

r_b is the base circle radius,

r_f is the roller radius, and

s is the cam rise, which is a function of cam angular displacement θ.

The general equation of the envelope is

$$F(x, y, \theta) = (x - r\cos\theta)^2 + (y - r\sin\theta)^2 - r_f^2 = 0\,.$$

It is now differentiated with respect to θ :

$$\frac{dF}{d\theta} = 2r\sin\theta(x - r\cos\theta) - 2r\cos\theta(y - r\sin\theta) = 0\,.$$

The rectangular coordinates of a point on the cam profile corresponding to a specific angle of cam rotation θ are obtained by solving these two equations simultaneously. The coordinates are

$$x = r\cos\theta + \frac{r_f}{\left[1 + \left(\frac{M}{N}\right)^2\right]^{1/2}} \tag{9-6}$$

$$y = \frac{xM + r\dfrac{ds}{d\theta}}{N}\,, \tag{9-7}$$

where

$$M = r\sin\theta - \frac{ds}{d\theta}\cos\theta$$

$$N = r\cos\theta - \frac{ds}{d\theta}\sin\theta\,,$$

and

$$\frac{ds}{d\theta} = \frac{dr}{d\theta}\,.$$

Here the proper sign choice may be determined by examining $\theta = 0$ when $x = r_b$. At this point

$$r = r_b + r_f$$

$$\frac{dr}{d\theta} = \frac{ds}{d\theta} = 0\,,$$

and

$$x = r_f + r_b \pm r_f\,.$$

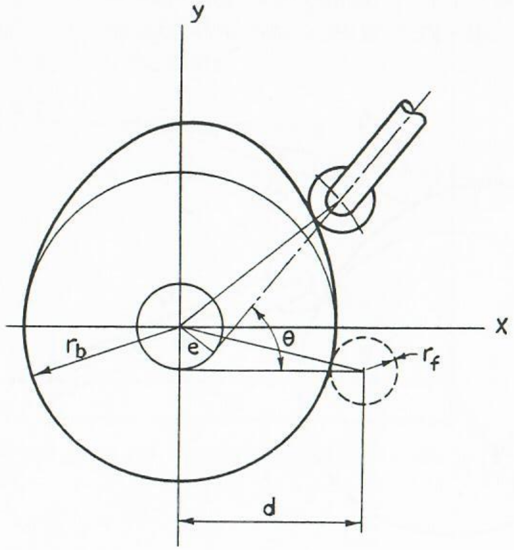

Fig. 9-11. Translating cam with offset roller follower.

Only the lower sign is justified in the above equation; thus the negative sign should be used to establish the inner envelope. The outer envelope is not required unless a grooved cam is used.

Translating Cams with Offset Roller Followers. The roller follower of this type of cam, Fig. 9-11, moves radially along a line that is offset from the cam center by an amount e.

The general equation of the envelope is

$$F(x, y, \theta) = [\,x - e\sin\theta - (d + s)\cos\theta\,]^2 + [\,y + e\cos\theta - (d + s)\sin\theta\,]^2 - r_f^2 = 0\,,$$

where

$$d = [\,(r_b + r_f)^2 - e^2\,]^{1/2}.$$

Differentiating F with respect to θ, setting the result equal to zero, and solving for x and y, we obtain the profile coordinates

$$x = e\sin\theta + (d + s)\cos\theta \pm \frac{r_f}{\left[1 + \left(\frac{U}{V}\right)^2\right]^{1/2}} \tag{9-8}$$

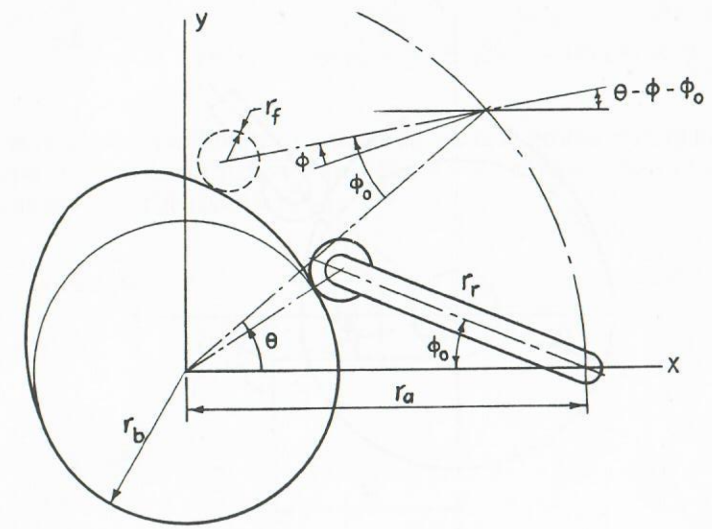

Fig. 9-12. Swinging roller–follower cam.

$$y = \frac{xU + (d+s)\dfrac{ds}{d\theta}}{V}, \tag{9-9}$$

where

$$U = (d+s)\sin\theta - \left(e + \frac{ds}{d\theta}\right)\cos\theta$$

$$V = (d+s)\cos\theta + \left(e + \frac{ds}{d\theta}\right)\sin\theta .$$

Again, the lower sign is associated with the inner envelope.

Swinging Cams with Roller Followers. As shown by the geometry of Fig. 9-12, the initial position of the follower before lift starts is designated by ϕ_0

$$\phi_0 = \cos^{-1}\frac{r_r^2 + r_a^2 - (r_b + r_f)^2}{2r_a r_r},$$

where r_r is the length of the roller–follower arm, and r_a is the distance between the pivot point of the swinging follower and the center of the cam.

The implicit equation of the family is

$$F(x, y, \theta) = (x - r_a\cos\theta + r_r\cos\alpha)^2 + (y - r_a\sin\theta + r_r\sin\alpha)^2 - r_f^2 = 0,$$

where

$$\alpha = \theta - \phi - \phi_0 .$$

Then

$$\frac{dF}{d\theta} = 2(x - r_a\cos\theta + r_r\cos\alpha)\left[r_a\sin\theta - r_r\sin\alpha\left(1 - \frac{d\phi}{d\theta}\right)\right] + 2(y - r_a\sin\theta + r_r\sin\alpha)\left[-r_a\cos\theta + r_r\cos\alpha\left(1 - \frac{d\phi}{d\theta}\right)\right] = 0 .$$

Solving these two equations simultaneously gives the profile coordinates

$$x = r_a\cos\theta - r_r\cos\alpha \pm \frac{r_f}{\left[1 + \left(\dfrac{P}{Q}\right)^2\right]^{1/2}} \tag{9-10}$$

$$y = \frac{xP}{Q}, \tag{9-11}$$

where

$$p = r_a\sin\theta - r_r\left(1 - \frac{d\phi}{d\theta}\right)\sin\alpha$$

$$Q = r_a\cos\theta - r_r\left(1 - \frac{d\phi}{d\theta}\right)\cos\alpha .$$

The negative sign gives the inner cam profile.

Translating Cams with Flat-Face Followers. Fig. 9-13 shows a translating cam with a flat-face follower. The general equation of the family of lines forming the envelope is governed by a straight line

$$y = mx + b\,,$$

where m is the slope, and b is the y-intercept. In this case

$$m = \cos\theta$$

$$b = \frac{(r_b + s)}{\sin\theta}\,.$$

Also

$$x = (r_b + s)\cos\theta$$

$$y = (r_b + s)\sin\theta\,.$$

Hence,

$$y = \frac{r_b + s - x\cos\theta}{\sin\theta}\,.$$

Therefore,

$$F(x, y, \theta) = y\sin\theta + x\cos\theta - (r_b + s) = 0$$

$$\frac{dF}{d\theta} = y\cos\theta - x\sin\theta - \frac{ds}{d\theta} = 0\,.$$

Solving these two equations yields the profile coordinates

$$x = (r_b + s)\cos\theta - \frac{ds}{d\theta}\sin\theta \tag{9-12}$$

$$y = (r_b + s)\sin\theta + \frac{ds}{d\theta}\cos\theta\,. \tag{9-13}$$

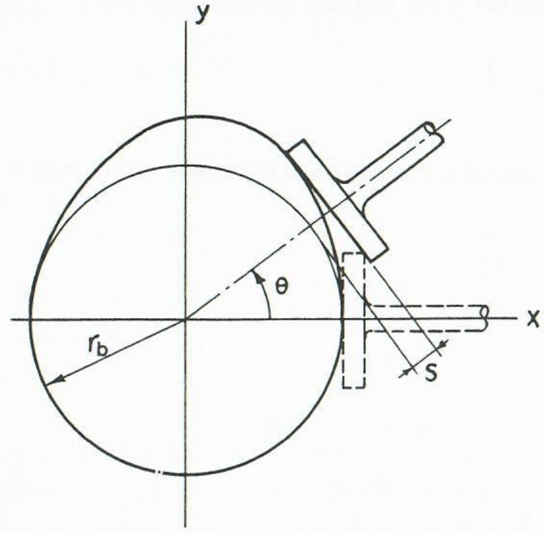

Fig. 9-13. Radial cam with flat-faced follower.

Cams with Flat-Face Centric Swinging Followers. The flat-face swinging-follower cams (Fig. 9-14) are of the centric type if the face, when extended, passes through the follower pivot.

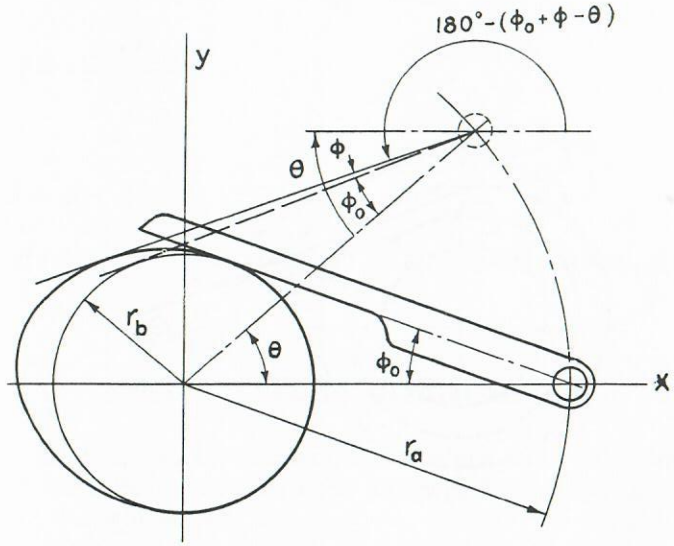

Fig. 9-14. Cam with swinging centric flat-faced follower.

The initial position of the follower before lift starts is

$$\phi_0 = \sin^{-1} \frac{r_b}{r_a} .$$

The general form of the equation of the family is the straight line

$$y = mx + b ,$$

where the slope is

$$m = - \tan (\phi - \theta + \phi_o),$$

and the y-axis intercept is

$$b = r_a [\sin \theta + \cos \theta \tan (\phi - \theta + \phi_o)] ,$$

in which we have used the relationships $x = r_a \cos \theta$ and $y = r_a \sin \theta$. Therefore,

$$F(x, y, \theta) = y + \tan (\phi - \theta + \phi_o)(x - r_a \cos \theta) - r_a \sin \theta = 0.$$

Setting $\dfrac{dF}{d\theta} = 0$ and solving simultaneously in the usual manner, we obtain

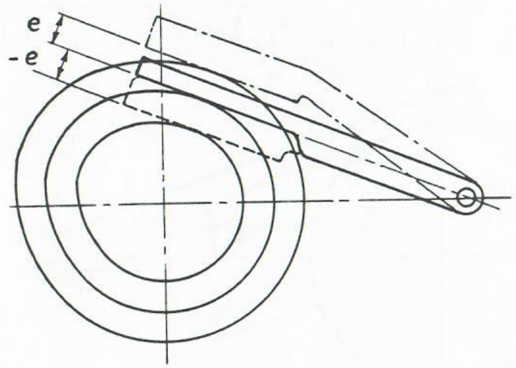

Fig. 9-15. Cam with swinging eccentric flat-faced follower.

$$x = r_a(\cos \theta + A \cos \beta) \tag{9-14}$$

$$y = r_a(\sin \theta + A \sin \beta) , \tag{9-15}$$

where

$$A = \frac{\cos (\theta + \beta)}{\dfrac{d\phi}{d\theta} - 1}$$

$$\beta = \phi_o + \phi - \theta .$$

Cams with Flat–Face Eccentric Swinging Followers. A swinging flat-face follower of the eccentric type is shown in Fig. 9-15. The amount of eccentricity (or offset) e is the distance between a line through the cam pivot and the follower face. The distance e is considered positive or negative, depending on whether the effect of e is to increase or to decrease the size of the in–line follower cam. The profile coordinates are

$$x = r_a(\cos \theta + A \cos \beta) + e \sin \beta \tag{9-16}$$

$$y = r_a(\sin \theta + A \sin \beta) + e \cos \beta , \tag{9-17}$$

where

$$A = \frac{\cos (\theta + \beta)}{\dfrac{d\phi}{d\theta} - 1}$$

$$\beta = \phi_o + \phi - \theta .$$

Note that when $e = 0$, Eqs. (9-16) and (9-17) degenerate to become Eqs. (9-14) and (9-15).

9-3. CUTTER COORDINATES

For cam manufacture, the location of the milling cutter or of the grinding wheel, i.e., the trace of the cutter or the grinder path, should be known in order to produce the right cam profile.

If the milling cutter size is different from that of the follower, the geometry required to describe the cutter center is a circle representing the cutter and the polar coordinates ρ and ψ locating the cutter center from point O. As shown in Fig. 9-10, the circle is tangent to the cam profile and has radius r_c. To locate the cutter center in the x–y coordinate system, we must first determine the projections of ρ on the x-axis and on the y-axis. These projectional components are designated as x_c and y_c. The mathematical expressions of x_c and y_c for various types of cam–follower arrangements are given in Table 9-1.

Table 9-1 Summary of Cutter-Center Equations for Various Types of Cam and Follower

Translating Roller	In-Line (Fig. 9-5)	$x_c = x + \frac{r_c}{r_f}(r\cos\theta - x)$ $y_c = y + \frac{r_c}{r_f}(r\sin\theta - y)$
	Offset (Fig. 9-6)	$x_c = x + \frac{r_c}{r_f}(x_f - x)$ $y_c = y + \frac{r_c}{r_f}(y_f - y)$ $x_f = e\sin\theta + (d+s)\cos\theta$ $y_f = -e\cos\theta + (d+s)\sin\theta$ $d = [(r_b + r_f)^2 - e^2]^{1/2}$
Swinging Roller	(Fig. 9-7)	$x_c = x + \frac{r_c}{r_f}(x_f - x)$ $y_c = y + \frac{r_c}{r_f}(y_f - y)$ $x_f = r_a\cos\theta - r_r\cos\alpha$ $y_f = r_a\sin\theta - r_r\sin\alpha$ $\alpha = \theta - \phi - \phi_o$
Translating Flat-Face	(Fig. 9-8)	$x_c = x + r_c\cos\theta$ $y_c = y + r_c\sin\theta$

Table 9-1 Cont'd.

Swinging Flat-Face	In-line (Fig. 9-9)	$x_c = x + r_c\sin\alpha$ $y_c = y + r_c\cos\alpha$ $\alpha = \phi - \theta + \phi_o$
	Offset (Fig. 9-10)	$x_c = x + x_c\sin\alpha$ $y_c = y + r_c\cos\alpha$ $\alpha = \phi - \theta + \phi_o$

Using the following coordinate conversions

$$\rho = (x_c^2 + y_c^2)^{1/2}$$

$$r_c = \tan^{-1}\left(\frac{y_c}{x_c}\right),$$

polar coordinate equations of the cutter center can also be obtained.

Example 9-4

Consider a swinging flat-face follower with $r_a = 7$, $e = 0.5$, (assumed positive), and $r_b = 2$. When the cam rotates 180° clockwise, the follower is to swing 18° by cycloidal motion. Determine the cam profile coordinates for a sample position, say, $\theta = 30°$.

The equation for cycloidal motion is

$$\phi = h\left(\frac{\theta}{\beta} - \frac{1}{2\pi}\sin\frac{2\pi\theta}{\beta}\right).$$

Hence,

$$\frac{d\phi}{d\theta} = \frac{h}{\beta}\left(1 - \cos\frac{2\pi\theta}{\beta}\right),$$