University of Nebraska-Lincoln
Department of Physics
PHYS 442

Pendulum Experiment

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INTRODUCTION

The periodicity of a bob on a string was first measured by Galileo Galilei in 1583 when he determined its period with his pulse [1]. In this lab report we aim to measure the period of a lead weight attached to a nylon fishing line (Berkley® Trilene) and determine if the period is dependent on the initial amplitude. A major overarching goal of this experiment is to handle systematic and random errors inherent in the experiment. In the following section, we will outline several models to see which theory best describes our data. After we establish the theory, we will give a more detailed experimental description of our apparatus and our measurement technique. Finally, we present our results and discuss which theory is in agreement with our data.

THEORY

The three models we will outline in this section are the simple pendulum, the damped pendulum, and the physical pendulum. We will compare these theories to our experimental data.

Simple Pendulum

A simple pendulum is one where you have a point mass attached to a massless string. The force of gravity ensures tension in the string and produces oscillatory motion, see Figure 1.

To get the equation of motion (EOM) for this system we sum all the torques acting on the bob. In our experiment, we will consider only small angles of period of the pendulum and thus use the first order approximation for $\sin\theta \approx \theta$. Here is the following EOM, using Newton's notation for differentiation, where I, l, θ, m, g is the moment of inertia of the system, the length of the string, the angle from equilibrium position, the mass of the bob, Earth's gravitational constant, respectively,

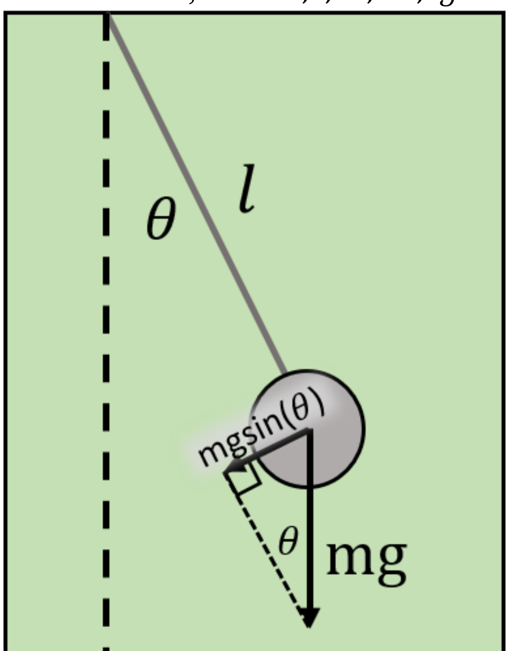


Figure 1. Massless string attached to a bob. The string is under tension due to the force of gravity.

$$\sum \tau = I\ddot{\theta} = ml^2\ddot{\theta} = -lmg\sin\theta \approx -lmg\theta. \quad (1)$$

This equation simplifies to,

$$\ddot{\theta} + \frac{g}{l}\theta = 0. \quad (2)$$

which is a linear 2nd order differential equation whose solution is a periodic sinusoidal solution, and the period of this function is given by,

$$T = 2\pi\sqrt{\frac{l}{g}}. \quad (3)$$

This model of our pendulum's period is what we will first compare all our data to in this report. Since we are using the small angle approximation, there will be potentially some angular dependance on our value for the period at larger angles. We explore this experimentally in the results section.

Damped Pendulum

The bob moving through the air will lead to a different EOM because the air applies an additional torque resisting the motion of the bob. We presume that the force of the air on the bob is proportional and opposite in direction to the velocity of the bob, $F_{drag} = -cv = -cl\dot{\theta}$. Summing torques again leads to the following EOM where c is the resistivity of air,

$$\ddot{\theta} - \frac{c}{m}\dot{\theta} + \frac{g}{l}\theta = 0\,. \tag{4}$$

Now we solve this differential equation for the underdamped case $\left[\frac{g}{l} > \left(\frac{c}{2m}\right)^2\right]$, because that is the only case we observe in our experiment,

$$\theta = \theta_0\, e^{\frac{-c}{2m}t} \cos\left(\sqrt{\frac{g}{l} - \left(\frac{c}{2m}\right)^2}\; t\right). \tag{5}$$

The phase is zero because the bob is released from rest at θ_0, the initial amplitude. Which yields a period of,

$$T = \frac{2\pi}{\sqrt{\frac{g}{l} - \left(\frac{c}{2m}\right)^2}}\,. \tag{6}$$

Physical Pendulum

In the above theories we assumed the mass of the string/fishing line to be massless. However, it does of course have some mass. To get the period we follow the same procedure as for the simple pendulum and sum the torques. The difference from the simple pendulum is that the moment of inertia and the distance to the center of mass has changed because we treat the fishing line as having mass. In that case, the period for the physical pendulum is,

$$T = 2\pi\sqrt{\frac{I_{CM}}{MgL_{CM}}}\,. \tag{7}$$

Where,

$$I_{CM} = \frac{1}{3}M_{line}{L_{CM}}^2 + \frac{2}{5}M_{bob}R^2 + M_{bob}(L_{CM} + R)^2, \tag{8}$$

and the moment of inertia for a cylinder rotating about one end with a sphere attached is, I_{CM}. Here L_{CM} is the distance from the point of rotation to the center of mass and R is the radius of the bob. M_{line} and M_{bob} is the fishing line mass and the mass of the bob, respectively. We calculated the moment of inertia by summing the moments of a cylinder about one axis and a sphere about the same axis of rotation, hence the additional last term. We note that if you replace $\frac{g}{l}$ in (6) with the inverse of the term inside the radical from (7) you obtain the period for a *damped physical pendulum*.

EXPERIMENTAL DESCRIPTION

We constructed our pendulum by cutting a piece of fishing line to length and tying it to a C-clamp, see Figure 2a. At the other end of the string, we tied it to the hook on our lead bob. Next, we added a rail near the bottom of its swing to ensure it was only swinging in a plane. To carry out measurements we need to measure the initial angle that the fishing line makes with respect to the rest position. So, we let it damp out and lined up the protractor with the fishing line then carefully taped it to the clamp, see Figure 2b. We added a lead brick so that we consistently pull the bob back to our desired angle every time.

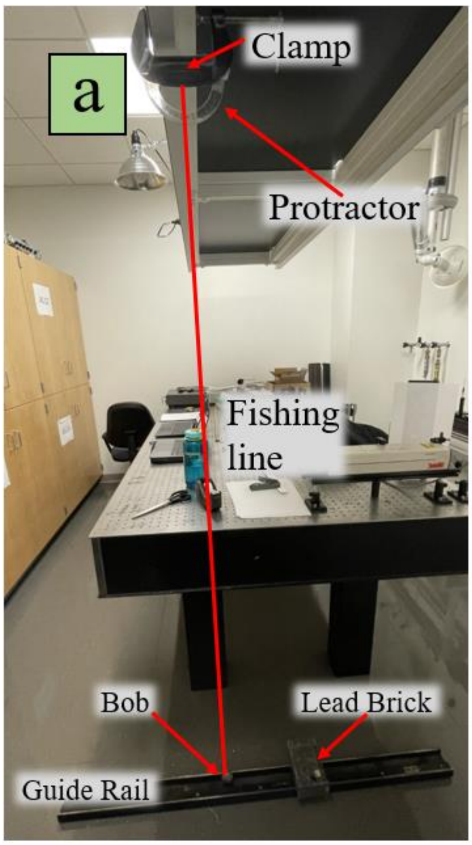


Figure 2a. Here is our full experimental set up. The top features a C-clamp where the string and protractor are attached to. The bottom is where our bob, rail, and lead brick rest.

Figure 2b. Picture of the protractor with our rest position of the pendulum at 0 degrees. For small angles, the knot did not move.

At the beginning of the experiment, we made careful decisions about the dimensions of our experiment. We chose the length of our fishing line to be as long as possible so that we could maximize the arc length and easily observable large swings since we are oscillating at small angles. Then we chose the heaviest bob available to minimize the effects due to it being physical pendulum.

Measurements

Before measuring the period of the pendulum, I will first state all the active components of our experiment. In Table 1, we report the mass of the bob, hook length of the bob, radius of the bob, g, and the last three columns are lengths of the fishing wire throughout the experiment.

Mass (grams)	Hook Length (cm)	Radius (cm)	g $\left(\frac{m}{s^2}\right)$	Length (1/10) (cm)	Length (100) (cm)	Anharmonicity Length (cm)
315.76 (1)	0.5(5)	1.887(25)	9.80119 (0)	186.45(30)	186.69(30)	142.61(30)

Table 1. Active component values in our pendulum experiment with the uncertainties.

Bob Mass - We measured the mass of our bob using a mechanical balance scale and a standard weight set. The uncertainty comes from the fact that the resolution of our measurement is dependent on the smallest weight we could balance it with. Once we got to the point where adding the smallest weight (10mg) unbalanced the scale, we stopped. So, our uncertainty in the measurement is 10mg.

Hook Length - Our hook length has a large uncertainty because it changed throughout the experiment, and we only measured it at the very end. Over time, due to unfortunate events, the hook length decreased from the maximally extended length to completely flat. So, we said the hook was half the maximally extended length and the uncertainty is half the hook length. Luckily, we took pictures of the hook length throughout the experiment, see Figure 3.

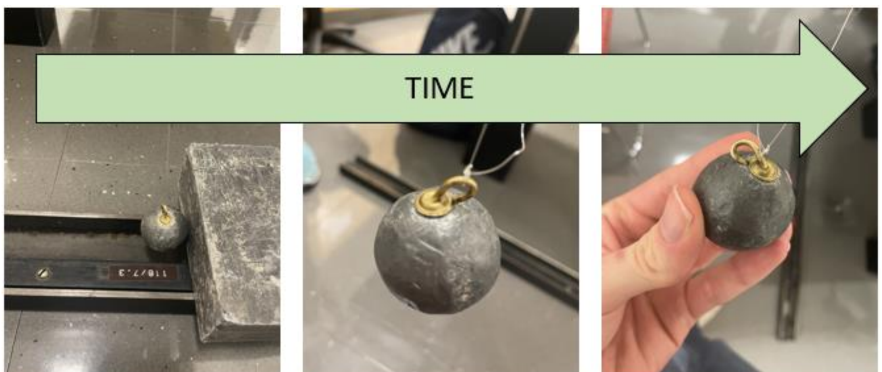


Figure 3. Shows the evolution of our hook length. The last picture shows the hook after it crashed into the wall from being releasing 90 degrees. Also, the first picture highlights what max amplitude looks like after one oscillation.

Bob Radius – We observed that the bob is not a perfect sphere and is rather asymmetrical. Using calipers, we measured the diameter of the bob along two directions perpendicular to one another. We then took the average and sample standard deviation of those values to be our nominal value and error.

g – Using WolframAlpha's local gravity calculator [2] we found Earth's gravity in Lincoln, Nebraska to be the value in Table 1. We quote no error because it was a not provided from the widget. It is also an order of magnitude more precise than (assuming it is correct up to the number of digits provided) than our length measurement.

Length 1/10 - The first day we assembled our pendulum we measured it to be this length. We took 10 measurements and obtained the mean and standard error.

Length 100 – After a few days passed we came to the lab gather the 100-period (see below for what that means) data and noticed it our bob was resting on our guide rail. Then, we tried to cut a new line to the length of the old one. We got close and used the same error from our first length measurements.

Anharmonicity Length – Unfortunately, we had to cut new line because our pendulum line snapped during the preliminary tests on how the period changes at different amplitudes (hence the name). In Figure 3, you can see the damage to the hook caused by this.

Period Data

Our experiment for measuring the period is divided into 3 categories: 1 period, 10 periods, and 100 periods. The initial angle for all these categories was maintained to be 5 degrees. We defined a period as a full period swing where the bob returns to where we released it at 5 degrees.

1 Period: The pendulum swung for only 1 period then recorded the time with a stopwatch. We repeated this measurement 10 times.

10 Periods: The pendulum swung 10 times then recorded the time with a stopwatch. We repeated this measurement 10 times. This whole process was repeated 9 times to get a total of 9 datasets with 10 points each.

100 Periods: The pendulum swung 100 times then recorded the time. We repeated this measurement 10 times to arrive at a single dataset with 10 points.

Sources of Error

For this report we categorize 3 types of errors present in this experiment: random error, and systematic error.

Random error **-** Random error is defined by error that varies from measurement to measurement. Random error is characterized by the width of our histograms (presented later). The largest expected source of random error that enters the experiment is when measuring the period. Since we must watch the bob reach its maximum amplitude and make a judgment call on when to stop the stopwatch. Another effect is that our bob is not perfectly spherical and every time we pull it back the center of mass shifts slightly between measurements. But I think we can safely assume this is a negligible effect.

Systematic errors – Two, of the many, sources of systematic error that we will explore in this report are outlined in the theory: are damping due to friction and the physical pendulum. To test the physical pendulum model, we will look at what the theory predicts the period to be and compare with the data. Another source of systematic error is our $\sin\theta \approx \theta$ low angle approximation. All these errors are legitimate, but it is unclear how much they affect our experiment. Measurements of the components of our experiment also introduce systematic error since we cannot measure length and mass with infinite precision. For these we are constrained by the tools we used to measure them.

Human errors – We define human error as major human blunder in the measurement process. An example would be miscounting the number of periods.

Data for the Anharmonicity of a Pendulum

To test the anharmonicity, we conducted an experiment where we take the bob to large angles where the $\sin\theta \approx \theta$ approximation fails and let the pendulum swing for 10 periods. We repeated this experiment 5 times for each given angle.

Since our fishing line snapped during preliminary tests of this experiment, we needed to cut new line. An error in our first experimental set up was the center of the protractor is not parallel with the axis of rotation of the pendulum. Another oversight was the axis of rotation of our pendulum would slide around the C-clamp at high angles.

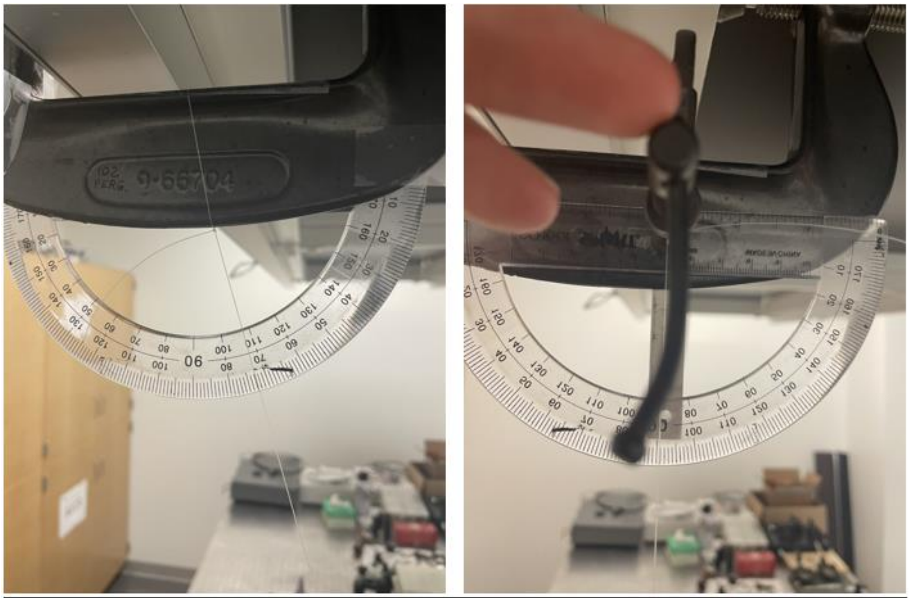


Figure 4. On the left you can see the knot slipping and moving along the C-clamp. Notice also that the center of the protractor is not in line with the knot so it would be impossible to make accurate angle readings. On the right, is our new set up that fixes these issues.

In Figure 4, you can see our new pendulum set up fixed both issues by putting the axis of rotation on a cylinder, so it doesn't slide around, and we lined up the protractor with the axis of rotation.

Data for Damped Oscillator

To measure the damping and get a rough estimate for the resistivity of air, c, we gathered some data on how much the pendulum damped after 10 periods. Since the pendulum is damped, and thus changes the amplitude of pendulum, it would add error in are anharmonicity experiment (we don't want the amplitude to change over time because we are seeing how the period changes due at different amplitudes). At a range of angles we measured how many degrees the pendulum damped after 10 periods and Table 2 showcases those values. Using our smallest angle to try to stay in the small angle approximation, we can estimate c by using the solution to the damped harmonic oscillator.

Data for Physical Pendulum

To get the moment of inertia of the physical pendulum I used (8). We did not have data on the mass of the fishing line but it can be calculated by using the density [3], length and radius [4] of the monofilament fishing line. And the values from Table 1 are needed for the full calculation. Using the same uncertainty package in python we propagated the error.

RESULTS

1 Period

The distribution of the data follows a normal distribution with the peak at the mean which is 2.677. We were able to measure the time in period with a precision of a tenth of a second which is represented by the standard deviation. The standard deviation of the mean is calculated as $\frac{\sigma}{\sqrt{n}}$, where σ is the standard deviation and *n* is the number of points in the dataset.

To calculate the analytical value for the period we use the small angle approximation and length of the fishing line, the period is given by (3). Where $l = S + H + R$, and S is the fishing line string length, H is the hook length, and R is the radius of the bob. The uncertainty in l is given by $\delta l = \delta S + \delta H + \delta R$, the $\boldsymbol{\delta}$ represents the uncertainty in the corresponding quantities, given in Table 1.

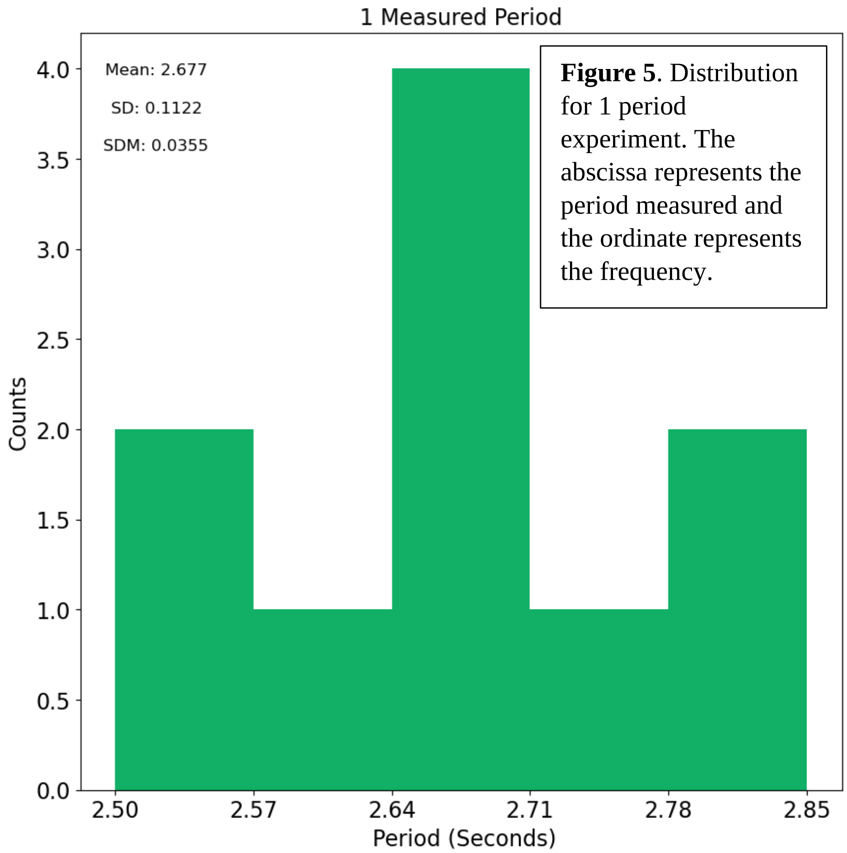


Figure 5. Distribution for 1 period experiment. The abscissa represents the period measured and the ordinate represents the frequency.

δT is calculated using the uncertainties package in python [5]. Using those values and (3) we report the analytical value of the period to be 2.758 (4) seconds. We also calculate the Z-value with respect to the distribution whose standard deviation is given by $\sqrt{\delta T^2 + SE^2}$, where SE is the standard error of our 1 Period distribution. Table 3 summarizes our 1 period experiment. The Z-value is calculated by (9) which tells us how many standard deviations away the measured period, $T_{measured}$, is from the theory T_{theory},

$$Z = \frac{T_{theory} - T_{measured}}{\sqrt{\delta T^2 + SE^2}}. \quad (9)$$

Mean (s)	SD (s)	SDM (s)	Z-value	Probability
2.677	0.112	0.035	2.27	0.97

Table 2. Finalized summary of our 1 period values

Based on this Z-value we calculate the probability of agreement using Table C2 in Bevington [6].

The last column tells that we have a 97% chance of producing a Z-value less than or equal to 2.27σ from the simple pendulum theoretical value. Therefore, it is unlikely (3% chance) that a simple pendulum model is in agreement with our data.

10 Periods

Table 4 summarizes the results for our 10-period experiment. It is important to note that we removed one data point that was 13 standard deviations away from the mean from Run 2. After doing that we get the following: The 10-period data was collected on the same day as the 1 period data. The analytical value is assumed to be the same as that reported in the 1 period data section. Moreover, the Z-values and the probabilities are calculated based on that value. Figure 6 shows the lowest and highest SDM out of our 9 runs. The rest of the histograms are put in the appendix.

The average SD for all our datasets for the 10-period experiment is 0.007, and the ratio of the SD for the 1 period data and this average SD is 16.23. We expect this behavior because we measured the total time for all 10 periods and then divided this total time by 10 to get a single period.

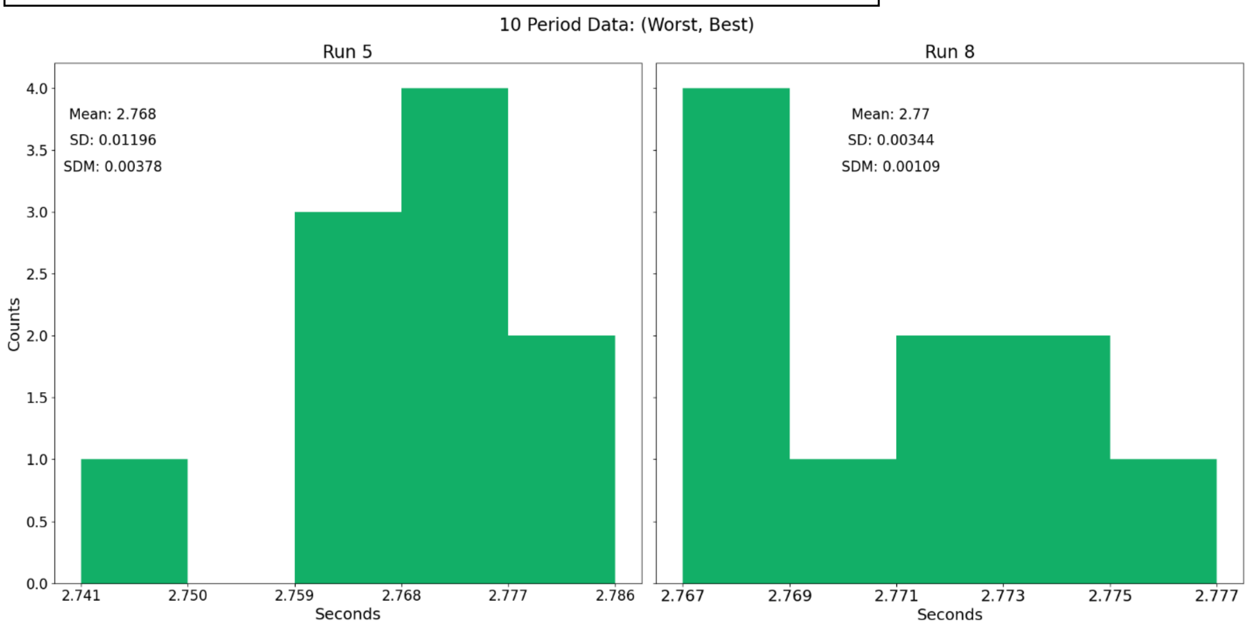
Runs	Mean (s)	SD (s)	SDM (s)	Z-value	Probability
1	2.762	0.008	0.002	0.80	0.57
2	2.768	0.008	0.003	2.13	0.96
3	2.766	0.009	0.003	1.63	0.89
4	2.763	0.005	0.002	1.21	0.99
5	2.768	0.011	0.004	1.82	0.93
6	2.767	0.007	0.002	2.10	0.96
7	2.769	0.005	0.002	2.65	0.99
8	2.770	0.003	0.001	2.99	0.99
9	2.776	0.005	0.002	4.12	0.99

Table 4. (Above) data from all 9 runs

Figure 6. (Below) Best and worst histogram based on their SDM

Furthermore, since the random error distributes uniformly for the 1 period and 10 period data. Thus, the standard deviation goes down by an order of magnitude when we divide the total time by 10 to get the period for a single period.

We also plotted the means of our 9 datasets, Figure 7, by the definition of SDM we expect the standard deviation of this distribution of the means to be equal to the SDM. Since



this value varies for all datasets, we calculate the mean of the SDM's and compare it to the standard deviation of the mean distribution. Mean of SDM's = 0.002 and the SD of the mean distribution is 0.004.

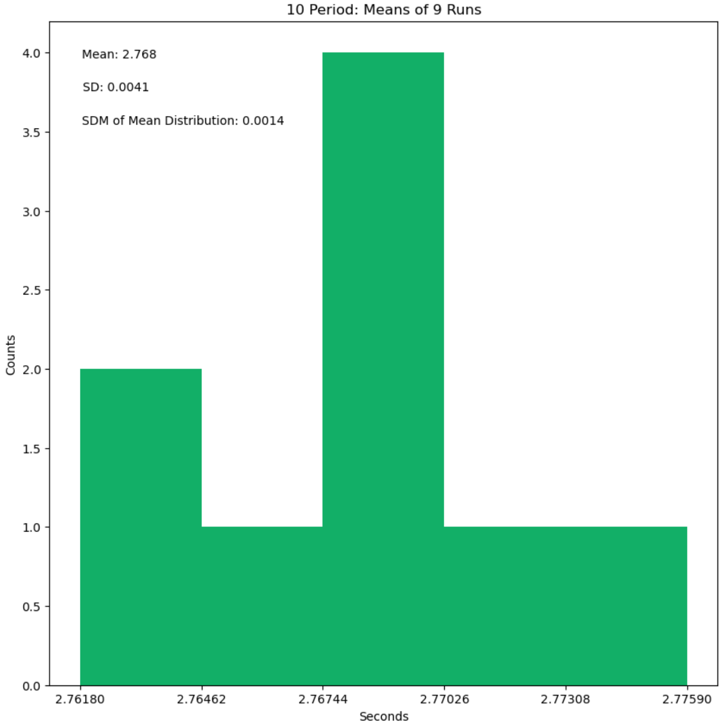


Figure 7. Mean of each of the 9 runs binned into a single histogram.

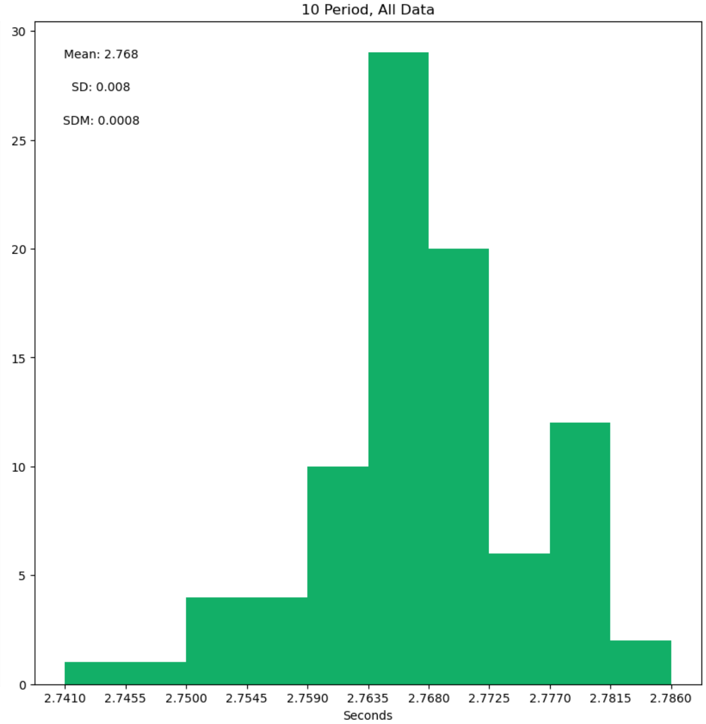


Figure 8. Histogram of all the data points from the 9 runs.

We combined all our 10 period datasets to get one dataset with 89 points, the results for this data along with the distribution of the data shown in Figure 8.

Mean	SD	SDM	Z value	Probability
2.768	0.008	0.0008	2.4	0.98

Table 4. Finalized summary of all data from the 9 Runs.

We expect the random error to distribute uniformly around all the datasets, which can be noticed here. The 10 period datasets have a mean standard deviation of 0.007 and the standard deviation of the 89-point dataset 0.008. We also expect the SDM of the 89-point dataset to go down by factor $\sqrt{8.9} \approx 3$, the mean of SDM's for 10 periods dataset is 0.002 and therefore 0.0008, is approximately 3 times the SDM for the 89-point dataset.

We combined the 1 period and 10 periods datasets to report a single value, the equation for the reported value is

$$X_{avg} = \frac{\sum_j \frac{x_j}{\sigma_j^2}}{\sum_j \frac{1}{\sigma_j^2}}, \qquad (10)$$

where x_j represent the means for the individual datasets and σ_j is the corresponding standard deviation. The total uncertainty is given by,

$$\sigma_{tot} = \sqrt{\frac{1}{\sum_j(\frac{1}{\sigma_j^2})}}. \qquad (11)$$

1-10 Period	Z-value	Probability
2.768(2)	2.25	0.97

Table 5. Combined 1 and 10

The Table 5 summarizes the results for this combined value. The Z-value and probability are calculated based on the true value and uncertainty used for 10 and 1 period data.

100 periods

As we conducted the 100-period experiment on a different day, we found that the length of our string stretched, but we did not measure the elongation of the string in that period. We cut out a new string and conducted this experiment with this new length.

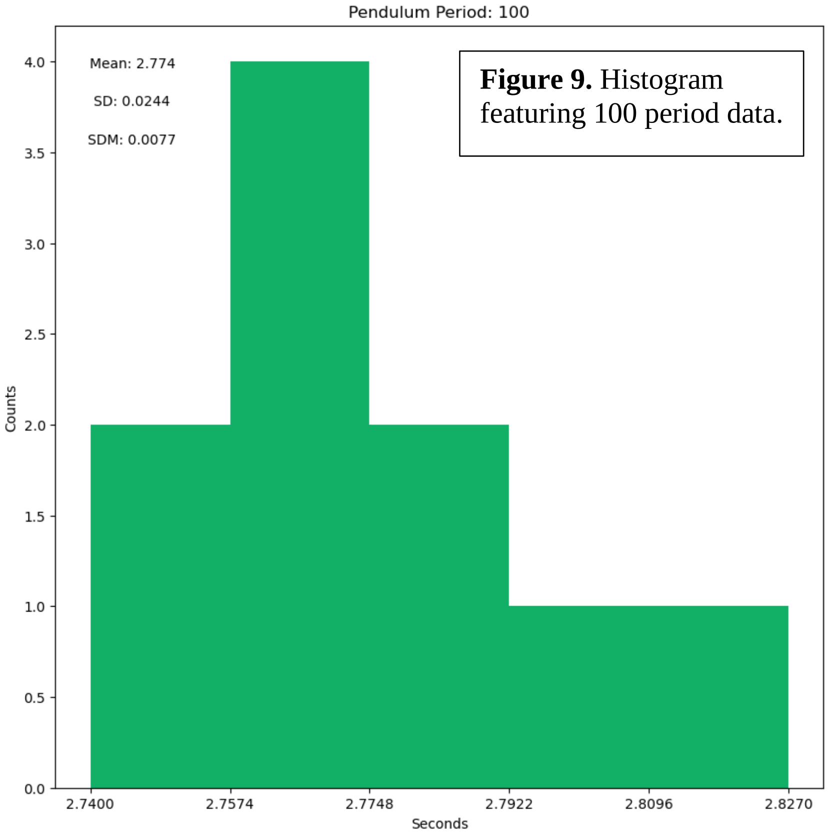


Figure 9. Histogram featuring 100 period data.

Our mean values are reported in Table 1 along with the error. We will not be combining the results of 100 period with the 1 period and 10 period results due to a change length of the fishing line. The Table 6 summarizes our results.

100 Period	Z-value	Probability
2.774(8)	1.62	0.95

Table 6. 100 period data.

Note that we would expect our standard deviation to decrease by 2 orders of magnitude as compared to the 1 period dataset but that does not happen and is evidence of human error involved in the process. Possibly we could have miscounted the number of periods which would through off our SDM. We do not expect any systematic error to cause this because we do not see such behavior creeping in the 10-period dataset.

Anharmonicity

The data we used to measure the anharmonicity of the pendulum is given in the Table 7.

Initial Amplitude (Degrees)	Mean	SD	SDM	Decrease in Amplitude (degrees)
60	2.600	0.018	0.008	6
55	2.570	0.017	0.008	6
45	2.509	0.013	0.005	4.5
35	2.461	0.003	0.001	2
25	2.429	0.010	0.004	1.5

15	2.411	0.012	0.005	1

Table 7. Summary of anharmonicity data.

The uncertainty in amplitude is the number of degrees the amplitude fell by during the 10-period interval. To obtain a horizontal error bar in both direction we redefine our angle by the following equation,

$$x_i' = x_i - \frac{\delta x_i}{2}\,, \tag{12}$$

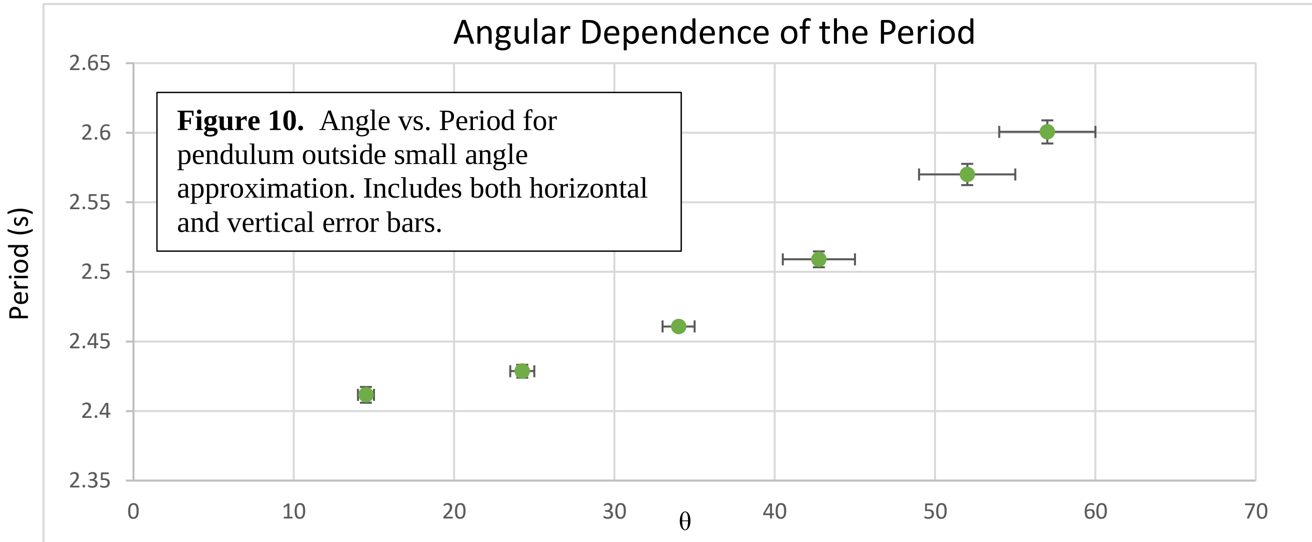


Figure 10. Angle vs. Period for pendulum outside small angle approximation. Includes both horizontal and vertical error bars.

where x_i' represents the redefined angle which will be used as the abscissa in the plot, x_i represents the initial amplitude defined in Table 7, and δx_i represents the decrease in amplitude due to air drag. By applying equation (11) we can define a horizontal error bar in both directions instead of just one. The vertical error bars are taken to be the SDM.

To decide whether our data is anharmonic we fit the curve to a parabola, to do that we first convert our horizontal error bars into vertical error bars. To achieve we first ignore the horizontal error bars and do a $y = ax + b$ fit to the data using Bevington's fit. To convert horizontal error bars to vertical we use the equation,

$$\delta y_i' = a\,\delta x_i. \tag{13}$$

Where $\delta y_i'$ represents the vertical error bar contributed by the horizontal error bar δx_i. The final vertical error bar is given by,

$$\delta y_i^{final} = \sqrt{\delta y_i'^2 + \delta y_i^2}\,. \tag{14}$$

Where δy_i represents the vertical error bar (SDM) is given in Table 7.

Figure 12 shows the final plot after converting horizontal error bars to vertical error bars. The next job is to fit this curve to a parabola of the form $y = ax^2 + b$ and optimize the reduced χ^2 to be as close as to 1 which Figure 12 also features in the orange line. We used trial and error method to achieve this. The final equation we arrived at is $y = 6.08 \times 10^{-5}x^2 + 2.3905$, with a reduced χ^2 of 1.03.

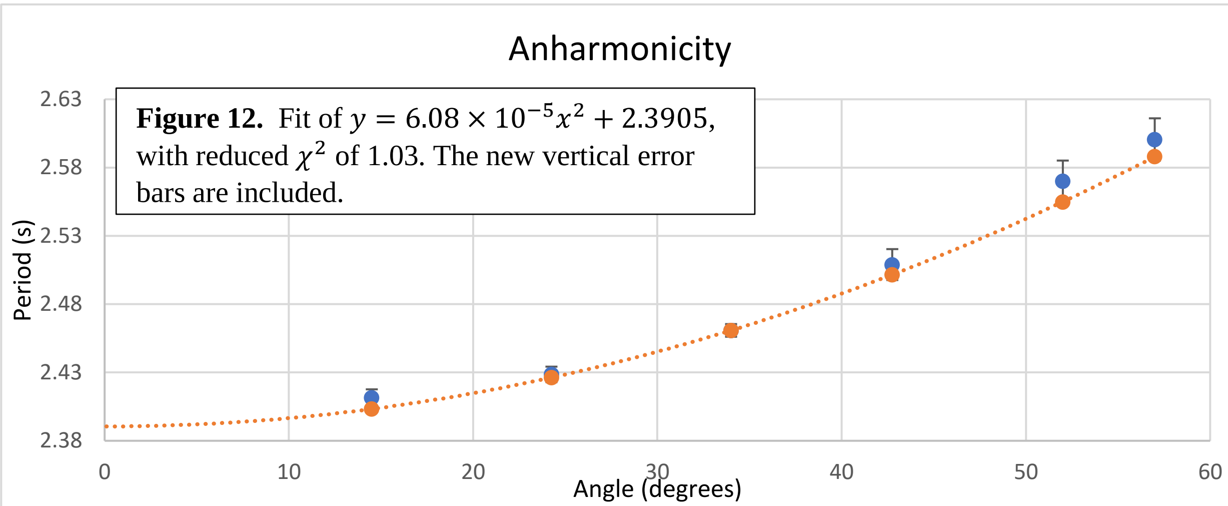


Figure 12. Fit of $y = 6.08 \times 10^{-5}x^2 + 2.3905$, with reduced χ^2 of 1.03. The new vertical error bars are included.

Since the reduced χ^2 is close to 1 we conclude the fit to be good and therefore our data is anharmonic for the length of the string we conducted this experiment with.

Damping Coefficient

To calculate the damping coefficient, we used the data in Table X. This gives us the how much the amplitude damped after 10 periods. To stay in the small angle approximation, we are forced to use the damping at 15 degrees (1.58% error for $\sin(\theta) \approx \theta$). All that is left to do is solve (5) for the exponent (the cosine will be 1 because we are at $10T_A$, where T_A is the period of the pendulum). Using the period from our anharmonicity data (keep in mind the pendulum has a different length than our other data sets), we solve for c,

$$c = \frac{2m}{10T_A} \ln\left(\frac{15}{14}\right) \approx 0.0018, \quad (15)$$

where we measured $T_A = 2.4116$ seconds.

DISCUSSION

In this section, we finally explore how much of an effect the damped pendulum and the damped physical pendulum have on our period. The calculation of "c", to measure the damping, was done with a pendulum with a different length than our other data. So we must compare that experimental period, to our different theories. Table 8 shows the different periods calculated by theory and measured in the experiment for this length given in Table 1.

Pendula	Measured	Simple	Damped	Damped Physical
Period (s)	2.412(5)	**2.416**(4)	**2.416**(4)	**2.416**(4)

Table 8. Shows the period of our pendulum compared to the different theories

The simple, damped, and physically damped pendulums all give the same value up 3 significant digits. Thus, the effect is negligible and does not contribute to the error in our other data. So we can safely use a simple pendulum approximation and not consider, damping or it being a physical pendulum.

	Measured (1-10 Combined)	Simple Pendulum
Period (s)	2.768(2)	2.758(4)

Table 9. Final theory and quoted measured value

Table 9 presents our final measured value from the 10 and 1 period data with our best theory we can compare it to. We do not compare it to the damped cases because it does not significantly change the period and we do don't have data for damping at the length of fishing line used in the 1-10 runs.

In conclusion, we did not correctly account for the errors in our experiment since our best theory is 2.25 standard deviations away from the mean. Which gives only a 3% of our data and error agreeing with the theoretical value. Either we underestimated our random error or there is a large systematic error that is unaccounted for.

Human error is likely the culprit for our differences in measured and theory. A way to test this would be to program a pendulum swinging with a defined period and replicate the experiment with the simulated pendulum. Some other effects we didn't consider is the stretchiness of the fishing line, the buoyancy of the air, the angular deviation Lincoln is from vertical (0.001 degrees)[2], and more.

References:

[1] https://www.britannica.com/technology/pendulum

[2] https://www.wolframalpha.com/widgets/view.jsp?id=e856809e0d522d3153e2e7e8ec263bf2

[3] https://www.gameandfishmag.com/editorial/which-fishing-line-is-right-for-you/

[4] https://www.wildlifedepartment.com/ooj/choosing-right-fishing-line#:~:text=10%2Dpound%20monofilament%2C%20which%20is,008%20inches

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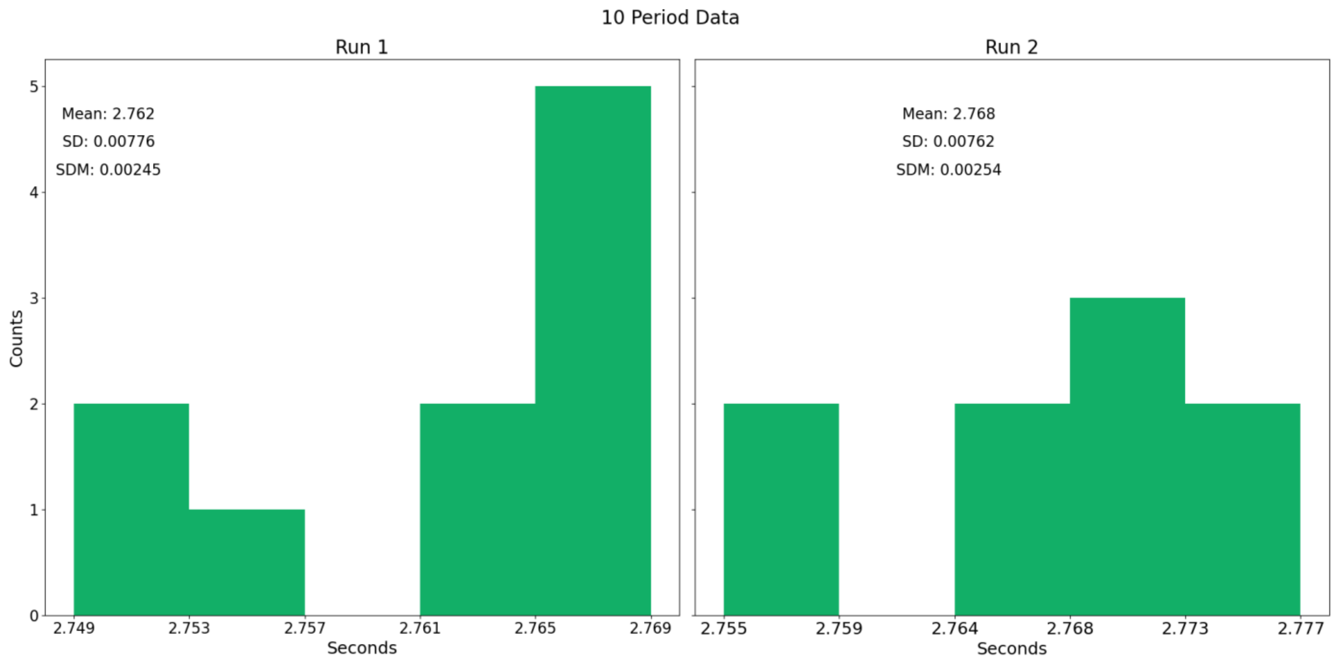
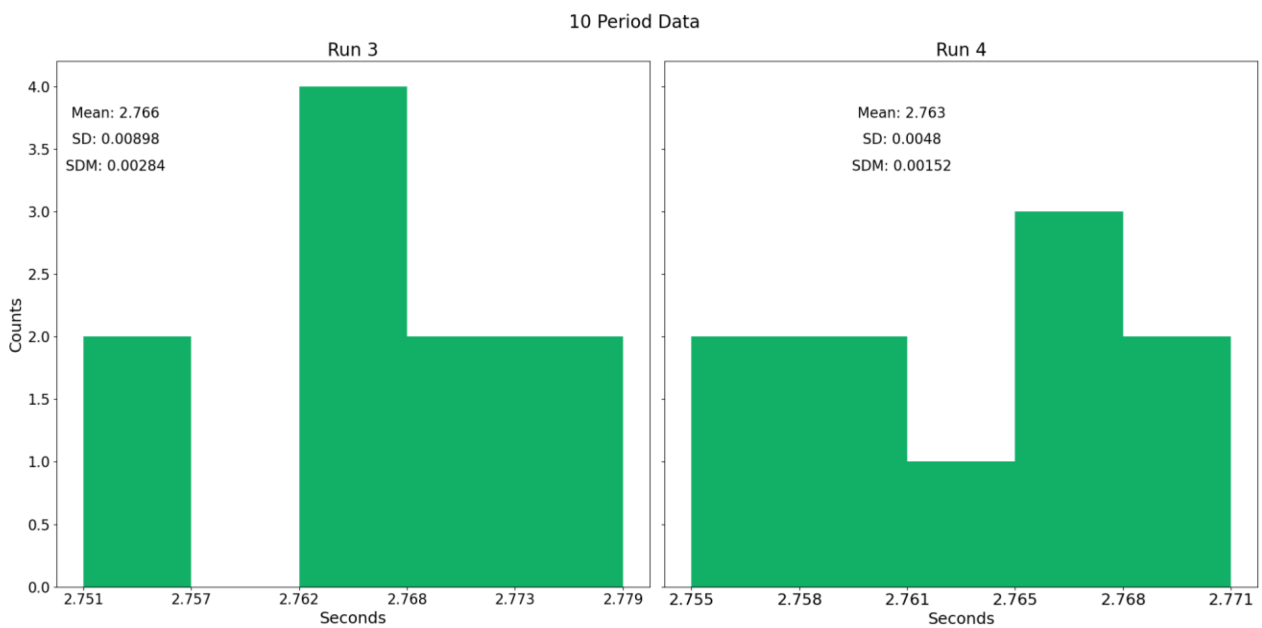
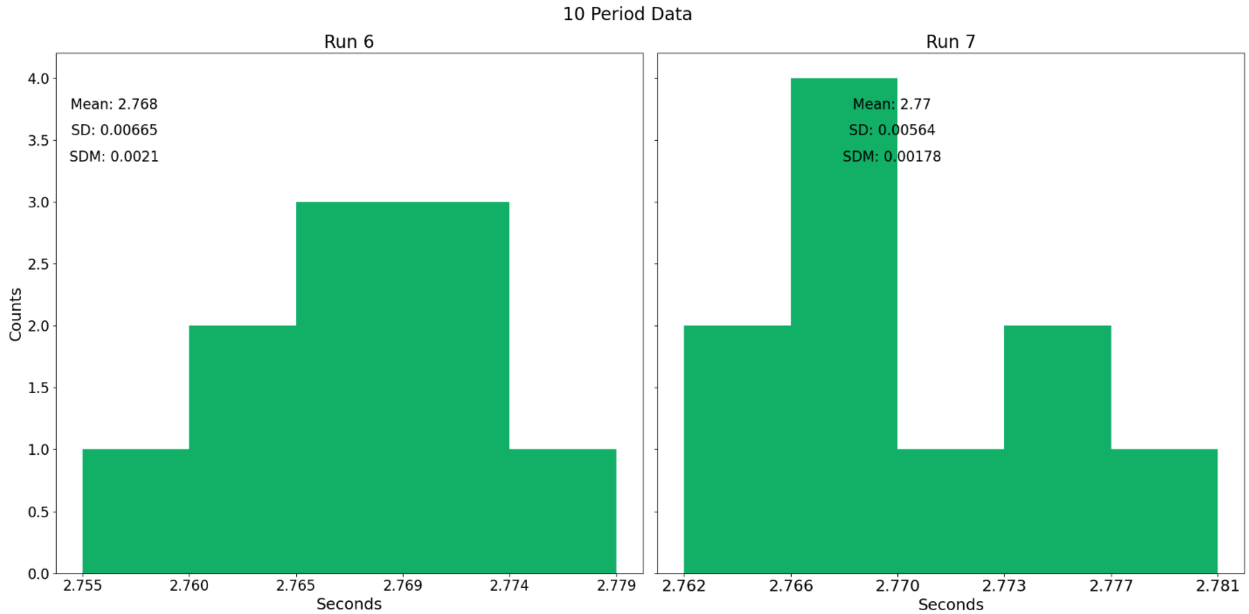
[6] Data Reduction and Error Analysis for the Physical Sciences by Philip R. Bevington and D. Keith Robinson. Published 1969. ISBN-13: 978-0072472271

Appendix:

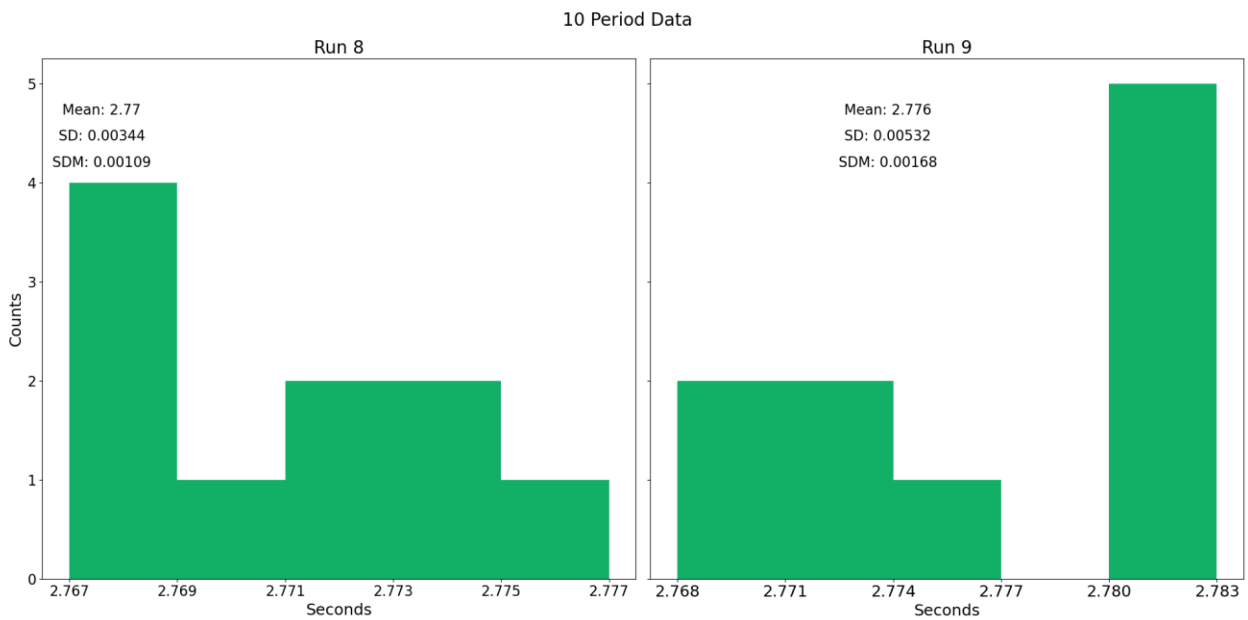
10 Period Runs 1-4, 6-9,







10 Period Data
Run 6
Run 7
Mean: 2.768
SD: 0.00665
SDM: 0.0021
Mean: 2.77
SD: 0.00564
SDM: 0.00178
Counts
Seconds
Seconds



10 Period Data
Run 8
Run 9
Mean: 2.77
SD: 0.00344
SDM: 0.00109
Mean: 2.776
SD: 0.00532
SDM: 0.00168
Counts
Seconds
Seconds