sis in quantum

- vectors have components, Vector $\vec{A}$: $A_x = \hat{i} \cdot \vec{A}$, $A_y = \hat{j} \cdot \vec{A}$

↳ [imagine you have a different set of axes to describe $\vec{A}$] → $A_x' = \hat{i}' \cdot \vec{A}$, $A_y' = \hat{j} \cdot \vec{A}$

but both are the same vector.

[They just have different bases] $\vec{A} = A_x \hat{i} + A_y \hat{j} = A_x' \hat{i}' + A_y \hat{j}'$

(All normalizable functions)

- In quantum mechanics we have a vector that lives in Hilbert space.

↳ when we get a solution we get n states that are mutually orthogonal (or you can make them that way by Gram-schmidt process) that form a basis.

↳ we have a set of mutually orthogonal vectors that span Hilbert Space. (in Hilbert then)

$|\alpha\rangle \Rightarrow \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\langle f|g\rangle = \int_{-\infty}^{\infty} f^* g \, dx$, $\langle f|f\rangle = \delta$

↳ the vector $|S(t)\rangle$ has components inside Hilbert Space but $\Psi(x,t)$ is the x-component in Hilbert Space

↳ Ψ can be 3 dimensional

$\boxed{\langle x|S(t)\rangle = \Psi(x,t)}$, remember that represents a dot product $\langle f|g\rangle = \int f^* g \, dx = \delta$, 1 if $f = g$, 0 otherwise bec/ they are $\perp$!!!

OR

$\hat{x}|x\rangle = x|x\rangle$

Momentum space → $\boxed{\Phi(p,t) = \langle p|S(t)\rangle}$, $\hat{p}|p\rangle = p|p\rangle$

$\boxed{c_n = \langle n|S(t)\rangle}$ where $|n\rangle$ for the n^{th} eigenfunction of $\hat{H}$

$|S(t)\rangle \to \underbrace{\int \Psi(y,t)\, \delta(x-y)\, dy}_{\Psi(x,t)} = \underbrace{\int \Phi(p,t) \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} dx}_{\Psi(x,t)} = \underbrace{\sum c_n e^{-iE_nt/\hbar} \psi_n(x)}_{\Psi(x,t)}$

↳ $\langle \delta(x-y)|\Psi\rangle$

Matrices (linear transformations) Overview

↳ $|\alpha\rangle \to |\alpha'\rangle = \hat{T}|\alpha\rangle$ (transforms $|\alpha\rangle$ into another vector)

↳ $\hat{T}(a|\alpha\rangle + b|\beta\rangle) = a(\hat{T}|\alpha\rangle) + b(\hat{T}|\beta\rangle)$ (linearity)

if these are the basis vecs — $\hat{T}$ transforms them to a new basis!

Suppose taking every vector and multiplying it, rotating it etc...
↳ you transform each of the basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, \dots\}$

$$\hat{T}|e_1\rangle = T_{11}|e_1\rangle + T_{21}|e_2\rangle + \dots + T_{n1}|e_n\rangle$$, transforms $|e_1\rangle$ to a general linear combo of the other basis vecs

$$\hat{T}|e_2\rangle = T_{12}|e_1\rangle + T_{22}|e_2\rangle + \dots + T_{n2}|e_n\rangle$$

$$\hat{T}|e_n\rangle = T_{1n}|e_1\rangle + T_{2n}|e_2\rangle + \dots + T_{nn}|e_n\rangle$$

↳ [more compactly] → $\hat{T}|e_j\rangle = \sum_{i=1}^{n} T_{ij}|e_i\rangle \quad (j = 1, 2, \dots, n)$

(arbitrary vector) → $|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \dots a_n|e_n\rangle = \sum_{j=1}^{n} a_j|e_j\rangle$

[Transform $|\alpha\rangle$] → $\hat{T}|\alpha\rangle = \sum_{j=1}^{n} a_j(\hat{T}|e_j\rangle) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_j T_{ij}|e_i\rangle = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} T_{ij}a_j\right)|e_i\rangle$

compare!

[looks like $\hat{T}$ takes a vector w/ components $a_1, a_2, \dots a_n$ into a vector] → $a_i' = \sum_{j=1}^{n} T_{ij}a_j$

↳ from $\underbrace{\hat{T}|e_j\rangle}_{\text{new transformed vector}} = \sum_{i=1}^{n} T_{ij}|e_i\rangle$ dot each side with $|e_i\rangle$

$|e_j'\rangle = \hat{T}|e_j\rangle$

↓

$\langle e_i|e_j'\rangle = \langle e_i|\hat{T}|e_j\rangle$

$= T_{ij}$ (looking at RHS)

$$\boxed{T_{ij} = \langle e_i|\hat{T}|e_j\rangle}$$

* really awesome, look where $\hat{i}$ and $\hat{j}$ go
* then make those the columns of your matrix *

where $|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \dots + a_n|e_n\rangle$

$$\hat{T} = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

where $|e_1\rangle$ goes (first column); where $|e_n\rangle$ goes (last column)

$$\to |\alpha'\rangle = \hat{T}|\alpha\rangle = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & & & \vdots \\ T_{n1} & \cdots & & T_{nn} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$= a_1\begin{pmatrix} T_{11} \\ T_{21} \\ \vdots \\ T_{n1} \end{pmatrix} + a_2\begin{pmatrix} T_{12} \\ T_{22} \\ \vdots \\ T_{n2} \end{pmatrix} + \dots + a_n\begin{pmatrix} T_{1n} \\ T_{2n} \\ \vdots \\ T_{nn} \end{pmatrix}$$

$$= a_1\hat{T}|e_1\rangle \quad a_2\hat{T}|e_2\rangle + \dots + a_n\hat{T}|e_n\rangle$$

where $|e_1\rangle$ goes; where $|e_2\rangle$ goes

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{matrix} a_1|e_1\rangle \\ a_2|e_2\rangle \end{matrix}$$

$$= a_1\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} + a_2\begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_1T_{11} + a_2T_{12} \\ a_1T_{21} + a_2T_{22} \end{bmatrix}$$

Dirac notation

$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$, $\langle\alpha| = |\alpha\rangle^\dagger = (a_1^* \; a_2^* \cdots)$ and $\langle\alpha|\beta\rangle = a_1 b_1 + a_2 b_2 + \cdots$

[Lets say we have] $\rightarrow |V\rangle = \sum_{i=1} V_i |i\rangle$, $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, ...

[Dot it w/ $|j\rangle$] $\rightarrow \langle j|V\rangle = \sum_i V_i \underbrace{\langle j|i\rangle}_{\delta_{ji}} = V_j$ cool!

[Identity operator] $\rightarrow |V\rangle = \sum |i\rangle V_i \rightarrow \langle i|V\rangle = V_i$ ← dot w/ $|i\rangle$

outer product and it has to equal 1 or not transform $|V\rangle$ at all because look at the results

$\hookrightarrow |V\rangle = \sum |i\rangle\langle i|V\rangle = \left(\sum |i\rangle\langle i|\right)|V\rangle$

$\hookrightarrow$ calculate: $|1\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(1\,0\,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $|2\rangle\langle 2| = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0\,1\,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, etc..

for this 3D case: $\sum |i\rangle\langle i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$ = identity matrix

"component in dir of $|\alpha\rangle$"

[For non basis (orthonormal) vectors its the projection matrix where it defined] $\rightarrow$ $\boxed{\hat{P} = |\alpha\rangle\langle\alpha|}$, lets see it in action,

$\hat{P}|\beta\rangle = |\alpha\rangle\langle\alpha|\beta\rangle$, $|\beta\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

x-component of β along the vector $|\alpha\rangle$ in that dir too

$\hookrightarrow (|\alpha\rangle\langle\alpha|)|\beta\rangle$, (lets say we want to project onto x-axis) $\rightarrow |\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow |\alpha\rangle\langle\alpha| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = a|\alpha\rangle$

[We can also define a vector] $\rightarrow |\beta\rangle = \sum c_n |e_n\rangle$ where $c_n = \langle e_n|\beta\rangle$

(ex) $\vec{\beta} = (\vec{V}\cdot\hat{i})\hat{i} + (\vec{V}\cdot\hat{j})\hat{j} + (\vec{V}\cdot\hat{k})\hat{k}$

$\hat{H} = \varepsilon(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$ where $|s\rangle = a|1\rangle + b|2\rangle$, $a^2 + b^2 = 1$

$\hat{H}|s\rangle = E|s\rangle = Ea|1\rangle + Eb|2\rangle$ and $\hat{H}|s\rangle = \varepsilon(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)(a|1\rangle + b|2\rangle)$

$\hookrightarrow = \varepsilon(a|1\rangle - 0 + 0 + a|2\rangle + 0 - b|2\rangle + b|1\rangle + 0) = \varepsilon((a+b)|1\rangle + (a-b)|2\rangle)$

Then, $\begin{bmatrix} \varepsilon(a+b) \\ \varepsilon(a-b) \end{bmatrix} = \begin{bmatrix} Ea \\ Eb \end{bmatrix} \rightarrow \begin{matrix} Ea = \varepsilon(a+b) \\ Eb = \varepsilon(a-b) \end{matrix}$ ⇝ now solve for E $\rightarrow \boxed{E = \pm\sqrt{2}\,\varepsilon}$

$\hookrightarrow$ plug back in $\pm\sqrt{2}\,\varepsilon a = \varepsilon(a+b) \rightarrow a(\pm\sqrt{2} - 1) = b$

$|s_\pm\rangle = a|1\rangle + (\pm\sqrt{2} - 1)a|2\rangle = a\begin{bmatrix} 1 \\ \pm\sqrt{2} - 1 \end{bmatrix}$

chop up the bra and ket of $\langle\alpha|\beta\rangle$ into $\langle\alpha|$ and $|\beta\rangle$

$\langle f| = \int f^* [\dots] dx$ instruction to integrate (in function space)

[In vector space] $\rightarrow$ $|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\langle\beta| = (b_1^*, b_2^*, b_3^*, \dots b_n^*)$

$\hat{P} \equiv |\alpha\rangle\langle\alpha| = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \underbrace{(a_1^*, a_2^*, \dots a_n^*) \begin{bmatrix} \vdots \end{bmatrix}}_{\text{scalar}}$ [waiting to pick out any portion of any other vector that lies along $|\alpha\rangle$]

$\hookrightarrow \hat{P}|\beta\rangle = (\langle\alpha|\beta\rangle)|\alpha\rangle$ (projection operator) onto 1D subspace spanned by $|\alpha\rangle$

[Also in Dirac notation] $\rightarrow$ $\langle e_m|e_n\rangle = \delta_{mn}$ then $\sum_n |e_n\rangle\langle e_n| = 1$ (identity operator)

$\hookrightarrow \sum(\underbrace{\langle e_n|\alpha\rangle}_{\text{scalar } a_1, a_2, \text{ etc.}})|e_n\rangle = |\alpha\rangle$

$\hookrightarrow \hat{A}\hat{A}^{-1} = |e_n\rangle\langle e_n| = \hat{I}$

Hermitian conjugate

- for scalar $a^\dagger = a^*$

conjugate (bec/ it can be imaginary) transpose (for dot product)

- for vector $[|\psi\rangle]^\dagger = \langle\psi|$ and $[\langle\phi|]^\dagger = |\phi\rangle$ hermitian conjugate

- for operator (more complicated first) $\rightarrow$ $\langle\phi|\hat{A}\psi\rangle = \langle\hat{A}^\dagger\phi|\psi\rangle$

Here is how it is consistent w/ our def of hermitian conjugate

$\rightarrow$ another definition $\langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi\rangle^*$

$\hat{A} = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}$ then $\hat{A}^\dagger = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}$

Check $|\phi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ and $|\psi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$ use definition $\langle\phi|\hat{A}\psi\rangle = \langle\hat{A}^\dagger\phi|\psi\rangle$

$\left\langle \begin{bmatrix} a \\ b \end{bmatrix} \middle| \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \middle| \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle$

$\left\langle \begin{bmatrix} a \\ b \end{bmatrix} \middle| \begin{bmatrix} c \\ ci + d \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} a - bi \\ b \end{bmatrix} \middle| \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle$

$[a^* \; b^*] \begin{bmatrix} c \\ ci + d \end{bmatrix} = [(a^* + b^* i) \; b^*] \begin{bmatrix} c \\ d \end{bmatrix}$

$a^* c + b^* d + b^* ci = a^* c + b^* d + b^* ci$ ✓

operators & linear transformations to vectors in Hilbert Space

$|\beta\rangle = \hat{Q}|\alpha\rangle$, operators are linear transformations on Hilbert Space

$\hookrightarrow \left[|\alpha\rangle = \sum a_n |e_n\rangle, \ |\beta\rangle = \sum b_n |e_n\rangle \right]$, $\left[a_n = \langle e_n|\alpha\rangle , \ b_n = \langle e_n|\beta\rangle \right]$

[The elements of the matrix is] $\rightarrow \langle e_m | \underbrace{\hat{Q} | e_n\rangle}_{\text{where the original basis is sent to}} \equiv Q_{mn}$

$\hookrightarrow |\beta\rangle = \hat{Q}|\alpha\rangle \rightarrow \sum b_n |e_n\rangle = \sum a_n \hat{Q}|e_n\rangle$

[take inner product w/ $|e_m\rangle$] $\rightarrow \sum b_n \underbrace{\langle e_m|e_n\rangle}_{\delta_{mn}} = \sum a_n \langle e_m | \overbrace{\hat{Q}|e_n\rangle}^{\text{transforming the } e_n \text{ basis}}$

$\boxed{b_m = \sum_n Q_{mn} a_n}$ → the $\hat{Q}$ matrix tells you how the components of $|\alpha\rangle$ transform to $|\beta\rangle$

Example (doing Quantum in the language of linear algebra)

a system with states $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\hookrightarrow |S\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, $|a|^2 + |b|^2 = 1$ (most general state)

Define $H = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$ (hamiltonian), h and g are real constants
hermitian matrix

[started w/ schrodinger in position space now its generalized to a state vector in Hilbert space]

[If the system starts out at $t=0$ in state $|1\rangle$, whats it state at t?]

Schrodinger equation
$i\hbar \frac{d}{dt}|S(t)\rangle = \hat{H}|S(t)\rangle \rightarrow \hat{H}|s\rangle = E|s\rangle$ (time ind schrodinger)

determinant is 0

[Solve the eigenvalue equation ⟶] $\det\left(\begin{bmatrix} h-E & g \\ g & h-E \end{bmatrix}\right) = (h-E)^2 - g^2 = 0 \rightarrow h - E = \mp g \rightarrow \boxed{E_{\pm} = h \pm g}$ eigenvalues

[now get your eigenvectors!] eigenvalues $E_+ = h+g$, $E_- = h-g$ $\rightarrow \underbrace{\begin{pmatrix} h & g \\ g & h \end{pmatrix}}_{H} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{|S\rangle} = \underbrace{(h \pm g)}_{E} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{|S\rangle} \Rightarrow h\alpha + g\beta = (h \pm g)\alpha$

mult out
$\hookrightarrow \beta = \pm\alpha$

$\beta^2 + \alpha^2 = 1 \rightarrow \alpha^2 = \frac{1}{2} \rightarrow \alpha = \frac{1}{\sqrt{2}}$

$$\boxed{|S_{\pm}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}}$$

$\beta = \pm\alpha = \pm\frac{1}{\sqrt{2}}$

Now use initial condition $\rightarrow |S(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ (Said it started in state $|1\rangle$)

$2/\sqrt{2}$ in top

$$|S_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm1\end{pmatrix} \rightarrow \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}_{|S_+\rangle} + \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}}_{|S_-\rangle} = \underbrace{\sqrt{2}\begin{pmatrix}1\\0\end{pmatrix}}_{|1\rangle}$$

$$\hookrightarrow \boxed{\frac{1}{\sqrt{2}}\left(|S_+\rangle + |S_-\rangle\right) = \begin{pmatrix}1\\0\end{pmatrix}}$$

$\leftarrow$ getting $\psi(x,0) = \sum c_n \psi_n$ $\leftarrow$ eigenstates

$$\hookrightarrow |S(0)\rangle = \frac{1}{\sqrt{2}}\left(|S_+\rangle + |S_-\rangle\right)$$ now we tack on the $e^{-i\frac{E_n t}{\hbar}}$

$$|S(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i(h+g)t/\hbar}|S_+\rangle + e^{-i(h-g)t/\hbar}|S_-\rangle\right]$$

$$= \frac{1}{2}e^{-iht/\hbar}\left[e^{-igt/\hbar}\begin{pmatrix}1\\1\end{pmatrix} + e^{igt/\hbar}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

$$= \frac{1}{2}e^{-iht/\hbar}\begin{pmatrix}e^{-igt/\hbar} + e^{igt/\hbar}\\ e^{-igt/\hbar} - e^{igt/\hbar}\end{pmatrix} = e^{-iht/\hbar}\begin{pmatrix}\cos(gt/\hbar)\\ -i\sin(gt/\hbar)\end{pmatrix}$$

$\hat{X}$ operator $\rightarrow \begin{cases} x & \text{in position basis} \\ i\hbar\frac{\partial}{\partial p} & \text{in momentum basis}\end{cases}$

$\hat{P}$ operator $\rightarrow \begin{cases} -i\hbar\,\partial/\partial x & \text{in position basis} \\ p & \text{in momentum basis}\end{cases}$

Finding the matrix form $\rightarrow \hat{H} = \varepsilon\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right)$

$$|1\rangle\langle 1| = \begin{pmatrix}1&0\\0&0\end{pmatrix},\quad |2\rangle\langle 2| = \begin{pmatrix}0&0\\0&1\end{pmatrix},\quad |1\rangle\langle 2| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0&1\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix},\quad |2\rangle\langle 1| = \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$\hookrightarrow \hat{H} = \begin{pmatrix}\varepsilon&\varepsilon\\\varepsilon&-\varepsilon\end{pmatrix} \rightarrow \hat{H}|s\rangle = E|s\rangle \rightarrow \underbrace{\begin{pmatrix}\varepsilon-E&\varepsilon\\\varepsilon&-\varepsilon-E\end{pmatrix}}_{\text{take det}}\begin{pmatrix}a\\b\end{pmatrix} = 0$$

$-\varepsilon^2 - \varepsilon E + \varepsilon E + E^2$

$$(\varepsilon-E)(-\varepsilon-E) = \varepsilon^2 \rightarrow E^2 = 2\varepsilon^2$$

$$\boxed{E = \pm\sqrt{2}\,\varepsilon}$$

changing basis in dirac notation

energy $1 = \sum |n\rangle\langle n|$

For $|x\rangle\langle x|$ in position space $1 = \int |x\rangle\langle x|\, dx$, momentum space $1 = \int |p\rangle\langle p|\, dp$

Now $|S(t)\rangle$:

gets component of $S(t)$ along x, in x direction

- $|S(t)\rangle = \int |x\rangle\langle x|S(t)\rangle\, dx \equiv \int |x\rangle\, \Psi(x,t)\, dx$
- $|S(t)\rangle = \int |p\rangle\langle p|S(t)\rangle\, dp \equiv \int |p\rangle\, \Phi(p,t)\, dp$

Lets derive the transformation using these

$\Phi(p,t) = \langle p|S(t)\rangle = \langle p| \left(\int |x\rangle\langle x|\, dx\right) |S(t)\rangle = \int \langle p|x\rangle\langle x|S(t)\rangle$

$\Phi(p,t) = \int \langle p|x\rangle\, \Psi(x,t)\, dx$, $\langle x|p\rangle$ is the momentum eigenstate in the position basis

$\langle x|p\rangle^* = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$, which is f_p^* for $\hat{p} f_p = p f_p$

$$\hookrightarrow \boxed{\Phi(p,t) = \int \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\, \Psi(x,t)\, dx}$$

Dirac notation is nice

$\langle x|\hat{x}|S(t)\rangle = x\, \Psi(x,t)$ and $\langle p|\hat{x}|S(t)\rangle = i\hbar \frac{\partial \Phi}{\partial p}$

Angular Momentum

$\vec{L} = \vec{r} \times \vec{p}$ → written out → $L_x = yP_z - zP_y$, $L_y = zP_x - xP_z$, $L_z = xP_y - yP_x$

↳ corresponding operators then → $P_x = -i\hbar \frac{\partial}{\partial x}$, etc... (no hats on operators here) reduce clutter

[Lets check the commuter relations] → $[L_x, L_y] = [yP_z - zP_y, zP_x - xP_z] =$ $[yP_z, zP_x]$ − ① $[yP_z, xP_z]$ − ② $[zP_y, zP_x]$ + ③ $[zP_y, xP_z]$ ④

② $y\hat{P}_z x\hat{P}_z - x\hat{P}_z y\hat{P}_z = 0$, ③ $zP_y zP_x - zP_x zP_y = 0$

① $yP_z(zP_x) - zP_x(yP_z) = yP_x(P_z z - zP_z) = yP_x[P_z, z]$

④ $[zP_y, xP_z] = xP_y[z, P_z]$

reverse terms w/ a minus sign & know $[P_z, z] = i\hbar$

↳ $[L_x, L_y] = yP_x[P_z, z] + xP_y[z, P_z]$

$$\boxed{[L_x, L_y] = i\hbar(xP_y - yP_x) = i\hbar L_z}$$ the others $$\boxed{\begin{aligned}[L_y, L_z] &= i\hbar L_x \\ [L_z, L_x] &= i\hbar L_y\end{aligned}}$$

Uncertainty

$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\langle i\hbar L_z\rangle\right)^2 = \frac{\hbar^2}{4}\langle L_z\rangle^2$ → $\boxed{\sigma_{L_x}\sigma_{L_y} \geq \frac{\hbar}{2}|\langle L_z\rangle|}$

you cant find states that are simultaneously eigenfunctions of L_x and L_y

$L^2 \equiv L_x^2 + L_y^2 + L_z^2$ does commute.

use identity! $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

$[L^2, L_x] = \cancel{[L_x^2, L_x]}^{0} + [L_y^2, L_x] + [L_z^2, L_x]$

$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$

$= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_y)L_z$ → $\boxed{[L^2, L_x] = 0}$

↳ then $[L^2, \vec{L}] = 0$ as well

Eignstates on the back ⟶

Eigenstates

we can measure $|L|^2$ and $\vec{L}$, w/ arbitrary ξ

Since $[L^2, \vec{L}] = 0$, we can find simultaneous eigenstates

↳ $L^2 f = \lambda f$, $L_z f = \mu f$ ← measure both simultaneously w/ $\sigma = 0$

Define a ladder operator: $L_\pm \equiv L_x \pm i L_y$

↳ [check commutator relations] → $[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x)$

↳ $[L_z, L_\pm] = i\hbar L_y \pm \hbar L_x = \hbar(iL_y \pm L_x) = \pm\hbar(L_x \pm iL_y)$ (works out if you mult)

↳ $\boxed{[L_z, L_\pm] = \pm\hbar L_\pm}$ and since $[L^2, \vec{L}] = 0 \rightarrow \boxed{[L^2, L_\pm] = 0}$

Now, carefully:

from $[L^2, L_\pm] = 0 \rightarrow L^2 L_\pm = L_\pm L^2 \rightarrow L^2(L_\pm f) = L_\pm(L^2 f)$ and if $L^2 f = \lambda f$, $L_z f = \mu f$

↳ $L_\pm(L^2 f) = L_\pm(\lambda f) = \lambda(L_\pm f) \rightarrow \boxed{L^2(L_\pm f) = \lambda(L_\pm f)}$

↳ this says that the eigenfunction f of L^2 also has another eigenfunction

$\boxed{L^2 \text{ other eigenfunction is } L_\pm f \text{ w/ the same } \lambda}$

Now carefully again but w/ $[L_z, L_\pm] = \pm\hbar L_\pm$!

↳ $L_z L_\pm f - L_\pm L_z f = \pm\hbar L_\pm f$

↳ $L_z L_\pm f = \pm\hbar L_\pm f + L_\pm(\mu f) = (\mu \pm \hbar)(L_\pm f)$ ← $L_z f = \mu f$

↳ $\boxed{L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)}$, $L_\pm f$ is an eigenfunction of L_z w/ eigenvalue of $\mu \pm \hbar$

L_+ = raising operator bec it increases the eigenvalue of L_z by $\hbar$

L_- = lowering operator bec it decreases the eigenvalue of L_z by $\hbar$

Important fact

$\langle L^2 \rangle = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle$, $\langle L_x^2 \rangle, \langle L_y^2 \rangle \geq 0$

you can keep increasing L_z (in turn decreasing L_x, L_y) but you cant go past the total so

$\boxed{L_+ f_t = 0}$, f_t for top rung continued

eigenvalue for top rung

$L_+ f_t = 0$ for L_z, $L_z f_t = \overbrace{\hbar l}\, f_t$, for L^2, $L^2 f_t = \lambda f_t$

Now some math trickery;

commutator for $[L_x, L_y]$

$$L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i\overbrace{(L_xL_y - L_yL_x)}$$

↑ add $L_z^2 - L_z^2$

$$L_\pm L_\mp = L^2 - L_z^2 \mp i(i\hbar L_z)$$

or $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$ now use eigen equation

$$L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t = (0 + \hbar^2 l^2 + \hbar^2 l) f_t = \hbar^2 l(l+1) f_t$$

↳ $\boxed{\lambda = \hbar^2 l(l+1)}$

$L^2 f_b = \lambda f_b$

Same argument for bottom rung, $L_- f_b = 0$, let $L_z f_b = \hbar \bar{l} f_b$

↳ use again; $L^2 f_b = (L_+ L_- + L_z^2 - \hbar L_z) f_b = (0 + \hbar^2 \bar{l}^2 - \hbar^2 \bar{l}) f_b = \hbar^2 \bar{l}(\bar{l}-1) f_b$

$\boxed{\lambda = \hbar^2 \bar{l}(\bar{l}-1)}$

bottom rung can't be higher than the top one so!

(two solutions) $\bar{l} = l+1$ or $\boxed{\bar{l} = -l}$ only soln

WAIT! → $l(l+1) = \bar{l}(\bar{l}-1)$

↳ from $L_z f_b = \hbar \bar{l} f_b$, $L_z f_t = \hbar l f_b$] Let $L_z f = m\hbar f$, where $m = -l, \dots l$ (bottom rung, top rung)

we go from $-l$ to l in N integer steps

total # steps

$N = 2l$ → $l = N/2$ so l must be an integer or half integer

↳ $\boxed{L^2 f_l^m = \hbar^2 l(l+1) f_l^m \quad \text{and} \quad L_z f_l^m = \hbar m f_l^m}$

$l = 0, 1/2, 1, 3/2$
$m = -l, -l+1, \dots, l-1, l$

Normalization

$$\hat{L}_\pm f_\ell^m = A_\ell^m f_\ell^{m\pm1}$$

$$\langle \hat{L}_+ f_\ell^m | \hat{L}_+ f_\ell^m \rangle = \langle f_\ell^m | \hat{L}_- \hat{L}_+ f_\ell^m \rangle = \langle f_\ell^m | (L^2 - \hat{L}_z^2 - \hbar \hat{L}_z) f_\ell^m \rangle$$

$$= (\ell(\ell+1)\hbar^2 - m^2\hbar^2 - m\hbar^2) \langle f_\ell^m | f_\ell^m \rangle$$

Spin

- Classically an object has two kinds of angular momentum (arising from the same physical principles): Orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ and spin $\vec{S} = I\vec{\omega}$ motion about the COM
- electrons have intrinsic angular momentum, thus it doesn't depend on any coordinates. (they also have extrinsic angular momentum)

Theory of Spin

$[S_x, S_y] = i\hbar S_z \quad [S_z, S_x] = i\hbar S_y$

Start w/ the commutators of spin like L: $[S_y, S_z] = i\hbar S_x$

spin aren't functions so use dirac notation

From

$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m$ →

$L_z f_\ell^m = \hbar m f_\ell^m$ →

$L_\pm f_\ell^m =$

$$\boxed{\begin{gathered} S^2 |sm\rangle = \hbar^2 s(s+1)|sm\rangle \quad \text{and} \quad S_z|sm\rangle = \hbar m_s |sm\rangle \\ S_\pm |sm\rangle = \hbar\sqrt{s(s+1) - m_s(m_s \pm 1)}\,|s(m\pm 1)\rangle, \quad S_\pm \equiv S_x \pm i S_y \end{gathered}}$$

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and $m = -s, -s+1, \dots, s-1, s$

Spin 1/2, spin is an unchanging amount that all particles have

Two eigenstates: spin up $|\frac{1}{2}\,\frac{1}{2}\rangle$ and spin down $|\frac{1}{2}\,(-\frac{1}{2})\rangle$

[With these vectors the general state of a spin-½ particle can be represented] → [This is a particles spin state, if the particle is moving around then you have to deal w/ its position state Ψ]

a two element column matrix or spinor: $\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\underset{\text{spin up}}{\chi_+} + b\underset{\text{spin down}}{\chi_-}$ w/ $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Notation reiteration, $|sm\rangle$ is a vector that lives out in Hilbert space, a spinor χ is a set of components of a vector w/ respect to the basis $|\frac{1}{2}\,\frac{1}{2}\rangle, |\frac{1}{2}\,(-\frac{1}{2})\rangle$ (for spin 1/2

Apply our S^2 operator on χ

$S^2\chi_+ = \frac{3}{4}\hbar^2\chi_+$ and $S^2\chi_- = \frac{3}{4}\hbar^2\chi_-$

Derive S^2

$S^2 = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$

$\begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} c \\ e \end{pmatrix} = \begin{pmatrix} \frac{3}{4}\hbar^2 \\ 0 \end{pmatrix}, \; c = \frac{3}{4}\hbar^2, \; e = 0$

$\begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{4}\hbar^2 \end{pmatrix}, \; d = 0, \; f = \frac{3}{4}\hbar^2$

Thus: $\boxed{S^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

Similarly w/ S_z;

$S_z \chi_+ = \frac{\hbar}{2}\chi_+$ and $S_z = -\frac{\hbar}{2}\chi_-$, $m = \frac{1}{2}$

? why $m = -1/2$

maybe! actually!!! ↓ of the values

$m = -s$ is spin down $= -\frac{1}{2}$

$m = +s$ is spin up $= \frac{1}{2}$

↳ $\boxed{S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$

↳ $S_\pm |s\,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\,|s\,(m\pm1)\rangle$

↳ $S_+\chi_- = \hbar\sqrt{\frac{3}{4} - \left(\underbrace{-\tfrac{1}{2}\left(-\tfrac{1}{2}+1\right)}_{1/2}\right)}\,\chi_+ = \hbar\sqrt{\frac{4}{4}}\,\chi_+ = \hbar\chi_+$

↳ $S_-\chi_+ = \hbar\sqrt{\frac{3}{4} - \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)}_{-1/2}} = \hbar\sqrt{\frac{4}{4}}\,\chi_+ = \hbar\chi_-$

$S_+\chi_+ = 0$

$S_-\chi_- = 0$

$S_x = \frac{1}{2}(S_+ + S_-)$

↳ $S_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $S_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ now, $S_\pm = S_x \pm iS_y \rightarrow S_y = \frac{1}{2i}(S_+ - S_-)$

Thus! $\boxed{S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \text{ and } S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}$

Compactly! $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

[If you solve the eigenvalue eq for $\vec{S}_z$] → $\chi_+ = \begin{pmatrix}1\\ 0\end{pmatrix}$, (eigenvalue $+\frac{\hbar}{2}$) and $\chi_- = \begin{pmatrix}0\\ 1\end{pmatrix}$, (eigenvalue $-\frac{\hbar}{2}$)

[if you measure S_z on a particle in the general state χ you could get $\frac{\hbar}{2}$ w/ a probability of $|a|^2$ or $-\frac{\hbar}{2}$ w/ probability $|b|^2$ & $|a|^2 + |b|^2 = 1$]

Lets measure S_x and find possible results; $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $S_x|v\rangle = \pm\frac{\hbar}{2}|v\rangle$

↳ $\begin{vmatrix}-\lambda & \hbar/2\\ \hbar/2 & -\lambda\end{vmatrix} = 0 \rightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2 \rightarrow \lambda = \pm\frac{\hbar}{2}$ (possible values for S_x are same for S_z)

eigenspinors: $\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \rightarrow \begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \pm\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ so $\beta = \pm\alpha$

#normalized

↳ [$\chi_+^{(x)} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$, (eigenvalue $+\frac{\hbar}{2}$) and $\chi_-^{(x)} = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$, (eigenvalue $-\frac{\hbar}{2}$)]

↳ $\chi = \frac{(a+b)}{\sqrt{2}}\chi_+^{(x)} + \frac{(a-b)}{\sqrt{2}}\chi_-^{(x)}$

the eigenvalue

if you measure S_x the probability of getting $\frac{\hbar}{2}$ is $\frac{1}{2}|a+b|^2$

and for $-\frac{\hbar}{2}$ is $\frac{1}{2}|a-b|^2$

note $\langle S_x\rangle = \chi^\dagger S_x \chi$

looks like $\Psi = \sum c_n \psi_n$ and $\chi = \begin{pmatrix}a\\ b\end{pmatrix} = a\begin{pmatrix}1\\ 0\end{pmatrix} + b\begin{pmatrix}0\\ 1\end{pmatrix}$ where $\begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\end{pmatrix}$ are your eigenvectors for

$|c_n|^2$: probability

$\frac{\hbar}{2}$ probability of measuring in up state

not always $\begin{pmatrix}1\\ 0\end{pmatrix}$ just for S_z

Adding spin $\chi_1^2 + \chi_2^2 = 1, \quad \chi_2 = \frac{1}{\sqrt{2}}$

combination of two spin-1/2 particles can carry a total spin of 1 or 0 $\hookrightarrow$ then $S^2|1\,0\rangle = 2\hbar^2|1\,0\rangle$

look at S^2 eigenvalues: $\swarrow$ use $S = S^{(1)} + S^{(2)}$

$$S^2 = (S^{(1)} + S^{(2)})\cdot(S^{(1)} + S^{(2)}) = (S^{(1)})^2 + (S^{(2)})^2 + 2S^{(1)}\cdot S^{(2)}$$

$$\hookrightarrow S^{(1)}\cdot S^{(2)}|\uparrow\downarrow\rangle = (S_x^{(1)}|\uparrow\rangle)(S_x^{(2)}|\downarrow\rangle) + (S_y^{(1)}|\uparrow\rangle)(S_y^{(2)}|\downarrow\rangle) + (S_z^{(1)}|\uparrow\rangle)(S_z^{(2)}|\downarrow\rangle)$$

Now use: $S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\left| \chi_+ = |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle ;\ \chi_- = \begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle \right|$

quick show: $\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}|\downarrow\rangle$ same for rest;

$$= \frac{\hbar}{2}|\downarrow\rangle\frac{\hbar}{2}|\uparrow\rangle + \frac{i\hbar}{2}|\downarrow\rangle\frac{-i\hbar}{2}|\uparrow\rangle + \frac{\hbar}{2}|\uparrow\rangle\frac{-\hbar}{2}|\downarrow\rangle = \boxed{\frac{\hbar^2}{4}(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)}$$

$$S^{(1)}\cdot S^{(2)}|\uparrow\downarrow\rangle = \frac{\hbar^2}{4}(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$$

$\swarrow$ by similar method

$$S^{(1)}\cdot S^{(2)}|\downarrow\uparrow\rangle = \frac{\hbar^2}{4}(2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

these sub

$$\hookrightarrow S^{(1)}\cdot S^{(2)}|1\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle + 2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

think like you plugging the $|1\,0\rangle$ eig or just work it out

$$= \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{\hbar^2}{4}|1\,0\rangle$$

$(s(s+1))\hbar^2$
$s=1 \rightarrow 2\hbar^2$

$$S^{(1)}\cdot S^{(2)}|0\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(2|\uparrow\downarrow\rangle - |\uparrow\downarrow\rangle - 2|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{-3\hbar^2}{4}|0\,0\rangle$$

now

$$\hookrightarrow S^2|1\,0\rangle = \Big(\underset{\uparrow}{S^{(1)}}{}^2 + S^{(2)2} + 2\underbrace{S^{(1)}\cdot S^{(2)}}_{\hbar^2/4}\Big)|1\,0\rangle = \left(\frac{3}{4}\hbar^2 + \frac{3\hbar^2}{4} + \frac{2\hbar^2}{4}\right)|1\,0\rangle = \boxed{2\hbar^2|1\,0\rangle}$$

$S^2\chi_+ = \frac{3}{4}\hbar^2\chi_+$

*important note, combined spin 1/2 system findings to get the integers 1 and 0

generally you get every spin from $(s_1 + s_2)$ down to $(s_1 - s_2)$ or $(s_2 - s_1)$ if $s_2 > s_1$

ex) $s_1 = 3/2$ w/ $s_2 = 2$

you get $7/2, 5/2, 3/2$, or $1/2$

Electron in a Magnetic Field

$\vec{\mu} = \vec{m}$ in griffiths

magnetic dipole moment $\vec{\mu} = \gamma \vec{S}$ where γ is the gyromagnetic ratio of its magnetic dipole moment to its angular momentum

$\vec{N} = \vec{m} \times \vec{B}$ (torque of dipole in E & M)

$U = -\vec{m} \cdot \vec{B}$ (energy of torque)

$\hat{H} = -\gamma \vec{B} \cdot \hat{S}$ (spin matrix) (neglects kinetic energy) ← Lorentz force isn't derivable from a potential energy function

Larmor Precession

- a particle of spin 1/2 at rest in a B-field $\vec{B} = B_0 \hat{z}$

↳ $\hat{H} = -\gamma B_0 \hat{S}_z$, recall $\hat{S}_z$ was derived $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for spin 1/2 system

$$\boxed{\hat{H} = -\gamma B_0 S_z = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$\hat{H}\chi_+ = E_+ \chi_+$ since $\hat{H}$ is just S_z it has the same eigenstates of S_z

$\Rightarrow \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = E_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \boxed{E_+ = \frac{-\gamma B_0 \hbar}{2} \text{ and } E_- = \frac{\gamma B_0 \hbar}{2}}$

General soln to the schrodinger eq:

$$i\hbar \frac{\partial \chi}{\partial t} = \hat{H}\chi, \quad \chi = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix}$$

a and b are determined by initial condition → $\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}$, $|a|^2 + |b|^2 = 1$

rewrite $a = \cos(\alpha/2)$, $b = \sin(\alpha/2)$

Now we calculate the expectation value of $\vec{S}$ as a function of time, need $\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle$

$\langle S_x \rangle = \chi(t)^\dagger \hat{S}_x \chi(t)$, χ is a vector and $\langle \chi | = [\chi_+^* \ \chi_-^*]$ ← conjugate transpose which is hermitian conjugate

$\int \psi^* \hat{S}_x \psi \, dx \rightarrow \langle \psi | \hat{S}_x | \psi \rangle$ but now ψ is a vector & unneeded

$$\langle S_x \rangle = \left(\cos(\tfrac{\alpha}{2}) e^{-i\gamma B_0 t/2} \quad \sin(\tfrac{\alpha}{2}) e^{i\gamma B_0 t/2}\right) \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{pmatrix} = \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t)$$

$\langle S_y \rangle = \chi^\dagger S_y \chi = -\frac{\hbar}{2}\sin\alpha \sin(\gamma B_0 t)$

$\langle S_z \rangle = \frac{\hbar}{2}\cos\alpha$

larmor frequency: $\omega = \gamma B_0$

$\langle \vec{S} \rangle$ is tilted at constant angle α to the z-axis and precesses about the field at the larmor frequency

Adding angular momentum

- we have two particles

↳ first is in state $|s_1 m_1\rangle$ and second is in $|s_2 m_2\rangle$

↳ composite state $|s_1 s_2 m_1 m_2\rangle$

$S^{(1)2}|s_1 s_2 m_1 m_2\rangle = s_1(s_1+1)\hbar^2|s_1 s_2 m_1 m_2\rangle$

$S^{(2)2}|s_1 s_2 m_1 m_2\rangle = s_2(s_2+1)\hbar^2|s_1 s_2 m_1 m_2\rangle$

What is $\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$

picks out m_2 (the second one)

$S_z|s_1 s_2 m_1 m_2\rangle = S_z^{(1)}|s_1 s_2 m_1 m_2\rangle + S_z^{(2)}|s_1 s_2 m_1 m_2\rangle = \hbar(m_1+m_2)|s_1 s_2 m_1 m_2\rangle$

Better example

- consider 2 particles, proton + electron in ground state $(m = m_1 + m_2)$

↳ we can have the following states, all the possibilities →

$|\uparrow\uparrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\rangle$, $m=1$

$|\uparrow\downarrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}\rangle$; $m=0$

$|\downarrow\uparrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}\rangle$, $m=0$

$|\downarrow\downarrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}\rangle$, $m=-1$

m is supposed to advance in integer steps from $-s$ to s, theres an extra state of $m=0$

To get around this lets apply the lowering operator to $|\uparrow\uparrow\rangle$, lowering operator steps down m

Here $S_- = S_-^{(1)} + S_-^{(2)}$, where superscript (1), (2) tells you what m it acts on

$$S_-|\uparrow\uparrow\rangle = \underbrace{(S_-^{(1)}|\uparrow\rangle)}_{\text{acts on } m_1}|\uparrow\rangle + |\uparrow\rangle\underbrace{(S_-^{(2)}|\uparrow\rangle)}_{\text{acts on } m_2}, \quad |\uparrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= \hbar|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle(\hbar|\downarrow\rangle) = \hbar(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$$

total s, total m

$|1\,1\rangle = |\uparrow\uparrow\rangle$

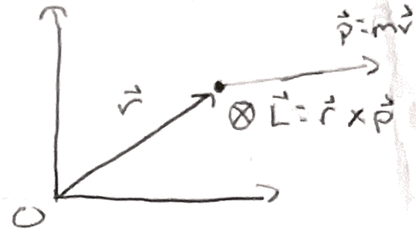
normalize

$|1\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ (lowered m by 1 yay!)

$|1\,-1\rangle = |\downarrow\downarrow\rangle$

s=1 is a triplet

is $s=0$ & $m=0$ $\quad |0\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ s=0 singlet



Precession and Spin

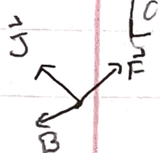
$$\vec{L} = \vec{r} \times \vec{p}$$

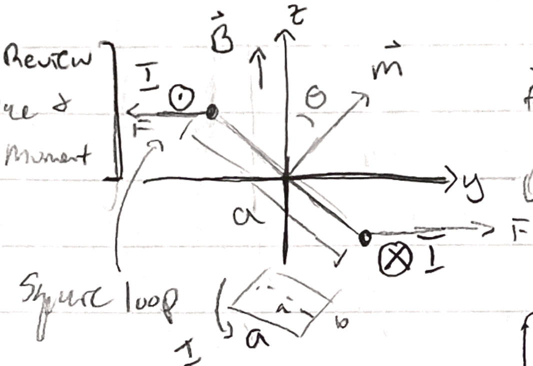
[Quick Review of angular momentum and Torque] → $\frac{d}{dt}L = \frac{d}{dt}(\vec{r} \times \vec{p}) = \cancel{\dot{\vec{r}} \times \vec{p}} + \vec{r} \times \underbrace{\dot{\vec{p}}}_{F}$ parallel

$$\dot{\vec{L}} = \vec{r} \times \vec{F} \equiv \vec{\tau} = \frac{d\vec{L}}{dt}$$

(rate of change of a particles angular momentum about origin O is equal to the net torque about O)

[Quick Review of Torque & dipole moment]





these forces cause a τ in $\vec{r} \times \vec{F}$

$\vec{\tau} = aF\sin\theta\,\hat{x}$ *if you do each separately & add you get this

$\vec{F} = I\int d\vec{l} \times \vec{B}$ ← along torque segments

$|\vec{F}| = IbB$ $\quad d\vec{l} \perp \vec{B}$

$\vec{\tau} = IabB\sin\theta\,\hat{x} = \mu B\sin\theta\,\hat{x}$

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$

For a tiny loop

$\vec{\mu} = I\vec{A}$ → [Find I] → $\lambda = \frac{dQ}{dl}$, $dQ = \lambda dl = \lambda v dt$ → $\frac{dQ}{dt} = \boxed{\lambda v = I}$

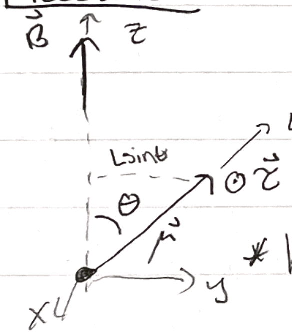
Precession

$\vec{\mu} = \frac{Qv}{2\pi R}\pi R^2 = \frac{1}{2}QvR$ and $\vec{L} = \vec{r} \times \vec{p} = MvR$

$\boxed{\vec{\mu} = \frac{Q}{2m}\vec{L}}$ and $\vec{L} \Leftrightarrow \vec{S}$ or spin angular momentum!

$\vec{L}$ of rotating disk

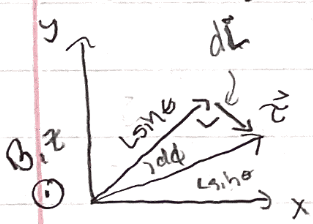
make z axis along B-field



* keep the precession classical for now * (just $\vec{\mu}$ in a $\vec{B}$ field)

$\vec{\tau} = \vec{\mu} \times \vec{B} = \frac{q}{2m}\vec{L} \times \vec{B}$, $\quad \vec{L} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ L_x & L_y & L_z \\ 0 & 0 & B_0 \end{vmatrix} = L_y B_0\,\hat{x} - L_x B_0\,\hat{y}$ (for direction)

$\vec{\tau} \perp \vec{L}$ and $\vec{B}$



take magnitude! $|\vec{L} \times \vec{B}| = \sqrt{L_y^2 B_0^2 + L_x^2 B_0^2} = \underbrace{\sqrt{(L_x^2 + L_y^2)}}_{L\sin\theta} B_0$

$\tau = \frac{q}{2m}\cancel{L\sin\theta}B_0 = \frac{dL}{dt} = \cancel{L\sin\theta}\,\omega$

$dL = L\sin\theta\, d\phi$

$$\boxed{\omega = \frac{q}{2m}B_0}$$

Precession of a spin 1/2 particle

$\vec{\mu} = \gamma \vec{S}$ just like $\vec{\mu} = \frac{q}{2m}\vec{L}$ but $\gamma \neq \frac{q}{2m}$ but multiple

[When this dipole is in B field it experiences a torque] $\vec{\tau} = \vec{\mu} \times \vec{B}$, the energy associated w/ the torque is

$$H = -\vec{\mu}\cdot\vec{B}$$

what spin matrix you use depends on $\vec{\mu}$

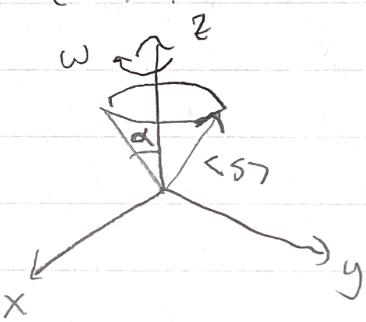
Now $\hat{H} = -\gamma \vec{B}\cdot\hat{S}$

Say $\vec{B} = B_0\hat{k}$ $\to$ $\hat{H} = -\gamma B_0 S_z = \frac{-\gamma B_0 \hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

[eigenstates are the same as S_z, χ_+, χ_-]; eigenvalues are $E_+ = -\gamma B_0 \hbar/2$, $E_- = \gamma B_0 \hbar/2$

Solve: $i\hbar \frac{\partial \chi}{\partial t} = \hat{H}\chi \to \chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar}$

$\chi(t) = \begin{pmatrix} a\, e^{i\gamma B_0 t/2} \\ b\, e^{-i\gamma B_0 t/2}\end{pmatrix}$, $\chi(0) = \begin{pmatrix} a \\ b\end{pmatrix}$, $|a|^2 + |b|^2 = 1$, so $a = \cos(\frac{\alpha}{2})$, $b = \sin(\frac{\alpha}{2})$



$\langle S_x\rangle = \chi^\dagger S_x \chi = \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t)$

$\langle S_y\rangle = \chi^\dagger S_y \chi = -\frac{\hbar}{2}\sin(\alpha)\sin(\gamma B_0 t)$

$\langle S_z\rangle = \chi^\dagger S_z \chi = \frac{\hbar}{2}\cos\alpha$