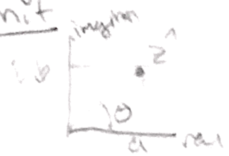
imaginary shit $z = a+ib = |z|[\cos\theta + i\sin\theta] = |z|e^{i\theta}$ $z^2 = |z|^2 e^{i2\theta}$

$\hookrightarrow$ complex conjugate $z^* = a - ib$




$(e^{ix})^* = e^{-ix}$

$e^{ix} \cdot e^{-ix} = 1$

$$\boxed{\begin{array}{ccc} \lambda = \frac{h}{p} & k = 2\pi/\lambda & p = \hbar k \\ f = \frac{E}{h} & \omega = 2\pi f & E = \hbar\omega \end{array}}$$

# Quantum Mechanics Postulates

wave function must be continuous, slope must be continuous except when boundary is $\infty$

→ $\Psi(\vec{r},t)$ complex wave function describes state of function.

→ $|\Psi(\vec{r},t)|^2$ probability of finding a particle in a small volume $dV$

$\hookrightarrow \Psi(\vec{r},t)\Psi^*(\vec{r},t)\,dV$

---

**Free particle as PLANE WAVES**

$$\Psi(x,t) = Ae^{i(kx-\omega t)} = Ae^{\frac{i}{\hbar}(px-Et)}$$ ← using $k = \frac{p}{\hbar}$, $\omega = \frac{E}{\hbar}$

↑ a particle with definite momentum and energy in $+x$ direction

← e's cancel cause of complex conjugate

$|\Psi(x)|^2 = \Psi(x,t)\cdot\Psi^*(x,t) = |A|^2$ $\int_{-\infty}^{\infty}\Psi(x,t)\Psi^*(x,t)\,dx = 1$ $\int_{-\infty}^{\infty}|A|^2 dx = 1$ $A \to 0$?

↑ prob density is constant for plane wave.

idea of where it comes from

Schrödinger Eq justifying, for a simple free particle, which should give a wave shape at any particular time, specified by $\Psi(x)$ is that of a simple de Broglie wave such as $\Psi(x) = A\sin(kx)$, $A$ = amplitude. Lets take some derivatives.

$\frac{d\Psi}{dx} = kA\cos kx$ $\frac{d^2\Psi}{dx^2} = -k^2 A\sin kx = -k^2\Psi(x)$ → $U=0$ for free particle but generally

$E = K + U$

$K = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$ ⟶ $\frac{d^2\Psi}{dx^2} = -k^2\Psi(x) = \frac{-2m}{\hbar^2}K\Psi(x) = \frac{-2m}{\hbar^2}(E-U)\Psi(x)$

from that $\boxed{\frac{-\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + U(x)\Psi(x) = E\Psi(x)}$ ← time ind shrodinger eqe

Full schrodinger! $\boxed{i\hbar\Psi(x,t) = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V\Psi(x,t)}$ Full schrodinger

---

**SPLIT SPACE TIME S.E.**

Seperating schrodinger, time and space → product of two functions. (start simple, plane wave)

$\Psi(x,t) = Ae^{\frac{i}{\hbar}(px-Et)}$ splitting $= Ae^{(\frac{ip}{\hbar}x)}e^{(\frac{-iE}{\hbar}t)} = \psi(x)\phi(t) = \Psi(x,0)$ time and space part defined

important if $V(x)$ is time independent

(time ind) $\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$, (time dependent) $i\hbar\frac{\phi(t)}{dt} = E\phi(t) \to \boxed{\phi(t) = e^{\frac{-iE}{\hbar}t}}$

↓ ↓ ↓ ↓ pull out $\psi(x)$

Using this time ind.

def. $\hat{H} = \underbrace{\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}}_{\text{kinetic energy operator}} + \underbrace{V(x)}_{\text{potential energy operator}}$ → $\hat{H}\psi(x) = E\psi(x)$ → $\psi(x)$ - "eigenfunction" of $\hat{H}$, $E$ - "eigenvalue" of $\hat{H}$

---

**Schrodinger Variations**

$\hat{H}\psi(x) = E\psi(x)$ ← time ind w/ $\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

time ind. $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$

time dep. $i\hbar\frac{d\phi}{dt} = E\phi(t) \to \phi = e^{\frac{-iE}{\hbar}t}$

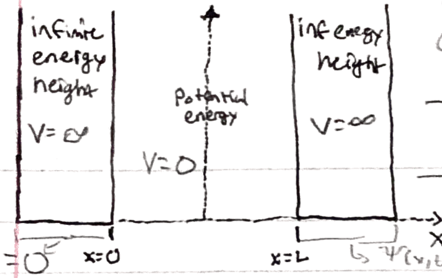
$\Psi(x,t) = \psi(x)\phi(t)$

$i\hbar\Psi(x,t) = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V\Psi(x,t)$

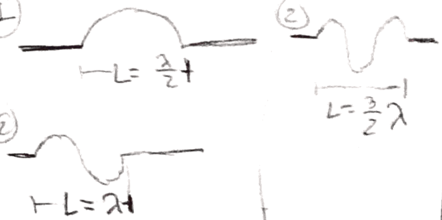
$$\Psi(x,t) = \int_{-\infty}^{\infty}A(p)\exp\left(i\frac{(px-Et)}{\hbar}\right)$$

$\boxed{\text{Infinetly potential well}}$

imagine wave function is a de Broglie wave




$\Psi(x,t)=0$ $\quad$ $\hookrightarrow \Psi(x,t)=0$




$\lambda_n=\frac{2L}{n}$ or $L=\frac{\lambda_n \cdot n}{2}$

Momentum: $p_n=\frac{h}{\lambda_n}=n\frac{h}{2L}$

Kinetic Energy: $E_n=\frac{p_n^2}{2m}=$

$= n^2\frac{h^2}{8mL^2}$ (Quantization of Energy)

Boundary has to be continous so $\boxed{\Psi(0,t)=\Psi(L,t)=0}$ ①

$\boxed{\text{Checking the uncertainty princ.}}$ $\Delta x \sim L$

$\Delta p=\sqrt{\langle p^2\rangle}\sim \frac{h}{2L}$ ($n=1$, first one) $\rightarrow \Delta x\Delta p \sim L\frac{h}{2L}=\frac{h}{2}$ and is definetly $\frac{h}{2}>\frac{h}{4\pi}$ ($\Delta x\Delta p \geq \hbar/2$)

---

1D inf □ well

④ solving ODE: assume $\psi(x)=e^{rx}\rightarrow e^{rx}\left(\frac{\hbar^2}{2m}r^2+E\right)=0$, $r^2=-\frac{2mE}{\hbar^2}$

④.1
$$V(x)=\begin{cases}\infty & x<0,\ x>L\\ 0 & 0<x<L\end{cases}$$

$r=\pm i\frac{\sqrt{2mE}}{\hbar}\rightarrow \psi(x)=c_1e^{i\frac{\sqrt{2mE}}{\hbar}x}+c_2e^{-i\frac{\sqrt{2mE}}{\hbar}x}$

we know $\Psi(0)=0\rightarrow 0=c_1+c_2\rightarrow c_1=-c_2$

for $x<0$ and $x>L$: ② $\Psi(x)=0$ $\quad$ using $e^{i\theta}=\cos\theta+i\sin\theta$ and $c_1=-c_2$ you find

for $0\leq x\leq L$: (time ind and $V=0$) $\quad$ $\psi(x)=-2ic_2\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right)=A\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right)$

③ $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=E\psi(x)$

*notice $p=\hbar k$ and $p=\sqrt{2mE}$ so $k=\frac{\sqrt{2mE}}{\hbar}$ ← E is kinetic energy. ⑤

---

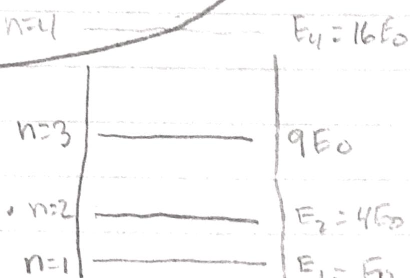
$\boxed{\text{last boundary cond}}$ ⑥ $\Psi(L)=0\rightarrow \Psi(x)=A\sin(kx)\rightarrow 0=A\sin(kL)$

either $A=0$ meaning $\Psi=0$ everywhere, $\Psi^2=0$ everywhere, meaning particle isnt anywhere. Or! $\sin(kL)=0$ which happens at ⑦ $kL=\pi,2\pi,3\pi,\ldots\rightarrow kL=n\pi$

Using $k=\frac{2\pi}{\lambda}$: $\frac{2\pi}{\lambda}L=n\pi\rightarrow \boxed{L=\frac{n\lambda}{2}}$ (identical for $\lambda$ of standing waves in a string of length $L$ fixed at both ends!)

**Solution to SE, for a particle trapped in a linear region of length L, is a series of de Broglie waves!!**

from $\hbar^2k^2=2mE$, $k$ happens only at certain values.




kinetic energy $\quad E_n=\frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 \overset{\text{not } \hbar}{=} \frac{h^2n^2}{8mL^2}$ so the energy is quantized.

$n=1$ is ground state, $n>1$ is excited state.

---

$\boxed{\text{To complet the solution we must use this to find } A}$ $\rightarrow \int_{-\infty}^{\infty}|\psi(x)|^2dx=1$, $\boxed{\text{integrate over parts where particle can be}}$ $\rightarrow \int_0^L A^2\sin^2\left(\frac{n\pi}{L}x\right)dx=1$, $A=\sqrt{\frac{2}{L}}$ then

$$\boxed{\Psi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\quad n=1,2,3,\ldots}$$

$\psi\cdot\psi^*$ ← complex conjugate is itself.

⑧ $k=\frac{\sqrt{2mE}}{\hbar}$, $kL=n\pi$, $L=\frac{n\lambda}{2}$, $E_n=\frac{h^2n^2}{8mL^2}$, $p_n=n\frac{h}{2L}$ and

# Operators and Expectation values

**Momentum operator.** Plane wave → $\Psi(x,t) = A\, e^{i\frac{(px - Et)}{\hbar}}$

$$\frac{\partial \Psi(x,t)}{\partial x} = i\frac{p}{\hbar}\Psi(x,t) \rightarrow -i\hbar\frac{\partial}{\partial x}\Psi(x,t) = p\Psi(x,t)$$

$\hat{p}\,\Psi(x,t) = p\Psi(x,t)$ where $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

operator

tons of of operaters for everything

$\boxed{\text{Expectation value of an Operater}}$

$$\langle g(x)\rangle = \int_{-\infty}^{\infty}\Psi^*\,\hat{g}(x)\,\Psi(x)\,dx$$

operator

or average (expectation value) of something like $x$

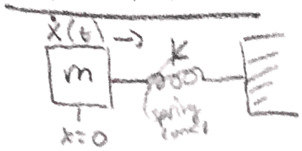
ex expectation value of $p$ in $\infty$ well

$$\langle p_x\rangle = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\left(-i\hbar\frac{d}{dx}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) = 0 \text{ by symm}$$

$$\langle p^2\rangle = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\left(-\hbar^2\frac{d^2}{dx^2}\right)\left[\sin\left(\frac{n\pi x}{L}\right)\right]dx = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Consistent w/ $\frac{\langle p_x^2\rangle_n}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}$

Simple Harm osc




$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$\omega^2 = \frac{k}{m}$

$\omega = \frac{2\pi}{T}$

$\hat{p} = -i\hbar\frac{\partial}{\partial x}$, $V(x) = \frac{1}{2}m\omega^2x^2$

# Simple Harmonic Oscillator

→ potential is oscillator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2} + \frac{1}{2}m\omega^2x^2$$

so $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x)$ ← solve ODE

$\psi(x) = Ae^{-ax^2}$, $\frac{d\psi(x)}{dx} = -2ax(Ae^{-ax^2})$, $\frac{d^2\psi(x)}{dx^2} = -2a(Ae^{-ax^2}) - 2ax(-2ax)Ae^{-ax^2}$

$$\frac{d^2\psi(x)}{dx^2} = (-2a + 4a^2x^2)Ae^{-ax^2}$$

$$-\frac{\hbar^2}{2m}(-2a + 4a^2x^2)\psi(x) + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x)$$

$E = -\frac{\hbar^2}{2m}(-2a + 4a^2x^2) + \frac{1}{2}m\omega^2x^2$, Energy constant only when x dep terms cancel eachother out

$-\frac{\hbar^2}{2m}(4a^2x^2) + \frac{1}{2}m\omega^2x^2 \rightarrow \boxed{a = \frac{m\omega}{2\hbar} \text{ and } E = \frac{\hbar^2}{m}a = \frac{1}{2}\hbar\omega}$

ground state eigenfunction. $\psi_0(x) = A_0\exp\left[-\frac{m\omega x^2}{2\hbar}\right]$

$$P = \int_{-\infty}^{\infty}\psi_0^*(x)\psi_0(x)dx = \int_{-\infty}^{\infty}A_0^2\exp\left[\frac{-m\omega x^2}{\hbar}\right]dx = 1$$

Gauss integral: $\int_{-\infty}^{\infty}\exp\left[\frac{-x^2}{2\sigma^2}\right]dx = \sigma\sqrt{2\pi}$, $\frac{1}{2\sigma^2} = \frac{m\omega}{\hbar}$

and $A_0^2(\sigma\sqrt{2\pi}) = 1 \rightarrow A_0^2 = \frac{1}{\sigma\sqrt{2\pi}} = \sqrt{\frac{m\omega}{\pi\hbar}}$ so $A_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega x^2}{2\hbar}\right]}$, $\psi_1(x) = A_1\sqrt{\frac{m\omega}{\hbar}}\, x\, \exp\left[\frac{-m\omega x^2}{2\hbar}\right]$

$\psi_n(x) = Af_n(x)e^{-ax^2}$ $\qquad \psi_2(x) = A_2\underbrace{\left(1 - \frac{2m\omega x^2}{\hbar}\right)}_{\text{polynomial}}\exp\left[-\frac{m\omega x^2}{2\hbar}\right]$

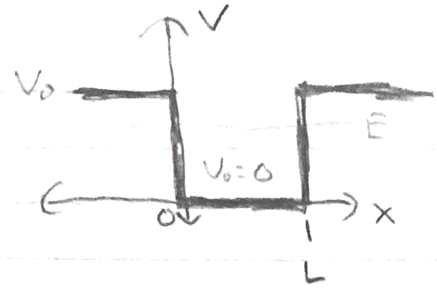
$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ $\qquad n = 0, 1, 2 \ldots$

$E_0 = \frac{1}{2}\hbar\omega$

# Harmonic oscillators and other Potentials

## Quantum v.2

### Finite Square Well




$E < V_0$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\begin{cases} V(x) = 0 & \rightarrow 0 < x < L \\ V(x) = V_0 & \rightarrow x > L \text{ and } x < 0 \end{cases}$$

* when doing Eulers, used bounds to take a term out now we can't do that.

for inside well $0 \le x \le L$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

(from before) $\psi(x) = C_1 e^{i\frac{\sqrt{2mE}}{\hbar}x} + C_2 e^{-i\frac{\sqrt{2mE}}{\hbar}x}$

$$\psi(x) = C\cos(kx) + D\sin(kx) \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

for outside well $x < 0$, $x > L$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E\psi(x) \Rightarrow +\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = (V_0 - E)\psi(x)$$

$E < V_0$

$$k' = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$x < 0$ $\quad E < V_0$, $\psi_0(x) = Ce^{k'x} + De^{-k'x}$

goes from $0 \rightarrow -\infty$, so we can't have $|\psi(x)|^2$ blow up at $-\infty$ so $e^{-k'(-\infty)}$ has to go thus

$$\boxed{\psi_0 = Ce^{k'x}}$$

$x > L$ $\quad \psi_L(x) = Fe^{k'x} + Ge^{-k'x}$ goes from $L \rightarrow +\infty$, can't have blow up so $e^{k'x}$ has to go

$$\boxed{\psi_L(x) = Ge^{-k'x}}$$

| | | |
|---|---|---|
| $\psi_0(x) = Ce^{k'x}$ | $\psi_L = Ge^{-k'x}$ | $\psi(x) = C\cos(kx) + D\sin(kx)$ |
| $\frac{d\psi_0(x)}{dx} = k'Ce^{k'x}$ | $\frac{d\psi_L(x)}{dx} = -k'Ge^{-k'x}$ | $\frac{d\psi(x)}{dx} = -kC\sin(kx) + kD\cos(kx)$ |

$$\begin{cases} \text{at } x = 0 & \psi_0(0) = \psi(0) \\ \text{at } x = L & \psi_L(L) = \psi(L) \end{cases}$$

and $\int_{-\infty}^{\infty} |\psi|^2 dx = 1$

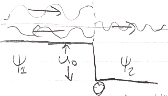
$$\begin{cases} \text{at } x = 0 & \frac{d\psi_0(0)}{dx} = \frac{d\psi(0)}{dx} \\ \text{at } x = L & \frac{d\psi_L(L)}{dx} = \frac{d\psi(L)}{dx} \end{cases}$$

$$\int_{-\infty}^{0}\psi_0^2\,dx + \int_0^L \psi^2\,dx + \int_L^{\infty}\psi_L^2\,dx = 1$$

solve E.

# STEPS AND BOUNDARIES | Pt. 2

## $E > U_0$



E

General Solution:

$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$ $\quad k_1 = \frac{\sqrt{2m(E-U_0)}}{\hbar}$

$\psi_2(x) = Ce^{ik_2x}$ $\quad k_2 = \frac{\sqrt{2mE}}{\hbar}$

reflection for a down step → phase shift 180°

$[\psi_1 = \psi_2$ at $x=0$, $\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx}$ $x=0]$

$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$, $\frac{C}{A} = \frac{2k_1}{k_1 + k_2}$

---

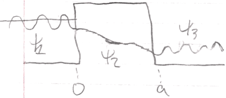
$E < U_0$ *more notes for solving boundary cond.

$\frac{B}{A} = \frac{ik_1 + k_2}{ik_1 - k_2}$, $\frac{C}{A} = \frac{2ik_1}{ik_1 - k_2}$ $\quad B = \left|\frac{B}{A}\right|^2 = 1$, $\left|\frac{C}{A}\right|^2 =$ non zero

all reflected, some transmitted but comes back later, causing a phase shift.

phase shift

## Finite Step



whole plane wave → incident, reflected

$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}$ $\quad k_1 = \frac{\sqrt{2mE}}{\hbar}$

$\psi_2 = Ce^{k_2x} + De^{-k_2x}$ $\quad k_2 = \frac{\sqrt{2m(U_0-}}{\hbar}$

some transmitted, some reflected

$\psi_3 = Fe^{ik_3x}$ ← only goes forwa

steps extra notes/ references

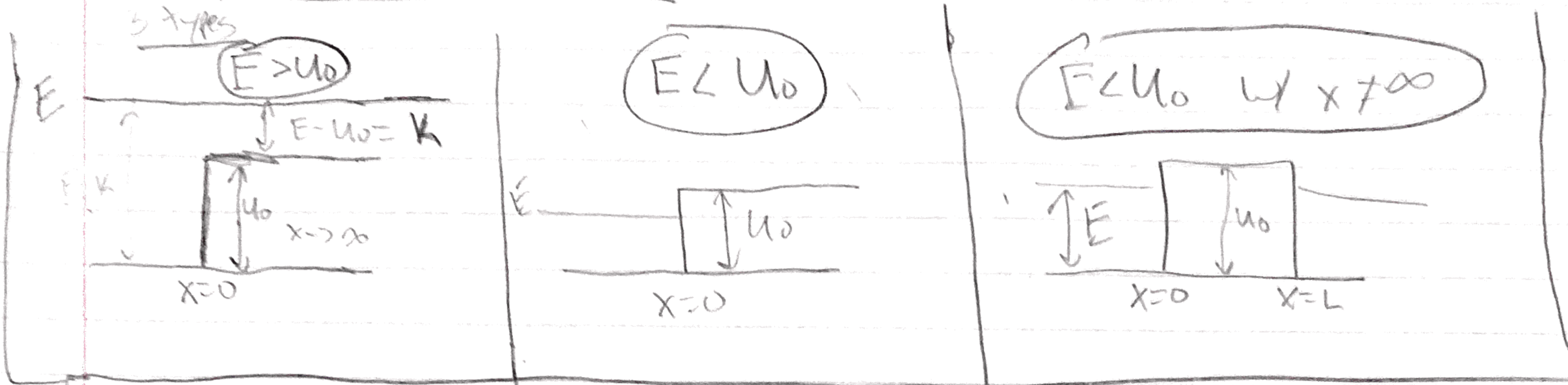
BOHR & RUTHEFORD ON BACK

* Particle is not confined like for ∞ well particle. So it's like a plane wave → $\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{i(kx-\omega t)}$

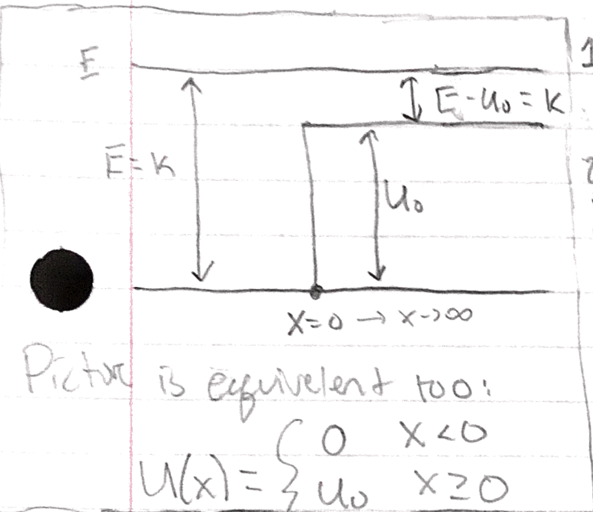
$E_n = \hbar\omega$

# Steps and Barriers

3 types




**Starting w/ $E > U_0$**




Picture is equivalent too:

$$U(x) = \begin{cases} 0 & x<0 \\ U_0 & x \geq 0 \end{cases}$$

E = fixed total energy of particle

$U_0$ = constant potential

time independent

1. write Hamiltonian: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$

[before step particle has E=K, after $K = E - U_0$]

2. General Solutions for wave function (plane wave):

$\boxed{x<0}$ $\boxed{\psi_0(x) = A\sin(k_0x) + B\cos(k_0x) = Ae^{ik_0x} + Be^{-ik_0x}}$

↳ $\boxed{k_0 = \sqrt{\frac{2mE}{\hbar^2}}}$

$\boxed{x>0}$ $\boxed{\psi_1(x) = Ce^{ik_1x} + \cancel{De^{-ik_1x}}, \; k_1 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}}$ 0

↳ BUT! $D=0$: particles are coming from $-\infty$ and always from the left, so its impossible to have a particle coming from the left AFTER $x=0$.

↳ for $x<0$ there can be particles going to left so $B \neq 0$!

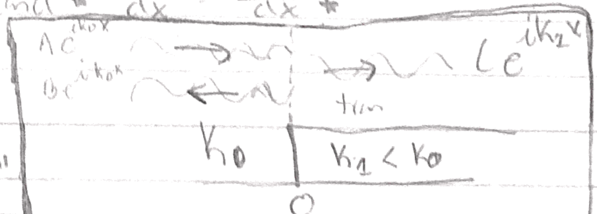
3. Apply Boundary Conditions: $\psi_0(0) \overset{①}{=} \psi_1(0)$ and $\frac{d\psi_0(0)}{dx} \overset{②}{=} \frac{d\psi_1(0)}{dx}$

① $A+B=C$ , ② $(A-B)k_0 = Ck_1$

[you get] → $\frac{B}{A} = \frac{k_0-k_1}{k_0+k_1}$ and $\frac{C}{A} = \frac{2k_0}{k_0+k_1}$

Probability Density

$\left|\frac{B}{A}\right|^2 = \left(\frac{k_0-k_1}{k_0+k_1}\right)^2$ and $\left|\frac{C}{A}\right|^2 = \left(\frac{2k_0}{k_0+k_1}\right)^2$




$V_0 = \frac{p_0}{m} = \frac{\hbar k_0}{m}$ , $V_1 = \frac{p_1}{m} = \frac{\hbar k_1}{m}$ ← group velocity

4. Group velocity and Reflection and transmission: R = reflection coeff , T = transmission coeff

Reflection

$V_0R = V_0\left|\frac{B}{A}\right|^2$ ← probability of reflected side

amount Reflected = percent that's reflected

$R = \left|\frac{B}{A}\right|^2$

$R + T = 1$

Transmission

$V_0T = V_1\left|\frac{C}{A}\right|^2$ ← probability of penetrated side

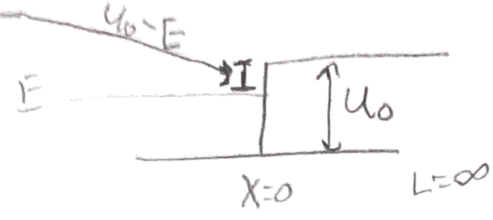
amount of whole wave for transmitted = whole penetrated wave • probability of finding a particle there = fraction of penetrated wave

$T = \frac{V_1}{V_0}\left|\frac{C}{A}\right|^2$

$\frac{C}{A} = \frac{\text{Penetrated wave}}{\text{incident wave}}$

$\frac{B}{A} = \frac{\text{reflected wave}}{\text{Incident wave}}$

$\left|\frac{X}{X}\right|^2$ = percent of wave finding the




But it can penetrate and come back!

## $E < U_0$

no tunneling, can't get past infinity

All particle wave will be reflected:
Gen Sol'n for a plane wave

| |
|---|
| 100% reflection |
| 0% transmission |

$$\Psi(x) = C_1 e^{ikx} + C_2 e^{-ikx} \quad \text{] general}$$

for $x<0$: just normal general plane wave

$$\hookrightarrow \boxed{\Psi_0(x) = A e^{ikx} + B e^{-ikx}} = A\cos(kx) + B\sin(kx) \leftarrow k \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

for $x>0$: real exponents bc/ finite potential

$$\Psi_1(x) = C e^{k_1 x} + D e^{-k_1 x} \leftarrow k_1$$

no particle past potential if $E < U_0$

*except for as $x \to \infty$ $\Psi(x) \to 0$ not blow up, C term makes it blow up
so $C = 0$

$$\boxed{\Psi_1(x) = D e^{-k_1 x}} \qquad k_1 = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)}$$

How far does it penetrate and then come back? → we are in $x>0$ teritory so → $\Psi_1(x) = D e^{-k_1 x}$, $|\Psi_1(x)|^2 \sim e^{-2k_1 x}$

lets say $\Delta x$ is distance in which probability $|\Psi(x)|^2$ drops by $1/e$. ↳ range $\Delta x$

→ $\Delta x$ happens when $|\Psi_1(x)|^2 = e^{-1}$

$$e^{-1} \sim e^{-2k_1 \Delta x} \rightarrow \Delta x = \frac{1}{2k_1} = \frac{1}{2}\frac{\hbar}{\sqrt{2m(U_0 - E)}}$$

can't do this but if it does it in $\Delta t$ from $\Delta E \Delta t \sim \hbar$ then it can.

in order for it to enter $x>0$ must gain $U_0 - E$ to just reach hump but also → some K so it can move

to reach top of step

* borrowed energy is $(U_0 - E) + K = \Delta E$ — K so it can move after step

$$\boxed{\Delta t = \frac{\hbar}{U_0 - E + K}}$$

→ penetration distance

$\frac{1}{2}mv^2 = K$ particle moves w/ ↘ $v = \sqrt{\frac{2K}{m}}$

distance it can go

$$\rightarrow \Delta x = \frac{1}{2} v \Delta t = \frac{1}{2}\sqrt{\frac{2K}{m}}\left(\frac{\hbar}{U_0 - E + K}\right)$$

↑ 1/2 because it has to come back as far as it goes

differentiate w/ respect to $\frac{dx}{dK} = 0$

$$\Delta x_{max} = \frac{1}{2}\frac{\hbar}{\sqrt{2m(U_0 - E)}}$$

as $K \to 0$ $\Delta x \to 0$
as $\Delta x \to 0$ as $K \to \infty$ $\Delta x$ [illegible]
so there's a max, $\Delta x_{max}$