Chapter 4

- goal is to show the Kepler problem of planetary orbits can be solved w/ Lagrangian Mech
  ↳ first consider the general solution for motion in a 1D potential $V(q)$
  ↑deg of freedom, w/ $L(q,\dot{q})$

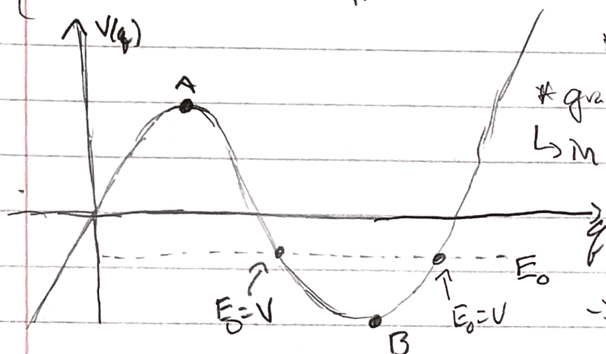
Motion of "Generic" one-dimensional system

- any 1-Deg of freed problem can be solved by expressing the answer as an integral
  ↳ $t(q)$, inverse of what one means by a sol'n $(q(t))$, $q(t)$ can't always be expressed analytically.

information about the sol'n to any 1D problem can be deduced from graphing the potential ($V(q)$ wrt q) then considers what happens as func of ever, E increase

$$E = T+V = \frac{1}{2}m\dot{q}^2 + V(q)$$

$$\dot{q} = \pm\sqrt{2T} = \pm\sqrt{2(E-V(q))/m}$$



*while $E > V$ T is positive and non-zero

*graph of how the potential changes as a func of gen coord.
↳ in our potential $V(q)$ for small energies E_0, system behaves like SHO, when $E_0 = V$, $\dot{q}=0$, $T=0$ (approximate parabola)
↳ called turning points
→ for higher engies its no longer a parabola

rearranging from above

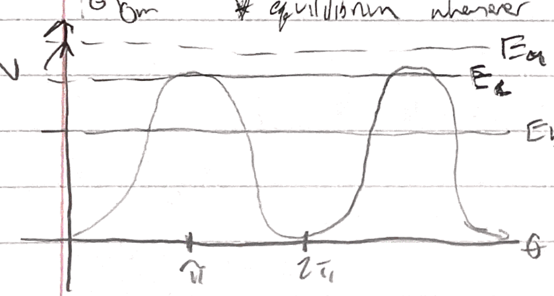
$$\frac{dq}{\pm\sqrt{2(E-V(q))}} = dt \Rightarrow t = \int_0^q \frac{dq}{\sqrt{2(E-V(q))}}$$

Grandfather clock

$V(\Theta) = mgl(1-\cos\Theta)$ (scaled time) $\Rightarrow E = \frac{\dot{\Theta}^2}{2} + (1-\cos\Theta)$

Unstable $\frac{d^2V}{d\Theta^2} < 0$

*equilibrium whenever the generalized force (torque) is 0, stable $\frac{d^2V}{d\Theta^2} > 0$



- Ea means θ can increase w/out limit (rotation) doesn't fall back down
- Eb it oscillates between 2 points
- Ec Period infinite just between rotation + parall

[multivariable taylor series] $P_3(x) = f(x_0) + \frac{\partial f}{\partial x_1}(x_0)\cdot V_1 + \frac{\partial f}{\partial x_2}(x_0)V_2 + O(V^2)$

$X - X_0 = V$

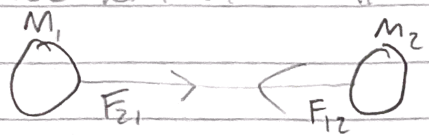
$V_1 = U_2$ for bottom

$\vec{r}_1 = \vec{r}_1(a) + \varepsilon a$

$\vec{r}_2 = \vec{r}_2(a) + \varepsilon a$

Central Forces

- applying tools earlier in the chapter to solve the motion due to gravity
- we want a central force that depends only on the distance "r" between them

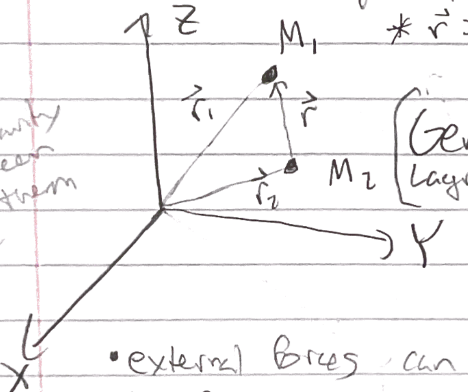


$V = V(r)$ for this type of force

motion of the center of mass

- you can predict the motion of the center of mass of an exploding artillery shell without knowing the internal forces released, center of mass of a system behaves as a single particle under the action of external gravitational forces

$* \ \vec{r} = \vec{r}_1 - \vec{r}_2$



not gravity between them

[General Lagrangian] $L = \frac{1}{2}M_1\dot{\vec{r}}_1^2 + \frac{1}{2}M_2\dot{\vec{r}}_2^2 - V_{ext,1}(\vec{r}_1) - V_{ext,2}(\vec{r}_2) - V_{12}(\vec{r}_1, \vec{r}_2)$

(V_{12})

- interactions between 1 and 2 can only depend on the difference in their positions so $V(\vec{r})$
- external forces can only depend on the particles absolute positions, think about falling in gravity / external (F_{ext}) to both, it will depend on their absolute positions.

$\boxed{\text{No external Forces}}$ → $V_{ext} = 0$, only force is between particle 1 and particle 2

↳ [follows that L doesn't change by translating the origin, so we can]

$\vec{r}_1 \Rightarrow \vec{r}_1 + \vec{a}$

$\vec{r}_2 \Rightarrow \vec{r}_2 + \vec{a}$

{can do this since it doesn't change the position}

$L = L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2) = L(\vec{r}_1 + \vec{a}, \vec{r}_2 + a, \dot{\vec{r}}_1, \dot{\vec{r}}_2)$ [translational invariance]

[make an infinitesimal translation $\varepsilon\vec{a}$, make ε arbitrarily small] → $L(\vec{r}_1 + \varepsilon\vec{a}, \vec{r}_2 + \varepsilon\vec{a}, \dot{\vec{r}}_1, \dot{\vec{r}}_2)$

[Taylor expand this in ε] $= L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2) + \varepsilon\vec{a}\cdot(\vec{\nabla}_1 + \vec{\nabla}_2)L + O[\varepsilon^2]$

$\underbrace{\begin{cases} \vec{r}_1 \to \vec{r}_1 + \varepsilon a \\ x = x_0 + V \end{cases}}_{\text{both variables}}$

∇_1 come from partial derivatives and factoring out L

are changed by $\varepsilon\vec{a}$ so can also pull that out

$\vec{\nabla}_1 = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1}\right)$, $\nabla_2 = \ldots$

$\nabla_1 L = -\nabla_1 V$ (since T depends only on $\dot{q}$)
$\nabla_2 L = -\nabla_2 V$

$-\nabla_1 V =$ force on particle 1,
$-\nabla_2 V =$ force on particle 2, by definition a 3rd law pair so it has to be 0

Now we have $= L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2) + \varepsilon \vec{a} \cdot (\vec{\nabla}_1 + \vec{\nabla}_2) L$

↳ lets look at $\left[\vec{\nabla}_1 L + \vec{\nabla}_2 L\right]$ the change in L for $\vec{r}_1$, the change in L for $\vec{r}_2$

Since external forces are 0, L is invariant under coord transformations, we can choose an $\vec{a}$, such that the $\nabla_1 L + \nabla_2 L = 0$

debate able !!! [↳ I think you bring origin to center, $\vec{a}$ here] so $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ are equal and opposite?

look at $\frac{\partial L}{\partial x_1} + \frac{\partial L}{\partial x_2} = 0$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{dP_{1x}}{dt}$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = \frac{dP_{2x}}{dt}$

$\frac{d}{dt}(P_{1x} + P_{2x}) = 0 \rightarrow \boxed{\vec{P}_{tot} = \text{const} = \vec{P}_1 + \vec{P}_2}$

$\frac{d\vec{P}_{tot}}{dt} = \frac{d}{dt}\left[m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2\right] = 0$ lets redefine and look at this eq w/ regard to a R_{cm}

[quick derivation] $\vec{P}_{cm} = M\dot{\vec{R}}_{cm} = (m_1 + m_2)\dot{\vec{R}}_{cm}$ and $P_{cm} = m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{R}}_2$

$\dot{\vec{R}}_{cm} = \frac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{M} \xrightarrow{\text{integrate}} \boxed{R_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}}$ *

now make from $r = r_1 - r_2$

$\begin{bmatrix} r_2 = r_1 - r \\ r_1 = r - r_2 \end{bmatrix} \xrightarrow[r_1 \text{ and } r_2]{\text{solve for}} \boxed{\begin{aligned} \vec{r}_1 &= \vec{R}_{cm} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 &= \vec{R}_{cm} - \frac{m_1}{M}\vec{r} \end{aligned}}$ $M = m_1 + m_2$

$T = \frac{1}{2}m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2 \dot{\vec{r}}_2^{\,2} = \frac{1}{2}M\dot{\vec{R}}_{cm}^2 + \frac{1}{2}\underbrace{\left(\frac{m_1 m_2}{m_1 + m_2}\right)}_{\mu = \text{reduced mass}}\dot{\vec{r}}^{\,2}$

$L = \underbrace{\frac{1}{2}M\dot{\vec{R}}_{cm}^2}_{\text{motion of CM}} + \underbrace{\frac{1}{2}\mu\dot{\vec{r}}^{\,2}}_{\text{motion about CM}} - V(r)$, $V_{12}(|\vec{r}_1 - \vec{r}_2|) = V(r)$

We can treat the $\dot{R}_{cm} = \text{constant} = 0$ think about gravity between Sun and Earth, $\dot{R}_{cm} = 0$.

$\boxed{L = \frac{1}{2}\mu\dot{r}^2 - V(r)}$

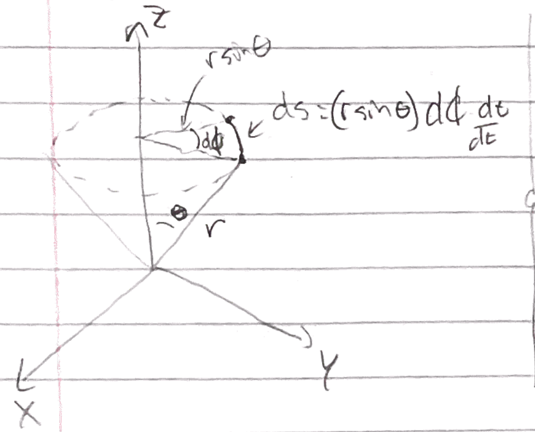
Angular Momentum is conserved for central forces

$L = \frac{1}{2}\mu \dot{\vec{r}}^2 - V(r)$ no external forces $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1 - \vec{r}_2)$

translational invariance

$V(\vec{r}_1 - \vec{r}_2) = V(|\vec{r}_1 - \vec{r}_2|) = V(r)$ only central force = rotational invariance

"central force" means that the interaction between the particles doesnt depend on their absolute spacial position or orientation but only dist between them

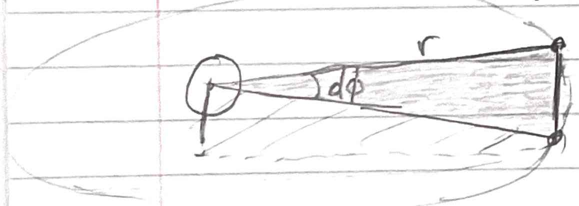


$\left(\frac{ds}{dt}\right)^2 = \dot{r}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, do inside Lagrangian but spherical

$$\boxed{\begin{aligned} &L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - V(r) \\ &\frac{\partial L}{\partial \phi} = 0 \qquad \frac{\partial L}{\partial \dot{\phi}} = \text{constant} = \mu r^2 \dot{\phi}\sin^2\theta = l_z \end{aligned}}$$

l_z is the z-component of the angular momentum
just like l_z is a constant, you could reorient so $\vec{l}$ has x, y, z components $(0, 0, l_z)$ like in my drawing

$dr = r\dot{\phi}dt$



2 consequences of angular momentum conservation
a) entire motion takes place in a plane
b) "Equal Areas swept in Equal time"

a) $\vec{l} = \text{constant} = \vec{r} \times \vec{p}$, $\vec{p}$ and $\vec{r}$ must be in a plane $\perp \vec{l}$ for all time

$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \underbrace{\dot{\vec{r}} \times \vec{p}}_{\vec{v} \times \vec{v} = 0} + \vec{r} \times \frac{d\vec{p}}{dt}$] $\frac{d\vec{p}}{dt}$ ~~total net force~~, for central forces w/ no external forces $= 0$

$\vec{l} = \vec{r} \times \vec{p} = \text{constant}$

↳ since $\vec{r} \times \vec{p}$ are in same plane, all motion is in the same plane

so $\boxed{\dot{\theta} = 0}$, then $\theta = \text{constant}$, lets say its $\theta = \pi/2$, makes $\vec{l}$ along z

↳ $l_z = \mu r^2 \dot{\phi}$ $(\sin\theta = 1)$

$A = \frac{1}{2}(r\sin\theta)^2 d\phi = \frac{1}{2}(r\sin\theta)^2 \dot{\phi}\, dt$

$\frac{dA}{dt} = \frac{l_z}{2\mu} = \text{constant}$ ✓ a) and b) proven

Look back at the Lagrangian

now we have, $\dot{\theta} = 0$, $\theta = \frac{\pi}{2}$, $\dot{\phi} = \frac{l}{\mu r^2}$

? * dont substitute $\dot{\phi}$ first into the Lagrangian
↳ ELE are derived under the fact of independent variations in each coordinate, if you eliminate it first then you dont have that

$$H = E = 2T - L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) + U(r)$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{l^2}{2\mu r^2} + V(r)}_{V_{eff}}$$

, $\frac{l^2}{2\mu r^2}$ comes from the kinetic energy and is responsible for the centrifugal force

$\boxed{\text{ELE}}$ $\boxed{\mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{dV}{dr}}$

Gravitational Attraction

- [potential energy between 2 bodies] $V(r) = -\underbrace{GM_1M_2}_{k} \frac{1}{r}$ | define $\frac{\ell^2}{2\mu} \equiv \beta$

↳ $V_{eff} = \frac{\ell^2}{2\mu r^2} + V(r) = \boxed{-\frac{k}{r} + \frac{\beta}{r^2} = V_{eff}}$

* derived this V_{eff} from central forces, and conservation laws

$L = T - V = \frac{1}{2}\mu\dot{r}^2 - V_{eff} \xrightarrow{EOM} \mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{k}{r^2}$

* inverse square law of gravity is opposed by the "centrifugal force"

Getting a better solution

- [make the substitution] $\boxed{u = \frac{1}{r}}$ $\xrightarrow[\text{about inverse}]{\text{w/ our assumptions}}$ $\ell = \mu r^2\dot{\phi}$ ↳ $\mu r^2 d\phi = \ell\, dt$ → $\boxed{\frac{d}{dt} = \frac{\ell}{\mu}u^2\frac{d}{d\phi}}$ using this

↳ rearranging this $\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{\ell}{\mu}\frac{du}{d\phi}$

↳ [recall E] → $E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + \underbrace{V(r)}_{-\frac{k}{r}}$

↳ turns into $\boxed{E = \frac{\ell^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right] - ku}$ E is a constant so $\frac{dE}{d\phi} = 0$

↳ [using $dE/d\phi = 0$] → $\boxed{\frac{d^2u}{d\phi^2} + u = \frac{\mu k}{\ell^2}}$ → $\boxed{u = \frac{1}{r} = \frac{\mu k}{\ell^2} + A\cos\phi}$

- a SHO, solve by inspection

$u(\phi + 2\pi) = $ [illegible]

$T = 2\pi$

put into here

rewriting the sol'n to a more familiar look.

$p \equiv \frac{\ell^2}{\mu k}$ and $\varepsilon \equiv pA$ $\xrightarrow{\text{these defs}}$ $pu = \frac{p}{r} = 1 + \varepsilon\cos\phi \Rightarrow p = r + \varepsilon r\cos\phi$

if $\varepsilon = 0$ then its a circle $r = p$

↳ [switch to cartesian] $p = \sqrt{x^2+y^2} + \varepsilon x \xrightarrow[\text{around}]{\text{square and move}} (1-\varepsilon^2)x^2 + 2\varepsilon p x + y^2 - p^2 = 0$ 4.53

$\varepsilon = 1$; parabola
$\varepsilon > 1$; hyperbola

ε is the eccentricity of the ellipse | get from

$a = \frac{p}{1-\varepsilon^2}$ a, is the semi major axis

Total Energy: $E = \frac{\mu k^2}{2\lambda^2}(\varepsilon^2 - 1) = \frac{k}{2p}(\varepsilon^2 - 1) = \boxed{-\frac{k}{2a} = E}$

start w/ eq 4.53 : complete the square $(1-\varepsilon^2)\left(x + \frac{\varepsilon p}{1-\varepsilon^2}\right)^2 + y^2 = \frac{p^2}{1-\varepsilon^2}$ → $\frac{(x-x_c)^2}{a^2} + \frac{y^2}{b^2}$

x_c = ellipse centered at $= -\frac{\varepsilon p}{1-\varepsilon^2}$

$a = \frac{p}{1-\varepsilon^2}$, $b = \frac{p}{\sqrt{1-\varepsilon^2}}$, [turning points is when $v = 0$] $\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = 0$ → motion about sun r_{max} = aphelion, r_{min} = perihelion

MORE ON BACK

↳ $\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{\ell^2}{\mu r^2}\frac{dr}{d\phi}$, $r(\phi) = p(1+\varepsilon\cos\phi)^{-1}$

$\frac{dr}{dt} = 0 = \frac{p\varepsilon\sin\phi}{(1+\varepsilon\cos\phi)^2} \to \phi = 0, \pi$

$r(\phi=0) = \frac{p}{1+\varepsilon} = a(1-\varepsilon) = r_{min}$

$r(\phi=\pi) = \frac{p}{1-\varepsilon} = a(1+\varepsilon) = r_{max}$

Normally to find t we would use our integral definition $\to \frac{\tau}{2} = \sqrt{\frac{\mu}{2}}\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{K - U(r) - \ell^2/2\mu r^2}}$ * easier way tho

1) $\frac{dA}{dt} = \frac{dA}{dt} \to \int_0^{A=ab\pi} \frac{dA}{\underbrace{dA/dt}_{\text{constant}}} = \int_0^T dt \to \boxed{T = \frac{2\mu}{\ell}\pi ab = \frac{2\mu}{\ell}\frac{\pi p^2}{(1-\varepsilon^2)^{3/2}} = 2\pi\sqrt{\frac{\mu}{k}}\sqrt{a^3}}$

$\varepsilon > 1$ hyperbola ! hyperbolic orbit only possible if the force is repulsive

↳ scattering of positively charged particles