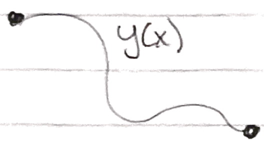
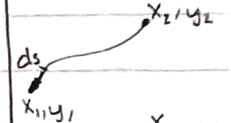
Variational Calculus

the language of modern theoretical physics



• $y(x)$ is an unknown curve

explaining with an example!



[lets say you want to extremize the path between $(x_1,y_1),(x_2,y_2)$]

$$S = \int_{x_1}^{x_2} ds \quad , \quad ds = \sqrt{dx^2+dy^2}$$

In the example we are trying to extremize the integral $S = \int_{x_1}^{x_2} dx \sqrt{1+\left(\frac{dy}{dx}\right)^2}$, We can see that the integral, I, has a general form!

since the value of I depends on the choice of the curve y(x)

I is a function of y

$$\boxed{I[y] = \int F(y, y', x)\, dx}$$

← now extremize this integral

[there's an ∞ amount of possible curves we want to find the extrema]

lets look at small variations in the function F, δy , $\delta \frac{dy}{dx} = \frac{d}{dx}\delta y$

↳ we are varying these parts of F as we go along the curve, but we want the varying to be 0 at the end so we obtain our curve between x_0, x_1 → $\delta y(x_0) = 0$ and $\delta y(x_1) = 0$

we want to set $\delta I = 0$ for extrema

Lets calculate a small variance in $I[y]$ → $\delta I = I[y+\delta y] - I[y]$

$$\delta I = \int \underbrace{F\Big(y+\delta y,\ \overset{y'}{\frac{dy}{dx}} + \delta \overset{y'}{\frac{dy}{dx}},\ x\Big)}_{} dx - \int F\Big(y, \overset{y'}{\frac{dy}{dx}}, x\Big)\, dx$$

Taylor Expand this, the δ terms are small so keep only higher order terms

$$F(y+\delta y,\ y'+\delta y',\ x) = F(y, y', x) + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \text{higher order terms}$$

$$\delta I = \int \Big(\frac{\partial F}{\partial y}\delta y + \underbrace{\frac{\partial F}{\partial y'}\delta y'}_{\text{integrate by parts}}\Big)\, dx$$

reminder $\int u\, dv = uv\Big|_{b_1}^{b_2} - \int v\, du$

$u = \frac{\partial F}{\partial y'}$

$du = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$

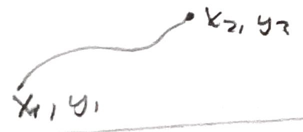
$\int dv = \int \frac{d}{dx}\delta y\, dx \overset{\text{F.T.C}}{=} \delta y$ ← indefinite integral, bounds are evaluated here

$dv = \delta y'\, dx = \frac{d}{dx}\delta y\, dx$

$$\delta I = \int \left[\frac{\partial F}{\partial y} + 0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \delta y\, dx$$

*for an extrema $\delta I = 0$ or $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

← Euler Lagrange equation pops out!



example from earlier of path between

$$S[y] = \int_{0=x_1}^{1=x_2} \underbrace{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}_{F}\, dx \rightarrow \frac{\partial F}{\partial y} = 0 \text{ and } \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$$

$$0 - \frac{d}{dx}\underbrace{\left(\frac{\partial F}{\partial y'}\right)}_{\text{constant}} = 0 \rightarrow \alpha = \frac{y'}{\sqrt{1+(y')^2}},$$ square both sides solve for y'

$$y' = \frac{\alpha}{\sqrt{1-\alpha^2}} = \text{constant} \quad \text{so} \quad y = \overbrace{y'}^{\text{constant}} x + \overset{b}{\text{constant}}$$

Lets call that integral S and S is called the action so if

$S[q] = \int L(q, \dot{q}, t)\, dt$, then the Lagrangian minimizes the action

$$\delta S = 0 \rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

ex cannon ball being fired at $t = t_0$, $t = t_1$ it strikes the ground

$$L = \frac{1}{2}m\dot{y}^2 - mgy, \quad \delta L = L(y + \delta y) - L(y) : \quad \frac{\partial L}{\partial y} = -mg, \quad \frac{\partial L}{\partial y'} = m\dot{y}, \quad \frac{\partial^2 L}{\partial y^2} = 0, \quad \frac{\partial^2 L}{\partial \dot{y}^2} = m$$

↳ taylor expand $L(q + \delta q, \dot{q} + \delta\dot{q}, t) \rightarrow L(y, \dot{y}, t) + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y' + \frac{1}{2}\left(\delta y^2 \frac{\partial^2 L}{\partial y^2} + \delta y'^2 \frac{\partial^2 L}{\partial y'^2}\right) + 2\delta y\, \delta y' \frac{\partial^2 L}{\partial y \partial y'}$

↳ $\delta S = \int -mg\,\delta y + m\dot{y}\,\delta\dot{y} + \frac{1}{2}m\,\delta\dot{y}^2\, dt$

rewrite ↳ $\delta S = \int \frac{1}{2}m\left[2\dot{y}\left(\frac{d}{dt}\delta y\right) + \left(\frac{d}{dt}\delta y\right)^2\right] - mg\,\delta y$

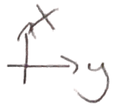
deriv $\frac{d}{dt}$ ↳ $\frac{d}{dt}(\dot{y}\,\delta y) = \dot{y}\frac{d}{dt}\delta y + \ddot{y}\,\delta y \rightarrow \dot{y}\frac{d}{dt}\delta y = \frac{d}{dt}(\dot{y}\,\delta y) - \ddot{y}\,\delta y$

$$\delta S = \int \underbrace{-m(\ddot{y} + g)\,\delta y}_{\text{this just has to be } 0} + \underbrace{m\frac{d}{dt}(\dot{y}\,\delta y)}_{\text{integrating this will depend only on end points so it } = 0} + (\delta \dot{y})^2\, dt$$ by F.T.C

$$\ddot{y} = -g$$

↳ if this true lets look at $\frac{m}{2}\int(\delta\dot{y})^2\, dt \geq 0$ by def of the δ

so $\ddot{y} = g$ minimizes the action



Lagrangian Examples

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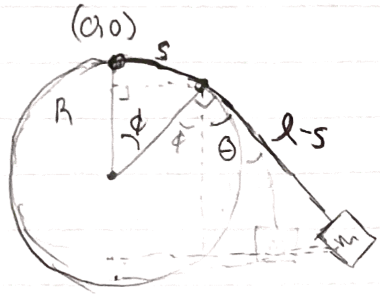

$$x_m = (R\sin\phi) + (l-s)\sin\theta$$
$$y_m = -[(l-s)\cos\theta + R(1-\cos\phi)]$$

$s = R\phi$ and $\theta + \phi = 90°$

$T = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2)$ and $V = -mgy$ ... calculate lagrangian

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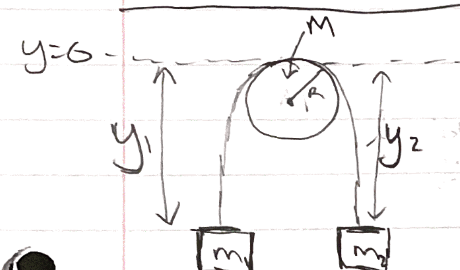


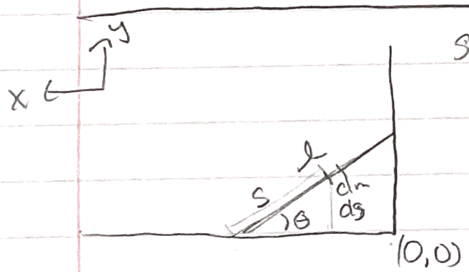
$y_1 = R\theta$, $\dot{y}_1 = R\dot{\theta}$

$y_1 + y_2 = l$ and $y_1 = R\theta$

$\dot{y}_2 = -\dot{y}_1$ and $\dot{\theta} = \frac{\dot{y}_1}{R}$

$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}I\dot{\theta}^2$

$V = m_1 g y_1 + m_2 g y_2$ + (constant potential of pulley)



s = center of mass of ladder

$y = s\sin\theta$

$x = l\cos\theta - s\cos\theta$

$dT = \underbrace{\frac{1}{2}dm}_{\text{little bits of the ladder}}(\dot{x}^2 + \dot{y}^2)$ $\quad \frac{dm}{ds} = \frac{M}{l}$, then integrate from 0 to l

Least Action Examples/explanation

$$I[y] = \int F(y, y', x)\,dx \rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}} = 0$$ [extremizes this integral]

Generally! $S[q] = \int L(q, \dot{q}, t)\,dt \qquad \delta s = 0 \rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$

* this is for each degree of freedom of the system, if they are coupled and you didnt pick the least amount of deg of freedom then use Lagrange Multiplers to find the relationship between them. [can also use them to learn about constraint forces]

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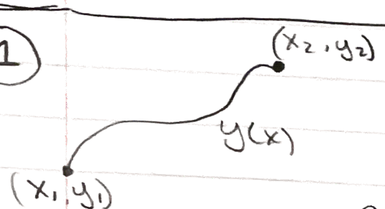
(1)



• find some curve, $y(x)$ that extremizes the distance between the two points.

$$S = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \underbrace{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}}_{F(y,\, y',\, x)}$$ use $\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}} = 0}$

$\frac{\partial F}{\partial y} = 0$ so $\frac{\partial F}{\partial \dot{y}} = \text{const.} = \frac{y'}{\sqrt{1+(y')^2}} = \alpha$

$\hookrightarrow y' = \pm \frac{\alpha}{\sqrt{1-\alpha^2}} = m \rightarrow \frac{dy}{dx} = m \rightarrow \boxed{y = mx + b}$ eq'n of line

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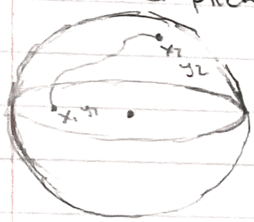
(2) more complicated.



$$ds^2 = \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2$$

$$I = \int \rho\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2} = \int_{\theta_1}^{\theta_2} \rho\, d\theta \underbrace{\sqrt{1 + \sin^2\theta\, \dot{\phi}^2}}_{F(\theta,\, \phi,\, \dot{\phi})}$$

$$\frac{\partial F}{\partial \phi} = 0, \quad \frac{\partial F}{\partial \dot{\phi}} = \frac{\dot{\phi}\sin^2\theta}{\sqrt{1+\sin^2\theta\,\dot{\phi}^2}} = \text{constant} = C \rightarrow \dot{\phi} = \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}}$$

integrate w/ some fancy substitutes $\qquad \phi = \arccos(\beta\cot\theta) + \phi_0$

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generally

$$\delta s = \int\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q\, dt$$

and if your δq's depend on eachother use Lagrange Multipliers where $G(q) = \text{constant} = C$

sum over different q's

$$\delta s = \int\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \underbrace{\lambda\frac{\partial G}{\partial q}}_{\text{double check}}\right]\delta q\, dt$$

Second form of Eulers Equation

↳ this is if F has form $f(y, y')$ no explicit x dependence.
ie $\frac{\partial F}{\partial x} = 0$.

Chain rule $\quad \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \underbrace{\frac{\partial f}{\partial y'} y''}$

also $\quad \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = y'' \frac{\partial f}{\partial y'} + y' \frac{d}{dx}\frac{\partial f}{\partial y'}$

Solve for this and put it here

$$\underbrace{\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right)} = \frac{df}{dx} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}y' + y' \frac{d}{dx}\frac{\partial f}{\partial y'} = \underbrace{\frac{df}{dx} - \frac{\partial f}{\partial x}} - \underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)}_{=0 \text{ Eulers eq.}} y'$$

only these terms left

$$\underbrace{\frac{\partial f}{\partial x}}_{0 \text{ by assumption of } f} - \frac{d}{dx}\left(f - y' \frac{\partial f}{\partial y'}\right) = 0$$

so just the inside is a constant

$$\boxed{f - y' \frac{\partial f}{\partial y'} = \text{constant}}$$

Lagrange multipliers

Recall from the integral we want to extremize or in physics case, action

$$\delta I = \delta S = \int \underbrace{\left[\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial \dot{y}}\right]}_{\frac{\delta L}{\delta q} \text{ for notation}} \underbrace{\delta y}_{\delta q}\, dt$$

for generality $\delta y \Rightarrow \delta q$

$$\delta S = \int \frac{\delta L}{\delta q}\, \delta q\, dt$$

* Its been proved that vanishing the variational derivative is <u>sufficient condition</u> for an extremum of the integral, so if $\frac{\delta L}{\delta q} \to 0$, $\delta S \to 0$ thus an extrema

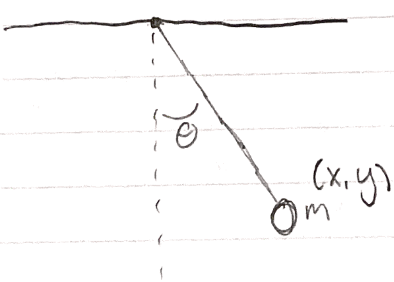
<u>Lets understand this,</u>

for an x and y case!

$$\delta S = \int \left(\frac{\delta L}{\delta x}\delta x + \frac{\delta L}{\delta y}\delta y\right) dt$$

- if δx and δy <u>are independent</u>, then if you make <u>$\frac{\delta L}{\delta x}$ and $\frac{\delta L}{\delta y} \to 0$</u> then <u>$\delta S \to 0$</u>
- if q_k's <u>cant be varied independently</u> then we have to resort to <u>Lagrange multipliers</u>, to restore their independence of δq_k's, usually one tries to choose coordinates so that they are independent and each one expresses a different <u>degree of freedom</u>. For non holonomic problem's Lagrange Multipliers cant be avoided.

<u>Example of how it can go wrong!</u>



Blindly apply ELE: $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

$\hookrightarrow m\ddot{x} = 0$ and $m\ddot{y} + mg = 0$] But this is the EOM for an object in freefa

So we didnt correctly describe the system, we <u>forgot a constraint</u>

$\hookrightarrow$ constraints describe the system in this case x and y are related

$x^2 + y^2 = \ell^2$

Review of Lagrange Multipliers:

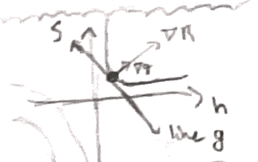
Labor \$20/h
Steel \$2000/ton
Budget = \$20,000

$20h + 2000s = 20{,}000$

Function of Revenue:

$R(h,s) = 100\,h^{2/3} s^{1/3}$

$L = R - \lambda(B-g)$ here! woah!



R will make a bunch of contours in this space.

To maximize our Revenue subject to … we want the point just as they touch. This means if you increase your hrs or steel just by a little bit then you fall out of your budget. So its an extremum.

$\nabla R = \lambda \nabla g$, $\nabla R = \begin{bmatrix} \frac{\partial R}{\partial h} \\ \frac{\partial R}{\partial s} \end{bmatrix}$, $\nabla g = \begin{bmatrix} \partial g/\partial h \\ \partial g/\partial s \end{bmatrix}$

$\frac{\partial R}{\partial h} = \lambda \frac{\partial g}{\partial h}$
$\frac{\partial R}{\partial s} = \lambda \frac{\partial g}{\partial s}$

soln: $h = 200s$, plug into constraint

then $s = 10/3$, $h = 2000/3$

Lagrange Multipliers

→ for dependent generalized coordinates

sign wrong here?

$$\delta S = \int \left\{ \frac{\delta L}{\delta x}\delta x + \frac{\delta L}{\delta y}\delta y \right\} dt \quad \text{where} \quad \frac{\delta L}{\delta q} = +\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

assume the constraint connecting them is of the type $G(x,y) = \text{constant} = C$

↳ G is assumed to be function of the coordinates and time, not velocities.

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial y}\delta y = \overbrace{0}^{\delta \text{constant} = 0} \quad (\text{variations in } G\text{, chain rule})$$

So we have $\delta S = \int \delta L\, dt$, What if we sub in $\delta(L + \overbrace{\lambda(G-C)}^{G-C=0}) = \delta L$

↳ then using the variation of $\lambda\,\delta G$ from above we can sub into δS

$$\delta S = \int \left[\left(\frac{\delta L}{\delta x} + \lambda\frac{\partial G}{\partial x} \right)\delta x + \left(\frac{\delta L}{\delta y} + \lambda\frac{\partial G}{\partial y} \right)\delta y \right] dt \quad , \quad \lambda(t)$$

by doing this it also makes coeff of δy vanishes, why?

make both 0 w/ λ

• now choose a λ so this vanishes bec/ we are trying to make $\delta S \to 0$

Then: $\frac{\delta L}{\delta x} + \lambda\frac{\partial G}{\partial x} = 0 \qquad G(x,y) = \text{constant}$

$\frac{\delta L}{\delta y} + \lambda\frac{\partial G}{\partial y} = 0$

Back to Pendulum!

does affect the intrepetation of λ

$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$ and $G(x,y) = \sqrt{x^2+y^2} = \ell$

* check signs

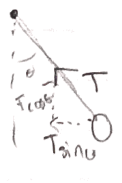
sloppy?

$\frac{\delta L}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 + m\ddot{x}$, $\quad \frac{\delta L}{\delta x} + \lambda\frac{\partial G}{\partial x} = 0$, $\quad \frac{\partial G}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\ell}$

$\frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = (-mg) - m\ddot{y}$ $\quad \frac{\delta L}{\delta y} + \lambda\frac{\partial G}{\partial y} = 0$ $\quad \frac{\partial G}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{\ell}$

$m\ddot{x} = \lambda\frac{x}{\ell}$

$-mg - m\ddot{y} = \lambda\frac{y}{\ell}$



$F_x = -T\sin\theta = -T\frac{x}{\ell}$, $F_x = m\ddot{x}$

$F_y = -mg + T\frac{y}{\ell}$, $F_y = m\ddot{y}$

Generalizing the result of Lagrange multipliers

$N_k \equiv \sum_i \vec{N}_i \cdot \frac{\partial \vec{r}}{\partial q_k}$ then $\delta W = N \delta q_1 + N_c \delta q_2 = 0$

$\vec{N}_i$ = constraint force on particle i

d'Alembert; $\frac{\delta T}{\delta q_k} = \frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} = -\sum \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}$

$\dot{\vec{p}}_i = -\vec{\nabla} V + \vec{N}_i$ (conservative and non-conservative forces)

$\frac{\delta T}{\delta q_k} = -\sum \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}$, plug in move potential part into ELE to get $\frac{\delta L}{\delta q_k}$

$\frac{\delta L}{\delta q_k} = -N_k \qquad k = 1, \dots, N$

N = # of coordinates
N_c = # of constraints
$N_0 = N - N_c$ = # of deg freedom

N_E = # equations = $N + N_c = N_0 + 2N_c$

From Lagrange multiples we generally formed $\frac{\delta L}{\delta q_k} = \underbrace{-\lambda \frac{\partial G}{\partial q_k}}_{} = -N_k$

Lagrange multipliers are proportional to constraint forces, obvious from understanding of Lagrange multiples in general because of

$$\nabla h = \lambda \underbrace{\nabla g}_{\text{constraint}}$$

Brachistochrone

$V = \frac{d \to \text{falls}}{t}$ → $dt = \frac{ds}{V}$, $ds^2 = dx^2 + dy^2$
and the speed at the bottom is $V = \sqrt{2gy}$

$t = \int \frac{ds}{V} = \int \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx$ thus our Function = $F(y, y', x) = \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}$

Use ELE to extremize it, or find the path of least action, minimize t

$\frac{\partial F}{\partial y} = \frac{-1}{2}\sqrt{\frac{1+y'^2}{2g}}\frac{1}{y^{3/2}}$ | $\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}, \frac{1}{\sqrt{2gy}} =$ constant bec $\frac{d}{dx}\frac{\partial F}{\partial y'} = 0$

mult each side by $\sqrt{2g}$ y(x) stw

$$-\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}} = \frac{d}{dx}\frac{y'}{\sqrt{y(1+y'^2)}} \xrightarrow[\text{shift down}]{\text{take derivite and do algebra}} -\frac{1}{2y} = \frac{y''}{1+y'^2} \to 1+2yy''+y'^2=0$$

Multiply each side by y', $y' + 2yy'y'' + y'^3 = 0 \to \frac{d}{dx}(y + yy'^2) = 0$

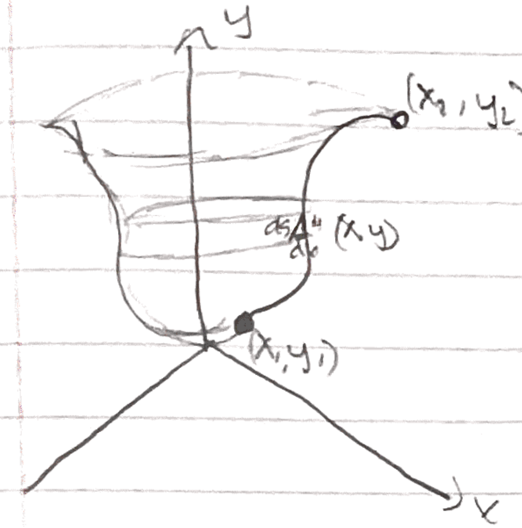
$$y + yy'^2 = C \to y' = \sqrt{\frac{C-y}{y}} \to x = \int\sqrt{\frac{y}{C-y}}\,dy + D$$

Say $y = C\sin^2(t)$, $dy = 2C\sin t\cos t\,dt \to x = \int\sqrt{\frac{C\sin^2 t}{C - C\sin t}}\,2C\sin t\cos t\,dt$

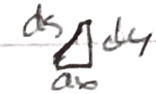
trig IBS $x = 2C\int \sin^2 t\,dt = \left(\frac{t}{2} - \frac{1}{4}\sin(2t)\right)2C + D$, if it starts at $(0,0)$ then $D=0$

$y = C\sin^2 t = \frac{C}{2} - \frac{C}{2}\cos(2t)$ this is the parameterized curve

More examples of minimizing action.



find a curve that when rotated about the y-axis it forms a minimum surface area



$$dA = 2\pi x \, ds$$

$$dA = \underbrace{2\pi x \sqrt{1+(y')^2}}_{f(y, y', x)} \, dx$$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{d}{dx}\left(\frac{x y'}{\sqrt{1+y'^2}}\right) = 0 \quad \text{so} \quad \frac{x y'}{\sqrt{1+y^2}} = \text{const} = C$$

$$x^2 (y')^2 = c^2 (1+(y')^2) \rightarrow \frac{dy}{dx} = \frac{c}{\sqrt{x^2-c^2}} \rightarrow y = c \int \frac{dx}{\sqrt{x^2-c^2}}$$

$$y = c \cosh^{-1}\left(\frac{x}{c}\right) + b$$