Getting F_y and F_x

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{dp_y/dt}{\gamma - \frac{\gamma\beta}{c}\frac{dx}{dt}} = \boxed{\frac{F_y}{\gamma(1-\frac{\beta}{c}u_x)}}$$

$$\boxed{\bar{F}_z = \frac{F_z}{\gamma(1-\beta\frac{u_x}{c})}}$$

$$\boxed{\text{For } d\bar{p}_x \quad p^\mu = m\eta^\mu \to \bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu \quad \hookrightarrow \bar{p}^\mu = \Lambda^\mu_\nu p^\nu \to \bar{p}^1 = \bar{p}_x = \gamma p_x - \gamma\beta p^0}$$

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma dp_x - \gamma\beta dp^0}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{1-\frac{\beta}{c}\frac{dx}{dt}} = \frac{F_x - \beta\frac{dp^0}{dt}}{1-\frac{\beta}{c}u_x}, \quad p^0 c = \gamma mc^2 = E$$

$$\bar{F}_x = \frac{F_x - \frac{\beta}{c}\left(\frac{dE}{dt}\right)}{1-\frac{\beta u_x}{c}} = \boxed{\frac{F_x - \beta(\vec{F}\cdot\vec{u})/c}{1-\frac{\beta u_x}{c}}}$$

If particle is stationary in S then $\vec{u} = 0$!

$$\bar{F}_\perp = \frac{1}{\gamma}F_\perp, \quad \bar{F}_\parallel = F_\parallel, \quad \bar{F}_z = \frac{F_z}{\gamma}, \quad \bar{F}_y = \frac{F_y}{\gamma}$$

$$\bar{F}_x = F_x$$

Proper Force

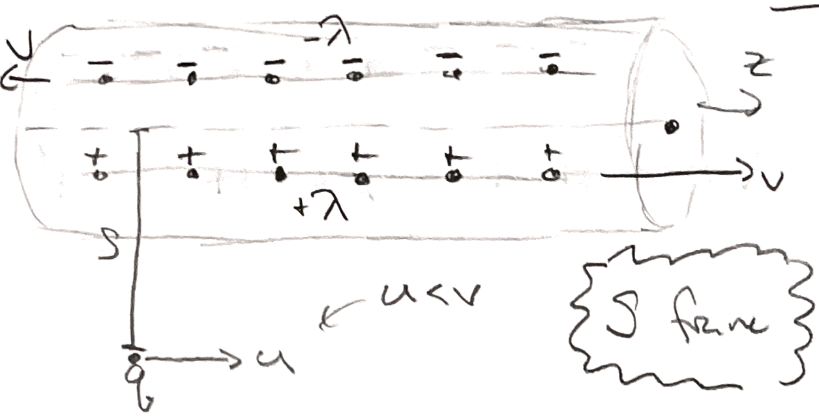
$$K^\mu = \frac{dp^\mu}{d\tau} \text{ (Minkowski Force)} \to \vec{K} = \frac{dt}{d\tau}\frac{d\vec{p}}{dt} = \gamma\vec{F}$$

$$\hookrightarrow K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{dt}$$

Relativistic ElectroDynamics

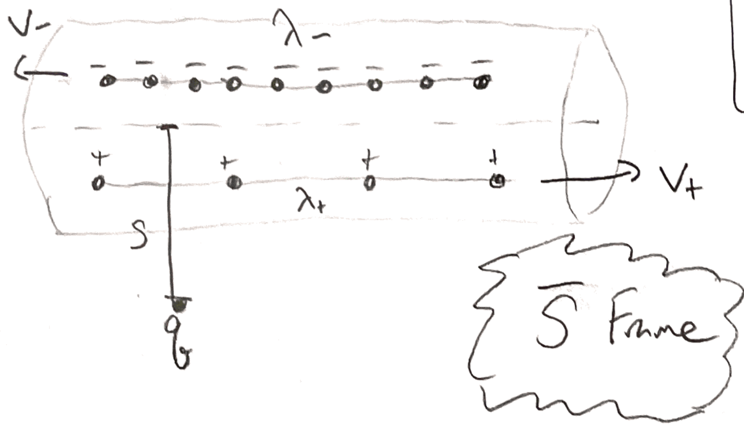
Consider this wire:

Net current: $I = 2\lambda v$



No electric field, as $Q_{encl} = 0$ by gausses law, which is also relativistic

← watcher a current carrying wire, w/ a charge moving



[we are now in a reference frame where q is at rest]

→ we need to transform v, u

$v_{AB} = v$, $\to v_{BC} = u$

$$V_{AC} = \frac{V_{AB} + V_{BC}}{1 + \frac{V_{AB}V_{BC}}{c^2}}$$

v = velocity of $\bar{S}$ relative to S

Review the Einstein velocity addition rule:

definition

$$\bar{u} = \frac{u - v}{1 - \frac{uv}{c^2}},$$

u = speed of object in S. $\bar{u}$ = speed of object in $\bar{S}$

v = speed of relative frames

def — our sys — $v \to u$ in this case

[In $\bar{S}$ frame we are measuring the new velocity of the line charges] → [S frame measured them to be the same speed v and relative frame velocity is u]

→ u = speed of line charge in S

$\bar{u}$ = speed of line charge in $\bar{S}$

speed of λ_+ — speed of q or frames

$$\hookrightarrow v_+ = \frac{v - u}{1 - \frac{uv}{c^2}}, \quad v_- = \frac{-v - u}{1 + \frac{uv}{c^2}} \quad \text{but } -v_- \to -v_- = \frac{v + u}{1 + \frac{uv}{c^2}}$$

[Just care about magnitude for lorentz contraction] → $\boxed{v_\pm = \frac{v \mp u}{1 \mp \frac{uv}{c^2}}}$

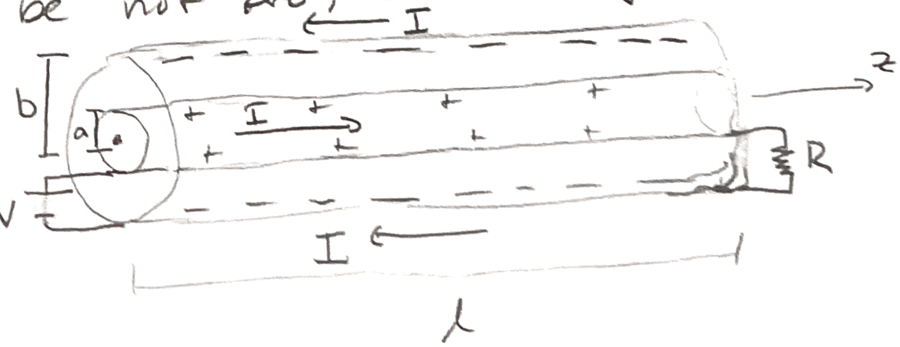
$v_- > v_+$

$v_- > v_+$ so lorentz contraction is more severe.

Center of Energy Relation

$$\vec{P} = \frac{E}{c^2}\frac{dR_e}{dt}, \quad R_e = \frac{1}{E}\sum E_i r_i$$

We found momentum stored in the fields of a coaxial cable to be not zero, even though cable is at rest.



momentum in the fields is

$$\vec{P} = \mu_0 \varepsilon_0 \int \vec{S}\, d\tau$$

$$= \frac{IVl}{c^2}\hat{z}$$

In relativity

$$R_e = \frac{1}{E}\left(E_0 \vec{R}_0 + E_R l \hat{z}\right)$$

energy in resistor

center of energy of E_0 = rest of energy ← doesn't change in time

$$\rightarrow \frac{dR_e}{dt} = \frac{(dE_R/dt)\, l}{E}\hat{z} = \frac{IVl}{E}\hat{z}$$

power of resistor

$$\vec{P} = \frac{E}{c^2}\frac{IVl}{E}\hat{z} = \frac{IVl}{c^2}\hat{z} \checkmark$$

momentum of fields is just looking at energy being transported

$\lambda_{0,} = \frac{q}{L_{0,+}}$, $\gamma_+ = \frac{1}{\sqrt{1+\frac{v_+^2}{c^2}}}$, $\gamma_- = \frac{1}{\sqrt{1+\frac{v_-^2}{c^2}}}$ (Rest length is in S frame so contract in $\bar{S}$)

$\lambda_+ = \frac{q}{L_+}$, $L_+ = \frac{1}{\gamma_+} L_0$, $L_- = \frac{1}{\gamma_-} L_0$

rest length in S

↳ $\lambda_+ = \gamma_+ \frac{q}{L_0} = \gamma_+ \lambda_0$ and $\lambda_- = \frac{-q}{L_-} \rightarrow \lambda_- = -\gamma_- \frac{q}{L_0} = -\gamma_- \lambda_0$

$S \rightarrow \bar{S}$

↳ $\boxed{\lambda_\pm = \pm \gamma_\pm \lambda_0}$ and $v_- > v_+$ so its contracted more leading to an overall negative charge

* λ = charge density of the positive line its own rest system

↳ in S though, the positive line charge is moving w/ speed V

So $\boxed{\lambda = \gamma \lambda_0}$, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ velocity between the frames , due to length contraction since $\lambda = \frac{q}{L_0}$, $L_0 = \frac{1}{\gamma} L_{0,0}$

length in S ; rest length in positive line sys

$S \quad \bar{S}$

$$\gamma_\pm = \frac{1}{\sqrt{1-\frac{v_\pm^2}{c^2}}} , \quad v_\pm = \frac{v \mp u}{1 \mp \frac{vu}{c^2}} \rightarrow \gamma_\pm = \frac{1}{\sqrt{1 - \frac{1}{c^2}\frac{(v \mp u)^2}{(1 \mp \frac{vu}{c^2})^2}}}$$

algebra

$$\gamma_\pm = \frac{c^2 \mp uv}{\sqrt{(c^2 \mp uv)^2 - c^2 (v \mp u)^2}} = \gamma \frac{1 \mp \frac{uv}{c^2}}{\sqrt{1-u^2/c^2}}$$

Net line charge in $\bar{S}$:

$\lambda_0 = \frac{1}{\gamma}\lambda$

$$\lambda_{tot} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \lambda \frac{1 - uv}{c^2\sqrt{1-u^2/c^2}} - \lambda \frac{1+uv}{c^2\sqrt{1-\frac{u^2}{c^2}}} = \frac{-2\lambda u v}{c^2\sqrt{1-\frac{u^2}{c^2}}}$$

$\boxed{\lambda_{tot} = \frac{-2\lambda u v}{c^2\sqrt{1-u^2/c^2}}} \rightarrow E = \frac{\lambda_{tot}}{2\pi\varepsilon_0}\frac{1}{s}$, theres an $\bar{F} = qE$ in $\bar{S}$ on q

↳ $\bar{F} = -\frac{\lambda v}{2\pi\varepsilon_0 c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}$, $F_\perp = \frac{1}{\gamma}\bar{F}_\perp \rightarrow \bar{F} = -\frac{\lambda v}{\pi\varepsilon_0 c^2}\frac{qu}{s}$

$\lambda v = I/2$ and $c^2 = \frac{1}{\mu_0\varepsilon_0}$ $\rightarrow \boxed{-qu \underbrace{\frac{\mu_0 I}{2\pi s}}_{\text{B field}} = F}$

Potential and Fields

i) $\nabla\cdot E = \rho/\varepsilon_0$ ii) $\nabla\cdot B = 0$ iii) $\nabla\times E = -\frac{\partial B}{\partial t}$ iv) $\nabla\times B = \mu_0 J + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$

What is the general solution to maxwells eq given ρ and J. In static case Coulomb & Biot-Savart provided answer.

Step 1: Turn fields into potentials

$E \neq -\nabla V$ bec $\nabla\times E \neq 0$ but B remains divergenceless

$\boxed{B = \vec\nabla\times\vec A} \rightarrow \vec\nabla\times\vec E = -\frac{\partial}{\partial t}(\vec\nabla\times\vec A) \Rightarrow \nabla\times\left(E + \frac{\partial A}{\partial t}\right) = 0$ ← this is the curl thats 0

$\hookrightarrow \boxed{\vec E + \frac{\partial\vec A}{\partial t} = -\vec\nabla V}$ → apply gauss's law $\nabla\cdot E = \frac{\rho}{\varepsilon_0}$ → $\nabla^2 V + \frac{\partial}{\partial t}\vec\nabla\cdot\vec A = \frac{-\rho}{\varepsilon_0}$

Now use $\nabla\times B = \mu_0 J + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$; use $\nabla\times\nabla\times A = \nabla(\nabla\cdot A) - \nabla^2 A$

$\hookrightarrow \nabla\times\nabla\times A = \mu_0 J - \mu_0\varepsilon_0\nabla\frac{\partial V}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}$,

$\hookrightarrow \nabla^2\vec A - \mu_0\varepsilon_0\frac{\partial^2\vec A}{\partial t^2} - \vec\nabla\left(\vec\nabla\cdot\vec A + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec J$ Maxwells equations

$\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot A) = \frac{-\rho}{\varepsilon_0}$

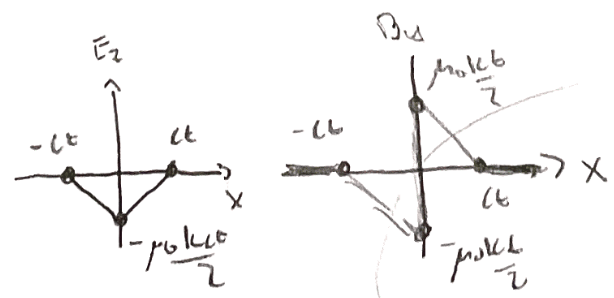
Ex) Find the charge & current distributions that would give rise to

$V = 0$ and $A = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2\hat z & |x| < ct \\ 0 & |x| > ct \end{cases}$, $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

$\hookrightarrow \vec E = \frac{-\partial\vec A}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\hat z$

$\vec B = \vec\nabla\times\vec A = \frac{-\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\hat y = \pm\frac{\mu_0 k}{2c}(ct - |x|)\hat y$ x>0 for plus, x<0 for minus } $x < ct$

$\vec E = \vec B = 0$ } for $x > ct$



discontinuity

$\vec\nabla\cdot\vec E = 0$ and $\vec\nabla\cdot\vec B = 0$

$\vec\nabla\times\vec E = \mp\frac{\mu_0 k}{2}\hat y$

$\vec\nabla\times\vec B = -\frac{\mu_0 k}{2c}\hat z$

$\frac{\partial\vec E}{\partial t} = -\frac{\mu_0 kc}{2}\hat z$

$\frac{\partial\vec B}{\partial t} = \pm\frac{\mu_0 k}{2}\hat y$

From boundary conditions $\frac{1}{\mu_0}B_1^{\parallel} - \frac{1}{\mu_0}B_2^{\parallel} = \vec K\times\hat n$

$kt\,\hat y = \vec K\times\hat x$

$\boxed{\vec K = kt\,\hat z}$

Lorentz Force in Potential Form

$$\vec{F} = q(\vec{E} \times \vec{v} \times \vec{B}) = q\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A})\right)$$

↳ $\vec{\nabla}(\vec{v} \cdot \vec{A}) = \vec{v} \times (\vec{\nabla} \times \vec{A}) + (\vec{v} \cdot \vec{\nabla})\vec{A}$

↳ $$\frac{d\vec{p}}{dt} = -q\left(\underbrace{\frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A}}_{\text{convective derivative}} + \nabla(V - \vec{v} \cdot \vec{A})\right)$$

$$d\vec{A} = \vec{A}(\vec{r} + \vec{v}\,dt, t + dt) - \vec{A}(\vec{r}, t) = \frac{\partial A}{\partial x} v_x\,dt + \frac{\partial \vec{A}}{\partial y} v_y\,dt + \frac{\partial \vec{A}}{\partial z} v_z\,dt + \frac{\partial \vec{A}}{\partial t}$$

$$= \frac{\partial \vec{A}}{\partial x}dx + \frac{\partial \vec{A}}{\partial y}dy + \frac{\partial \vec{A}}{\partial z}dz + \frac{\partial \vec{A}}{\partial t}dt \quad , \quad dx = v_x\,dt, \text{ etc....}$$

$\boxed{\text{Divide by } dt}$ → $\frac{d\vec{A}}{dt} = \frac{\partial A}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A}$ → $v_x \frac{\partial}{\partial x}\vec{A}$

↳ $$\frac{d}{dt}(\vec{p} \cdot q\vec{A}) = -\vec{\nabla}\big(\underbrace{q(V - \vec{v} \cdot \vec{A})}_{U}\big)$$

$\frac{d\vec{p}}{dt} = -\vec{\nabla} U$ → $\vec{P}_{canonical} = \vec{p} + q\vec{A}$

$U = q(V - \vec{v} \cdot \vec{A})$

$\vec{A}$ is potential momentum per unit charge

$$\frac{d}{dt}(T + qV) = \frac{\partial}{\partial t}\left[q(V - \vec{v} \cdot \vec{A})\right]$$

Gauge Transformations

The potential formulation is ugly but we reduced 6 problems (finding E and B) down to 4 (V, and A_x, A_y, A_z)

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

Dont uniquely define the potential bcc they are defined by derivatives

we will work at this gauge freedom now,

Suppose we have two potentials $(V, \vec{A})$ and $(V', \vec{A}')$ → these potentials correspond to the same E + B field → how much do they differ → $\vec{A}' = \vec{A} + \vec{\alpha}$, $V' = V + \beta$

Since both A and A' both give the same $\vec{B}$ the curls have to be equal → $\vec{\nabla} \times \vec{\alpha} = 0$, $\vec{\alpha} = \vec{\nabla}\lambda$ →

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$
$$= -\vec{\nabla} V' - \underbrace{\vec{\nabla}\beta} - \frac{\partial A'}{\partial t} - \underbrace{\frac{\partial \vec{\alpha}}{\partial t}}$$

have to add to 0 to give same $\vec{E}$

So $\vec{\nabla}\beta + \frac{\partial \vec{\alpha}}{\partial t} = 0$ → $\nabla\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0$

derivatives of position

By taking derivatives of position we get 0 for β and $\frac{\partial \lambda}{\partial t}$ so $\beta + \frac{\partial \lambda}{\partial t}$ is ind. of position → $\beta = -\frac{\partial \lambda}{\partial t} + k(t)$ →

$$\boxed{\begin{aligned} A' &= A + \nabla\lambda \\ V' &= V - \frac{\partial \lambda}{\partial t} \end{aligned}}$$

absorb $k(t)$ into λ

Coulomb gauge

$\vec{\nabla} \cdot \vec{A} = 0$ → $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho}{\varepsilon_0}$ → $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ → $V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}', t)}{r} d\tau'$

↳ $\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 \vec{J} + \mu_0 \varepsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right)$

Lorenz gauge

$\vec{\nabla} \cdot \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$ → $\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$

$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$

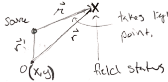
$$\Box^2 = \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}$$ → $\Box^2 V = -\frac{\rho}{\varepsilon_0}$, $\Box^2 \vec{A} = -\mu_0 \vec{J}$

Retarded Potentials

$$\boxed{\begin{aligned}\Box^2 V &= -\frac{\rho}{\varepsilon_0}\\ \Box^2 \vec{A} &= -\mu_0 \vec{J}\end{aligned}}$$

$\left[\text{In static case}\right] \to \nabla^2 V = -\frac{\rho}{\varepsilon_0}$, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ $\to \left[\text{their Solutions}\right] \to V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau'$, $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$

$\mathscr{r}$ = distance from source point $\vec{r}\,'$ to the field point $\vec{r}$



- EM waves travel at the speed of light. c — RIGHT Now — takes light $\frac{\mathscr{r}}{c}$ to get to the point.

- if we want to know the status of the field at a point, we need the EM wave that came $\frac{\mathscr{r}}{c}$ seconds ago

$\to$ $\boxed{t_r = t - \frac{\mathscr{r}}{c}}$ ← time it takes for light to travel

t_r: the time when light was emitted from source $\frac{\mathscr{r}}{c}$ seconds ago; t: current time

The solutions are therefore:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

$\rho(\vec{r}\,', t_r)$ is the point $\vec{r}\,'$ from origin when light left the source

t_r = when light left the source

t now $- \frac{\mathscr{r}}{c}$ time it took for light to travel

$\left[\text{Now these must satisfy the wave eq and meet the Lorentz gauge } \nabla\cdot\vec{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right]$ $\mathscr{r} = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

depends on $\vec{r}$ in two places, ρ and $\mathscr{r}$

$\left[\text{Calculate gradient } V\right] \to \vec{\nabla} V = \frac{1}{4\pi\varepsilon_0}\int \vec{\nabla}\left(\frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}}\right) d\tau' = \frac{1}{4\pi\varepsilon_0}\int\left[(\vec{\nabla}\rho)\frac{1}{\mathscr{r}} + \rho\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$ $\boxed{\text{so } \nabla(\mathscr{r}^n) = n\mathscr{r}^{n-1}\hat{\mathscr{r}}}$

$t - \frac{\mathscr{r}}{c}$

use $\nabla(fg) = f(\nabla g) + g(\nabla f)$ and $\frac{\partial}{\partial x}(\mathscr{r}^n) = n\mathscr{r}^{n-1}\frac{\partial \mathscr{r}}{\partial x} = n\mathscr{r}^{n-1}\frac{1}{2}\frac{1}{\mathscr{r}}2\mathscr{r}_x = \hat{\mathscr{r}}$

$\frac{\partial}{\partial x}\rho = \frac{d\rho}{dt_r}\frac{\partial t_r}{\partial x}$

$$\vec{\nabla}\rho = \frac{d\rho}{dt_r}\vec{\nabla}t_r + \frac{d\rho}{d\vec{r}\,'}\cancel{\vec{\nabla}\vec{r}\,'} = \dot{\rho}\,\vec{\nabla}t_r = \dot{\rho}\,\vec{\nabla}\left(t - \frac{\mathscr{r}}{c}\right) = -\frac{\dot{\rho}}{c}\vec{\nabla}\mathscr{r} \quad \to \vec{\nabla}\mathscr{r} = \hat{\mathscr{r}}$$

doesn't change

$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\hat{\mathscr{r}}/\mathscr{r}^2$

$\to \vec{\nabla}V = \frac{1}{4\pi\varepsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$ $\to$ take the divergence use $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$ twice

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int -\frac{1}{c}\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot(\nabla\dot{\rho}) + \dot{\rho}\,\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)\right] - \left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\rho + \rho\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau$$

$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\,\vec{\nabla}\mathscr{r} = -\frac{1}{c}\ddot{\rho}\,\hat{\mathscr{r}}$ and $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}$ and $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\,\delta^3(\mathscr{r})\,d\tau = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\varepsilon_0}\rho(\vec{r},t)$$

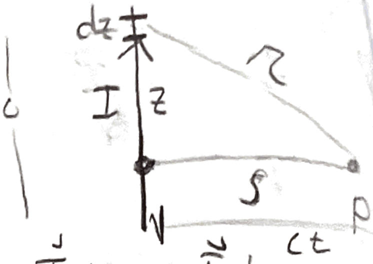
split integral up and pull out $\frac{\partial^2}{\partial t^2}$ ta da!!

Example 10.2

$\vec{r}\,' = s'\hat{s} + z'\hat{z}$, $\vec{r} = s\hat{s} + z\hat{z}$

$$I(t) = \begin{cases} 0 & t \le 0 \\ I_0 & t > 0 \end{cases}$$

[current for an infinetly long wire] constant current turned on abruptly at $t=0$



$\vec{J}\,d\tau \to \vec{I}\,dz$

no ρ so no $V(\vec{r},t)$ but $\vec{A}$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \to \vec{A}(s,t) = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{+\infty} \frac{I(t_r)}{\mathscr{r}}\,dz$$

[Potential at point P depends on the retarded time t_r. $t_r = t - \frac{\mathscr{r}}{c}$]

$\frac{s}{c}$ = time for news to arrive to point P

$ct_r = ct - \sqrt{z^2+s^2}$

$z^2+s^2 \le (ct)^2$

$t_r < 0$ then $I(t_r) = 0$ which happens when

$$\vec{A}(s,t) = \frac{\mu_0}{4\pi}\int_{-\sqrt{(ct)^2-s^2}}^{\sqrt{(ct)^2-s^2}} \frac{I_0}{\mathscr{r}}\,dz = \frac{\mu_0}{4\pi}\int_{-\sqrt{(ct)^2-s^2}}^{\sqrt{(ct)^2-s^2}} \frac{I_0}{\sqrt{s^2+z^2}}\,dz$$

Integrate from $z=0$ to the point

[When $ct = \sqrt{z_{any}^2+s^2}$, the light reaches the length of the hypotenuse ~~at we integrating~~, Anything more than it or our point z_{any} ~~at we~~ I is still 0. If z is on at $\sqrt{(ct)^2-s^2}$ then its certainly on for $z=0$ than by symmetry its on
↳ or if light reached at the top then at the bottom $z = -\sqrt{(ct)^2-s^2}$]

Now integrate

$$\vec{A}(s,t) = \frac{\mu_0 I_0}{4\pi}\,2\int_0^{\sqrt{(ct)^2-s^2}} \frac{dz}{\sqrt{s^2+z^2}} = \frac{\mu_0 I_0}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)\hat{z}$$

$$\vec{E} = -\frac{\partial A}{\partial t} = \frac{-\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-s^2}}\,\hat{z}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial \vec{A}}{\partial s}\,\hat{\phi} = \frac{\mu_0 I_0}{2\pi s}\,\frac{ct}{\sqrt{(ct)^2-s^2}}\,\hat{\phi}$$

as $t \to \infty$
$\vec{E} = 0$ and
$\vec{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}$

We want our current to depend on the retarded time, because our $\vec{A}$ depends on what the current was t_r seconds ago

10.3 Point Charges

→ Liénard Wiechert Potentials

• calculate the retarded potentials of a point charge w/ a specific trajectory

↳ $\vec{w}(t)$ ≡ position of q at time t

can pull out since its <u>one</u> charge

↳ $V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\eta}\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{1}{\eta}\underline{\int \rho(\vec{r},t_r)\,d\tau'}$

cant evaluate this integral to q bec its <u>time dependent</u>

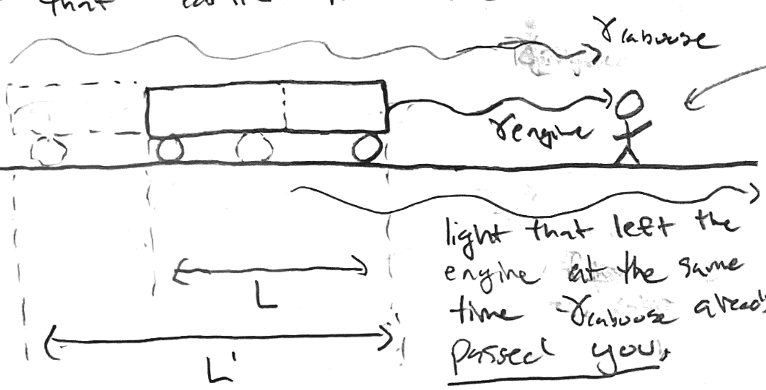
↳ $\int \rho(\vec{r}\,',t_r)\,d\tau' = \frac{q}{1-\hat{\eta}\cdot\frac{\vec{v}}{c}}$

<u>Proof</u> (geometrical effect)

• a train coming towards you looks a little longer

↳ <u>light from the caboose</u> <u>left earlier</u> than the light you recieve simultaneously from <u>the engine</u>

↳ at that earlier time the train was farther away



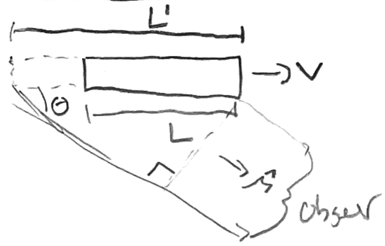
as the human you recieve light from the whole train, γ_{engine} & $\gamma_{caboose}$ arrive at <u>the same time</u>. But! Since light moves at c, $\gamma_{caboose}$ came from earlier

$t_{caboose} = \frac{L'}{c}$ is the same time it takes the train to move a distance $L'-L$

↳ time it takes for the caboose light to reach the engine

$\frac{L'}{c} = \frac{L'-L}{v} \rightarrow L' = \frac{L}{1-v/c}$, train going away looks shorter by $L' = \frac{L}{1+v/c}$

If from an angle:

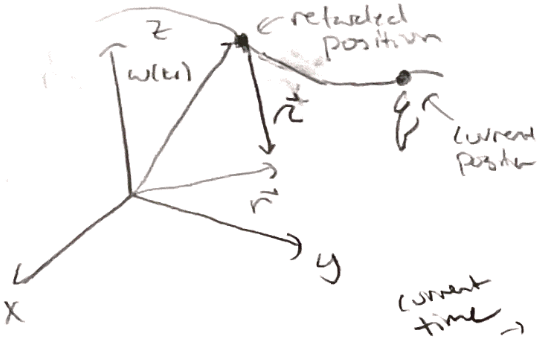


* extra distance light from the caboose must cover is $L'\cos\theta$

↳ train still moves a distance $L'-L$

$\frac{L'\cos\theta}{c} = \frac{L'-L}{v} \rightarrow L' = \frac{L}{1-\frac{v\cos\theta}{c}}$

$\hookrightarrow L' = \dfrac{L}{1-\frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}}$ → Volume only distorted in one direction → $\boxed{\tau' = \dfrac{\tau}{1-\hat{\mathscr{r}}\cdot\frac{\vec{v}}{c}}}$



→ $w(t_r)$ is the particles trajectory at the time when the news left the source

→ $\vec{r}$ is the distance from the origin to field point P

$|\vec{r} - w(t_r)|$ = distance news must travel

current time → $t - t_r$ = time it takes for light to travel

$\hookrightarrow \underbrace{|\vec{r} - \vec{w}(t_r)|}_{\vec{\mathscr{r}}} = c(t - t_r)$

$$\hookrightarrow V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}}\int \rho(r', t_r)\,d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}}\frac{q_s}{1-\frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}} = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{qc}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}}$$

$I = \rho v$

$$\hookrightarrow \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\rho(\vec{r}', t_r)\vec{v}(t_r)}{\mathscr{r}}\,d\tau' = \frac{\mu_0}{4\pi}\frac{\vec{v}}{\mathscr{r}}\int \rho(\vec{r}, t_r)\,d\tau'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}(t_r)}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}} = \boxed{\frac{\vec{v}(t_r)}{c^2}V(\vec{r},t)}$$

Lienard-Wiechert Potentials

Example Potential of a point charge w/ constant velocity

- Say a particle passes the origin at $t=0$

$\vec{w}(t) = \vec{v}t$ is its trajectory

$\hookrightarrow |\vec{r} - \vec{v}t_r| = c(t - t_r) \rightarrow r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$

$$\hookrightarrow t_r = \frac{(c^2t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

if $v=0$, $t_r = t \pm \frac{r}{c}$

retarded time is $t - r/c$

Also need: $\mathscr{r} = c(t - t_r)$ and $\hat{\mathscr{r}} = \dfrac{\vec{r} - vt_r}{c(t-t_r)}$ ← vector ← magnitude

$$\mathscr{r}\left(1 - \frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}\right) = \ldots \text{mean} = \frac{1}{c}\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2 - c^2t^2)}$$

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}}}$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{q_s c\vec{v}}{\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}}}$$

[Calculating $\vec{E}$ and $\vec{B}$ of a point charge
Using Lienard - Wiechart potentials]

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(\eta c - \vec{\eta}\cdot\vec{v})} \quad , \quad \vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

[Hard because η depends on t_r] $\rightarrow$ $\vec{\eta} = \vec{r} - \vec{\omega}(t_r)$ and $|\vec{r} - \vec{\omega}(t_r)| = c(t - t_r)$

$\vec{v} = \dot{\vec{\omega}}(t_r)$

Gradient of V: $\nabla V = \frac{qc}{4\pi\varepsilon_0}\frac{-1}{(\eta c - \vec{\eta}\cdot\vec{v})^2}\vec{\nabla}(\eta c - \vec{\eta}\cdot\vec{v})$

$\hookrightarrow \nabla\eta = \nabla(c(t - t_r)) = -c\nabla t_r$

$$\boxed{\nabla(\vec{\eta}\cdot\vec{v}) = \underset{①}{(\vec{\eta}\cdot\vec{\nabla})\vec{v}} + \underset{②}{(\vec{v}\cdot\vec{\nabla})\vec{\eta}} + \underset{③}{\vec{\eta}\times(\vec{\nabla}\times\vec{v})} + \underset{④}{\vec{v}\times(\vec{\nabla}\times\vec{\eta})}}$$

① $(\vec{\eta}\cdot\vec{\nabla})\vec{v} = \left(\eta_x\frac{\partial}{\partial x} + \eta_y\frac{\partial}{\partial y} + \eta_z\frac{\partial}{\partial z}\right)\vec{v}(t_r) = \eta_x\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial x} + \frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial y} + \eta_z\frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial z}$

$= \vec{a}(\vec{\eta}\cdot\vec{\nabla}(t_r))$, $\vec{a} = \dot{\vec{v}} = \frac{d\vec{v}}{dt_r}$

② $(\vec{v}\cdot\vec{\nabla})\vec{\eta} = (\vec{v}\cdot\vec{\nabla})\vec{r} - (\vec{v}\cdot\vec{\nabla})\vec{\omega}(t_r)$

$(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z})\vec{\omega}(t_r)$
$= v_x\frac{d\omega}{dt_r}\frac{\partial t_r}{\partial x} + v_y\frac{d\omega}{dt_r}\frac{\partial t_r}{\partial y} + v_z\frac{d\omega}{dt_r}\frac{\partial t_r}{\partial z}$
$= \dot{\omega}(\vec{v}\cdot\vec{\nabla}(t_r)) = \vec{v}(\vec{v}\cdot\vec{\nabla}(t_r))$

$(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z})(x\hat{x} + y\hat{y} + z\hat{z}) = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} = \vec{v}$

③ $\vec{\nabla}\times\vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} = \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{x} +$

$\left(\underbrace{\frac{dv_x}{dt_r}}_{\text{comp of } \vec{a}}\underbrace{\frac{\partial t_r}{\partial z}}_{\text{comp of } \nabla t_r} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{y} + \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{z} = -\vec{a}\times\vec{\nabla}(t_r)$

by same argument

$\vec{\nabla}\times\vec{\eta} = \vec{\nabla}\times\vec{r} - \vec{\nabla}\times\vec{\omega} \Rightarrow \vec{\nabla}\times\vec{\omega} = -\vec{v}\times\nabla(t_r)$

$\vec{\nabla}(\vec{\eta}\cdot\vec{v}) = \vec{a}(\vec{\eta}\cdot\nabla t_r) + \vec{v} - \vec{v}(\vec{v}\cdot\vec{\nabla}(t_r)) - \vec{\eta}\times(\vec{a}\times\vec{\nabla}(t_r)) + \vec{v}\times(\vec{v}\times\vec{\nabla}t_r)$

$= \vec{v} + (\vec{\eta}\cdot\vec{a} - v^2)\nabla t_r$ use BAC - CAB for latter

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})^2} \vec{\nabla}(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})$$

w/ $\nabla \mathscr{R} = -c\nabla t_r$ and $\vec{\nabla}(\vec{\mathscr{R}}\cdot\vec{v}) = \vec{v} + (\vec{\mathscr{R}}\cdot\vec{a} - v^2)\vec{\nabla} t_r$

↳ $$\boxed{\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})^2}\left[\vec{v} + (c^2 - v^2 + \vec{\mathscr{R}}\cdot\vec{a})\vec{\nabla} t_r\right]}$$

Find $\nabla(t_r)$ → $-c\vec{\nabla}(t_r) = \nabla\mathscr{R} = \nabla\sqrt{\vec{\mathscr{R}}\cdot\vec{\mathscr{R}}} = \frac{1}{2\sqrt{\vec{\mathscr{R}}\cdot\vec{\mathscr{R}}}}\nabla(\vec{\mathscr{R}}\cdot\vec{\mathscr{R}})$

use product rule here

$$\boxed{= \frac{1}{\mathscr{R}}\left[(\vec{\mathscr{R}}\cdot\nabla)\vec{\mathscr{R}} + \vec{\mathscr{R}}\times(\vec{\nabla}\times\vec{\mathscr{R}})\right]}$$ ← cancel chain terms

$(\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{\mathscr{R}} = \vec{\mathscr{R}} - \vec{v}(\vec{\mathscr{R}}\cdot\nabla t_r)$ (same as $(\vec{v}\cdot\vec{\nabla})\vec{\mathscr{R}} = (\vec{v}\cdot\vec{\nabla})\vec{r} - (\vec{v}\cdot\vec{\nabla})\vec{w}$)

↳ $(\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{r} - (\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{w}$

$\vec{\mathscr{R}} - \vec{v}(\vec{\mathscr{R}}\cdot\nabla t_r)$ look at $(\vec{v}\cdot\vec{\nabla})\vec{w} = \vec{v}(\vec{v}\cdot\nabla t_r)$

Now: $\nabla\times\vec{\mathscr{R}} = \overset{0}{\cancel{\vec{\nabla}\times\vec{r}}} - \vec{\nabla}\times\vec{w}$ ⇐ look at $\vec{\nabla}\times\vec{v} = -\vec{a}\times\vec{\nabla} t_r$

$= \vec{v}\times\vec{\nabla} t_r$

use BAC-CAB

↳ $-c\vec{\nabla} t_r = \frac{1}{\mathscr{R}}\left[\vec{\mathscr{R}} - \vec{v}(\vec{\mathscr{R}}\cdot\vec{\nabla} t_r) + \vec{\mathscr{R}}\times(\vec{v}\times\vec{\nabla} t_r)\right] = \frac{1}{\mathscr{R}}\left[\vec{\mathscr{R}} - (\vec{\mathscr{R}}\cdot\vec{v})\vec{\nabla} t_r\right]$

↳ $$\nabla t_r = \frac{-\vec{\mathscr{R}}}{\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v}}$$

↳ $$\nabla V = \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})^2}\left[\vec{v} + (c^2 - v^2 + \vec{\mathscr{R}}\cdot\vec{a})\frac{-\vec{\mathscr{R}}}{\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v}}\right]$$

$$= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})^3}\left[(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})\vec{v} - (c^2 - v^2 + \vec{\mathscr{R}}\cdot\vec{a})\vec{\mathscr{R}}\right]$$

$$\frac{\partial\vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})^3}\left[(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})\left(-\vec{v} + \frac{\mathscr{R}\vec{a}}{c}\right) + \frac{\mathscr{R}}{c}(c^2 - v^2 - \vec{\mathscr{R}}\cdot\vec{a})\vec{v}\right]$$

let $\vec{u} = c\hat{\mathscr{R}} - \vec{v}$

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{R}}{(\vec{\mathscr{R}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{R}}\times(\vec{u}\times\vec{a})\right]}$$

Calculated   Calculated

Now finding $\vec{B}$

$$\vec{\nabla}\times\vec{A} = \frac{1}{c^2}\vec{\nabla}\times(V\vec{v}) = \frac{1}{c^2}\left[V(\vec{\nabla}\times\vec{v}) - \vec{v}\times(\vec{\nabla}V)\right]$$

$$\vec{\nabla}\times\vec{A} = -\frac{1}{c}\frac{q}{4\pi\varepsilon_0}\frac{1}{(\vec{u}\cdot\vec{\imath})^3}\vec{\imath}\times\underbrace{\left[(c^2-v^2)\vec{v} + (\vec{\imath}\cdot\vec{a})\vec{v} + (\vec{\imath}\cdot\vec{u})\vec{a}\right]}_{\text{similar term in brackets of } \vec{E}}$$

↳ rewrite bracket term in $\vec{E}$
as BAC - CAB..
↳ it'll be the same term as $\vec{E}$ but w/ $\vec{u}$'s

$\hat{\imath} = v + u/c$, $\hat{\imath}\times[\ \ ]$ carry in → ends up making $\vec{v}$'s into $\vec{u}$'s

$$\boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\imath}\times\vec{E}(\vec{r},t)}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{\imath}}{(\vec{\imath}\cdot\vec{u})^3}\left[\underbrace{(c^2-v^2)\vec{u}}_{\substack{\text{velocity field} \\ \text{"generalized Coulomb"}}} + \underbrace{\vec{\imath}\times(\vec{u}\times\vec{a})}_{\substack{\text{acceleration field} \\ \text{or} \\ \text{radiation field}}}\right] , \quad \vec{u} = c\hat{\imath} - \vec{v}$$

(Ex) 460 Calculate E and B of a moving point charge w/ constant velocity $(\vec a = 0)$

$$\vec E(\vec r, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{R}}{(\vec{\mathscr{R}}\cdot\vec u)^3}\left[(c^2 - v^2)\vec u + \vec{\mathscr{R}}\times\vec u\times\vec a^{\,\to 0}\right]$$

$$\hookrightarrow \vec E(\vec r, t) = \frac{q}{4\pi\epsilon_0}\frac{(c^2-v^2)\mathscr{R}}{(\vec{\mathscr{R}}\cdot\vec u)^3}\vec u, \quad \vec u = c\hat{\mathscr{R}} - \vec v$$

$$\mathscr{R}\vec u = c\vec{\mathscr{R}} - \mathscr{R}\vec v, \quad \vec{\mathscr{R}} = \vec r - \vec w(t_r) \text{ and } \mathscr{R} = c(t - t_r)$$

$$\mathscr{R}\vec u = c(\vec r - \vec v t_r) - c(t - t_r)\vec v = c(\vec r - \vec v t)$$

in problem 7.16 $\hookrightarrow$ Found that $\vec{\mathscr{R}}\cdot\vec u = \mathscr{R}c - \vec{\mathscr{R}}\cdot\vec v = \sqrt{(c^2 t - \vec r\cdot\vec v)^2 + (c^2 - v^2)(r^2 - c^2t^2)}$

$$= Rc\sqrt{1 - v^2\sin^2\theta/c^2}, \quad \vec R \equiv \underbrace{\vec r}_{\text{field point now}} - \underbrace{\vec v t}_{\text{where charge is now}}$$ (vector from present location to field point)

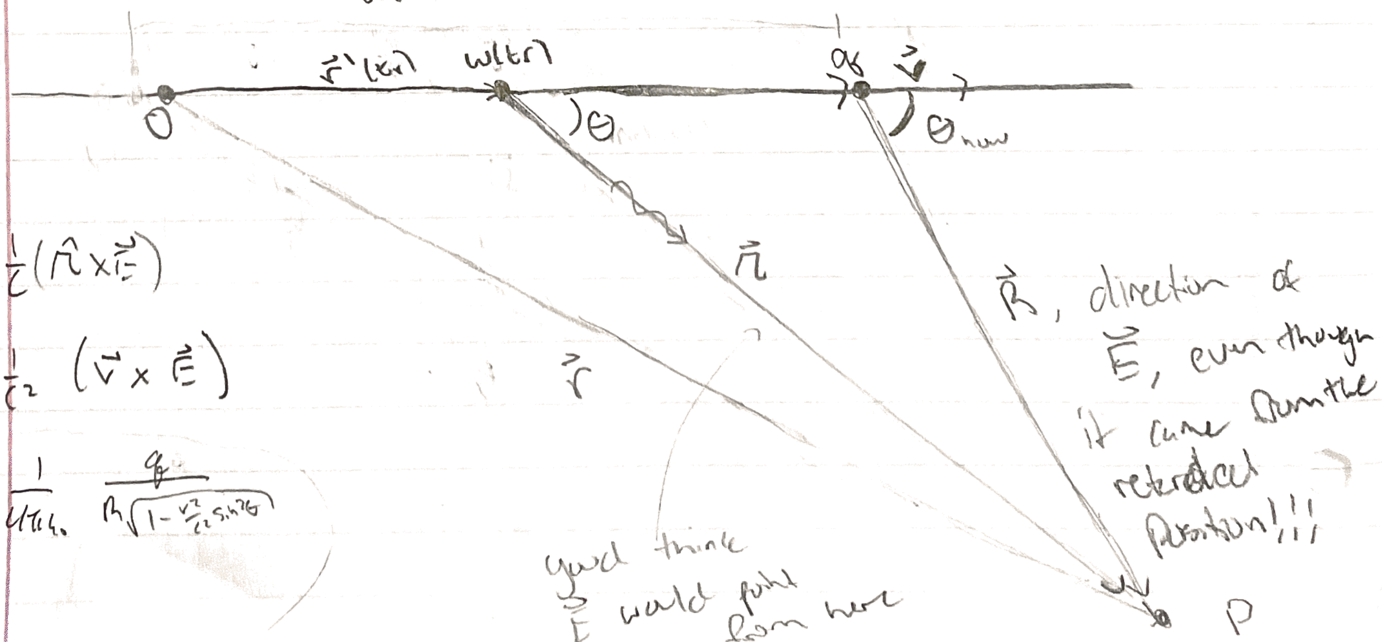
$$\vec E(\vec r, t) = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{(1 - v^2\sin^2\theta/c^2)^{3/2}}\frac{\hat R}{R^2}$$

reduced field $v < c$

$\theta = 0$: $\vec E = k(1 - v^2/c^2)\frac{\hat R}{R^2}$

$\theta = \frac{\pi}{2}$: $E = k\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\frac{\hat R}{R^2}$ enhanced field

$$w(t) = \vec v(t)$$



$$\vec B = \frac{1}{c}(\hat{\mathscr{R}}\times\vec E)$$

$$= \frac{1}{c^2}(\vec v\times\vec E)$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}}$$