THE HOLY GRAIL OF Laplace Functions

Laplace Transform Overview/Summary

Definition: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$ ← changes from t-world to s-world
← when computing you have to take limit as goes to ∞

From this you get all the relations:
$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$, $\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

*Can't do Laplace Transforms of e^{t^2} cause it grows faster than e^{at},

$\boxed{\text{*needs to at least be piecewise continuous}}$

Laplace Transform of Derivitives:

$\mathcal{L}\{f'\} = \int_0^\infty f'(t)e^{-st}\,dt \rightarrow$ let: $u = e^{-st}$ $\quad w = f$
$du = -se^{-st}dt \quad dw = f'\,dt$

$\int_0^\infty f'(t)e^{-st}dt = \underbrace{f(t)e^{-st}}_{w\ u}\Big|_0^\infty - \int_0^\infty -s\,f(t)e^{-st}dt = 0 - e^0 \cdot f(0) + s\boxed{\int_0^\infty f(t)e^{-st}dt} \rightarrow \mathcal{L}\{f\}$

so:
$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$ * for higher derivitives $\rightsquigarrow s\mathcal{L}\{f'\} - f'(0)$ just take derivitive and plug in ex now plug in $\mathcal{L}\{f'\}$ eqn

Big Example, I will skip steps for space.
$y'' - 2y' + 2y = 5\cos(2t) + 10\sin(2t)$ with $y(0) = -1$ and $y'(0) = -2$ and Initial conditions

① $\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 5\mathcal{L}\{\cos(2t)\} + 10\mathcal{L}\{\sin(2t)\}$ * plug in $\mathcal{L}\{y''\} + \mathcal{L}\{y'\}$ formulas

② $\mathcal{L}\{y\}(s^2 \overset{(s-1)^2}{-} 2s + 1) + s = \frac{5s}{s^2+4} + \frac{10 \cdot 2}{s^2+4} \Rightarrow \mathcal{L}\{y\} = \frac{-s}{(s-1)^2} + \frac{5s+20}{(s-1)^2(s^2+4)}$ * now use Partial Frac Dec

③ $\frac{-s^3+s+20}{(s^2+4)(s-1)^2} = \frac{As+B}{s^2+4} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$, $A = 7$, $D = 4$, $C = -3$, $B = -8$

④ $\mathcal{L}\{y\} = \frac{7s-8}{s^2+4} - \frac{3}{s-1} + \frac{4}{(s-1)^2} = \frac{7s}{s^2+4} - \frac{4 \cdot 2}{s^2+2^2} - \frac{3}{s-1} + \frac{4!}{(s-1)^2}$

⑤ $y(t) = 7\cos(2t) - 4\sin(2t) - 3e^t + 4te^t$

First Translation Thereom! remember $\mathcal{L}\{f(t)\} = F(s)$ $\qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

t-space

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

ex) $\mathcal{L}\{t^3e^{-2t}\} \Rightarrow a = -2$, $F(s+2)$ and $f(t) = t^3 \rightarrow F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t^3\} = \frac{6}{s^4}$ so $F(s+2) = \frac{6}{(s+2)^4}$

s-space

ex) $\mathcal{L}\{y(t)\} = \frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2}$ (so $a = -2$) so! $F(s+2) = \frac{1}{(s+2)^2}$ but we want $F(s)$ so we can invert

so subsitute $\bar{s} = s+2 \rightarrow F(\bar{s}) = \frac{1}{\bar{s}^2}$ inverting we get $\mathcal{L}^{-1}\{F(\bar{s})\} = t$ where $\mathcal{L}^{-1}\{F(\bar{s})\} = f(t) = t$

putting together $\boxed{e^{-2t} \cdot t = \mathcal{L}^{-1}\{F(s+2)\} = y(t)}$

Lets say you get $\mathcal{L}\{y\} = \frac{s}{(s+1)^2+4}$, (so $a = -1$) lets substitute, $\bar{s} = s+1$, $\mathcal{L}\{y(t)\} = \frac{\bar{s}-1}{\bar{s}^2+4} = F(\bar{s})$

$F = \frac{\bar{s}}{\bar{s}^2+4} - \frac{1}{\bar{s}^2+4} = \frac{\bar{s}}{\bar{s}^2+4} - \frac{1}{2} \cdot \frac{2}{\bar{s}^2+4} \Rightarrow f(t) = \cos(2t) - \frac{1}{2}\sin(2t)$ now use FTT

grabbing $f(t)$ part by using substitution

$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \rightarrow y(t) = e^{-t}[\cos(2t) - \frac{1}{2}\sin(2t)]$
$e^{at}f(t) = \mathcal{L}^{-1}\{F(s+1)\} = y(t)$

Functions with switches

Heaviside Function: (unit step function)

$$H(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases} \rightarrow f(t)\cdot H(t) = \begin{cases} 0\cdot \cancel{f(t)}^{\,0}, & t<0 \\ 1\cdot f(t), & t\geq 0 \end{cases}$$

→ if you have $f(t) - [f(t)H(t)] = \begin{cases} f(t), & t<0 \\ 0, & t\geq 0 \end{cases}$

→ if you have $f_1(t) + [f_2(t) - f_1(t)]H(t) = \begin{cases} f_1(t), & t<0 \\ f_2(t), & t\geq 0 \end{cases}$

$$H(t-a) = \begin{cases} 0, & t-a<0 \rightarrow t<a \\ 1, & t-a\geq 0 \rightarrow t\geq a \end{cases} \qquad \boxed{\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}}$$

ex $y''+4y = \begin{cases} 12\cdot 0, & t<\pi \\ 12\cdot 1, & t\geq \pi \end{cases} = 12H(t-\pi)$ and $y(0)=1$, $y'(0)=0$

$s^2\mathcal{L}\{y\} - s + 4\mathcal{L}\{y\} = \mathcal{L}\{12H(t-\pi)\} = 12\frac{e^{-\pi s}}{s}$ ← dont know how to do yet!

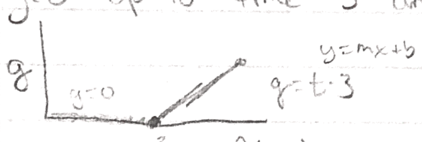
$\mathcal{L}\{y\} = \frac{s}{s^2+4} + \frac{12}{s(s^2+4)}e^{-\pi s} \rightarrow y(t) = \cos(2t) + \mathcal{L}^{-1}\left\{\frac{12}{s(s^2+4)}e^{-\pi s}\right\}$

Second Translation Therem!

$$\boxed{\begin{aligned} &\text{form ① } \mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s) \\ &\text{form ② } \mathcal{L}\{f(t)H(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\} \end{aligned}}$$

ex 1)

- $g=0$ up to time 3 and has a slope of 1 after time 3
-



$g = \begin{cases} 0, & t<3 \\ t-3 & t\geq 3 \end{cases} = \mathcal{L}\{(t-3)H(t-3)\} = e^{-3s}F(s)$ $\boxed{f(t-3)}$!!! $a=3$

$\mathcal{L}\{f(t)\}$ $f(t-3) = t-3$ using $\bar{t} = t-3 \Rightarrow f(\bar{t}) = \bar{t} \rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = F(s)$

so $F(s) = \mathcal{L}\{\bar{t}\} \rightarrow F(s) = \frac{1}{s^2}$ so $\boxed{\mathcal{L}\{g\} = \mathcal{L}\{f(t-3)H(t-3)\} = e^{-3s}\cdot\frac{1}{s^2}}$

How to get Form 2 from form 1

let: $\bar{f}(t) = f(t-a)$ then $\bar{f}(t+a) = f(t)$

Substituting in this ↗ into form 1: $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$

$$\boxed{\mathcal{L}\{\bar{f}(t)H(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}}$$

using form 2 on ex 1

① $\mathcal{L}\{(t-3)H(t-3)\}$, $a=3$, $\boxed{f(t)=t-3}$

$\mathcal{L}\{f(t)H(t-3)\} = e^{-3s}\mathcal{L}\{f(t+3)\}$ → So what does this equal?

② $\mathcal{L}\{f(t+3)\} = \mathcal{L}\{(t+3)-3\} = \mathcal{L}\{t\} = \frac{1}{s^2}$

$\boxed{f(t+3) = (t+3)-3}$

so $\mathcal{L}\{(t-3)H(t-3)\} = \mathcal{L}\{f(t)H(t-3)\} = e^{-3s}\mathcal{L}\{f(t+3)\}$

$= \boxed{e^{-3s}\cdot\frac{1}{s^2}}$

HOLY Grail of Laplace Transforms

Part 2! :)

Inverting Transforms:

$$y'' + 4y = \begin{cases} 0, & t < \pi \\ 12, & t > \pi \end{cases} = 12H(t-\pi) \quad \text{with } y(0)=1, \; y'(0)=0$$

we got: $\mathcal{L}\{y\} = \frac{s}{s^2+4} + \frac{12}{s(s^2+4)} e^{-\pi s} \rightarrow y(t) = \cos(2t) + \mathcal{L}^{-1}\left\{\frac{12}{s(s^2+4)} e^{-\pi s}\right\}$

need to do: $\mathcal{L}^{-1}\left\{\frac{12}{s(s^2+4)} e^{-\pi s}\right\}$ — this is some $\mathcal{L}\{?\} = e^{-\pi s} \cdot \frac{12}{s(s^2+4)}$

that part

our forms: $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as} F(s)$ ← always use to go to t world!

or $\mathcal{L}\{f(t)H(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

a cool check — but whats

$a = \pi$

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t-\pi)H(t-\pi)\}\} = f(t-\pi)H(t-\pi) = e^{-\pi s}F(s)$$

$\mathcal{L}^{-1}\left\{\frac{12}{s(s^2+4)} e^{-\pi s}\right\}$ is in the form $F(s) \cdot e^{-as}$ so $F(s) = \frac{12}{s(s^2+4)} = \mathcal{L}\{f(t)\}$

$F(s): \frac{12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+2C}{s^2+4} \Rightarrow f(t) = A + B\cos(2t) + C\sin(2t)$

$A = 3, \; C = 0, \; B = -3$ so $f(t) = 3 - 3\cos(2t) + 0$

$f(t-\pi) = 3 - 3\cos(2t - 2\pi) = 3 - 3\cos(2t)$

Thus: $y(t) = \cos(2t) + [3 - 3\cos(2t)]H(t-\pi)$

or

$$y(t) = \begin{cases} \cos 2t & t < \pi \\ 3 - 2\cos 2t & t \geq \pi \end{cases}$$

The Dirac Delta Function

Suppose we want to change mv by $\mathbf{1}$ unit by applying a force that acts on a short interval $0 \le t \le \Delta t$ and we want it to change by <u>1 unit</u> so

$$F = ma \rightarrow m\frac{dv}{dt} = F \rightarrow F = \frac{d(mv)}{dt} = \frac{dp}{dt}$$

$$F = \frac{d(mv)}{dt} \rightarrow \int_0^{\Delta t} F(t)\,dt = \int_0^{\Delta t} d(mv) = \Delta mv = 1$$

- The point is to model short impulses so the force is 0 except when its hit, t_0
- So lets say $\delta(t)$ is 0 for all time $t > 0$
  - * $\delta(t)$ isnt a function if it was $\int_0^{\Delta b} \underbrace{\delta(t)}_{0,\ \text{always } 0} dt = 0$ but we want that equal to 1

So it has two properties: $\delta(t) = 0,\ t \neq 0$, $\int_a^b \delta(t)\,dt = 1$ as long as 0 is between a and b

generalizing $\delta(t - t_0) = 0 \quad t \neq t_0$, $\int_a^b \delta(t - t_0)\,dt = 1 \quad a \le t_0 \le b$ * last time $t_0 = 0$

* This means $F = \delta(t - t_0) = \Delta mv = 1$ at $t = t_0$ *

$\delta(t - t_0) = 0 \quad t \neq 0$, $\int_a^b \delta(t - t_0)\,dt = 1 \quad a \le t_0 \le b$

Lets attach a function to δ so:

$$\int_0^\infty \delta(t - t_0) f(t)\,dt \rightarrow \int_0^\infty \delta(t-t_0) \underbrace{f(t_0)}_{\text{legal}} dt$$

$f(t_0)$ is wrong everywhere $\delta = 0$ and right where it needs to be so it works

$f(t_0)\int_0^\infty \delta(t - t_0)\,dt = 1$ so

$f(t_0) = 1$, if you multiply any function by $\delta(t - t_0)$ you pull out $f(t_0)$

$$\boxed{\int_0^\infty \delta(t - t_0) f(t)\,dt = f(t_0)}$$

Using what we just learned

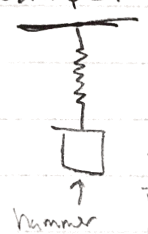
$$\mathcal{L}\{\delta(t - t_0)\} = \int_0^\infty e^{-st}\delta(t - t_0)\,dt = e^{-st}\Big|_{t = t_0} = e^{-st_0}$$

so $s\mathcal{L}\{H(t - t_0)\} = \mathcal{L}\{\delta(t - t_0)\}$

$\boxed{\text{So}}$

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$$

example:



$my'' + ky = -\delta(t) \quad y(0) = 0,\ y'(0) = 0$

$y'' + w^2 y = -\frac{1}{m}\delta(t)$ Let $Y = \mathcal{L}\{y\}$, $\mathcal{L}\{y''\} = s^2 Y$

$\mathcal{L}\{\delta(t)\} = e^{-s \cdot 0} = 1$

$s^2 Y + w^2 Y = -\frac{1}{m} \rightarrow Y = -\frac{1}{m} \cdot \frac{1}{s^2 + w^2} = \frac{-1}{mw} \cdot \frac{w}{s^2 + w^2}$

$\boxed{y = -\frac{1}{mw}\sin(wt)}$

notice:

$y' = v = -\frac{1}{mw} \cdot w\cos(wt) = -\frac{1}{m}\cos(wt)$

$mv = -\cos(wt)$ and $mv\big|_{t=0} = -1$ unit upwards

immediatly after hammer blow

example 2: take unit mass so $m = 1$, $k = 4$, no damping and mass is pulled down 1 unit. Hit the mass at $t = 1$ up hard enough to change mv by 4 units

$$\underbrace{y'' + 4y}_{(\frac{k}{m} = 4)} = \underbrace{-4\delta(t - 1)}_{\Delta mv \text{ by } 4 \text{ upwards}} \qquad \underbrace{y(0) = -1}_{\text{pulled down}},\ \underbrace{y'(0) = 0}_{\text{released}}$$

1) take laplace transform and solve for $Y = \frac{s}{s^2+4} - e^{-s}\left(\frac{4}{s^2+4}\right)$

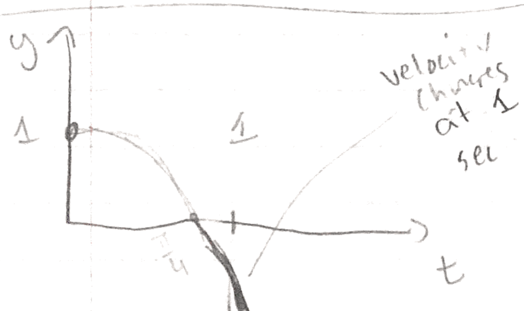
$y = \cos(2t) - \mathcal{L}^{-1}\left\{e^{-s}\frac{4}{s^2+4}\right\} \rightarrow e^{-s} \cdot F(s)$ 2nd TT

$F(s) = \frac{4}{s^2+4} \rightarrow f(t) = 2\sin(2t)$

$\mathcal{L}^{-1}\{e^{-s}F(s)\} = f(t-1)H(t-1) = 2\sin(2t-2)H(t-1)$

so $y = \cos(2t) - 2\sin(2t-2)H(t-1)$

switch



this side gets turned on

7.1 Laplace Transform Math

2, 13, 21, 27, 29, 31, 43

1. $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$

$$f(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$

$$\mathcal{L}\{-1\} + \mathcal{L}\{1\} = \int_0^1 e^{-st}(-1)\,dt + \int_1^\infty e^{-st}(1)\,dt$$

$u = -st$

$du = -s\,dt$

$dt = \frac{1}{-s}du$

$$= \frac{1}{s}e^{-st}\Big|_0^1 + \left(-\frac{1}{s}e^{-st}\right)\Big|_1^\infty$$

$$= \frac{1}{s}e^{-s} - \frac{1}{s} + 0 + \frac{1}{s}e^{-s} = \frac{1}{s}\left(2e^{-s} - 1\right)$$

$$\mathcal{L}(te^{4t}) = \int_0^\infty te^{4t} \cdot e^{-st}\, dt = \int_0^\infty te^{(4-s)t}\, dt$$

$wu - \int w\, du$

$u = t \qquad w = \frac{1}{4-s}e^{(4-s)t}$

$du = dt \qquad dw = e^{(4-s)t}\, dt$

$$\frac{t}{4-s}e^{(4-s)t} - \int_0^\infty \frac{1}{4-s}e^{(4-s)t}\, dt$$

$\infty \cdot 0 = 0$

$$\frac{t}{4-s}e^{(4-s)t}\Big|_0^\infty - \left(\frac{1}{(4-s)^2}e^{(4-s)t}\Big|_0^\infty\right) = \frac{t}{4-s}e^{(4-s)t} - \frac{1}{(4-s)^2}\left[\lim_{t\to\infty} e^{(4-s)t} - 1\right]$$

$$\frac{t}{4-s}e^{-(s-4)t}\Big|_0^\infty + \frac{1}{(4-s)^2}\left[+1\right] \qquad \underbrace{\frac{t}{4-s}e^{-(s-4)t}\Big|_0^\infty}_{} + \frac{1}{(4-s)^2}$$

$$\lim_{t\to\infty} \frac{t}{e^{(s-4)t}} \stackrel{L}{=} \frac{1}{(s-4)e^{(s-4)t}} = 0$$

$$0 - 0 + \frac{1}{(4-s)^2}$$

so $\mathcal{L}(te^{4t}) = \frac{1}{(4-s)^2}$

$$\int_0^{\infty} 5\sin(3t)\, e^{-st}\, dt = 0 + \frac{15}{s}\int_0^{\infty} \cos(3t)\, e^{-st}\, dt$$

$u = 5\sin(3t)$ $\quad\quad$ $w = -\frac{1}{s} e^{-st}$

$du = 15\cos(3t)$ $\quad\quad$ $dw = e^{-st}\, dt$

$$-\frac{5\sin(3t)}{s} e^{-st}\Big|_0^{\infty \to 0} + \int + \frac{15}{s}\cos(3t)\, e^{-st}\, dt$$

$$-\frac{5\sin(\cancel{3\cdot 0})}{s}\cdot 1 \to 0$$

$u = \frac{15}{s}\cos(3t)$ $\quad\quad$ $w = -\frac{1}{s} e^{-st}$

$du = -\frac{45}{s}\sin(3t)$ $\quad\quad$ $dw = e^{-st}\, dt$

$$-\frac{15}{s^2}\cos(3t)\, e^{-st}\Big|_0^{\infty} - \int + \frac{45}{s^2}\sin(3t)\, e^{-st}\, dt$$

$$-\frac{15}{s^2}\cdot 1\cdot 1 - \frac{9}{s^2}\underbrace{\int 5\sin(3t)\, e^{-st}\, dt}_{\mathcal{L}(f)} = \underbrace{\int 5\sin(3t)\, e^{-st}\, dt}_{\mathcal{L}(f)}$$

$$-\frac{15}{s^2} = \mathcal{L}(f)\left(\frac{9}{s^2} + 1\right)$$

$$\frac{9+s^2}{s^2}$$

$$\mathcal{L}(f) = -\frac{15}{s^2+9}$$