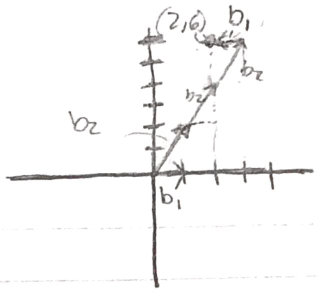
if you cant know these by inspection you can do linear combin, see example below.



Coordinate Spaces

span & lin ind.

B is a basis, $B = \{b_1, b_2\}$

in normal cartesian a point $(2,6)$ isnt the same for B

lets say; $b_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $b_2 = \begin{bmatrix}1\\2\end{bmatrix}$, you need 3 b_2's and -1 b_1 to get to the same point. The coord for B is $(-1, 3)$

in vectors,

$B = \{b_1, b_2\} \rightarrow b_1 = \begin{bmatrix}1\\0\end{bmatrix}, b_2 = \begin{bmatrix}1\\2\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}2\\6\end{bmatrix} = -1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} \leftarrow [\vec{X}]_B = \begin{bmatrix}-1\\3\end{bmatrix}$

$= \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix}$

thus

$[\vec{X}]_B = \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$

$\boxed{\text{2 ways}}$ let $b_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $b_2 = \begin{bmatrix}1\\2\end{bmatrix}$ find $\vec{x}$ if $[x]_B = \begin{bmatrix}-2\\3\end{bmatrix}$

$\vec{x} = -2b_1 + 3b_2 = \begin{bmatrix}1\\6\end{bmatrix}$

2nd way $b_1 = \begin{bmatrix}3\\1\end{bmatrix}, b_2 = \begin{bmatrix}-1\\1\end{bmatrix}, \vec{x} = \begin{bmatrix}-1\\5\end{bmatrix}$ find coord vectr $[X]_B$ of x relative to B

form of $Ax=b \rightarrow Ac = x$

$[\vec{X}]_B = \begin{bmatrix}c_1\\c_2\end{bmatrix}$ when $c_1b_1 + c_2b_2 = \vec{x} \rightarrow [\vec{b_1}\ \vec{b_2} \mid \vec{x}] \rightarrow \begin{bmatrix}3 & -1 & -1\\1 & 1 & 5\end{bmatrix}$ REF and

$3c_1 = 3$, $c_1 = 1$, $c_2 = 4$ $\quad 1b_1 + 4b_2 = 1x \rightarrow [\vec{X}]_B = \begin{bmatrix}1\\4\end{bmatrix}$

Dimensions

$\rightarrow$ basis is $\{1, x, x^2, \dots x^n\}$ $\quad$ 2 vectors so only $\mathbb{R}^2$ if lin ind $\rightarrow$ dim(s) = 2

ex $\dim(\mathbb{R}^n) = n$, $\dim(P^n) = n+1$, $S = \text{span}\{u, v\}$

dimension of column space

$\dim(\text{Col}(A)) = 3$, 3 vectors, 3 pivot colums

$A = \begin{bmatrix}1 & 1 & 1 & 1\\2 & 2 & 3 & 3\\3 & 0 & 3 & 2\\1 & 2 & 6 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & -2 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$ so $\text{Col}(A) = \left\{\begin{bmatrix}1\\2\\3\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\2\end{bmatrix}, \begin{bmatrix}1\\3\\2\\1\end{bmatrix}\right\}$

↑ ↑ ↑ Pivot colum

dimension of Nul Space $\leftarrow$ found by $Ax=0$ + parametric form, $\dim(\text{Nul}(A)) = \#$ free var.

$[A\,0] \rightarrow \begin{bmatrix}1 & 0 & 3 & 0 & 0\\0 & 1 & -2 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ $\quad x_1 = 3x_3$, $x_2 = 2x_3$, x_3 is free $x_4 = 0$

so $\text{Nul}(A) = \left\{\begin{bmatrix}-3\\2\\1\\0\end{bmatrix}\right\}$ dim = 1

Theres only pivots and non pivots (free var) so $(\#\text{ pivot}) + (\#\text{ free}) =$ total # colums

thus $\dim(\text{col}(A)) + \dim(\text{Nul}(A)) = n$

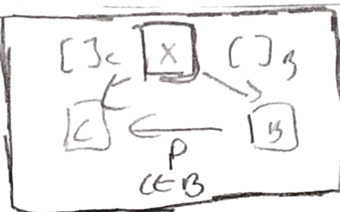
dim (Row(A)) = dim Col(A) (a.s)

if I were to increase rank I would have to delete a row and put number which would increase pivots

Row space like column space is # rows, the basis is found by Row reducing $\rightarrow$ and taking rows that arent all 0's in the REF matrix.

(added dim Row correlates with adding a pivot coordinates of $[x]_B$ will be $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$

then goes if you invert cart to B



Change of Basis

(B's basis vectors)

On the back, there was point $\underset{\text{cartesian}}{\begin{bmatrix} 2 \\ 6 \end{bmatrix}}$, $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

→ We said $\vec{x} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \underbrace{n_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + n_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{linear combos of } b}$ → solve by aug $\left[\begin{array}{cc|c} b_1 & b_2 & \vec{x} \\ 1 & 1 & 2 \\ 0 & 2 & 6 \end{array}\right]$ → $n_1 = -1$, $n_2 = 3$

so $[x]_B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ But whats a matrix that always takes you from B to Cartesian?

we want to do } $[x]_{cart} = [[x]_B]_C = [-1b_1 + 3b_2]_C = -[b_1]_C + 3[b_2]_C = [[b_1]_C \ [b_2]_C] \underset{[x]_B}{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}$

* b_1 in cartesian is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and b_2 in cart is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *

then $[x]_{cart} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 2 \\ 6 \end{bmatrix}_C$ and if you want to go from C→B → take inverse of $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$

$\underset{C\leftarrow B}{P}$

$\begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\underset{B\leftarrow C}{P}$

Generally if you do $[c_1 \ c_2 \,|\, b_1 \ b_2] \to [I \,|\, \underset{C\leftarrow B}{P}]$

always works

$[b_1 \ b_2 \,|\, c_1 \ c_2] \to [I \,|\, \underset{B\leftarrow C}{P}]$

---

ex $B = \{1 - 2t + t^2,\ 3 - 5t + 4t^2,\ 2t + 3t^2\}$ $C = \{1, t, t^2\}$

$\boxed{\begin{array}{ll} b_1 = 1 - 2t + t^2, & c_1 = 1 \\ b_2 = 3 - 5t + 4t^2, & c_2 = t \\ b_3 = 2t + 3t^2, & c_3 = t^2 \end{array}}$

from above we will get to the point of

$[x]_C = [[b_1]_C \ [b_2]_C \ [b_3]_C][x]_B$

But whats b_1 in C coordinates?

$x_1 \underset{c_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + x_2 \underset{c_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + x_3 \underset{c_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = \underset{b_1}{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$ → can do aug but you can read it off $x_1 = 1$, $x_2 = -2$, $x_3 = 1$, $[b_1]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

since its going to be one to one like that

$[b_1]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $[b_2]_C = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$, $[b_3]_C = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ → $[x]_C = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix} [x]_B$

Whats $-1 + 2t$ in B coordinates? could take inverse of ↑ or lin combo!

$c_1 \underset{b_1}{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} + c_2 \underset{b_2}{\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}} + c_3 \underset{b_3}{\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ → do aug and find constants will be $c_1 = 5$, $c_2 = -2$, $c_3 = 1$

$[x]_B = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

Transformations + Change of Basis

T
R
A
N
S
F
O
R
M
A
T
I
O
N
S

$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4) \to A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ ← $T(x) = Ax$

$\hookrightarrow \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \mapsto \begin{matrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{matrix}$

$T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$

↳ find matrix of T relative to $B = \{1, t, t^2\}$

$\begin{matrix} a_0 \\ a_1 \\ a_2 \end{matrix} \xrightarrow[{[T]_B}]{\text{mult by}} \begin{bmatrix} 3a_0 \\ 5a_0 - 2a_1 \\ 4a_1 + a_2 \end{bmatrix} \to [T]_B = \left[[T(b_1)]_B \; [T(b_2)]_B \; [T(b_3)]_B \right] = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

Find B matrix relative to C, $B = \{\underset{b_1}{1 - 2t + t^2}, \underset{b_2}{3 - 5t + 4t^2}, \underset{b_3}{2t + 3t^2}\}$, $C = \{1, t, t^2\}$

$b_1 = 1 - 2t + t^2 \quad c_1 = 1$

$b_2 = 3 - 5t + 4t^2 \quad c_2 = t$

$b_3 = 2t + 3t^2 \quad c_3 = t^2$

$x_1 \overset{c_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + x_2 \overset{c_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + x_3 \overset{c_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = \overset{b_1}{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [b_1]_C$

$[b_1]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ likewise $[b_2]_C = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$, $[b_3]_C = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

$[X]_C = \left[[b_1]_C \; [b_2]_C \; [b_3]_C \right] [X]_B$

$= \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$

* ~~you can't do →~~ $\cancel{\begin{matrix} 1 \\ t \\ t^2 \end{matrix} \mapsto \begin{matrix} 1 - 2t + t^2 \\ 3 - 5t + 4t^2 \\ 2t + 3t^2 \end{matrix}}$

$[c_1 \; c_2 \mid b_1 \; b_2] \to [I \mid \underset{C \leftarrow B}{P}]$, $[b_1 \; b_2 \mid c_1 \; c_2] \to [I \mid \underset{B \leftarrow C}{P}]$

Find B matrix for transformation $x \mapsto Ax$ when $B = \{b_1, b_2\}$

$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $B = P^{-1}AP$, $P = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ eigen vects

equivalent

$\Lambda = S^{-1} A S \Rightarrow A = PDP^{-1} \rightarrow D = P^{-1} A P$

$\|v\| = \sqrt{v \cdot v}$, $\text{Row}\, A^{\perp} = \text{Nul}\, A$, $\text{Col}\, A^{\perp} = \text{Nul}\, A^T$, $\cos\phi = \frac{u \cdot v}{\|u\|\|v\|}$

$\rightarrow$ if orthogonal basis B, $c_1 = \frac{x \cdot b_1}{b_1 \cdot b_1}$, $[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_j \end{bmatrix}$

$\hat{y} = \text{proj}_L y$ = proj of y onto $L = \frac{y \cdot L}{L \cdot L} L$

ortho normal!

- if U has $\perp$ colums $U^T U = I$, $\text{proj}_W y = (U U^T) y$
- projecting onto a line onto a space
  - $\hookrightarrow$ $\text{proj}_W y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \ldots$ $W(u_1, u_2, \ldots)$

Gram schmidt, $\Rightarrow$ $v_1 = x_1$, $v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$, $v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$

QR Factorization $\rightarrow$ $A = QR$, cols of Q are orthnomal basis of A, R is upper triangular

$\hookrightarrow R = Q^T A$

least squares $-$ $A^T A \hat{x} = A^T b \rightarrow \hat{x} = (A^T A)^{-1} A^T b$, $\hat{x} = R^{-1} Q^T b$

Symmetric: $A^T = A \leftarrow$ if diagonalize this normalize P since they'll be orthogonal set. (or make them by Gram schmidt)

$\hookrightarrow A = PDP^{-1}$ since orthonormal matrix $\boxed{A = PDP^T}$

$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \ldots$ spectral decomp

Just look at Notes dude.

$P_B[x]_B = x$

Termonology notes

→ change of coord matrix from $B \to$ standard basis in $\mathbb{R}^n$.

→ $B = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$ → $P_B = [b_1 \;\; b_2] = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}$

→ finding $[x]_B$ given $B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$, $x = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

∘ could do lin combo or use

↳ $P_B[x]_B = \vec{x}$ → $\cancel{P_B^{-1}P_B}[x]_B = P_B^{-1}x$

$c_1[\;] + c_2[\;] + \dots$

↓

$[\{\;\}\{\;\}\dots]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$P_B = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}$ → $P_B^{-1} = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix}$

$[x]_B = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$

$b_1 = 6c_1 - 2c_2$

$b_2 = 9c_1 - 4c_2$

$[b_1]_C = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ $[b_2]_C = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$

$[x]_C = \underbrace{\begin{bmatrix} 6 & 9 \\ 2 & -4 \end{bmatrix}}_{P_{C \leftarrow B}}[x]_B$

"Operator Martix's"

$\boxed{ex}$ P_3, polynomials of degree 3, function $\partial : P \to P$ by $\partial(f(t)) = \frac{df}{dt}$

Basis $B = \{t^3, t^2, t, 1\}$

Find B matrix for ∂, find the matrix that performs the derivitive,

p(t) $\quad\quad\quad\quad\quad \partial$ $\quad$ ← a, b, c, d method just looks nice not general method.

$$\begin{matrix} a\,t^3 \\ + \\ b\,t^2 \\ + \\ c\,t \\ + \\ d\cdot 1 \end{matrix} \to \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\begin{matrix} \leftarrow t^3 \\ \leftarrow t^2 \\ \leftarrow t \\ \leftarrow 1 \end{matrix} \quad\quad \begin{matrix} d\cdot 0\cdot t^3 \\ + \\ a\cdot 3\,t^2 \\ + \\ b\cdot 2\,t \\ + \\ c\cdot 1 \end{matrix} \leftrightarrow \begin{bmatrix} 0d \\ 3a \\ 2b \\ 1c \end{bmatrix} \sim \begin{bmatrix} 0 \\ 3a \\ 2b \\ 1c \end{bmatrix} = a\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{that parametric vector equation becomes}}$ →

$$= \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} \leftarrow t^3 \\ \leftarrow t^2 \\ \leftarrow t \\ \leftarrow 1\text{'s} \end{matrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

↑ B matrix for D

$\cancel{a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}}$

$\cancel{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}}, \cancel{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}}, \cancel{\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}}, \cancel{\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}}$

$\boxed{ex\,2}$

$$B = \left\{ \overset{b_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \overset{b_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{b_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \overset{b_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

~~using the~~

columns of first Basis

$$F(x) = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} X$$

$$F(b_1) = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{F(x)} \begin{bmatrix} 4 & 0 \\ -2 & 0 \end{bmatrix} \to F(b_1) = 4b_1 - 2b_3$$

$$F(b_2) = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{F(x)} \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix} \to F(b_2) = 4b_2 - 2b_4$$

$$F(b_3) = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{F(x)} \begin{bmatrix} -6 & 0 \\ 3 & 0 \end{bmatrix} \to F(b_3) = -6b_1 + 3b_3$$

$$F(b_4) = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{F(x)} \begin{bmatrix} 0 & -6 \\ 0 & 3 \end{bmatrix} \to F(b_4) = -6b_2 + 3b_4$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \xrightarrow{F(x)} \begin{bmatrix} 4b_1 - 2b_3 \\ 4b_2 - 2b_4 \\ -6b_1 + 3b_3 \\ -6b_2 + 3b_4 \end{bmatrix} = b_1\begin{bmatrix} 4 \\ 0 \\ -6 \\ 0 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 4 \\ 0 \\ -6 \end{bmatrix} + b_3\begin{bmatrix} -2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + b_4\begin{bmatrix} 0 \\ -2 \\ 0 \\ 3 \end{bmatrix}$$

thus

$$\begin{bmatrix} 4 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 \\ -6 & 0 & 3 & 0 \\ 0 & -6 & 0 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$\begin{bmatrix}1&1\\0&2\end{bmatrix}$ $\frac{1}{2}\begin{bmatrix}2&-1\\0&1\end{bmatrix}$ $\begin{bmatrix}1&-1/2\\0&1/2\end{bmatrix}\begin{bmatrix}2\\6\end{bmatrix}$ = $\begin{bmatrix}-1\\3\end{bmatrix}$

two bases $B = \{b_1, b_2\}$ $C = \{c_1, c_2\}$

Book ex given — change of basis

$b_1 = 4c_1 + c_2$, $b_2 = -6c_1 + c_2$

Suppose $[X]_B = \begin{bmatrix}3\\1\end{bmatrix} \to X = 3b_1 + b_2$ find $[X]_C$

Book $[X]_C = \big[[X]_B\big]_C = [3b_1 + b_2]_C = 3[b_1]_C + [b_2]_C = \big[[b_1]_C\ [b_2]_C\big]\begin{bmatrix}3\\1\end{bmatrix}$

↑ means finding C coordinates with respect to B coordinates of X

the coordinates of $[X]_B = \begin{bmatrix}3\\1\end{bmatrix}$ that is $3b_1$'s and 1 b_2

old example side by side

$\boxed{x_1 = \vec{i},\ x_2 = \vec{j}}$

given $b_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $b_2 = \begin{bmatrix}1\\2\end{bmatrix}$, $b_1 = 1x_1 + 0x_2$, $b_2 = 1x_1 + 2x_2$

$[X]_B = \begin{bmatrix}-1\\3\end{bmatrix}$ find its cartesian counterpart.

$[X]_C = \big[[X_B]\big]_C = [-b_1 + 3b_2]_C = -[b_1]_C + 3[b_2]_C =$

(we already know it in cartesian) $\big[[b_1]_C\ [b_2]_C\big]\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2\\6\end{bmatrix}$, $[X]_C = \begin{bmatrix}2\\6\end{bmatrix}$

$P_{C\leftarrow B}$

To go from Cartesian to B →

you do inverse of $A = \begin{bmatrix}1&1\\0&2\end{bmatrix} \to A^{-1} = \begin{bmatrix}1&-1/2\\0&1/2\end{bmatrix}$

then do $A^{-1}\begin{bmatrix}2\\6\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$

$P_{B\leftarrow C}$

$b_1 = 4c_1 + c_2$ $b_2 = -6c_1 + c_2$

Book $\big[[b_1]_C\ [b_2]_C\big]\begin{bmatrix}3\\1\end{bmatrix}$,

$P_{C\leftarrow B}$ $\begin{bmatrix}4&-6\\1&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix}$ $[X]_C = \begin{bmatrix}6\\4\end{bmatrix}$

with cheapo method

$P_{C\leftarrow B}$, $[c_1\ c_2 \mid b_1\ b_2] \to [I \mid P_{C\leftarrow B}]$ } if you have all just find these

$P_{B\leftarrow C}$, $[b_1\ b_2 \mid c_1\ c_2] \to [I \mid P_{B\leftarrow C}]$

Diagonalization $A = PDP^{-1}$ in terms of linear transformations

$A = [T(e_1) \cdots T(e_n)]$ $\quad T(x) = A\vec{x}$

(ex) let $T: P_2 \to P_2$ given by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$

Idea: T w/ linear transformation $S: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$S: \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \longmapsto \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix}$ so $A = [S(e_1) \ S(e_2) \ S(e_3)]$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Apply T to $\vec{p} = a_0 + a_1 t + a_2 t^2$

1. Convert $\vec{p}$ to vector $\vec{x} = [p]_B$ $\to B = \{1, t, t^2\}$
2. Map $\vec{x} \longmapsto A \cdot \vec{x}$
3. Convert back to polynomial

$$\begin{tikzcd}
\vec{p} \arrow[r] \arrow[d, "1"] & T(\vec{p}) \\
\vec{x} = [\vec{p}]_B \arrow[r, "2"'] & A \cdot \vec{x} = [T(\vec{p})]_B \arrow[u, "3"']
\end{tikzcd}$$

(ex) to compute $T(\vec{p})$ for $\vec{p} = 3 - 6t + 10t^2$

1. $\vec{x} = [p]_B$ where $B = \{1, t, t^2\}$ $\therefore$ $\vec{x} = \begin{bmatrix} 3 \\ -6 \\ 10 \end{bmatrix} = [\vec{p}]_B$
2. $[T(\vec{p})]_B = A \cdot \vec{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \\ 10 \end{bmatrix} = \begin{bmatrix} -6 \\ 20 \\ 0 \end{bmatrix}$ $\quad$ 3. $\to T(\vec{p}) = -6 + 20t$

definition $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$

- $A = [T]_{\mathcal{B}} = \Big[[T(\vec{b}_1)]_{\mathcal{B}} \ \cdots \ [T(\vec{b}_n)]_{\mathcal{B}} \Big]$

- satisfies $[T(\vec{v})]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{B}}$

ex let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis for $\mathbb{P}_3$
find $\mathcal{B}$- matrix for the linear transformation $T: \mathbb{P}_3 \to \mathbb{P}_3$

* $T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = (a_0 - a_3) + (a_1 + 4a_2)t + (3a_2 - a_3)t^2 + a_3 t^3$

~~Solution~~ $[T]_{\mathcal{B}} = \Big[[T(1)]_{\mathcal{B}} \ [T(t)]_{\mathcal{B}} \ [T(t^2)]_{\mathcal{B}} \ [T(t^3)]_{\mathcal{B}} \Big]$

$a_0 = 1$ rest of a's are 0 so just 1

($a_1 = 1$ rest are 0 so just t)

($a_2 = 1$ rest are 0) so $4t + 3t^2$

$a_3 = 1$ ($-1 - t^2 + t^3$)

constant

$$\begin{array}{c} \\ 1 \\ t \\ t^2 \\ t^3 \end{array} [T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or you can think about it like $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \longmapsto \begin{bmatrix} a_0 - a_3 \\ a_1 + 4a_2 \\ 3a_2 - a_3 \\ a_3 \end{bmatrix}$

echelon, diagonal, pivot in every column, rank = 4
T is invertable