$$\frac{d}{dx}\int_c^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

ex) $$\frac{d}{dx}\int_2^{\sin x} \frac{t}{e^t+3}\,dt = \frac{\sin x}{e^{\sin x}+3}\cdot \cos x = \frac{\sin x\cdot(\cos x)}{e^{\sin x}+3}$$

Intuition of Integrals
(simple)

Start with a "derivitive"

so $f(x)$ will be a higher degree

The distance between the y-values of $f(x)$ is the integral or area under the $f'(x)$ graph

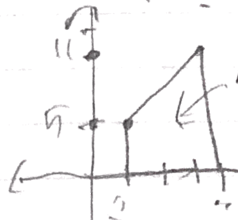
ex) $f(x) = x^2+3x$ $\quad f'(x) = 2x+3$

$\int_1^4 (2x+3)\,dx = ?$ well its the are under that graph

But also the diffrence of y-values of the anti-derivitive

$$\int_1^4 2x+3 = x^2+3x\Big|_1^4 \quad \text{so} \quad f(4)-f(1) = 24$$
$$(4^2+3(4))-(1^2+3(1))$$

Same as



$A = \frac{1}{2}(5+11)(3) = 24$

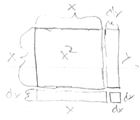
Same but calculatable because it's a geometric shape, higher order not so

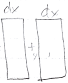
Same answer

$$2 = e^{k(1)}$$

$$\ln 2 = \ln e^{k(1)}$$
$$\ln 2 = k(1)\ln e$$
$$\ln 2 = k(1)$$
$$k = \ln 2$$

$$f(x) = x^2$$



$df = x$  $+ \; dx^2$ — super small

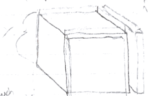
$$df = 2x\,dx$$

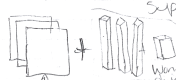
The rate at which the area is growing $\dfrac{df}{dx} = 2x$

$$f'(x) = 2x$$

$$f(x) = x^3$$

increase are by small amount



$df =$ 

Volume of one

Super small cause of dx raised to high power
Wont survive dividing by dx and letting it go to zero

how much the area changes

So $\dfrac{df}{dy} = \dfrac{3x^2\,dx}{dx}$

the small nudge

$$\frac{df}{dx} = 3x^2$$

$x \cdot x \cdot dx$

$x^2 dx$

so Volume of 3 is $3x^2 dx$

* what differentiates the different volume laws integrand?

Distance = Right − Left
Distance = Top − bottom

Area by Integration

$\int_a^b$ (height of rectangle)(width of rectangle)   * one is d(whatever)

Volume by Integration

* one is d(whatever)

Volume $\int_a^b \pi r^2 h$ ← cylindrical ← notation as you add up a lot of parts (cylinders) and $dx \to \frac{1}{\infty}$

usually dx

Volume $\int_a^b l \cdot w \cdot h$

example?

Volume by revolution

→ What is the difference

Volume = $\int_a^b \pi r^2 dr$ radius is a function
what makes it go to 3D

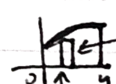
Washers $V = \pi(R^2 - r^2)h$

Volume by Crossection

Volume = $\int_a^b A(x)\,dx$

sq: s^2
Δ: $\frac{1}{2}bh$
$\square$: bh
semicircle: $\frac{1}{2}\pi r^2$
qrt circle: $\frac{1}{4}\pi r^2$

← ex) $y = \sqrt{x}+1$, x-axis, $x = 4$



side

base goes to 0

$\int_0^4 (\sqrt{x}+1)^2 dx$

Arc Length

$\int_a^b \sqrt{1 + f'(x)^2}\,dx$
(On a regular curve)

length = $\sqrt{\Delta x^2 + \Delta y^2}$
$\sqrt{(x'(t))^2 + (y'(t))^2}$ ⇒ $\int_a^b \sqrt{x't^2 + y't^2}\,dt$
(Parametric)

*mass of object if forced $x^2 \frac{gm}{cm}$ then find a small distance and cancel length for mass then $\int$ it all

8.3 Polar

Conversion from Rectangular to Polar

(radius, θ)

$r^2 = x^2 + y^2$ use $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$= \sin^{-1}\left(\frac{y}{r}\right)$
$= \cos^{-1}\left(\frac{x}{r}\right)$

Conversion from Polar to Rectangular

$x = r\cos\theta$
$y = r\sin\theta$

$\boxed{\int_a^b \sqrt{r'^2 + r^2}\, d\theta \quad \text{Arc length of polar graph}}$

Area in a Polar Curve

$\frac{\text{arc measure}}{2\pi} \cdot \pi r^2 \Rightarrow \frac{\text{arc measure}}{2} \cdot r^2 \Rightarrow \int_a^b \frac{r^2}{2}\, d\theta$

$d\theta$ $f(\theta)$

Slope $= \frac{r'\sin(\theta) + r\cos(\theta)}{r'\cos(\theta) - r\sin(\theta)}$ * (8.3 sheet notes)

Density $\int_a^b (\text{Density})(\text{Area})$ ← ~~why does not use area?~~

Circle $\int_a^b D(x) \cdot 2\pi r\, dr$
$A = l \cdot w$

Springs $W = \int_a^b kx\, dx$ (b ← pulled/compressed, a ← rest pos)

$k = \frac{F}{\text{distance pulled back}}$

Lift an object $\int_a^b \text{mass}(\text{distance lifted})\, dy$ or

---

Work to lift a liquid

a = lowest liquid value
b = highest liquid value

$\int_a^b (\text{Density})(\text{area}) \cdot (\text{Distance lifted})\, dy$

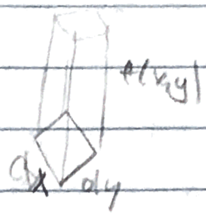
volume of crosssection with dy as thickness

$W = \int_a^b \text{Force} \cdot \text{Dism}$

volume · Dens

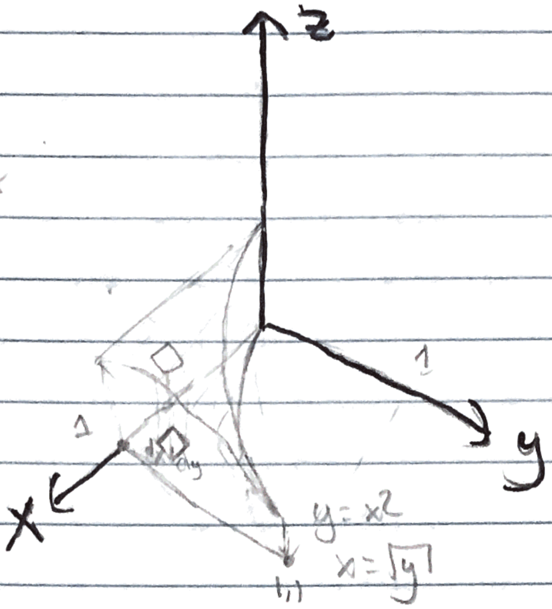
Iterated Integrals

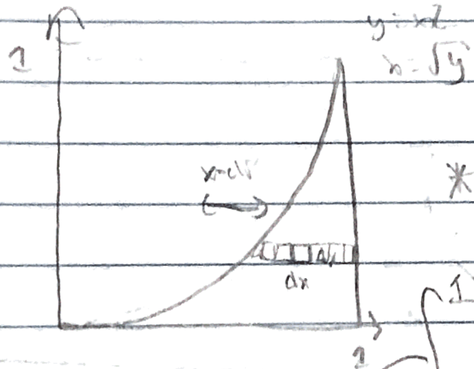
ex) $z = xy^2$



$\hookrightarrow (xy^2)\,dx\,dy$

* the top part flat its xy^2



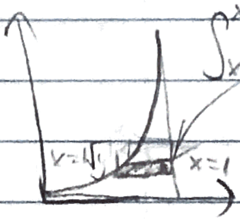


* lets sum them up in the x-direction first, holding y-constant

$$\int_0^1 \int_{x=\sqrt{y}}^{x=1} xy^2\,dx\,dy$$

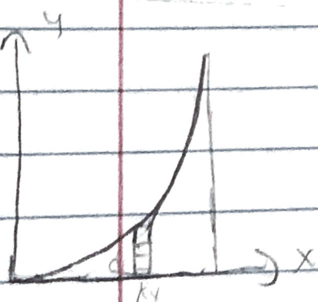
$\hookrightarrow$ gives volume above for arbitrary x-direction

now add up dy in the y-dir



$\int_{x=\sqrt{y}}^{x=1} xy^2\,dx\,dy$ you added up all those squares in the x-direction.

*It's A VOLUME

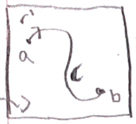


$$\int_0^1 \int_0^{x^2} xy^2\,dy\,dx$$

Line Integral

$$\boxed{\int_a^b F(\vec{r})\cdot d\vec{r}}$$

Vector Field



*Some small vector r' in vector Field F along a curve "C"

*orient axis so r' is straight up

Now dot them you get the amount of $F(t)$ along r'

a — b so

$$\boxed{\int_a^b F(\vec{r})\,d\vec{r} = \int_a^b F(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}$$

Quickly if your path doesn't matter then $f(b)-f(a)$ = line integral

$\boxed{\vec{F} = \nabla f}$ then its path independent, proof of this formula

$$\int_C \nabla f\cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_a^b\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}\right)dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt$$

by fundemental therom of Calc then $f(b)-f(a)$ = line integral

example $\int 4xz+2y\,dx$ from $(2,1,0)$ to $(4,0,2)$, so $\vec{F} = \langle 4xz+2y, 0, 0\rangle$ * or just parameterize :)

$x = 2+(4-2)t$ [using $x = x_0+(x_f-x_0)t$] $0\le t\le 1$ = $2 = 2+2t$, $t_0 = 0$ $4 = 2+2t$ $t_f = 1$

$y = 0+(0-1)t$ $z = 0+(2-0)t$ so $\vec{r} = \langle 2+2t, 1-t, 2t\rangle$, dr or $\vec{r}\,'(t) = \langle 2,-1,2\rangle dt$

$\int_0^1 \vec{F}\cdot d\vec{r} \Rightarrow \int_0^1 \langle 4xz+2y,0,0\rangle\langle 2,-1,2\rangle dt$ dot out, plug $x = 2+2t$ etc + compute

or another $\int_0^1 4xz+2y\,dx$ $x = 2+2t$, $dx = 2dt$ plug x,y,z's in

Curl $F = \langle M,N\rangle$ $\boxed{N_x - M_y = 0 \quad M_y = N_x}$ its path independent

* intuitively its how much does it cause a vector field rotation CCW, CW

3D curl
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \frac{\partial}{\partial x} & F_1 \\ \hat{j} & \frac{\partial}{\partial y} & F_2 \\ \hat{k} & \frac{\partial}{\partial z} & F_3 \end{vmatrix}$$

Greene's Therom $\iint \text{curl}\,\vec{F}\,dA$ * $\vec{F} = \langle M,N\rangle$

$\oint M\,dx + N\,dy$, $\iint N_x - M_y\,dA$ | used in closed loops in a Vector Field where its not path independant

Flux simply its $\iint_{\text{every bit of our surface}} \vec{F}\cdot\vec{n}\cdot dA$ | * if $\vec{F}$ is constant and $\perp$ surface $= |\vec{F}|$(area of surface)

we split our surface into tiny bits, summing all our $\vec{F}$ & $\vec{n}$'s & dAs over the surface

Flat surface example: $\vec{F} = \langle x, x^2+y^2, z\rangle$ through a disc or $r=2$ in xz-plane: $\int_S \vec{F}\cdot d\vec{A} = \int_S \langle \overset{0}{x}, x^2+y^2, \overset{0}{z}\rangle\cdot\langle 0,1,0\rangle dA$ ($\perp$ xz plane)

$\int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta$

Sidenote Area of a region $= \frac{1}{2}\left[\oint x\,dy - \oint y\,dx\right]$; Area of a surface $= \iint_R |\vec{r}_x \times \vec{r}_y|\,dx\,dy$

Curvy surface example: $\iint_R \vec{F}\cdot\vec{n}\,dA$ $\vec{F} = \langle yz, x, z^2\rangle$ through parabolic cylinder $y = x^2$ $0\le x\le 1$ $0\le z\le 4$

$x = x$, $y = x^2$, $z = z$ $\Rightarrow \vec{r} = \langle x,y,z\rangle \Rightarrow \vec{r} = \langle x, x^2, z\rangle$ $\begin{bmatrix} r_x = \langle 1, 2x, 0\rangle \\ r_z = \langle 0,0,1\rangle \end{bmatrix}$, $\vec{n} = r_x \times r_y = \begin{vmatrix} \hat{i} & 1 & 0 \\ \hat{j} & 2x & 0 \\ \hat{k} & 0 & 1 \end{vmatrix} = \langle 2x, -1, 0\rangle$

(partial in to x, z etc...)

$$\int_0^4\int_0^1 \langle yz, x, z^2\rangle\cdot\langle 2x,-1,0\rangle\,dx\,dz$$

usually has $|r_x \times r_y|$ but it cancels out always! as far as we've learned

Over

if $\vec{F} = \nabla f$ then $\vec{F}$ is path independent $f(b) - f(a)$
potential function

$W = \int \vec{F} \cdot d\vec{r}$

Line integral → $\int_C 4xz + 2y\,dx$, c is line segment (2,1,0) to (4,0,2) : parameterize when paring

(see if you can do it without parametrizing

Curl

$\langle M, N \rangle$ $\quad M_y = N_x$

3D curl $\quad$ if curl $= 0$ then path independent

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \frac{\partial}{\partial x} & F_1 \\ \hat{j} & \frac{\partial}{\partial y} & F_2 \\ \hat{k} & \frac{\partial}{\partial z} & F_3 \end{vmatrix}$$

Greens therom

finding work done in a nonconservative field $\quad \vec{F} = \langle M(x,y), N(x,y) \rangle$

$\iint$ curl of f dA, $\quad \oint M\,dx + N\,dy$, $\quad \iint N_x - M_y\,dA$

Flux

- if $\vec{F}$ is constant and $\perp$ to surface then flux $= |\vec{F}| \cdot$ (area of surface)
  $\hat{n} \parallel F$ $\quad$ ex plane $2x - y + 3z = 10 \rightarrow \hat{n} = 2\hat{i} - \hat{j} + 3\hat{k}$ $\quad$ if scalar then $\hat{n} \parallel \vec{F}$

- Flux $= \int_S \vec{F} \cdot d\vec{A}$

Flux through a flat surface

ex $\quad \vec{F} = x\hat{i} + (x^2+y^2)\hat{j} + z\hat{k}$ through a disc of $r=2$ in xz-plane towards

$\int_S \vec{F} \cdot d\vec{A}$ $\quad d\vec{A} = \hat{j}\,dA$ $\quad \vec{F} \cdot d\vec{A} = \langle x, x^2+y^2, z \rangle \cdot \langle 0,1,0 \rangle$

$\hat{j}$ $\perp$ xz-plane $\quad \int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta$

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*IDK. $\quad \vec{F} = \langle 0, x \rangle$, $\quad \oint x\,dy = \oint \vec{F} \cdot d\vec{r}$, $\quad \iint dA =$ Area of R

$\vec{F} = \langle -y, 0 \rangle$ $\quad \oint 1\,dA$

more # parametrized

Area of R $\quad \frac{1}{2}\left[\oint x\,dy - \oint y\,dx\right]$

Surface Area

$S = \vec{r}(x,y)$ $\quad \vec{r}_x = \frac{\partial \vec{r}}{\partial x} = \langle \frac{\partial}{\partial x}x, \frac{\partial}{\partial x}y, \frac{\partial}{\partial x}z \rangle$

$\vec{r}_y = \frac{\partial \vec{r}}{\partial y} = \langle \frac{\partial}{\partial y}x, \frac{\partial}{\partial y}y, \frac{\partial}{\partial y}z \rangle$

$$\text{Area} = \iint_R |\vec{r}_x \times \vec{r}_y| \, dx\,dy$$

* taking areas of parallelograms on the surface & adding them up

~~Back to Flux~~

$\boxed{\begin{array}{l}\vec{r}\langle x,y,f(x,y)\rangle \text{ when ?} \\ \vec{r} = \langle f_x, -f_y, 1\rangle\end{array}}$?

$$\iint \vec{F}\cdot\vec{n}\, dA$$

example $\vec{F} = \langle yz, x, z^2\rangle$ through parabolic cylinder $y = x^2$ $0 \le x \le 1$, $0 \le z \le 4$

$$\iint \vec{F}\cdot\vec{n}\cdot dA$$

$x = x$ $\quad \vec{r} = \langle x, y, z\rangle$

$y = x^2$ $\quad \vec{r} = \langle x, x^2, z\rangle$

$z = z$ $\quad \boxed{r_x = \langle 1, 2x, 0\rangle, \; r_z = \langle 0,0,1\rangle}$

$\vec{n} = \frac{r_x \times r_z}{|r_x \times r_z|} = \frac{\langle 2x, -1, 0\rangle}{\sqrt{(2x)^2 + (-1)^2 + 0^2}}$, $\quad dA = |r_x \times r_z| \, dx\,dz$

So we have

$$\int_0^4\int_0^1 \langle y z, x, z^2\rangle \cdot \overbrace{\frac{\langle 2x, -1, 0\rangle}{\sqrt{4x^2+1}}}^{\vec{n}} \cdot \overbrace{\sqrt{4x^2+1}}^{dA} \, dx\,dz$$

$y \to x^2$

Parametrize

line $\vec{r} = \langle x_0 + (x_1 - x_0)t, \; y_0 + (y_1 - y_0)t, \; z_0 + (z_1 - z_0)t \rangle$

cylinder $\vec{r} = \langle r\cos\theta, r\sin\theta, z\rangle$

sphere $\vec{r} = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\rangle$

Taylor Polynomial

Intuition on the back! 3blue 1brown

ex) Find a Taylor Polynomial of degree 4 for

$f(x) = e^x$ near $x=0$

$$P_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$P_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2\frac{17}{24} \approx 2.7083$$

$$e^1 = 2.71828$$

Taylor Polynomial near $x=a$

$$P_n(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^n(a)(x-a)^n}{n!}$$

ex) Find a taylor degree 3 for $f(x) = \sqrt{x-2}$ near $x=3$

$P_n(x) =$ $\qquad$ $P_3 = 1 + \frac{1}{2}(x-3) - \frac{1}{4}\frac{(x-3)^2}{2!} + \frac{3}{8}\frac{(x-3)^3}{3!}$

$f(3) = \sqrt{3-2} = 1$

$f'(x) = \frac{1}{2}(x-2)^{-1/2}$ $\quad$ $f'(3) = \frac{1}{2}(3-2)^{-1/2} = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(x-2)^{-\frac{3}{2}}$ $\quad$ $f''(3) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(x-2)^{-5/2}$ $\quad$ $f'''(3) = \frac{3}{8}$

Plugging $x=a$ makes the $(x-a)^n$ behave as if you were at the origin, or a point that works well

for $\cos(x) = 1$ where the derivative repeats

$-\sin(0) = 0$
$-\cos(0) = -1$
$\sin(0) = 0$
$\cos(0) = 1$

$$P(x) = 1 + 0\frac{x^1}{1!} + \frac{-1x^2}{2!} + \frac{0x^3}{3!} + 1\frac{x^4}{4!} + \dots$$

$$\boxed{P(x) = f(0) + f'(0)\frac{x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots}$$

Starting away from 0

$$\boxed{P(x) = f(a) + f'(a)\frac{(x-a)^1}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \dots}$$

Taylor Polynomial Intuition

* Your matching how the curve changes for higher order derivatives of the function your approximation

ex

$\cos(x)$ at $x=0$ $\quad P(x) = c_0 + c_1 x + c_2 x^2$

$P(0) = c_0 + c_1(0) + c_2(0)^2$
$P(0) = c_0 + 0$
↑ you want this to be $\underline{1}$ cause $\cos(0)=1$

$\cos(0) = 1$ This value is it because if you want an apprx to be good at all at $x=0$ you want your $P(0)=1$ Function + approximate to be equal at that value

*Plus plugging in 0 for x cancels all terms

So...

$P(x) = 1 + c_1 x + c_2 x^2$

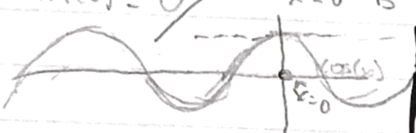
$\cos(0) = 1$ $\quad P(0) = 1$

$\frac{dP}{dx}(x) = c_1 + 2c_2 x$

same thing

$\frac{d(\cos)}{dx} = -\sin(0) = 0$ So the slope of the tangent line at $x=0$ is 0

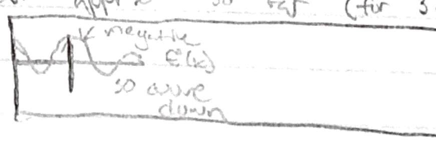
$P'(0) = c_1 + 0$



*To further better approximate $\cos(x)$ we want to match the $\frac{dP}{dx}(x)$ with $\frac{d(\cos)}{dx}(x)$ at $x=0$

So now we have for our best apprx so far (for 3 term taylor series)

$P(x) = 1 + 0 + c_2 x^2$ ⟵⟶ Now c_2



$\frac{d^2(\cos)}{dx^2}(0) = -\cos(0) = -1$

to match the $\frac{d^2(\cos)}{dx^2} = -1$ at $x=0$

$P''(x) = 2c_2$ ⟵ we need $2c_2 = -1$ so $= -\frac{1}{2}$

$P''(x) = 2(-\frac{1}{2})$

$= -1$ ⟵ what we want for the best approx

Now

$P(x) = 1 + 0x + -\frac{1}{2}x^2$

$P(x) = 1 - \frac{x^2}{2}$

To get better estimations match high $f'''''(x)$ with $p'''''(x)$

once you get to x^3, x^4, x^5, x^6, x^7 to cancel these → when taking the $d'''''(x)$ you have to divide by $n!$ to make it equal to the derivative of your f...

$a>0$ $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p>1$] proof below
diverges if $p \le 1$]

Simple "proof" for convergence

$\int_1^\infty \frac{1}{x^2}\,dx$ converges because it follows the trend of $p>1$ so $\frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \frac{1}{x^{100}}$ etc... which gets increasingly small, finite value

But for values of $p \le 1$ then it never reaches a finite value
ex $\int \frac{1}{\sqrt{x}}$, $\int \frac{1}{\sqrt[3]{x^2}}$

Proof

$p>0$ $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$

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if $p=1$ $\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \Rightarrow \ln x \Big|_1^b \Rightarrow \ln b - \cancel{\ln 1}$

so $\lim_{b\to\infty} \ln(b)$ which is just ∞
so the area under graph is ∞, divergence

- - - - - - - - - - - - - - - - - - - -

for $p \neq 1$

$\lim_{b\to\infty} \int_1^b x^{-p}\,dx$ this is $\frac{x^{1-p}}{1-p}\Big|_1^b = \frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}$

so $\lim_{b\to\infty}$ of $\boxed{\frac{b^{1-p}}{1-p}} - \frac{1}{1-p}$ ← 1 to any power is just 1

↖ this is the only part affected by the limit

$\frac{1}{1-p} \lim_{b\to\infty} b^{1-p}$

If b approaches ∞ then exponent is what matters (if its + or −)
$1-p>0$ or $p<1$ if going to ∞ and the exponent is positive it will go **BIG**

- - - - - - - - - - - - - - - - - - - -

$1-p<0$ or $p>0$ in this case its going to be $\frac{1}{b}$ to some exponent so as "b" goes ∞ it approaches 0

* or equal cause $\frac{1}{x}$ diverges ($\ln(\infty)$ is ∞)

If $1-p>0$ then it gets huge and keeps on to ∞
Divergence

In Conclusin

$$\lim_{b\to\infty}\int_1^b x^{-p} = \frac{1}{1-p}\left[\lim_{b\to\infty} b^{(1-p)}\right]$$

If $1-p<0$ then it approachs 0
Convergence

$\infty^{\text{negative}} = \frac{1}{\infty} = 0$

$\infty^{\text{posit}} = \infty$

Divergence $\int_a^\infty \frac{1}{x^p}\,dx$	$1-p\geq 0$ $1\geq p$ $p\leq 1$
Convergence	$1<p$ $p>1$

Divergence	$p\geq 1$	$\int_0^1 \frac{1}{x^p}\,dx$
Convergence	$p<1$	

For $\int_0^1 \frac{1}{x^p}\,dx$ Converge if $p<1$
Diverge if $p\geq 1$

$$-\int_1^0 x^{-p}\,dx \Rightarrow \lim_{b\to 0}\int_1^b x^{-p}\,dx$$

$\underline{p=1}$ you get $\ln x\Big|_1^b$ $\ln 0 - \ln 1 \Rightarrow \ln 0$ is $-\infty$
Divergence

$\underline{p\neq 1}$ $\lim_{b\to 0}\frac{x^{1-p}}{1-p}\Big|_1^b = \frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}$

$\boxed{\frac{1}{1-p}\lim_{b\to 0} b^{1-p}}$ So if $1-p<0$ then its $\frac{1}{0}$ so its undefined and doesnt reach a defintive answer DIVERGence

If $1-p<0$ then its 0^{positive} which is 0 so it converges to 0

Divergence: $p\geq 1$
Convergence: $p<1$

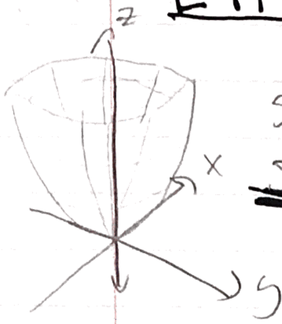
TWO Positive Parabolas

Paraboloid

Elliptical

$$z = x^2 + y^2$$

putting different values here obviously disrupts the symmetry



so now x and y are changing by the same amount, the square and the output being z so you dont get a cylinder

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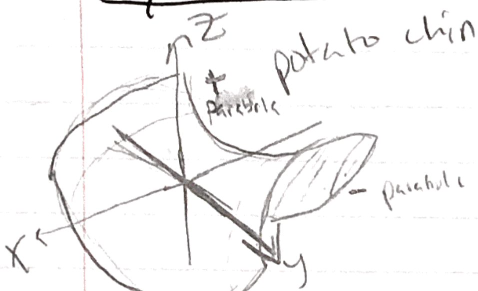
Hyperbolic Paraboloid

$$z = -x^2 + y^2$$

(sign can obviously go to any)



Its the same as an elliptical paraboloid but one is negative

E.P. is 2 positive parabola's connected in 3D space

* The parabolas are turned 90° from ea

H.P is 1 positive parabola, 1 negative parabola connected in 3D space

If true why?

Thoughts on why:

like connecting points on a graph but now its connecting graphs in 3D space

---

Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

* if $b=c=a$ its a sphere

The a, b, c stretch and pull the sphere by the constant factor of either x, y, z

blue = z
green = y
red = x

3D equations and why their graphs make sense

PLANES

always 0

$x = z \rightarrow y = 0$ so in (x, y, z) then $x = z$

$\left.\begin{matrix} y = z \\ x = y \end{matrix}\right\}$ same rule applies for these two

QUESTION: for $x = y$ (or any) z ISN'T zero (or any) (?) | just think about it being extruded out in the z direction?

Answer: It hugs that axis, definition of being zero

General **Plane:** $ax + by + c = z$ or without sparkles $x + y = z$

$\rightarrow x + y = z$ no axis hugging cause $[x, y, \text{or } z] \neq 0$

Cylindrical: $x^2 + y^2 = c^2$

just a circle in x, y world but in 3 dimensions you extrude in z directions you get cylinder

Question: What is z's value, not 0 because it has a height

Answer:

Parabolic: $y = ax^2$

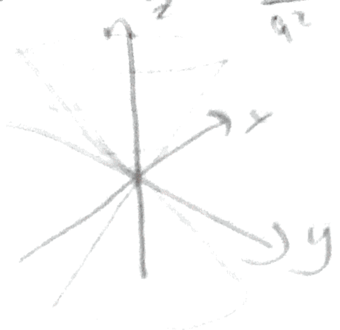
Same thing as a cylinder but a parabola

On the back

3D equations cont...

(Hyperboloid of a point?)

Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ or $x^2 + y^2 = z^2$



How to know, when x, y are 0, $z = 0$

also is [illegible] with circular equations circles of various sizes

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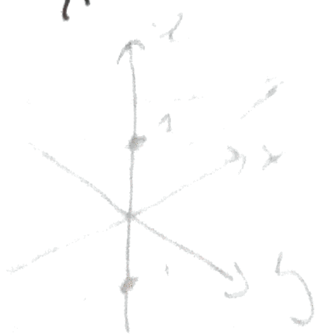
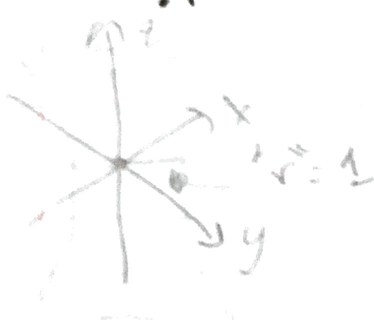
Hyperboloid 2 sheets: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$



When x and $y = 0$, $z = 1, -1$

Hyperboloid of 1 sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



x and y can't be $= 0$

*look at unit vectors

Vector Proofs

Dot product Proof, Projection Proof etc...

[2] $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$

[1] $\vec{a} \cdot \vec{b} = |a||b| \cos\theta$

[4] $\text{proj}_{\vec{b}} \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|b|^2} \cdot \vec{b}$

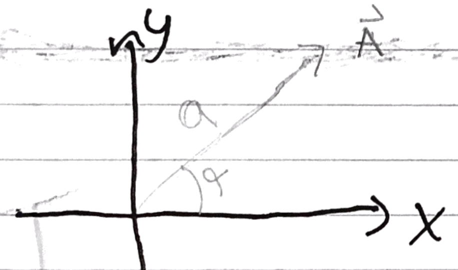
[3] $\text{comp}_{\vec{b}} \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|b|}$

$\vec{v} - V_{par} = V_{prp}$

$V_{par} = \text{proj}$

---

[1]



- $\vec{A} = \underbrace{a\cos(\alpha)}_{x \text{ component}}, \underbrace{a\sin(\alpha)}_{y \text{ component}}$

Now do this with a vector $\vec{b}$

- $\vec{B} = \underbrace{b\cos(\beta)}_{x \text{ component}}, \underbrace{b\sin(\beta)}_{y \text{ component}}$

Dot Product

(multiply the x and y components and add them together)

- $\vec{A} \cdot \vec{B} = a \cdot b \cos(\alpha)\cos(\beta) + a \cdot b \sin(\alpha)\sin(\beta)$
- $= a \cdot b\left([\cos(\alpha)\cos(\beta)] + [\sin(\alpha)\sin(\beta)]\right)$

Dot Product proof 1

The trig Identity: $\cos(\alpha - \beta) = [\cos(\alpha)\cos(\beta)] + [\sin(\alpha)\sin(\beta)]$

so $\vec{A} \cdot \vec{B} = a \cdot b \cos(\alpha - \beta)$ — the angle between two vectors

---

[2] $\begin{bmatrix} \vec{A} = a_1\hat{i} + a_2\hat{j} \\ \vec{B} = b_1\hat{i} + b_2\hat{j} \end{bmatrix}$ $\vec{A} \cdot \vec{B} = (a_1\hat{i} + a_2\hat{j})(b_1\hat{i} + b_2\hat{j})$

Now multiply it out

Dot Product Proof 2

$\left[\vec{A} \cdot \vec{B} = a_1 b_1 \overset{1}{(\hat{i} \cdot \hat{i})} + a_1 b_2 (\hat{i} \cdot \hat{j}) + a_2 b_1 (\hat{i} \cdot \hat{j}) + a_2 b_2 \overset{1}{(\hat{j} \cdot \hat{j})}\right]$

- $\hat{i}$ is a unit vector of length 1
- $\hat{j}$ is a unit vector of length 1

$\hat{i} \cdot \hat{j} = 0$ they are $\perp$

* $a \cdot b \cos(90°) = 0$ and $a \cdot b \cos(0°) = 1$

with this now it (un vectors the vectors)

$$\boxed{\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2}$$

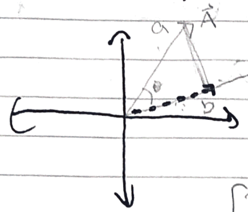
(3) $\text{Comp}_{\vec{b}}\, \vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|b|}$

[4] $\text{proj}_{\vec{b}}\, \vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|b|^2}\cdot\vec{b}$

Magnetude only — no direction (vector version is projection)

Component Proof

[3]



(dotted line)

[1] $\text{Comp}_{\vec{B}}\, \vec{A} = |a| \cos\theta$

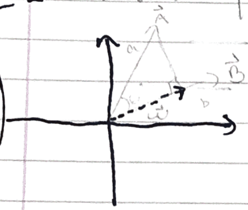
we know $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|a||b|}$

[2] so... (substitute part 1) $\text{com}_{\vec{B}}\, \vec{A} = \cancel{|a|}\, \dfrac{\vec{A}\cdot\vec{B}}{\cancel{|a|}\,|b|}$

$$\boxed{\text{com}_{\vec{B}}\, \vec{A} = \frac{\vec{A}\cdot\vec{B}}{|b|}}$$

Projection Proof

[4] Recall: $\dfrac{\vec{a}}{||a||}$ = unit vector with a length of 1 because without $||a||$ it would just be $\vec{a}$ with its length [redacted]



$\text{proj}_{\vec{B}}\, \vec{A} = \vec{w}$

We want $\vec{w} = |\vec{w}|\, \hat{w}$

[1] Take the $\text{comp}_{\vec{B}}\, \vec{A} = |\vec{w}| \Rightarrow |\vec{w}| = \dfrac{\vec{A}\cdot\vec{B}}{|b|}$

[2] $\hat{w} = \hat{b} = \dfrac{b}{|b|}$

[3] Since we want $|\vec{w}|\hat{w} = \text{proj}_{\vec{B}}\, \vec{A}$

$\text{proj}_{\vec{B}}\, \vec{A} = \dfrac{\vec{A}\cdot\vec{B}}{|b|} \cdot \dfrac{b}{|b|}$

$$\vec{w} = \text{proj}_{\vec{B}}\, \vec{A} = \frac{\vec{A}\cdot\vec{B}}{|b|^2}\cdot b$$

* $\text{comp}_{\vec{B}}$ * is important!

The difference is $\text{comp}_{\vec{B}}\, \vec{A}$ is a scalar and $\text{proj}_{\vec{B}}\, \vec{A}$ is a vector

so $\text{proj}_{\vec{B}}\, \vec{A} = [\text{comp}_{\vec{B}}\, \vec{A} \cdot \dfrac{b}{|b|}]$ → the unit vector

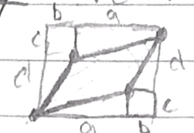
Cross Product

[1] Matrices: A matrix is a linear transformation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

- ac tell where horz comp goes and bd for y comp
- performs a transformation for the vector $\begin{bmatrix} x \\ y \end{bmatrix}$

* $\hat{\imath}$ can have a vertical component after a transformation (thats the c value)

[2] Determinents: Tell you the area of a parralellogram made by 2 vectors negative if $\hat{\imath}$ $\hat{\jmath}$ the matrix transformation flips $\hat{\imath}$ and $\hat{\jmath}$



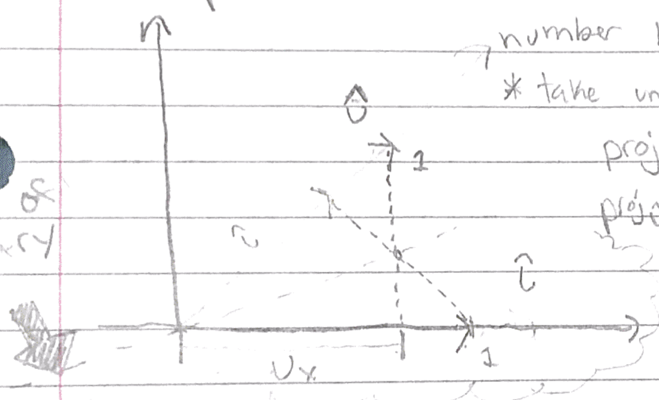
$\det\left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\right) = 2$

$(-1)(-1) - (-1)(1) = 2$

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)(c+d) - ac - bd - 2bc = \boxed{ad - bc}$

---

[3] Dot product: definition is $\vec{v} \cdot \vec{w} = \text{comp}_{\vec{v}}\vec{w}\,(|\vec{v}|)$ $\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}|} \cdot |\vec{v}|\right)$ $\text{comp}_{\vec{v}}\vec{w}$



number line: 1D, to get to it you need a transformation [? ?]

* take unit vectors $\hat{u}$, $\hat{\imath}$, $\hat{\jmath}$

$\text{proj}_{\hat{\imath}}\hat{u} = \hat{u}_x$ x-component of $\hat{u}$

$\text{proj}_{\hat{u}}\hat{\imath} = \hat{\imath}$ [So!!! $U_x = \hat{\imath}$]

which $= \hat{u}$

Lets say $\hat{u}$ was $3\hat{u}$, to get U_x you multiply by 3 or $|\hat{u}|$!! (DOT PRODUCT)

so $\vec{v} \cdot \vec{w} = (\text{component})(|\text{vector}|$ you projected onto$)$

---

[4] How can this transformation be described? [for this vector] !!!!!!!

- we know its a matrix [ ] : 1 by 2 because it goes to the number line
  : So the vertical comp of $\hat{\imath}$ DNE, just the length
- goes from length $\hat{\imath}$ and $\hat{\jmath}$ to the component $= U_x$ by symmetry

$[U_x \; U_y]$ $\Rightarrow$ so transformation for vector $\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow [U_x \; U_y]\begin{bmatrix} x \\ y \end{bmatrix}$

$[U_x \; U_y]\begin{bmatrix} x \\ y \end{bmatrix} = U_x(x) + U_y(y)$ [Wait... we multiplied the component by vector?]

dot product $\begin{bmatrix} U_x \\ U_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = [U_x \; U_y]\begin{bmatrix} x \\ y \end{bmatrix}$ (This is called Duality and it is important for cross product proof)

goes to vector of size $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

NOW For Cross Product Proof

Lets have a function that takes a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and goes 3D to 1D
so a 3D matrix (so it transforms a vector) and goes 3D to numberline and spits out that vector
[important note: it takes a $\vec{v}$ and $\vec{w}$]

- Back up! 2D cross product was the determinant $\vec{v} \times \vec{w} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ = area of parallelogram

A guess thats wrong: just up 2D to 3D: $\vec{u} \times \vec{v} \times \vec{w} = \det\left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right)$ → gives a 3D parallelogram (Gets us rally close though)
This spits out a number we want a vector!!

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

variable, fixed vectors

this means you plug a vector in and get a 3D parallelogram (sly key to answer)

3D→1D so it transforms Matrix!!

Once you know its linear you can describe it as matrix multiplication

$$\begin{bmatrix} ? & ? & ? \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

$\boxed{\text{Duality says its the same as dot product}}$

$$\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

(vector)

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

Dot product = [area of 3D parallelogram but first column is a variable, other is a vector]

$$p_1 x + p_2 y + p_3 z = x(v_2 w_3 - v_3 w_2) + y(v_3 w_1 - v_1 w_3) + z(v_1 w_2 - v_2 w_1)$$

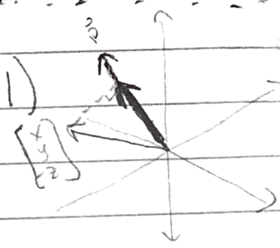
$p_1 = v_2 w_3 - v_3 w_2$
$p_2 = v_3 w_1 - v_1 w_3$
$p_3 = v_1 w_2 - v_2 w_1$

coordinates of a vector that is ⊥ to parallelogram or xy plane

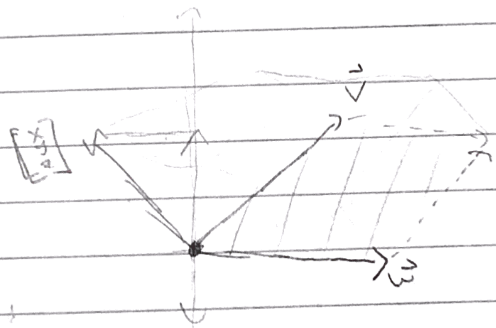
$$\boxed{\hat{i}(v_2 w_3 - v_3 w_2) + \hat{j}(v_3 w_1 - v_1 w_3) + \hat{k}(v_1 w_2 - v_2 w_1)}$$

Geometric understanding

Recall: $\vec{p} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (\text{length of component})(|\vec{p}|)$



Now the volume or $\det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$



⊥ to parallelogram

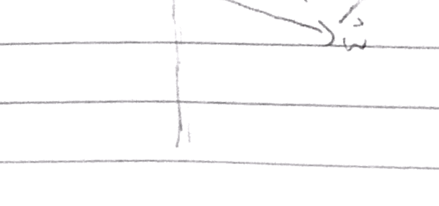
$$(\text{Area of parallelogram})\left(\text{comp}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

→ gives a 3D parallelogram

THIS IS THE SAME THING AS...
Dot Product

* where dot product is − is where right hand rule of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\vec{v}$, $\vec{w}$ is negative also

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \vec{p} = \left(\text{comp}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)(|\vec{p}|)$$



WHERE $|\vec{p}|$ = area of parallelogram

Unified Integral Methods

1. U-sub
2. Integration by Parts
3. Partial Fractions
4. Trig Sub

U-Substitution:

you cant expand this out so u-sub is helpful

$$\int (x+3)^{17} \qquad u = x+3 \qquad du = 1dx$$

$$\int u^{17}\,du = \frac{1}{18}u^{18} + C = \frac{1}{18}(x+3)^{18} + C$$

Integration by Parts:

U if it can go to zero or broken down

dw if its not u (Lo

$\boxed{wu - \int w\,du}$ * you will know if it doesnt deduce when you do it.

weird one

$$\int \underbrace{x^2}_{dw}\,\underbrace{\ln(x)}_{u}\,\underbrace{dx}_{dw}$$

$u = \ln(x)$ $\qquad$ $w = \frac{1}{3}x^3$

$du = \frac{1}{x}dx$ $\qquad$ $dw = x^2\,dx$

$$\left(\tfrac{1}{3}x^3\right)\left(\ln(x)\right) - \int \left(\tfrac{1}{3}x^3\right)\left(\tfrac{1}{x}\right)dx$$

$$\frac{x^3\ln(x)}{3} - \frac{1}{3}\int x^2\,dx \Rightarrow \frac{x^3\ln(x)}{3} - \frac{1}{9}x^3 + C$$

Partial Fractions:

This is a lot of algebra to make it easy to intgrate!

$$\int \frac{x-5}{x^2+4x+3}\,dx \Rightarrow \left(\frac{(x-5)}{(x+3)(x+1)} = \frac{A}{(x+3)} + \frac{B}{(x+1)}\right)(x+3)(x+1)$$

$\Rightarrow (x-5) = A(x+1) + B(x+3)$

$x-5 = Ax + A + Bx + 3B$: Now put terms of orders of x; x^2, x^0, x^{19}, etc in their own equation

constants: $-5 = A + 3B$ $\quad$ elimination $\Rightarrow -6 = 2B$

x's: $1 = A + B$ $\quad$ Plug in $\boxed{B=-3}$ $\boxed{A=4}$

Back to integration

$$\int \frac{4}{x+3} + \frac{-3}{x+1}$$

$$\boxed{4\ln|x+3| - 3\ln|x+1|}$$

What we found is *

$$\frac{x-5}{x^2+4x+3} = \frac{4}{x+3} - \frac{3}{x+1}$$

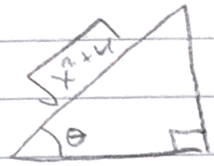
* if $\frac{k}{(x-r)^n} = \frac{A}{x-r} + \frac{B}{(x-r)^2} + \frac{C}{(x-r)^3} + \dots + \frac{A_n}{(x-r}$

also if factor is like $\frac{k}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ only a power of 1 differe

Trig Sub!

Use a triangle to set up a pythagorean relationship
-If unsure that the relationship is correct check!

$\int \frac{x^2}{4+x^2}\,dx \rightarrow (2)^2+(x)^2$



$x^2+4=c^2$

so we got the right sides!

2

• $\cos\theta = \frac{2}{\sqrt{x^2+4}} \Rightarrow \sqrt{x^2+4} = \frac{1}{\cos\theta}\cdot 2 \Rightarrow 2\sec\theta = \sqrt{x^2+4}$

$\Rightarrow 4\sec^2\theta = x^2+4$ We got 1 component now we need $\boxed{x \text{ or } x^2}$ $\boxed{dx}$

• $\tan\theta = \frac{x}{2} \Rightarrow x = 2\tan\theta$ Now we just need $\boxed{dx}$

• $dx = 2\sec^2\theta\,d\theta$

$x = 2\tan\theta$
$dx = 2\sec^2\theta\,d\theta$
$4+x^2 = 4\sec^2\theta$

$\int \frac{(x)^2}{4+x^2}\,dx$ — $x \to 2\tan\theta$, $dx \to 2\sec^2\theta\,d\theta$, $4+x^2 \to 4\sec^2\theta$

$\int \frac{(\cancel{2}\tan\theta)^2}{(\cancel{4}\sec^2\theta)}(2\sec^2\theta\,d\theta) \Rightarrow 2\int \tan^2\theta\,d\theta$

$\sin^2\theta+\cos^2\theta=1$
$\tan^2\theta+1=\sec^2\theta \Rightarrow \tan^2\theta = \sec^2\theta-1$

$2\int \tan^2\theta\,d\theta$

$2\int(\sec^2\theta-1)\,d\theta \Rightarrow 2\left[\tan\theta-\theta\right]+C$

What is $\tan\theta$ and θ?! ← GO BACK TO TRIANGLE

$\tan\theta = \frac{x}{2}$
$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

$2\left[\frac{x}{2} - \tan^{-1}\left(\frac{x}{2}\right)\right]+C$

Trig Derivitives!

$\frac{d}{dx}\sin(x) = \cos(x)$
$\frac{d}{dx}\cos(x) = -\sin(x)$
$\frac{d}{dx}\tan(x) = \sec^2(x)$
$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
$\frac{d}{dx}\cot(x) = -\csc^2(x)$

Integrals of trig

$\int \cos(u)\,du = \sin(u)+C$
$\int \sin(u)\,du = -\cos(u)+C$
$\int \sec^2(u)\,du = \tan(u)+C$
$\int \sec(u)\tan(u) = \sec(u)+C$
$\int \csc^2(u)\,du = -\cot(u)+C$
$\int \csc(u)\cot(u)\,du = -\csc(u)+C$