Linear Algebra

(1,1) → (1,5)

Matrixes can be used to rewrite and manipulate sys of linear eqs

ex: $x_1 + 3x_2 - x_3 = 4$
$x_1 + 4x_3 = 2$
$x_2 + x_3 = -1$

$$\rightarrow \begin{array}{cccc} & & & \\ \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 1 & 0 & 4 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \\ x_1 \quad x_2 \quad x_3 \quad RHS \end{array}$$

← augmented matrix bec of RHS

You can manipulate using row reduction algorithms (scale + add rows, swap rows) (echelon form and reduced echelon form example)

① swap

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \rightarrow$$

② start with top then subtract a row to zero out far left entry

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

← mult by (-1) add to this →

③

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

← NOW MOVE TO this row to cancel out this guy

④ multiply row 2 by $-3/2$ to cancel out 3 when you add

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

← IN ECHElon form

to get in reduced we need all these (pivots) to be 1 and 0's above them.

⑤ to get zeros above pivots start at bottom and add multiples of that row to above ones to get zeros in the column

* this ensures you don't mess up above rows!

⑥

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

make these zero by (row3)(-2) + row 2 etc until →

⑦

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -72 \\ 0 & 2 & -4 & 4 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

⑦ doing this by going to each pivot (3, 2, 1) and getting zeros on top yields →

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

← reduced echelon form (very useful for solving problems)

To use REF (reduced echelon form) looks like this!

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \end{array} \qquad \begin{array}{l} x_1 = 5x_3 + 1 \\ x_2 = 4 - x_3 \end{array}$$

x_1 and x_2 are static or basic variables while x_3 can be anything

* no soln if you get like $0 \neq 13$

Vectors

seeing if you can combine vectors by scaling them then adding them to equal another vector

ex

$a_1 = \begin{bmatrix}1\\-2\\-5\end{bmatrix}$ $a_2 = \begin{bmatrix}2\\5\\6\end{bmatrix}$ $b_1 = \begin{bmatrix}7\\4\\-3\end{bmatrix}$

if you can linear combo it then let x be the scales to get to b

$x_1\begin{bmatrix}1\\-2\\-5\end{bmatrix} + x_2\begin{bmatrix}2\\5\\6\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$ multiply in you get → $\begin{bmatrix}x_1\\-2x_1\\-5x_1\end{bmatrix} + \begin{bmatrix}2x_2\\5x_2\\6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$ using rules of adding vecs

In order for it to be a linear combo this row has to equal

$\begin{bmatrix}x_1 + 2x_2\\-2x_1 + 5x_2\\-5x_1 + 6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$ x compon, y compon, z compon

we can use our matrix to solve → $\begin{bmatrix}1 & 2 & | & 7\\-2 & 5 & | & 4\\-5 & 6 & | & -3\end{bmatrix}$ (REF) → $\begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$

a_1 a_2 a_3 (OBSERVE) this is a shortcut to finding combinations

solving: $x_1 + 0 = 3$, $0 + x_2 = 2$, $0 = 0$ thus $x_1 = 3$, $x_2 = 2$ → $3\begin{bmatrix}1\\-2\\-5\end{bmatrix} + 2\begin{bmatrix}2\\5\\6\end{bmatrix} = \begin{bmatrix}7\\4\\3\end{bmatrix}$ notice $3(1) + 2(2) = 7$ that's what we are doing with x

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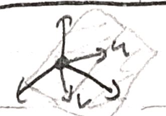
Span is all combinations to get that vector



random example: compute A

$A = \begin{bmatrix}2 & 3\\-1 & 5\\6 & -2\end{bmatrix}$ $x = \begin{pmatrix}x_1\\x_2\end{pmatrix}$

$x_1\begin{bmatrix}2\\-1\\6\end{bmatrix} + x_2\begin{bmatrix}3\\5\\-2\end{bmatrix}$

helpful fact, obvious but just added for clarity

$Ax=b$, another way to look at these problems! A being matrix (a is a vector)

$[a_1, a_2]\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1 a_1 + x_2 a_2$

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does $Ax=b$ have a soln, only if linear combo → $A = \begin{bmatrix}1 & 3 & 4\\-4 & 2 & -6\\-3 & -2 & -7\end{bmatrix}$ $b = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$, now reduce $Ax=b$ yields $\begin{bmatrix}1 & 3 & 4 & b_1\\0 & 14 & 10 & b_2+4b_1\\0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3\end{bmatrix}$

(only consistent if bottom b combos equal 0)

Homogenous solutions $Ax=0$, $x=0$ is trivial solution, we want non trivial → ex: $\begin{bmatrix}1 & 0 & -4/3 & | & 0\\0 & 1 & 0 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix}$ $x_1 = \frac{4}{3}x_3$ ← free (Non trivial; $x_2 = 0$ ← basic

ex2: $x_1 - 3x_2 - 2x_3 = 0$, $x_1 = 3x_2 + 2x_3$

$x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3x_2 + 2x_3\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}0\\0\\0\end{bmatrix} + x_2\begin{bmatrix}3\\1\\0\end{bmatrix} + x_3\begin{bmatrix}2\\0\\1\end{bmatrix}$ ← useful for parametric eq

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Parametric vector eq: $x = su + tv$, best shown w/ example: know W = all soln, V_h = homogenous soln, p is constant scale

aka Soln to non homogenous

$A = \begin{bmatrix}3 & 5 & -4\\-3 & -2 & 4\\6 & 1 & 8\end{bmatrix}$ $b = \begin{bmatrix}7\\-1\\-4\end{bmatrix}$ Put in augment an REF $Ax=b$ → $\begin{bmatrix}1 & 0 & -4/3 & -1\\0 & 1 & 0 & 2\\0 & 0 & 0 & 0\end{bmatrix}$ → $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, $0=0$, x_3 is free

so: $x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-1 + \frac{4}{3}x_3\\2 + 0x_3\\0 + x_3\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix} + \begin{bmatrix}\frac{4}{3}x_3\\0\\x_3\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix} + x_3\begin{bmatrix}4/3\\0\\1\end{bmatrix}$

substitute basic variables (most important part !!!!!!!!)

p, v

$p_1\begin{bmatrix}3\\-3\\6\end{bmatrix} + p_2\begin{bmatrix}5\\-2\\1\end{bmatrix} + p_3\begin{bmatrix}-4\\4\\8\end{bmatrix} = \begin{bmatrix}7\\-1\\-4\end{bmatrix}$!! !

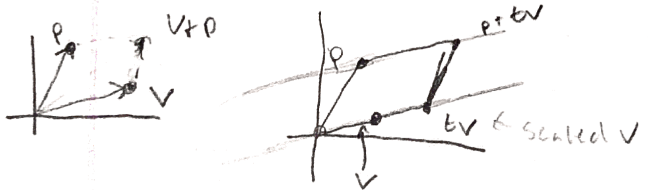
$p = \begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix}$ idk but ask prof

warning might be wrong

so $x = p + x_3 v$ or $x = p + tv$ this describes all soln of $Ax=b$ in par form

and $Ax=0$ mean $x = tv$, soln of $Ax=b$ is obtained by adding p to V_h (→ $Ax=0$)

let w be all soln then $W = p + V_h$



Applications

$\underline{x_1} C_3H_8 + \underline{x_2} O_2 \longrightarrow \underline{x_3} CO_2 + \underline{x_4} H_2O \equiv x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} = x_3\begin{bmatrix}1\\0\\2\end{bmatrix} + x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$ $\begin{bmatrix}C\\H\\O\end{bmatrix}$

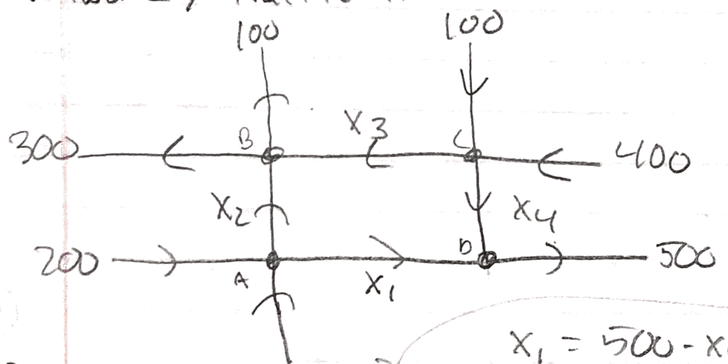
bring these over but make negative!

$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\-2\end{bmatrix} + x_4\begin{bmatrix}0\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ← good so you have conservation of mass

now combine row reduce and solve

$x_1 = \frac{1}{4}x_4 \quad x_2 = \frac{5}{4}x_4 \quad x_3 = \frac{3}{4}x_4 \quad x_4$ is free can't have fractions $x_4 = 4$

Network / traffic flow



A: $400 = x_1 + x_2$
$400 = x_2 + x_3$
$500 = x_3 + x_4$
$500 = x_1 + x_4$

$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -400 \\ 0 & 1 & 1 & 0 & -400 \\ 0 & 0 & 1 & 1 & -500 \\ 1 & 0 & 0 & 1 & -500 \end{array}\right]$

row reduce! $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -500 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

professor example confusing just look at flow in vs flow out to set up probs

$x_1 = 500 - x_4$
$x_2 = x_4 - 100$ ← got by inspecting this
$x_3 = 500 - x_4$

$X = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}500 - x_4\\x_4 - 100\\500 - x_4\\x_4\end{bmatrix} = \begin{bmatrix}500\\-100\\500\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}$

negative solutions dont make sense so! $100 \le x_4 \le 500$

Linear independance ✱ like homogenous soln in part 1.5 but vector form

→ linear independant if its only trivial soln → dependant if more than one

ex $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $v_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$ $v_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$, determine if its linearly independant

$\left[\begin{array}{cccc} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ x_1, x_2 are basic x_3 is free thus its dependent $a > 0$

$b = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $x_1 = 2x_3 \quad x_2 = -x_3$ set $a = x_3$, $2aV_1 - aV_2 + aV_3 = 0$ are all soln

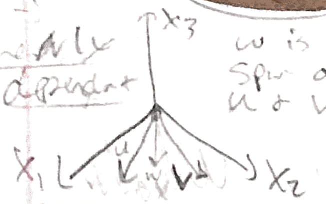
separate note:
- you know its dependent if $V_1 = V_2 + V_3$ ← cause ~~this~~ is if its spanning that means you could add vectors to get to that point.
- generally $V_1 = V_2c_2 + V_3c_3 + \dots + V_nc_n \rightarrow 0 = -V_1 + V_2c_2 + V_3c_3 + \dots + V_nc_n$ this is why you do $\underline{Ax = 0}$
  non zero scale

→ For Matrix's its obviously the same process

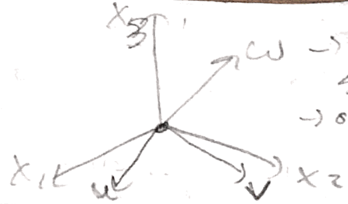
(warning) linear independent/dom linear combo

$\begin{bmatrix}\vdots\end{bmatrix} + \begin{bmatrix}\vdots\end{bmatrix} + \begin{bmatrix}\vdots\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \not\equiv \begin{bmatrix}\vdots\end{bmatrix} + \begin{bmatrix}\vdots\end{bmatrix} = \begin{bmatrix}\vdots\end{bmatrix}$

Linearly dependent



w is in span of u + v



w → not in span so its independent; → only way to make it work is for it to be 0, that's why you do $A x = \vec{0}$ — if its only 0 it means its striking out.

$\boxed{\text{Linear independance w/ vectors}}$

w/one vector(v) $xv = 0$ if $v \neq 0$ only soln is $x = 0$, so independent
if $v = 0$ then dependent cause all x are a soln

a) $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ $\boxed{a}$ notice multiples of eachother so $\vec{v}_2 = 2\vec{v}_1 \Rightarrow \underset{x_1}{-2}\vec{v}_1 + \underset{x_2}{\vec{v}_2} = \vec{0}$ so this is a sol'n, dependent

b) $v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2 \\ 38.4 \end{bmatrix}$ not scalars, assume c and d satisfy $cv_1 + dv_2 = 0$ if $c \neq 0$ $v_1 = -\frac{d}{c} v_2$ but their not scalars of eachother so c has to be zero, similarly d is zero, ∴ independent

$\boxed{\text{2 or more vectors}}$

- dependent if linear combo $\boxed{ex}$ $u = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, not scalars so independent

⤳ if theres more vectors than components of the vectors then its dependent since there will be a free var

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In all 3 ways to think of Linear systems:

①

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 4 \\ x_1 - x_2 + x_3 &= 2 \\ 3x_1 + x_2 &= 6 \end{aligned}$$

② aug to solve ...

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 1 & -1 & 1 & 5 \\ 3 & 1 & 0 & 6 \end{array}\right] = (A\ \vec{b})$$

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{b}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad A\vec{x} = \vec{b}$$

Matrix's and Transformations a different way of thinking about it.

$m \times n$ matrix

Consider $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ $\rightsquigarrow [A\ \vec{b}] = \begin{bmatrix} 1 & 0 & 5/2 & 1/2 & 4 \\ 0 & 1 & 3 & -1/3 & 11/3 \end{bmatrix}$

thus $x_1 = 4 - \frac{5}{2}x_3 - \frac{1}{2}x_4$, $x_2 = \frac{11}{3} - 3x_3 + \frac{1}{3}x_4$

gen sol'n for x, that is gen soln for the scale to get A to b

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 5/2\,x_3 - 1/2\,x_4 \\ 11/3 - 3x_3 + 1/3\,x_4 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 11/3 \\ 0 \\ 0 \end{bmatrix}}_{p} + \underbrace{x_3 \begin{bmatrix} -5/2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1/2 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}}_{V_h}$$

using all soln $= p + V_h$

* we know $A\vec{x} = \vec{0}$ $\rightarrow$ $\vec{x} = x_3[\ ] + x_4[\ ]$

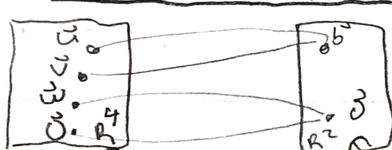
now pick some (x_3, x_4)'s to get unique soln $\vec{u}$, $\vec{v}$ for $A\vec{x} = \vec{b}$

do same for $A\vec{x} = \vec{0}$, $\vec{w}$.

$\Rightarrow$ $A\vec{u} = \vec{b}$ $A\vec{0} = \vec{0}$ multiplication in A transforms $\vec{u}$ into $\vec{b}$

$A\vec{v} = \vec{b}$ $A\vec{w} = 0$ transforms $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ into $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, also $\vec{v}$ into $\vec{b}$, $\vec{w}$ into 0

$\boxed{\text{we can think about this in terms of a function}}$



A transformation "T" from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\vec{x} \in \mathbb{R}^n$ a vector

rember we had 4 columns $= n$ 2 rows $= m$ thus $\mathbb{R}^n \to \mathbb{R}^m$

$T: \vec{x} \to T(\vec{x})$ "image of x"

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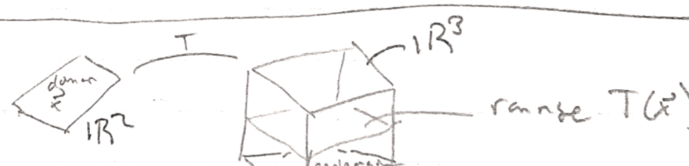
given $T(\vec{x}) = A\vec{x}$

$A = \begin{bmatrix} & 0 & \\ 0 & 1 & \\ 0 & 0 & \end{bmatrix}$ $T(\vec{x}) = A\vec{x}$ looks like $\mathbb{R}^n \to \mathbb{R}^m$



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ex Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

(a) Find $T(\vec{u})$, the image $\vec{u}$ under T

$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 + 3 \\ 6 - 5 \\ -2 - 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 9 \end{bmatrix}$

(b) find vector $\vec{x}$ whose image under T is $\vec{b}$

$T(x) = b \Rightarrow A\vec{x} = \vec{b} \rightarrow (A\ \vec{b}) \rightarrow \begin{bmatrix} 1 & 0 & 4.5 \\ 0 & 1 & .5 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow \vec{x} = \begin{bmatrix} 4.5 \\ .5 \end{bmatrix}$

x_1 x_2

(c) its unique no free var

(d) is $\vec{c}$ in range T?

$T(x) = \vec{c} \rightarrow A\vec{x} = \vec{c}$

$[A\ \vec{c}] = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$ $\Rightarrow$ no solution

so no c is not in range.

$T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if

$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

$T(c \cdot \vec{u}) = c\, T(\vec{u})$

$\boxed{\text{Facts}}$ $T(\vec{0}) = \vec{0}$

and $T(c\vec{u} + d\vec{v}) = c\,T(\vec{u}) + d\,T(\vec{v})$

$T(c_1 \vec{v}_1 + \dots + c_p v_p) = c_1 T(\vec{v}_1) + \dots c_p T_p(\vec{v}_p)$

any liner combo!

ex

$T(\vec{x}) = r \cdot \vec{x}$ where r is scaler. If $0 \le r \le 1$ then T is contrac

$r > 1$ then dilata

Let $r=3$ show T is a liner transform

$T(\vec{x}) = 3\vec{x}$ realize $T(\vec{x}) = 3\vec{x} \to T(\vec{u}) = 3\vec{u}$

① check: $T(\vec{u}+\vec{v}) = 3(\vec{u}+\vec{v}) = 3\vec{u} + 3\vec{v}$ so $\to$ $T(\vec{u}) + T(\vec{v})$

② show $T(c \cdot \vec{u}) = c\,T(\vec{u})$:

$T(c \cdot \vec{u}) = 3(c \cdot \vec{u}) = c(3\vec{u}) = c\,T(\vec{u})$ ✓ (both satisfies rules!)

ex 2 Find images of u under T →

$T(\vec{x}) = A\vec{x} \to T(\vec{u}) = A \cdot \vec{u}$

$\boxed{1.9}$ Linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ is some matrix transformation $T(\vec{x}) = A\vec{x}$, A tells us what t does

ex $T\left(\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\vec{e}_1}\right) = \begin{bmatrix}5\\-7\\2\end{bmatrix}$ and $T\left(\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\vec{e}_2}\right) = \begin{bmatrix}-3\\8\\2\end{bmatrix}$ let $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ find $T(\vec{x})$

Weird example

$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 \to T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) = x_1 \begin{bmatrix}5\\-7\\2\end{bmatrix} + x_2 \begin{bmatrix}-3\\8\\2\end{bmatrix}$

$T(\vec{x}) = \begin{bmatrix}5x_1 - 3x_2\\ -7x_1 + 8x_2\\ 2x_1 + 2x_2\end{bmatrix}$

Matrix way: $T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$

recall: $x_1 \vec{v}_1 + x_2 \vec{v}_2 = A \cdot x = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ so

$T(\vec{x}) = \underbrace{\begin{bmatrix}T(\vec{e}_1) & T(\vec{e}_2)\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{X}$

really check this

$\vec{e} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is part of the identity matrix $I_n = \begin{bmatrix}1 & 0 & 0 & \dots & 0\\0 & 1 & 0 & & 0\\0 & 0 & \ddots & & \vdots\\ \vdots & \vdots & & \ddots & 0\\0 & 0 & \dots & 0 & 1\end{bmatrix} \to I_n \cdot \vec{X} = \vec{X}$

$I_{n=2}$, $\vec{e}_j = \begin{bmatrix}0\\ \vdots\\ j\\ \vdots\\ 0\end{bmatrix}$ the j^{th} e corresponds to "that one down"

$T(\vec{x}) = A \cdot \vec{x}$ $A = \begin{bmatrix}T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n)\end{bmatrix}$

$\vec{x} = \begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix}$ $\vec{x} = \underbrace{x_1 \vec{e}_1}_{\text{first spot 1, rest 0}} + \underbrace{x_2 \vec{e}_2}_{\text{second spot 1, rest 0}} + \dots + \underbrace{x_n \vec{e}_n}_{n^{th}\text{ spot 1, rest 0}}$ → can do this thus $\underline{I_n \cdot \vec{x} = \vec{x}}$

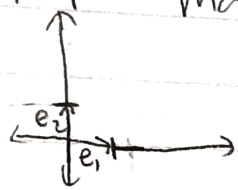
$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots \vec{e}_n x_n) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \dots + x_n T(\vec{e}_n)$

$T(\vec{x}) = \underbrace{\begin{bmatrix}T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n)\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix}}_{X} = A\vec{x}$

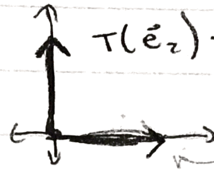
* I think the $I_n \cdot \vec{x} = \vec{x}$ is useful bec/ it lets you plot things and thats how you transform!!

1.9 matrix transformations pt. 2

remember



$T(\vec{x}) = 3\vec{x}$



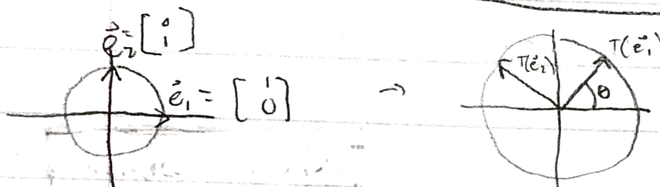
$T(\vec{e}_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$T(\vec{e}_1) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ← note $= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3I_2$

$I_2 \cdot \vec{x} = \vec{x}$

$T(\vec{x}) = A\vec{x} \Rightarrow T(\vec{x}) = A\vec{x} = 3I_2 \cdot \vec{x} \Rightarrow T(\vec{x}) = 3\vec{x}$



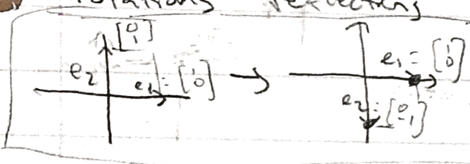
$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}_{y}^{x}, \quad T(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

* rotation by $\frac{\pi}{2} \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
↳ $\sin(\frac{\pi}{2}) = 1$
$\cos(\frac{\pi}{2}) = 0$

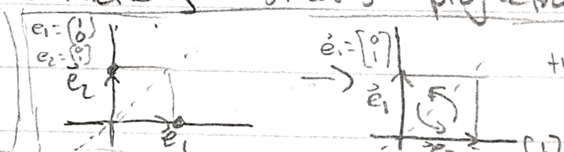
$T(\vec{x}) = A \cdot \vec{x}$

rotations, reflections, expansions, contractions, shears, projections



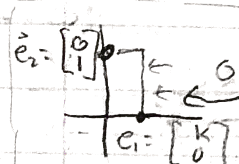
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

reflection about x-axis



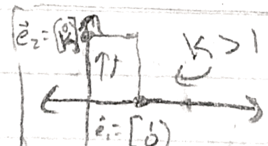
thus $X_2 = X_1$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



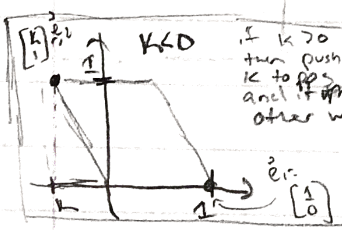
for $k > 1$ it stretches horz

$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$



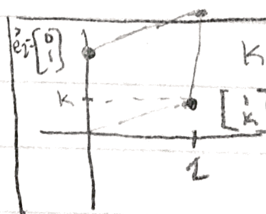
if k $0 < k < 1$ then shrinks vertically

$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$



if $k > 0$ then push k to pos and it moves other way

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$



$k > 0$ if $k < 0$ then push it down

$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

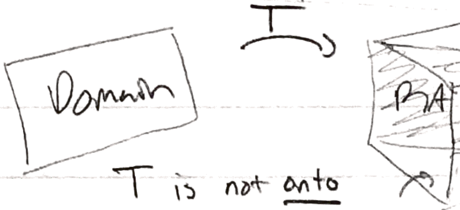
* just follow where each e_1 and e_2 go!

Familiar Questions → is $A\vec{x} = \vec{b}$ consistent? → is b in range of T

→ can b be written as a linear combo of columns of A? → is b in range of T

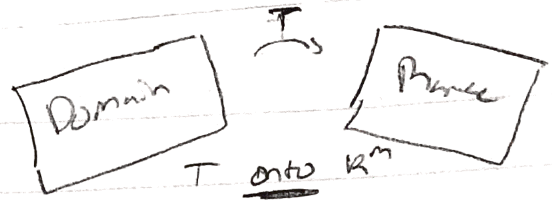
- Range is all the outputs
- co domain is everything else

existence



T is not onto $\mathbb{R}^m$

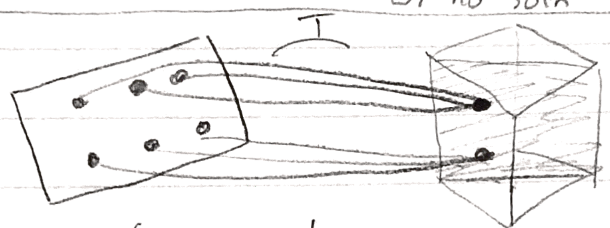
* there's some $T(x)=b$ w/ no sol'n



T onto $\mathbb{R}^m$

* all $T(x)=b$ has sol'n

uniques



not one to one

* there's not unique sol'n

T is one to one

* only unique sol'n

Theorems 4-12 Let A be $m \times n$, $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$

(a) For each $\vec{b} \to \mathbb{R}^m$ the eq $A\vec{x} = \vec{b}$ is consistent
(b) each $\vec{b} \to \mathbb{R}^m$ is linear combo of columns of A
(c) A has pivot pos in every row
(d) columns of A span $\mathbb{R}^m$
(e) T is onto

$\boxed{\text{Not Onto: if some } T(x)=b \text{ has no sol'n}}$

$\boxed{T(x) = Ax}$

Theorem 11-12

(a) $b \to \mathbb{R}^m$ the eq $Ax = b$ has at most one sol'n
(b) columns of A are linearly independent
(c) $A\vec{x} = 0$ has only trivial sol'n $\vec{x} = 0$
(d) A has pivot pos'n in every column
(e) T is one to one

$\boxed{T(x)=b \text{ has unique sol'n or none at all}}$

is eq $Ax = b$ consistent for every b? → Is T onto?	is T onto? does A have piv pos in each row
$Ax = b$ have at most one sol'n → Is T one to one?	is T one to one? does A have piv pos in each column
Do columns of A span $\mathbb{R}^m$ is T onto?	
Are columns of A linearly independent? is T one to one?	

ex: $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$

show T is one to one

$T(\vec{e}_1) = T(1, 0) = (3, 5, 1)$

$T(\vec{e}_2) = T(0, 1) = (1, 7, 3)$

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$

$Ax = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$

that linearly independent, if it was the two vectors would be scal.

Matrix multiplication

- if a matrix does some operation to space (transformation), multiplying them performs them in order. *like when you do that to a vector.

calculation → $AB = A\begin{bmatrix} b_1 & b_2 & b_3 \cdots \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 \cdots Ab_n \end{bmatrix}$ (columns of b; using th.3)

$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}$ (2x3) $\quad B = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 6 \\ -1 & 4 & 0 \end{bmatrix}$ (3x3)

$$\rightarrow = \left[\begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix} \right]$$

can do this cause think about it like components and what would go with what, i comp go w/ i, j w/ j etc... WARNING → until you add it together this analogy is good

RULE Notice the # rows of B have to be the same as columns of A.

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} & \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix} \end{bmatrix}$$

Notice the final form is the size of $(2 \times 3) \cdot (3 \times 3)$ outside

$$= \begin{bmatrix} 1 & 8 & 17 \\ -1 & 22 & 15 \end{bmatrix}$$

By thinking about transformations (3b1b) you can see $AB \neq BA$ and $A(BC) = (AB)C$

Identity matrix → Let say I have a matrix A, and to get to A_2 I do a row operation that's the same as: doing that operation to an identity matrix and multiplying by that identity matrix. Call it E

$I_m = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ square with only 1's on diagonals

$A_2 = E \cdot A$ (E is like instructions of a single transformation)

ex to get something into REF → start with $A \rightarrow REF(A)$

$A \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots A_n$ ↑REF
(E_1, E_2, E_n)

$REF(A) = E_n E_2 E_1 A$

in order to get to A from REF you do it backwards.

Identity matrix does like nothing, no transformation. eq to mult by 1.

Inverses: idea; start w/ A, whatever steps take you to I is the inverse.

to calc $\begin{bmatrix} A \mid I \end{bmatrix}$ → [reduce A to REF] → [then right hand side will be the inverse]

then $A \cdot A^{-1} = I$

(can only invert if its square and det ≠ 0 ie, pivot in every row $\begin{bmatrix} \# & & \\ & \# & \\ & & \# \end{bmatrix}$ one to one, $Ax = b$ has soln for every b etc...)

~~$A^{-1}Ax = b \cdot A^{-1} \rightarrow x = b \cdot A^{-1}$ good for computers~~

$$A^{-1}Ax = A^{-1}b$$
$$x = A^{-1}b$$

lets you compute $Ax=b$, $Ax=b_2$ etc.. really easy, good for computers → splits $Ax=b$ into 2 problems

LU Factorization

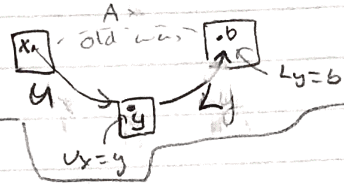
examples

lets say $A = L \cdot U$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}$, $U = \begin{bmatrix} \square & * & * & * & * \\ 0 & \square & * & * & * \\ 0 & 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ echelon form of A

USED TO SOLVE $Ax=b$ by solving

$Ly=b$
$Ux=y$

note $b = Ly = \underbrace{LUx} = Ax$



to solve $Ax=b$ with L and U → $A = [\ ] = [L][U]$

$Ly=b$ → $[L \mid b]$ → row reduce to solve y, then $[U \mid y]$ → row reduce to get x.

How to get L+U:

start with A → ① get in echelon form (not reduced) by adding only multiples of rows above to rows below.

→ for each operation get the corresponding E (do that operation on identity matrix)

this will be U, and $\boxed{U = E_n \cdots E_2 E_1 \cdot A}$

so $A = \underbrace{(E_n \cdots E_2 E_1)^{-1}}_{L} \cdot U = \underbrace{E_n^{-1} \cdots E_2^{-1} E_1^{-1}}_{L} U$

to combine: do ex: $E_1^{-1} E_2^{-1} E_3^{-1} I$ all on Identity matrix, essentially doing all your operations backwards starting with E_3^{-1} then E_2^{-1} etc..

ex of backward lets say to get E_3 it was $R_3 \to R_3 - 5R_2$,
E_3^{-1} is $R_3 \to R_3 + 5R_2$, do this on Identity matrix till done.

Determinants

in 2D area of parallelogram. $ad-bc$ | lets say $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

generally: $\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

cross out column + row its in → use whats left

alternating sign

* this determinant of A is done by jus picking a row or colum + going down by doin this

Fast way: get in echelon form, warning if you switch rows make negative if you scale by n it scales det() by n (when in echelon form mult the diagonals)

rules: $\det(AB) = \det(A)\det(B)$