Chapter 5 — Noethers Theorem and Hamiltonian dynamics

Noethers Theorem ~ general symettry leading to conservation laws ~

• continuous family of transformations (like a rotation) for coordinates of the system
↳ we want it to be when 0 the identity transformation
↳ call this transformation s
↳ if $q(t)$ is a solution to orginal sol'n let $\underline{Q(0,t) = q(t)}$ (with $s=0$)
↳ $Q(s,t)$ is the solution in the transformed case

$$\boxed{L(Q(s,t), \dot{Q}(s,t), t) = L(q, \dot{q}, t)}$$ ← definition of invariance

• if L is invarient of the transformation s, then $\boxed{\frac{d}{ds} L(Q(s,t), \dot{Q}(s,t)) = 0}$

Chain Rule! $\frac{dL}{ds} = \frac{\partial L}{\partial Q}\frac{dQ}{ds} + \frac{\partial L}{\partial \dot{Q}}\frac{d\dot{Q}}{ds}$ and ELE; $\frac{\partial L}{\partial Q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}}$

↳ $0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}}\frac{dQ}{ds} + \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}}\frac{dQ}{ds}$ ← pulled $\frac{d}{dt}$ out → $0 = \frac{d}{dt}\underbrace{\left[\frac{\partial L}{\partial \dot{Q}}\frac{dQ}{ds}\right]}_{\text{conserved quantity}} = 0$ $= I$

$I \equiv p \left.\frac{dQ}{ds}\right|_{s=0}$ = a constant — for convinience cuz it dont matter

↳ (for more than one coordinate) → $I_j(q_1, q_2, \dots q_N, \dot{q}_1, \dot{q}_2, \dots \dot{q}_N) \equiv \sum_{k=1}^{N} p_k \left.\frac{dQ_k}{ds_j}\right|_{s=0}$ = constant

ex: for spacial rotations, $I_1 I_2 I_3$ are components of total angular momentum

HAMILTONIAN DYNAMICS

→ changing from the variables $q_k, \dot{q}_k$ to q_k, p_k (Hamiltonian) $\quad (L(q,\dot{q},t) \to H(q,p,t))$
↳ New ELE, Hamiltonian equation, Hamiltons principle holds for independent variations in q, p unlike $q, \dot{q}$

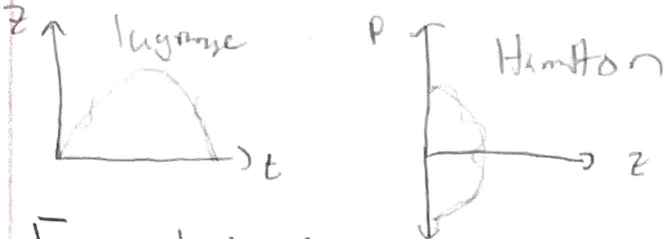
Definition: $H = p\dot{q} - L(q,\dot{q})$, $p = \frac{\partial L}{\partial \dot{q}}$

$\delta S = \int \delta L \, dt = 0$ w/ $\delta L = \dot{q}\,\delta p + p\,\delta\dot{q} - \delta H$ w/ $\delta H = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p$

$$\boxed{\delta L = \left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q + \frac{d}{dt}(p\,\delta q)}$$ ← each of these coeff will vanish which is a req of action

quick proof! (total differential) $dH = \dot{q}\,dp + \cancel{p\,d\dot{q}} - \frac{\partial L}{\partial q}dq - \cancel{\frac{\partial L}{\partial \dot{q}}d\dot{q}}$ ← $d\dot{q}$ terms vanish bec $p = \frac{\partial L}{\partial \dot{q}}$

looking at $\boxed{\left.\frac{\partial H}{\partial p}\right|_q = \dot{q}}$ and $\boxed{\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \overset{\text{ELE}}{=} -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -\dot{p}}$



From last step we can conclude $dH = \dot{q}\,dp - \dot{p}\,dq$

Hamiltons Equations of Motion

$H = \sum p_k \dot{q}_k - L$, Recall: $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

$$\boxed{\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = \frac{-\partial H}{\partial q_k} \qquad \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}}$$

Steps to solving w/ Hamiltonian dynamics!

1. identify L
2. for each q_k find $p_k = \frac{\partial L}{\partial \dot{q}_k}$
3. determine $H = p_k \dot{q}_k - L$
4. invert 2 and find $\dot{q}_k$ and plug into 3
6. use Hamiltons eq of motion

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ex SHO $\rightarrow$ $L = \underbrace{\tfrac{1}{2} m \dot{x}^2}_{T} - \underbrace{\tfrac{1}{2} k x^2}_{V}$

- $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = \frac{p_x}{m}$
- $H = p_x \dot{x} - L = p_x \dot{x} - \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \underbrace{\frac{p_x^2}{2m}}_{T} + \underbrace{\frac{1}{2} k x^2}_{V}$

$\dot{x} = \frac{\partial H}{\partial p_x} = p_x/m \rightarrow \ddot{x} = \frac{\dot{p}_x}{m}$

$\dot{p}_x = \frac{-\partial H}{\partial x} = -kx$ combining: $\boxed{\ddot{x} + \frac{k}{m} x = 0}$

Legendre Transformation

- Basic idea is to transform from one set of variables to another

Lets say we have some function $A(x,y)$ and call $B(x,y,z) \equiv yz - A(x,y)$ ← for convenience later

z is a new variable that will gain a definition from what we want to do

Take a differential
$$dB = \underbrace{z\,dy + y\,dz}_{\text{product rule/chain rule}} \underbrace{- \left.\frac{\partial A}{\partial x}\right|_y dx - \left.\frac{\partial A}{\partial y}\right|_x dy}_{\text{chain rule}}$$

Regroup the dy, dx, dz's
$$dB = \underbrace{\left(z - \left.\frac{\partial A}{\partial y}\right|_x\right)}_{\text{define } z \text{ so this term is } 0} dy + y\,dz - \frac{\partial A}{\partial x}dx \rightarrow \boxed{z \equiv \left.\frac{\partial A}{\partial y}\right|_x}$$

and from the general chain rule
$$\rightarrow dB = \underbrace{\left.\frac{\partial B}{\partial y}\right|_y dy}_{\text{goes to } 0} + \left.\frac{\partial B}{\partial z}\right|_x dz + \left.\frac{\partial B}{\partial x}\right|_z dx \rightarrow \boxed{\begin{array}{l}\left.\frac{\partial B}{\partial x}\right|_z = -\left.\frac{\partial A}{\partial y}\right|_x \\ \left.\frac{\partial B}{\partial z}\right|_x = y\end{array}}$$

To compute $B = yz - A(x,y)$ we have to invert the relationship for $z = (\partial A/\partial y)$, solving for y and then substituting into $B(x,y,z) \rightarrow B(x, y(x,z), z) = B(x,z)$

$y = \left.\frac{\partial B}{\partial z}\right|_x$ so if you have B it can be inverted

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ex $A(x,y) = (1+x^2)y^2$ w/ $B = yz - A(x,y) = yz - (1+x^2)y^2$

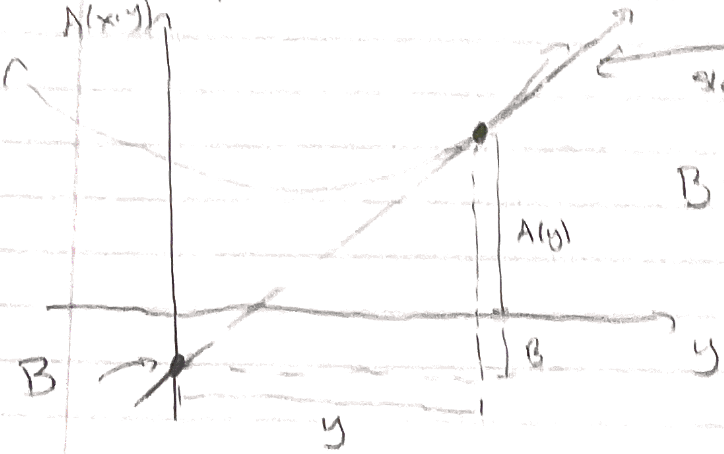
What is z, from definition above $z = \frac{\partial A}{\partial y} = 2y(1+x^2)$ but we want to solve for y though, so B is a function of x and our new variable z

$$B(x,z) = \left(\frac{z}{2(1+x^2)}\right)z - (1+x^2)\left(\frac{z}{2(1+x^2)}\right)^2 = \frac{z^2}{2(1+x^2)}\left(1 - \frac{1}{2}\right) = \frac{z^2}{4(1+x^2)}$$

Recovering $A(x,y)$ from $B(x,z)$: then $A(x,y) = yz - B(x,z)$

$y = \frac{\partial B}{\partial z} = \frac{z}{2(1+x^2)} \xrightarrow{\text{solve for } z} z = (1+x^2)2y$ plug in and you recover

Graphical Interpretation



$z = \frac{\partial A}{\partial y}$ slope of this line.

$B = zy - A \rightarrow A = \underbrace{z}_{\text{slope}}\underbrace{y}_{\text{variable}} + \underbrace{B}_{\text{intercept}}$

For this to work A has to have a second derivative
Moreover for higher order A (x^3, x^4) this transformation can have more than one solution so only works for x^2 looking stuff

Dot Product deep

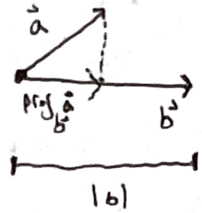
formulas: $\vec{r}\cdot\vec{s} = |r||s|\cos\theta$

$\vec{r}\cdot\vec{s} = \sum_i r_i s_i$

→ [geometrically it means] = $(\text{projection}_{\vec{b}}\,\vec{a})(\text{length of } \vec{b})$ ~~divide by unit vector of b~~ dot product ↓ $a\cdot b \cdot \frac{b}{|b|}$ [unit vector in direction of $\vec{b}$]

↳ mathematically

$\text{proj}_{\vec{b}}\,\vec{a} = \frac{a\cdot b}{|b|}\cdot\frac{b}{|b|} \rightarrow |b|\,\text{proj}_{\vec{b}}\,\vec{a} =$



[also times unit vector of $\vec{b}$: $\frac{b}{|b|}$]

↳ (scalar · vector)(vector of length 1) = dot product

Linear algebra and transpose vectors

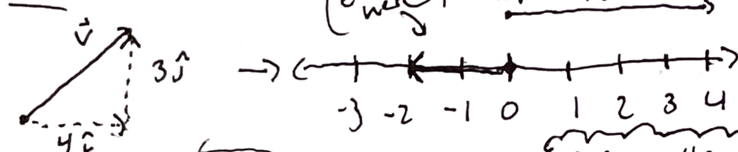
→ $\begin{bmatrix} r_1 & r_2 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = s_1 r_1 + r_2 s_2$

[its a matrix that forces it onto a line] operates onto $\vec{s}$

(ex) lets say you have a linear transformation $L(\vec{v})$ that takes $\hat{\imath}$ to $+1$ and $\hat{\jmath}$ to -2

follow where a vector w/ coordinates $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ go

Before



you go $4\hat{\imath}$ and $-6\hat{\jmath}$

** matrix vector multiplication

$\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = (4\cdot 1) + (3\cdot -2) = -2$

looks like a vector tipped on its side

Basic ideas

~~geometrically~~:

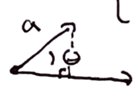
- linear transformations from multiple dimensions to just 1D, the number line
- $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \rightarrow L(\vec{r}) \rightarrow$ [1D output number]
- transformations are determined by where it takes $\hat{\imath}$ and $\hat{\jmath}$ *(both just a number)*

↳ [do transformation] → [$\hat{\imath}$ where lands | $\hat{\jmath}$ where lands]

- connection between these transformations and vectors themselves (transpose of a vector)

an added note to projection idea

⤳ recall [$\vec{a}\cdot\vec{b} = |a||b|\cos(\theta)$] (derived on back)



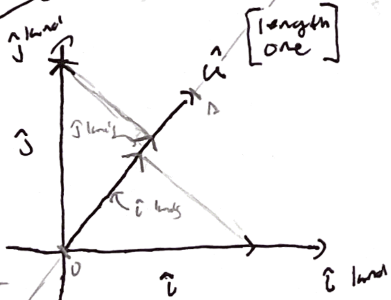
$\frac{a\cdot b}{|b|} = |a|\cos\theta$

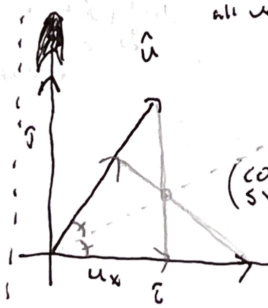
* or the component of a in the dir of b * (which is the projection of a onto b!)

Unit Vector example

*where they land is columns of matrix *

all unit vec





line of symmetry

(COMPLETLEY symmetrical!) → [projection of $\hat{\imath}$ onto $\hat{u}$ is the exact same as $\hat{u}$ onto $\hat{\imath}$]

so → $\hat{\imath}$ goes to $\hat{u}_x$ (same for y) $\hat{\jmath}$ goes to $\hat{u}_y$ → $\begin{bmatrix} u_x & u_y \end{bmatrix}$

[works if $\hat{u}$ is scaled just scale so its like $\hat{u}$ is a $\vec{v}$]

THESE ARE COMPONENTS OF A VECTOR!!!! WOW!

→ $\begin{bmatrix} u_x & u_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x u_x + y u_y$

(transformation vector, takes $\hat{\imath}$ + $\hat{\jmath}$ to components of $\vec{u}$)

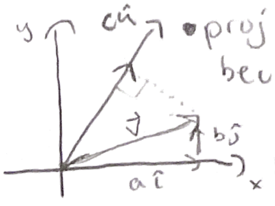
$\vec{\jmath}$ $\vec{v}$

now non-unit vectors → ex $3\hat{u}$ → $\begin{bmatrix} 3u_x & 3u_y \end{bmatrix}$

→ so $\hat{\imath}$ and $\hat{\jmath}$ go 3 times their projection

→ since just $\hat{\imath}$ and $\hat{\jmath}$ can describe any vector, it works for any vector

THIS IS IT!!



- project then scale $\hat{\imath}$ and $\hat{\jmath}$ bec $\begin{bmatrix} cu_x & cu_y \end{bmatrix}$, scale by c

$c = |\vec{u}|$, $|\vec{u}|\hat{u}$ is vector

project → $|\vec{v}|\cos\theta$ · (scale = c)

→ projection transformation, takes vector and brings it onto number line ←(span of that vector) side note $\begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \leftarrow$ (matrix vector multiplication)

Usually $Ax = b$ spits out a vector b

Ex $Ax = b \rightarrow Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_2 \end{bmatrix} = \begin{bmatrix} (a_{11} \cdot x_1) + (a_{12} \cdot x_2) + \cdots + (a_{1n} x_n) \\ (a_{21} \cdot x_1) + (a_{22} \cdot x_2) + \cdots + (a_{2n} x_n) \\ \vdots \\ \underset{\overset{\uparrow}{x_1}}{(a_{m1} \cdot x_1)} + \underset{\overset{\uparrow}{x_2}}{(a_{m2} \cdot x_2)} + \cdots + \underset{\overset{\uparrow}{x_n}}{(a_{mn} x_n)} \end{bmatrix}$

* notice columns of x_1
* collapses to a vector

~> if you only have one row it collapses to a point

$\begin{bmatrix} a_{11} x_1 + \cdots + a_{1n} x_n \end{bmatrix} \rightarrow \begin{bmatrix} \text{number} \end{bmatrix} \leftarrow$ a point!

{just like if these were vectors!}

~> also you can split it up! $\begin{bmatrix} a_{11} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underset{\overset{\uparrow}{\text{scaler}}}{x_1} a_1 + \underset{\overset{\uparrow}{\text{scaler}}}{x_2} a_2 + \underset{\overset{\uparrow}{\text{scaler}}}{x_n} a_n$