The monopole & dipole terms

• the expansion is dominated by $V_{mon} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ at large r.

• If the total charge is zero, the dominant term is the dipole

↑ why isn't dominated mon

$$V_{dip}(\vec r) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int \underbrace{r'\cos\alpha}_{\hat r\cdot\vec r'}\,\rho(\vec r')\,d\tau$$, α is the angle between $\vec r$ and $\vec r'$

$$V_{dip}(\vec r) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\,\hat r\cdot\int \vec r'\rho(\vec r')\,d\tau \quad , \quad \vec p = \int \vec r'\rho(\vec r')\,d\tau$$

$$V_{dip}(\vec r) = \frac{1}{4\pi\varepsilon_0}\frac{\vec p\cdot\hat r}{r^2}$$

[↳ for a collection of point charges] $\rightarrow \vec p = \sum_{i=1}^{n} q_i \vec r_i'$

For dipole (q & $-q$): $\vec p = q\vec r_+' - q\vec r_-' = q\vec d$ ↖ vector from the negative charge to the positive charge

---

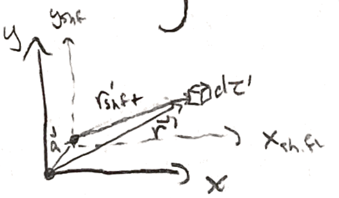
Moving the origin:

• lets say we displace the origin by an amount $\vec a$

$$\vec P_{shift} = \int \vec r'_{shift}\,\rho(\vec r')\,d\tau' = \int(\vec r' - \vec a)\rho(\vec r')\,d\tau' = \int \vec r'\rho(\vec r')\,d\tau' - \vec a\int\rho(\vec r')\,d\tau'$$

↑ why w/ respect to old $\vec r'$

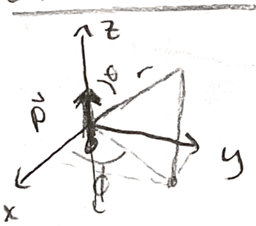
$$\vec P_{shift} = \vec p - Q\vec a$$



If $Q=0$ then:

$$\vec P_{shift} = \vec p$$

Electric Field of a dipole



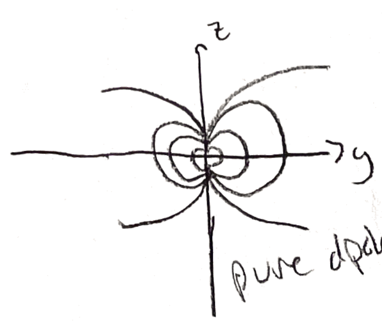
$$V_{dip}(r,\theta) = \frac{\hat r\cdot\vec p}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$ now do $-\vec\nabla V_{dip}$ in spherical

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \quad , \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} \quad , \quad E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

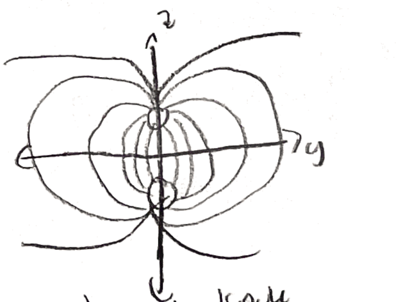
$$\vec E_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat r + \sin\theta\,\hat\theta\right)$$

↳ only holds for a physical dipole if $r \gg d$

↑ shortening d or long r



pure dipole



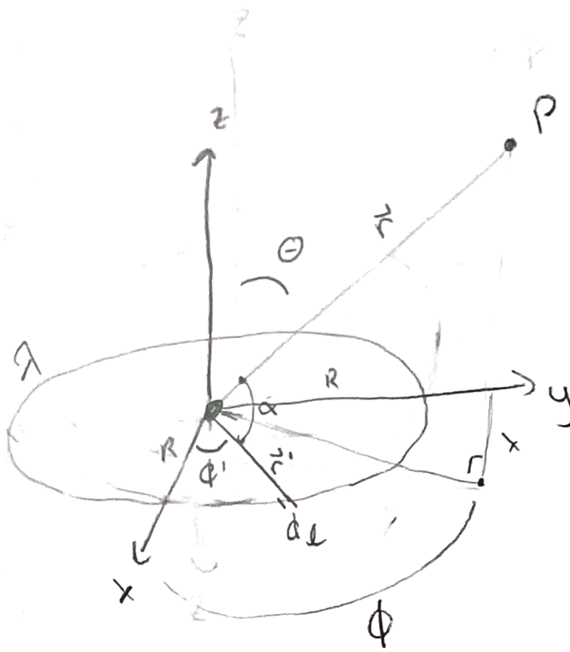
physical dipole

E+M dipole stuff

looking at the first few terms of

$$V(\vec r) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(\vec r')\,d\tau'$$

angle between $\vec r$ and $\vec r'$



Looking at first 3:

$$V_{mono} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\underbrace{\int \lambda(\vec r')\,d\ell'}_{Q = 2\pi r'\lambda} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

$$V_{dip}(\vec r) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^{2\pi} r'\cos\alpha\,\lambda\,\vec r'\,d\phi' \quad , \quad |r'| = R$$

we know: $\vec r = r\sin\theta\cos\phi\,\hat x + r\sin\theta\sin\phi\,\hat y + r\cos\theta\,\hat z$

$\vec r' = R\cos\phi'\,\hat x + R\sin\phi'\,\hat y$

also: $\vec r\cdot\vec r' = r\underbrace{r'}_{R}\cos\alpha$, we can get $\cos\alpha$ in terms of ϕ'

$$\vec r\cdot\vec r' = rR(\underbrace{\sin\theta\cos\phi\cos\phi' + \sin\theta\sin\phi\sin\phi'}_{\cos\alpha})$$

$$\vec r\cdot\vec r' = rR\cos\alpha$$

$$V_{dip}(\vec r) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^2}\int_0^{2\pi}(r')^2(\sin\theta\cos\phi\,\underbrace{\cos\phi'}\, + \sin\theta\sin\phi\,\underbrace{\sin\phi'})\,d\phi' = 0$$

essentially $\underbrace{\int_0^{2\pi}\cos\phi'\,d\phi'}_{=0} + \underbrace{\int_0^{2\pi}\sin\phi'\,d\phi'}_{=0}$

Quadripole:

$$V_{quad} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int (\overset{R}{r'})^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\lambda\,\underbrace{d\ell'}_{R\,d\phi'}$$

, to compute, plug that expression in for $\cos\alpha$ again

* answer is it has a quadripole term *

Field at a distance d from a uniformly charged sphere

$$\oint \vec E\cdot d\vec a = \frac{Q_{encl}}{\varepsilon_0}$$

for $r>a$: $4\pi r^2 E = \frac{Q}{\varepsilon_0} \rightarrow \vec E = \frac{Q}{4\pi\varepsilon_0 r^2}\hat r$

$r<a$: $4\pi r^2 E = \frac{Q_{encl}}{\varepsilon_0}$, $\int\rho\,d\tau \quad \int\rho\,d\tau$

$Q_{encl} = \gamma Q$

$\boxed{\vec E = \frac{Qr}{4\pi\varepsilon_0 a^3}\hat r}$

$\cancel{\rho}\,\frac{4\pi}{3}r^3 = \gamma\frac{4\pi}{3}a^3\cancel{\rho}$ $\quad \gamma = \frac{r^3}{a^3}$ ✓

?

(does it match with) $\rightarrow \vec E_{dip} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat r + \sin\theta\,\hat\theta)$

$= \frac{2p}{4\pi\varepsilon_0 r^3}$

arization

↳ What happens when a piece of dielectric is placed in an electric field?

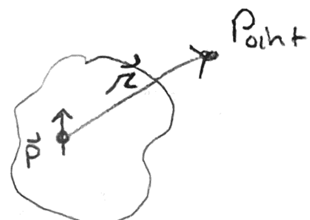
↳ If made up of neutral atoms then the field will induce a tiny dipole in all the atoms, (pointing in same dir as field

↳ if polar molecules each dipole will experience a torque tending to line up along the field direction,

* both produce a lot of little dipoles pointing along $\vec{E}$ *

define $\vec{P} \equiv$ dipole moment per unit volume (polarization

Material w/ lots of tiny dipoles lined up',



[What is the field produced by this object?] → For a single dipole $\vec{p}$ → $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \hat{\mathscr{r}}}{\mathscr{r}^2}$

$\vec{P} = \frac{\vec{p}}{d\tau'}$ ← tiny dipole
← per tiny vol it has

→ $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{vol} \frac{\vec{P}(\vec{r}') \cdot \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$ ✓

$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$, $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$, $\frac{\partial}{\partial r'}\left(\frac{1}{r-r'}\right)\hat{r} = \frac{\partial}{\partial r'}\left((r-r')^{-1}\right)\hat{r} = -1 \cdot (r-r')^{-2} \cdot (-1)\hat{r}$

$V = \frac{1}{4\pi\varepsilon_0} \int \vec{P} \cdot \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$

Integrate by parts (5) in front cover!: $\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} f)$

↳ $V = \frac{1}{4\pi\varepsilon_0}\left[\int \vec{\nabla}' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) d\tau' - \int \frac{1}{\mathscr{r}}(\vec{\nabla}' \cdot \vec{P}) d\tau\right]$

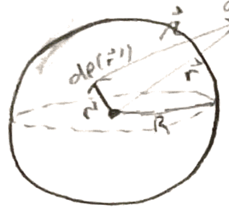
Divergence theorem: $\int \vec{\nabla} \cdot \vec{A}\, d\tau = \oint \vec{A} \cdot d\vec{a}$ | $\sigma_b \equiv \vec{P} \cdot \hat{n}$

$$V = \underbrace{\frac{1}{4\pi\varepsilon_0} \oint_S \frac{1}{\mathscr{r}} \vec{P} \cdot d\vec{a}'}_{\text{Surface}} - \underbrace{\frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{\mathscr{r}}(\vec{\nabla}' \cdot \vec{P}) d\tau'}_{\text{Volume}}$$

$\rho_b \equiv -\vec{\nabla} \cdot \vec{P}$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\sigma_b}{\mathscr{r}} da' + \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_b}{\mathscr{r}} d\tau'}$$

Polarization example:



It has a (polarization Density) of $\vec{P} = k\vec{r}$

From: $V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_b}{\mathcal{r}}\, d\tau' + \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_b}{\mathcal{r}}\, da'$

Calculate $\rho_b = -\vec{\nabla}\cdot\vec{P}$ and $\sigma_b = \vec{P}\cdot\hat{n}$:

$\sigma_b = k\vec{r}\cdot\hat{r}$, $\vec{r} = \vec{r}'$ in our case $\sigma_b = kr'$ but $r' = R$ (surface)

$\hookrightarrow \boxed{\sigma_b = kR}$

$\vec{P}_r = ?$ $\quad \vec{P} = k\vec{r}$, $\vec{P}\cdot\hat{r} = P_{r'} = kr'$

$\rho_b = -\vec{\nabla}'\cdot\vec{P} = -\frac{1}{r'^2}\frac{\partial}{\partial r}(r'^2 V_r) = -\frac{1}{(r')^2}\frac{\partial}{\partial r'}\left((r')^2 kr'\right) = \boxed{-3k}$

Find the field inside and outside the sphere;

Inside

$\frac{Q_{encl}}{\varepsilon_0} = \oint \vec{E}\cdot d\vec{a} = \oint E\,da = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0}$

$\hookrightarrow Q_{enc} = \int \rho_b\, d\tau = -3k\int_0^r 4\pi (r')^2\, dr' = -4\pi k r^3$

$\hookrightarrow 4\pi r^2 E = \frac{-4\pi k r^3}{\varepsilon_0} \rightarrow \boxed{\vec{E} = \frac{-kr}{\varepsilon_0}\hat{r}}$

Outside $Q_{encl} = 0$

every charge has an opposite equivalent

check tho

$\overbrace{-4\pi k R^3}^{P\ \text{contribut}}$, $\oint \sigma_b\, da = \int kR\, da = \int kR\,(R^2 \sin\theta\, d\phi\, d\theta)$

$= kR^3 4\pi$

$-4\pi k R^3 + kR^3 4\pi = 0$ ✓

ectric - insulator

electrons on tight leash (can only move a little bit)

→ 2 mechanisms by which $\vec{E}$ fields can distort the charge distribution, of a dielectric atom or molecule.

↳ stretching

↳ rotating

ffect of an E field on a neutral atom:

→ $\vec{E}$ field will pull the positive nucleus and push the electron cloud away (if strong enough it ionizes it)

weaker $\vec{E}$-field leaves the atom polarized (+ charge slightly shifted one way and negative shifted the other)

→ now it has a tiny dipole moment $\vec{p}$ (same dir as $\vec{E}$)

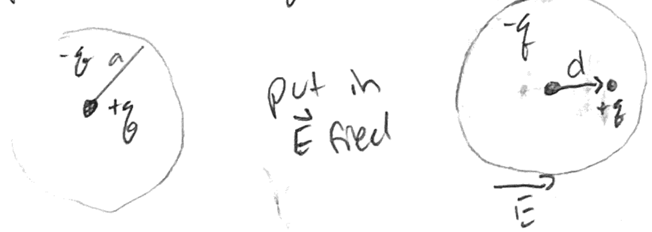
↳ typically $\vec{p} \simeq \alpha \vec{E}$

atomic polarizability (depends on detailed structure of the atom)

Example w/ an atom

spherical cloud $-q$, center q

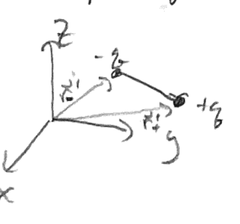
*nucleus shifted, such small displacement we can assume cloud doesn't shift * moves til it balances out



$\overleftarrow{E_e}$ = field produced by cloud, cancels $\vec{E}$ when in equilibrium

Field at a distance d from the center of a uniformly charged sphere:

$\vec{p} = q\vec{r}_+' - q\vec{r}_-' = q(\vec{r}_+' - \vec{r}_-') \Rightarrow p = qd$



$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3} \Rightarrow \boxed{E_e = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3}}$

in our case →

$\vec{r}_-' = 0$

$\vec{r}_+' = d\hat{r}$

rearrange: $p = \underbrace{4\pi\varepsilon_0 a^3}_{\alpha = 3\varepsilon_0 V_{ol}} E_e$

Molecules

Some are more polarizable in some directions than others

For example, CO_2 is more polarizable along axis of molecule & less perpindicular. If field is at an angle do

$$\vec{p} = \alpha_\perp \vec{E}_\perp + \alpha_{\parallel} \vec{E}_{\parallel} \quad \text{(resolve into components)}$$

For a general asymetric molecule you have,

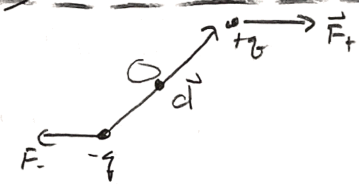
$$\left.\begin{aligned} P_x &= \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z \\ P_y &= \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z \\ P_z &= \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z \end{aligned}\right\}$$

polarizability tensor

values depend on axis but can always choose principle axes

Alignment of polar molecules

↳ built in dipoles, not induced



what about the other side?

Put water in a uniform field

$$\vec{N} = \tau_{orq} = (\vec{r}_+ \times F_+) + (r_- \times F_-)$$

$$\vec{N} = \left(\frac{\vec{d}}{2}\right) \times (q\vec{E}) + \left(\frac{-\vec{d}}{2}\right) \times (-q\vec{E}) = \underbrace{q\vec{d}}_{\vec{p}} \times \vec{E} = \vec{p} \times \vec{E}$$

In a non-uniform field:

$$\text{Total Force} = \vec{F} = \vec{F}_+ + \vec{F}_- = q(\vec{E}_+ + \vec{E}_-) = q\Delta\vec{E}$$

← difference in the field at the plus end and the field at the minus end

assume $\Delta\vec{E} = d\vec{E}$, small change → from $dT = \vec{\nabla} T \cdot d\vec{\ell}$

$$\Delta E_x \equiv (\vec{\nabla} E_x) \cdot \vec{d}$$

$$\Delta E_x \hat{x} + \Delta E_y \hat{y} + \Delta E_z \hat{z} = \Delta \vec{E}$$

?

$$(\vec{\nabla} E_x \cdot \vec{d})\hat{x} + (\vec{\nabla} E_y \cdot \vec{d})\hat{y} + (\vec{\nabla} E_z \cdot \vec{d})\hat{z} \rightarrow$$

$$\hookrightarrow \Delta\vec{E} = (\vec{d} \cdot \vec{\nabla})\vec{E}$$

$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$

$$\vec{F} = -\vec{\nabla} V$$

$$= -\vec{\nabla}(-\vec{p} \cdot \vec{E})$$

$$\vec{\nabla} \cdot \vec{p} = 0$$

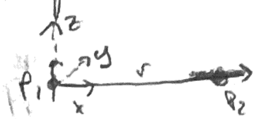
rque Example

(perfect dipoles) we need: $\vec{E}_{dip} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

$\vec{N}_2 = \vec{p}_2 \times \vec{E}_1$ (the torque on p_2 due to the field of p_1)

$\theta = \pi/2$

$\vec{E}_1 = \frac{p_1}{4\pi\epsilon_0 r^3}(\hat{\theta})$ and



so $\vec{p}_2 = p_2\hat{x}$ (magnitude + direction)

$= p_2 \frac{p_1}{4\pi\epsilon_0 r^3}(\underbrace{\hat{x}\times\overset{-\hat{z}}{\hat{\theta}}}_{\hat{y}}) \rightarrow \boxed{\vec{N}_2 = \frac{p_2 p_1}{4\pi\epsilon_0 r^3}\hat{y}}$

* net torque not zero because it is defined in a different coordinate system *

$\vec{N}_1 = \vec{p}_1 \times \vec{E}_2$ (torque on $\vec{p}_1$ due to the field of p_2)

Find $\vec{E}_{dip}$ at the position of p_1's center

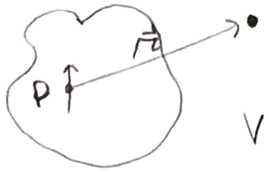
$\vec{E}_{dip} = \frac{p_2}{4\pi\epsilon_0 r^3}(2\cos(\pi)\,\hat{r} + \sin(\pi)\,\hat{\theta}) = -\frac{2p_2}{4\pi\epsilon_0 r^3}\hat{r} \leftarrow \hat{z} = -\frac{2p_2}{4\pi\epsilon_0 r^3}\hat{z}$

$\vec{N}_1 = \vec{p}_1 \times \vec{E}_1 = p_1(-\hat{x}) \times \frac{2p_2}{4\pi\epsilon_0 r^3}(-\hat{z}) = \boxed{\frac{2p_1 p_2}{4\pi\epsilon_0 r^3}\hat{y} = \vec{N}_1}$

Polarization

- putting a piece of dielectric in a field (the field the polarization itself causes)

$$\vec{P} = \vec{p}\, d\tau \rightarrow$$



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

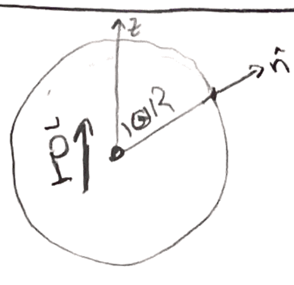
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{Vol} \frac{\vec{P}(\vec{r}\,')\cdot\hat{r}}{r^2}\, d\tau'$$

Doing some math to this!

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\sigma_b}{r}\, da' + \frac{1}{4\pi\varepsilon_0} \int_{Vol} \frac{\rho_b}{r}\, d\tau' \quad \text{where} \quad \sigma_b = \vec{P}\cdot\hat{n}, \quad \rho_b = -\vec{\nabla}\cdot\vec{P}}$$

Examples

Find the $\vec{E}$ field of a uniformly polarized sphere



- choose $\hat{z}$ to coincide w/ the polarization

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\left(0 + 0 + \frac{\partial}{\partial z}P\right) = 0$$

$$\vec{P} = P\hat{z}$$

(This just like a charged spherical shell)

$$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$$

($\hat{z}\cdot\hat{r}$)

We calculated this using laplaces equation in example 3.9, pg 149 $\rightarrow$

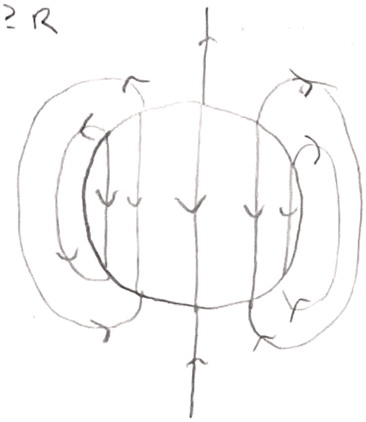
$$V(r,\theta) = \begin{cases} \dfrac{P}{3\varepsilon_0}\, r\cos\theta & r \le R \\ \dfrac{P}{3\varepsilon_0} \dfrac{R^3}{r^2}\cos\theta & r \ge R \end{cases}$$

($z = r\cos\theta$)

Inside

$$\vec{E} = -\vec{\nabla}V = -\frac{P}{3\varepsilon_0}\hat{z} = -\frac{1}{3\varepsilon_0}\vec{P} \quad \text{(constant)}$$

($\frac{\partial}{\partial z}\hat{z}$)



outside

$$V = \frac{P}{3\varepsilon_0} \frac{R^3}{r^2}\cos\theta, \qquad \text{(perfect dipole)} \quad V_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

equal if $\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$

Better approach

(this what we really have here) $\rightarrow$ (two superimposed spheres w/ one $+q$ and other $-q$)

(inside)

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \rightarrow 4\pi r_+^2 E_+ = \frac{4}{3}\pi r_+^3 \rho_+$$

(same radius within positive circle)

$$E_+ = \frac{\rho_+ r_+}{3\varepsilon_0}$$

By same logic! $E_- = \frac{\rho_- r_-}{3\varepsilon_0} \rightarrow \vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\varepsilon_0}\underbrace{(\vec{r}_+ - \vec{r}_-)}_{-d}$

$$\vec{E} = -\frac{\vec{d}\rho}{3\varepsilon_0}, \quad \rho = \frac{q}{\frac{4}{3}\pi R^3} \rightarrow \vec{E} = -\frac{1}{4\pi\varepsilon_0}\frac{q\vec{d}}{R^3}$$

but $\vec{p} = q\vec{d} = \left(\frac{4}{3}\pi R^3\right)\vec{P} \rightarrow \boxed{\vec{E} = -\frac{1}{3\varepsilon_0}\vec{P}}$

Field inside a Dielectric

equations used have been for pure dipoles, not physical ones
↳ we have mostly been dealing w/ physical dipoles

Outside we are far away. r is much bigger than d

Inside we aren't far away, what do we do?

→ microscopic field is fantastically complicated
↳ what we do is average it, like when calculating density

Average over regions large enough to contain many thousand atoms.

Calculating the macroscopic field at $\vec{r}$ within a dielectric:

the macroscopic field over an appropriate volume

↳ must average the true microscopic field over an appropriate volume (lets say a few thousand atoms)

[In the dielectric we have our "point" $\vec{r}$ that is a sphere] → [Macroscopic field consists of two parts avg field due to charges ↳ outside & inside] → $\vec{E} = \vec{E}_{out} + \vec{E}_{in}$

is r from the center of $\vec{r}$?

[$\vec{E}_{out}$ is the field at $\vec{r}$ due to the dipoles outside sphere] → [far enough away we can use the perfect dipole approx felt by $\vec{r}$] → $V_{out} = \frac{1}{4\pi\varepsilon_0}\int_{outside} \frac{\vec{P}(\vec{r}\,')\cdot\hat{r}}{r^2}\, d\tau'$

↳ bc/ Prob 3.47(d), proved that the avg field (over a sphere) produced by charges outside is equal to the field they produced at the center

[For $\vec{E}_{in}$ we cant do the same trick (too close)] → [but all we need is the average field] → $\vec{E}_{in} = -\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}}{R^3}$ | total dipole moment which is $\vec{p} = (\text{Volume})\,\vec{P}$

↳ $\vec{E}_{in} = -\frac{1}{3\varepsilon_0}\vec{P}$, big $\vec{P}$, doesnt vary a lot over this volume.

[Since $\vec{P}$ doesnt vary much, we can say its uniform, then we can use result we found for $\vec{E}$ on the inside & compare] → Remarkably $\vec{E} = -\frac{1}{3\varepsilon_0}\vec{P}$ page 175

So we have V_{out} given by that integral, for a sphere the E field inside is $-\frac{1}{3\varepsilon_0}\vec{P}$, the same as the average we said the inside, since they are equivalent we can just use the whole integral for

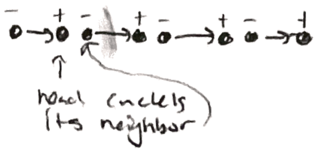
$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}(\vec{r}\,')\cdot\hat{r}}{r^2}\, d\tau'$$

Physical interpretation of bound charges

- σ_b and ρ_b the "bound charges", we dont understand their physical meaning yet
- They represent accumulation of charge

Origin of σ_b

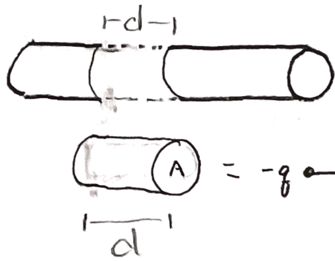
[Imagine a bunch of tiny dipoles]



your left with

*net charge at the ends are "bound charges" can't be removed

[Now lets examine a tube of dielectric material parallel to $\vec{P}$]



$= -q \bullet\!\!-\!\!\bullet +q$

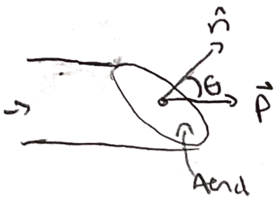
$q = PA$

and

$p_{\text{little}} = (PA)\,d$, dipole

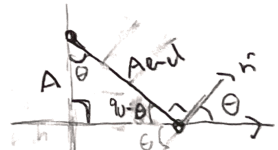
By definition: $\sigma_b = \frac{\text{charge}}{\text{Area}} = \frac{q}{A}$

[Make it more general for an oblique cut] $\rightarrow$



[now its] $\rightarrow \sigma_b = \frac{q}{A_{end}}$,

what is Aend?



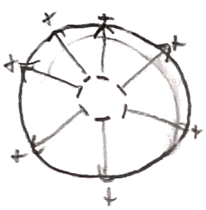
$\frac{A}{A_{end}} = \cos\theta$

$A_{end} = \frac{A}{\cos\theta}$

Thus: $\sigma_b = \frac{q}{A}\cos\theta = P\cos\theta = \vec{P}\cdot\hat{n}$ ← definition of dot product

If polarization is non uniform we now got accumulations of bound charges within the material, and the surface.

*key is the charge pile up from the middle is left upon the surface and its equal & opposite (pile up I mean this idea)
[if lined up P_{dir} if not $\cos(\theta)$ proportion]



$$\underbrace{\int \rho_b\, d\tau}_{\text{net charge on the inside}} = -\underbrace{\oint \vec{P}\cdot d\vec{a}}_{\text{negative surface charge}} = \underbrace{-\int_V (\vec{\nabla}\cdot\vec{P})\, d\tau}_{\text{divergence theorem}}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

factor ε_0 has been extracted to make

electric displacement

We have found $\rho_b = -\vec{\nabla}\cdot\vec{P}$ and $\sigma_b = \vec{P}\cdot\hat{n}$ which is the bound charge due to polarization

↳ put it all together: $\rho = \rho_b + \rho_f$ (ρ_f is free charge on a conductor for example)

→ ρ_f is any charge that is not a result of polarization

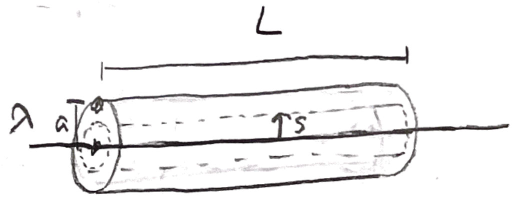
Gauss's Law: $\frac{\rho}{\varepsilon_0} = \vec{\nabla}\cdot\vec{E}$ → and $\rho = \rho_b + \rho_f = -\vec{\nabla}\cdot\vec{P} + \rho_f$

$\uparrow$ total field

↳ $\rho_f = \vec{\nabla}\cdot(\underbrace{\varepsilon_0\vec{E} + \vec{P}}_{D})$, D is known as the electric displacement

↳ integral form $\oint \vec{D}\cdot d\vec{a} = Q_{fencl}$ (total free charge enclosed in the volume)

(ex) a long straight wire w/ λ, is surrounded by rubber insulation of radius a, Find the electric displacement $\vec{D}$.



• we enclose the free charge λ

$\oint \vec{D}\cdot d\vec{a} = Q_{fencl}$ → $\oint D\, da = Q_{fencl}$

constant, uniform λ

$D\oint da = \int \lambda\, dl$

$D(2\pi s L) = \lambda L$ → $\vec{D} = \frac{\lambda}{2\pi s}\hat{s}$

the density of dipoles is 0, theres no charge

$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

• outside the insulation $\vec{P} = 0$, so it holds like normal Gauss law since

↳ $\vec{E} = \frac{1}{\varepsilon_0}\vec{D} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$ for $s > a$

A deceptive parallel

$\vec{D}(\vec{r}) \neq \frac{1}{4\pi}\int \frac{\hat{r}}{r^2}\rho_f(\vec{r}')d\tau'$, the parallel between $\vec{E}$ + $\vec{D}$ is more subtle

- divergence alone is insufficient to determine a vector field, you also need to know the curl

$\vec{\nabla}\times\vec{D} = \varepsilon_0(\vec{\nabla}\times\vec{E}) + \vec{\nabla}\times\vec{P} = \vec{\nabla}\times\vec{P}$, curl of $\vec{E}$ is always zero for electrostatic fields

↳ look for symetry (spherical, cylindrical, plane symetry) then you can use the Gauss law. $\vec{\nabla}\times\vec{P}$ is evidently 0 in these configurations,

↳ if symetry is gone you'll have to think of another approach. (can't assume that $\vec{D}$ is determined exclusively by the free charge)

Boundary conditions

From $E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$ → $D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$ ← says $\vec{P}_{\perp} = 0$

$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$ → you get → $D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$

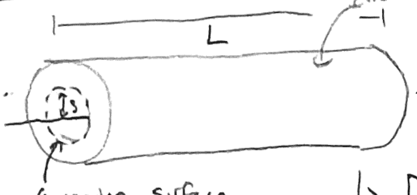
factor ε_0 has been extracted to make ~ dimensionless

...ctric Displacement

$\rho_{tot} = \rho_{bound} + \rho_{free}$ and $\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho_{tot} = \rho_b + \rho_f = -\vec{\nabla}\cdot\vec{P} + \rho_f$

$\hookrightarrow \vec{\nabla}\cdot\underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{\vec{D}} = \rho_f \rightarrow \boxed{\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}}$ with $\boxed{\oint \vec{D}\cdot d\vec{a} = Q_{f\,encl}}$

Examples



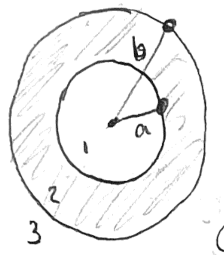
Find $\vec{D}$: [enclose the free charge λ] $\rightarrow \oint \vec{D}\cdot d\vec{a} = Q_{f\,encl}$

$\int \lambda\, dl$

$\hookrightarrow$ $\vec{E}$ and $\vec{P}$ point in the same direction for uniform λ (think neutral atom)

$\hookrightarrow D(2\pi S L) = \lambda L \rightarrow \boxed{\vec{D} = \frac{\lambda}{2\pi S}\hat{s}} \leftarrow$ holds for inside insulation and outside, outside $\vec{P} = 0$

inside $\vec{E}$ can't be determined since we don't know P $\rightarrow$



$\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$

Calculate bound charges and use gauss's law

$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right)\right) = \boxed{\frac{-k}{r^2}}$

$\sigma_b = \vec{P}\cdot\hat{n}\Big|_{r=?}$ $\Rightarrow$ there is two $\rightarrow \sigma_{ba} = \frac{k}{r}\hat{r}\cdot\overset{\hat{n}}{(-\hat{r})} = -\frac{k}{r}\Big|_{r=a} = \boxed{-\frac{k}{a}}$, $\boxed{\sigma_{bb} = \frac{k}{b}}$

getting charges

$q_{\rho} = \int \rho_b\, d\tau = 4\pi \int_a^r -\frac{k}{r^2} r^2\, dr = -4\pi k(r-a)$

$q_{ba} = (4\pi a^2)\sigma_{ba} = -4\pi k a$

$q_{bb} = (4\pi b^2)\sigma_{bb} = 4\pi k b$

add all charge $\rightarrow -4\pi k(b-a) + 4\pi k(b-a) = 0$ ✓

$r<a$: $E_1 = 0$

$a<r<b$: $\oint \vec{E}_2\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \rightarrow E_2 4\pi r^2 = \frac{Q_{encl}}{\varepsilon_0}$

$E_2 4\pi r^2 = \frac{1}{\varepsilon_0}\left(-4\pi k(r-a) - 4\pi k a\right)$

$E_2 \cancel{4\pi} r^2 = \frac{4\pi k}{\varepsilon_0}\left(\cancel{a} - r - \cancel{a}\right) = -\frac{4\pi k}{\varepsilon_0} r$

$\boxed{\vec{E}_2 = \frac{-k}{\varepsilon_0}\frac{1}{r}\hat{r}}$ and $r>b$ $Q_{encl} = 0$

Fast way with $\vec{D}$! $\vec{D} = \varepsilon_0\vec{E} + \vec{P} = 0 \leftarrow$ because no free charge $\rightarrow \oint \vec{D}\cdot d\vec{a} = Q_f = 0$

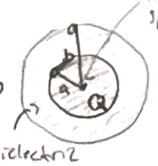
$\vec{P} = -\varepsilon_0\vec{E} \rightarrow \boxed{-\frac{k}{\varepsilon_0 r}\hat{r} = \vec{E}}$, only place $P \neq 0$ is inside dielectric boom!

Linear dielectrics

↳ in this case: $\vec{P} = \varepsilon_0 \chi_e \vec{E} \rightarrow \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi_e)\vec{E} = \varepsilon \vec{E}$

and $\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}$

[Full example of this on other sheet, just quick calculation to show its use] →



* Find V at the center *

↳ we need to find $\vec{E}$ first

↳ could get bound charges from $\vec{P}$

↳ but to get $\vec{P}$ we need to know $\vec{E}$ $(\vec{P} = \varepsilon_0 \chi_e \vec{E})$

from $\oint \vec{D} \cdot d\vec{a} = Q_f = Q$

$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$ for all points $r > a$

$\underline{r<a}$: $\boxed{\vec{E} = 0}$

$\underline{a<r<b}$: $\vec{D} = \varepsilon \vec{E} \rightarrow \boxed{\vec{E} = \frac{Q}{4\pi \varepsilon r^2}\hat{r}}$

$\underline{r>b}$: $\vec{P} = 0 \rightarrow \vec{D} = \varepsilon_0 \vec{E} \rightarrow \boxed{\vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2}\hat{r}}$

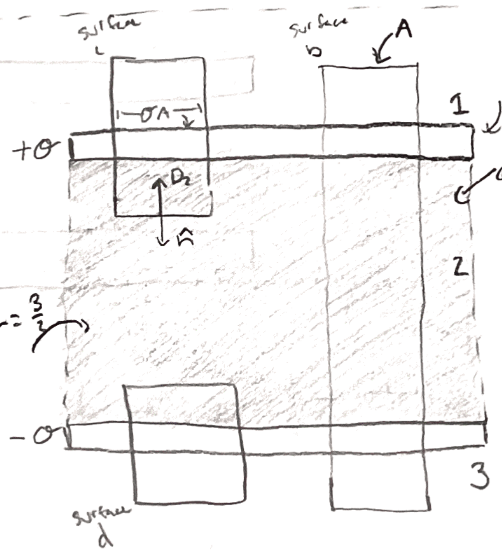
$$\boxed{\vec{E} = \begin{cases} 0 & r<a \\ \frac{Q}{4\pi \varepsilon r^2}\hat{r} & a<r<b \\ \frac{Q}{4\pi \varepsilon_0 r^2}\hat{r} & r>b \end{cases}}$$

↙ could calc $\vec{P}$ from $\vec{E}$

now then get bound charges

$$V_{center} = -\int_{\infty}^{0} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{b} \frac{Q}{4\pi \varepsilon_0 r^2}\,dr - \int_{b}^{a} \frac{Q}{4\pi \varepsilon r^2}\,dr - \int_{a}^{0} 0\,dr = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right)$$

reference ↗ ∞



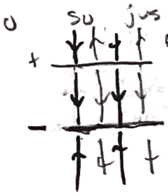
sides don't contribute ↘

$\oint_b \vec{D} \cdot d\vec{a} = Q_f = 0$

$D_1 A - D_3 A = 0 \rightarrow D_1 = D_3$

$\boxed{D_1 = D_3 = 0} \rightarrow \boxed{E_1 = E_3 = 0}$

$P = 0$



$\oint_c \vec{D} \cdot d\vec{a} = \cancel{D_1 A}^{\,0} - D_2 A = \underbrace{\sigma A}_{\text{free charge}} \rightarrow D_2 = -\sigma$ and $\vec{D}_2 = \varepsilon \vec{E}_2$

$\vec{D}_2 = -\sigma \hat{z}$

↳ $\boxed{-\frac{\sigma}{\varepsilon}\hat{z} = \vec{E}_2}$

$\boxed{\vec{E}_2 = -\frac{2}{3\varepsilon_0}\sigma \hat{z}}$

$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} \quad \frac{1}{\varepsilon_r} = \frac{\varepsilon_0}{\varepsilon} = \frac{2}{3} \rightarrow \frac{1}{\varepsilon} = \frac{2}{3\varepsilon_0}$

...r Dielectrics

factor ε_0 has been extracted to make χ dimensionless

...vided $\vec{E}$ is not too strong $\vec{P} \propto \vec{E}$ so $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ ← electric susceptibility

χ_e depends on the microscopic structure, temperature, etc.

- $\vec{E}$ is the total field, it may be due to free charges and the polarization

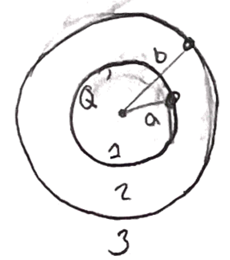
↳ for instance if we put a dielectric into an external field $\vec{E}_0$, we can't compute $\vec{P}$ w/ $\vec{P} = \varepsilon_0 \chi_e \vec{E}$, the external field will polarize the material, this sets up its own field (then it contributes to the total field), which then modifies the polarization... (infinite regress, but breaking out first easy example later).

(In order to solve problems we start with displacement) → $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E}$

↳ $\vec{D} = \varepsilon \vec{E}$ where $\varepsilon \equiv \varepsilon_0 (1 + \chi_e)$ (permittivity), ε_0 = (permittivity of free space)

and $1 + \chi_e = \dfrac{\varepsilon}{\varepsilon_0} = \varepsilon_r$ or (relative permittivity) or (dielectric constant)

Example [metal sphere of radius a carries a charge Q] and [it is surrounded by a linear dielectric material of permittivity ε] Find V at the center.



Find $\vec{E}$ then V. find $\vec{D}$ bec $Q_f = Q$

$\oint \vec{D} \cdot d\vec{a} = Q_f$ 3 regions $\begin{cases} r<a: & Q_f = 0 \text{ so } \vec{D} = 0 \\ a<r<b: & Q_f = Q \to D\, 4\pi r^2 = Q \to \vec{D} = \dfrac{Q}{4\pi r^2}\hat{r} \\ r>b: & Q_f = Q \to D\, 4\pi r^2 = Q \Rightarrow \vec{D} = \dfrac{Q}{4\pi r^2}\hat{r} \end{cases}$

$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$ where $\vec{P}$ is a linear dielectric

$\boxed{r<a \to \vec{E} = 0}$, $\boxed{r>b \to \vec{P} = 0 \text{ so } \vec{D} = \varepsilon_0 \vec{E} \text{ or } \vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}}$ no charge out there

$\boxed{a<r<b \to \text{this in the dielectric so } \vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E} \text{ so } \vec{E} = \dfrac{Q}{4\pi\varepsilon r^2}\hat{r}}$

$$V = \underbrace{-\int_{\infty}^{0} \vec{E} \cdot d\vec{l}}_{\text{from } \infty \text{ to the center}} = \underbrace{-\int_{\infty}^{b} \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right) dr}_{\text{from } \infty \text{ to surface}} \underbrace{- \int_{b}^{a} \left(\frac{Q}{4\pi\varepsilon r^2}\right) dr}_{\text{from surface to}} \underbrace{- \int_{a}^{0} 0\, dr}_{\text{inside sphere}}$$

Potential at the center relative to ∞ (0 potential)

$$= \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} + \frac{1}{\varepsilon b}\right)$$

(We can calc $\vec{P}$) → $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \dfrac{\varepsilon_0 \chi_e Q}{4\pi \varepsilon r^2}\hat{r}$

$\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$ and $\sigma_b = \vec{P} \cdot \hat{n} = \begin{cases} \dfrac{\varepsilon_0}{\varepsilon} \chi_e \dfrac{Q}{4\pi b^2} & \text{(outer surface)} \\ -\dfrac{\varepsilon_0}{\varepsilon} \chi_e \dfrac{Q}{4\pi a^2} & \text{(inner surface)} \end{cases}$