Electrostatics

- goal → lets say we have some source charges, what force do they exert on a "test charge" Q

Principle of Superposition

* only works bec em force $\propto q$ and not q^2 *

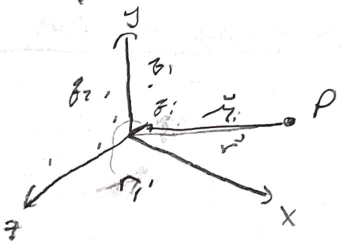
* interaction between any two charges is unaffected by the presence of others

↳ To determine the force on Q, you can first compute $\vec{F}_1$ due to q_1 alone then $\vec{F}_2$ due to q_2 etc...

↳ Then you can do $\vec{F}_{tot} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots$

Electromagnetic "News" travels at the speed of light, so what concerns Q is the position, velocity and acceleration q had at some earlier time.
This is hard we start w/ Electrostatics first (all source charges are stationary)

Coulombs Law (for source charge that is stationary)



$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$$, $\vec{r} = \vec{r} - \vec{r}'$

location of Q ($\vec{r}$), location of q ($\vec{r}'$)

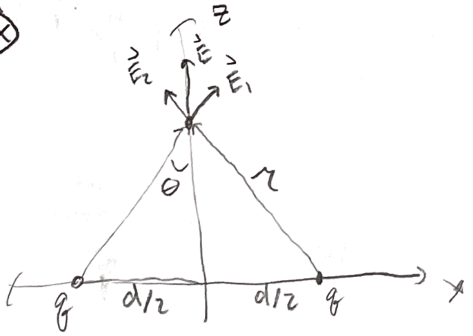
Electric Field

Lets say $\vec{F} = \vec{F}_1 + \vec{F}_2 + \dots = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2}\hat{r}_2 + \frac{q_3}{r_3}\hat{r}_3 + \dots\right)$

↳ $\vec{F} = Q\vec{E}$ → $\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum \frac{q_i}{r_i^2}\hat{r}_i}$ electric field of the source charges

The electric field is a vector quanity that varies from point to point, and is determined by the configuration of source charges; physically, $\vec{E}(\vec{r})$ is the force per unit charge that would be exerted on a test charge placed at point P.

ex



$$E_z = 2\,\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta$$, only vertical components contribute

$$r = \sqrt{z^2 + (d/2)^2} \quad \text{and} \quad \cos\theta = \frac{z}{r}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left[z^2 + (d/2)^2\right]^{3/2}}\hat{z}$$

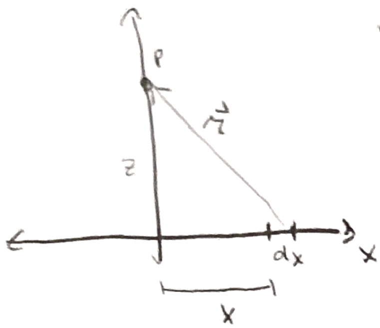
Continuous charge distributions

different charge densities

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\, dq \quad , \quad dq \to \lambda\, dl' \to \sigma\, da' \to \rho\, d\tau'$$

$\mathscr{r}$ is the vector from dq to the field point $\vec{r}$.

ex



rod length = 2L w/ uniform charge λ

$\vec{r} = z\hat{z}$, $r' = x\hat{x}$, $dl = dx$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = z\hat{z} - x\hat{x}$, $\mathscr{r} = \sqrt{x^2+z^2}$, $\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{z\hat{z} - x\hat{x}}{\sqrt{x^2+z^2}}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L} \frac{\lambda}{z^2+x^2}\, \frac{z\hat{z} - x\hat{x}}{\sqrt{x^2+z^2}}\, dx$$

$L \to \infty$

$E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[z\hat{z}\int_{-L}^{L} \frac{dx}{(z^2+x^2)^{3/2}} - \hat{x}\int_{-L}^{L}\frac{x\, dx}{(z^2+x^2)^{3/2}}\right] = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{z}}$$

2.2 Divergence and Curl of Electrostatic Fields

- integrals of computing the force on a charge Q through $E(\vec{r}) = -\int -$ can be hard, so there are tricks w/ the divergence + curl of the field.

[Point charge at the origin] $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ (page 66, 67 for field lines understanding)

Flux

$$\Phi_{\vec{E}} = \int_{\text{surface}} \vec{E}\cdot d\vec{a}$$

measure of the number of field lines passing through a surface.

[Field strength $\propto$ density of field lines]

$\vec{E}\cdot d\vec{a} \propto$ # of lines passing through $d\vec{a}$

→ dot product picks out the component of $d\vec{a}$ along the direction of $\vec{E}$

Through a closed surface !!!

↳ since number of field lines tells you how strong the field is, total flux is a measure of the total charge inside

- charge outside would go through the surface and out the other, so it wouldn't contribute
- Charge inside would have field lines going out of surface, unless it was a +, − inside

Gauss's Law

* surface are of sphere goes up by r^2
* E field lines goes down as $1/r^2$

$$\oint \vec{E}\cdot d\vec{a} = \int_{\text{sphere}} \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\, \hat{r}\right) = \boxed{\frac{q}{\varepsilon_0}}$$

[for a bunch of charges scattered] → $\vec{E} = \underbrace{\sum_{i=1}^{n}\vec{E}_i}_{\text{principle of superposition}}$ → $\oint \vec{E}\cdot d\vec{a} = \sum_{i=1}^{n}\oint \vec{E}_i\cdot d\vec{a} = \sum_{i=1}^{n}\frac{q_i}{\varepsilon_0} = \frac{Q_{encl}}{\varepsilon_0}$ ← charges enclosed within the surface

Differential form by divergence theorem

$$\oint_S \vec{E}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{E}\, d\tau \quad , \quad Q_{encl} = \int_V \rho\, d\tau \;\; (\rho:\text{ charge density}) \quad \to \quad \frac{Q_{enc}}{\varepsilon_0} = \int_V \vec{\nabla}\cdot\vec{E}\, d\tau$$

$$\hookrightarrow \int_V \frac{\rho}{\varepsilon_0}\, d\tau = \int_V \vec{\nabla}\cdot\vec{E}\, d\tau \to \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}} \quad \text{differential form}$$

[Div]ergence of E

example on Page 71, 72

③

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\vec{r}')\,d\tau' \rightarrow \vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\rho(r')\,d\tau'$$

(We use the formula) $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})$, the divergence of $\frac{\hat{r}}{r^2} = 0$ but its integral is 4π defines the dirac delta function

$$\hookrightarrow \vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int 4\pi\,\delta^3(\vec{r}-\vec{r}')\rho(r')\,d\tau' = \boxed{\frac{\rho(\vec{r})}{\varepsilon_0}}$$

$$\left\{ \int_V \vec{\nabla}\cdot\vec{E}\,d\tau = \oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int_V \rho\,d\tau = \frac{Q_{encl}}{\varepsilon_0} \right\}$$

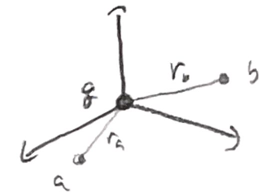
[Important example problems on page 71, 72, 73, 74]

Curl of $\vec{E}$

$$\to d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr$$

(Curl of a point charge) $\rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$, $\int_a^b \vec{E}\cdot d\vec{l}$

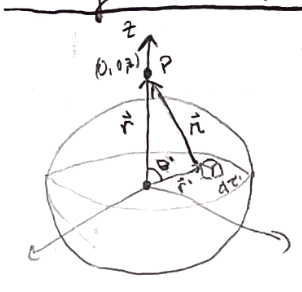


$$\int_a^b \vec{E}\cdot d\vec{l} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\Big|_{r_a}^{r_b} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

for a closed path point $a = b$, $\oint \vec{E}\cdot d\vec{l} = 0$, $r_a = r_b$

if $\boxed{\oint \vec{E}\cdot d\vec{l} = 0 = \vec{\nabla}\times\vec{E} = 0}$ by stokes theorem; $\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint \vec{v}\cdot d\vec{l}$

Finding $\vec{E}$ examples

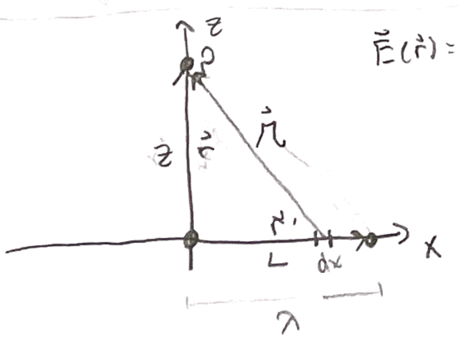


$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\overbrace{\rho(\vec{r}')\,d\tau'}^{dq}$$

length of r is a constant, z — dot product — integration over r'

$$|\vec{\mathscr{r}}|^2 = (\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}') = (\vec{r}\cdot\vec{r}) - 2\vec{r}\cdot\vec{r}' + ((\vec{r}'\cdot\vec{r}') = z - 2z\,r'\cos\theta' + (r')^2$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\mathscr{r}|} = \frac{(\vec{r}-\vec{r}')}{\sqrt{z - 2zr'\cos\theta' + (r')^2}} = \frac{z\hat{z} - r'\hat{r}}{|\mathscr{r}|}$$



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\overbrace{\lambda\,dx}^{dq}$$

$$\mathscr{r} = \sqrt{x^2+z^2}, \quad \hat{\mathscr{r}} = \frac{-x\hat{x} + z\hat{z}}{\sqrt{x^2+z^2}}$$

$$\rightarrow E(\vec{r}) = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{-x\hat{x}+z\hat{z}}{(x^2+z^2)^{3/2}}\,dx$$

integrate

Electric Potential

- $\vec{E}$ is special, its curl is always 0, $\vec{E} = y\hat{x}$ wouldn't exist, or at least be a electros[tatic]
  by stokes theorem $\oint \vec{E} \cdot d\vec{l} = 0$

- Since $\vec{\nabla} \times \vec{E} = 0$ then

from fund. theorem gradient

$$V(\vec{r}) \equiv -\int_{origin}^{\vec{r}} \vec{E} \cdot d\vec{l} \quad , \quad V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b (\vec{\nabla} V) \, d\vec{l}$$

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

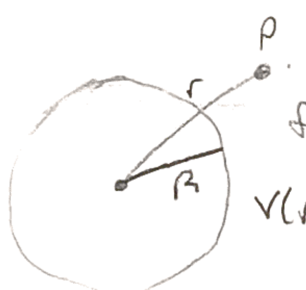
Curl info

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \quad , \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} \quad , \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

ex uniform plane

$$V(z) = -\int_{\substack{ref \\ point}}^{z} \frac{1}{2\varepsilon_0} \sigma \, dz = -\frac{1}{2\varepsilon_0} \sigma (z - \text{ref point}) \quad , \quad V = \frac{\text{Joules}}{\text{Coulombs}} = \text{volt}$$

ex Find potential inside + outside a sphere



$r > R$

from gauss, $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

$$V(r) = -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0} \frac{q}{r'^2} dr' = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

$r < R$

from infinity to surface of sphere

from surface to inside

$$V(r) = \frac{-1}{4\pi\varepsilon_0} \int_{\infty}^{R} \frac{q}{(r')^2} dr' - \int_R^r 0 \, dr'$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{R}$$

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examples

Poisson's Equation and Laplace's Equation

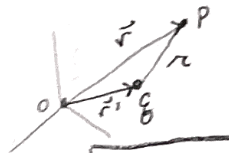
$\vec{E} = -\vec{\nabla} V$ what are $\vec{\nabla}\cdot\vec{E}$ & $\vec{\nabla}\times\vec{E}$ in terms of the potential

$\nabla \times E = 0$

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \rightarrow \vec{\nabla}\cdot(-\vec{\nabla} V) = -\nabla^2 V \rightarrow \nabla^2 V = \frac{-\rho}{\varepsilon_0}$, places w/ no charge $\nabla^2 V = 0$

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ and $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{(r')^2}dr' = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r'}\right)\Bigg|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

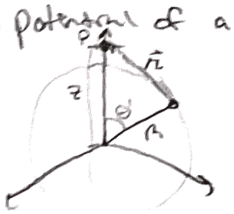


$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}} \leftarrow$

Superposition Principle $\rightarrow$ $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$

$$\hookrightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathscr{r}}dq = \boxed{\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')}{\mathscr{r}}d\tau' \text{ or } V = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(r')}{\mathscr{r}}dl' \text{ and } V = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(r')}{\mathscr{r}}da'}$$

ex) potential of a shell



$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma}{\mathscr{r}}da'$, $\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{R^2\sin\theta'\,d\theta'\,d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}} = \frac{R\sigma}{2\varepsilon_0 z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

$z > R$ (outside)

$V(z) = \frac{R\sigma}{2\varepsilon_0 z}\left[(R+z)-(z-R)\right] = \frac{R^2\sigma}{\varepsilon_0 z}$

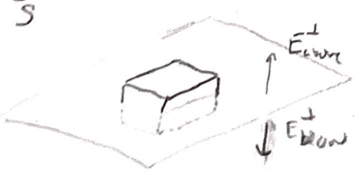
$z < R$ (inside)

$V(z) = \frac{R\sigma}{2\varepsilon_0 z}\left[(R+z)-(R-z)\right] = \frac{R\sigma}{\varepsilon}$

$q = 4\pi R^2\sigma$

$$V(r) = \begin{cases}\frac{1}{4\pi\varepsilon_0}\frac{q}{r} & r \geq R \\ \frac{1}{4\pi\varepsilon_0}\frac{q}{R} & r \leq R\end{cases}$$

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc} = \frac{\sigma A}{\varepsilon_0} \rightarrow 2AE = \frac{\sigma A}{\varepsilon_0} \rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

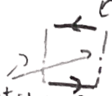


Lets look at only the $\perp$ component:

$E^{\perp}_{above} = \frac{\sigma}{2\varepsilon_0}$ (up) , $E^{\perp}_{below} = \frac{-\sigma}{2\varepsilon_0}$ (down) $\rightarrow E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$

Lets look at only the $\parallel$ component:

for a loop $\oint\vec{E}\cdot d\vec{l} = 0$, we only care about



these bec, its the parallel part

dont contribute

thus $E^{\parallel}_{above} = E^{\parallel}_{below}$ so this can hold

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

Potential Boundary condition

$$V_{above} - V_{below} = -\int_c^b \vec{E} \cdot d\vec{l} = 0$$

from $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ $\rightarrow$ $\nabla V_{above} - \nabla V_{below} = -\frac{\sigma}{\varepsilon_0}\hat{n}$

dot each side w/ $\hat{n}$

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

normal derivative, rate of change in the direction $\perp$ to the surface

Work and Energy

- have a few q's and you want to move Q from a to b
- $\vec{F} = Q\vec{E}$

minus sign cuz you exert force in opposition to the electrical force

if $V(a) = V(\infty)$ then $V = W/Q$

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q\int_c^b \vec{E} \cdot d\vec{l} = Q[V(b) - V(a)] \rightarrow V(b) - V(a) = \frac{W}{Q}$$

work to assemble a collection of charges

- q_1 takes no work since there is no charges to fight against
- $W_2 = q_2 V_1(\vec{r}_2) = \frac{1}{4\pi\varepsilon_0} q_2\left(\frac{q_1}{r_{12}}\right) \rightarrow W_3 = q_3 V_{1,2}(\vec{r}_3) = \frac{1}{4\pi\varepsilon_0} q_3\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$

place we are putting q_2

potential due to both charges

potential of q_1

distance between q_1 and q_2

$\hookrightarrow W_4 = \frac{1}{4\pi\varepsilon_0} q_4\left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right)$

$$W_{tot} = W_1 + W_2 + W_3 + W_4 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right)$$

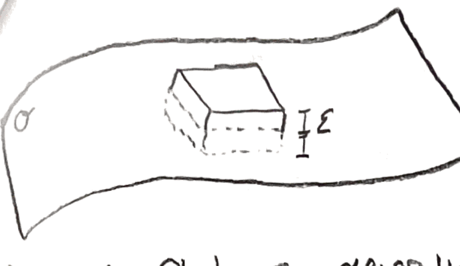
$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{r_{ij}}$$, $j > i$ is to remind you to not count the same pair twice

$$W = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{r_{ij}}$$ ← count em twice & divide by two ← now you can add em up in any order

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \sum_{j\neq i}^{n}\left(\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}\right)$$ ← can pull out q_i bec its like a double for loop ↳ up once in first sum then loop the second sum

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$$

ailing down this boundary condition shit



$Q_{encl} = \sigma A$

$$\oint_S \vec{E}\cdot d\vec{a} = \oint E\,dA = 2EA = \frac{\sigma}{\varepsilon_0}A$$

↑ flux goes out bottom and top of box

$\hookrightarrow E = \frac{\sigma}{2\varepsilon_0}$

* its not flat so generally the sides contribute also to the net flux. But this is a limiting procedure, as $\varepsilon \to 0$, $\Phi_{side} \to 0$ ✱

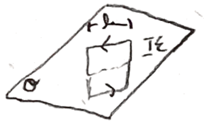
↑ • E field of the sheet, so lets look at the E field at the bottom and top

$E^{\perp}_{top} = \frac{\sigma}{2\varepsilon_0}$, $E^{\perp}_{bott} = \frac{-\sigma}{2\varepsilon_0}$

$\boxed{E^{\perp}_{top} - E^{\perp}_{bot} = \frac{\sigma}{\varepsilon_0}}$ ← how much the E field is discontinous at a boundary

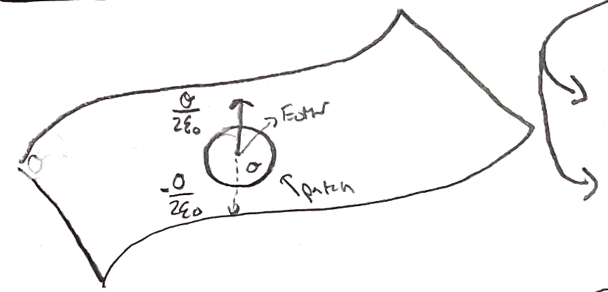
* $E^{\parallel}$ though? $\oint \vec{E}\cdot d\vec{l} = 0$, for a closed loop, Conservative field.

condition in order for $\oint \vec{E}\cdot d\vec{l} = 0$ ↓



$$\int_{below}^{above} \vec{E}\cdot d\vec{l} = \left(E^{\parallel}_{top} - E^{\parallel}_{bot}\right) l \Rightarrow E^{\parallel}_{top} = E^{\parallel}_{bot}$$

Surface charge + force on a conductor



$\pm\frac{\sigma}{2\varepsilon_0}$ ↓

$\vec{E}_{tot} = \vec{E}_{patch} + \vec{E}_{other}$

removed contribution of patch ↓

$$\left.\begin{aligned} \vec{E}_{above} &= \vec{E}_{other} + \frac{\sigma}{2\varepsilon_0}\hat{n} \\ \vec{E}_{below} &= \vec{E}_{other} - \frac{\sigma}{2\varepsilon_0}\hat{n} \end{aligned}\right] \quad E_{other} = E_{average} = \frac{1}{2}\left(E_{above} + E_{below}\right)$$

• In the presence of an E field a surface charge will experience a force, $f = \sigma E$, force per unit area

$\vec{F} = Q\vec{E} = \sigma A\vec{E}$

→ E field is discontinous at a surface charge

↳ discontinous by $\frac{\sigma}{\varepsilon_0}\hat{n}$

$\rightarrow \vec{f} = \sigma\vec{E}_{other} = \sigma\frac{1}{2}\left(E_{above} + E_{below}\right)$

$\vec{E}_{below} = 0$ for a conductor

$\vec{E}_{above} = \frac{\sigma}{\varepsilon_0}\hat{n}$ just outside

$$\vec{f} = \sigma\frac{\sigma}{2\varepsilon_0}\hat{n} = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$

...gy of a continuous distribution

$\int f \frac{dg}{dx} dx = -\int \frac{df}{dx} g + fg\Big|_a^b$

$W = \frac{1}{2}\int \rho V d\tau$

↳ rearrange: $\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$ → $W = \frac{\varepsilon_0}{2}\int (\underset{\frac{dg}{dx}}{\vec{\nabla}\cdot\vec{E}})\underset{f}{V} d\tau$

$W = \frac{\varepsilon_0}{2}\left(-\int \vec{E}\cdot(\nabla V)\, d\tau + \oint V\vec{E}\, d\vec{a}\right)$ ← integration by parts, vector edition

$W = \frac{\varepsilon_0}{2}\left(\int E^2 d\tau + \oint V\vec{E}\cdot d\vec{a}\right)$

↳ from the continuous version of $W = \frac{1}{2}\sum q_i V(r_i)$, you just want to count the number of charges you have, so the continuous version would be just the object w/ the charge,

↳ if you include more you split up integral, where ρ = value and $\rho = 0$ ← extra area

$$\underset{V}{\int} E^2 d\tau + \oint V\vec{E}\cdot d\vec{a} \rightarrow$$

↑ as V goes up this only increases

↑ so this has to decrease for the growth in first term.

$E \sim \frac{1}{r^2}$, $V \sim \frac{1}{r}$, surface area ~ r^2

so $\oint V\vec{E}\cdot d\vec{a} \sim \frac{1}{r}$ goes down by $1/r$ has to go down be E^2 only increasing

↳ $\int E^2 d\tau$ goes up by $1/r$ → $\left(\frac{1}{r^4} r^3\right)$

→ integrate over all space surface integral goes to 0

$$\boxed{W = \frac{\varepsilon_0}{2}\int E^2 d\tau}$$

pg 96 + 97 for more talk about what this equation means!

• surface integral as $r \to \infty$ goes to 0, its away from the source, however the E^2 always contains the ~~source~~ charge

---

interaction energy between two charges! $\int (E_1 + E_2)^2 d\tau \rightarrow \int E_1^2 + \int E_2^2 + 2\int \underbrace{\vec{E}_1\cdot\vec{E}_2}_{\text{interaction energy}} d\tau$

problem 2.37

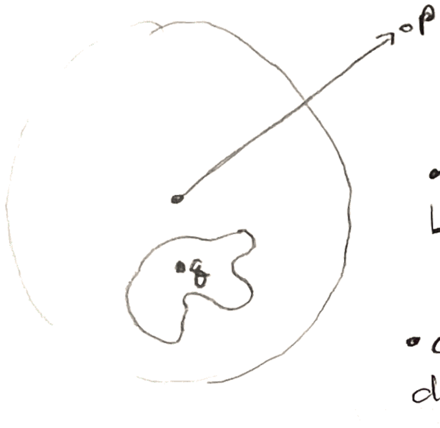
Conductors

- $\vec{E} = 0$ inside a conductor, if there were any field, electrons would arrange themselves to cancel out

↳ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = 0$ so $\rho = 0$

↳ net charge resides on surface

↳ E is $\perp$ to the surface always, otherwise charges would get a component & move



E field of the outside sphere is just

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

- $+q$ induces a charge $-q$ on the wall
  ↳ distributes itself in such a way to cancel the field of q
- conductor feels the $-q$ on the cavity wall displaces over the surface
  ↳ spreads uniformly over the surface

Cavity surrounded by a conductor,
→ field within the cavity $= 0$, any field line begins and ends on the cavity wall

---

Surface Charge & force on a conductor

- Boundary condition; $\vec{E} = \frac{\sigma}{\varepsilon_0} \hat{n} \rightarrow \sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$, force per unit area $= \vec{f} = \sigma \vec{E}$

$\vec{E} = \vec{E}_{patch} + \vec{E}_{other}$ [field due to the patch and the field from everything else]

$$\left.\begin{aligned} \vec{E}_{above} &= \vec{E}_{other} + \frac{\sigma}{2\varepsilon_0}\hat{n} \\ \vec{E}_{below} &= \vec{E}_{other} - \frac{\sigma}{2\varepsilon_0}\hat{n} \end{aligned}\right\} \vec{E}_{other} = \frac{1}{2}(E_{above} + E_{below}) = \vec{E}_{average}$$

- conductor → the field on inside is 0 and $\frac{\sigma}{\varepsilon_0}\hat{n}$ on surface

$$\vec{f} = \sigma \vec{E}_{avg} = \sigma\left(\frac{\sigma}{2\varepsilon_0}\right)\hat{n} = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$

~~average of E field just outside,~~ $E_{above} - E_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ (not same E_{above} & E_{below} ~~just net charge~~, this has no E_{other})

$\frac{F}{A} = \sigma E_{av}$

citors

$$V = V_+ - V_- = -\int_-^+ \vec{E}\cdot d\vec{\ell}$$

- calculating field would be a nightmare, we know $E \propto Q$ by Coulomb's law
- $E \propto Q$ and so is V $\rightarrow$ $C \equiv \frac{Q}{V}$

(ex) find capacitance of two concentric spherical shells, w/ radius a & b

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (field between the spheres)

$V = -\int_a^b \vec{E}\cdot d\vec{\ell} = \frac{-Q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}\,dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \rightarrow C = \frac{Q}{V} = 4\pi\varepsilon_0\frac{ab}{(b-a)}$

- To charge up a capacitor you have to remove electrons from the positive plate & carry them to the negative plate.

↳ fight $\vec{E}$ field which is trying to bring it back

↳ Work to charge a capacitor to a final Q?

$V = \frac{q}{C}$, $W = QV \rightarrow dW = V\,dq \rightarrow dW = \frac{q}{V}\,dq$

$W = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$

Chapter 3 potentials

- calculating E-field is hard
- Calculating V is easier

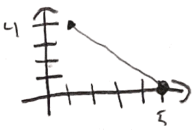
→ [more fruitful in differential form] → $\nabla^2 V = \frac{-\rho}{\varepsilon_0}$

↳ [concern ourself in regions w/ $\rho = 0$] → $\nabla^2 V = 0$ (places w no charge) → $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

One dimension

↳ $\frac{d^2 V}{dx^2} = 0$ → $V = mx + b$, needs 2 boundary conditions

$V(1) = 4$, $V(5) = 0$] $m = -1$, $b = 5$ → $V = -x + 5$



$V(x+a)$ and $V(x-a)$ → $V(x) = \frac{1}{2}(V(x+a) + V(x-a))$

1. $V(x)$ is the average of $V(x+a)$ and $V(x-a)$

↳ Check; $V(x) = \frac{1}{2}(m(x+a) + b + m(x-a) + b) = \frac{1}{2}(2mx + 2b) = mx + b$ ✓

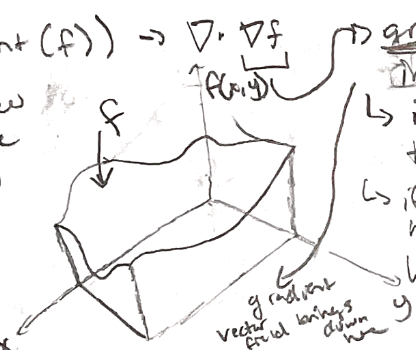
↳ Laplaces Equations tells you to assign the average values to the left and to the right of x

↳ tolerates no local maxima or minima, etreme values of V must occur at the end points

↳ If there were a local maximum, V would be greater at that point than on either side, which cant be an average

divergence(gradient (f)) → $\nabla \cdot \nabla f$ → gradient gives you slope of steepest incline

divergence tells you to think of the arrows as water flow. If water is going away div is positive



↳ if theres a hill, the gradient field tells you to go toward top of hill

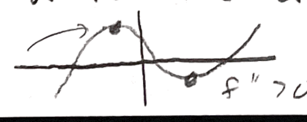
↳ if theres a bowl, the vectors tell you to walk away from, (the dir to increase the value the most rapidly)

↳ when the divergence is positive, is when the vector field is moving away, this comes from the gradient, on the function $f(x,y)$ that corresponds to a bowl

↳ neg div, vector field converging, corresponds to a hill

* kind of a measure of how much of a minimum point is this x, y

analogous to;



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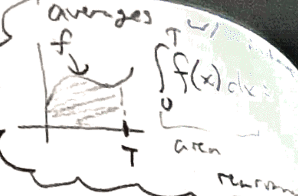
Two dimensions

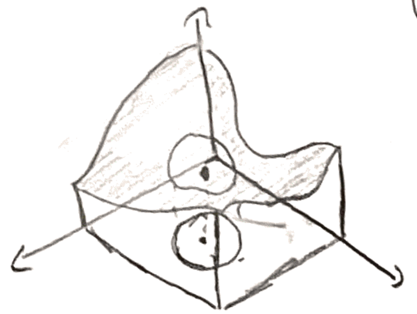
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = \frac{1}{2\pi R}\oint_{\text{circle}} V\,dl$$

[a loop around the point + averaged (what you have to do for div)]

averages



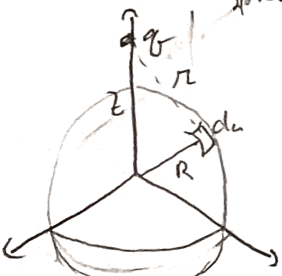


Three Dimensions

$$V(\vec{r}) = \frac{1}{4\pi R^2}\oint_{\text{sphere}} V\,da$$

↳ extreme value for V must occur at the boundaries.

Average Potential over a spherical surface of a single charge located outside the sphere

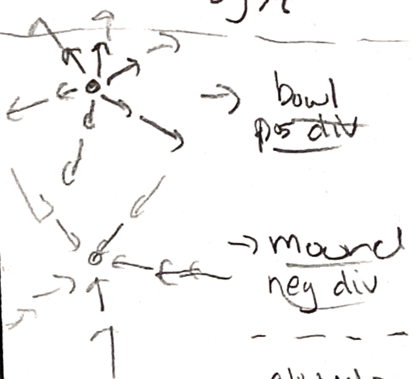


Potential at a point on the surface is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$r^2 = z^2 + R^2 - 2zR\cos\theta$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int \frac{R^2\sin\theta}{r}\,d\theta\,d\phi$$



What is this averaging business?

- comes from how the divergence works (div acts on grad f), grad tells you which direction f is steep

Vector Field

↳ each point has a vector w/ magnitude + direction

[Divergence tells you how much a vector field flows into or out of a point]

div = + → flowing out, div = − flowing in

↳ div > 0 if you have ⇉⇉ | divergence is just a number representing this in or out flow

$$\text{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v}, \quad \vec{v} = \vec{\nabla}V$$

Our vector field $\vec{v}$ comes from $\vec{\nabla}V$, this tells you the direction in which V is changing the most.

Proof

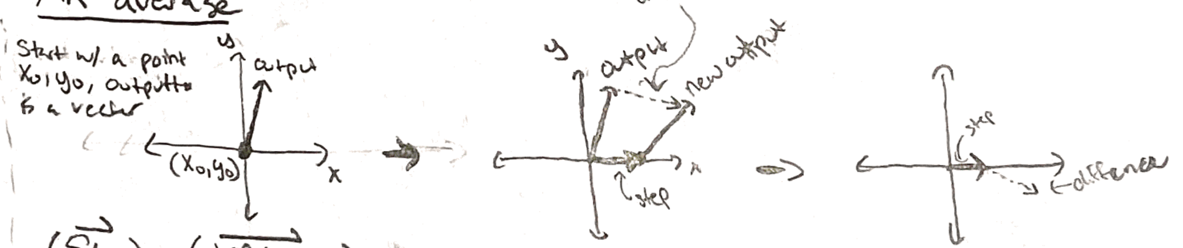
$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy \rightarrow dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y}\right)\cdot(dx\,\hat{x} + dy\,\hat{y}) = \vec{\nabla}T\cdot d\vec{r}$$

↳ $dT = \vec{\nabla}T\cdot d\vec{r} = |\vec{\nabla}T||d\vec{r}|\cos\theta$, dT is greatest when $\cos\theta = 1$ or ALONG the Direction of $\vec{\nabla}T$

Dot product tells you how much two vectors are aligned

An average

Start w/ a point x_0, y_0, output is a vector



$(\overrightarrow{\text{Step}})\cdot(\overrightarrow{\text{difference}})$ ← take average of these over all possible step directions $= \text{div}(\vec{v})$

positive dot product / positive divergence

- If a step in that direction causes a change to the vector in that same direction this corresponds to outward flow
- step · difference = − then it corresponds to inward flow

① $\vec{\nabla}V$ = vector field where each vector points in the direction of greatest descent

↳ if there is a mound then all the vectors point up that mound

↳ if there is a bowl then all the vectors point away from the bottom of the bowl

② $\vec{\nabla}\cdot\vec{\nabla}V$: Scalar valued function which tells you how much the vector field is moving away or toward

OR if we trace back, how much its a mound or a bowl

Divergence is also the average of these over all possible directions at (x_0, y_0)

↳ each difference vector dotted w/ the step vector tells you how aligned they are,

aligned = + dot product = flow out = positive divergence (= bowl) ; opposite = − dot product = flow in = negative divergence (= hill)

Boundary Conditions for Laplace's Equations

Laplaces equation on its own doesn't determine V

↳ BOUNDARY CONDITIONS are needed, but how many

- you also dont want redundant boundary conditions

↳ Boundary conditions uniquely define V (like $y = mx + b$)

First uniqueness theorem: the solution to Laplaces eq in some volume Vol is uniquely determined if V is specified on the boundary surface S

→ $\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$ (suppose two solutions to Laplace eq)

↳ look at difference $V_3 \equiv V_1 - V_2$

$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$ obeys laplaces equation

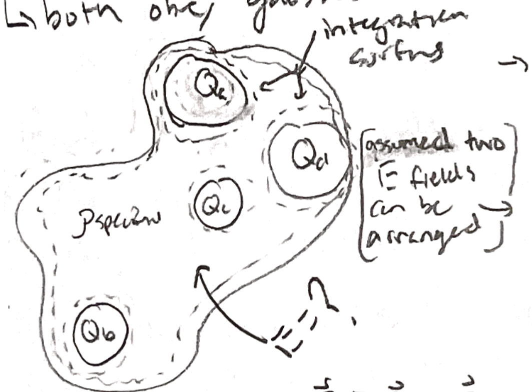
V_3 takes a value 0 on all boundaries, since V_1 and V_2 are equal there

↳ All extrema occur on boundaries, max + min of V_3 are both 0 for it to be 0, or $V_1 = V_2$

Second uniqueness theorem: In a volume Vol surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely defined if the total charge on each conductor is given.

Proof: Supose there are two fields satisfying the conditions of the problem. We are looking at if the the $\vec{E}$ field is uniquely determined inside these surfaces. Is there a few ways the charges can arrange themselves on each conductor, each leading to a new field?

↳ both obey gausses law in differential form → $\vec{\nabla} \cdot \vec{E}_1 = \frac{\rho}{\varepsilon_0}$, $\vec{\nabla} \cdot \vec{E}_2 = \frac{\rho}{\varepsilon_0}$



→ $\oint \vec{E}_1 \cdot d\vec{a} = \frac{Q_i}{\varepsilon_0}$ ith conductor surface, big ones two smaller ones $\qquad \oint \vec{E}_2 \cdot d\vec{a} = \frac{Q_i}{\varepsilon_0}$

$\oint \vec{E}_1 \cdot d\vec{a} = \frac{Q_{tot}}{\varepsilon_0}$ outer boundary $\qquad \oint \vec{E}_2 \cdot d\vec{a} = \frac{Q_{tot}}{\varepsilon_0}$ outer boundary

Look at difference: $\vec{E}_3 \equiv \vec{E}_1 - \vec{E}_2$, $\vec{\nabla} \cdot \vec{E}_3 = 0$ from → then in integral form $\oint \vec{E}_3 \cdot d\vec{a} = 0$ over each boundary

↳ We know each conductor is an equipotential (charges zing uniformly) → $V_3 =$ constant, not necissarly same constant

$\vec{\nabla} \cdot (V_3 \vec{E}_3) = V_3 (\cancelto{0}{\vec{\nabla} \cdot \vec{E}_3}) + \vec{E}_3 \cdot \cancelto{-\vec{E}_3}{\vec{\nabla} V_3} = -(E_3)^2$

Now integrate over Vol: $\oint_V \vec{\nabla} \cdot (V_3 \vec{E}_3)\, d\tau = \oint V_3 \vec{E}_3 \cdot d\vec{a} = -\int_V (E_3)^2\, d\tau$, make outer boundary ∞ so $V_3 = 0$

all the surfaces

↳ $\boxed{\int_V (E_3)^2\, d\tau = 0}$ → the only way this integral can vanish is if $E_3 = 0$ → $\vec{E}_3 = 0 = \vec{E}_1 - \vec{E}_2$ → $\vec{E}_1 = \vec{E}_2$

$V_3 \oint \vec{E}_3 \cdot d\vec{a}$, $E_3 = E_1 - E_2$

$\oint \vec{E}_1 \cdot d\vec{a} = \frac{Q}{\varepsilon_0} = \oint \vec{E}_2 \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$

A UNIQUE SOLUTION!

↳ $E_3 = 0$

Laplaces Equation Revisit

$\nabla^2 V = 0$, [no charge in the area we are looking]

→ positive charges field is positive divergence a hill (wait wasnt it 0?)

→ negative acts like a bowl, negative divergence

↳ where $\vec{\nabla}\cdot(\vec{\nabla}V) = 0$ is when theres none of ... so no hill or bowl possible,

Proof its like an average!

$$V_{average} = \frac{1}{4\pi R^2} \oint V(R)\, da$$

$$= \frac{1}{4\pi} \oint V \sin\theta\, d\theta\, d\phi \; ; \left(\frac{1}{\cancel{R^2}} \cancel{R^2}\right)$$

$$\frac{dV_{avg}}{dR} = \frac{1}{4\pi} \oint \frac{\partial V}{\partial R} \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi} \oint \vec{\nabla}V \cdot \hat{r} \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi R^2} \oint (\vec{\nabla}V) \cdot (R^2 \sin\theta\, d\theta\, d\phi\, \hat{r})$$

$$= \frac{1}{4\pi R^2} \oint \vec{\nabla}V \cdot d\vec{a}$$

$$= \frac{1}{4\pi R^2} \oint \vec{\nabla}\cdot\vec{\nabla}V\, d\tau$$

$$\frac{dV_{avg}}{dR} = \frac{1}{4\pi R^2} \oint \nabla^2 V\, d\tau = 0$$

$\frac{dV_{avg}}{dR} = 0$, V_{avg} doesnt change as R increases

Redo uniqueness theorem #1

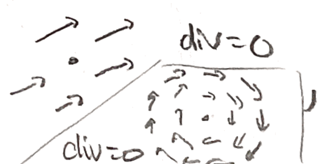
- assume two solutions with the same boundary conditions exist

$$\left.\begin{array}{l} \nabla^2 V_1 = 0 \\ \nabla^2 V_2 = 0 \end{array}\right\} V_3 = V_1 - V_2 \rightarrow \nabla^2 V_3 = 0$$

thus no max or min for V_3 and $V_1 = V_2$

$\vec{\nabla}\cdot\vec{\nabla}V$; what does this mean, well if the divergence is 0 then the amount going into a region = amount going out

↳ divergence of an incompressible fluid = 0 cant create or destroy mass



No points as sources or sinks because no charge there!

More notes on it if needed...

Method of Images

point charge held above an infinite grounded conducting plane

- What is the potential in the region above the plane

↳ q will induce an amount of neg charge on the plane

[How can we possibly calculate the potential when we dont know how much charge is induced or how its distributed?]

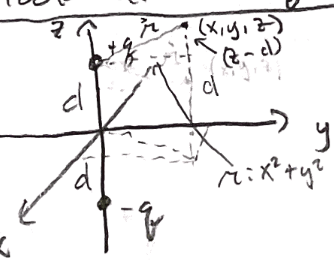
↳ Solve Poissons Equation. (in the region $z>0$ w/ q at $(0,0,d)$)

Boundary Conditions

1. $V=0$ when $z=0$ (the conducting plane is grounded)
2. $V\to0$ far from the charge $(x^2+y^2+z^2 \gg d^2$

[Its obvious we cant get good headway on this equation, but if we find a distribution w/ the same boundary conditions and satisfies Poissons equation in the ROI, then its a solution.]

we look at an equivalent charge distribution



here $(0,0,d)$, $-q$ at $(0,0,-d)$ and no conducting plane.

Using $V = \sum \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$

$r: x^2+y^2+(z-d)^2$, $r = d\hat{z}$, $r' = x\hat{x}+y\hat{y}+z\hat{z}$, $r = r-r'$, works.

But notice! when $z=0$, $V=0$ and $V\to0$ for $x^2+y^2+z^2 \gg d^2$!

↳ the only charge in the region $z>0$ is q at $(0,0,d)$. These are precisely the same conditions for the original problem.

By the uniqueness theorem (the solution to Laplaces eq in some volume is uniquely defined if V is specified on the boundary surface S.

Conclusion: The potential of a point charge above an infinite grounded conductor

Induced Surface charge

normal derivative at the surface

- if we know V we can calculate $\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} \rightarrow \sigma = -\varepsilon_0 \frac{\partial V}{\partial z}\Big|_0$

$\hookrightarrow \frac{\partial V}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right]\Big|_{z=0}$

$\hookrightarrow \boxed{\sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+z^2)^{3/2}}}$

Compute total induced charge $\rightarrow$ $Q = \int \sigma da$, $\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}}$

$\hookrightarrow Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r\,dr\,d\phi = \frac{qd}{\sqrt{r^2+d^2}}\Big|_0^{\infty} = -q$, total induced charge on the plane is $-q$, duh!

Force and Energy

q is attracted toward the plane bec of the induced $-q$ on the plane

$F = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$ (force in the analog problem

$\hookrightarrow$ not everything is the same, Energy isnt the same,

$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i) \rightarrow W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$ (0,0,d)

$\hookrightarrow$ but for a single charge & conducting plane the energy is half this,

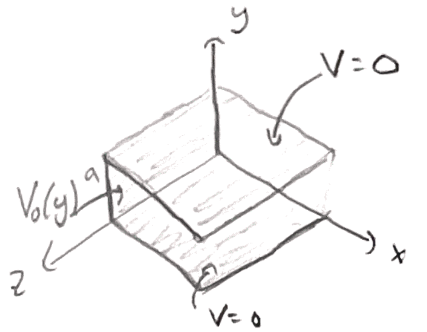
- One has two charges, so if Energy is in the field, the one w/ one charge has half the energy

Or by

$W = \int_{\infty}^{d} \vec{F}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}dz = \frac{1}{4\pi\varepsilon_0}\left(\frac{-q^2}{4z}\right)\Big|_{\infty}^{d} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$ ✓

- as I move q toward the conductor I do work only on q, it is it true that induced charge is moving over the conductor, costs nothing because the whole conductor is at $V=0$

ration of variables



inf plats so it looks the same everywhere in z

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Boundary Cond

1) $V=0$, when $y=0$
2) $V=0$, when $y=a$
3) $V=V_0(y)$ when $x=0$
4) $V \to 0$ as $x \to \infty$

First

$V(x,y) = X(x)\,Y(y)$ → plugging in $Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$

$$\hookrightarrow \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{f(x)} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{g(y)} = 0$$

(move x freely w/out changing g) (same here)

→ so the only way two seperated functions added together = 0 (always) is when they are constants otherwise you could change f and it would no longer be 0.

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2 \quad \text{w/ } C_1 + C_2 = 0,$$

one of these has to be negative or both equal 0

$$\frac{d^2X}{dx^2} = k^2 X, \quad \frac{d^2Y}{dy^2} = -k^2 Y \xrightarrow{\text{Solve the ODE}} \begin{aligned} X(x) &= Ae^{kx} + Be^{-kx} \\ Y(y) &= C\sin(ky) + D\cos(ky) \end{aligned}$$

$$\hookrightarrow V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

(now apply boundary conditions)

(for $V \to 0$ as $x \to \infty$) → $A=0$ other wise V blows up $V(x,y) = Be^{-kx}(C\sin(ky) + D\cos(ky))$

(V=0 when y=0) → $D=0$ $V(x,y) = Be^{-kx}(\sin(ky)) = Ce^{-kx}\sin(ky)$

(V=0 when y=a) → $\sin(ka) = 0$ → $k = \frac{n\pi}{a}$ $(n = 1, 2, 3, 4, 5, \ldots)$

if we chose C_1 to be pos & C_2 be negative, if ~~X were sinusoidal~~, couldnt be 0 at ∞

$$V(x,y) = Ce^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)$$

Apply

$$V_0(y) = C\sin\left(\frac{n\pi}{a}y\right)$$

, Darn we dont have a way to generally fit the boundary cond at $x=0$,

→

[So far we have] $V(x,y) = Ce^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi y}{a}\right)$ and cant fit $x=0$, $V=V_0(y)$

We do have an inf # of solutions:

$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \dots$ if $V_1, V_2, V_3,$ satisfy $\nabla V_n = 0$ so do their sum

$\hookrightarrow \nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \dots = 0 + 0 + 0 + \dots = 0$

$\hookrightarrow V(x,y) = \sum_{n=1}^{\infty} C_n e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$, this solution is more general, now can we apply the boundary cond?

$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$ → need to choose the right set of C_n's to do the trick

But how? By Dirchlet Theorem, we can find an appropriate set of C_n to "make" $V_0(y)$ [any funct can be built w/ inf sin waves]

Fouriers Trick: mult each side by $\sin\left(\frac{n'\pi y}{a}\right)$ & integrate from 0 to a

$$\sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy}_{\text{we can evaluate this}} = \int_0^a V_0(y)\sin(n'\pi y/a)\,dy$$

$\hookrightarrow \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \begin{cases} 0 & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$

$\sum_{n=1}^{\infty} C_n \frac{a}{2}\delta_{n'n} = \frac{a}{2}C_{n'}$ you only get $C_{n'}$ out when $n=n'$

$$\hookrightarrow \boxed{C_{n'} = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy}$$

* dropped the n' for ease later when using *

Now we can apply $x=0$, $V=V_0$:

$$C_n = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy = \frac{2V_0}{n\pi}(1-\cos(n\pi)) = \begin{cases} 0 & n \text{ is even} \\ \frac{4V_0}{n\pi} & n \text{ is odd} \end{cases}$$

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\dots} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a)$$

[Sum can be solved]

$$V(x,y) = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin\left(\frac{\pi y}{a}\right)}{\sinh\left(\frac{\pi x}{a}\right)}\right)$$

3.2 Laplace in spherical coordinates

pg 143,

Laplace in spherical: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$

assume V is independent of $\phi \rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$

↳ $V(r,\theta) = R(r)\Theta(\theta)$

↳ Put in and divide by V $\rightarrow \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{constant}} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\text{constant}} = 0$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) \quad \text{and} \quad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

↳ constants are equal & opposite & $l(l+1)$ is chosen to simplify math

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R \rightarrow \boxed{R(r) = Ar^l + \frac{B}{r^{l+1}}}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta \rightarrow \Theta = P_l(\cos\theta) \quad \text{(legendre polynomials)}$$

and $P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$ (polynomials listed on pg 142)

$$V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta) \rightarrow V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Example: $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R

Find V inside of the sphere:

at $r=0$ the potential blows up for all l so $B_l = 0 \rightarrow V(r,\theta) = \sum_{l=0} A_l r^l P_l(\cos\theta)$

at $r=R$, $V(R,\theta) = V_0(\theta) \rightarrow V(R,\theta) = \sum_{l=0} A_l R^l P_l(\cos\theta) = V_0(\theta)$

[Just like sin's constitute a complete list of functions Legendre Polynomials do too!] only on the interval $-1 \le x \le 1$, $0 \le \theta \le \pi$

mult each side by $P_{l'}(\cos\theta)\sin\theta$ $\rightarrow A_l R^l \int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \int_0^\pi V_0(\theta)P_{l'}(\cos\theta)\overbrace{\sin\theta\,d\theta}^{\text{part of integration}}$

and $\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases}$

get rid of R to R side

↳ $\boxed{A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta}$

Usually easier by eyeball!

Suppose $V_0(\theta) = k\sin^2(\theta/2) = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right] \rightarrow A_0 = \frac{k}{2}$, $A_1 = \frac{-k}{2R}$, $A_{l\,\text{rest}} = 0$

↳ $V(r,\theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^l}{R}P_1(\cos\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)$