12.1.4

# Structure of Spacetime

[Notation Change] $\rightarrow$ $x^0 \equiv ct$ , $\beta \equiv \frac{v}{c}$ , change $t$ to be scaled by $c$
↳ $ct$ is how far light travels in $t$ seconds

[More notation] $\rightarrow$ $x^1 = x$, $x^2 = y$, $x^3 = z$

Old Lorentz transformation $\rightarrow$ $\bar{t} = \gamma(t - \frac{v}{c^2}x)$ $\rightarrow$ $c\bar{t} = \bar{x}^0 = \gamma(\underbrace{ct}_{x^0} - \underbrace{\tfrac{v}{c}}_{\beta}\underbrace{x}_{x^1})$

$\bar{x} = \gamma(x - vt)$ $\rightarrow$

$\bar{t} = \gamma(t - \frac{v}{c^2}x) \rightarrow \bar{x}^0 = \gamma(x^0 - \beta x^1)$ and $\bar{x}^2 = x^2$, $\bar{x}^3 = x^3$

$\bar{x}^1 = \gamma(x^1 - \beta x^0)$

we have a system of equations now

Matrix

$$\begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{bmatrix} = \begin{array}{cccc} x^0 & x^1 & x^2 & x^3 \\ \end{array} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \rightarrow \boxed{\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu}$$

$\mu$ = row
$\nu$ = column

this this

[The structure of $\bar{x}^\mu$ is most general and for not just x-axis motion just need write $\Lambda^\mu{}_\nu$]

↳ $\bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu a^\nu$ $\rightarrow$ transformation along x-axis $\rightarrow$ $\bar{a}^0 = \gamma(a^0 - \beta a^1)$, $\bar{a}^2 = a^2$
$\bar{a}^1 = \gamma(a^1 - \beta a^0)$, $\bar{a}^3 = a^3$

↳ 4-vector, any set of four components that transform in the same way

4 vector dot product: for $\bar{a}^\mu, \bar{b}^\mu$

$-a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3$ $\rightarrow$ $-\bar{a}^0\bar{b}^0 + \bar{a}^1\bar{b}^1 + \bar{a}^2\bar{b}^2 + \bar{a}^3\bar{b}^3 = -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3$

contravariant: $a^\mu$

covariant: $a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$

$a_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} a^\nu$ where $g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Minkowski Metric

# Double Checking

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{array}{l} \mu = \text{row} \\ \nu = \text{column} \end{array}$$

$$\bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu a^\nu \rightarrow \begin{array}{l} \bar{a}^0 = \gamma a^0 - \gamma\beta a^1 = -(-\gamma a^0 + \gamma\beta a^1) \\ \bar{a}^1 = -\gamma\beta a^0 + \gamma a^1 \end{array}$$

$$\begin{bmatrix} \text{angle for } b \\ b \end{bmatrix} \rightarrow \begin{array}{l} \bar{b}^0 = \gamma b^0 - \gamma\beta b^1 \\ \bar{b}^1 = -\gamma\beta a^0 + \gamma a^1 \end{array}$$

Play w/ later

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w/ $a_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} a^\nu$

↳ scalar product $\sum_{\mu=0}^{3} a^\mu b_\mu$ or $a^\mu b_\mu$

$\bar{x}^\mu$ is an element of a vector (transformed one)

$a^\mu$ is the lement of a four vector (time, x, y, z)

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$a^\mu a_\mu = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$

if positive: spacelike, if negative: timelike, if 0 light like

Event A: $(x_A^0, x_A^1, x_A^2, x_A^3)$

Event B: $(x_B^0, x_B^1, x_B^2, x_B^3)$

$\Delta x^\mu \equiv x_A^\mu - x_B^\mu$ (displacement 4 vector)

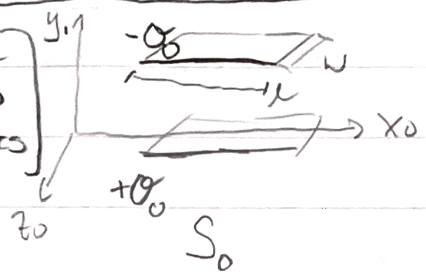
magnitude of displacement

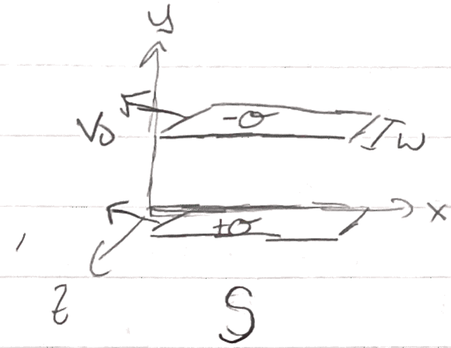
↳ $\Delta x^\mu \Delta x_\mu$ = invariant interval = $-(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$

$= -c^2 t^2 + d^2$ { t = difference in time between the two events
d = spatial separation

## How the Fields Transform

[consider the field of two uniform plates]



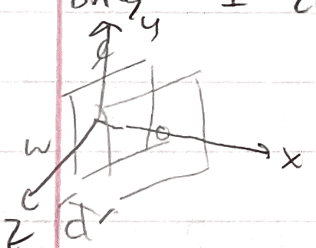

$\vec{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{y}$ and $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{y}$ only diffrce is $\sigma, \sigma_0$

w = unchanged

l = length contracted $\rightarrow$ $\gamma_0 = \frac{1}{\sqrt{1-(v_0/c)^2}}$ $\rightarrow$ $\boxed{\sigma = \gamma_0 \sigma_0}$

↳ $\boxed{\vec{E}^{\perp} = \gamma_0 \vec{E}_0}$ * components E ⊥ to direction of motion of )

only ⊥ changes




now d is all that gets contracted

↳ l, w and σ are the same

↳ field doesnt depend on d ↳ $E^{\parallel} = E_0^{\parallel}$

which is the ∥ component in our first situation!

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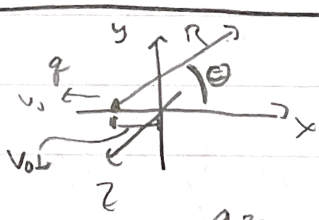
Electric field of a point charge




$\vec{E}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_0^2}\hat{r}_0$ ← field in $S_0$ → or

$E_{x_0} = \frac{1}{4\pi\varepsilon_0}\frac{q x_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}$, $E_{y_0} = \frac{q y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}\frac{1}{4\pi\varepsilon_0}$, $E_{z_0} = \frac{q z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}$ z

↳ [Transform Using] → $\vec{E}_\perp = \gamma_0 \vec{E}_0^{\perp}$, $\vec{E}_\parallel = E_0^{\parallel}$, [moves in x direction so y, z are the ⊥] → $E_x = E_{x_0}$, $E_y = \gamma_0 E_{y_0}$ and $E_z = \gamma_0 E_{z_0}$

[Transform to moving coord] → $x_0 = \gamma_0(x + v_0 t) = \gamma_0 R_x$, $y_0 = y$ and $z_0 = z$ cuz its $\vec{r} - \vec{v}t$ (w(t))

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\gamma_0 q \vec{R}}{(\gamma_0^2 R_x^2 + R_y^2 + R_z^2)^{3/2}}$, $\sqrt{R_y^2+R_z^2} = R\sin\theta$, $R_x = R\cos\theta$ ] spherical coord w/ x as z

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\gamma_0 q \vec{R}}{(\gamma_0^2 R^2 \cos^2\theta + R^2 \sin^2\theta)^{3/2}}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q(1 - v_0^2/c^2)\,\hat{R}}{(1 - (\frac{v_0}{c})^2 \sin^2\theta)^{3/2}} \frac{1}{R^2}$$ Same as w/ retarded potentials.

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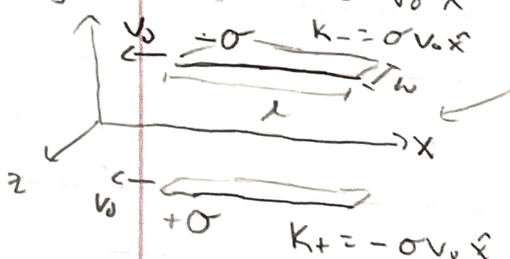
Not most general transformation, we need one w/ B-field

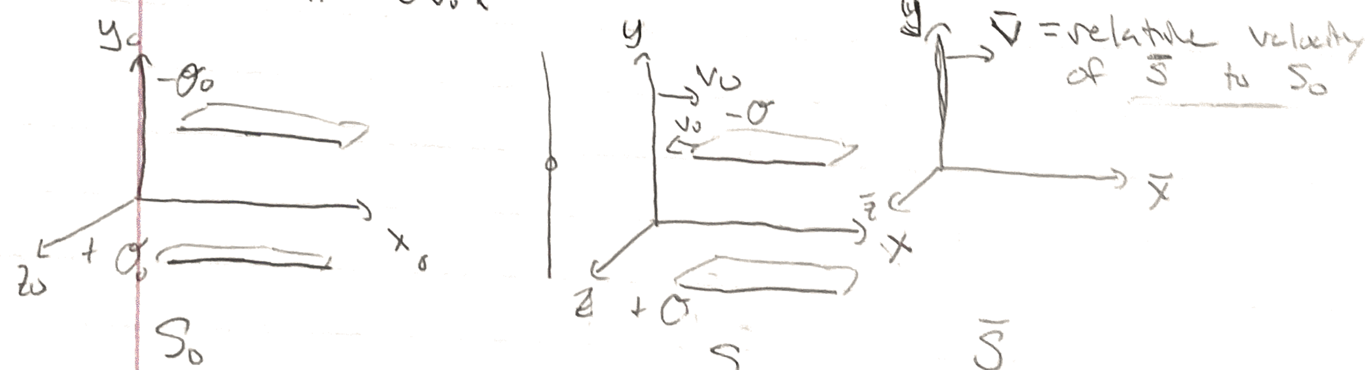
$E_y = \sigma/\varepsilon_0$

$\vec{K}_\pm = \mp \sigma v_0 \hat{x}$ → due to $\sigma$ moving in $-x$ dir




inside the plate the B-field is $B_z = -\mu_0 \sigma v_0$

RH surface current, but under it flips to both point in z-dir, (each $B = \frac{\mu_0}{2}\sigma v_0$)




In the third sys → $\bar{E}_y = \frac{\bar{\sigma}}{\varepsilon_0}$, $\bar{B}_z = -\mu_0 \bar{\sigma} \bar{v}$ fields in third system ref.

* Similar concept to the wires,
$S_0$ = like the frame of rest $\lambda_0$
$S$ = observer of neutral wire
$\bar{S}$ = charge moving relative to wire

but in this case it is S

In this case

velocity of object in $S_0$ → $\bar{v}$ → target quantity

$$v_s = \frac{v_{s_0} - v_R}{1 - \frac{v_{s_0} v_R}{c^2}} \rightarrow v = \frac{\bar{v} - v_0}{1 - \frac{\bar{v} v_0}{c^2}} \rightarrow \bar{v} = \frac{v + v_0}{1 + \frac{v v_0}{c^2}}$$

$v_s$: velocity of object measured in S → just call $v$

$v_R$: relative velocity between $S_0$ + S → $v_0$

$S_0$ = moving w/ surface current

$S$ = main frame seeing plates move

$\bar{S}$ = frame moving, measured in $S$.

$$\bar{V} = \frac{V + V_0}{1 + \frac{VV_0}{c^2}}$$

[We have the following] $\rightarrow$ $E_y = \sigma/\varepsilon_0$, $B_z = -\mu_0 \sigma V_0$

$\bar{E}_y = \bar{\sigma}/\varepsilon_0$, $\bar{B}_z = -\mu_0 \bar{\sigma} \bar{V}$

$\sigma = \gamma_0 \sigma_0$ and $\bar{\sigma} = \bar{\gamma} \sigma_0$, $\bar{\gamma} = \frac{1}{\sqrt{1+(\bar{v}/c)^2}}$

$\hookrightarrow \bar{E}_y = \frac{\bar{\sigma}}{\varepsilon_0} = \frac{\bar{\gamma}}{\gamma_0} \frac{\sigma}{\varepsilon_0}$, $\bar{B}_z = -\frac{\bar{\gamma}}{\gamma_0} \mu_0 \sigma \bar{V}$

$\hookrightarrow \frac{\bar{\gamma}}{\gamma_0} = \gamma\left(1 + \frac{VV_0}{c}\right)$ w/ algebra, $\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$

$B_z = -\mu_0 \sigma V_0$

$\hookrightarrow \bar{E}_y = \gamma\left(1 + \frac{VV_0}{c^2}\right)\frac{\sigma}{\varepsilon_0} = \gamma\left(E_y - \frac{V}{c^2 \varepsilon_0 \mu_0} B_z\right)$

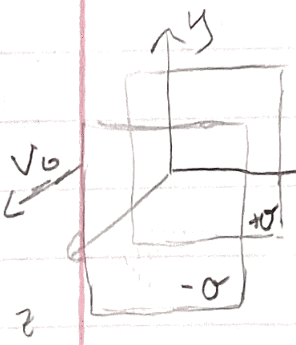
$\hookrightarrow \bar{B}_z = -\gamma\left(1 + \frac{VV_0}{c^2}\right)\mu_0 \sigma \left(\frac{V + V_0}{1 + VV_0/c^2}\right) = \gamma\left(B_z - \mu_0 \varepsilon_0 V E_y\right)$

or

$\boxed{\bar{E}_y = \gamma(E_y - VB_z)}$ ← transforming $E_y$ to a moving frame $\bar{E}_y$

$\boxed{\bar{B}_z = \gamma\left(B_z - \frac{V}{c^2} E_y\right)}$ ← transforming $B_z$, $E_y$ to moving frame $\bar{B}_z$

To do $E_z$, $B_y$: set up plates like this



[fields in S are] $\rightarrow$ $E_z = \frac{\sigma}{\varepsilon_0}$, $B_y = \mu_0 \sigma V_0$ (by RHR)

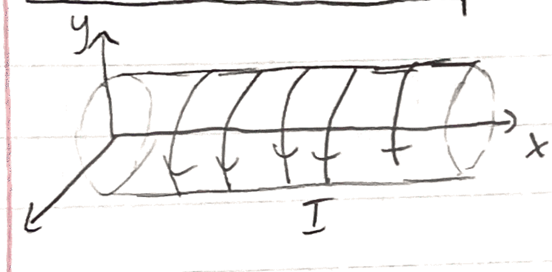
So $E_y \rightarrow E_z$ and $B_z \rightarrow -B_y$

$$\boxed{\begin{aligned}\bar{E}_z &= \gamma(E_z + VB_y) \\ \bar{B}_y &= \gamma\left(B_y + \frac{V}{c^2} E_z\right)\end{aligned}}$$

But the parallel component $E_x$ is always the same so $\rightarrow \bar{E}_x = E_x$

↳ no magnetic in a unique way so cant deduce transformation

use different example:




$B_x = \mu_0 n I$

In system $\bar{S}$ $\rightarrow \bar{n} = \gamma n \rightarrow \frac{d\bar{q}}{d\bar{t}} = \frac{1}{\gamma}\frac{dq}{dt}$

(rest frame the $\uparrow$ S) $\gamma$ = (what frame says something else sees $\uparrow$ $\bar{S}$) $\rightarrow \frac{1}{dt}\frac{1}{\gamma} = \frac{1}{d\bar{t}}$

↳ $\bar{B}_x = \mu_0 \bar{n} \bar{I} = \mu_0 n I = B_x$ so its the same

$$\boxed{\begin{aligned} &\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y) \\ &\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{aligned}}$$

# Relativistic Potentials

$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$ , $\vec{B} = \vec{\nabla} \times \vec{A}$

$A^\mu = (V/c, A_x, A_y, A_z)$ (4vector)

$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$ , differentation w.r.t covarient vectors → $\boxed{x_0 = -x^0}$ changes sign of zeroth component

## Evaluating a few terms

$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial (ct)} - \frac{1}{c}\frac{\partial V}{\partial x} = -\frac{1}{c}\left(\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}V\right)_x = \frac{E_x}{c}$

$\mu = 0, \nu = 1$ , $\nu = 2$, $\nu = 3$ gets you $y$ + $z$ components

$\mu = 1, \nu = 2$: $F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\vec{\nabla} \times \vec{A})_z = B_z$

$F^{23}$ and $F^{31}$ get you $\vec{B} = \vec{\nabla} \times \vec{A}$

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$\left.\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu\right]$ maxwells eq , $\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ → taken care of in potential formula

plugged this in and swapped $\frac{\partial}{\partial x}$

$\frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right) = \mu_0 J^\mu$

↳ $A^\mu \to A^{\mu'} = A^\mu + \frac{\partial \lambda}{\partial x_\lambda}$ ← gauge invariance , doesn't change $F^{\mu\nu}$

↳ use lorentz gauge → $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$ → $\frac{\partial A^\mu}{\partial x^\mu} = 0$

$$\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu}A^\mu = -\mu_0 J^\mu$$

$$\hookrightarrow \boxed{\Box^2 A^\mu = -\mu_0 J^\mu}$$

# The Field Tensor

$$\bar{E}_x = E_x \qquad \bar{B}_x = B_x$$
$$\bar{E}_y = \gamma(E_y - vB_z) \quad \text{and} \quad \bar{B}_y = \gamma(B_y + \tfrac{v}{c^2}E_z)$$
$$\bar{E}_z = \gamma(E_z + vB_y) \qquad \bar{B}_z = \gamma(B_z - \tfrac{v}{c^2}E_y)$$

↑ transformation like an anti-symmetric second rank tensor

$$\hookrightarrow t^{\mu\nu} = \begin{bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{bmatrix} \quad \text{vs} \quad \rightarrow t^{\mu\nu} = \begin{bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{bmatrix}$$

Short hand: $\bar{t}^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\sigma} t^{\lambda\sigma}$

Recall! $\bar{a}^{\mu} = \Lambda^{\overset{\mu \leftarrow \text{row}}{}}{}_{\underset{\text{column}}{\nu}} a^{\nu}$ and $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

ex calc $\bar{t}^{01}$:

$$\bar{t}^{01} = \Lambda^{0}{}_{\lambda} \Lambda^{1}{}_{\sigma} t^{\lambda\sigma} = \Lambda^{0}{}_{0} \Lambda^{1}{}_{\sigma} t^{0\sigma} + \Lambda^{0}{}_{1} \Lambda^{1}{}_{\sigma} t^{1\sigma}$$
$$+ \underset{=0}{\underset{\uparrow}{\Lambda^{0}{}_{2}}} \Lambda^{1}{}_{\sigma} t^{2\sigma} + \underset{=0}{\underset{\uparrow}{\Lambda^{0}{}_{3}}} \Lambda^{1}{}_{\sigma} t^{3\sigma}$$

$$\bar{t}^{01} = \Lambda^{0}{}_{0} \Lambda^{1}{}_{\sigma} t^{0\sigma} + \Lambda^{0}{}_{1} \Lambda^{1}{}_{\sigma} t^{1\sigma}$$

$$= \Lambda^{0}{}_{0} \Lambda^{1}{}_{0} t^{00} + \Lambda^{0}{}_{0} \Lambda^{1}{}_{1} t^{01} + \Lambda^{0}{}_{0} \overset{=0}{\overset{\downarrow}{\Lambda^{1}{}_{2}}} t^{02} + \Lambda^{0}{}_{0} \overset{=0}{\overset{\downarrow}{\Lambda^{1}{}_{3}}} t^{03}$$
$$+ \Lambda^{0}{}_{1} \Lambda^{1}{}_{0} t^{10} + \Lambda^{0}{}_{1} \Lambda^{1}{}_{1} t^{11} + \Lambda^{0}{}_{1} \underset{=0}{\underset{\uparrow}{\Lambda^{1}{}_{2}}} t^{12} + \Lambda^{0}{}_{1} \underset{=0}{\underset{\uparrow}{\Lambda^{1}{}_{3}}} t^{13}$$

So: $\bar{t}^{01} = \Lambda^{0}{}_{0} \Lambda^{1}{}_{0} t^{00} + \Lambda^{0}{}_{0} \Lambda^{1}{}_{1} t^{01} + \Lambda^{0}{}_{1} \Lambda^{1}{}_{0} t^{10} + \Lambda^{0}{}_{1} \Lambda^{1}{}_{1} t^{11}$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad t^{\lambda\nu} = \begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}$$

$\hookrightarrow \bar{t}^{01} = \Lambda^0_{\ 0}\Lambda^1_{\ 0} t^{00} + \Lambda^0_{\ 0}\Lambda^1_{\ 1} t^{01} + \Lambda^0_{\ 1}\Lambda^1_{\ 0} t^{10} + \Lambda^0_{\ 1}\Lambda^1_{\ 1} t^{11}$

well $t^{00} = 0$, $t^{11} = 0$ and $t^{01} = -t^{10}$

$$\bar{t}^{01} = \left(\Lambda^0_{\ 0}\Lambda^1_{\ 1} - \Lambda^0_{\ 1}\Lambda^1_{\ 0}\right) t^{01} = \underbrace{\left(\gamma^2 - (\gamma\beta)^2\right)}_{=1} t^{01}$$

$\hookrightarrow \boxed{\bar{t}^{01} = t^{01}}$ all the others are:

$\bar{t}^{01} = t^{01}$, $\bar{t}^{02} = \gamma(t^{02} - \beta t^{12})$, $\bar{t}^{03} = \gamma(t^{03} + \beta t^{31})$

$\bar{t}^{23} = t^{23}$, $\bar{t}^{31} = \gamma(t^{31} + \beta t^{03})$, $\bar{t}^{12} = \gamma(t^{12} - \beta t^{02})$

These our are exact E, B transformation equations

$\hookrightarrow$ make $F^{\mu\nu}$, $F^{01} = \frac{E_x}{c}$, $F^{02} = \frac{E_y}{c}$, $F^{03} = \frac{E_z}{c}$

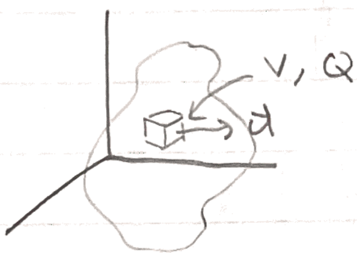
$F^{12} = B_z$, $F^{13} = B_y$, $F^{23} = B_x$

$$\hookrightarrow F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

# Electrodynamics in Tensor Notation

- reformulating Maxwells Eq & Lorentz force law in tensors

**Determining how $\rho$ and $J$ transform:**




$\rho = \frac{Q}{V}$ and $\vec{J} = \rho \vec{u}$

proper charge density $\rightarrow$ $\rho_0 = \frac{Q}{V_0}$

$V = \sqrt{1 - u^2/c^2}\, V_0$ (lorentz contracted)

$\hookrightarrow \rho = \frac{\rho_0}{\sqrt{1-\frac{u^2}{c^2}}}$ , $\vec{J} = \rho_0 \frac{\vec{u}}{\sqrt{1-u^2/c^2}}$ $\rightarrow$ analogous to 4 vector $\eta = \frac{\vec{u}}{\sqrt{1-u^2/c^2}}$

$\hookrightarrow J^\mu = \rho_0 \eta^\mu$ , $J^\mu = (c\rho, J_x, J_y, J_z)$

$\boxed{\text{Continuity Equation}}$ $\rightarrow$ $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ , $\vec{\nabla} \cdot \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum \frac{\partial J^i}{\partial x^i}$

and $\frac{\partial \rho}{\partial t} = \frac{1}{c} \frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}$ adding over $\rightarrow$ $\boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$

$$\boxed{\begin{matrix}\text{Maxwells}\\ \text{Equations}\end{matrix}} \rightarrow \quad \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad \text{and} \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

$\mu = 0$:
$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$$
$$= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}(\nabla \cdot E)$$
$$= \mu_0 J^0 = \mu_0 c\rho \rightarrow \boxed{\nabla \cdot E = \frac{\rho}{\varepsilon_0}}$$

$\mu = 1$:
$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$$
$$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \quad \text{and } \mu_0 J^1 = \mu_0 J_x$$

$\mu = 2$:
$$\frac{\partial F^{2\nu}}{\partial x^\nu} = \frac{\partial F^{20}}{\partial x^0} + \frac{\partial F^{21}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{23}}{\partial x^3}$$
$$= -\frac{1}{c^2}\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}, \quad \mu_0 J^2 = \mu_0 J_y$$

$\mu = 3$:
$$\frac{\partial F^{3\nu}}{\partial x^\nu} = -\frac{1}{c^2}\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}, \quad \mu J^3 = \mu_0 J_z$$

$$\hookrightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

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$\mu = 0$:
$$\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3}$$
$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

$\mu = 1$:
$$\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = 0$$

w/ $\mu = 2$ & $\mu = 3$ like above! $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

$$\vec{K} = \frac{1}{\sqrt{1-u^2/c^2}} \vec{F}$$

$$\vec{\eta} = \frac{1}{\sqrt{1-u^2/c^2}} \vec{u}$$

## Minkowski Force

$$K^\mu = \frac{dp^\mu}{d\tau} \quad , \left[\begin{matrix}\text{Force on a} \\ \text{charge particle}\end{matrix}\right] \rightarrow \boxed{K^\mu = q\,\eta_\nu F^{\mu\nu}}$$

lower indice!

$\mu = 1$

$$K^1 = q\,\eta_\nu F^{1\nu} = q\left(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13}\right)$$

$$= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(-\frac{E_x}{c}\right) + 0 + \frac{u_y}{\sqrt{1-u^2/c^2}} B_z + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right]$$

$$= \frac{q}{\sqrt{1-u^2/c^2}}\left[\vec{E} + (\vec{u}\times\vec{B})\right]_x$$

$\mu = 2, \mu = 3$ $\rightarrow$ $\vec{K} = \frac{q}{\sqrt{1-u^2/c^2}}\left[\vec{E} + (\vec{u}\times\vec{B})\right]$

$$\hookrightarrow \boxed{\vec{F} = q\left(\vec{E} + (\vec{u}\times\vec{B})\right)}$$

$\mu = 0$

$$K^0 = q\,\eta_\nu F^{0\nu} = q\left(-\eta_0 F^{00} + \eta_1 F^{01} + \eta_2 F^{02} + \eta_3 F^{03}\right)$$

$$K^0 = q\left(0 + \frac{u_x}{\sqrt{1-u^2/c^2}}\left(\frac{E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}\left(\frac{E_y}{c}\right) + \frac{u_z}{\sqrt{1-u^2/c^2}}\left(\frac{E_z}{c}\right)\right) = \frac{1}{c}\frac{\partial E}{\partial \tau}$$

$$\frac{q}{c}\frac{1}{\sqrt{1-u^2/c^2}}(\vec{u}\cdot\vec{E}) = \frac{1}{c}\frac{\partial E}{\partial \tau}$$

$$q(\vec{u}\cdot\vec{E}) = \underbrace{\frac{\partial E}{\partial t}}_{\text{energy}} = \frac{dW}{dt}$$

$$\boxed{\text{Power} = \vec{u}\cdot\underbrace{(q\vec{E})}_{\text{Force}}}$$