Rotating Reference Frames

Symplified Frame

S_0 = inertial reference frame w/ acceleration A (capital letters for acc and velocity relative to S_0)

S = non-inertial reference frame (ex) frame fixed in a railroad car that is moving relative to S_0 w/ velocity V, $A = \dot{V}$

Suppose a passenger is playing catch w/ a ball of mass m consider the motion of the ball measured relative to S_0

Since S_0 is inertial

$$\boxed{F = m\ddot{r}_0}$$

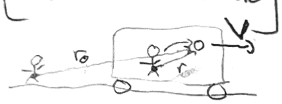
balls position relative to S_0

F is the net force, gravity, air resistance etc...

Now consider the balls motion relative to the accelerating frame S

$$\boxed{\dot{r}_0 = \dot{r} + V}$$

Lets understand why this is the case →



r_0 = position of ball in inertial frame

r = position of ball in non-inertial frame

V = velocity of car

So its: (balls velocity relative to ground) = (balls velocity relative to car) + (cars velocity rel to ground)

* makes sense because the persons inside velocity & plus the cars velocity is what person on ground would see *

Now we can take that equation and differentiate it → and solved for

$$\boxed{\ddot{r} = \ddot{r}_0 - A}$$

multiply by m →

$m\ddot{r}_0$ = inertial frames total net force

$$\boxed{m\ddot{r} = F - mA}$$

eq 9.4

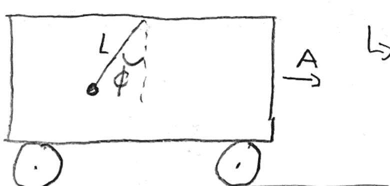
looks totally like a $F = ma$ for the non-inertial frame except for the extra term $-mA$. Called the inertial Force

→ inertial force is the force you experience when you accelerate really fast and get pushed into your seat

→ another example Standing on a bus when it suddenly breaks, A is backward so you feel the $+mA$ forward and you fall on your face

Example • a pendulum inside a car → find the equilibrium angle ϕ_{eq} and the frequency of small oscillations about ϕ_{eq}

* in an inertial frame there is just two forces, Tension and weight. $F = T + mg$



↳ if we choose to work in a non inertial frame we need $-mA$ term

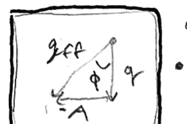
$$\boxed{m\ddot{r} = (T + mg) - mA}$$

→ rearrange

$$m\ddot{r} = T + m\underbrace{(g - A)}_{g_{eff} = g - A}$$

(no A) • normally it reaches equilibrium

but once there is A it'll be



• if at rest $\ddot{r} = 0$

• then $T = -mg_{eff}$ → then from our picture

$$\boxed{\phi_{eq} = \arctan\left(\frac{A}{g}\right)}$$

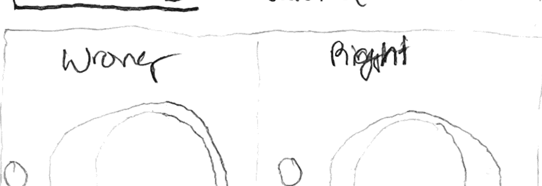
↳ Small amplitude frequency for $\omega = \sqrt{\frac{g}{L}}$ but here $g = g_{eff} = \sqrt{g^2 + A^2}$ →

$$\boxed{\omega = \sqrt{\frac{\sqrt{g^2 + A^2}}{L}}}$$

why do this in non-inertial? you could do it in inertial, finding ϕ_{eq} isnt too bad you know $T + mg = mA$ but freq of small oscillations is very difficult

TIDES

a beautiful result of $\boxed{m\ddot{r} = F - mA}$



a single bulge would cause one tide per day when there is two.

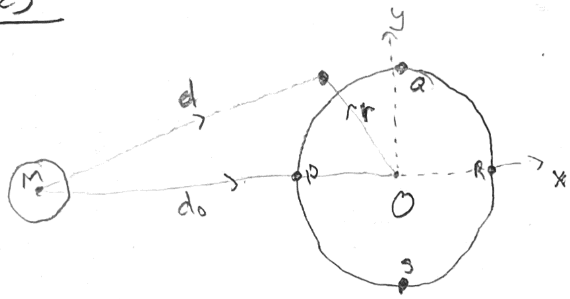
• an object on earth feels the force of the moon as if it were coming from earths center

• An object on the moons side is pulled by the moon w/ a force that is slightly greater than it would be at the center

• On the other side are pulled by the moon that is slightly weaker than it would be which causes another bulge

math explanation on back

Tides



Using $m\ddot{r} = F - mA$ we want to figure out the 3

[Forces on mass m near earths surface is]

- $m_E g$ (gravity of the earth)
- $-\frac{GM_m m}{d^2}\hat{d}$ (gravity of moon)
- $F_{non\,grav}$ (like buoyant force for ex)

$$\left[F = mg - \frac{G(M_m)m}{d^2}\hat{d} + F_{ng}\right]$$

→ [But what is A? Well this is ~~the~~ car's acceleration it experienced. The car in this situation is the Earth and the acceleration or force it it experiences is from the moon]

$$\left[A = -\frac{GM_m}{d_o^2}\hat{d}_o\right]$$

* M_m = mass of moon *

[putting it together]

$$\boxed{m\ddot{r} = \left[mg - \frac{GM_m m}{d^2}\hat{d} + F_{ng}\right] + \frac{GM_m m}{d_o^2}\hat{d}}$$

[Combine M_m terms] → eq 9.12

$$\underbrace{F_{tidal} = -GM_m m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_o}{d_o^2}\right)}_{\text{entire effect of the moon on the motion of any object near earth}}$$

$$\Rightarrow m\ddot{r} = F_{tidal} + F_{non\,grav}$$

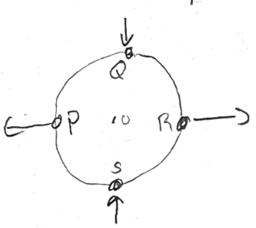
[Lets anayze this equation]

At point P, d and d_o lie in the same direction, $d = \vec{MP}$, $d_o = \vec{MO}$, but $d_o > d$ ← distance of earth surface to moon

distance from moon to earth

So the first term dominates and the tidal force is toward the moon

At point R, $d_o < d$ so the second term dominates but is negative so it points away from the moon



[So thus you get your outward bulges]

[Could do for point Q, S but just know they almost cancel except for y-component which gives rise to both those ↓ ↑.]

Newtons Second Law in rotating frame S

- Assume angular velocity $\vec{\Omega}$ of S relative to S_0 is constant, (good approximation for axis fixed to Earth)

[S_0 = inertial frame, S = non inertial frame] → [Consider a particle of mass m and position r] → [In the inertial frame S_0 the particle obeys Newtons 2nd law] → $\boxed{F = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0}}$

[Now we need to find $\vec{F}$ in the non-inertial frame S] → [use relation now we have $(d^2r/dt^2)_{S_0}$] → $\boxed{\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}}$ now differentiate.

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\underbrace{\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]}_{\vec{Q}}$$

(apply 9.30 again) → $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega}\times\vec{Q}$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_S\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right) + \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

, use dot notation for non-inertial frame $\left(\frac{d\vec{Q}}{dt}\right)_S \equiv \dot{\vec{Q}}$

* Ω is constant

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

* $\dot{r}, \ddot{r}$ are derivatives evaluated w.r.t the rotating frame S

↳ [apply newtons Law and solve for $\ddot{\vec{r}}$] $\boxed{m\ddot{\vec{r}} = \vec{F}_{apparent} = \vec{F} + 2m\dot{\vec{r}}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}}$

last 3 terms arent forces but purely consequences of the rotation of the body but we can pretend they are forces

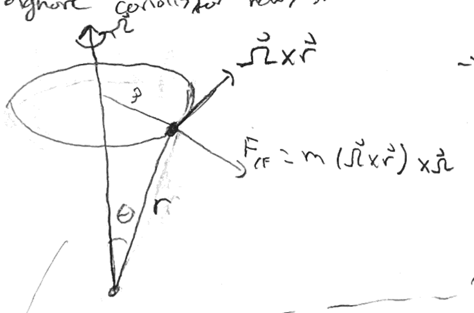
$\left[\vec{F}_{cor} = +2m\dot{\vec{r}}\times\vec{\Omega}\right]$ coriolis force

T could be plusses got conflicting results

* check signs of cross product flip them

$\left[\vec{F}_{centrifugal\ force} = +m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}\right]$ centrifugal force

- ignore coriolis for now, smaller than F_{cf}.



$\hat{n} = \perp$ to $\hat{\rho}$; crossing again gets you back to $\hat{\rho}$

→ $F_{CF} = m(\vec{\Omega}\times\vec{r})\times\Omega = (m\Omega r\sin\theta)\times\Omega = m\Omega^2 r\sin\theta\hat{\rho} = \boxed{m\Omega^2\rho\hat{\rho}}$

[A body in free fall to Earths Surface]

$\vec{F}_{eff} = \vec{F}_{grav} + \vec{F}_{cf} = m\vec{g}_0 + m\Omega^2 R\sin\theta\hat{\rho}$

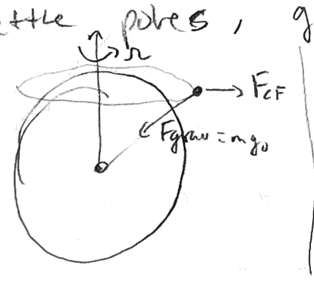
pulling out m, $\vec{g} = \vec{g}_0 + \Omega^2 R\sin\theta\hat{\rho}$

$\sin^2\theta$ bec $\hat{r} = \sin\theta\hat{\rho}$

Now $\boxed{g_{rad} = g \text{ in the direction of } -\hat{r} = g_0 - \Omega^2 R\sin^2\theta}$

(more like 0.5x cuz of Earth) the value of g at the equator is 0.3% less than at the poles

- zero at the poles, $g_0 \approx 9.8\,m/s^2$,



g also has a tangential acceleration, $g_{tan} = \Omega^2 R\sin\theta\cos\theta$

- zero at poles and equator, max at latitude of 45°

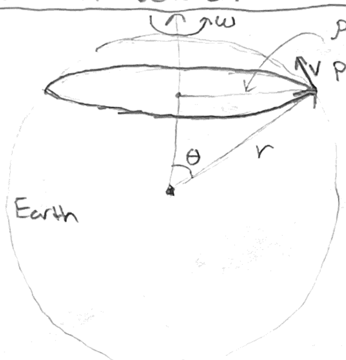
$$\alpha_{max} = \frac{g_{tan}}{g_{rad}} = \frac{\Omega^2 R}{2g_0} = \frac{0.0034}{2\cdot 9.8} = 0.0017\,\text{rad} \approx 0.1°$$

angle between $\vec{g}$ and radial direction

Angular Velocity Vector

- rotating reference frames and [Eulers Theorem] * hard to prove but fairly natural * motion of any body relative to a fixed point O. is a rotation about some axis through O.
- concerned w/ rate of rotation or angular velocity, $\omega = \frac{d\theta}{dt}$ and $\boxed{\vec{\omega} = \omega \cdot \hat{u}}$
- to get the direction of $\hat{u}$ use the RHR and curl fingers in direction of spinning
- $\vec{\omega}$ can change in direction and magnitude

Relation between $\vec{\omega}$ and the linear velocity on the body



- θ is the colatitude
- As Earth rotates about its axis "P" is dragged with radius $\rho = r\sin\theta$

$V = \underbrace{\omega}_{\text{angles/sec}} \underbrace{\rho}_{\text{distance}} = \omega r \sin\theta \Rightarrow \boxed{\vec{v} = \vec{\omega} \times \vec{r}}$

- relation for any vector fixed in the rotating body

[if $\hat{e}$ is a unit vector fixed in Earth then] $\Rightarrow$ $\boxed{\frac{de}{dt} = \vec{\omega} \times e}$ ← its rate of change as seen by the non-rotating frame

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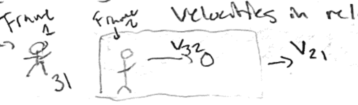
[you can add $\vec{\omega}$] $\Rightarrow$ $V_{31} = V_{32} + V_{21}$ $\xrightarrow{\text{now lets say they are rotating w/ same } \vec{r}}$ $\vec{\omega}_{31} \times \vec{r} = (\vec{\omega}_{32} \times r) + (\vec{\omega}_{21} \times r) = (\vec{\omega}_{32} + \vec{\omega}_{21}) \times r$

velocities in relative frames



[Must be true for any r] $\rightarrow$ $\boxed{\vec{\omega}_{31} = \vec{\omega}_{32} + \vec{\omega}_{21}}$

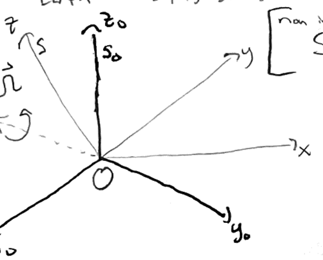
$\vec{\omega}$ for angular velocity, like spinning top

$\vec{\Omega}$ for angular velocity of noninertial, rotating ref frame, relative to which we are calculating the motion of objects

- $\vec{A}$ and $\vec{V}$ for accel and vel of non-inertial frame as well

Time derivatives of rotating ref frame S_0 = inertial frame, S = non-inertial frame w/ angular velocity $\vec{\Omega}$ rel to S

$\vec{\Omega}_{Earth} = \frac{2\pi \text{ rad}}{24\text{hrs} \cdot 3600\text{s}} = 7.3 \cdot 10^{-5}$ rad/s ← for a frame attached to earth that rotates every 24 hrs



[non inertial S is rotating w/ $\vec{\Omega}$ relative to inertial S_0] ← they share a common origin O (center of Earth), only motion S rel to S_0 is rotation $\vec{\Omega}$

↳ (ex) O could be the center of Earth, S is fixed in the Earth, and S_0 w/ same origin but different axis direction

Consider a vector $\vec{Q}$, could be velocity or position of a ball, net force on an object etc.

Relate $\frac{dQ}{dt}$ measured in inertial S_0 to the corresponding rate in non inertial S, $\left(\frac{dQ}{dt}\right)_{S_0}$ [rate of change of Q rel to S_0], $\left(\frac{dQ}{dt}\right)_S$ [rate of change of Q rel to S] ← rotating frame

$\vec{Q} = Q_1\hat{e}_1 + Q_2\hat{e}_2 + Q_3\hat{e}_3$ ← for example these vectors could be along $\hat{x}, \hat{y}, \hat{z}$. So in frame S they are fixed but in S_0 they are rotating

[Differentiate as seen in frame S] $\left(\frac{dQ}{dt}\right)_S = \sum \frac{dQ_i}{dt} e_i$ compare w/ $\frac{de}{dt} = \vec{\omega} \times e$

[differentiate as seen in frame S_0] $\left(\frac{dQ}{dt}\right)_{S_0} = \sum \frac{dQ_i}{dt} e_i + \sum Q_i \left(\frac{de_i}{dt}\right)_{S_0}$

[e_i is fixed in S which is rotating w/ angular velocity Ω relative to S_0]

↳ $\left(\frac{de_i}{dt}\right)_{S_0} = \vec{\Omega} \times \vec{e}_i \rightarrow \sum Q_i \left(\frac{de_i}{dt}\right)_{S_0} = \sum Q_i(\vec{\Omega} \times \hat{e}_i) = \vec{\Omega} \times \sum Q_i e_i = \vec{\Omega} \times \vec{Q}$

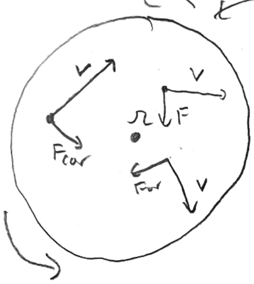
$\boxed{\underbrace{\left(\frac{dQ}{dt}\right)_{S_0}}_{\text{inertial frame}} = \underbrace{\left(\frac{dQ}{dt}\right)_S}_{\text{non inertial}} + \vec{\Omega} \times \vec{Q}}$ ← [relates derivative of any vector $\vec{Q}$ measured in inertial frame S_0, to the corresponding derivative in non-inertial frame S]

The Coriolis Force

$F_{cor} = 2m\dot{\vec{r}} \times \vec{\Omega} = 2m\vec{v} \times \vec{\Omega}$, $\vec{v}$ is the objects velocity relative to the rotating frame

↳ F_{cor} is always $\perp$ to the velocity of an object

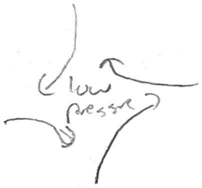
A turn table



← tends to deflect objects to the right in Northern hemisphere
if $\vec{\Omega}$ polarity flipped, like in Southern Hemisphere the world deflect to the left

* See John R Taylor example about a puck on a turn table *

a cyclone gets started by coriolis



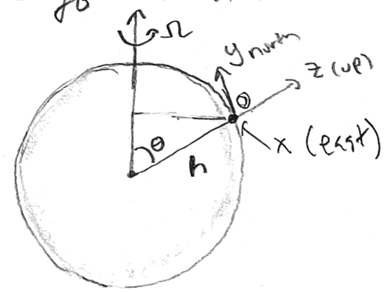
Free fall w/ coriolis & Centrifugal

r radius to object rel to surface of earth

$m\ddot{\vec{r}} = mg_0 + F_{cf} + F_{cor} = mg_0 + m(\vec{r} \times \vec{\Omega}) \times \vec{\Omega} + 2m\dot{\vec{r}} \times \vec{\Omega}$

$\ddot{\vec{r}} = (\vec{r} \times \vec{\Omega}) \times \vec{\Omega} + 2\dot{\vec{r}} \times \vec{\Omega} + g_0$, we defined $g = g_0 + \underline{(R \times \Omega) \times \Omega} = \vec{g}_0 + \Omega^2 R \sin\theta$

$\boxed{\ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}} \times \vec{\Omega}}$ ← [the net force on object doesn't depend on its position r] → [lets move the origin to the surface then] →



↳ [now position r is relative to this new O.] , $\dot{r} = (\dot{x}, \dot{y}, \dot{z})$

$\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta)$ then $\dot{\vec{r}} \times \vec{\Omega} = (\dot{y}\Omega\cos(\theta) - \dot{z}\Omega\sin\theta, -\dot{x}\Omega\cos\theta, \dot{x}\Omega\sin\theta)$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + 2\begin{bmatrix} \dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta \\ -\dot{x}\Omega\cos\theta - \\ \dot{x}\Omega\sin\theta \end{bmatrix} \rightarrow$$

make approx $\Omega \approx 0$
$\ddot{x} = 0 \rightarrow x = 0$
$\ddot{y} = 0 \rightarrow y = 0$
$\ddot{z} = -g \rightarrow z = h - \frac{1}{2}gt^2$

zeroth order, no coriolis effects shown

Next approximation

[since zeroth order is a good approx sub it in for original equation you get]

$\ddot{x} = 2\Omega g t \sin\theta \rightarrow x = \frac{1}{3}\Omega g t^3 \sin\theta$ ← coriolis causes the objects to deflect in the x-direction (east)

$\ddot{y} = 0 \longrightarrow y = 0$

$\ddot{z} = -g \rightarrow z = h - \frac{1}{2}gt^2$

if an object dropped 100m it would deflect ≈ 2.2 cm

• All the information contained in the mass distribution that is needed to solve rotational dynamics problems is contained in the <u>moment of inertia tensor.</u>

↳ it relates $\vec{\omega}$ to $\vec{L}$

Motion of Rigid Bodies

→ quick preliminary info

inertial frame: $r'_k = \begin{bmatrix} r'_1 \\ r'_2 \\ r'_3 \end{bmatrix}$

rotating coord sys: $r_k = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

$\vec{r}'\cdot\hat{\imath} \rightarrow$ inertial frame vector that lies along the $\hat{\imath}$ direction (rotating unit vector $\hat{\imath}$)

$$U = \begin{bmatrix} \hat{\imath}'\cdot\hat{\imath} & \hat{\imath}'\cdot\hat{\jmath} & \hat{\imath}'\cdot\hat{k} \\ \hat{\jmath}'\cdot\hat{\imath} & \hat{\jmath}'\cdot\hat{\jmath} & \hat{\jmath}'\cdot\hat{k} \\ \hat{k}'\cdot\hat{\imath} & \hat{k}'\cdot\hat{\jmath} & \hat{k}'\cdot\hat{k} \end{bmatrix} \rightarrow$$

whole vector transforms the inertial frame to the rotating one

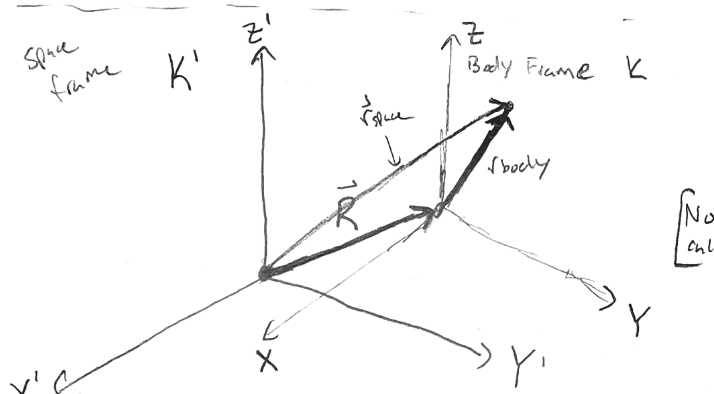
↓ function of time

$\boxed{\vec{r}' = U\vec{r}}$ inertial frame to rotating frame

$\vec{r} = U^{-1}\vec{r}'$

take the time derivative: $\vec{V}'_{fixed} = \dot{U}r + U\vec{V}_{rot} \rightarrow \vec{V}' = \dot{U}(\underbrace{U^{-1}U}_{I})r + Uv = \boxed{\dot{U}(U^{-1})r' + Uv = V'}$

[Also recall the proof of] $\boxed{\vec{V}' = \vec{V} + \vec{\omega}\times\vec{r}}$



using this equation

$$\boxed{\vec{V}_{space} = \dot{\vec{R}} + \vec{\omega}\times\vec{r}_{body}}$$

[Now we can calculate T] $T = \frac{1}{2}\sum_i m_i V^2_{i\,space} = \underbrace{\left(\sum m_i\right)}_{\text{total mass } M}\frac{1}{2}\dot{\vec{R}}^2 + \dot{\vec{R}}\cdot\vec{\omega}\times\underbrace{\sum m_i r_i}_{M\vec{r}_{cm}} + \frac{1}{2}\sum m_i(\vec{\omega}\times\vec{r}_i)^2$

$T = T_{translational} + T_{rotational}$

$T_{rotational} = \frac{1}{2}\sum m_i(\vec{\omega}\times\vec{r}_i)^2$

$T_{translational} = \frac{1}{2}M\dot{\vec{R}}^2$

[lets make the origin of the body coordinate system to be the center of mass so $r_{cm} = 0$]

[Focus on Rotational Part] → Vector identity

$(\omega\times r)(\omega\times r) = \omega\cdot(r\times(\omega\times r)) = \omega\cdot(r^2\omega - (\omega\cdot r)r) = r^2(\omega\cdot\omega) - (\omega\cdot r)(\omega\cdot r)$

$\vec{a}\cdot\vec{b} = \sum_\alpha a_\alpha b_\alpha$

also $\vec{a}\cdot\vec{b} = \sum_{i,j} a_i b_j \delta_{ij}$

being summed by same 1 to 3

$$\hookrightarrow T_{rot} = \frac{1}{2}\underbrace{\sum_i}_{\text{pulled out}} m_i\left[\underbrace{\sum_{\alpha=1}^{3} r_{i,\alpha}^2}_{r^2}\underbrace{\sum_{\beta=1}^{3}\omega_\beta^2}_{\omega\cdot\omega} - \underbrace{\sum_{\alpha=1}^{3} r_{i,\alpha}\omega_\alpha}_{\omega\cdot r}\underbrace{\sum_{\beta=1}^{3} r_{i,\beta}\omega_\beta}_{\omega\cdot r}\right]$$

$$\hookrightarrow \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} r_{i,\alpha}^2\underbrace{\omega_\beta^2}_{\sum_{\beta,\gamma=1}\omega_\beta\omega_\gamma\delta_{\beta\gamma}} \rightarrow \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\sum_{\gamma=1}^{3} r_{i,\alpha}^2\,\omega_\beta\omega_\gamma\delta_{\beta\gamma} \xrightarrow{\text{swap indices}} \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\sum_{\gamma=1}^{3} r_{i,\gamma}^2\,\omega_\beta\omega_\alpha\delta_{\beta\alpha}$$

swapped

$$\hookrightarrow \underbrace{\sum_{\gamma=1}^{3} r_{i,\gamma}^2}_{r_i^2}\sum_{\alpha,\beta}^{3}\omega_\beta\omega_\alpha\delta_{\beta\alpha} \rightarrow r_i^2\sum_{\alpha,\beta}\omega_\beta\omega_\alpha\delta_{\beta\alpha}$$

and RHS $\sum_{\alpha,\beta} r_{i\alpha}r_{i\beta}\omega_\alpha\omega_\beta$ pullout $\omega_\alpha\omega_\beta$

$r_i^2 - x_i^2$

$$\boxed{T = \frac{1}{2}\sum_{i=1} m_i\left[\sum_{\alpha,\beta=1}^{3}\underbrace{\left(r_i^2\delta_{\beta\alpha} - r_{i,\alpha}r_{i,\beta}\right)}_{I_{\alpha\beta}}\omega_\alpha\omega_\beta\right]}$$

α = row
β = column

$r_i^2 = x_i^2 + y_i^2 + z_i^2$

1 = x
2 = y
3 = z

Moment of inertia tensor

$$I = \begin{bmatrix} \sum m_i (y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i x_i y_i & \sum m_i (x_i^2 + z_i^2) & -\sum m_i y_i z_i \\ -\sum m_i x_i z_i & -\sum m_i y_i z_i & \sum m_i (x_i^2 + y_i^2) \end{bmatrix}.$$

$$T_{rotation} = \frac{1}{2} [\omega_1 \ \omega_2 \ \omega_3] \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Further

[For each mass point]

$$dI = dm \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} \quad \text{and} \quad dm = \rho \, dx\,dy\,dz$$

ex $I_{33} = \iiint \rho(x^2+y^2)\,dx\,dy\,dz$

$$U^{-1} I U = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = D$$

λ_1, λ_2, λ_3 are eigenvectors that are the direction cosines of the principal axes

---

I is symmetric, $I^T = I$ it is always possible then to find an orthogonal matrix U that will diagonalize I. This transformation is called the principle-axis-transformation.

$$I \to I' = U I U^T = \begin{bmatrix} I_1' & 0 & 0 \\ 0 & I_2' & 0 \\ 0 & 0 & I_3' \end{bmatrix}$$

I_k' are the principle moments are eigenvalues, they are the rotational inertia of the body around each of the axis. I_k must be positive

[What if we displace the rotational axis by a constant vector $\vec{a}$]

$$I_{\vec{a}} = I_{cm} + M(a^2 \delta_{\alpha\beta} - a_\alpha a_\beta)$$

$I_{\alpha\beta}$ of $\vec{a}$

, if its parallel then $I_{\vec{a}} = I_{cm} + Ma^2$

Moment
consider

Moment of inertia Tensor

considered rotations only about the z-axis, and gained some useful formulas for those that told us about the angular momentum of the object and its direction.

↳ Specifically for z-axis → $\vec{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$, It can also tell you about the kinetic energy in the form of $T = \frac{1}{2} I_z \omega^2$, its an analog to mass or resistance to movement, inertia

↳ [if L is a constant the $I = \frac{L}{\omega}$ holds] → [describes motion of a figure skater, as $I \downarrow$, $\omega \uparrow$] → [As Newtons second law but rotations!] → $\tau_{sign} = I\alpha$ $\quad \tau = r \times F$

Angular Momentum for an arbitrary angular velocity

• $\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$ • two kinds of rotation, rotation about a fixed point and rotation relative to a CM

$$\vec{L} = \sum m_\alpha r_\alpha \times v_\alpha = \sum m_\alpha r_\alpha \times (\omega \times r_\alpha)$$

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \begin{bmatrix} \omega_x(y^2+z^2) - xy\,\omega_y - xz\,\omega_z \\ -\omega_x(yx) + (z^2+x^2)\omega_y - yz\,\omega_z \\ -\omega_x(zx) - zy\,\omega_y + (x^2+y^2)\omega_z \end{bmatrix} = \omega_x \begin{bmatrix} y^2+z^2 \\ -xy \\ zx \end{bmatrix} + \omega_y \begin{bmatrix} -xy \\ x^2+z^2 \\ -zy \end{bmatrix} + \omega_z \begin{bmatrix} -xz \\ -yz \\ x^2+y^2 \end{bmatrix}$$

$$m_\alpha \vec{r}_\alpha \times \vec{\omega} \times \vec{r}_\alpha = \omega_x \underbrace{\begin{bmatrix} m_\alpha(y_\alpha^2+z_\alpha^2) \\ -m_\alpha x y_\alpha \\ m_\alpha x_\alpha z_\alpha \end{bmatrix}}_{\text{Scale } \omega_x \text{ a certain way}} + \omega_y \underbrace{\begin{bmatrix} -m_\alpha x_\alpha y_\alpha \\ m_\alpha(x_\alpha^2+y_\alpha^2) \\ -m_\alpha z_\alpha y_\alpha \end{bmatrix}}_{\text{Scale } \omega_y \text{ a certain way}} + \omega_z \underbrace{\begin{bmatrix} -m_\alpha x_\alpha z_\alpha \\ -m_\alpha y_\alpha z_\alpha \\ m_\alpha(x_\alpha^2+y_\alpha^2) \end{bmatrix}}_{\text{Scale } \omega_z \text{ a certain way}}$$

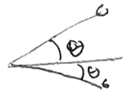
$\boxed{\text{as a matrix}}$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \overbrace{\begin{bmatrix} m_\alpha(y_\alpha^2+z_\alpha^2) & -m_\alpha x_\alpha y_\alpha & -m_\alpha x_\alpha z_\alpha \\ -m_\alpha(x_\alpha y_\alpha) & m_\alpha(x_\alpha^2+z_\alpha^2) & -m_\alpha y_\alpha z_\alpha \\ m_\alpha(x_\alpha z_\alpha) & -m_\alpha z_\alpha y_\alpha & m_\alpha(x_\alpha^2+y_\alpha^2) \end{bmatrix}}^{\text{moment of inertia tensor! cool}} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

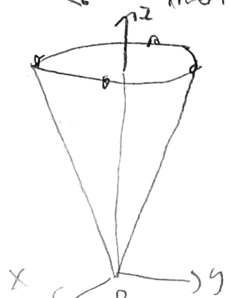
/ I is symetric so I^T is the same

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Last problem on the back; all the cross terms are 0.

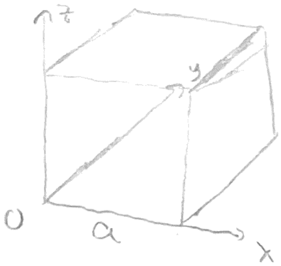
→ just like the mass symetry example  , two masses mirror cancel the moment of inertia. $x=0$ symmetry and $y=0$ symmetry

$$I = \frac{3}{20} M \begin{bmatrix} R^2+4h^2 & 0 & 0 \\ 0 & R^2+4h^2 & 0 \\ 0 & 0 & 2R^2 \end{bmatrix}$$



cross terms, look for symmetry arguments or calculate it.

Inertia tensor for a solid Cube



a) a solid cube of Mass M rotating about corner O

$$I_{xx} = \int_0^a dx \int_0^a dy \int_0^a dz\, \rho(y^2+z^2) \quad , \quad \rho = \frac{M}{\text{Volume}} = \frac{M}{a^3} = \text{constant}$$

$$I_{xx} = \int_0^a\int_0^a\int_0^a \rho(y^2+z^2)\,dz\,dy\,dx = \rho\int_0^a\int_0^a y^2 z\Big|_0^a + \frac{1}{3}z^3\Big|_0^a\,dy\,dx$$

$$I_{xx} = \rho\int_0^a\int_0^a ay^2 + \frac{1}{3}a^3\,dy\,dx = \rho\int_0^a \frac{ay^3}{3}\Big|_0^a + \frac{1}{3}a^3y\Big|_0^a\,dx = \rho a\left(\frac{a^4}{3}+\frac{1}{3}a^4\right)$$

$\boxed{I_{xx} = \frac{2}{3}\rho a^5 = \frac{2}{3}Ma^2}$ and by symmetry $\boxed{I_{yy}}$ and $\boxed{I_{zz} = \frac{2}{3}Ma^2}$

$$I_{xy} = \int_0^a\int_0^a\int_0^a -\rho xy\,dz\,dy\,dx = -\rho a\int_0^a x\frac{y^2}{2}\Big|_0^a dx = -\rho a\int_0^a \frac{a^2}{2}x\,dx = -\rho a\left(\frac{a^4}{4}\right) = -\frac{Ma^2}{4}$$

by symmetry $\boxed{I_{xy} = I_{yx} = I_{zx} = I_{zy} = I_{yz} = I_{xz} = -\frac{Ma^2}{4}}$

$$\boxed{I_{cube} = \begin{bmatrix} \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 \end{bmatrix} = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}}$$

If $\vec{\omega} = \begin{bmatrix} w \\ 0 \\ 0 \end{bmatrix}$ then $\vec{L} = I\vec{\omega}$

$$\vec{L} = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{bmatrix} w \\ 0 \\ 0 \end{bmatrix} = \frac{Ma^2}{12}\begin{bmatrix} 8w \\ -3w \\ -3w \end{bmatrix}$$

$\boxed{\vec{L} = Ma^2 w\left(\frac{2}{3}, -\frac{1}{4}, -\frac{1}{4}\right)}$

if the cube is rotating about its main diagonal then we get

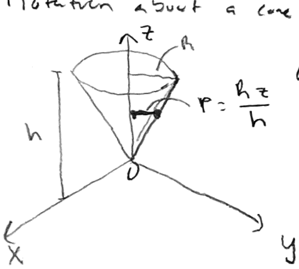
$\hat{u}$ in direction of the rotation along the diagonal $\rightarrow \hat{u} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \vec{\omega} = |\omega|\,\hat{u} = \frac{w}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

then $\vec{L} = I\vec{\omega} = \frac{Ma^2}{12}\frac{w}{\sqrt{3}}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{Ma^2}{12}\frac{w}{\sqrt{3}}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \boxed{\frac{Ma^2}{6}\vec{\omega} = \vec{L}}$ $\vec{L}$ in same dir as $\vec{\omega}$

b) If the cube is rotating about its center then we have to move O to the center. Now all of the integrals run from $-\frac{a}{2}, \frac{a}{2}$. If you do so then

$I_{xx} = \frac{1}{6}Ma^2$, $I_{xy} = 0$ = all off diagonals thus $\boxed{I = \frac{1}{6}Ma^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \rightarrow L = I\omega = \frac{1}{6}Ma^2\vec{\omega}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ does nothing

Rotation about a cone



$$I_{zz} = \mu\iiint (x^2+y^2)\,dx\,dy\,dz = \int_0^h\int_0^{2\pi}\int_0^{r=\frac{Rz}{h}} \mu\, p^2\,dp\,d\phi\,dz = 2\pi\mu\int_0^h\int_0^{\frac{Rz}{h}} p^3\,dp\,dz$$

dont forget p again!

density $\rightarrow$ to not get confused w/ radius p

$$\mu = \frac{M}{V} = \frac{3M}{\pi R^2 h}$$

$$\hookrightarrow I_{zz} = 2\pi\mu\int_0^h \frac{p^4}{4}\Big|_0^{\frac{Rz}{h}} dz = 2\pi\mu\int_0^h \frac{R^4z^4}{4h^4}\,dz = 2\pi\mu\frac{R^4z^5}{20h^4}\Big|_0^h = \frac{2\pi\mu R^4 h}{20}$$

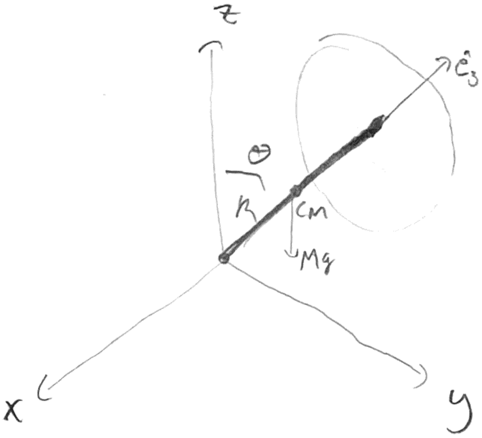
$$\hookrightarrow \boxed{I_{zz} = \frac{3}{10}MR^2}$$

Since of rotational symmetry $\Rightarrow I_{xx} = I_{yy} = \int_V dV(y^2+z^2)\mu = \underbrace{\int_V \mu y^2\,dV}_{①} + \underbrace{\int_V \mu z^2\,dV}_{②}$

① $\int_0^h\int_0^{2\pi}\int_0^{\frac{Rz}{h}} \mu p^3\sin^2\phi\,dp\,d\phi\,dz$

② $\int_0^h\int_0^{2\pi}\int_0^{\frac{Rz}{h}} \mu z^2 p\,dp\,d\phi\,dz = \frac{3\cdot 4h^2}{20}M$ thus $\boxed{I_{xx} = I_{yy} = \frac{3}{20}M(R^2+4h^2)}$

cession of a weak torque



[$\hat{e}_2, \hat{e}_1$ are $\perp$ to $\hat{e}_3$], [Mg produces a torque that causes L to change]

Due to the choice of axis, $I' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

[First suppose gravity is turned off so its just spinning about $\vec{\omega} = \omega \hat{e}_3$] $\rightarrow \vec{L} = \lambda_3 \vec{\omega} = \lambda_3 \omega e_3$ (ω has only 3rd component)

[Switch gravity back on, which will cause a torque, $\vec{\tau} = \vec{R} \times M\vec{g}$, $\Rightarrow \tau = RMg\sin\theta$ (direction $\perp$ to $\hat{e}_3$ and $\hat{z}$)] $\rightarrow$ Existence of a torque implies there will be a change in momentum. $\vec{\tau} = \dot{\vec{L}}$

↳ a changing $\vec{L}$ implies $\vec{\omega}$ starts to change and [ω_1 and ω_2 are now no longer 0] $\longrightarrow$ If torque is small then ω_1, ω_2 are small so its a good approximation

↳ $\lambda_3 \omega \dot{e}_3 = \vec{R} \times M\vec{g}$ and $\vec{R} = Re_3$ and $\vec{g} = -g\hat{z}$ plug in

↳ $\dot{e}_3 = \dfrac{Re_3}{\lambda_3 \omega} \times (-Mg)\hat{z} = \dfrac{Mg\hat{z}}{\lambda_3 \omega} \times \overset{\text{constant}}{Re_3} = \underbrace{\dfrac{RMg}{\lambda_3 \omega}\hat{z}}_{\vec{\Omega}} \times e_3$

this says that the axis of the top e_3 rotates w/ angular velocity Ω about the vertical $\hat{z}$] torque by gravity causes it to process

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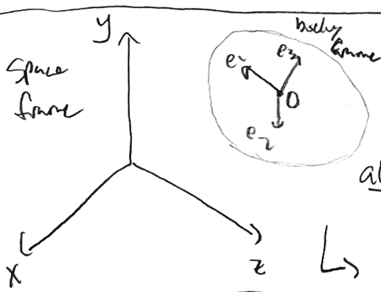


[If we have the principle axis then] $\rightarrow \vec{L} = \begin{bmatrix} \lambda_1 \omega_1 \\ \lambda_2 \omega_2 \\ \lambda_3 \omega_3 \end{bmatrix}$, [In space frame if a torque is acting on the body] $\rightarrow \left(\dfrac{dL}{dt}\right)_{space} = \vec{\tau}$

also: $\left(\dfrac{dL}{dt}\right)_{space} = \left(\dfrac{dL}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = \dot{L} + \vec{\omega} \times \vec{L}$ and subbing into

↳ $\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\tau}$

$$\begin{aligned} \tau_1 &= \lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_1)\omega_2 \omega_3 \\ \tau_2 &= \lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3 \omega_1 \\ \tau_3 &= \lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1 \omega_2 \end{aligned}$$

EULER'S Equations

or if $\frac{dL}{dt} = 0$ or no net torques $\left(\frac{dL}{dt}\right)_{space} = 0 = \left(\frac{dL}{dt}\right)_{body} + \vec{\omega} \times$

$L_{body} = \begin{bmatrix} \lambda_1 \omega_1 \\ \lambda_2 \omega_2 \\ \lambda_3 \omega_3 \end{bmatrix}$ then

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} &= \omega_3 L_2 - \omega_2 L_3 = \omega_2 \omega_3 (I_2 - I_3) \\ I_2 \frac{d\omega_2}{dt} &= \omega_1 L_3 - \omega_3 L_1 = \omega_1 \omega_3 (I_3 - I_1) \\ I_3 \frac{d\omega_3}{dt} &= \omega_2 L_1 - \omega_1 L_2 = \omega_1 \omega_2 (I_1 - I_2) \end{aligned}$$

For eulers equations, often times the torque is very complicated so more useful when torque is 0.

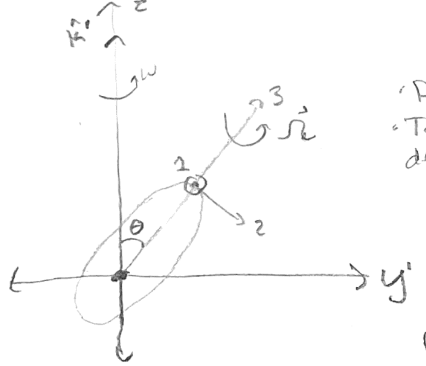
[Ex of the spinning top the moment of inertia is along the diagonal $I_1 = I_2$] → thus $\lambda_3 \dot{\omega}_3 = \tau_3$ ← but the torque for gravity is always $\perp e_3$ so τ_3 is 0. there is no component of torque along e_3

thus $\lambda_3 \dot{\omega}_3 = 0$

and $\boxed{\omega_3 \text{ is a constant}}$

Spinning top again

- most general motion possible for a torque-free top (symmetric)
- top spins around 3 axis w/ constant angular velocity
- top is tilted by an angle θ
- Define z' to be along $\vec{L}$
- Take a snapshot and define y' space axis to be in the plane momentarily defined by the 3-axis



$\hat{k}' = \cos\theta\,\hat{k} - \sin\theta\,\hat{j}$ → [at $\theta = 0$: $\hat{k} = 1$ and $\hat{j} = 0$] [at $\theta = 90$: $\hat{k} = 0$, $\hat{j} = -1$] so $\boxed{\hat{k}' = \cos\theta\,\hat{k} - \sin\theta\,\hat{j}}$

$L = r \times p$ if r is a constant vector

$\frac{dL}{dt} = r \times \dot{p} = \tau$ if $\tau = 0$ L is a constant

[No torques on it (gravity is being neglected)] → L is fixed & no torque — choose $\boxed{\vec{L} = L\hat{k}'}$

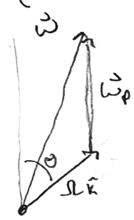
T is fixed & no work is being done

Now for a mind fuck

- part of the contribution to $\vec{\omega}$ is due to the (angular momentum) spin around $\hat{e}_3$, $\vec{\Omega}$. (part of angular momentum is in $\vec{\Omega}$)

↳ [You will see this part after multiplying w/ $\vec{I}$ and $I_3 \Omega \hat{k}$ ← here] → [For our picture at an instance $I_3 \Omega \hat{k}$ lies in $z'y'$ plane]

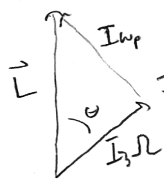
↳ [since we have an $\vec{\omega}$ and this $\Omega\hat{k}$ there has to be another ω_p to bring us to $\vec{\omega}$]



- multiplying $\vec{\omega}_p$ by I gets us a missing piece of angular momentum ← recall $I_1 = I_2$

$\boxed{\text{First}}$ $\vec{\omega}_p = \omega_{p,1}\hat{i} + \omega_{p,2}\hat{j} + \omega_{p,3}\hat{k}$ → [Now $L = \vec{I}\vec{\omega}$] → $\vec{I}\vec{\omega}_p \equiv I_1(\omega_{p,1}\hat{i} + \omega_{p,2}\hat{j}) + I_3\omega_{p,3}\hat{k} = L\hat{k}' - I_3\Omega\hat{k}$

is the



$\vec{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$ $I_1 = I_2$

↳ $\vec{I}\vec{\omega}_p = \underbrace{L\hat{k}'}_{L\cos\theta\,\hat{k} - L\sin\theta\,\hat{j}} - I_3\Omega\hat{k} = (L\cos\theta - I_3\Omega)\hat{k} - L\sin\theta\,\hat{j}$

Then $\omega_{p,1} = 0$

$I_1\omega_{p,2} = -L\sin\theta$

$I_3\omega_{p,3} = L\cos\theta - I_3\Omega$

① $\omega_{p,2} = \left(-\frac{L}{I_1}\sin\theta\right)$

② $\omega_{p,3} = \left(\frac{L}{I_3}\cos\theta - \Omega\right)\hat{k}$

Recall $T = \frac{1}{2}\omega^T I \omega$

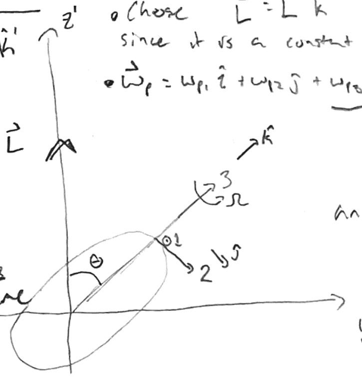
$T = \frac{1}{2}\left[0,\ -\frac{L}{I_1}\sin\theta,\ \frac{L}{I_3}\cos\theta\right] \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{L}{I_1}\sin\theta \\ \frac{L}{I_3}\cos\theta \end{bmatrix}$

$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \omega_{p,1} \\ \omega_{p,2} \\ \omega_{p,3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix}$

$-\Omega + \Omega$ from

...view

- $\hat{k}' = \cos\theta\,\hat{k} - \sin\theta\,\hat{j}$
- Chose $\vec{L} = L\hat{k}'$ since it is a constant
- $\vec{\omega}_p = \omega_{p1}\hat{i} + \omega_{p2}\hat{j} + \omega_{p3}\hat{k}$

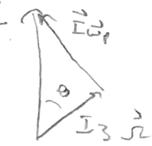


Axis shown for principal axis to get $\vec{L}$ this way comes out too, or prime axis for body frame

and $I_1 = I_2$ (diagonal matrix)

Angular momentum again

$\vec{\omega}_p$ ← missing piece bc we know there's an $\vec{\omega}$ and $\vec{\Omega}$ so there has to be a $\vec{\omega}_p$ to get to $\vec{\omega}$



$I_3\Omega + I\omega_p = L$

$I\omega_p = L - I_3\Omega$

$\boxed{\vec{\omega}_p + \Omega\hat{k} = \vec{\omega}}$

$$I\vec{\omega}_p = I_1(\omega_{p1}\hat{i} + \omega_{p2}\hat{j}) + I_3\omega_{p3}\hat{k}$$

and

$$I\vec{\omega}_p = L\hat{k}' - I_3\Omega\hat{k}$$

$\hat{k}' = \cos\theta\,\hat{k} - \sin\theta\,\hat{j}$ → $I\vec{\omega}_p = (L\cos\theta - I_3\Omega)\hat{k} - L\sin\theta\,\hat{j}$

then $\begin{bmatrix} I_1\omega_{p2} = -L\sin\theta \\ \omega_{p1} = 0 \quad \text{and} \quad I_3\omega_{p3} = L\cos\theta - I_3\Omega \end{bmatrix}$ ← Solve for $\omega_{p1}, \omega_{p2}, \omega_{p3}$ then use $\vec{\omega} = \vec{\omega}_p + \Omega\hat{k}$

Now we can plug into $T = \frac{1}{2}\omega^T I \omega \Rightarrow T = \frac{1}{2}\left[0, \frac{-L}{I_1}\sin\theta, \frac{L}{I_3}\cos\theta\right]\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{L}{I_1}\sin\theta \\ \frac{L}{I_1}\cos\theta \end{bmatrix}$

$$\boxed{T = I_1\omega_2^2 + I_3\omega_3^2 = \frac{L^2}{2}\left[\frac{\sin^2\theta}{I_1} + \frac{\cos^2\theta}{I_3}\right]}$$

T is a constant then θ is a constant

$$\left.\begin{aligned} I\omega_p &= -L\sin\theta\,\hat{j} + (L\cos\theta - I_3\Omega)\hat{k} \\ I\omega_p &= I_1(\cancel{\omega_{p1}}^{0}\hat{i} + \omega_{p2}\hat{j}) + I_3\omega_{p3}\hat{k} \end{aligned}\right\} \begin{bmatrix} L\cos\theta = I_3(\Omega + \omega_{p3}) \\ -L\sin\theta = I_1\omega_{p2} \end{bmatrix}$$

← you can solve again come back

*constant θ means $\omega_{py'} = 0$ otherwise θ would change

→ $\omega_{p1} = 0$

$\hat{j}$ $\quad \omega_{p2} = -\frac{L}{I_1}\sin\theta$

$\hat{k}$ $\quad \omega_{p3} = \frac{L}{I_3}\cos\theta - \Omega$

w/ $\hat{k} = \cos\theta\,\hat{k}' - \sin\theta\,\hat{j}'$

$\hat{j} = \cos\theta\,\hat{j}' - \sin\theta\,\hat{k}'$

↔ [transforming $\vec{\omega}_p$] ⟶

$$\omega_{py'} = 0 = L\sin\theta\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right) - \Omega\sin\theta$$

$$\omega_{pz'} = L\left[\frac{\sin^2\theta}{I_1} + \frac{\cos^2\theta}{I_3}\right] - \Omega\cos\theta$$

→ [Solve for Ω] → $\boxed{\Omega = L\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\cos\theta}$ now look in space frame

$\omega_{px'} = 0$

$\omega_{py'} = 0$

$\omega_{pz'} = \text{value}$ ⟶

pg 297 for proof

↳ So $|\omega_p| = \vec{\omega}_p \cdot \hat{k}' = L\left(\frac{\sin^2\theta}{I_1} - \frac{\cos^2\theta}{I_3}\right) - \Omega\cos\theta = L\left(\frac{\sin^2\theta}{I_1} - \frac{\cos^2\theta}{I_3}\right) - L\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\cos^2\theta = \boxed{\frac{L}{I_1} = |\vec{\omega}_p|}$

Calculating the Precession Rate

if we define $\Omega_p = \left(\frac{I_3}{I_1} - 1\right)\omega_3$

Use Euler's equations:

1) $I_1\frac{d\omega_1}{dt} = \omega_2\omega_3(I_2 - I_3)$

2) $I_2\frac{d\omega_2}{dt} = \omega_1\omega_3(I_3 - I_1)$

3) $I_3\frac{d\omega_3}{dt} = \omega_1\omega_2\underbrace{(I_1 - I_2)}_{=0 \;\; I_1 = I_2} = 0$

$$\frac{d\omega_1}{dt} + \left(\frac{I_3}{I_1} - 1\right)\omega_3\omega_2 = 0$$

$$\frac{d\omega_2}{dt} - \left(\frac{I_3}{I_1} - 1\right)\omega_3\omega_1 = 0$$

→ take derivatives and plug in →

$$\begin{bmatrix} \frac{d^2\omega_1}{dt^2} + \Omega_p^2\omega_1 = 0 \\ \frac{d^2\omega_2}{dt^2} + \Omega_p^2\omega_2 = 0 \end{bmatrix}$$

→ $\omega_1 = A\sin(\Omega_p t + \phi)$

$\omega_1^2 + \omega_2^2 = A^2$

→ $\omega_2 = A\cos(\Omega_p t + \phi)$

- $\omega_1(t=0) = 0$ to make consistent w/ earlier def → $\phi = 0$, $|\vec{\omega}| = \underbrace{\omega_1^2 + \omega_2^2}_{A^2} + \omega_3^2$

ω_3 constant $= \frac{I_1}{I_3 - I_1}\Omega_p$ = const

more notes after could be important?

Euler Angles

[can convert between space and body coordinates and vice versa]

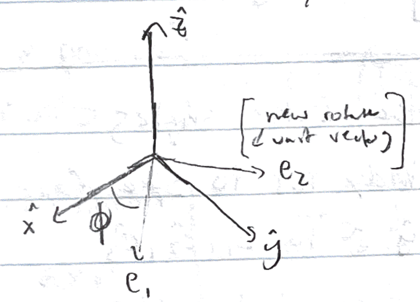
$$\overset{\text{space}}{r'} = U \overset{\text{body}}{r}$$

$$\overset{\text{body}}{r} = U^{-1} \overset{\text{space}}{r'}$$

U must be a function of 3 params Θ, Φ, ψ

[We are going to use these 3 angles to define a sequence of rotations acting on the body coord which finally converts this coordinate sys into one that has an arbitrary orientation w.r.t the space coord.]

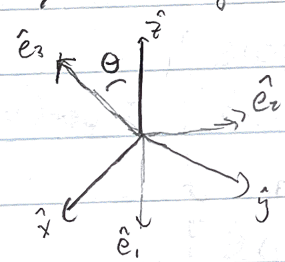
(Step a) Start w/ your space and body coordinates aligned and rotate by an angle ϕ about the z-axis



$$U_1^{-1} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \text{new} \\ \text{vectors} \\ \text{after} \\ \text{rot} \end{bmatrix} = U_1^{-1} \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix}$$

$\hat{x} \to \hat{e}_1$, $\hat{y} \to e_2$, $\hat{z} \to \hat{e}_3$

(Step B) take e_3 which is aligned w/ $\hat{z}$ and rotate it by the angle Θ.

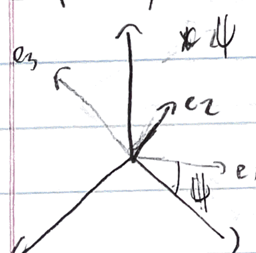


$$U_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

rotating about $\hat{e}_1$ axis; when $\hat{e}_2$ gets sent to; when $\hat{e}_3$ gets sent to

(Step C) rotate about the the $\hat{e}_1, \hat{e}_2$ plane

* ψ is wrt the last rotation * pg 302 for details



$$U_3^{-1} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = U_3 \begin{bmatrix} \text{last unit vectors from part B} \end{bmatrix}$$

- $U^{-1} = U_3^{-1} U_2^{-1} U_1^{-1}$

- multiply out:

$$U = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi \\ \sin\theta\sin\psi & \sin\theta\cos\psi \end{bmatrix}$$

last column: $\begin{bmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{bmatrix}$

- Calculating ω in terms of the Euler angles
- find angular velocity of the body rel to space frame

$\omega_{step\,a} = \dot\phi \hat z$ we go from $\hat x, \hat y, \hat z$ to $\hat e_1, \hat e_2, \hat e_3$

$\omega_{step\,b} = \dot\theta e_2'$ but have the intermediate step

$\omega_{step\,c} = \dot\psi e_3$ where we get $\hat e_1', \hat e_2', \hat e_3'$

$$\left[\begin{matrix} (\hat x, \hat y, \hat z) & \rightarrow (\hat e_1', \hat e_2', \hat e_3') & \rightarrow (\hat e_1, \hat e_2, \hat e_3) \\ \phi & \theta & \psi \end{matrix}\right]$$

$\omega = \omega_a + \omega_b + \omega_c = \dot\phi \hat z + \dot\theta \hat e_2' + \dot\psi \hat e_3$

↳ we want in terms of just $\hat e_1, \hat e_2, \hat e_3$

But we can do some tricks

$$\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$$

- we want in terms of just $\hat{e}_1, \hat{e}_2, \hat{e}_3$ but if we just consider the symmetric case where $I_{xx} = I_{yy}$ we don't have to do this.

↳ [The point B] → (w/ $I_{xx} = I_{yy}$ any two ⊥ axis in the) → (plane of $\hat{e}_1, \hat{e}_2$ are also principle axis) SO!

[We can just get to the point where $\hat{e}_1', \hat{e}_2', \hat{e}_3$]

notice

$$\hat{z} = \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1' \rightarrow \boxed{\vec{\omega} = (-\dot{\phi}\sin\theta)\hat{e}_1' + \dot{\theta}\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3}$$

(w/ principle axis) $\rightarrow L = (I_{xx}\omega_1, I_{yy}\omega_2, I_{zz}\omega_3)$

[w/ principle axis] $\rightarrow T = \frac{1}{2}I_{xx}\omega_1^2 + \frac{1}{2}I_{yy}\omega_2^2 + \frac{1}{2}I_{zz}\omega_3^2$

$$T = \frac{1}{2}I_{xx}(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_{zz}(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

* $\vec{\omega}$ is not an integrable function

$$\vec{\omega}_{body} = \begin{pmatrix} \dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta \\ -\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\sin\theta \\ \dot{\psi} + \dot{\phi}\cos\theta \end{pmatrix}$$

$$\vec{\omega}_{space} = \begin{bmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\sin\theta \\ \dot{\theta}\sin\phi - \dot{\psi}\cos\phi\sin\theta \\ \dot{\phi} + \dot{\psi}\cos\theta \end{bmatrix}$$

(Ce)nter of mass

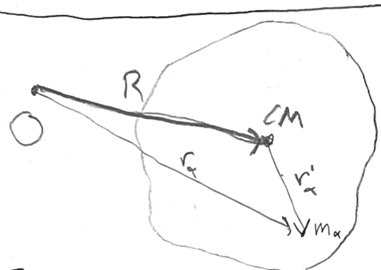
$$R_{cm} = \frac{1}{M_{tot}} \cdot (m_1 r_1 + \dots + m_N r_N) = \frac{1}{M_{tot}} \sum_{\alpha=1}^{N} m_\alpha r_\alpha$$

R_{cm} is the weighted average of the positions $r_1 \to r_N$ where r_α is weighted by m_α

$$P_{cm} = M_{tot} \dot{R}_{cm} \quad \text{and} \quad R_{cm} = \frac{1}{M_{tot}} \int r\, dm = \frac{1}{M_{tot}} \int \underset{\text{density}}{\rho}\, r\, dV$$

If M_{tot} is the total mass and R_{cm} is the position of the center mass then

$$\boxed{\vec{F}_{external} = M_{tot} \ddot{\vec{R}}_{cm}}$$



$$\vec{r}_\alpha = \vec{R} + \vec{r}'_\alpha \quad \to \quad \dot{\vec{r}}_\alpha = \dot{\vec{R}} + \dot{\vec{r}}'_\alpha$$

$$\vec{l}_\alpha = r_\alpha \times p_\alpha = r_\alpha \times m_\alpha \dot{r}_\alpha$$

$$\vec{L} = \sum \vec{l}_\alpha = \sum r_\alpha \times m_\alpha \dot{r}_\alpha = \sum (R + r'_\alpha) \times m_\alpha (\dot{R} + \dot{r}'_\alpha)$$

FOIL it

$$\vec{L} = \sum R \times m_\alpha \dot{R} + \sum R \times m_\alpha \dot{r}'_\alpha + \sum r'_\alpha \times m_\alpha \dot{R} + \sum r'_\alpha \times m_\alpha \dot{r}'_\alpha$$

$$\hookrightarrow \vec{L} = R \times M\dot{R} + R \times \sum m_\alpha \dot{r}'_\alpha + \underbrace{\left(\sum m_\alpha r'_\alpha\right)}_{\text{center of mass relative to the center of mass which makes } r'_\alpha = 0} \times \dot{R} + \sum r'_\alpha \times m_\alpha \dot{r}'_\alpha$$

then $m_\alpha \dot{r}'_\alpha$ is also zero if you differentiate

$$\vec{L} = \underbrace{\vec{R} \times \vec{P}}_{\text{motion of center of mass}} + \underbrace{\sum r'_\alpha \times m_\alpha \dot{r}'_\alpha}_{\text{motion relative to the center of mass}}$$

(ex) Motion of a planet around our sun. This guy asserts the total angular momentum of the planet is the angular momentum of the orbital motion and its angular momentum about its center of mass.

$$\boxed{L = L_{orb} + L_{spin}}$$

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{r}_\alpha^2 \quad \text{and} \quad \dot{r}_\alpha^2 = \dot{R}^2 + \dot{r}'^2_\alpha + 2\dot{R}\dot{r}'_\alpha \quad \to \quad T = \underbrace{\frac{1}{2} \sum m_\alpha \dot{R}^2}_{\text{motion of CM}} + \underbrace{\frac{1}{2} \sum m_\alpha \dot{r}'^2_\alpha}_{\text{motion about CM}} + \underbrace{\dot{R} \cdot \sum m_\alpha \dot{r}'_\alpha}_{=0}$$

$= 0$, find the center of mass rel to CM which makes $r_\alpha = 0$

$\longrightarrow$ potential Energy on the back

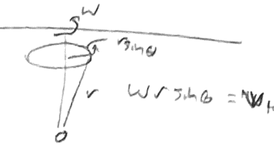
$$U = \underbrace{U^{ext}}_{\text{potentials due to external forces}} + \underbrace{U^{int}}_{\text{internal energy}}$$

gravity →

$U^{int} = \sum_{\alpha<\beta} U_{\alpha\beta}(r_{\alpha\beta})$ ← distance between particles α and β

[Sum of the potential energies for all pairs of particles]

[For a rigid body all of the interparticle distances are fixed. Therefore U^{int} is a constant, and can be ignored.]



$$L = \sum l_\alpha = \sum r_\alpha \times m_\alpha v_\alpha$$

→ v_α are the velocities by which the pieces are being carried in circles by $\vec{\omega}$, then we know $\boxed{v_\alpha = \omega \times r_\alpha}$

$\omega r \sin\theta = v_\alpha$

[Lets say: $\omega = (0, 0, \omega)$ and $r_\alpha = (x_\alpha, y_\alpha, z_\alpha)$] → $\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha = (-\omega y_\alpha, \omega x_\alpha, 0)$

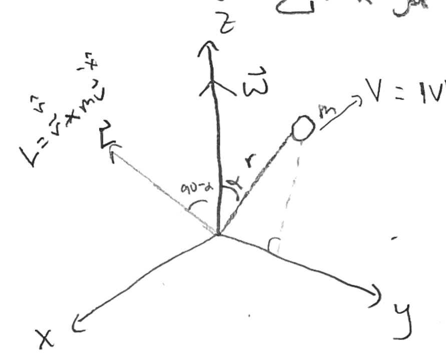
Finally: $l_\alpha = m_\alpha r_\alpha \times v_\alpha = m_\alpha \omega(-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2)$

We want angular momentum to look like normal momentum, mv so $I_z = \sum m_\alpha \rho_\alpha^2$ (like mass)

[Lets calculate components of L] → $L_z = \sum m_\alpha \underbrace{(x_\alpha^2 + y_\alpha^2)}_{\rho_\alpha^2 = \text{distance from z-axis}} \omega = \underbrace{\sum m_\alpha \rho_\alpha^2}_{I_z = \text{usually an integral}} \omega = I_z \omega$

↳ [and T is] $T = \frac{1}{2}\sum m_\alpha v_\alpha^2 = \frac{1}{2}\sum m_\alpha \rho_\alpha^2 \omega^2 = \frac{1}{2} I_z \omega^2$ ($v_\alpha = \rho_\alpha \omega_\alpha$)

[Now for L_x and L_y] $L_x = -\sum m_\alpha x_\alpha z_\alpha \omega$, $L_y = -\sum m_\alpha y_\alpha z_\alpha \omega$] non zero, even though $\vec{\omega}$ points in the z-dir L may be in a different direction. Thus $\cancel{\vec{L} = I\vec{\omega}}$ is generally not true!



$V = |V|(-\hat{x})$ → $\vec{L} = \vec{r} \times m\vec{v}$, then clearly $L_y \neq 0$

10.2 Taylor

- L will also sweep around the axis, then $\dot{L} \neq 0$ ← $\dot{L}$ dir is $+\hat{x}$
- Torque is required to keep the object steady, if $\dot{L}$ is $+\hat{x}$ then τ $-\hat{x}$ dir counterclockwise w/ $\vec{\omega}$

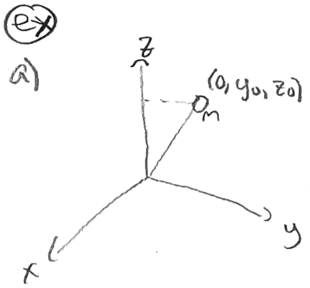
Streamline notation: $L_x = \underbrace{I_{xz}}\omega$ and $L_y = \underbrace{I_{yz}}\omega$ → products of inertia of the body

$I_z \to I_{zz} = \sum m_\alpha \rho_\alpha^2 = \sum m_\alpha (x_\alpha^2 + y_\alpha^2)$

Thus $\vec{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$

ex

a)

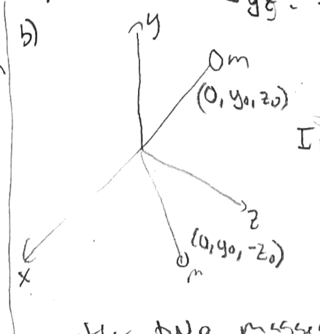


$x = 0$ so $I_{xz} = 0$

$I_{zy} = -z_0 y_0 m$

$I_{zz} = m y_0^2$

b)



$I_{xz} = -\sum_{\alpha=1} m_\alpha x_\alpha z_\alpha = 0 + 0$

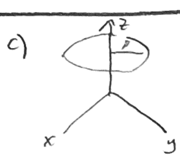
I

$I_{yz} = -\sum m_\alpha y_\alpha z_\alpha = -m[y_0 z_0 + y_0(-z_0)] = 0$ woah!

$I_{zz} = m(0 + y_0^2 + 0 + y_0^2) = m y_0^2$

the two masses mirrored cancel one another!

c)



every mass can be paired w/ one across from it at $(-x_\alpha, -y_\alpha, z_\alpha)$ to cancel it so $= 0$

$I_{xz} = -\sum m_\alpha x_\alpha y_\alpha = 0$

$I_{yz} = -\sum m_\alpha y_\alpha z_\alpha =$ Same z_α but have opposite y_α

$I_{zz} = \sum m_\alpha \underbrace{(x_\alpha^2 + y_\alpha^2)}_{\text{constant } \rho_0^2} = M\rho_0^2$