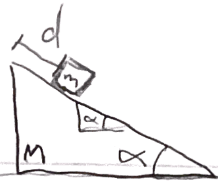
W/ inclined plane grad Gcel

Virtual Work



[Lets pick some natural variables from our block sliding problem] → [lets choose the position of the inclined plane (x) and d, the distance from the top, This coordinate is defined w/ respect to an accelerating ref frame]

↳ [Lets look at the kinetic energies] → For inclined plane → $\boxed{T_{IP} = \frac{1}{2} M \dot{X}^2}$ * using d and X

V_y is the y comp

Small block: V_x and $V_y = ?$ for V_x it is $\dot{X}$ + X component of $\dot{d}$

$$\left.\begin{aligned} V_x(SB) &= \dot{X} + \dot{d}\cos(\alpha) \\ V_y(SB) &= -\dot{d}\sin(\alpha) \end{aligned}\right\} \quad T_{SB} = \frac{1}{2} m (V_x^2 + V_y^2) = \frac{1}{2} m (\dot{X}^2 + 2\dot{d}\dot{X}\cos(\alpha) + \dot{d}^2)$$

$$T_{tot} = T_{SB} + T_{IP} = \frac{1}{2}(m+M)\dot{X}^2 + \frac{1}{2} m (\dot{d}^2 + 2\dot{d}\dot{X}\cos(\alpha))$$

[now for some virtual shift]

lets make some displacements δd and δX, they are virtual bec they are imaginary

[Reqs for virtual displacements] ① time is fixed ② displacements are infentesimal ③ no change in time dependency ④ as many possible virtual displacements needed to describe the motion ⑤ obey constraints of problem

↳ so ⑤ you cant do a δY for the inclined plane

[multiply by dt]

$$\left.\begin{aligned} \delta \vec{r}_{SB} &= (\delta X + \delta d \cos\alpha)\hat{i} - \delta d \sin\alpha\, \hat{j} \\ \delta \vec{R}_{IP} &= \delta X \hat{i} \end{aligned}\right\}$$ representing infentesimal vectors in a cartesian system

↳ [Recall $W = \int \vec{F} \cdot d\vec{r}$] → $\delta W = \vec{F} \cdot \delta \vec{r}$ ← no integration necessary bec the displacement is an infentesimal one

Notice! We dont have to take into account the constraint forces bec they act ⊥ to our displacements. ← go to zero in dot product :), (lets look at gravitational non constraint forces)

$\delta W(SB) = mg\sin(\alpha)\,\delta d$ (mgh form), $\delta W(IP) = 0$ ← gravity doesnt do work on it

↳ get from $F_g = -mg\hat{j}$ (small block) $\quad F_g \cdot \delta\vec{r} = -mg\,\delta d \sin(\alpha) = \delta W$

remember $\delta W = \vec{F} \cdot \delta r$

As always in classical mechanics, the heart of the dynamics lies in $\vec{F} = m\vec{a} = \dot{\vec{p}}$ (if mass isn't changing)

$$\boxed{\delta W - \dot{\vec{p}} \cdot \delta \vec{r} = 0 \qquad \text{[d'Alemberts Principle]}}$$

Mathematical identity: $\dot{\vec{p}} \cdot \delta \vec{r} = \frac{d(\vec{p} \cdot \delta \vec{r})}{dt} - \vec{p} \cdot \frac{d(\delta \vec{r})}{dt}$????????

Using chain rule of $\vec{r}(d, x)$: $\boxed{\vec{p} \cdot \delta \vec{r} = \vec{p} \cdot \frac{\partial \vec{r}}{\partial d} \delta d + \vec{p} \cdot \frac{\partial \vec{r}}{\partial x} \delta x}$

↳ finding $\dot{\vec{r}} = \frac{\partial \vec{r}}{\partial d} \dot{d} + \frac{\partial \vec{r}}{\partial x} \dot{x}$

↳ $\vec{p} \cdot \delta \dot{\vec{r}} = \vec{p} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{d}} \delta d + \vec{p} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{x}} \delta x$ ← since $\frac{\partial \vec{r}}{\partial d}$ is a partial w/out respect to time we can replace it w/ $\frac{\partial \dot{\vec{r}}}{\partial \dot{d}}$

→ also $\vec{p} = m\dot{\vec{r}}$ → $p_x = \frac{\partial T}{\partial \dot{x}}$, $p_y = \frac{\partial T}{\partial \dot{y}}$, $p_z = \frac{\partial T}{\partial \dot{z}}$

Using $\frac{\partial T}{\partial \dot{d}} = \underbrace{\frac{\partial T}{\partial \dot{x}}}_{p_x} \underbrace{\frac{\partial \dot{x}}{\partial \dot{d}}}_{r_x} + \underbrace{\frac{\partial T}{\partial \dot{y}}}_{p_y} \underbrace{\frac{\partial \dot{y}}{\partial \dot{d}}}_{r_y} + \underbrace{\frac{\partial T}{\partial \dot{z}}}_{p_z} \underbrace{\frac{\partial \dot{z}}{\partial \dot{d}}}_{r_z}$ and for $\frac{\partial T}{\partial \dot{x}}$

$\vec{p} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{d}} = \frac{\partial T}{\partial \dot{d}}$ and $\vec{p} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}}$ then!

$$\boxed{\vec{p} \cdot \delta \vec{r} = \frac{\partial T}{\partial \dot{d}} \delta d + \frac{\partial T}{\partial \dot{x}} \delta x}$$

more on this later

Quick use from earlier: $m_g m \alpha \, \delta d = \delta W(SB) = \frac{d\left(\frac{\partial T}{\partial \dot{d}}\right)}{dt} \delta d + \frac{d\left(\frac{\partial T}{\partial \dot{x}}\right)}{dt} \delta x$

Virtual Work to Lagrangian

Motivation: in the sliding incline plane (IP) with a block on it (SB), there were 2 objects that can be described with (x_{IP}, y_{IP}) and (x_{SB}, y_{SB}) but there were two constraints, The IP didnt have acceleration in y-direction or any movement at all and the SB moved $\perp$ to the IP's plane.
Before we had 4 parameters but now we can explain all the motion w/ just two  (x, d)

↳ In generally in d Dimensions: $D \cdot N - J$ = # deg of freedom to describe the system

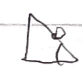
(N = # of objects, J = constraints)

$$\boxed{\text{Call theses generalized coordinates } \{q_k\}}$$

$\vec{r}_i = \vec{r}_i(\{q_k\}, \{\dot{q}_k\}, t)$, lets probe these generalized coordinates w/ small changes denoted δ

$$\boxed{\delta \vec{r}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\delta q_k + \frac{\partial \vec{r}_i}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\delta t} \leftarrow \text{chain rule}$$

~~if $\{\dot{q}_k\}$ is constant~~ ⇒ holonomic → constraints dont depend on $\dot{q}_k$ & each hold for all time

for holonomic $\dot{q}_k$ doesnt appear in constraints  $\tan\alpha = \frac{y}{x}$, no $\dot{d}$ or $\dot{x}$

If $\{\vec{r}_i\}$ and $\{q_k\}$ dont depend on time its called schleronomic

If there is time dependence its called rheonomic

For our case its schleronomic so $\frac{\partial \vec{r}_i}{\partial \dot{q}_k} \to 0$ and time goes to zero

$$\boxed{\delta \vec{r}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\delta q_k}$$

1) Virtual displacements have to obey all constraints
2) displacements occur at a fixed time (purely mathematical)
3) $\{\dot{q}_k\}$ held fixed during displacement

$\delta W = \sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_i (\vec{F}_i^{\,non\,con} + \vec{F}_i^{\,con}) \cdot \delta \vec{r}_i$

constraint force is like Normal force

NET $\boxed{\vec{F}^{\,con} \cdot \delta\vec{r}_i = 0}$

We consider cases when there are no virtual work by the constraining forces

$\delta W = \sum_{i,k} \vec{F}_i^{\,non\,con} \cdot \frac{\partial \vec{r}_i}{\partial q_k}\delta q_k \Rightarrow$ define $f_k = \vec{F}^{\,non\,con} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \frac{\vec{F}^{\,non\,con} \cdot \partial \vec{r}_i}{\partial q_k} = \frac{\delta W}{\delta q_k}$

f_k is the component of the force acting along k^{th} generalized coordinate

• Going back to δW lets try to get f_k in there

$$\delta W = \sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i \vec{F}_i^{\,non\,con} \cdot \delta \vec{r}_i$$

if mass are constant

since they are equal;

$$\boxed{\sum_i \left(\vec{F}_i^{\,non\,con} - \dot{\vec{p}}_i \right) \cdot \delta \vec{r}_i = 0 \quad \text{d'Alemberts principle}}$$

Constraint forces not needed!!

↳ This is a fundemental step saying the dynamics ie rate of change of the momentum depends only on the non-constraint forces

Recall: $f_k = \frac{\delta W}{\delta q_k} \rightarrow \delta W = \underbrace{\sum_k f_k \delta q_k}_{k} = \overbrace{\sum_{i,k} \vec{F}_i^{\,non\,con} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k}^{\text{another def of } \delta W} = \sum_{i,k} \overbrace{\dot{\vec{p}}_i}^{F=\dot{p}} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k$

← relate these →

$$\boxed{f_k = \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}}$$

← analog of $F=ma$ that depends on non constraint forces

↳ it is the weighted sum of the time derivtives of $\vec{p}$ of the partizles in the system

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With these ideas in hand lets look at Energy

KE: $T = \sum \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = T(\{q_k\}, \{\dot{q}_k\}, t)$ useful bcc $\Delta T = W_{net}$

Our general $\delta \vec{r}_i$ vector w $\{\dot{q}_k\}$ = constant! (and divide by dt)

generally $\dot{\vec{r}}_i = \sum_k \left(\frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial \dot{q}_k} \ddot{q}_k \right) + \frac{\partial \vec{r}_i}{\partial t} \rightarrow$ (W these constraints) $\boxed{\dot{\vec{r}}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}}$

* this shown w the constraints *

Lets take a $\partial \dot{q}_k$ and change our indices above to j so its not confusing

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \cdot \underbrace{\frac{\partial \dot{q}_j}{\partial \dot{q}_k}} \rightarrow \text{then } \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}$$

for holonomic constraints!

Kronecker delta function when $j=k \rightarrow 1$ and $j \neq k \rightarrow 0$

Back to Energy lets look at partials of T

$$\frac{\partial T}{\partial q_k} = \sum_i m_i \overbrace{\dot{\vec{r}}_i}^{p} \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$$

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} \overset{\text{for holonomic}}{\Longrightarrow} \frac{\partial T}{\partial \dot{q}_k} = \sum_i \vec{p}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

Using $\frac{\partial T}{\partial q_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$ and $\frac{\partial T}{\partial \dot{q}_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k}$

Lets take a time derivites of $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \left(\dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}\right)$

product rule

Proof $\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_k} = \sum_\ell \left(\frac{\partial^2 \vec{r}_i}{\partial q_\ell \partial q_k}\dot{q}_\ell + \cancel{\frac{\partial^2 \vec{r}_i}{\partial \dot{q}_\ell \partial q_k}\ddot{q}_\ell}\right) + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_k}$

[other indice for this chain rule] 0 for holonomic $\cancel{\ddot{q}_\ell = 0}$ ← I think

Its this or $\frac{\partial \vec{r}_i}{\partial \dot{q}_\ell} = 0$

↳ $\frac{\partial}{\partial q_k}\underbrace{\left(\sum_\ell \frac{\partial \vec{r}_i}{\partial q_\ell}\dot{q}_\ell + \frac{\partial \vec{r}_i}{\partial t}\right)}_{\dot{\vec{r}}_i}$ ← look at previous derivatives for $\dot{\vec{r}}_i$

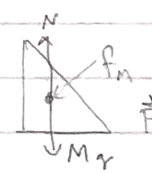
↳ thus $\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_k} = \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \left(\dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \underbrace{\vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}}_{\frac{\partial T}{\partial q_k}}\right)$$

by d'Alemberts principle

then $\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \underbrace{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}}_{\text{determined from KE}} = \underbrace{f_k}_{\text{determined by non constraint forces}}$ ← $f_k = \sum \dot{\vec{p}}_i \frac{\partial \vec{r}_i}{\partial q_k} = \sum \vec{F}_i^{\,\text{non con.}} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$

$$\boxed{f_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}}$$



$\vec{F}_m = f_m \sin\alpha\,\hat{i} + (f_m\cos\alpha - mg)\,\hat{j}$

ex 

2 constraints: $y_M = 0$ and $\tan\alpha = \frac{y_m}{x_m}$, So the 4 coordinates $(x_m, y_m), (X_M, Y_M)$ become

$X_M = x$, $y_M = 0$, $\dot{X}_M = \dot{x}$, $\dot{y}_M = 0$, $\dot{y}_m = -\dot{d}\sin\alpha$

$x_m = x + d\sin\alpha$, $y_m = h - d\sin\alpha$, $\dot{x}_m = \dot{x} + \dot{d}\cos\alpha$

① $f_k = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \frac{\delta W}{\delta q_k}$; $\vec{r}_M = X\hat{i}$, $\vec{r}_m = (x + d\cos\alpha)\hat{i} + (h - d\sin\alpha)\hat{j}$

$\vec{F}_M = -f_m\sin\alpha\,\hat{i} + (N - Mg - f_m\cos\alpha)\hat{j}$

same ① $f_x = \vec{F}_M \cdot \frac{\partial \vec{r}_M}{\partial x} + \vec{F}_m \cdot \frac{\partial \vec{r}_m}{\partial x} = \vec{F}_M \cdot \hat{i} + \vec{F}_m \cdot \hat{i} = -\overset{(\text{on } M)}{f_m\sin\alpha} + \boxed{\overset{(\text{on } m)}{f_m\sin\alpha} = 0}$

$f_d = \vec{F}_M \cdot \cancelto{0}{\frac{\partial \vec{r}_M}{\partial d}} + \vec{F}_m \cdot \frac{\partial \vec{r}_m}{\partial d}$ = works out to = $\boxed{mg\sin\alpha}$

$\delta W_x = f_x\,\delta x = 0$

$\delta W_d = f_d\,\delta d = mg\sin\alpha\,\delta d$

↳ $T = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) + \frac{1}{2}M(\dot{y}_M^2 + \dot{X}_M^2) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{d}^2 + \frac{1}{2}m\dot{x}^2 + m\cos\alpha\,\dot{x}\dot{d}$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \cancelto{0}{\frac{\partial T}{\partial x}} = f_x = 0$ (from above) → $\frac{\partial T}{\partial \dot{x}} = (m+M)\dot{x} + m\cos\alpha\,\dot{d}$ ← $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)$ → $\boxed{(m+M)\ddot{x} + m\cos\alpha\,\ddot{d} = 0}$ eq 1

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{d}}\right) - \frac{\partial T}{\partial d} = f_d = mg\sin\alpha = \frac{d}{dt}(m\dot{d} + m\cos\alpha\,\dot{x}) = m\ddot{d} + m\cos\alpha\,\ddot{x}$ → $\boxed{\cancel{m}\cos\alpha\,\ddot{x} + \ddot{d} = \cancel{m}g\sin\alpha}$ eq 2

$\ddot{x} = -\frac{m\cos\alpha}{m+M}\ddot{d}$ eq 1

$\ddot{d} = \frac{(m+M)g\sin\alpha}{M + m\sin^2\alpha}$

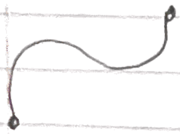
$\ddot{y}_m = -\ddot{d}\sin\alpha$

$\ddot{x}_m = \ddot{x} + \ddot{d}\cos\alpha$

recover original equations

Found f_k for kinetic eq then plugged and chugged

Final Stretch for Lagrangian



$U(q_k, \dot{q}_k, t)^{\nearrow 0}$ for our conditions

$\vec{F} = -\vec{\nabla} U$, $\vec{F}\cdot\hat{i} = \frac{-\partial U}{\partial x_i}$

$W = \int \sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_i \int \vec{\nabla}_i U_i(\{\vec{r}_i\}) \cdot d\vec{r}_i$, holonomic: $d\vec{r}_i = \sum \frac{\partial \vec{r}_i}{\partial q_k} dq_k$

$= -\sum_k \int \left(\sum_i \vec{\nabla}_i U \cdot \frac{\partial \vec{r}_i}{\partial q_k}\right) dq_k = -\sum_k \int_{q_{k0}}^{q_{kf}} \underbrace{\frac{\partial U}{\partial q_k}}_{} dq_k$

talkin' about potential work

$W = -\sum_k \int f_k \, dq_k$ $\qquad f_k = \frac{dW}{dq_k} = \frac{dU}{dq_k}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = f_k = \overbrace{\frac{-\partial U}{\partial q_k}}^{f_k} + \overbrace{\frac{\partial U}{\partial \dot{q}_k}}^{\text{its just adding } 0}$$

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_k}\right) - \frac{\partial (T-U)}{\partial q_k} = 0$$

$L = T - U \rightarrow$ $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0}$

if $\frac{\partial L}{\partial q_k} = 0$ then its cyclical, $\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{q}_k}}_{\text{constant}}\Big) = 0$ $\rightarrow L \rightarrow L - p_k \dot{q}_k$

Justification for why $\sum_k \int \sum_i \nabla_i V \frac{\partial \vec{r}_i}{\partial q_k} dq_k = \sum_k \int \frac{\partial U}{\partial q_k} dq_k$ look at Lagrangian overview or Calc 3 line intergral notes you can see if LHS has that form then you can right the RHS..

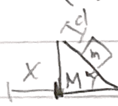
Lagrangian overview and assumptions

[helpful guide to vocab and some important ideas underlying the Lagrangian]

We assume $\vec{r}$ can be written as a set of generalized coordinates and velocities, it can also have time dependence

[Sub i because you can have many particle systems]

$\vec{r}_i = \vec{r}_i(q_k, \dot{q}_k, t)$ these would be the coordinates for a problem

ex  , X and d would be the generalized coordinates
↳ notice its only two instead of 4 (x, y for IP and SB). This is because we have constraints in our problem of the two objects

↳ constraints reduce the number of coordinates needed to describe the motion

this is a constant
↳ ex $\begin{bmatrix} X_M = X & X_m = X + d\sin\alpha \\ y_M = 0 & y_m = h - d\sin\alpha \end{bmatrix}$ [now 4 coords are 2 X and d]

* types of constraints *

holonomic: constraints dont depend on $\dot{q}_k$, notice how $\dot{X}$ or $\dot{d}$ dont appear in the above set of coordinates

schleronomic: no explicit time dependence in the constraints, notice how t doesnt apper in above, this is also holonomic

rhenomic: if there is time dependence its rhenomic

Mathematically recall we have $\vec{r}_i = \vec{r}_i(q_k, \dot{q}_k, t)$

$$\boxed{\frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_k}\frac{\partial q_k}{\partial t} + \frac{\partial \vec{r}_i}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial t} + \frac{\partial \vec{r}_i}{\partial t}\frac{\partial t}{\partial t}^{1}}$$ generally w/ constraints

holonomic: $\frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + 0 \cdot \ddot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$

schleronomic: $\frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + 0 \cdot \ddot{q}_k + 0$

* if it is holonomic the derivative comes out to this but remember this is about the equations of the generalized coordinates *

Line integral: $\int_a^b F(\vec{r})\cdot d\vec{r}$

Proof if you have $\vec{F} = \nabla f$ its path ind.:

$\int_a^b F(\vec{r})\,d\vec{r} = \int_a^b F(\vec{r}(t))\cdot \vec{r}'(t)\,dt$ if $\vec{F} = \nabla f$ then $\rightarrow \boxed{\int_a^b \nabla f(\vec{r})\cdot \vec{r}'(t)\,dt}$

↳ [dot those together] $\rightarrow \int_a^b \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\right)dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt$

then by the Fund Theorem of Calc $\Rightarrow f(b) - f(a) =$ line integral

- Another component of the Lagrangian it assumes the constraint force does no work.

Constraint force: Its the force that constrains the problem. Constraint forces act to maintain the constraint. They point in a direction perpendicular to the movement of the parts of the system. This means they do no net work.

$$dW = \sum \left(F_i^{non\,con} + \cancel{F_i^{con}}^{\,0}\right) d\vec{r}_i$$

[this leads to a generalized force f_k]

$$\boxed{f_k = \vec{F}^{non\,con} \cdot \frac{\delta \vec{r}_i}{\delta q_k} = \frac{\delta W}{\delta q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}}$$

↑ generalized force if constraint force does no work

think about like gravity $\frac{d}{dx}$ mgx is F

f_k is the component of force acting along the q_k component

[Notice the kinetic energy T]

General form of T for holonomic is $T = \frac{1}{2}\sum m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = T(q_k, \dot{q}_k, t)$

[Notice the non-constraint force from above is in terms of non-constraint force.

↳ If its also conservative then you can write it as a change in potential $\vec{F}_i = -\vec{\nabla}_i V(r_i)$

↳ by definition of conservative forces is they are path independent so the work done if you start at a point and move it around and come back will be 0.

$$\Delta W \equiv \oint \vec{F}\,d\vec{r}_i = 0$$

* see calc 3 notes on line integrals and path independence squeezed at the top *

↳ [W/ the proof at top that in order to be a conservative force $\Rightarrow F = \nabla f$ / potential function] you can write down the Euler lagrange eq

In all,

- the relationship between coordinates is apart of the constraints
- generalized force is the net non-constraint force, if conservative then $f_k = \partial V/\partial q_k$ which leads to ELE

[It also needs to be holonomic or path independent / it won't be ↳]

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0}$$

* requires potential to be holonomic and path independent *

The Hamiltonian

$L = L(q, \dot{q}, t)$

(total derivative) $\dfrac{dL}{dt} = \dfrac{\partial L}{\partial q}\dfrac{\partial q}{\partial t} + \dfrac{\partial L}{\partial \dot{q}}\dfrac{\partial \dot{q}}{\partial t} + \dfrac{\partial L}{\partial t}$ ← not same thing as $\frac{\partial L}{\partial t}$

Euler-Lagrange eq: $\dfrac{d}{dt}\overbrace{\dfrac{\partial L}{\partial \dot{q}}}^{p} = \overbrace{\dfrac{\partial L}{\partial q}}^{F} \rightarrow \dot{p} = F$

~~When deriving the ELE didn't we assume $\frac{\partial L}{\partial q} = 0$?~~ no, we assumed no ~~explicit time dependence~~ dependence on $\dot{q}$ for $V(q, \dot{q}, t)$

$\dfrac{dL}{dt} = \underbrace{\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}}\dot{q} + \dfrac{\partial L}{\partial \dot{q}}\ddot{q}}_{\text{looks like product rule}} + \dfrac{\partial L}{\partial t} \Rightarrow \dfrac{d}{dt}\left(\dot{q}\dfrac{\partial L}{\partial \dot{q}}\right)$

$\dfrac{dL}{dt} = \dfrac{d}{dt}\left(\dot{q}\dfrac{\partial L}{\partial \dot{q}}\right) + \dfrac{\partial L}{\partial t} \longrightarrow 0 = \dfrac{d}{dt}\left(\dot{q}\dfrac{\partial L}{\partial \dot{q}}\right) - \dfrac{dL}{dt} + \dfrac{\partial L}{\partial t}$

$0 = \dfrac{d}{dt}\underbrace{\left(\dot{q}\dfrac{\partial L}{\partial \dot{q}} - L\right)}_{\text{Halmitonian}} + \dfrac{\partial L}{\partial t}$

If this is 0 then H is conserved moreover if L doesn't have any explicit time dependence (ex $L = \frac{1}{2}mv^2 t + V$ ← explicit time dependence)

$\boxed{H = p\dot{q} - L}$

H = T+V proofs

lets say: $L = \frac{1}{2}m\dot{q}^2 - V(q)$ ← no $\dot{q}$ dependence true for conservative force

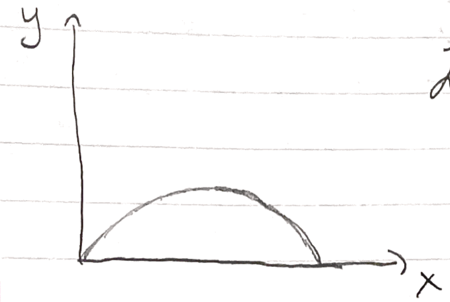
no time depend

$\dfrac{\partial L}{\partial \dot{q}} = m\dot{q} \rightarrow \dot{q}\dfrac{\partial L}{\partial \dot{q}} = m\dot{q}^2$

$$H = m\dot{q}^2 - \tfrac{1}{2}m\dot{q}^2 + V(q)$$
$$= \tfrac{1}{2}m\dot{q}^2 + V(q)$$

$\boxed{H = T+V}$

Lagrangian applied to projectile motion



$$\mathcal{L} = T - U$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$ [Euler-Lagrange equation]

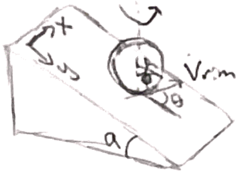
$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgh = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

(ma)

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} \qquad \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\ddot{y}}_{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)} \qquad \Bigg| \qquad \underbrace{\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad \frac{\partial \mathcal{L}}{\partial y} = -mg}_{\frac{\partial \mathcal{L}}{\partial q}}$$

ELE 1: $m\ddot{x} = 0 \rightarrow \ddot{x} = 0 \rightarrow \dot{x} = V_{0x} \rightarrow \boxed{x_f = V_{0x}t + x_i}$

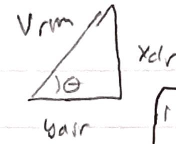
ELE 2: $m\ddot{y} = -mg \rightarrow \ddot{y} = -g \rightarrow \dot{y} = -g t + V_{0y} \rightarrow \boxed{y = -\tfrac{1}{2}gt^2 + V_{0y}t + y_i}$



ϕ for rolling
θ for turning

y tells you where along the wedge you are → sin a ~~gets~~ tells you the vertical part

non-holonomic non-integrable constraint



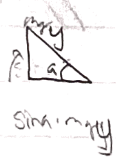
Rolling w/out slipping means $V_{rim} = R\dot{\phi}$, components of this $\boxed{\begin{aligned}\dot{x} &= R\dot{\phi}\sin\theta \\ \dot{y} &= R\dot{\phi}\cos\theta\end{aligned}}$

↳ These cant be integrated to get an equation of constraint.
• They dont lead to eq's linking y and the other coordinates, thus its impossible to describe the penny w/ just 2 coordinates despite it having only 2 deg of freedom.

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\phi}^2 + \frac{1}{2}I_2\dot{\theta}^2$$

I_1 [moment of inertia about center of the penny] I_2 [moment of inertia about vertical axis]

$$T = \frac{1}{2}mR^2\dot{\phi}^2 + \frac{1}{4}mR^2\dot{\phi}^2 + \frac{1}{8}mR^2\dot{\theta}^2 \rightarrow \boxed{T = \frac{3}{4}mR^2\dot{\phi}^2 + \frac{1}{8}mR^2\dot{\theta}^2}$$



$V = mgy\sin a$, y depends on potential but also sin a because thats the vertical part of the wedge,

$$L = \frac{mR^2}{2}\left(\frac{3}{2}\dot{\phi}^2 + \frac{1}{4}\dot{\theta}^2\right) - mgy\sin a$$

↑ notice only y no x → so lets use our y constraint

Now we want to use our constraint $dy = R\,d\phi\cos\theta \rightarrow dy - R\cos\theta\, d\phi = 0$ (G)

↳ our prescription $G(y,\theta,\phi)$

$\delta L' \rightarrow \delta L + \lambda\,\delta G$ and $\delta G = \frac{\partial G}{\partial y}\delta y + \frac{\partial G}{\partial \theta}\delta\theta + \frac{\partial G}{\partial \phi}\delta\phi$

$\delta L' \rightarrow \delta L + \lambda(\delta y - R\cos\theta\,\delta\phi)$

$$\delta s = \int\left[\left(\frac{\delta L}{\delta y} + \lambda\frac{\partial G}{\partial y}\right)\delta y + \left(\frac{\delta L}{\delta \phi} + \lambda\frac{\partial G}{\partial \phi}\right)\delta\phi + \left(\frac{\delta L}{\delta \theta} + \lambda\frac{\partial G}{\partial \theta}\right)\delta\theta\right] dt$$

Our $\frac{\partial G}{\partial y}$ is whatever is multiplied by δy so 1 in our constraint
→ for $\frac{\partial G}{\partial \phi}$ its $(-R\cos\theta)$ and $\frac{\partial G}{\partial \theta}$ its 0.

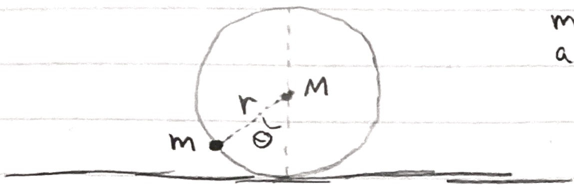
$$\delta s = \int\left[\left(\frac{\delta L}{\delta y} + \lambda\right)\delta y + \left(\frac{\delta L}{\delta \phi} - \lambda R\cos\theta\right)\delta\phi + \left(\frac{\delta L}{\delta \theta}\right)\delta\theta\right] dt$$

now you can set these to be 0 and solve, After you solve these you can find x(t) and y(t) from the constraint equation.

WOW!

Rolling without slipping

(1)



mass m is fixed to the end of a wheel that rolls w/out slipping

M at center

non-holonomic constraint but its integrable so its OK

• Rolling w/out slipping constraint: $\dot{x}_M = R\dot{\theta}$, from the distance θ arcs out as it rolls, $S = R\theta$, $x_M = R\theta \to \dot{x}_M = R\dot{\theta}$, if it slipped it wouldn't do that

m traces out cycloid: $(x_m, y_m) = R(\theta - \sin\theta, 1 - \cos\theta)$

$\hookrightarrow (\dot{x}_m, \dot{y}_m) = R\dot{\theta}(1 - \cos\theta, \sin\theta)$ $\quad T = \frac{1}{2}M\dot{x}_M^2 + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2)$

$T = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2\left[(1-\cos\theta)^2 + \sin^2\theta\right] = R^2\dot{\theta}^2\left[m(1-\cos\theta) + \frac{M}{2}\right]$

$U = mgy_m + Mgy_M = mgR(1-\cos\theta) + MgR = -mgR\cos\theta$

now just E.L.E....