How to solve Ordinary Diff eq

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- first order
- und coef
- var par
- homogenous

first order form: $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

$$y = e^{-\int \frac{a_0(x)}{a_1(x)}dx}\left[\int \frac{g(x)}{a_1(x)} e^{\int \frac{a_0(x)}{a_1(x)}dx} dx\right]$$

very cool!

Differential equations usually come in the form $Ly = g$: no x's

- homogenous sol'n $\to Ly = 0$ this means no "x"'s $y''^2 + y' + 3 = 0$
- particular sol'n $Ly = g$ this means solve for that right side.

$\boxed{y_h = \text{homogenous sol'n},\ y_p = \text{particular solution} \quad y = y_h + y_p \ \{ y_h = c_1y_1 + c_2y_2 + \ldots}$

* for first order there is one method.

First order

first order $\to \frac{dy}{dx} + f(x)y = g(x)$ $\quad \frac{dy}{dx} + 2y = 4x$

first solve homogenous! $\frac{dy}{dx} = -2y \to y_1 = e^{-2x}$

whole sol'n looks like: $y = \overset{\text{something}}{u} \cdot y_1 = ue^{-2x} \to \frac{dy}{dx} = -2ue^{-2x} + e^{-2x}\frac{du}{dt}$

plug into main! $-2ue^{-2x} + e^{-2x}\frac{du}{dt} + 2(\overbrace{ue^{-2x}}^{y}) = 4x$ * always cancel *

solve for u! $\frac{du}{dt} = 4xe^{2x}$ $\quad s = 4x,\ w = \frac{1}{2}e^{2x},\ ds = 4dx,\ dw = e^{2x}$ $\quad 2xe^{2x} - \int \frac{1}{2}4e^{2x}dx = u \to \boxed{u = e^{2x}(2x-1) + C}$

$$\boxed{y = u \cdot y_1 \to y = e^{-2x}(e^{2x}(2x-1) + C)}$$

HIGHER ORDER HIGHER ORDER HIGHER ORDER HIGHER ORDER

$\boxed{y_h \text{ is easy!}}$ assume $y = e^{rx}$ so ex) $y''' + y'' + y = 0$ never zero can just do this

$y' = re^{rx},\ y'' = r^2e^{rx},\ y''' = r^3e^{rx}$ etc... $\quad r^3e^{rx} + r^2e^{rx} + e^{rx} = 0 \to e^{rx}(r^3 + r^2 + 1) = 0$

* if you get $(r-1)^2 \to y_h = c_1e^x + c_2xe^x$ $r = 1$ twice so $\boxed{y_1 = e^{3ix},\ y_2 = e^{-3ix}}$

* if you get imaginary roots use $\{e^{i\theta} = \cos\theta + i\sin\theta\}$ ex: $r = \pm 3i$ plug in

$y = c_1e^{3ix} + c_2e^{-3ix} = c_1\cos(3x) + c_1 i\sin(3x) + c_2\cos(3x) - c_2 i\sin(3x)$ factor sin & cos

$\hookrightarrow y = \cos(3x)(c_1 + c_2) + \sin(3x)(c_1 i - c_2 i) \Rightarrow y = k_1\cos(3x) + k_2\sin(3x)$

$\boxed{\text{solve } y_p}$ $\quad y_p = u_1y_1 + u_2y_2 \to u_2' = y_1\frac{g}{W} \quad u_1' = -y_2\frac{g}{W}$ ← wronskian

integrate u's

homogenous solution

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (\ ) - (\ )$$

g is the right side

now for und coeff

$$L(y_p + y_h) = L(y_p) + L(y_h) = g + 0$$

quick justification

$$L(y_p) = g$$

* Undetermined coefficients works by guessing what the sol'n will look like!

(Full description other page)

Another way to solve full ODE's is with Laplace Transforms. also on other page

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Euler's method: $\frac{dy}{dx} = 2t - y + 2$ $\quad y(0) = 2$ $\quad$ step size $= .1$

$m_e = 2(0) - 2 + 2 = 0$

(1) $y^+ = y(0) + m_e \Delta t = 2 + 0(.1) = 2$

$m^+ = 2(.1) - 2 + 2 = .2$

$m = \frac{m_e + m^+}{2} = .1$

$y = y(0) + m \Delta t = 2 + (.1)(.1) = 2.01$

$y(.1) = 2.01$

(2) $m_e = 2(.1) - 2.01 + 2 = .19$

$y^+ = 2.01 + .19(.1) = 2.029$

$m^+ = 2(.2) - 2.029 + 2 = .371$

all to get m_{av} to put in update formula

$m = \frac{m_e + m^+}{2} = .2805$

$y = 2.01 + (.2805)(.1) = 2.03805$

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* also to check if y_n isn't double dipping

$$W = \begin{vmatrix} y_1 & y_1' & y_1'' \\ y_2 & y_2' & y_2'' \\ y_3 & y_3' & y_3'' \end{vmatrix}$$

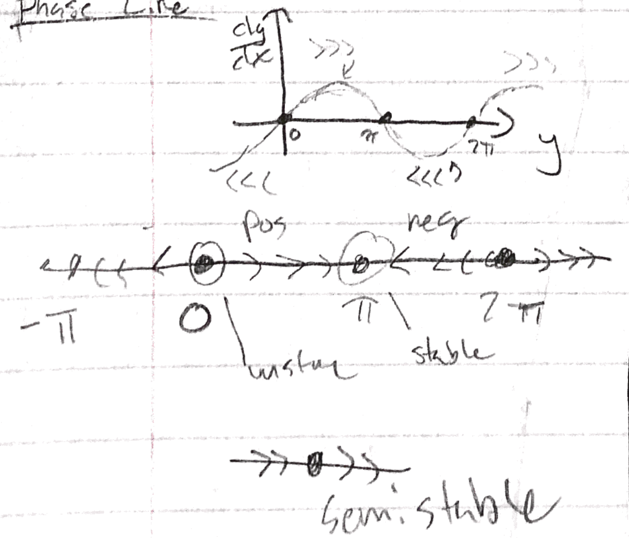
as many as u want.

Not all ODE sol'n are soln for all!

ex) $\frac{dy}{dx} = 1 + y^2$ $(y(0) = 0)$ $\quad y(x) = \tan(x)$ is sol'n

but $\tan(x)$ blows up at $\frac{\pi}{2}$

Phase Line:



natural growth: $\frac{dP}{dt} = rP \rightarrow P = P_0 e^{rt}$

radioactive decay: $\frac{dA}{dt} = -kA \Rightarrow A = A_0 e^{-kt}$

Newton Cooling: $\frac{dT}{dt} = k(T_e - T) \rightarrow T = T_e + (T_0 - T_e)e^{-kt}$

Chemical Mixing: $\frac{dA}{dt} = F_{in} C_{in} - F_{out} C_{out}$ (flow, concentration; usually form $\frac{A(t)}{V(t)}$ volume) differential operator $\to$ $Ly = g$ $\qquad$ $Ly = 0$

ex: $y''' + 4y'' = 3e^{-4x}$ $\qquad$ ex $y'' - y = 0$

Solving for y_p in $y = y_p + y_h$

$3y = 3$ ex.

* you get y_h from $y = e^{rx}$ and then abstracted to: ex $y'' = r^2, y' = r$

if $(r+1)^2 \to y_1 = e^{-x}$ and $y_2 = xe^{-x}$, for imaginary use $e^{i\theta} = \cos\theta + i\sin\theta$

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the goal to solve for y_p is to assume g looks like Ly
(ex: $y'' + y = 5e^{3x}$ since $y'' + y = g(x)$ assume its sol'n $y_p = Ae^{3x}$)
* note this is the idea, minor adjustments needed between problems.

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simple example: $y' + 4y = 4e^{-2x}$

for $Ly = 0 \to r + 4 = 0$ so $y_1 = e^{-4x}$ and $y_h = C_1e^{-4x}$

Lets try: $y_p = Ae^{-2x}$ $\to$ goal is to try to get $4e^{-2x}$ after you do all of the $y' + 4y$ stuff, so you know it'll look like (something)e^{-2x}

$y_p' = -2Ae^{-2x}$ and $y_p = Ae^{-2x}$ so

$\to Ly_p = \underbrace{-2Ae^{-2x}}_{y'} + \underbrace{4Ae^{-2x}}_{4y} \to 2Ae^{-2x}$ and $g = 4e^{-2x}$

thus $Ly_p = g \to 2Ae^{-2x} = 4e^{-2x} \to A = 2$ so $\boxed{y_p = 2e^{-2x}}$

example 2 * the same thing but added complexity! *

$y'' + 4y' + 4y = 4e^{-2x} \to Ly = 0$ stuff $\to r^2 + 4r + 4 = 0 \to (r+2)^2 = 0$ $r = -2, s = 2$ (how many times its repeated)

so: $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ $\to y_h = C_1e^{-2x} + C_2xe^{-2x}$

first try: $y_p = Ae^{-2x}$ $\to$ if you plug that in, it makes $Ly = 0$, because its a solution to $Ly = 0$

ok try: $y_p = Axe^{-2x} \to$ same thing, its part of $Ly = 0$, WE WANT $Ly = g$!!!!

well: $y_p = Ax^2e^{-2x}$ $\to$ not part of $Ly = 0$ so it works,

after some evaluation..... $Ly_p = 2Ae^{-2x}$ and $Ly = g = 4e^{-2x}$

then $2Ae^{-2x} = 4e^{-2x} \to A = 2$ so $y_p = 2e^{-2x}$

LESSON LEARNED = if y_p is a part of the solutions y_h

KEEP going up in orders of x until it aint

* this is only the case for orders of "s"

example 3: $\boxed{y'' - y' + y = 2\sin(3x)}$

NOTE: a first guess would be $y_p = A\sin(3x)$, its not since when you take DERIVITIVES of $\sin(x)$ you get $\sin(x)$ and $\cos(x)$

not always 3 obviously

SO then probably looks like $y_p = A\cos(3x) + B\sin(3x)$

plugging in: $y_p'' = -9A\cos(3x) - 9B\sin(3x)$, $y_p' = -3A\sin(3x) + 3B\cos(3x)$ and $y_p = A\cos(3x) + B\sin(3x)$

doing $y_p'' - y_p' + y_p \rightarrow \underbrace{-9A\cos(3x) - 9B\sin(3x)}_{y''} + \underbrace{3A\sin(3x) - 3B\cos(3x)}_{y'} + \underbrace{A\cos(3x) + B\sin(3x)}_{y}$

Factor: $Ly_p = (-8A - 3B)\cos(3x) + (3A - 8B)\sin(3x)$

we know this equals $g(x) = 2\sin(3x) + 0\cos(3x)$ you'll see why we say this

$(-8A - 3B)\cos(3x) + (3A - 8B)\sin(3x) = 2\sin(3x) + 0\cos(3x)$

so $-8A - 3B = 0$ and $3A - 8B = 2$ solve: $A = \frac{6}{73}$ and $B = \frac{-16}{73}$

$\boxed{y_p = \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x)}$

The same technique from example 3 can be applied to equations of $y'' + 4y' - 2y = 2x^2 - 3x + 6$ $\rightarrow$ answer $y_p = -x^2 - \frac{5}{2}x - 9$

$\hookrightarrow$ assume $\boxed{y_p = Ax^2 + Bx + C}$

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example 4 $y'' - 2y' - 3y = (4x - 5) + 6xe^{2x}$ $\rightsquigarrow Ly = 0 \rightarrow y_h = c_1e^{-x} + c_2e^{3x}$

* theres a linear part and an exponentials

$g(x) = g_1(x) + g_2(x)$: "polynomial" + "exponentials"

$g_1(x)$ = polynomial $= Ax + B$

$g_2(x)$ = exponential $= Cxe^{2x} + De^{2x}$

why does it have two exp? Bec/ the derivitive of $y = xe^{2x} \Rightarrow y' = xe^{2x} + e^{2x}$ (product rule...)

$y_p = Ax + B + Cxe^{2x} + De^{2x}$

do $y_p'' - y_p' + y = \ldots$

group like terms... you get..

$x - 2A - 3B - 3Cxe^{2x} + (2C - 3D)e^{2x} = 4x - 5 + 6xe^{2x}$

*Since its second Derivive do product rule again for more realistic guess, but you will just group constants into the form $Cxe^{2x} + De^{2x}$

up the x's and the constants and the xe^{2x}

$-3A = 4$ and $-2A - 3B = -5$ and $-3C = 6$ and $2C - 3D = 0$

Solve for A, B, C, D

$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$

would of done 2 easier ones

$y_{p_1} = Ax + B \rightarrow y'' - 2y' - 3y = 4x - 5$

$y_{p_2} = Cxe^{2x} + De^{2x} \rightarrow y'' - 2y' - 3y = 6xe^{2x}$

* works for orders higher than just 2 (has to be linear th

if it is add orders of x

examples of y_p's

a) $Ly = 4e^{-2x} \rightarrow y_p = Ae^{-2x}$ * check if Ae^{-2x} is apart of sol'n $Ly=0$!

b) $Ly = 2\sin(3x) \rightarrow y_p = A\cos(3x) + B\sin(3x)$ ← bec/ contained in derivitives

c) $Ly = 2x^2 - 3x + 6 \rightarrow y_p = Ax^2 + Bx + C$

d) $Ly = (4x-5) + 6xe^{2x} \rightarrow y_p = (Ax+B) + Cxe^{2x} + De^{2x}$ ← bec/ $\frac{dy}{dx}$ of $xe^{2x} = xe^{2x} + e^{2x}$

c) $Ly = x\cos(x) \rightarrow y_p = (Ax+B)\cos(x) + (Cx+D)\sin(x)$

d) $Ly = 5x^3e^{-x} - 7e^{-x} \rightarrow y_p = (Ax^3 + Bx^2 + Cx + D)e^{-x}$ no constant in front of e^{-x} bec/ it could be combined

if $y'' + 4y = x\cos(x)$ and $y_h = C_1\cos(2x) + C_2\sin(2x)$

↳ $r^2 + 4 = 0$ complex stuff... $e^{i\theta} = \cos(\theta) + i\sin\theta$

*Not A Duplication!

e) $Ly = 3x^2 - 5\sin(2x) + 7xe^{6x} \rightarrow y_p = (Ax^2 + Bx + C) + D\cos(2x) + E\sin(2x) + (Fx+G)e^{6x}$

$(Fx+G)e^{6x}$
$= Fxe^{6x} + Ge^{6x}$

f) $Ly = 6x^2 + 2 - 12e^{3x}$, lets say, $y_h = C_1e^{3x} + C_2xe^{3x}$

xe^{3x} is also a sol'n!

y_p IS NOT! $\neq Ax^2 + Bx + C + De^{3x}$

$y_p = Ax^2 + Bx + C + Dx^2e^{3x}$

De^{3x} is apart of $Ly=0$! So you have to go up in x's

G) fourth order? big boi! $y'''' + y''' = 1 - x^2e^{-x} \rightarrow y_h = C_1 + C_2x + C_3x^2 + C_4e^{-x}$

* normal assumption duplications are eliminated if,

$y_p = A + (Bx^2 + Cx + D)e^{-x}$ but duplications!!

$A \rightarrow Ax \rightarrow Ax^2 \rightarrow \boxed{Ax^3}$

multiply by x so...

$(Bx^3 + Cx^2 + Dx)e^{-x}$

NOT SURE WHY!

My guess is since C_4 is multiplied by e^{-x} you just go up an order

Spring systems ODE

undamped unforced oscillators

$W = mg = 32$ lbs so $m = 1$ slug the $g = 32\ ft/s$

↳ stretches a spring 2 ft.

so $F_g = F_s = kx \to mg = kx$ so $32 = k \cdot 2$ $k = 16$

$x'' + \omega^2 x = 0 \to x'' + \frac{k}{m} x = 0 \to x'' + 16x = 0$

$r^2 + 16 = 0$ $r = \pm 4i$

$x(t) = C_1 \cos(4t) + C_2 \sin(4t)$

released from a point 1 foot above eq. pos.

$X(0) = 1$

and at a upward velocity of $V = 2ft/s$

$X'(0) = 2$

$1 = C_1 \cos(0) + C_2 \sin(0) \to C_1 = 1$

$x' \to -2 = -4C_1 \sin(0) + 4C_2 \cos(0) \to C_2 = -\frac{1}{2}$

so $x(t) = \cos(4t) - \frac{1}{2}\sin(4t)$

$A = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}$

$T = \frac{2\pi}{\omega}$ $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{1}} = 4$ so $T = \frac{\pi}{2}$

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undamped forced motion Resonance

$x'' + \omega^2 x = F_0 \sin(\gamma t)$ $X(0) = 0$, $x'(0) = 0 \to$ so $X_h = C_1\cos(\omega t) + C_2 \sin(\omega t)$

$y_p = F_0 \sin(\gamma t)$ $y_{(x)} = A\cos(\gamma t) + B\sin(\gamma t)$ now do $y''_{(x)}$ and $y_{(x)}$ and plug into $x'' + \omega^2 x = F_0 \sin(\gamma t)$

you get $A = 0$ and $B = \frac{F_0}{(\omega^2 - \gamma^2)}$

Therefore $X_p = \frac{F_0}{\omega^2 - \gamma^2}\sin(\gamma t)$ so $X = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2}\sin(\gamma t)$

Say initial conditions are $X(0) = 0$ and $X'(0) = 0$ so only motion comes from $F_0 \sin(\gamma t)$

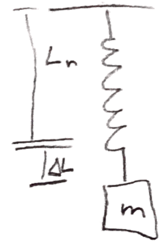
yields $C_1 = 0$ and $C_2 = -\gamma \frac{F_0}{\omega(\omega^2 - \gamma^2)}$ so $\boxed{X = \frac{F_0}{\omega(\omega^2 - \gamma^2)}(-\gamma \sin(\omega t) + \omega \sin(\gamma t))}$

if $\gamma \to \omega$ you get pure resonance

so $\lim_{\gamma \to \omega} F_0 \frac{-\gamma\sin(\omega t) + \omega \sin(\gamma t)}{\omega(\omega^2 - \gamma^2)}$ use L'Hopital rule $\to \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$

blows up!

Derivation of the equation



$\downarrow +$

$ma = mx''$ contains force from gravity causing it to pull the spring and extra distance "x"

energy from $\downarrow$ clampening force

$F_{net} = F_g + F_s + F_d + F_a$ ← applied force

(stretched from gravity) ← extra stretched

$mx'' = \underbrace{mg - kL_n}_{\text{cancel}} - kx - \beta x' + F_a$

↳ constant change in displacement

divide by $m \to x'' + \frac{k}{m}x + \frac{\beta}{m}x' = \frac{F_a}{m}$

let $\boxed{\omega = \sqrt{\frac{k}{m}}}$ and $\boxed{\lambda = \frac{\beta}{2m}}$

$$\boxed{\text{so } x'' + 2\lambda x' + \omega^2 x = \frac{F_a}{m} = f(t)}$$ ← applied force

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damped oscillators: $x'' + 2\lambda x' + \omega^2 x = 0$ using $x = e^{rt} \Rightarrow r^2 + 2\lambda r + \omega^2 = 0$

roots are $(\lambda > \omega) \to (r + \lambda) = \pm\sqrt{\lambda^2 - \omega^2}$ and $(\lambda < \omega) \to r + \lambda = \pm i\sqrt{\omega^2 - \lambda^2}$

real roots

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ex1

$x'' + 6x' + 25x = 0 \to r^2 + 6r + 25 = 0 \to (r+3)^2 = -(4)^2 \to \boxed{r = -3 \pm 4i}$

so $X_h = e^{-3t}(c_1\cos(4t) + c_2\sin(4t))$ lets say $x(0) = 0$ and $x'(0) = 1$

applying those $0 = e^{-3 \cdot 0}(c_1 \cdot \cos(0) + c_2\sin(0)) \to c_1 = 0$ then $c_2 = -\frac{1}{4}$

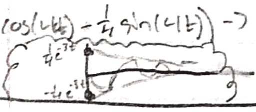
get c_2 from $x' = \dots$ using c_1 and c_2 in X_h

$x = e^{-3t}(0 \cdot \cos(4t) - \frac{1}{4}\sin(4t)) \to x_h = -\frac{1}{4}e^{-3t}\sin(4t)$

acts like amplitude so for peaks

$-\frac{1}{4}e^{-3t} = -\frac{1}{4}e^{-3t}\sin(4t) \to 1 = \sin(4t)$

$\frac{\pi}{2} = 4t$



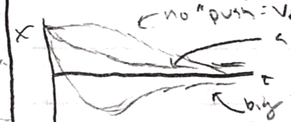
ex2

$x'' + 12x' + 25x = 0 \to x(0) = 1,\ x'(0) = V_0$ | roots: $r = -6 \pm \sqrt{11} \to x_h = c_1 e^{-(6-\sqrt{11})t} + c_2 e^{-(\dots}$

so $c_1 + c_2 = 1$ and $c_1 - c_2 = \frac{V_0 + 6}{\sqrt{11}}$

← no "push": $V_0 = 0 = x'(0) = 0$ so slope $= 0$

← little push



↖ big push (overdamped)

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forced case: $x'' + 2\lambda x' + \omega^2 x = f_0 \to \left\{\begin{matrix}\cos(\gamma t)\\ \sin(\gamma t)\\ e^{\lambda\gamma t}\end{matrix}\right\}$ $\lambda > 0$

What happens after a long time

$\lim_{t\to\infty} X_h = 0$ cause it'll lose all that energy we just care about the forced case

$X_{\text{steady state}} = X_p$ and $X_{cd} = A\cos(\gamma t) + B\sin(\gamma t)$ have $r = \pm i\gamma$

now that you have that, get X'_{cd} and X''_{cd} and set $= f_0\sin(\gamma t)$ to get A, B

Amplitude $= \sqrt{A^2 + B^2}$

$f(t)$ is like pulling up the system with the spring on

friday ask to explain the X_h $t\to\infty$ part

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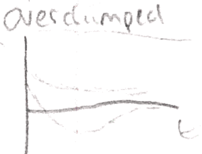
in just damped with roots $r + \lambda = \pm\sqrt{\lambda^2 - \omega^2}$

if $\lambda^2 > \omega^2$ so $\lambda^2 - \omega^2 > 0$: overdamped β big k

if $\lambda^2 = \omega^2$ critically damped → any change will cause oscillation

if $\lambda^2 < \omega^2$ underdamped (complex roots)

overdamped



critically damped

underdamped