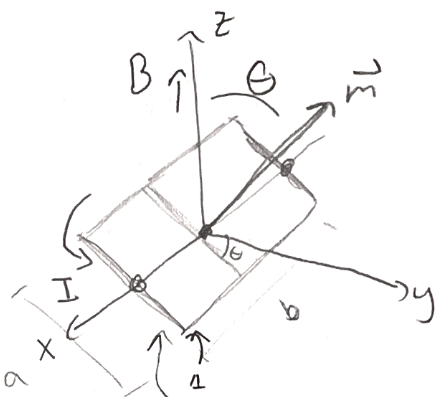
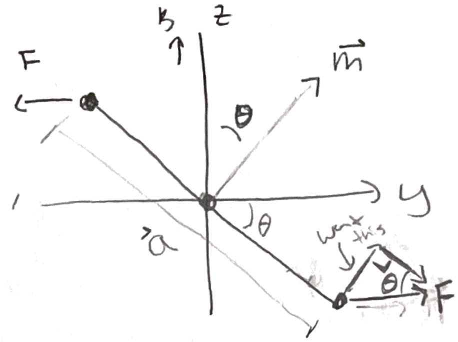
Magnetic Fields in Matter

- Some materials acquire a magnetization parallel to B (paramagnets)
- Some materials acquire a magnetization opposite to B (diamagnets)

Torques on Magnetic Dipoles

- think of the loop as build up of tiny loops (atoms)





By the Lorentz force, then RHR

$\vec{N} = \frac{\vec{a}}{2} \times \vec{F} + \frac{\vec{a}}{2} \times \vec{F} = \vec{a} \times \vec{F} = aF\sin\theta\,\hat{x}$

the force on these sides just stretch it check w/ RHR

$F_1 = \left|\int (\vec{I} \times \vec{B})\,dl\right| = IBb$, $I \perp B$,

$m = Iab$

↳ $\vec{N} = IabB\sin\theta\,\hat{x} = mB\sin\theta\,\hat{x} = \boxed{\vec{m} \times \vec{B} = \vec{N}}$

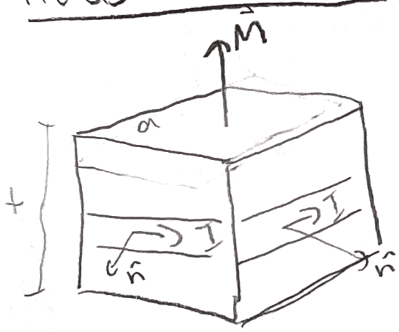
↳ torque B such to line up the dipole parallel to the field

$$\boxed{\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})}$$

Fields in matter

Since $m = Mat$, and $m = Ia$

$I = Mt$ and $K_b = I/t$ so $K_b = M$



$\vec{K}_b = \vec{M} \times \hat{n}$

Cross product for direction

For the argument for $\vec{J}_b$ look at final bonus problem

Free current & bound current from the magnetization

$\vec{J}_b = \vec{\nabla} \times \vec{M}$

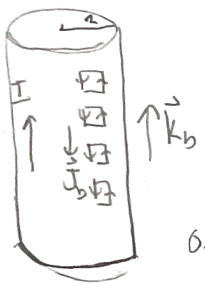
↳ $\vec{J} = \vec{J}_b + \vec{J}_f$

Amperes Law: $\vec{\nabla} \times \vec{B} = \mu_0 J = \mu_0 \left(\vec{\nabla} \times \vec{M} + \vec{J}_f \right) \rightarrow \vec{\nabla} \times \underbrace{\left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right)}_{\text{define as } \vec{H}} = \vec{J}_f$

Then: $\vec{\nabla} \times \vec{H} = \vec{J}_f$ or $\oint \vec{H} \cdot d\vec{l} = I_{f\,encl}$

total free current passing through an amperian loop

Ex copper rod is diamagnetic so dipoles will line up opposite to the field.



$H(2\pi s) = I_{f\,encl} = I \frac{\pi s^2}{\pi R^2} \rightarrow \boxed{\vec{H} = \frac{I}{2\pi R^2} s \,\hat{\phi} \qquad s \leq R}$

outside $H(2\pi s) = I \rightarrow \boxed{\vec{H} = \frac{I}{2\pi s} \hat{\phi} \qquad s \geq R}$

In outside region $\vec{M} = 0$ so $\vec{B} = \mu_0 \vec{H} \Rightarrow \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad s \geq R$

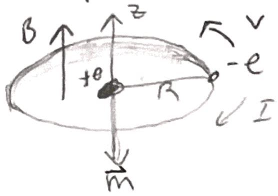
Effect of a magnetic field on atomic orbits

- assume electron orbit is a circle of radius R

↳ $I = \frac{-e}{T} = -\frac{ev}{2\pi R}$ and $\vec{m} = I\pi R^2 = -\frac{1}{2} evR\hat{z}$

- subject to a magnetic field the current loop will experience a torque, but its small

↳ it will affect the orbit by speeding up or slowing down the electron



↳ $F_e = ma \rightarrow \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = m_e \frac{v^2}{R}$

$\bar{v}$ is the new speed

↳ in a magnetic field theres an additional force on the electron $|-e(\vec{v}\times\vec{B})| = evB$ ($B \perp$ plane of orbit) $\sin\theta = \pi/2$

$\rightarrow \boxed{\underbrace{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2}}_{m_e\frac{v^2}{R}} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}}$

↳ $e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v)$

↳ $\frac{eBR}{m_e} = \frac{1}{\bar{v}}\underbrace{(\bar{v}+v)}_{\Delta v + \Delta v}\underbrace{(\bar{v}-v)}_{\Delta v}$, $\Delta v = \bar{v} - v$ is small $= 2\Delta v \rightarrow \boxed{\Delta v = \frac{eRB}{2m_e}}$

electron speeds up when B is on

↳ $\boxed{\Delta\vec{m} = -\frac{1}{2}e\Delta vR\hat{z} = -\frac{e^2R^2}{4m_e}\vec{B}}$ (change in $\vec{m}$ is opposite to $\vec{B}$)

(an electron circling the other way, would have a dipole moment pointing upward, but would be slowed down by the field, so change is opposite to $\vec{B}$)

(but such an orbit would be slowed down by the field)

- each atom picks up a dipole moment that is antiparallel to the field

↳ (Which is diamagnetism!)

Magnetization

- matter becomes magnetized, tiny dipoles w/ a net alignment in a dir

$\boxed{\vec{M} = \text{magnetic dipole moment per unit volume}}$ ← now just take as given we dont care how it got here

paramagnetism - dipoles associated w/ spins of unpaired electrons experience a torque making them line up parallel to field

diamagnetism - orbital speed of the electrons is altered in such a way to change the orbital dipole moment in the dir opposite to B

Field of a magnetized Object

[• we have a piece of magnetized material, w/ $\vec{M}$ whats the field?]

→ for a little $\vec{m}$ → $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\imath}}{\imath^2}$ ← vector potential for a single

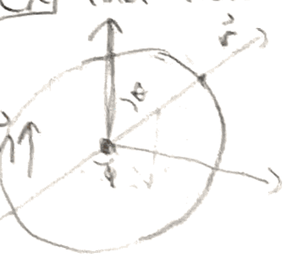
↳ then $\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}\,') \times \hat{\imath}}{\imath^2} d\tau'}$

↳ doing similar footwork w/ $\nabla' \frac{1}{\imath} = \frac{\hat{\imath}}{\imath^2}$, and integration by parts... Stokes theorem

↳ $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{\imath} \underbrace{\left(\vec{\nabla}' \times \vec{M}(\vec{r}\,')\right)}_{\vec{J}_b = \vec{\nabla} \times \vec{M}} d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\imath} \underbrace{\left(\vec{M}(\vec{r}\,') \times d\vec{a}'\right)}_{\vec{K}_b = \vec{M} \times \hat{n}}$

↳ $\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \underbrace{\int_V \frac{\vec{J}_b(\vec{r}\,')}{\imath} d\tau'}_{\text{potential of a volume current}} + \frac{\mu_0}{4\pi} \underbrace{\oint_S \frac{\vec{K}_b(\vec{r}\,')}{\imath} da'}_{\text{potential of a surface current}}}$

$\boxed{\text{ex}}$ find field of a uniformly magnetized sphere



$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$

$\vec{K}_b = \vec{M} \times \hat{n} = M \sin\theta \, \hat{\phi}$

(only surface current

[Recall the rotating shell
$\vec{K} = \sigma \vec{v} = \sigma \omega R \sin\theta \, \hat{\phi}$
by mapping $\sigma \vec{\omega} R = \vec{M}$]

Thus $\vec{B} = \frac{2}{3} \mu_0 \sigma R \vec{\omega}$ → $\boxed{\vec{B} = \frac{2}{3} \mu_0 \vec{M}}$

(naive) Parallel

$\vec\nabla \times \vec H = \vec J_f$ and $\vec\nabla \times \vec B = \mu_0 \vec J$ look similar but $\mu_0 \vec H$ is not just $\vec B$ but its source is $\vec J_f$. To describe a vector field you need the divergence and curl. From:

$\vec H = \frac{1}{\mu_0}\vec B - \vec M \Rightarrow \vec\nabla\cdot\vec H = -\vec\nabla\cdot\vec M$ only when $\vec\nabla\cdot\vec M = 0$ can you think about it like that.

- In a bar magnet w/ uniform $\vec M$ parallel to axis

↳ naive application: $\vec J_f = 0$ so $\vec H = 0$ thus $\vec B = \mu_0 \vec M$ inside & $\vec B = 0$ outside

This is obviously wrong. $\vec\nabla\cdot\vec M \neq 0$

Use $\vec H$ if there is symmetry otherwise dont assume $\vec H$ is zero bec there is no free current in sight.

Boundary conditions

$$H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below}) \quad , \quad \vec H^{\parallel}_{above} - \vec H^{\parallel}_{below} = \vec K_f \times \hat n$$

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Linear Media

- In paramagnetic & diamagnetic materials, magnetization is sustained by the field when B is removed M disappears. Often its proportional to the field

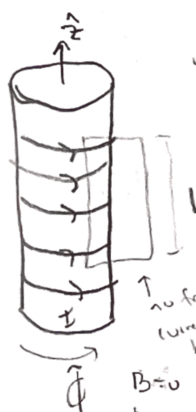
↳ $\vec M = \frac{1}{\mu_0}\chi_m \vec B$ but custom is done this way: $\boxed{\vec M = \chi_m \vec H}$

$\chi_m > 0$ paramagnet

$\chi_m < 0$ diamagnet

↳ $\vec B = \mu_0(\vec H + \vec M) = \mu_0(1+\chi_m)\vec H$ or $\boxed{\vec B = \mu \vec H \ , \ \mu \equiv \mu_0(1+\chi_m)}$

(ex) infinite solenoid filled w/ linear material w/ χ_m



$\oint \vec H \cdot d\vec l = \vec J_f$

$HL = nLI$

$\vec H = nI\hat z$

symmetry arguments & Ampere's law

no free current out here

$B=0$ outside & $M=0$ so $H=0$

then $\vec B = \mu n I \hat z$

$\hat z \times \hat r = \hat\phi$

↳ $\vec K_b = \vec M \times \hat n = (\chi_m \vec H)\times \hat n = \chi_m n I \hat\phi$

(if $\chi_m > 0$ it is in the same dir as I if $\chi_m < 0$ then its opposite & hurts the field)

$\vec J_b = \vec\nabla \times \vec M = \vec\nabla \times (\chi_m \vec H) = \chi_m \vec J_f$

$\vec\nabla \times \vec H = \vec J_f$

Divergence of $\vec H$ in linear media

$$\vec\nabla\cdot\vec H = -\vec\nabla\cdot\left(\frac{1}{\mu}\vec B\right) = \frac{1}{\mu}\vec\nabla\cdot\vec B + \vec B\cdot\vec\nabla\left(\frac{1}{\mu}\right) = \vec B\cdot\vec\nabla\left(\frac{1}{\mu}\right)$$

$\vec H$ is not divergenceless at points where μ is changing in Magnetic Fields

$\frac{d\Phi}{dt} \rightarrow \mathcal{E} = L\frac{dI}{dt}$

LI, where the inductance L is a constant of the system,
↳ the flux is proportional to the current

takes a certain amount of energy to start a current flowing in a circuit
→ work against the back emf to get the current going
↳ you get it back when you turn off the current

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt} \rightarrow \boxed{W = \frac{1}{2}LI^2}$$

Nicer way to write W:

$$\Phi = LI = \int \vec{B}\cdot d\vec{a} = \int(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$$

↳ then $W = \frac{1}{2}I\oint\vec{A}\cdot d\vec{l} = \frac{1}{2}\oint(\vec{A}\cdot\vec{I})\,dl$ (I and dl have same direction)

↳ [generalize to volume currents] → $W = \frac{1}{2}\int_V(\vec{A}\cdot\vec{J})\,d\tau$, use $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

↳ $W = \frac{1}{2\mu_0}\int\vec{A}\cdot(\vec{\nabla}\times\vec{B})\,d\tau$, product rule 6 transfers derivative from B to A

product Rule 6: $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\underbrace{(\vec{\nabla}\times\vec{A})}_{B} - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \rightarrow \vec{A}\cdot(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot\vec{B} - \vec{\nabla}\cdot(\vec{A}\times\vec{B})$

↳ $W = \frac{1}{2\mu_0}\left[\int B^2\,d\tau - \int\vec{\nabla}\cdot(\vec{A}\times\vec{B})\,d\tau\right] = \frac{1}{2\mu_0}\left[\int_V B^2\,d\tau - \oint_S(\vec{A}\times\vec{B})\cdot d\vec{a}\right]$

if integrated over all space surface integral is 0

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau}$$

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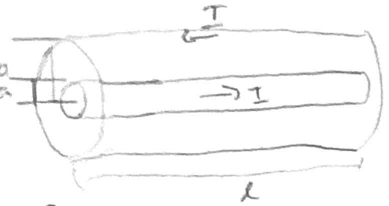


a long coaxial cable carries current I, current flows down surface of the inner radius a, and back along b of the outer cylinder

↳ find the magnetic energy stored in a section l

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ → between the cylinders $\boxed{\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}}$

$\frac{1}{\mu_0} B^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}$ (energy per unit volume)

Integrate a cylindrical shell: $2\pi l s\, ds$ → $\frac{\mu_0 I^2 l}{4\pi}\left(\frac{ds}{s}\right)$

↳ integrate from a to b: $\boxed{W = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a}\right)}$

also $W = \frac{1}{2} L I^2$

↳ $\frac{1}{2} L I^2 = \frac{\mu_0 l}{4\pi} I^2 \ln\left(\frac{b}{a}\right)$ → $\boxed{L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)}$

Electrodynamics

Ohms law → most materials the current density just depends on the force per unit charge.

↳ $\vec{J} = \sigma \vec{f}$ (to make current flow you have to push the charges)

conductive and resistivity is $1/\sigma = \rho$

The force that drives the charge is usually the electromagnetic force

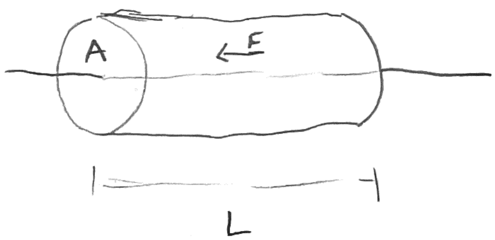
↳ $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \rightarrow \boxed{\vec{J} = \sigma \vec{E}}$

v is small

Example

a cylindrical resistor w/ conductivity σ. If we stipulate that the potential is constant over each end and the potential difference between the ends is V what is the current,

$\vec{J} = \frac{dI}{da_\perp} \rightarrow I = JA$ (if E is uniform then J is uniform)



↳ $I = \sigma E A = \frac{\sigma A}{L} V \rightarrow EL = V$

so $E = \frac{V}{L}$

This shows that the total current flowing is proportional to the difference between them

$\boxed{V = IR}$

Driving current around a circuit

Two forces that move current around a circuit

↳ f_s = source (battery) → can be many things, check out pg 304
(force per charge)

↳ E = electrostatic force

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint \vec{f_s} \cdot d\vec{l} \quad \text{bec} \quad \oint \vec{E} \cdot d\vec{l} = 0 \text{ for electrostatic forces}$$

(ex) an ideal source of emf (a resistanceless battery)

↳ take for granted that the net force on charges around a loop is 0

Then! $$V = -\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b \vec{f_s} \cdot d\vec{l} = \oint \vec{f_s} \cdot d\vec{l} = \mathcal{E}$$

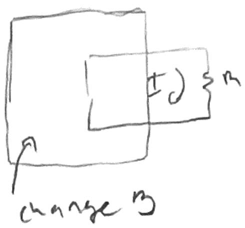
$\vec{E} = -\vec{f_s}$ since $\vec{f} = 0$

↑ can make close loop since f_s is 0 everywhere outside the source

* function of a battery is to establish & maintain a voltage difference equal to the electromotive force

ctromagnetic Induction

a changing magnetic field induces an electric field



by changing B you get the same current through the wire. What is this $\mathcal{E}$ then

before $\mathcal{E} = \oint \vec{f}_s \cdot d\vec{\ell} = -\frac{d\Phi}{dt}$

↓

f mag before

↓

He says $\vec{f}_s = \vec{E}$

↳ $\oint \vec{E} \cdot d\vec{\ell} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$

$\int \vec{\nabla} \times \vec{E} \cdot d\vec{a} = -\int \frac{\partial \vec{B}}{dt} \cdot d\vec{a}$ ↳ $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Φ = positive to the left

positive current is CCW

↳ if $\mathcal{E}$ is negative then current flows clockwise

↳ if $\mathcal{E}$ is positive then current flows CCW

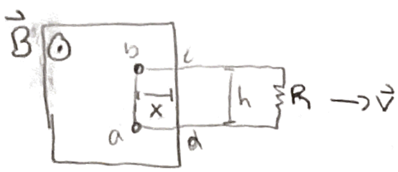
Lenz's law: Nature Abhors a change in flux

• the induced current will flow in such a way that the flux it produces tends to cancel the change

→ in the ring example the flux decreases so the wire sets up a current to maintain that B field (same dir)

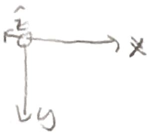
initially turned off but when turned on the ring wants to keep the B field off so it sets up a current opposite to the B that's on (current CW while solenoid is CCW) opposite currents repel & ring flies off

Motional emf (moving a wire through a B-field)



• The only force on the wire is that of the magnetic

$$\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l}$$



$\vec{B} = B_0 \hat{z}$

$\vec{V} = V_0 \hat{x}$

$\vec{F}/q = \vec{f}$

$\rightarrow \vec{F}_{mag} = Q(\vec{v} \times \vec{B})$, $\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ V_0 & 0 & 0 \\ 0 & 0 & B_0 \end{vmatrix} = -(V_0 B_0)\hat{y} = -V_0 B_0 \hat{y}$

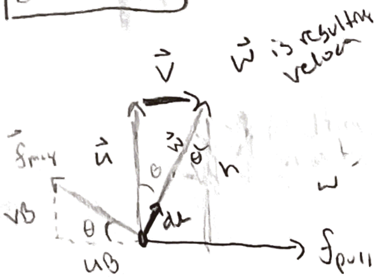
$\boxed{\vec{F}_{mag} = -q V_0 B_0 \hat{y}}$

current goes clockwise around the loop

$\mathcal{E} = \oint \frac{(-q V_0 B_0)}{q} \hat{y} \cdot d\vec{l}$ along "bc" and "ad", it'd be $\hat{y} \cdot \hat{x} = 0$ so just ab

$\rightarrow d\vec{l} = -\hat{y} dl$

$\boxed{\mathcal{E} = vBh}$



• the person has to pull the wire w/ some force but what is the force? Well since the charges are moving up and to the right they can be decomposed. This means f_{mag} has a vertical component & horizontal. My first calculation was a snapshot but generally v has x and y components u and V now.

↳ thus $\vec{f}_{mag}$ will gain one as well. $|\hat{F}_{mag} \cdot \hat{x}| = quB$ or $f_x = uB$

This is the same force the person has to pull so it moves at constant speed.

$f_{pull} = uB$ $\qquad f_{pull} = uB\hat{x}$ opposite to B field

the work per charge is then

$$\int \vec{f}_{pull} \cdot d\vec{l} = \int_0^h uB \sin\theta \frac{dh}{\cos\theta}$$

angle between $d\vec{l}$ and f_{mag}

bec $\rightarrow \frac{dh}{dl} = \cos\theta \rightarrow dh = dl\cos\theta$

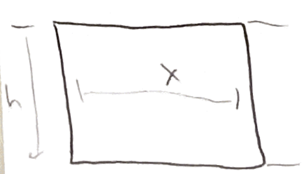
$dl = \frac{dh}{\cos\theta}$

↳ $(uB)\left(\frac{h}{\cos\theta}\right)\sin\theta = uBh\tan\theta = \boxed{Bhv}$ and $\tan\theta = \frac{v}{u}$

which is $\mathcal{E}$. Notice $\mathcal{E}$ came from integrating f_{mag} around the loop and the work came from integrating f_{pull} along its path

an alternative hide way to define $\mathcal{E}$

$\Phi = \int \vec{B} \cdot d\vec{a} = Bhx \rightarrow \frac{d\Phi}{dt} = Bh\frac{dx}{dt} = \underbrace{-Bhv}_{\mathcal{E}}$, x gets smaller over time so its negative



so $\frac{d\Phi}{dt} = -\mathcal{E}$ or $\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$

...well's Equations

... of the equations before Maxwell:

i) $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$

ii) $\vec{\nabla}\cdot\vec{B} = 0$

iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$

iv) $\vec{\nabla}\times B = \mu_0 \vec{J}$

There is a fatal inconsistency with these formulas, apply the divergence to ...

$\vec{\nabla}\cdot\underbrace{(\vec{\nabla}\times\vec{E})}_{\text{has to be } 0} = \vec{\nabla}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0}$ so this is ok

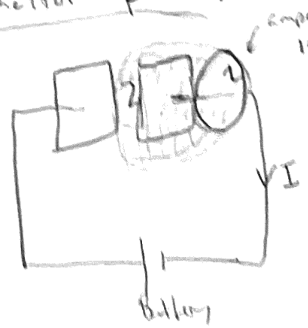
$\vec{\nabla}\cdot\underbrace{(\vec{\nabla}\times\vec{B})}_{\text{so this is always equal to } 0} = \mu_0(\vec{\nabla}\cdot\vec{J}) \Rightarrow 0 = \mu_0(\vec{\nabla}\cdot\vec{J})$ but the divergence of $\vec{J}$ in general is not

↳ To fix this look at the continuity eq: $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \overset{\text{gauss law}}{=} -\frac{\partial}{\partial t}(\varepsilon_0\vec{\nabla}\cdot\vec{E}) = -\vec{\nabla}\cdot\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

↳ $0 = \vec{\nabla}\cdot\left(\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$

↳ now $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}$

Capacitor paradox



The E field between two (infinite) plates is $\boxed{E = \frac{\sigma}{\varepsilon_0}}$

$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{A} \Rightarrow \frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{dQ}{dt} = \boxed{\frac{1}{\varepsilon_0 A}I}$

$\vec{\nabla}\times B = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial E}{\partial t} \xrightarrow{\text{integral form}} \oint\vec{B}\cdot d\vec{l} = \mu_0 I_{encl} + \mu_0\varepsilon_0\int\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$

↳ on surface 1, $E = 0$ and $I_{encl} = I$ so $\oint\vec{B}\cdot d\vec{l} = \mu_0 I$ ✓

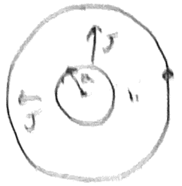
↳ on surface 2, $I_{encl} = 0$, $\int\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a} = \frac{I}{\varepsilon_0} \rightarrow \oint\vec{B}\cdot d\vec{l} = \mu_0 I$ ✓

example 2 | Two concentric spherical shells

↳ inner one (radius a) has $Q(t)$

↳ outer one (radius b) has $-Q(t)$

space in between is ohmic of conductivity σ



↳ $\vec{J} = \sigma\vec{E} = \frac{\sigma}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$, $I = -\dot{Q} = \int\vec{J}\cdot d\vec{a} = \frac{\sigma Q}{\varepsilon_0}$ (radial & spherical symmetry)

what currents are supposed to make B fields?

↳ $\vec{\nabla}\cdot\vec{B} = 0$ (maxwell eq) → integral form → $\oint\vec{B}\cdot d\vec{a} = 0 = B(4\pi r^2)$ so $\vec{B} = 0$

This is NOT a static configuration Q, E, J are functions of time

$\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{1}{4\pi}\frac{\dot{Q}}{r^2}\hat{r} = -\frac{1}{4\pi\varepsilon_0}\frac{\sigma Q}{r^2}\hat{r}$

cancels out $\vec{J}$ thus

$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{B} = 0$

The induced electric field

Faradays law! $\boxed{\vec{\nabla}\times\vec{E} = \frac{-\partial B}{\partial t}}$

↳ this a generalitation from the original curl definition $\boxed{\vec{\nabla}\times\vec{E} = 0}$ (electrostatics)

↳ divergence of $\vec{E}$ is still $\underbrace{\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}}_{\text{gauss law}}$ (need the curl & div to describe a field)

[For a pure Faraday field] → $\rho = 0$ and just $\frac{\partial B}{\partial t}$ → $\boxed{\vec{\nabla}\cdot\vec{E} = 0,\ \vec{\nabla}\times\vec{E} = \frac{-\partial B}{\partial t}}$

[Recognize: This is completley identical to magnetostatics w/ $\underbrace{\vec{\nabla}\cdot\vec{B}}_{\text{replace } B \text{ w/ } E} = 0,\ \vec{\nabla}\times\vec{B} = \underbrace{\mu_0\vec{J}}_{\text{replace } \mu_0\vec{J} \text{ w/ } -\frac{\partial B}{\partial t}}$] → [Faraday induced electric fields are determined by $-\partial B/\partial t$ in exactly the same way as magnetostatic fields are by $\mu_0\vec{J}$]

→ $\vec{E} = -\frac{1}{4\pi}\int \frac{\left(\frac{\partial B}{\partial t}\right)\times\hat{\eta}}{\eta^2}\,d\tau$

→ $\boxed{\vec{E} = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\vec{B}\times\hat{\eta}}{\eta^2}\,d\tau}$

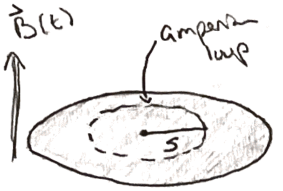
↳ for symetry: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$ → $\boxed{\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}}$

$\mu_0 I_{encl} = \mu_0\int \vec{J}\cdot d\vec{a}$

$= -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} = -\frac{\partial}{\partial t}\underbrace{\int \vec{B}\cdot d\vec{a}}_{\text{flux}}$

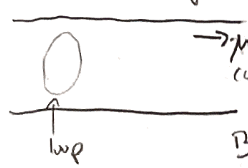
Example

• A uniform magnetic field $\vec{B}(t)$, pointing up fills the circular region. If B is changing w/ time what is the induced field?



Use $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi$: The direction of E is just like B bec they follow the same equations, B for a long straight wire. So its circumferential. (inside it w/ uniform J)

↓ analogous



→ $\mu_0\vec{J} = \mu_0 J_0\hat{z}$ (constant)

B → E

$\mu_0 J \to \frac{\partial B}{\partial t}$

→ $E(2\pi s) = -\frac{d}{dt}(\Phi) = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2\frac{dB}{dt}$

↳ $\boxed{\vec{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\phi}}$ if B increasing E runs clockwize as viewed from above

Example 2

• a rim of radius b there is line charge λ glued to the rim, but the rim is free to rotate.

• in radius a is a uniform magnetic field, B_0 pointing up. Some one turns B off, what happens?

• The changing B field induces an E field curling around the axis of the wheel (just like previous example)

[Faraday loop at radius b] → $\oint \vec{E}\cdot d\vec{l} = E2\pi b = -\pi a^2\frac{dB}{dt}$ or $\boxed{\vec{E} = -\frac{a^2}{2b}\frac{dB}{dt}\hat{\phi}}$

$\vec{N} = \vec{r}\times\vec{F} = \underbrace{b}_{r}\underbrace{\lambda E\,dl}_{qE}$ ⇒ $N = b\lambda\left(-\frac{a^2}{2b}\frac{dB}{dt}\right)\int dl = -b\lambda\pi a^2\frac{dB}{dt}$

b and E are ⊥

Maxwell's equations

1) $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$

2) $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$

3) $\vec{\nabla}\cdot\vec{B} = 0$

3) $\vec{\nabla}\times B - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$

$\vec{F} = q(\vec{E}+\vec{v}\times\vec{B})$

fields on the left
sources on the right

Maxwells equations in matter

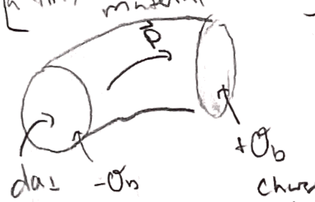
↳ more convenient way to write maxwells equations
↳ splitting free & Bound charges/current

We know

$\rho_b = -\vec{\nabla}\cdot\vec{p}$ and $\vec{J}_b = \vec{\nabla}\times\vec{M}$
$\sigma_b = \vec{p}\cdot\hat{n}$

New feature in the non-static case: Any change in the electric polarization involves a flow of bound charge. Call it $\vec{J}_p$ which should be included in the current.

[a tiny chunk of polarizable material]



→ Polarization induces a $\sigma_b = P$ at one end and $-\sigma_b$ at the other

↳ if P increases a bit then the charge on each increases giving a net current

$$dI = \frac{\partial}{\partial t}(\underset{\text{charge}}{\sigma_b\, da_\perp}) = \frac{\partial}{\partial t}(\underset{\text{charge}}{P\, da_\perp})$$ wait! → $\boxed{J_p = \frac{\partial\vec{P}}{\partial t}}$ then bec $\frac{dI}{da_\perp} = J$

polarization current

[Check w/ continuity eq] → $\vec{\nabla}\cdot\vec{J}_p = \vec{\nabla}\cdot\frac{\partial\vec{P}}{\partial t} = \frac{\partial}{\partial t}(\underbrace{\vec{\nabla}\cdot\vec{P}}_{-\rho_b}) = -\frac{\partial}{\partial t}\rho_b$ ✓

• a changing B field doesnt lead to an accumulation of charge or current
↳ $\vec{J}_b = \vec{\nabla}\times\vec{M}$ responds to variations in $\vec{M}$

Linear media:
$\vec{P} = \varepsilon_0\chi_e\vec{E}$, $\vec{M} = \chi_m\vec{H}$
$\vec{D} = \varepsilon\vec{E}$ and $\vec{H} = \frac{1}{\mu}\vec{B}$

Now seperate charge & current

$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P}$ and $\vec{J} = \underset{\text{free}}{\vec{J}_f} + \underset{\text{due to B}}{\vec{J}_b} + \underset{\text{due to E changes}}{\vec{J}_p}$

↳ Gauss law: $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}(\rho_f - \vec{\nabla}\cdot\vec{P})$ or $\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$, $\vec{D} = \varepsilon_0\vec{E}+\vec{P}$

↳ Ampere Law: $\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ or

$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}}$ w/ $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

$\boxed{\vec{\nabla}\cdot\vec{B} = 0 \quad\text{and}\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$ are unaffected since they dont involve p or J