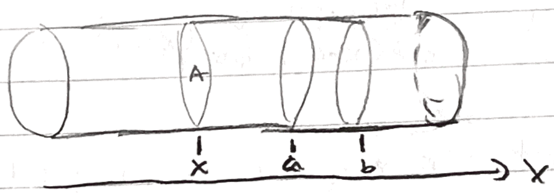
Conservation Laws

(ex) population rate of change must equal
+ migration rate in
- migrate rate out
+ birth rate
- death rate

• verbal expression of balance, same for energy, # of cars on freeway etc...

Quantifying:



$u(x,t)$ = linear density
↳ quantity in a width dx is $u(x,t)\,A\,dx$

$\phi(x,t)$ = flux of the quantity at x at time t ($A\phi$ = total amount crossing x)
↳ amount of quantity crossing the section at x at time t

[+ if flow is to right
- if flow is to left]

$f(x,t)$ = rate at which the quantity is created or destroyed within the section at x at time t.

[+ source
- sink] | $fA\,dx$ = amount created in a small width dx per unit time

Conservation Law Mathematically: in region between $x=a$ & $x=b$

Rate of change total quantity = Rate it flows in at $x=a$ − Rate it flows in at $x=b$ + Rate at which it was created in the region

↓

$$\frac{d}{dt}\int_a^b u(x,t)\,A\,dx = A\phi(a,t) - A\phi(b,t) + \int_a^b f(x,t)\,A\,dx$$

→ back

Simplifying w/ $\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b \frac{\partial u}{\partial t}\,dx$ if a & b are constants

$\phi(a,t) - \phi(b,t) = -\int_a^b \phi_x(x,t)\,dx$ (F, T, C)

↳ $\int_a^b u_t + \phi_x - f \, dx = 0$

Fundamental Conservation Law $\boxed{u_t + \phi_x = f}$

flux source

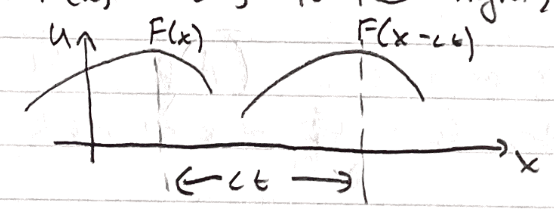
Advection Model: $\phi = cu \rightarrow u_t + cu_x = 0$

$\boxed{\text{Solution}}$ $u(x,t) = F(x-ct)$ thus $-cF' + cF' = 0$ ✓

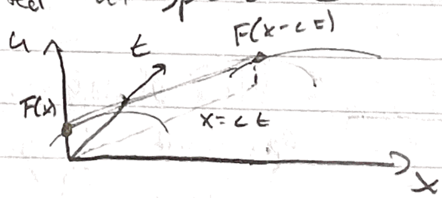
↳ $\frac{\partial u}{\partial t} = \frac{\partial F(x-ct)}{\partial(x-ct)}\frac{\partial}{\partial t}(x-ct) = -cF'$ and $c\frac{\partial u}{\partial x} = \frac{\partial F(x-ct)}{\partial(x-ct)}\frac{\partial}{\partial x}(x-ct) = cF'$

Right-traveling waves is what F is called

↳ F(x) moves to the right, undistorted at speed c



or



Ex Advection & Decay (Radioactive Element going through a tube)

$u_t + cu_x = -\lambda u$ quantity created from $\frac{du}{dt} = -\lambda u$ (radioactive decay eq)

$\frac{\partial u}{\partial t} = F_0^{-\gamma t}$

IVP for advection

$u_t + cu_x = 0$ and $u(x,0) = u_0(x)$

* F can be any function so lets make it our IV!

general solution is $u(x,t) = F(x-ct) = u_0(x-ct)$

Solving general advection eq:

$$u_t + cu_x + au = f$$

*advection eq propogates signals at speed c, transform to moving coordinate system

↳ let $\gamma = x - ct$ and $\tau = t$

$u(x,t) = U(x-ct, t)$ $u(x,t)$ in new variables $U(\gamma,\tau)$

↳ $u_t = U_\gamma \gamma_t + U_\tau \tau_t = -cU_\gamma + U_\tau$

$u_x = U_\gamma \gamma_x + U_\tau \tau_x = U_\gamma$

↳ gen eq: $U_\tau + aU = F(\gamma,\tau)$, $F(\gamma,\tau) = f(\gamma + c\tau, \tau)$

By way of this substitution it turns the eq into first order ODE.

Even more general: $u_t + cu_x = f(x,t,u)$ source term depends on u

↳ make sure transformation $\gamma = x - ct$, $\tau = t$

↳ taking derivatives from above again you just get

$$U_\tau = F(\gamma, \tau, U)$$

$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \to \frac{\partial}{\partial \tau}$ changing ind variables is a strategy for handeling eq w/ advection operators

For c having time & spatial dependence:

$u_t + c(x,t)\,u_x = f(x,t,u)$

↳ $c(x,t)$, speed of hetergerous mixture w/ now the speed depends on the location & time in the mixture

$\gamma = \gamma(x,t)$ and $\tau = t$

↳ $\gamma(x,t) = C$ is the general sol'n to $\frac{dx}{dt} = c(x,t)$

Ex $u_t + 2t\,u_x = 0$ → $c(x,t) = 2t$ and [Solving velocity eq so you have the correct moving coord sys]

↳ $\frac{dx}{dt} = 2t$ → $x - t^2 = C$ thus $\gamma = x - t^2$

Chain Rule $u_t = U_\gamma(-2t) + U_\tau$ $\quad \frac{d\tau}{dt} = 1$ and $u_x = U_\gamma$

$-2tU_\gamma + U_\tau + 2tU_\gamma = 0$ → $U_\tau = 0$

↳ solve: $U = g(\gamma)$ → $U(\gamma,\tau) = g(\gamma)$

$u(x-t^2, t) = g(x-t^2)$

↑

u is constant along this so

$u(x,t) = u(x-t^2, t) = g(x-t^2)$

Next: Ex 1.11

Example 1.11

[Solve advection eq in 1st quadrant] $\rightarrow$ $u_t + 2u_x = 0$, $u(x,0) = e^{-x}$, $u(0,t) = (1+t^2)^{-1}$

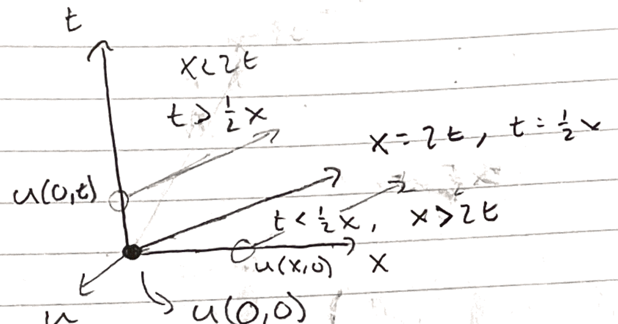
$x>0,\ t>0$

$x - ct,\ c = 2$

General Solution: $u(x,t) = F(x-2t)$

Determine F seperately in region $x > 2t$, $x < 2t$

$x - 2t > 0$ $\quad$ $x - 2t < 0$



* u is constant along this line *

Need to appy boundary condition $u(x,0)$ for $x > 2t$ because the characteristic line meets the x axis for $x > 2t$

$u(x,0) = F(x) = e^{-x} \longrightarrow u(x,t) = e^{-(x-2t)}$, $x > 2t$

$u(0,t) = F(-2t) = \frac{1}{1+t^2}$ $\quad 0 < x < 2t$

let $s = -2t \rightarrow F(s) = \frac{1}{1+s^2/4} \rightarrow F(x-2t) = \frac{1}{1+(x-2t)^2/4}$

Our Family of solutions:

$$\begin{cases} u(x,t) = e^{-(x-2t)}, & x > 2t \\ u(x,t) = \dfrac{1}{1+\frac{(x-2t)^2}{4}}, & x < 2t \end{cases}$$

PDE Classification of $\boxed{A u_{xx} + B u_{xy} + C u_{yy} = \Phi}$

[The type of PDE depends on] $D = B^2 - 4AC$ if $D>0$ hyperbolic, $D=0$ parabolic, $D<0$ elliptic

can change type ex $u_{xx} + x\, u_{yy} = 0$ ↑ depends on sign of x!

[Doing a transformation] $\gamma = \gamma(x,y)$ and $\eta = \eta(x,y)$ and $J = \begin{vmatrix} \gamma_x & \gamma_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$ for one to-one transform

Transform, generally

$u_x = u_\gamma \gamma_x + u_\eta \eta_x$ and $u_{xx} = u_{\gamma\gamma}\gamma_x^2 + 2u_{\gamma\eta}\gamma_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\gamma \gamma_{xx} + u_\eta \eta_{xx}$

$u_y = u_\gamma \gamma_x + u_\eta \eta_x$ and $u_{yy} = u_{\gamma\gamma}\gamma_y^2 + 2u_{\gamma\eta}\gamma_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\gamma\gamma_{yy} + u_\eta\eta_{yy}$

$u_{xy} = u_{\gamma\gamma}\gamma_x\gamma_y + u_{\gamma\eta}(\gamma_x\eta_y + \gamma_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\gamma\gamma_{xy} + u_\eta\eta_{xy}$

Substitute and move u_γ, u_η terms to Φ

$$\underbrace{(A\gamma_x^2 + B\gamma_x\gamma_y + C\gamma_y^2)}_{a}u_{\gamma\gamma} + \underbrace{(2A\gamma_x\eta_x + B(\gamma_x\eta_y + \gamma_y\eta_x) + 2C\gamma_y\eta_y)}_{b}u_{\gamma\eta} + \underbrace{(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2)}_{c}u_{\eta\eta}$$

* If you multiply out you'll see this matrix works, ew lin alg *

$$\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} = \begin{bmatrix} \gamma_x & \gamma_y \\ \eta_x & \eta_y \end{bmatrix}\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}\begin{bmatrix} \gamma_x & \gamma_y \\ \eta_x & \eta_y \end{bmatrix}^T \xrightarrow{\text{take the determinant}} \begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix} = \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix} J^2$$

$\hookrightarrow b^2 - 4ac = J^2(B^2 - 4AC) \rightarrow b^2 - 4ac = J^2 D$ (discriminant doesn't change sign under transformation) $J \neq 0$

Canonical Forms

$A u_{xx} + B u_{xy} + C u_{yy} = \Phi \rightarrow a u_{\gamma\gamma} + b u_{\gamma\eta} + c u_{\eta\eta} = \phi'$

[Hyperbolic] $D = B^2 - 4AC > 0$ and since $b^2 - 4ac = J^2 D$, $b^2 - 4AC > 0$

case 1: $a=0$ and $c=0$ $\Rightarrow$ $b u_{\gamma\eta} = \phi$ to get to this form

case 2: $b=0$ and $c=-a$ $\rightarrow$ $u_{\gamma\gamma} - u_{\eta\eta} = \phi/a$ to get to this form

[Parabolic] $D=0$ and $b^2 - 4ac = 0$

Case 1: $(a \text{ or } c) = 0$ and $b=0$ $\rightarrow$ $u_{\gamma\gamma} = \phi/a$

[Elliptic] $D<0$ and $b^2 - 4ac < 0$

case 1: $b=0$ and $c=a$ $\rightarrow$ $u_{\gamma\gamma} + u_{\eta\eta} = \phi/a$

...perbolic Equations

[to make $b^2-4ac>0$] $a=0=A\gamma_x^2+B\gamma_x\gamma_y+C\gamma_y^2 \rightarrow A\left(\frac{\gamma_x}{\gamma_y}\right)^2+B\left(\frac{\gamma_x}{\gamma_y}\right)^2+C=0$

$c=0=A\eta_x^2+B\eta_x\eta_y+C\eta_y^2 \rightarrow A\left(\frac{\eta_x}{\eta_y}\right)^2+B\left(\frac{\eta_x}{\eta_y}\right)^2+C=0$

[Quadratic Equation] $\frac{\gamma_x}{\gamma_y}=\mu$, roots: $\mu_1=\frac{-B+\sqrt{B^2-4AC}}{2A}$, $\frac{\eta_x}{\eta_y}=\mu_2=\frac{-B-\sqrt{B^2-4AC}}{2A}$ (make them distinct from eachother)

[For non constant γ_x, η_x etc.] $\rightarrow$ First order ODEs: $\gamma_x=-\mu_1\gamma_y=0$, $\eta_x=-\mu_2\eta_y=0$ [For constant coord transformation $\gamma=$ constant, $\eta=$ constant] $\rightarrow$ $d\gamma=\gamma_x dx+\gamma_y dy=0$

$\hookrightarrow \frac{dy}{dx}=-\frac{\gamma_x}{\gamma_y}$, $\frac{dy}{dx}=\frac{-\eta_x}{\eta_y}$

[Find quadratic in terms of dy/dx] $\rightarrow A\left(\frac{dy}{dx}\right)^2-B\left(\frac{dy}{dx}\right)+C=0$

$\hookrightarrow$ [Roots of this] $\rightarrow \frac{dy}{dx}=\frac{B+\sqrt{B^2-4AC}}{2A}$, $\frac{dy}{dx}=\frac{B-\sqrt{B^2-4AC}}{2A}$

[If A, B, C are CONSTANTS] $\rightarrow \boxed{y-\frac{B+\sqrt{B^2-4AC}}{2A}x=C_1=\gamma}$ and $\boxed{y-\frac{B-\sqrt{B^2-4AC}}{2A}x=C_2=\eta}$

$\hookrightarrow$ Reduces to $u_{\gamma\eta}=\phi/b$ and $b=2A\gamma_x\eta_x+B(\gamma_x\eta_y+\gamma_y\eta_x)+2C\gamma_y\eta_y=-\frac{D}{A}$

Second Canonical Form, $\alpha=\gamma+\eta$ and $\beta=\gamma-\eta \rightarrow u_{\alpha\alpha}-u_{\beta\beta}=\Psi$

Ex Wave eq: $u_{tt}-c^2u_{xx}=0$, $A=1$, $B=0$, $C=-c^2 \rightarrow D=4c^2>0$

$\hookrightarrow \gamma=x-\frac{0+\sqrt{0^2-4(1)(-c^2)}}{2}t=x-ct$, $\eta=x+ct$

$\hookrightarrow$ makes $a=0$, $c=0$ and $b=-D/A=-4c^2 \rightarrow u_{\gamma\eta}=0 \rightarrow u=f(\gamma)+g(\eta)$

$u=f(x-ct)+g(x+ct)$ ✓

Ex $(1-M^2)\phi_{xx}+\phi_{yy}=0 \rightarrow A=1-M^2$, $B=0$, $C=1$, $D=-4(1-M^2)$, $M^2>1$

$4(M^2-1)$

$y-\frac{0+\sqrt{4(M^2-1)}}{2(1-M^2)}x=y+\frac{1}{\sqrt{M^2-1}}x=\gamma$ and $y-\frac{x}{\sqrt{M^2-1}}=\eta$

$\hookrightarrow$ reduces to $u_{\gamma\eta}=0$ ✓

abolic Equations

- $D = 0 = B^2 - 4AC = 0$ same for $b^2 - 4ac = 0$ so $a = 0$ and $b = 0$

$a = 0 = A\gamma_x^2 + B\gamma_x\gamma_y + C\gamma_y^2$
$b = 0 = 2A\gamma_x\eta_x + B(\gamma_x\eta_y + \gamma_y\eta_x) + 2C\gamma_y\eta_y$
need to satisfy both of these conditions

↳ [start w/ $a = 0$] $\to A\left(\frac{\gamma_x}{\partial y}\right)^2 + B\left(\frac{\gamma_x}{\partial y}\right) + C = 0$ try to find $\gamma =$ const $\to d\gamma = 0 = \gamma_x dx + \gamma_y dy$

↳ $\frac{dy}{dx} = -\frac{\gamma_x}{\gamma_y} \to A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$

↳ w/ $D = 0 \to \frac{dy}{dx} = \frac{B}{2A}$, the other one is the same! thus we now use the $b = 0$ condition

$b = 0 = 2A\gamma_x\eta_x + B(\gamma_x\eta_y + \gamma_y\eta_y) + 2C\gamma_x\eta_y = 0 \to 2A\frac{\gamma_x}{\partial y}\eta_x + B\left(\frac{\gamma_x}{\partial y}\eta_y + \eta_x\right) + 2C\eta_y = 0$

↳ $2A\left(\frac{-B}{2A}\right)\eta_x + B\left(\frac{-B}{2A}\eta_y + \eta_x\right) + 2C\eta_y \to \underbrace{(B^2 - 4AC)}_{=0}\eta_y = 0$

↳ η_y can be anything so can η then

↳ $\boxed{\eta = x \text{ and } y - \frac{B}{2A}x = \gamma}$ $\to$ Jacobian is still $\neq 0$

↳ $w_{\eta\eta} = \phi/C$ and $C = A$ from plugging into other eq.
$w_{\gamma\gamma} = d/A$ (if we choose $C = 0$ instead of $a = 0$)

---

ex) $\alpha\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $A = \alpha$, $B = 0$, $C = 0$, $D = 0$

↳ $\frac{dt}{dx} = \frac{B}{2A} = 0 \to t =$ constant $\to \gamma = t$ and $\eta = x$

↳ $u_t = u_\gamma\gamma_t + u_\eta\eta_t = u_\gamma \to \boxed{u_{\eta\eta} = \frac{1}{\alpha}u_\gamma}$

Elliptic Equations $D = B^2 - 4AC < 0$

- $b = 0 = 2A\gamma_x\eta_x + B(\gamma_x\eta_y + \gamma_y\eta_x) + 2C\gamma_y\eta_y$ ⟶ γ & η equations are coupled to eachother
- $a = c$ or $a - c = 0 = A(\gamma_x^2 - \eta_x^2) + B(\gamma_x\gamma_y - \eta_x\eta_y) + C(\gamma_y^2 - \eta_y^2) = 0$

[Seperate coupled equations] → Do $(a-c) + ib$ → get terms like $A(\gamma_x^2 + 2i\gamma_x\eta_x - \eta_x^2)$

$$\downarrow$$

$$(\gamma_x + i\eta_x)^2$$

↳ [For all the terms] → $A(\gamma_x + i\eta_x)^2 + B(\gamma_x + i\eta_x)(\gamma_y + i\eta_y) + C(\gamma_y + i\eta_y)^2 = 0$

[Then we have] → $A\left(\frac{\gamma_x + i\eta_x}{\gamma_y + i\eta_y}\right)^2 + B\left(\frac{\gamma_x + i\eta_x}{\gamma_y + i\eta_y}\right) + C = 0$ now we can solve!

define $\alpha = \gamma + i\eta$, $\beta = \gamma - i\eta$

↳ $\frac{\gamma_x + i\eta_x}{\gamma_y + i\eta_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}$,

↳ [for each ± we get] → $\frac{\alpha_x}{\alpha_y} = \frac{-B + i\sqrt{4AC - B^2}}{2A}$ and $\frac{\beta_x}{\beta_y} = \frac{-B - i\sqrt{4AC - B^2}}{2A}$

[Find the Transformation] → $d\alpha = \alpha_x dx + \alpha_y dy = 0$, $\alpha =$ constant → $\frac{dy}{dx} = -\frac{\alpha_x}{\alpha_y}$ and $\frac{dy}{dx} = \frac{-\beta_x}{\beta_y}$

[Now solve for dy/dx] → $\frac{dy}{dx} = \frac{B - i\sqrt{4AC - B^2}}{2A}$ and $\frac{dy}{dx} = \frac{B + i\sqrt{4AC - B^2}}{2A}$

↳ $\boxed{y - \frac{B - i\sqrt{4AC - B^2}}{2A}x = \alpha \text{ and } y - \frac{B + i\sqrt{4AC - B^2}}{2A}x = \beta}$

↳ $\boxed{\gamma = \frac{\alpha + \beta}{2}, \quad \eta = \frac{\alpha - \beta}{2i}}$

(ex) $u_{xx} + x^2 u_{yy} = 0$, $A = 1$, $B = 0$, $C = x^2$, $D = -4x^2$, $x \neq 0$

$\frac{dy}{dx} = \frac{B - i\sqrt{4AC - B^2}}{2A} = -ix$, other one $\frac{dy}{dx} = ix$ → $y + i\frac{x^2}{2} = \alpha$ and $y - i\frac{x^2}{2} = \beta$

$\gamma = \frac{\alpha + \beta}{2} = y$ and $\eta = \frac{\alpha - \beta}{2i} = \frac{x^2}{2}$ (using transformation we got)

$$\boxed{u_{\gamma\gamma} + u_{\eta\eta} = -\frac{1}{2\eta}u_\eta}$$

2.1 Cauchy Problem for Heat Equation

$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-r^2} dr$

$$\boxed{\begin{aligned} u_t &= k u_{xx} \\ u(x,0) &= \phi(x) \end{aligned}}$$

initial temp in all of space is $\phi(x)$ (no boundary cond. yet) } Cauchy Problem

Solve for a Step Function $\phi(x)$ & construct solutions using that solution → Consider the problem

$w_t = k w_{xx}$

$w(x,0) = 0 \quad x<0$

$w(x,0) = u_0 \quad x>0$

} this is a step function

Dimensional Analysis Motivation

$y = -\frac{1}{2} g t^2 + v t \rightarrow \underbrace{\frac{y}{vt}}_{\pi_1} = -\frac{1}{2}\underbrace{\left(\frac{gt}{v}\right)}_{\pi_2} + 1$

↳ $\pi_1 = -\frac{1}{2}\pi_2 + 1 \rightarrow \boxed{\pi_1 = f(\pi_2), \ f = -\frac{1}{2}x + 1}$

* if we can find the dimensionless quantity in our heat ~~eq~~ we can try to write it in terms of them *

Lets look at our variables & constants of the problem & construct the dimensionless ones →

w ↑ temp; u_0 ↑ temp; k ↑ $\frac{\text{length}^2}{\text{time}}$; x ↑ length; t ↑ time

→ π_1 = cancel units of temp

π_2 = cancel units of k

↳ $\frac{w}{u_0}$ and $\frac{x}{\sqrt{4kt}}$

→ $\frac{w}{u_0} = f\left(\frac{x}{\sqrt{4kt}}\right)$

let $u_0 = 1$ and $z = \frac{x}{\sqrt{4kt}}$

$w = f(z)$ → compute partial derivatives

$w_t = f'(z) z_t = -\frac{1}{2} \frac{x}{\sqrt{4kt^3}} f'(z)$

$w_x = f'(z) z_x = \frac{1}{\sqrt{4kt}} f'(z)$

$w_{xx} = \frac{1}{4kt} f''(z)$

Plug in

$-\frac{1}{2} \frac{x}{\sqrt{4kt^3}} f'(z) = \frac{1}{4t} f''(z)$

→ $\boxed{f''(z) + 2z f'(z) = 0}$ → let $g = f'$: $g' + 2zg = 0 \rightarrow g = c_1 e^{-z^2}$

↳ $\boxed{f'(z) = c_1 e^{-z^2}} \rightarrow f(z) = c_1 \int_0^z e^{-r^2} dr + c_2 \rightarrow \boxed{w(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-r^2} dr + c_2}$

Recall

$\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$

Apply initial Conditions

$0 = w(x,0) = c_1 \int_0^{-\infty} e^{-r^2} dr + c_2 \quad (x<0)$ and $1 = w(x,0) = c_1 \int_0^{\infty} e^{-r^2} dr + c_2$

Solve and you get → $c_1 = \frac{1}{\sqrt{\pi}}$ and $c_2 = \frac{1}{2}$ → $\boxed{w(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-r^2} dr}$ or $w(x,t) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(x/\sqrt{4kt})$

To simplify Solution

↳ if w is a solution so is w_x → $0 = (w_t - k w_{xx})_x \rightarrow (w_x)_t - k(w_x)_{xx} \rightarrow \boxed{G = w_x}$

$w_x = G = \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial x} \int_0^{x/\sqrt{4kt}} e^{-r^2} dr$ → Second theorem of calc → $\frac{d}{dx}\int_0^{f(x)} h(t)\,dt = h(f(x)) \cdot f'(x)$

↳ $w_x = \boxed{\frac{1}{\sqrt{4\pi kt}} e^{-x^2/\sqrt{4kt}} = G(x,t)}$

$$\int_{-\infty}^{\infty} G(x,t)\,dx = 1$$

Recall

<u>Our original Problem</u>

$u_t = k u_{xx}$

$u(x,0) = \phi(x)$ ← Step func

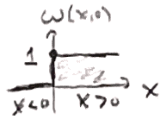
→

<u>Simplified Problem</u>

$w_t = k w_{xx}$

$w(x,0) = 0 \quad x<0$

$w(x,0) = 1 \quad x>0$



→

<u>Solution</u>

$$G(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$$

$$\int_{-\infty}^{\infty} G(x,t)\,dx = 1$$

$\boxed{G(x,t) \text{ is a solution for point source heat sources}}$ → $G(x-y,t)$ = temperature surface caused by an initial heat source at $x=y$, if we know the solution at one point we can use superposition to get the other solutions

$LG = \delta(x-y)$

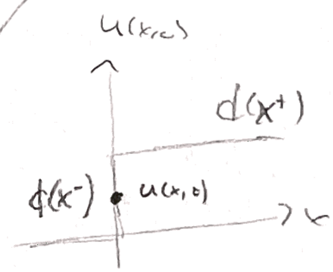
$Lu(x,0) = \phi(x)$

↳ we want all the initial temps, at $\phi(y)$ → $u(x,t) = \int_{-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{initial height of all temps}}}{\phi(y)}\, \underset{\substack{\uparrow \\ \text{point source solution at } x=y}}{G(x-y,t)}\,dy$ → Solution!

$$u(x,t) = \int_{-\infty}^{\infty} \phi(y) \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}}\,dy$$

works because

$$Lu = \int \phi(y)\, LG\, dy = \int \phi(y)\,\delta(x-y)\,dy = \phi(x)$$



<u>Theorem 2.1</u>

as $u(x,t) \to \phi(x)$ as $t \to 0^+$

$$u(x,t) = \frac{1}{2}\left(\phi(x^-) + \phi(x^+)\right)$$

<u>Poisson integral Representation</u>

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt})\,dr$$

, w/ the transformation $r = \frac{(x-y)}{\sqrt{4kt}}$, $dr = \frac{-dy}{\sqrt{4kt}}$

Ex Solve $u_t = k u_{xx}$, $u(x,0) = f(x)$ → $f(x) = \begin{cases} 2 & 0 \le x \le 1 \\ 0 & \end{cases}$

$f \neq 0$ only between $\underset{y=0}{0} \le x \le \underset{y=1}{1}$

↳ $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\text{all space}} e^{-r^2} f(\underbrace{x - r\sqrt{4kt}}_{y})\,dr$, $r = \frac{(x-y)}{\sqrt{4kt}}$ → $x = y + r\sqrt{4kt}$

$y=0 \to x = r\sqrt{4kt} \to r = \frac{x}{\sqrt{4kt}}$

$y=1 \to x = 1 + r\sqrt{4kt} \to r = \frac{x-1}{\sqrt{4kt}}$

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} 2e^{-r^2}\,dr = \frac{1}{\sqrt{\pi}}\left[\int_0^{\frac{x}{\sqrt{4kt}}} 2e^{-r^2}\,dr - \int_0^{\frac{(x-1)}{\sqrt{4kt}}} 2e^{-r^2}\,dr\right]$$

$$\boxed{u(x,t) = \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4kt}}\right)}$$

$G(x) = f(x) - F(x)$
$-F(x) = -f(x) + G(x)$

$$\int_0^x G'(x)\,dx = G(x) - G(0)$$

Cauchy Problem Wave Equation

$u_{tt} - c^2 u_{xx} = 0$ and $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$

Solution: $u(x,t) = F(x-ct) + G(x+ct)$ $\qquad G' - F' = g/c$

$u(x,0) = F(x) + G(x) = f(x)$ and $u_t = -cF'(x) + cG'(x) = g(x)$

$$(G(x) - F(x)) - (G(0) - F(0)) = \int_0^x (G' - F')\,dx = \int_0^x \frac{g(s)}{c}\,dx$$

$$G(x) - F(x) = \frac{1}{c}\int_0^x g(s)\,ds + G(0) - F(0)$$

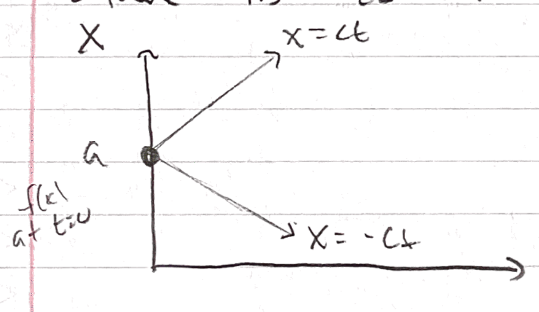
$$2G(x) = f(x) + \frac{1}{c}\int_0^x g(s)\,ds + G(0) - F(0)$$
$$2F(x) = f(x) - \frac{1}{c}\int_0^x g(s)\,ds - G(0) + F(0)$$

now plug $u(x,t) = F(x-ct) + G(x+ct)$

$$\boxed{u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds} \quad \cancel{- \frac{G(0)-F(0)}{2} + \frac{G(0)-F(0)}{2}}$$

Lets say we have a disturbance $f(x)$ at some point a
→ then its carried bec $u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

$x - ct$ ↑ initial condition



- the initial disturbance is carried along $x = ct$ & $x = -ct$
- atside this theres on wave
  → still have inside from $\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

2.4 Semi-infinite Domains

- originally we solved the heat eq $u_t = k u_{xx}$ on domain $-\infty \le x \le \infty$
- now study on domain $0 \le x \le \infty$
  - ↳ Boundary condition at $x=0$

The Problem:

$u_t = k u_{xx}$ $x>0, t>0$
$u(0,t) = 0$ $t>0$
$u(x,0) = \phi(x)$ $x>0$

We specify the temperature to be 0 at $x=0$ for all time

To solve we use the method of reflection across the boundary

$$\psi(x) = \begin{cases} \phi(x) & x>0 \\ -\phi(-x) & x<0 \end{cases}, \quad \psi(0)=0$$

*Physically we are attaching a bar from $-\infty < x < 0$ & giving it an initial temperature *
* negative of that in the original bar

[Consider the solution to this "odd" boundary condition]

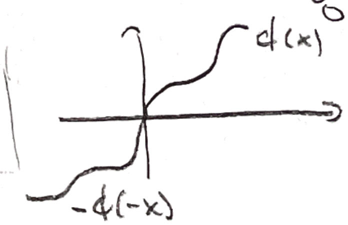
$$V(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,\psi(y)\,dy$$

defined in two areas

let $w = -y \to \int_{0}^{w=\infty} G(x+w,t)\,\phi(w)\,(-dw)$

$$V(x,t) = -\int_{-\infty}^{0} G(x-y,t)\,\phi(-y)\,dy + \int_{0}^{\infty} G(x-y,t)\,\phi(y)\,dy = \int_{0}^{-\infty} G(x-y,t)\,\phi(-y)\,dy + \int_{0}^{\infty} G(x-y,t)\,\phi(y)\,dy$$

$$\boxed{V(x,t) = \int_{0}^{\infty} \left(G(x-y,t) - G(x+y,t)\right)\phi(y)\,dy, \quad x \ge 0}$$



The Wave Equation (solved in same manner)

$u_{tt} = c^2 u_{xx}$ $x>0, t>0$
$u(0,t) = 0$ $t>0$
$u(x,0) = f(x)$, $u_t(x,0) = g(x)$ $x>0$

on back

If the boundary condition for heat eq. is replaced by $u_x(0,t) = 0$, solve by extending to an even function

Wave Equation $u_{tt} = c^2 u_{xx}$, $u(0,t) = 0$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $x > 0$

→ think of a fixed end on a string

Consider the PDE

$$v_{tt} - c^2 v_{xx} = 0 \qquad 0 < x < \infty,\ 0 < t < \infty$$

$$\left.\begin{aligned} v(x,0) &= \phi(x) \\ v_t(x,0) &= \psi(x) \end{aligned}\right\} x > 0 \quad \text{and} \quad v(0,t) = 0,\ t > 0$$

try reflection method

$$\phi_{odd}(x) = \begin{cases} \phi(x) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -\phi(-x) & \text{for } x < 0 \end{cases} \quad \text{and} \quad \psi_{odd}(x) = \begin{cases} \psi(x); & x > 0 \\ 0, & x = 0 \\ -\psi(-x), & x < 0 \end{cases}$$

same initial value as original!

Now we consider the IVP!

$u_{tt} - c^2 u_{xx} = 0$, $-\infty < x < \infty$, $0 < t < \infty$ ← all space!;

$u(x,0) = \phi_{odd}(x)$, $u_t(x,0) = \psi_{odd}(x)$

the initial value problem are odd

↳ solution, $u(x,t)$ will also be odd

$$\boxed{v(x,t) = u(x,t)\big|_{x \geq 0}}$$

For $x \geq 0$ (restriction here) and (d'Alembert Solution)

$$v(x,t) = \frac{1}{2}\left(\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(s)\,ds$$

Notice

1) $x \geq 0 \rightarrow x + ct > 0$

2) $\phi_{odd}(x+ct) = \phi(x+ct)$

3) $x - ct > 0 \rightarrow \phi_{odd}(x-ct) = \phi(x-ct)$

4) ↳ $x-ct$ to $x+ct$, $\psi_{odd}(s) = \psi(s)$

↳ for $x - ct > 0$, or $x > ct$ (ahead of leading signal)

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

(exactly d'Alembert's solution so works)

For $0 < x < ct$; → $x - ct < 0$

↳ x is still $x > 0$ but $x - ct < 0$ so negative in $\phi_{odd}(x-ct)$!!, ex) $\phi_{odd}(x-ct) = -\phi(ct-x)$

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\left(\int_{s=x-ct}^{0} -\psi(-s)\,ds + \int_0^{x+ct} \psi(s)\,ds\right)$$

let $s = -s$

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\left(\int_{ct-x}^{0} \psi(s)\,ds + \int_0^{x+ct} \psi(s)\,ds\right) = \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds$$

look we have solved it for the whole domain

$$\boxed{v(x,t) = \begin{cases} \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds & \text{for } x > ct \\ \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds & \text{for } 0 < x < ct \end{cases}}$$

2.5 PDE's w/ Source Terms

- **Duhamel's Principle**: Solution of a heat eq w/ a source & homogenous boundary conditions may be found by solving a homogenous heat eq w/ non-homogenous boundary conditions

*Same idea of how I solve first order ODE to take into account source terms

↳ solve homogenous → use homogenous & integrate

Consider the Heat Eq w/ a source term

$u_t - k u_{xx} = Q(x,t)$

$u(x,0) = 0 \qquad t>0$

- Lets say the initial heat is 0 to $Q = 0$ at $t=0$
- Apply the source heat $Q(x,0)$ to the rod at $t=0$

$V_t - k V_{xx} = 0$

$V(x,0) = Q(x,0) \qquad t>0$

More generally →

source is turned on at $t = s - \delta s$ and turned off at $t=s$

$w_t = k w_{xx}$

$w(x,s) = Q(x,s) \qquad t>s$

$w(x,t) \approx V(x,t-s) \rightarrow w(x,s) = V(x,0) \rightarrow \boxed{V(x,0) = Q(x,s)}$ its like the same initial cond

All w is saying is even at a later time w/ initial heat it will give same solution.

↳ $V_t - k V_{xx} = 0$, $V = V(x, t-s) \xrightarrow{\text{Solution}} u(x,t) = \int_0^t V(x,t-s)\,ds$

$V(x,0) = Q(x,s)$

Solution of $V(x,t) = \int_{\text{all}} G(x-y,t)\,Q(y,s)\,dy$

$$\boxed{u(x,t) = \int_0^t \int_{\text{all}} G(x-y,t-s)\,Q(y,s)\,dy\,ds}$$

adding up homogenous solutions subject to initial conditions at different times

$u(x,t) = u_1(x,t) + u_2(x,t) + \dots u_n(x,t)$

$u_1(x,0) = Q(x,t_1) + u_2(x,0) = Q(x,t_2) + \dots$

For Heat Eqn

$$u_t - ku_{xx} = f(x,t), \quad u(x,0) = \phi(x) \rightarrow \begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases} + \begin{cases} u_t - ku_{xx} = f(x,t) \\ u(x,0) = 0 \end{cases}$$

$$u = u_1 + u_2$$

$$\boxed{u(x,t) = \int_{\text{All}} G(x-y,t)\phi(y)\,dy + \int_0^t \int_{\text{All}} G(x-y,t-s) f(y,s)\,dy\,ds}$$

Wave eq

$f(x,t)$ is also an acceleration term

$$u_{tt} - c^2 u_{xx} = f(x,t)$$
$$u(x,0) = u_t(x,0) = 0$$

[u is the solution but u_t is the solution as well] $\rightarrow$ [$u_t(x,0)$ can therefore be thought of the initial accelerat.]

Consider

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for } t > s.$$
$$u(x,s) = 0, \quad u_t(x,s) = f(x,s)$$

let $w(x,\tau) = u(x,t)$, $\tau = t-s$

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 & \tau > 0 \\ w(x,\tau=0) = 0, \quad w_\tau(x,0) = f(x,s) \end{cases}$$

$\hookrightarrow$ solution: $w(x,\tau) = \frac{1}{2c}\int_{x-c\tau}^{x+c\tau} f(y,s)\,dy$

$$\boxed{u(x,t) = \int_0^t w(x,t-s)\,ds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds}$$