~~Free~~ Particle

$V(x) = 0$

$\hookrightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi$, $k \equiv \frac{\sqrt{2mE}}{\hbar}$

General sol'n time ind — time dep

$\psi(x) = Ae^{ikx} + Be^{-ikx} \rightarrow \psi(x,t) = Ae^{ik(x - \frac{\hbar k}{2m}t)} + Be^{-ik(x + \frac{\hbar k}{2m}t)}$

like a velocity

If you dont want the shape of your wave to change make $x \pm vt =$ constant

check it works

$k>0$ travels to the right

[Change of variables] $\rightarrow \psi_k(x,t) = Ae^{i(kx - \frac{\hbar k^2}{2m}t)}$, $k \equiv \pm\sqrt{2mE}/\hbar$

$k<0$ travelling to the left

By debrogle they carry a momentum: $p = \hbar k$

Speed of wave: coeff of t is a v when i's $\rightarrow V_{quantum} = \sqrt{\frac{E}{2m}} = \frac{\hbar|k|}{2m} = \frac{\text{coeff of } t}{\text{coeff of } x}$

Classically: $E = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{2E}{m}} = 2V_{quantum}$

- $\int_{-\infty}^{\infty}|\psi_k|^2 dx = |A|^2\int_{-\infty}^{\infty} dx = \infty$ (not normalizable)

$p = \hbar k$
$dp = \hbar dk$

- There is no such thing as a free particle w/ a definite energy

Before $k = n\cdot$(stuff)

$\psi(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\psi_n(x)\,\phi_n(t)}_{\psi_n(x,t)}$

$c_n = \int \psi_n^*(x)\,\psi(x,0)\,dx$
$c_k =$
really its also like this — just here it is used explicitly

Instead of $\psi_n \rightarrow \psi_k$ and k is continuous

$\hookrightarrow$ continuous range of Energies so a general sol'n is the superposition of all of them

$\psi(x,t) = \int_{\text{All space}} \psi_k(x,t)\,c_n(k)\,dk$

better notation

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\,e^{i(kx - \frac{\hbar k^2}{2m}t)}\,dk}$$

Normalizable for the correct range k's, range of Energy & speeds,

$\hookrightarrow$ wave packet

$\hookrightarrow$ Now find $\phi(k)$

General sol'n

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}\, dk$$

Just like $c_n = \int \psi_n \ldots$ we can apply initial conditions to the free particle

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk$$

← this is a fourier transform, we are saying $\psi(x,0)$ can be represented by adding ϕe^{ikx} linearly

We can invert a Fourier transform!

[Plancherel's theorem]

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0)\, e^{-ikx}\, dx$$

(proof online)

particles velocity → group velocity

speed this propagates

phase velocity → pick a point on a ripple and calculate its speed.

Lets determine the group & phase velocity for $\psi(x,t)$!

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \omega t)}\, dk \quad , \quad \omega = \frac{\hbar k^2}{2m} \quad , \quad p = \hbar k$$

To find the result lets assume $\phi(k)$ is narrowly peaked at some value k_0

↳ $\omega(k) \simeq \omega_0 + \omega_0'(k - k_0)$ and say $s \equiv k - k_0$

$$\psi(x,t) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k_0+s)\, e^{i((k_0+s)x - (\omega_0 + \omega_0' s)t)}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{i(\underset{1/m}{\uparrow}{k_0}x - \underset{1/s}{\uparrow}{\omega_0} t)} \int_{-\infty}^{\infty} \phi(k_0+s)\, e^{is(x - \omega_0' t)}\, ds$$

so $V_{phase} = \left.\frac{\omega}{k}\right|_{k=k_0}$, $\omega = \frac{\hbar k^2}{2m}$, $\frac{\omega}{k} = \frac{\hbar k}{2m}$

and $V_{group} = \left.\frac{d\omega}{dk}\right|_{k=k_0}$ $\quad d\omega/dk = \frac{\hbar k}{m}$

$V_{group} = 2 V_{phase}$

δ-fn Function Potential

$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$

$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$ and $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$, $\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$

Consider: $V(x) = -\alpha\,\delta(x)$

$\hookrightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi = E\psi$

<u>$x<0$, $V=0$:</u> bound state $E<0$, $\tilde{E} = E - (-V_0) = V_0 - |E|$ ($E = -|E|$)

$\frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi = k^2\psi$, $k \equiv \frac{\sqrt{-2mE}}{\hbar} = \frac{\sqrt{2m|E|}}{\hbar}$

$\hookrightarrow \psi(x) = Ae^{-kx} + Be^{kx}$ at $x = -\infty$ blows up so $A = 0$

$\boxed{\psi(x) = Be^{kx}}$ $x<0$

<u>$x>0$, $V=0$:</u> $V=0$ so same solution but at $x = \infty$ positive blows up

$\hookrightarrow \boxed{\psi(x) = Fe^{-kx}}$ $x>0$

there's a kink at $x=0$ where V blows up

<u>Boundary Conditions</u> $\rightarrow \psi(0) = \psi(0) \rightarrow B = F$

Trick to deal w/ this, integrate the schrödinger eq:

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{+\varepsilon}\frac{d^2\psi}{dx^2}\,dx + \int_{-\varepsilon}^{\varepsilon} V(x)\,\psi(x)\,dx = E\,\cancel{\int_{-\varepsilon}^{\varepsilon}\psi(x)\,dx}$$

goes to 0 as $\varepsilon \to 0$

only the potential term gets weird

$\left.\frac{d\psi}{dx}\right|_{-\varepsilon}^{\varepsilon}$

ordinarily this is zero in the limit $\varepsilon \to 0$, since the derivatives are cont but we have a delta funct potential so its just $\psi(0)$

$$\Delta\left(\frac{d\psi}{dx}\right) \equiv \lim_{\varepsilon\to 0}\left(\left.\frac{\partial\psi}{\partial x}\right|_{\varepsilon} - \left.\frac{\partial\psi}{\partial x}\right|_{-\varepsilon}\right) = \frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int V(x)\,\psi(x)\,dx$$

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

$\Delta\left(\frac{d\psi}{dx}\right) = -Bk - (Bk) = -2Bk = -\frac{2m\alpha}{\hbar^2}\psi(0) = -\frac{2m\alpha}{\hbar^2}B$

$B\,be^0$, $Be^{-0} = B$

$x>0$:

$\frac{d\psi}{dx} = -Bke^{-kx} \rightarrow \frac{d\psi}{dx} = -Bk$

$x<0$:

$\frac{d\psi}{dx} = Bke^{+kx} \rightarrow \frac{d\psi}{dx} = Bk$

$$\boxed{k = \frac{m\alpha}{\hbar^2} \qquad E = -\frac{\hbar^2k^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}}$$

Normalize ψ

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2B^2 \int_0^{\infty} e^{-2kx} dx = \frac{|B|^2}{k} = 1 \rightarrow B = \sqrt{k} = \frac{\sqrt{m\alpha}}{\hbar}$$

$$\hookrightarrow \psi(x) = \begin{cases} Be^{kx} & x \le 0 \\ Be^{-kx} & x \ge 0 \end{cases} \rightarrow \int_0^{\infty} B^2 e^{-2kx} dx + \int_{-\infty}^{0} B^2 e^{2kx} dx = 2B^2 \int_0^{\infty} e^{-2kx} dx$$

$$\boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha|x|}{\hbar^2}} \qquad E = -\frac{m\alpha^2}{2\hbar^2}}$$

, $|x|$ bec exp will always decay, always has neg exponent.

$\leftarrow$ always exactly one bound state

Scattering states $E>0$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \rightarrow k = \frac{\sqrt{2mE}}{\hbar} \rightarrow \psi(x) = Ae^{ikx} + Be^{-ikx}$$ $\leftarrow$ form of sinusoids

Sinusoids dont blow up at $x \to \infty$

$x<0$

$\psi(x) = Ae^{ikx} + Be^{-ikx}$ and

$x>0$

$\psi(x) = Fe^{ikx} + Ge^{-ikx}$

Boundary Conditions $\rightarrow$ at $x=0 \rightarrow F+G = A+B$

[Using limit argument from before we have] $\rightarrow$

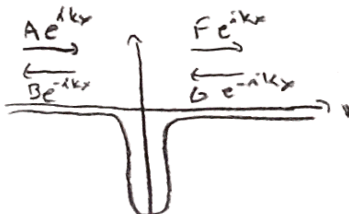
$\frac{d\psi}{dx} = ik(Fe^{ikx} - Ge^{-ikx})$, $x>0 \rightarrow \frac{d\psi}{dx} = ik(F-G)$

$\frac{d\psi}{dx} = ik(Ae^{ikx} - Be^{-ikx})$, $x<0 \rightarrow \frac{d\psi}{dx} = ik(A-B)$

evaluated at $x=0$ bec $\epsilon \to 0$ thats where the discontinuity is

$$\Delta\left(\frac{d\psi}{dx}\right) = ik(F-G-A+B), \quad \psi(0) = (A+B) = F+G$$

$$\hookrightarrow \underbrace{ik(F-G-A+B)}_{\Delta\frac{d\psi}{dx}} = -\frac{2m\alpha}{\hbar^2}\underbrace{(A+B)}_{\psi(0)}$$ $\leftarrow$ 2nd boundary condition $\rightarrow$ (do some algebra) $\rightarrow$ $F-G = A(1+2i\beta) - B(1-2i\beta)$, $\beta \equiv \frac{m\alpha}{\hbar^2 k}$



*lets say the particle comes from the left, so you will have a incident (A), reflected (B) and transmitted (F) * $\rightarrow$ $G=0$ we only have the wave coming from the left

$F = A(1+2i\beta) - B(1-2i\beta)$

$F = A+B$

$$\hookrightarrow \boxed{B = \frac{i\beta}{1-i\beta}A, \quad F = \frac{1}{1-i\beta}A}$$

$$\boxed{R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} \quad \text{and} \quad T \equiv \left|\frac{F}{A}\right|^2 = \frac{1}{1+\beta^2} \qquad R+T=1}$$

$$R = \frac{1}{1+\frac{2\hbar^2 E}{m\alpha^2}}$$

$$T = \frac{1}{1+\frac{m\alpha^2}{2\hbar^2 E}}$$

$R+T=1$

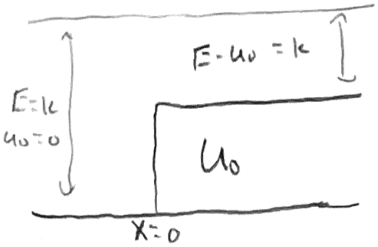
pg 69
joke lol
haha XD

$$\frac{V_T}{V_I} = \sqrt{\frac{E-V}{E}}, \quad V=0 \quad = 1$$

...ticle incident on a step

$E > U_0$

$$V(x) = \begin{cases} 0 & x<0 \\ U_0 & x \geq 0 \end{cases}, \quad -\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi$$



$x<0$: $V=0 \rightarrow -\frac{\hbar^2}{2m}\psi'' = E\psi \rightarrow \psi'' = -\frac{2m}{\hbar^2}E\psi$, $k_0 = \frac{\sqrt{2mE}}{\hbar}$

↳ $\psi_0(x) = A\sin(k_0x) + B\cos(k_0x) = Ae^{ik_0x} + Be^{-ik_0x}$

$x>0$: $V = U_0 \rightarrow -\frac{\hbar^2}{2m}\psi'' = (E-U_0)\psi \rightarrow \psi'' = -\frac{2m}{\hbar^2}(E-U_0)\psi$, $k_1 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$

(↳ more useful to write in complex exponential form) $\rightarrow \psi_1(x) = Ce^{ik_1x} + \cancel{De^{-ik_1x}}^{0}$, [particles only go in $+x$ dir after $x>0$]

Boundary conditions

$\psi_0(0) = \psi_1(0)$ and $\frac{d\psi_0(0)}{dx} = \frac{d\psi_1(0)}{dx} \rightarrow A+B = C$ and $(A-B)k_0 = Ck_1$

$$\boxed{\frac{k_1}{k_0} = \sqrt{1 - \frac{U_0}{E}}}$$

[A = incident amplitude
B = reflected amplitude
C = transmitted amplitude] $\rightarrow \left|\frac{B}{A}\right|^2 = \left|\frac{k_0-k_1}{k_0+k_1}\right|^2$ and $\left|\frac{C}{A}\right|^2 = \left|\frac{2k_0}{k_0+k_1}\right|^2$

probability density of being in a specific region

Group velocity, R and T

$V_0 = \frac{P_0}{m} = \frac{\hbar k_0}{m}$ = velocity of incident wave = group velocity

$V_1 = \frac{P_1}{m} = \frac{\hbar k_1}{m}$ =

↳ V_T = velocity of transmitted wave

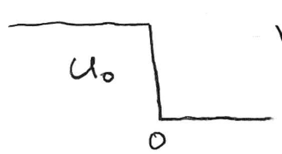
$V_0 R = V_0\left|\frac{B}{A}\right|^2 \rightarrow R = \left|\frac{B}{A}\right|^2 = \left(\frac{k_0-k_1}{k_0+k_1}\right)^2$

$\underbrace{V_0 T}_{V_0 \text{ that is transmitted}} = V_1 \left|\frac{C}{A}\right|^2 \rightarrow T = \frac{V_1}{V_0}\left|\frac{C}{A}\right|^2 = \frac{4k_0k_1}{(k_0+k_1)^2}$

V_T · prob density

$E > U_0$

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi$$



$$V(x) = \begin{cases} U_0 & x<0 \\ 0 & x>0 \end{cases}$$

$x<0$: $\psi_0(x) = Ae^{ik_0x} + Be^{-ik_0x}$, $k_0 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$

$x>0$: $\psi_1(x) = Ce^{ik_1x}$, $k_1 = \frac{\sqrt{2mE}}{\hbar}$

Boundary cond: $\psi_0(0) = \psi_1(0)$ and $\frac{d\psi_0(0)}{dx} = \frac{d\psi_1(0)}{dx} \rightarrow$ $A+B=C$, $(A-B)k_0 = Ck_1$] Same form

$\frac{B}{A} = \frac{k_0-k_1}{k_0+k_2}$, $k_0 < k_1$, goes down a hill & gains speed $\uparrow p = \hbar k$ and has a 180° phase shift

$\frac{C}{A} = \frac{2k_0}{k_0+k_1}$

$V_0 R = V_0\left|\frac{B}{A}\right|^2$ and $T = \frac{V_1}{V_0}\left|\frac{C}{A}\right|^2$

$= \frac{\cancel{V_{reflect}}^{R}}{\cancel{V_{inc}}}\left|\frac{B}{A}\right|^2$ and $T = \frac{V_{transmit}}{V_{inc}}\left|\frac{C}{A}\right|^2$

$E < U_0$ $\quad$ $x<0,\ V=0$

$V(x) = \begin{cases} 0 & x<0 \\ U_0 & x\geq 0 \end{cases}$ $\quad$ $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \psi'' = \frac{-2mE}{\hbar^2}\psi$

$\psi(x) = Ae^{ik_1 x} + Be^{-ik_1 x}$ $\quad$ $x<0$

$x>0$ $\quad V = U_0$ $\qquad\qquad -|E-U_0| = (E-U_0)$

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U_0\psi = E\psi \rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E-U_0)\psi \rightarrow \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}|E-U_0|\psi$

↳ (solutions are exponentials) $\rightarrow \psi = Ce^{k_2 x} + De^{-k_2 x}$, $k_2 = \frac{\sqrt{2m|E-U_0|}}{\hbar}$

we want decay

$\psi(0) = \psi(0) = A+B = D$ and $\frac{\partial}{\partial x}\psi(0) = \frac{\partial}{\partial x}\psi(0) = A-B = -i\frac{k_2}{k_1}D$ $\quad$ ✓ checked on white board

↳ $\frac{B}{A} = \frac{1+ik_2/k_1}{1-ik_2/k_1}$ and $\frac{C}{A} = \frac{2}{1-ik_2/k_1}$

$R = \left|\frac{B}{A}\right|^2 = 1$ $\quad$ then $T=0$

Calculating $|\psi|^2$ in $x<0$, $\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}$, $\psi^* = A^*e^{-ik_1x} + B^*e^{ik_1x}$

expand out B/A & real the bottom

$$B = A\left(\underbrace{\frac{k_1^2-k_2^2}{k_1^2+k_2^2}}_{x} + i\underbrace{\frac{2k_1k_2}{k_1^2+k_2^2}}_{y}\right) = Are^{+i\theta}, \quad \theta = \tan^{-1}\left(\frac{2k_1k_2}{k_1^2-k_2^2}\right)$$

$r = \sqrt{x^2+y^2}$, absorb into A

$\psi_1(x) = Ae^{ik_1x} + Ae^{-i(+k_1x-\theta)}$, do $\psi_1^*\psi_1$

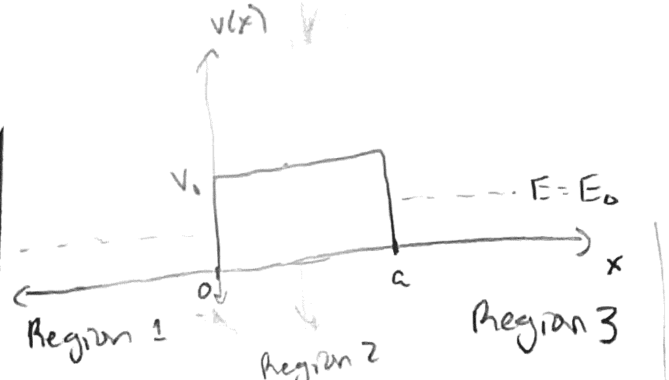
$$= \left(Ae^{-ik_1x} + Ae^{i(k_1x-\theta)}\right)\left(Ae^{ik_1x} + Ae^{-i(k_1x-\theta)}\right)$$

$$= 2A^2 + A^2e^{i(2k_1x-\theta)} + A^2e^{-i(2k_1x-\theta)}$$

$$= 2A^2\left[1+\cos(2k_1x-\theta)\right]$$

$P=0$ when $2k_1x - \theta = (2n+1)\pi$

Rectangular Potential Step



$\cancel{\text{SO. } (E-V_0) = -(E-V_0)}$

<u>Region 1</u>: $\psi_1 = Ae^{ikx} + Be^{-ikx}$, $k^2 = \frac{2mE}{\hbar^2}$

<u>Region 2</u>: $\frac{-\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2} + V_0\psi_2 = E\psi_2$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2} = (E-V_0)\psi_2 \quad \text{but } V_0 > E$$

$$\frac{\partial^2\psi_2}{\partial x^2} = \frac{-2m}{\hbar^2}(E-V_0)\psi_2$$

$$\frac{\partial^2\psi_2}{\partial x^2} = \underbrace{\frac{2m}{\hbar^2}(V_0-E)}_{k_2^2}\psi_2 \qquad E < V_0$$

<u>Region 2</u>: $\psi_2 = Ce^{ik_2x} + De^{-k_2x}$

<u>Region 3</u>: $\psi_3 = Fe^{+ikx}$, particle only comes from the left,

<u>Boundary Conditions</u>

① $\psi_1(0) = \psi_2(0)$ and ② $\psi_2(a) = \psi_3(a)$

③ $\frac{\partial\psi_1}{\partial x}(0) = \frac{\partial\psi_2}{\partial x}(0)$ and ④ $\frac{\partial\psi_2}{\partial x}(a) = \frac{\partial\psi_3}{\partial x}(a)$

① $A + B = C + D$ and ② $Ce^{k_2a} + De^{-k_2a} = Fe^{+ika}$

③ $ikA - ikB = k_2C - k_2D$ ④ $k_2Ce^{k_2a} - k_2De^{-k_2a} = +ikFe^{+ika}$

↳ $ik(A-B) = k_2(C-D)$

↳ $A - B = -i\frac{k_2}{k}(C-D)$

* find A in terms of C and D * ① $B = -A + C + D$

③ and ①

$$A - (-A + C + D) = i\frac{k_2}{k}(D-C)$$

$$2A - C - D = i\frac{k_2}{k}D - i\frac{k_2}{k}C$$

$$\boxed{2A = D\left(1 + \frac{ik_2}{k}\right) + C\left(1 - \frac{ik_2}{k}\right)}$$

② and ④

$$Ce^{k_2a} = Fe^{+ika} - De^{-k_2a}$$

$$Fk_2e^{ika} - 2Dk_2e^{-k_2a} = +ikFe^{+ika}$$

$$-2Dk_2e^{-k_2a} = -F(k_2 - ik)e^{ika}$$

$$\boxed{D = \frac{1}{2}\left(1 + \frac{ik}{k_2}\right)Fe^{ika + k_2a}}$$

now find D and C in terms of F

$$2A = D\left(1+\frac{ik_2}{k}\right) + C\left(1-\frac{ik_2}{k}\right)$$

$$D = \frac{1}{2}\left(1-\frac{ik}{k_2}\right)Fe^{ika+k_2a}$$

Do same algebra but for C:

$$C = \frac{1}{2}\left(1+\frac{ik}{k_2}\right)Fe^{ika-k_2a}$$

Now plug into top eq

$$2+i\left(\frac{k_2}{k}-\frac{k}{k_2}\right)$$

$$2+i\left(\frac{k}{k_2}-\frac{k_2}{k}\right)$$

$$2A = \frac{1}{2}\left(1+\frac{ik_2}{k}\right)\left(1-\frac{ik}{k_2}\right)Fe^{ika+k_2a} + \frac{1}{2}\left(1-\frac{ik_2}{k}\right)\left(1+\frac{ik}{k_2}\right)Fe^{ika-k_2a}$$

$$\left|\frac{A}{F}\right|^2 = \left[\frac{1}{4}\left(1+i\frac{k_2}{k}\right)\left(1-i\frac{k}{k_2}\right)e^{ika}e^{k_2a} + \frac{1}{4}\left(1-i\frac{k_2}{k}\right)\left(1+i\frac{k}{k_2}\right)e^{ika}e^{-k_2a}\right] \cdot$$

$$\left[\frac{1}{4}\left(1-i\frac{k_2}{k}\right)\left(1+i\frac{k}{k_2}\right)e^{-ika}e^{k_2a} + \frac{1}{4}\left(1+i\frac{k_2}{k}\right)\left(1-i\frac{k}{k_2}\right)e^{-ika}e^{-k_2a}\right]$$ expand out

$$= \underbrace{\frac{1}{16}\left(1+i\frac{k_2}{k}\right)\left(1-i\frac{k_2}{k}\right)\left(1+i\frac{k}{k_2}\right)\left(1-i\frac{k}{k_2}\right)e^{2k_2a}}_{\text{square terms}} + \underbrace{\frac{1}{16}\left(1-i\frac{k_2}{k}\right)\left(1+i\frac{k_2}{k}\right)\left(1-i\frac{k}{k_2}\right)\left(1+i\frac{k}{k_2}\right)e^{-2k_2a}}_{\text{square terms}}$$

$$+ \underbrace{\frac{1}{16}\left(1+i\frac{k_2}{k}\right)\left(1+i\frac{k_2}{k}\right)\left(1-i\frac{k}{k_2}\right)\left(1-i\frac{k}{k_2}\right)}_{\text{cross terms}} - \underbrace{\frac{1}{16}\left(1+i\frac{k_2}{k}\right)\left(1+i\frac{k_2}{k}\right)\left(1-i\frac{k}{k_2}\right)\left(1-i\frac{k}{k_2}\right)}_{\text{cross terms}}$$

pulled a negative out

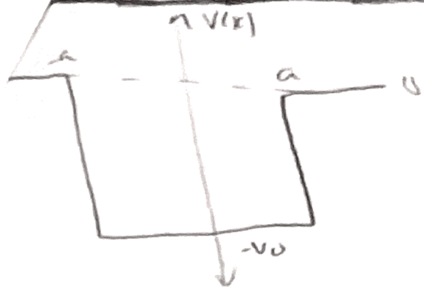
$$\left|\frac{A}{F}\right|^2 = \frac{1}{16}\left(1+\left(\frac{k_2}{k}\right)^2\right)\left(1+\left(\frac{k}{k_2}\right)^2\right)\underbrace{\left(e^{2k_2a}+e^{-2k_2a}\right)}_{2\cosh(2k_2a)}, \quad k_2^2 = V_0 - E, \quad k^2 = E$$

$$T^{-1} = \frac{1}{8}\left(\frac{V_0}{E}\right)\left(\frac{V_0}{V_0-E}\right)\cosh(2k_2a)$$ (my answer)

book:

$$T = \left|\frac{F}{A}\right|^2 = \left[1 + \frac{\sinh^2(k_2a)}{\frac{4E}{V_0}\left(1-\frac{E}{V_0}\right)}\right]^{-1}$$

...te Potential Well



$$V(x) = \begin{cases} -V_0 & -a \le x \le a \\ 0 & |x| > a \end{cases}$$

$E < 0 \Rightarrow$ bound state (stuck in turning points)

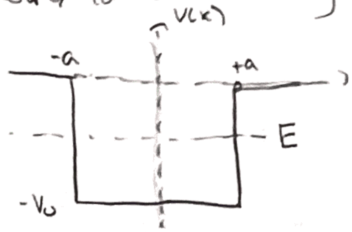
$E > 0 \Rightarrow$ scattering state
↳ doesn't get trapped

Harmonic oscillator only permitted bound states

Outside well $V=0$: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$ or $\frac{d^2\psi}{dx^2} = k^2\psi$, $k \equiv \frac{\sqrt{-2mE}}{\hbar}$] wait this is weird lets make it more mathematically rigorous

[If $V(x)$ rises higher than the particles total energy (on either side)] → bound state

[if E exceeds $V(x)$ on either side, particle goes from $-\infty$ to ∞] → scattering state



$V=0$ for $|x|>a$ and $V=-V_0$ in the well

note the difference from inf well, it had $V=0$ at the bottom of well
↳ if we wanted to make this an inf well, take $V_0 \to \infty \to V = -\infty$

[Energies in the inf well were measured w.r.t the bottom of the well $(E=0)$] → [finite well are measured w.r.t the bottom at $-V_0$]

→ we are interested in bound states, namely ones that are normalizable
↳ for this **E must be negative**

*if $E>0$, any solutions in the region $x>a$, where $V=0$, would be a plane wave, not normalizable
↳ $\boxed{-V_0 < E < 0}$ → E is negative, $E = -|E|$, for a bound state, $\tilde{E}$ must be measured w.r.t the bottom of the well, → $\tilde{E} = E-(-V_0) = V_0 - |E| > 0$ ← these $\tilde{E}$'s can be compared to inf well in limit

Schrodinger Equation

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$ → $\frac{d^2\psi}{dx^2} = \frac{-2m}{\hbar^2}(E - V(x))\psi = \alpha(x)\psi(x)$ [in region $|x|>a$ α is pos (real exponentials); in region $|x|<a$ α is neg (Airy functions)]

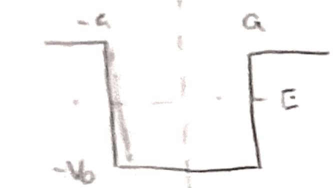
[The Potential is an even function, $V(-x) = V(x)$, boundstates are either symmetric / antisymmetric]

Even solutions ($\psi(-x) = \psi(x)$):

- $|x|<a$

$$\frac{d^2\psi}{dx^2} = \frac{-2m}{\hbar^2}(E-(-V_0))\psi = \frac{-2m}{\hbar^2}(V_0 - |E|)\psi \quad (V_0 > |E|), \quad k^2 \equiv \frac{2m}{\hbar^2}(V_0 - |E|) > 0, \; k>0$$

↳ $\psi'' = -k^2\psi$, the only possible even solution is $\boxed{\psi(x) = \cos(kx), \; |x|<a}$



Even solutions

$\boxed{\text{for } |x|<a : \ \psi(x) = \cos(kx)}$ (inside well), $k^2 = \frac{2m}{\hbar^2}(V_0 - |E|) > 0$, $k>0$

- $|x|>a \to \psi'' = -\frac{2m}{\hbar^2}(E-0)\psi = \frac{2m}{\hbar^2}|E|\psi \to \gamma^2 = \frac{2m|E|}{\hbar^2}$, $\gamma > 0$

↳ $\psi'' = \gamma^2\psi$ → solutions are exponentials, we need decay, otherwise not normalizable

↳ $\boxed{\psi(x) = Ae^{-\gamma x}, \ x>a}$ (outside well), (for $x<-a$ and $x>a$) → $\boxed{\psi(x) = Ae^{-\gamma|x|}, \ |x|>a}$

↳ note $\gamma^2 + k^2 = \frac{2mV_0}{\hbar^2}$, $|E|$ drops out,

[Define some unit free constants] → $\left[\eta \equiv ka, \ \xi \equiv \gamma a\right.$ and $\left. z_0^2 \equiv \frac{2mV_0a^2}{\hbar^2}\right]$ → multiply γ^2+k^2 by a^2 → $\boxed{\eta^2 + \xi^2 = z_0^2}$ → [a deep or wide potential ↳ z_0 is big; a shallow or narrow potential ↳ z_0 is small]

↳ solving for ξ is like solving for Energy → $\xi^2 = \gamma^2a^2 = \frac{2m|E|a^2}{\hbar^2} = \frac{2mV_0a^2}{\hbar^2}\frac{|E|}{V} = \boxed{z_0^2\frac{|E|}{V_0}}$

↳ $\left(\frac{\xi}{z_0}\right)^2 = \frac{|E|}{V_0}$

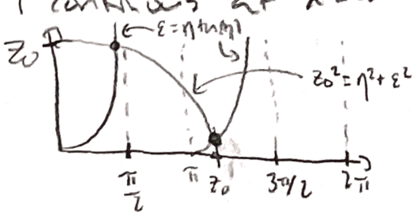
↳ η also encodes $|E|$ → $\eta^2 = k^2a^2 = \frac{2ma^2}{\hbar^2}(V_0 - |E|)$ and $\tilde{E} = V_0 - |E| = \eta^2 \frac{\hbar^2}{2ma^2}$ (energy relative to the bottom of potential)] characteristic energy of inf well

Boundary conditions

ψ continuous at $x=a$ → $\cos(ka) = Ae^{-\gamma a}$

ψ' continuous at $x=a$ → $-k\sin(ka) = -\gamma Ae^{-\gamma a}$

] divide: $k\tan(ka) = \gamma$ → mult each side by a → $\boxed{\eta\tan(\eta) = \xi}$



$\boxed{\text{Even solutions: } \eta^2 + \xi^2 = z_0^2, \ \xi = \eta\tan(\eta)}$, $\xi, \eta > 0$

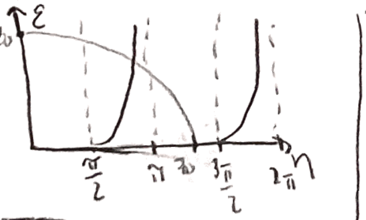
$$\psi(x) = \begin{cases} \sin(kx), & |x|<a \\ Ae^{-\gamma|x|}, & |x|>a \end{cases}$$

Odd solutions: Exact same thing but do odd side

ψ continuous at $x=a$ → $\sin(ka) = Ae^{-\gamma a}$

ψ' continuous at $x=a$ → $k\cos(ka) = -\gamma Ae^{-\gamma a}$

] divide $\xi = -\eta\cot(\eta)$, w/ $\eta^2 + \xi^2 = z_0^2$



Finite well scattering states! $\psi'' = -\frac{2m}{\hbar^2}(E - V(x))\psi$, now $E>0$

- $x<-a$, $V=0$ → $\psi(x) = Ae^{i\mu x} + Be^{-i\mu x}$, $\mu \equiv \frac{\sqrt{2mE}}{\hbar}$ (outside well) (TO THE LEFT)
- $|x|<a$, $V=-V_0$ → $\psi(x) = C\sin(\ell x) + D\cos(\ell x)$, $\ell \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$ (inside well)
- $x>a$, $V=0$ → $\psi(x) = Fe^{i\mu x}$, no incoming wave from the right (TO THE Right)

Boundary conditions

continuity of $\psi(x)$ at $-a$: $Ae^{-i\mu a} + Be^{i\mu a} = -C\sin(\ell a) + D\cos(\ell a)$

continuity of $\psi'(x)$ at $-a$: $i\mu[Ae^{-i\mu a} - Be^{i\mu a}] = \ell[C\cos(\ell a) + D\sin(\ell a)]$

continuity of $\psi(x)$ at $+a$: $C\sin(\ell a) + D\cos(\ell a) = Fe^{i\mu a}$

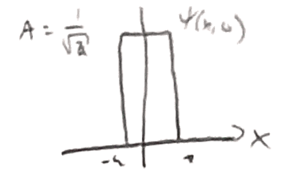
continuity of $\psi'(x)$ at $+a$: $\ell[C\cos(\ell a) - D\sin(\ell a)] = i\mu Fe^{i\mu a}$

A is incident amplitude
B is reflected amplitude
F is transmitted amplitude

$T = |F|^2/|A|^2$

...ed in Quantum

Localized free particle: $\Psi(x,0) = \begin{cases} A & \text{if } -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$



Normalize $A = \frac{1}{\sqrt{2a}}$

[Recall for free particle] $\rightarrow$ $\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk$ and $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx$

↳ [plug in for $t=0$] $\rightarrow$ $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{1}{\sqrt{2a}} e^{-ikx} dx = \frac{1}{2\sqrt{\pi a}} \frac{e^{-ik}}{-ik} \Big|_{-a}^{a}$

$= \frac{1}{\sqrt{\pi a}} \frac{\sin(ka)}{k} = \boxed{\sqrt{\frac{a}{\pi}} \operatorname{sinc}(ka) = \phi(k)}$

$\operatorname{sinc}(u) = \frac{\sin(u)}{u}$, $\lim_{c \to \infty} \operatorname{sinc}(cu) = \frac{1}{\pi}\delta(cu)$

$\operatorname{sinc}(0) = 1$, $\int_{-\infty}^{\infty} \operatorname{sinc}(u)\, du = \pi$

then $\boxed{\Psi(x,t) = \frac{1}{\pi}\sqrt{\frac{a}{2}} \int_{-\infty}^{\infty} \operatorname{sinc}(ka) e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk}$

check uncertainty

$\lim_{a \to 0} \phi(k) = \sqrt{\frac{a}{\pi}} \operatorname{sinc}(0) = \sqrt{\frac{a}{\pi}}$, constant number, $\Delta p \to \infty$, $\Delta x \to 0$

$\lim_{a \to \infty} \operatorname{sinc}(ka) = \pi\delta(ka) = \frac{\pi}{a}\delta(k)$ $\rightarrow$ $\lim_{a \to \infty} \phi(k) = \sqrt{\frac{\pi}{a}}\delta(k)$, $\Delta p \to 0$, $\Delta x \to \infty$

Full width half maximum;

$\phi(k) = \sqrt{\frac{a}{2\pi}} \frac{2a}{a^2+k^2} \rightarrow \sqrt{\frac{a}{2\pi}} \frac{2a}{a^2}$

max at $k=0$ ex: w/

half max! $\frac{\phi(k)}{\phi(0)} = \frac{1}{2}$ (point you're looking for; want it to be a half; max) $\rightarrow \frac{1}{2} = \frac{\frac{2a}{a^2+k^2}}{\frac{2}{a}} = \frac{2a}{a^2+k^2} = \frac{1}{a} \rightarrow k = \pm a$

$\boxed{\text{So width is } 2a}$