2.6 Laplace Transforms

Definition: $\mathcal{L}(u) = U(s) = \int_0^\infty u(t)\, e^{-st}\, dt$, $\mathcal{L}^{-1}(U) = u$

Derivitives: $\mathcal{L}(u') = sU - u(0)$ and $\mathcal{L}(u'') = s^2 U - s u(0) - u'(0)$

Quick ex: $u'' + u = 0$, $u(0) = 0$, $u'(0) = 1$

$\hookrightarrow s^2 U - 1 + U = 0 \to U = \frac{1}{s^2+1} \to \mathcal{L}^{-1}(U) = u(t) = \sin(t)$

Convolution Theorem: $\mathcal{L}(u * v) = UV$, $u * v \equiv \int_0^t u(t-\tau)\, v(\tau)\, d\tau$

$\mathcal{L}^{-1}(UV) = u * v$

Example: $u' + 3u = f(t)$, $u(0) = 1$

$$sU + 3U - 1 = \mathcal{L}(f(t)) = F(s)$$

$$U(s+3) = 1 + F(s) \to U = \frac{1}{s+3} + \frac{1}{s+3} F(s),$$

$$\mathcal{L}^{-1}(U) = u(t) = e^{-3t} + \underbrace{\mathcal{L}^{-1}\left(\frac{1}{s+3} F(s)\right)}_{e^{-3t} * f(t)} = \boxed{e^{-3t} + \int_0^t e^{-3(t-\tau)} f(\tau)\, d\tau}$$

PDE's

$$\mathcal{L}u = U(x,s) = \int_0^\infty u(x,t)\, e^{-st}\, dt$$

Derivitives: $\mathcal{L}(u_t) = sU(x,s) - u(x,0)$, $\mathcal{L}(u_{tt}) = s^2 U(x,s) - s u(x,0) - u_t(x,0)$

$\mathcal{L}(u_x) = U_x$, $\mathcal{L}(u_{xx}) = U_{xx}$

Example:

$u_t - u_{xx} = 0$, $x > 0$, $t > 0$

$u(x,0) = 0$, $x > 0$

$u(0,t) = 1$, $t > 0$ (boundary)

$\xrightarrow{\text{take Laplace transform}}$ $sU(x,s) - U_{xx}(x,s) = 0$ (ODE)

$\hookrightarrow$ solution: $\boxed{U(x,s) = a(s)\, e^{-\sqrt{s}\,x} + b(s)\, e^{\sqrt{s}\,x}}$

$\hookrightarrow$ [as $x \to \infty$ we want it not to blow up] $\to$ $U(x,s) = a(s)\, e^{-\sqrt{s}\,x}$ [and the $\mathcal{L}$ of boundary cond.] $\to$ $\mathcal{L}(u(x,0)) = 0$ (don't worry lol)

$\mathcal{L}(u(0,t)) = \frac{1}{s} \to a(s) = \frac{1}{s}$

$\hookrightarrow U(x,s) = \frac{1}{s} e^{-\sqrt{s}\,x} \to \boxed{\mathcal{L}^{-1}(U) = u(x,t) = \operatorname{erfc}\left(\frac{x}{\sqrt{4t}}\right)}$

Inverse Problems

$u_t = K u_{xx}$
$u(0,t) = f(t)$
$u(x,0) = 0 \quad x > 0$

trying to determine K
$\hookrightarrow -K\, u_x(0,t_0) = a$

Laplace transform of clarity
$\mathcal{L}(u_t) = sU - u(x,0)$
$\mathcal{L}(u_{xx}) = U_{xx}$

$\hookrightarrow \mathcal{L}(u_t) = K\mathcal{L}(u_{xx}) \rightarrow sU = KU_{xx} \rightarrow U_{xx} - \frac{s}{K}U = 0 \rightarrow U = a(s)e^{-\sqrt{\frac{s}{k}}x} + \cancel{b(s)e^{+\sqrt{\frac{s}{k}}x}}$ bounded $\to 0$

Boundary condition: $\mathcal{L}(u(0,t)) = F(t) \rightarrow F(t) = a(s) \Rightarrow U = F(t)e^{-\sqrt{\frac{s}{k}}x}$

$\hookrightarrow \mathcal{L}^{-1}(U) = \mathcal{L}^{-1}(F(t)) * \mathcal{L}^{-1}\left(e^{-\sqrt{\frac{s}{k}}x}\right)$, $\mathcal{L}^{-1}\left(e^{-\sqrt{\frac{s}{k}}x}\right) \overset{\text{using table}}{=} \frac{x}{2\sqrt{k}\sqrt{t^3}}e^{-\frac{x^2}{4kt}} = \frac{x}{\sqrt{4kt^3}}e^{-\frac{x^2}{4kt}}$

$$\boxed{u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi k(t-\tau)^3}} e^{-x^2/4(t-\tau)} f(\tau)\,d\tau}$$

Now just need to find the flux u_x, but can't just take derivatives w.r.t. x, you get $1/0$

To fix this recall: $G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$, $G_x = \frac{-2x}{4kt}\frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt} = \frac{-x}{2k\sqrt{4\pi kt^3}}e^{-x^2/4kt}$

$\hookrightarrow u(x,t) = -2k\int_0^t \frac{\partial}{\partial x} G(x,t-\tau) f(\tau)\,d\tau$
$G_\tau(x,t-\tau) = -\frac{\partial G}{\partial t}\underbrace{\frac{\partial t}{\partial \tau}}_{-1} = -G_t$, $G_t = kG_{xx}$
so $-G_\tau = kG_{xx}$

$\hookrightarrow u_x(x,t) = -2k\int_0^t \frac{\partial^2}{\partial x^2} G(x,t-\tau) f(\tau)\,d\tau = \boxed{2\int_0^t G_\tau(x,t-\tau) f(\tau)\,d\tau}$

Now integrate by parts:

$= \underbrace{2\,G(x,t-\tau)f(\tau)\Big|_0^t}_{=0 \text{ b/c } e^{-\infty} \text{ faster than } \frac{1}{x},\ f(0)=0} - \int_0^t G(x,t-\tau)\frac{\partial}{\partial\tau}f(\tau)\,d\tau \rightarrow \boxed{u_x = -2\int_0^t G(x,t-\tau)\frac{\partial f(\tau)}{\partial \tau}\,d\tau}$

$\hookrightarrow -K u_x(0,t_0) = a = +2k\int_0^{t_0} G(0,t_0-\tau)\frac{\partial f(\tau)}{\partial\tau}\,d\tau = +2k\int_0^{t_0}\frac{f'(\tau)}{\sqrt{4\pi k(t_0-\tau)}}\,d\tau$

$$\boxed{a = \sqrt{k}\int_0^{t_0}\frac{f'(\tau)}{\sqrt{\pi(t_0-\tau)}}\,d\tau}$$

2.7 Fourier Transforms

$\mathcal{F}u \equiv \hat{u}(\gamma) = \int_{-\infty}^{\infty} u(x)e^{i\gamma x}\,dx$ if $\int_{-\infty}^{\infty} |u|\,dx < \infty$ then $\hat{u}(\gamma)$ exists

Derivitives

$\mathcal{F}(u_{x^k}) = (-i\gamma)^k\,\hat{u}(\gamma)$

$\hookrightarrow \mathcal{F}u_x = (-i\gamma)\hat{u}(\gamma,t)$

$\hookrightarrow \mathcal{F}u_{xx} = (-i\gamma)^2\hat{u}(\gamma,t)$

$\hookrightarrow \mathcal{F}u_t = \hat{u}_t(\gamma,t)$

Multivariable Functions

$(\mathcal{F}u)(\gamma,t) \equiv \hat{u}(\gamma,t) = \int_{-\infty}^{\infty} u(x,t)\,e^{i\gamma x}\,dx$

Inversion Formula

$(\mathcal{F}^{-1}\hat{u})(x) \equiv u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\gamma)\,e^{-i\gamma x}\,d\gamma$

Example Calculate the Fourier Transform of $u(x) = e^{-ax^2}$, $a>0$

$\hookrightarrow \mathcal{F}u = \int_{-\infty}^{\infty} e^{-ax^2}e^{i\gamma x}\,dx = \hat{u}(\gamma) \rightarrow \frac{\partial \hat{u}(\gamma)}{\partial\gamma} = i\int_{-\infty}^{\infty} x\,e^{-ax^2}e^{i\gamma x}\,dx$

[Integrate by Parts] $\Rightarrow (wv)' = w\frac{dv}{dx} + \frac{dw}{dx}v$, $\frac{dw}{dx}v = -w\frac{dv}{dx} \rightarrow \int \frac{dw}{dx}v\,dx = -\int w\frac{dv}{dx}\,dx$, $\frac{dw}{dx} = xe^{-ax^2} \rightarrow w = \frac{1}{-2a}e^{-ax^2}$, $v = e^{i\gamma x} \Rightarrow \frac{dv}{dx} = i\gamma e^{i\gamma x}$

$\hat{u}_\gamma = -i\int_{-\infty}^{\infty} \frac{1}{-2a}e^{-ax^2}\,i\gamma e^{i\gamma x}\,dx = \frac{-\gamma}{2a}\int_{-\infty}^{\infty} e^{-ax^2}e^{i\gamma x}\,dx = \frac{-\gamma}{2a}\hat{u}(\gamma)$

$\hookrightarrow \hat{u}_\gamma = \frac{-\gamma}{2a}\hat{u}(\gamma)$ diff eq to solve $\rightarrow \hat{u}(\gamma) = Ce^{-\gamma^2/4a} \rightarrow \hat{u}(0) = \int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$

$\hat{u}(0) = C$

$\hookrightarrow \boxed{\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}\,e^{-\gamma^2/4a}}$

Convolution Theorem

$\mathcal{F}(u*v) = \hat{u}(\gamma)\hat{v}(\gamma)$ and $u*v = \mathcal{F}^{-1}(\hat{u}(\gamma)\hat{v}(\gamma))$

where $u*v = \int_{-\infty}^{\infty} u(x-y)v(y)\,dy$

Fourier Transform to the Heat Eq.

$u_t - k u_{xx} = 0$, $u(x,0) = f(x)$, $t > 0$

$\hookrightarrow \mathcal{F}(u_t) - \mathcal{F}(k u_{xx}) = 0 \rightarrow \hat{u}_t + \gamma^2 k \hat{u} = 0 \xrightarrow{\text{Solve}} \hat{u}(\gamma,t) = C e^{-\gamma^2 k t}$ (bounded)

$\hookrightarrow u(x,0) = f(x) \rightarrow \hat{u}(\gamma,0) = \hat{f}(\gamma) \rightarrow C = \hat{f}(\gamma) \rightarrow \hat{u}(\gamma,t) = \hat{f}(\gamma) e^{-\gamma^2 k t}$

$\hookrightarrow$ we found $\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}\, e^{-\gamma^2/4a}$,

$\mathcal{F}^{-1}(\hat{u}(\gamma,t)) = u(x,t) = \mathcal{F}^{-1}(\hat{f}(\gamma)) * \mathcal{F}^{-1}(e^{-\gamma^2 k t})$

$e^{-ax^2} = \sqrt{\frac{\pi}{a}}\, \mathcal{F}^{-1}(e^{-\gamma^2/4a}) \rightarrow kt = \frac{1}{4a} \rightarrow a = \frac{1}{4kt} \rightarrow \mathcal{F}^{-1}(e^{-\gamma^2/4a}) = \sqrt{\frac{a}{\pi}}\, e^{-ax^2}$

$\hookrightarrow \boxed{u(x,t) = f(x) * \sqrt{\frac{1}{4k\pi t}}\, e^{-\frac{x^2}{4kt}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}}\, e^{-(x-y)^2/4kt} f(y)\, dy}$

Laplaces Equation in a half plane

$u_{xx} + u_{yy} = 0$, $y > 0$, $u(x,0) = f(x)$ also bounded so solution remains as $y \to \infty$

$\boxed{\hat{u}_{yy} - \gamma^2 \hat{u} = 0} \xrightarrow{\text{Solve ODE}} \hat{u}(\gamma,y) = a(\gamma) e^{-\gamma y} + b(\gamma) e^{\gamma y} \rightarrow \hat{u} = \begin{cases} a(\gamma) e^{-\gamma y} & \gamma > 0 \\ b(\gamma) e^{\gamma y} & \gamma < 0 \end{cases}$

$\begin{cases} b(\gamma) = 0 & \text{if } \gamma > 0 \\ a(\gamma) = 0 & \text{if } \gamma < 0 \end{cases} \rightarrow \hat{u}(\gamma,y) = c(\gamma)\, e^{-|\gamma| y}$

Same thing

w/ $u(x,0) = f(x) \rightarrow \hat{u}(\gamma,0) = \hat{f}(\gamma) \rightarrow c(\gamma) = \hat{f}(\gamma) \rightarrow \boxed{\hat{u}(\gamma,y) = \hat{f}(\gamma)\, e^{-|\gamma| y}}$

$\rightarrow$ [use convolution theorem] $\rightarrow \mathcal{F}^{-1}(\hat{u}(\gamma,y)) = u(x,y) = \mathcal{F}^{-1}(\hat{f}(\gamma)\, e^{-|\gamma| y}) = \underbrace{\mathcal{F}^{-1}(\hat{f}(\gamma))}_{f(x)} * \underbrace{\mathcal{F}^{-1}(e^{-|\gamma| y})}_{\frac{y}{\pi}\frac{1}{x^2+y^2}}$

$$\boxed{u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)\, d\tau}{(x-\tau)^2 + y^2}}$$

2.3 Well posed Problems

Laplaces Equation: $u_{xx} + u_{yy} = 0$ $y > 0$, $u(x,0) = f(x)$, $u_y(x,0) = g(x)$

Special case: $f(x) = g(x) = 0$ the solution is $u(x,y) = 0$

Special case 2: $f(x) = \frac{1}{n}\cos(nx)$, $g(x) = 0$ the sol'n is $u(x,y) = \frac{1}{n}\cos(nx)\cosh(ny)$

↳ For large n we have only changed the boundary condition $f(x)$ by a small amount

But $u(x,y) \to \infty$ bec of the $\cosh(ny)$ term

A small change in Boundary Condition caused a large change in $u(x,y)$

↳ in real world this bad bec/ if we only know the data approximately then have a small change would make it impossible to model.

well posed if: 1) theres a solution 2) unique 3) stable

Consider: $u_t = k u_{xx}$, $u(x,0) = \phi(x)$ and $v_t = k v_{xx}$, $v(x,0) = \psi(x)$

↳ $|\phi(x) - \psi(x)| \leq \delta$ (are close) lets see what $\underset{w}{\underset{|||}{u-v}}$ turns out to be

↳ $w_t = k w_{xx}$, $w(x,0) = \phi(x) - \psi(x)$ δ = constant or small

↳ $w(x,t) = \int_{-\infty}^{\infty} (\phi(y) - \psi(y))\, G(x-y,t)\,dy \to |u-v| \leq \int_{-\infty}^{\infty} |\phi - \psi|\, |G(x-y,t)|\,dy$

↳ $|u-v| \leq \delta$, small change in initial cond produced small change in w

↳ for difusion

3.1 Orthogonal

heat flow in a finite bar

$l = \pi$, $k = 1$, ends are held at $0°$ degrees

↳ $u_t = u_{xx}$ $\quad 0 < x < \pi,\ t > 0$
$u(0,t) = u(\pi, t) = 0,\ t > 0$ } solution: $\boxed{u_n(x,t) = e^{-n^2 t} \sin(nx)}$

To know if a problem is well posed, we need $u(x,0) = f(x)$
* So we can see how much a small change in $f(x)$ effects u &

↳ $u(x,0) = f(x)$, $\quad 0 < x < \pi$

How can we solve it now w/ just $u_n(x,t) = e^{-n^2 t} \sin(nx)$?

↳ Inf sum of linear combos: $\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx)}$

↳ $\boxed{u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx)} = f(x)$

To solve a_n:

$\int_0^{\pi} \sin(nx)\sin(mx)\,dx = 0$ for $n \neq m$

for $n = m$ is non-zero

$$\int_0^{\pi} f(x) \sin(mx)\,dx = \int_0^{\pi} \sum_{n=1}^{\infty} a_n \sin(nx) \sin(mx)\,dx$$

$$= \int_0^{\pi} a_m \sin(mx)^2\,dx = \frac{\pi}{2} a_m$$

↳ $\boxed{a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx}$, $n = 1, 2, 3, \ldots$

Orthogonal Expansions (3.2)

[An inf series like the fourier series] → [on the interval $-\pi$ to π we have] →

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad n=0,1,2,\dots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \quad n=1,2,\dots$

ex fourier series of the step function

$f(x)=0, \quad -\pi \le x < 0, \quad f(x)=1, \quad 0 \le x \le \pi$

Calc coeff

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = 1$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx = \frac{1}{n\pi}\sin(nx)\Big|_0^{\pi} = 0$$

and $b_n = \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx = -\frac{1}{n\pi}\cos(nx)\Big|_0^{\pi}$

$$b_n = -\frac{1}{n\pi}\left(\cos(n\pi)-1\right) = \frac{1}{n\pi}\left(1-(-1)^{n+1}\right)$$

Thus: $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \dots\right)$

More general

in $\int_a^b |f(x)|^2\,dx < \infty$ and $\langle f|g\rangle = \int_a^b f(x)g(x)\,dx$

$\langle f|g\rangle = \langle g|f\rangle$
$\langle f, g+h\rangle = \langle f|g\rangle + \langle f|h\rangle$
$\langle f|\alpha g\rangle = \alpha\langle f|g\rangle$

and $|f| = \sqrt{\langle f|f\rangle}$, $\hat{f} = \frac{f(x)}{|f|}$

Expanding Arbitrary f

$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$ (does this converge or blow up? 3 types of errors

1. Pointwise Error: $E_N(x) \equiv f(x) - S_N(x) \equiv f(x) - \sum_{n=1}^{N} c_n f_n(x)$

* if for every x in $[a,b]$ we have $\lim_{N\to\infty} E_N(x) = 0$ then it converges

2. Mean Square Error: $e_N \equiv \int_a^b |f(x) - S_N(x)|^2\,dx \equiv \int_a^b \left|f(x) - \sum_{n=1}^{N} c_n f_n(x)\right|^2 dx$

* converges if $\lim_{N\to\infty} e_N = 0$ or $\lim_{N\to\infty} \|f - S_N\|^2 = 0$

3. Uniform Error: $\mathcal{E}_N = \sup_{a\le x\le b} |f(x) - S_N(x)| = \max_{a\le x\le b} \left|f(x) - \sum_{n=1}^{N} c_n f_n(x)\right|$

* converges uniformly if $\lim_{N\to\infty} \mathcal{E}_N = 0$

Ex

$\sum_{n=0}^{\infty} (-1)^n x^{2n} \quad -1<x<1$, each convergence tests needs to calc $S_N(x)$

$\hookrightarrow S_N(x) = \sum_{n=0}^{\infty} c_n f_n(x) = \sum_{n=0}^{N} (-1)^n x^{2n} = \sum_{n=0}^{N} (-x^2)^n = \frac{1-(-x^2)^{N+1}}{1+x^2}$

$S_N(x) = \frac{1+(-1)^{N+2} x^{2(N+1)}}{1+x^2}$ for $-1<x<1 \longrightarrow S_N(x) \to \frac{1}{1+x^2}$

Uniform Error

$\mathcal{E}_N = \max_{-1<x<1} \left| S_N - \frac{1}{1+x^2} \right| = \max_{-1<x<1} \frac{x^{2(N+1)}}{1+x^2}$ at $x=1$ $= \frac{1}{2}$ * doesn't go to zero as $N\to 0$

Mean Square

bc $(1+x^2)^2 = 4$

$E_n = \int_{-1}^{1} \left| S_N(x) - \frac{1}{1+x^2} \right|^2 dx = \int_{-1}^{1} \frac{x^{4N+4}}{(1+x^2)^2} dx < \underbrace{\int_{-1}^{1} x^{4N+4} dx}$

so if this is bigger & converges then so does

$\hookrightarrow \int_{-1}^{1} x^{4N+4} dx = \frac{2}{4N+5} \to 0$ as $N \to \infty$

So it converges in mean square sense

Suppose $f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$ converges in mean square sense

$\to$ find c_n: $\langle f | f_m \rangle = \int_a^b \sum_{n=1}^{\infty} c_n f_n f_m \, dx = \sum_{n=1}^{\infty} c_n \underbrace{\langle f_n | f_m \rangle}_{0 \text{ unless } n=m} = c_m \langle f_m | f_m \rangle$

$\hookrightarrow \boxed{c_n = \frac{1}{\|f_n\|^2} \langle f | f_n \rangle \quad n = 1, 2, \ldots}$

Theorem 3.8 (Best Approximation)

We have some function $f(x)$

$\hookrightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$\hookrightarrow C_n = \frac{1}{|f_n|^2}\langle f(x)|f_n(x)\rangle$

Each represent an expansion in $f(x)$

$f(x) = \sum_{n=1}^{\infty} C_n f_n(x)$ or $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$

What about the mean square?

orthonormal set $\langle f_n|f_n\rangle = 1$

Start w/ mean square error of fourier

$$\left|f - \sum_{n=1}^{N} a_n f_n\right|^2 = \langle f - \sum a_n f_n | f - \sum a_n f_n\rangle = \langle f|f\rangle - 2\sum a_n \langle f|f_n\rangle + \sum a_n^2$$

$$\langle f|f_n\rangle = \underset{1}{|f_n|^2} C_n$$

$$= \langle f|f\rangle - 2\sum a_n C_n + \sum a_n^2 + \sum C_n^2 - \sum C_n^2$$

$$= \underbrace{\langle f|f\rangle - \sum_{n=1}^{N} C_n^2}_{} + \sum_{n=1}^{N}(a_n - C_n)^2$$

$$|f - \sum C_n f_n|^2 = \langle f - \sum C_n f_n | f - \sum C_n f_n\rangle = \langle f|f\rangle - 2\sum C_n \overbrace{\langle f|f_n\rangle}^{C_n} + \sum C_n^2$$

$$= \langle f|f\rangle - \sum C_n^2$$

So

$$\boxed{\left|f - \sum_{n=1}^{\infty} a_n f_n\right|^2 = |f - \sum C_n f_n|^2 + \sum_{n=1}^{\infty}(a_n - C_n)^2}$$

bec if we $|f - \sum a_n f_n|^2$ is bigger than the C_n one, and converges, so does $|f - \sum C_n|^2$

$\hookrightarrow \left|f - \sum_{n=1}^{N} C_n f_n\right|^2 \le \left|f - \sum_{n=1}^{N} a_n f_n\right|^2 \leftarrow$ useful for convergence

$\hookrightarrow \underbrace{|f - \sum C_n f_n|^2}_{\text{always positive}} = |f|^2 - \sum C_n^2 \rightarrow |f|^2 \ge \sum C_n^2$ (Bessel inequality)

Proving an orthonormal sequence $\{f_n\}$ is complete

↳ completeness is equivalent that any f thats square integrable can be expanded in a fourier series where c_n are the coef

AND that series converges to f in the mean square sense

↳ Completeness is also true if $\sum_{n=1}^{\infty} c_n^2 = |f|^2$ is satisfied

Parsevals equality

Proof of Orthogonality of eigenfunctions

Green's Identity: $\langle y_2 | L y_1 \rangle - \langle y_1 | L y_2 \rangle = \langle y_2 | \lambda_1 y_1 \rangle - \langle y_1 | \lambda_2 y_2 \rangle$

$= (\lambda_1 - \lambda_2) \langle y_1 | y_2 \rangle = p(b) \bar{W}_b - p(a) \bar{W}(a)$

Calculate $W(a)$ + $W(b)$ using boundary cond's

$\alpha_1 y_1(a) + \alpha_2 y_1'(a) = 0$
$\alpha_1 y_2(a) + \alpha_2 y_2'(a) = 0$

← use boundary condition: y_1 and y_2 both have to satisfy the boundary cond

$$\begin{bmatrix} y_1(a) & y_1'(a) \\ y_2(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

← solve linear system w/ this set up!

so determinant of this matrix has to be 0

$\alpha_1, \alpha_2 \neq 0$

$\rightarrow y_1 y_2' - y_2 y_1' = 0 = W(a)$
and similarly for $W(b)$

$\rightarrow (\lambda_1 - \lambda_2) \langle y_1 | y_2 \rangle = 0$, $\lambda_1 \neq \lambda_2$ so $\boxed{\langle y_1 | y_2 \rangle = 0}$

Eigenvalues are Real

$Ly = \lambda y \rightarrow \langle y | Ly \rangle = \langle y | \lambda y \rangle = \langle \lambda y | y \rangle$

$\langle y | \lambda y \rangle = \langle \lambda y | y \rangle = \lambda \langle y | y \rangle = \bar{\lambda} \langle y | y \rangle$

has to be real

or $\lambda \neq \bar{\lambda}$

[An eigenvalue has a unique eigenfunction up to a constant multiple]

↳ Lagranges identity:

$$y_2 L y_1 - y_1 L y_2 = \frac{d}{dx}\left(p(x) W(x)\right)$$

let $Ly_1 = \lambda y_1$ and $Ly_2 = \lambda y_2$ so $= 0$

$$0 = \frac{d}{dx}\left(p(x) W(x)\right) \rightarrow p(x) W(x) = c$$

↳ [get c by boundary conditions on it] $\rightarrow$ $\alpha_1 y_1(a) + \alpha_2 y_1'(a) = 0$

$\alpha_1 y_2(a) + \alpha_2 y_2'(a) = 0$

↳ $\begin{bmatrix} y_1(a) & y_1'(a) \\ y_2(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow W(a) = 0$ so $c = 0$

$W(x) = y_1 y_2' - y_1' y_2 = 0 \rightarrow \boxed{\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = 0}$ so $y_1 =$ constant $\cdot\, y_2$

y_1 and y_2 are depe

~~eigenfunctions are real~~

Sturm-Liouville Problems

$u_t = (p(x)u_x)_x - q(x)u \qquad a<x<b,\ t>0$

$\alpha_1 u(a,t) + \alpha_2 u_x(a,t) = 0$

$\beta_1 u(b,t) + \beta_2 u_x(b,t) = 0$

$u(x,0) = f(x) \qquad a<x<b$

] contain all diff boundary conditions Dirichlet, Neumann, radiation w/ appropriate choices of α, β

Assume seperatable solution: $u(x,t) = y(x)\,g(t)$

$\hookrightarrow y g' = (p(x) g y_x)_x - q(x) y g = g(p(x) y_x)_x - q(x) y g$

$y \frac{1}{g} g' = (p(x) y_x)_x - q(x) y(x)$

$\hookrightarrow \frac{g'}{g} = \frac{(p(x) y_x)_x - q(x) y(x)}{y} = -\lambda$

$\hookrightarrow \boxed{-(p(x) y_x)_x + q(x) y = \lambda y}$ Sturm-Liouville diff eq

Boundary Equations

$\alpha_1 y(a) + \alpha_2 y_x(a) = 0$

$\beta_1 y(b) + \beta_2 y_x(b) = 0$

w/ $\alpha_1^2 + \alpha_2^2 \neq 0$

$\beta_1^2 + \beta_2^2 \neq 0$

on $a<x<b$

* if $[a,b]$ = infinite or $p(x_0) = 0 \rightarrow$ singular SLP;
* λ is called eigen value, solution is eigen function

ex $\boxed{\begin{array}{l} -y'' = \lambda y \quad 0<x<1 \\ y'(0)=0, \quad y(1)=0 \end{array}}$

for $\lambda > 0$ is only non trivial solution to this diff equation

$\lambda = \alpha^2$ to be explicit

↳ $y(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$y'(0) = \alpha B = 0 \rightarrow B = 0$

↳ $y(x) = A\cos(\alpha x)$

and w/ $y(1)=0$ → $A\cos\alpha = 0 \rightarrow \alpha = \frac{\pi}{2} + n\pi = \frac{\pi}{2}(2n+1)$

↳ $\lambda_n = \left(\frac{\pi}{2}(2n+1)\right)^2 \quad n = 0, 1, 2, \ldots,$

eigenvalue

eigenfunction

↳ $y_n(x) = \cos\left(\frac{\pi}{2}(2n+1)x\right), \quad n = 0, 1, 2, \ldots$

*all eigenfunctions solution from a SLP is complete

Notation: $Ly = -(p(x)y_x)_x + q(x)y \quad a<x<b$

↳ SLP: $\boxed{Ly = \lambda y \quad \text{w/} \quad \begin{array}{l} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{array} \quad \text{on } a<x<b}$

Recall: $\langle y_1 | y_2 \rangle = \int_a^b y_1(x) y_2(x)\, dx$

Wronskian: $W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$

Lagranges Identity: $y_2 L y_1 - y_1 L y_2 = \frac{d}{dx}\left(p(x)W(x)\right)$

Green's Identity: $(y_2, Ly_1) - (y_1, Ly_2) = \left(p(x)W(x)\right)\Big|_a^b = p(b)W_b - p(a)W_{(a)}$

↳ to prove do $\frac{d}{dx}(p(x)W(x)) = \frac{d}{dx}(py_1y_2') - \frac{d}{dx}(py_1'y_2)$ and expand out...

↳ integrate Lagranges identity + integrate by parts

4.1 Seperation of variables

Consider the problem:

$$\left.\begin{aligned} &u_t = u_{xx} \quad 0<x<\pi,\ t>0 \\ &u(0,t) = u(\pi,t) = 0, \quad t>0 \\ &u(x,0) = f(x); \quad 0<x<\pi \end{aligned}\right\} \quad \text{Solve by } u(x,t) = y(x)\,g(t).$$

see back

$\hookrightarrow\ y g' = g y'' \ \rightarrow\ \frac{1}{g} g' = \frac{1}{y} y'' = \overset{-\lambda}{\text{constant}},\ y(0) = y(\pi) = 0$ $\quad \cancel{g(0) = f(x)}\ \ 0<x<\pi$

$g' = -\lambda g$ and $y'' = -\lambda y$

consider cases $\lambda = 0$, $\lambda > 0$, $\lambda < 0$

$\hookrightarrow\ \boxed{g(t) = e^{-\lambda t}}$ now solve for y'' (nontrivial solutions)

$\underline{\lambda = 0}$: $y'' = 0 \rightarrow y(x) = Ax + B \rightarrow$ $y(0)=0 \rightarrow B=0$, $y(\pi)=0 \rightarrow A=0$ trivial solut.

$\underline{\lambda < 0}$: $\lambda = -\alpha^2 \rightarrow y'' = \alpha^2 y \rightarrow y(x) = Ae^{\alpha x} + Be^{-\alpha x}$ trivial solution

$y(0)=0$: $A + B = 0$

$y(\pi)=0$: $Ae^{\alpha\pi} + Be^{-\alpha\pi} = 0 \rightarrow -Be^{+\alpha\pi} + Be^{-\alpha\pi} = 0$ (only true if $B=0$, hence $A=0$)

$\underline{\lambda > 0}$: $\lambda = \alpha^2$, $y'' = -\alpha^2 y \Rightarrow y(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$y(0) = 0 \rightarrow A = 0$ $\quad$ if $B=0$ then trivial solution

$y(\pi) = 0 \rightarrow 0 = B\sin(\alpha\pi) \rightarrow \alpha\pi = n\pi$, $\boxed{\alpha = n}$

$\hookrightarrow\ \lambda = \lambda_n = \alpha^2 = n^2$, $n = 1, 2, \ldots$ and $y_n = B\sin(nx)$

$\hookrightarrow\ g_n = e^{-n^2 t}$, $\quad u(x,t) = g_n(t)\,y_n(x) = e^{-n^2 t}\sin(nx)$

Problem

$u_t = u_{xx} \quad 0<x<\pi$

$u(0,t)=u(\pi,t)=0$

$u(x,0)=f(x)$

found

$u(x,t)= e^{-n^2 t}\sin(nx)$

Total Solution: $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$

$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, $\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

More general Problem:

$$\boxed{\begin{array}{l} u_t = k u_{xx} \quad 0<x<\ell,\ t>0 \\ u(0,t)=0,\ u(\ell,t)=0 \\ u(x,0)=f(x) \end{array}} \rightarrow \frac{g'(t)}{kg} = \frac{y''(x)}{y} \rightarrow \begin{array}{l} g' = -\lambda k g \\ y'' = -\lambda y \end{array}$$

initial cond: $y(0)g(t)=0$, $y(\ell)g(t)=0$ $\rightarrow$ excluding case where $g(t)=0$ $\rightarrow$ $y(0)=0$, $y(\ell)=0$

$y(x)g(0)=f(x)$ — not just $g(0)$!

$\lambda = \alpha^2$

$y(0)=0=A$

$y'' + \alpha^2 y = 0 \rightarrow y(x) = A\cos(\alpha x) + B\sin(\alpha x) \rightarrow y(\ell)=0=B\sin(\alpha\ell)$

$\alpha\ell = n\pi \rightarrow \alpha = \frac{n\pi}{\ell} \rightarrow \lambda_n = \left(\frac{n\pi}{\ell}\right)^2$

$y_n = \sin\left(\frac{n\pi x}{\ell}\right)$ and

$g'(t) = -\lambda k g \rightarrow g(t) = e^{-\lambda k t} = e^{-n^2\pi^2 k t/\ell^2}$

$u_n = e^{-n^2\pi^2 k t/\ell^2}\sin\left(\frac{n\pi x}{\ell}\right) \rightarrow u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 k t/\ell^2}\sin\left(\frac{n\pi x}{\ell}\right)$

and $u(x,0) = f(x) = \sum C_n \sin\left(\frac{n\pi x}{\ell}\right) \rightarrow C_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

4.1 wave function

$$\boxed{\begin{array}{l} u_{tt} = c^2 u_{xx} \\ u(0,t)=0 \quad \underset{t>0}{} \text{ and } \quad u(x,0)=F(x) \qquad 0<x<l \\ u(l,t)=0 \quad \text{and} \quad u_t(x,0)=G(x) \end{array}}$$

Separate $u(x) = y(x)g(t)$

$\hookrightarrow \frac{g''}{c^2 g} = \frac{y''}{y} \rightarrow \begin{array}{l} g'' + c^2\lambda g = 0 \\ y'' = -\lambda y \end{array}$ w/ $y(0)=0,\ y(l)=0$

$\hookrightarrow y'' = -\lambda y,\ 0<x<l,\ y(0)=0,\ y(l)=0 \rightarrow \lambda_n = \frac{n^2\pi^2}{l^2},\ y_n = \sin\left(\frac{n\pi x}{l}\right)$

and $g(t) = g_n(t) = C\sin\left(\frac{n\pi ct}{l}\right) + D\cos\left(\frac{n\pi ct}{l}\right)$

$u_n = \left(c_n \sin\left(\frac{n\pi ct}{l}\right) + d_n \cos\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right)$

$u = \sum u_n \begin{cases} u(x,0) = F(x) = \sum d_n \sin\left(\frac{n\pi x}{l}\right) \\ \hookrightarrow d_n = \frac{2}{l}\int_0^l F(x)\sin\left(\frac{n\pi x}{l}\right)dx \\ u_t = \sum \frac{nc\pi}{l}\left(c_n\cos\left(\frac{n\pi ct}{l}\right) + d_n \sin\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right),\ u_t(x,0) = G \\ u_t(x,0) = G(x) = \sum \frac{nc\pi}{l} c_n \sin\left(\frac{n\pi x}{l}\right) \end{cases}$

$\hookrightarrow \underbrace{\frac{nc\pi}{l}c_n}_{\text{whole coeff!}} = \frac{2}{l}\int_0^l G(x)\sin\left(\frac{n\pi x}{l}\right)dx$

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Example 4.12

$u_t = k u_{xx} \quad 0<x<1, \; t>0$

$u(0,t)=0, \quad u(1,t)+u_x(1,t)=0$ $\rightarrow$ $u(x,t)=g(t)y(x)$

$u(x,0)=f(x) \quad 0<x<1$

Boundary cond

$y(0)g(t)=0$

$y(1)g(t)+y'(1)g(t)=0$

so just

$y(0)=0$ and $y(1)+y'(1)=0$

$\hookrightarrow \frac{g'(t)}{kg} = \frac{y''}{y} = -\lambda \rightarrow \begin{array}{l} g'(t)=-\lambda k g \\ y''=-\lambda y \end{array}$

Check sign of λ using Rayleigh Quotient

$$\lambda = \frac{\int_a^b (p(x)y_x^2 + q(x)y^2)dx - y p(x) y_x \Big|_a^b}{|y|^2} = \frac{\int_0^1 (y')^2 dx - yy'\Big|_0^1}{|y|^2}$$

$-yy'\Big|_0^1 = -y(1)y'(1) + \cancel{y(0)}^{0}y'(0) = -y(1)y'(1) = y(1)^2$ so $\lambda > 0$

$\hookrightarrow y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \rightarrow y(0)=0 \rightarrow A=0$

$\hookrightarrow y(x) = B\sin(\sqrt{\lambda}x)$ now $y(1)+y'(1)=0$

$\hookrightarrow B\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x) = 0$ $\quad B \neq 0$ so

$\hookrightarrow \sin(\sqrt{\lambda}x) + \sqrt{\lambda}\cos(\sqrt{\lambda}x) = 0 \rightarrow \sqrt{\lambda} = -\tan\sqrt{\lambda} \rightarrow$ leads to a λ_n but not solvable

$y_n = \sin(\sqrt{\lambda_n}x) \quad n=1,2,\dots$

$\int_0^1 \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_m}x)dx = 0$ for $m \neq n$

Time soln: $g = ce^{-\lambda k t} \rightarrow g_n = c_n e^{-\lambda_n k t}$

$\hookrightarrow u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \sin(\sqrt{\lambda_n}t)$

$$\sqrt{\lambda_n} = -\frac{\sin(\sqrt{\lambda_n})}{\cos(\sqrt{\lambda_n})}$$

Initial Cond:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n} x)$$

0 unless $m=n$

$$\hookrightarrow \int_0^1 f(x) \sin(\sqrt{\lambda_m} x)\,dx = \sum c_n \int_0^1 \sin(\sqrt{\lambda_n} x)\sin(\sqrt{\lambda_m} x)\,dx$$

for $\lambda_m \neq \lambda_n$

$$\frac{\sqrt{\lambda_n}\sin(\sqrt{\lambda_m})\cos(\sqrt{\lambda_n}) - \sqrt{\lambda_m}\cos(\sqrt{\lambda_m})\sin(\sqrt{\lambda_n})}{\lambda_m - \lambda_n}$$

use

$$= \frac{-\sin(\sqrt{\lambda_m})\sin(\sqrt{\lambda_n}) + \sin(\sqrt{\lambda_m})\sin(\sqrt{\lambda_n})}{\lambda_m - \lambda_n} = 0 \quad \text{for } m \neq n$$

for $\lambda_m = \lambda_n$

$$\int_0^1 f(x)\sin(\sqrt{\lambda_m} x)\,dx = c_m \int_0^1 \sin(\lambda_n x)^2\,dx$$

$$\hookrightarrow c_n = \frac{1}{\int_0^1 \sin^2(\sqrt{\lambda_n} x)\,dx} \int_0^1 f(x)\sin(\lambda_n x)\,dx \qquad n = 0,1,2,\ldots$$

4.3 Generalized SLP

$$-(p(x)y')' + q(x)y = \lambda r(x) y \quad a<x<b \quad \text{w/} \quad \begin{matrix}\alpha_1 y(a) + \alpha_2 y(a) = 0 \\ \beta_1 y(b) + \beta_2 y(b) = 0\end{matrix}$$

↑ we have a positive weight function

Using an integrating factor: consider $a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$

[Note any diff eq of this form can be turned into a SLP] → [SLP of form] → $(p(x)\frac{dy}{dx})' + q(x)y = F(x)$

[Multiply by an integrating factor] → $\mu(x)y'' + \mu(x)\frac{a_1(x)}{a_2(x)}y' + \mu(x)\frac{a_0(x)}{a_2(x)}y = \mu(x)\frac{f(x)}{a_2(x)}$ ↙ divide out a_2 1st

what does μ need to be to make into one derivative $(\sim)'$ → $\frac{d}{dx}\mu(x)y' = \underbrace{\frac{d\mu}{dx}}y' + \mu(x)y''$

$= \mu(x)\frac{a_1(x)}{a_2(x)}$ to be this diff eq

So! $\frac{d\mu}{dx} = \mu(x)\frac{a_1(x)}{a_2(x)}$ (integrating factor)

↳ $\boxed{\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx}}$ now multiply original by: $\boxed{\frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)}dx}}$

↳ (keep as $\mu(x)$) → $\frac{d}{dx}(\underbrace{\mu(x)}_{p(x)} y') + \underbrace{\mu(x)\frac{a_0(x)}{a_2(x)}}_{q(x)} y = \underbrace{\mu(x)\frac{f(x)}{a_2(x)}}_{F(x)}$

ex $x^2y'' + xy' + 2y = 0$

$\frac{1}{x^2}e^{\int \frac{1}{x}dx} = \frac{1}{x}$ → $0 = xy'' + y' + \frac{2y}{x}$ → $(xy')' + \frac{2}{x}y$ now in SLP

---

weighted inner product

$$\langle y_1, y_2 \rangle_w = \int_a^b y_1(x)y_2(x)r(x)\,dx \quad \text{and} \quad \|y\|_w = \sqrt{\langle y_1, y_2 \rangle_w}$$

$$f(x) = \sum c_n y_n(x) \quad \text{and} \quad c_n = \frac{\langle f \mid y_n \rangle_w}{\|y\|^2_w}$$

Periodic Boundary Conditions

Consider: $-y'' = \lambda y \quad -\pi < x < \pi$

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi)$$

* not an SLP because of the boundary conditions *

$\lambda = \alpha^2 \rightarrow \quad y'' + \alpha^2 y = 0 \rightarrow y(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$y(\pi) = y(-\pi) \longrightarrow \cancel{A\cos(\alpha\pi)} + B\sin(\alpha\pi) = \cancel{A\cos(\alpha\pi)} - B\sin(\alpha\pi)$

$\hookrightarrow 2B\sin(\alpha\pi) = 0$

$y'(\pi) = y'(-\pi) \rightarrow -A\alpha\sin(\alpha\pi) + \cancel{B\alpha\cos(\alpha\pi)} = A\alpha\sin(\alpha\pi) + \cancel{B\alpha\cos(\alpha}$

$\hookrightarrow 2A\sin(\alpha\pi) = 0$

$\hookrightarrow$ both satisfied by $\alpha\pi = n\pi \rightarrow \alpha = n$

$y_n(x) = a_n\cos(nx) + b_n\sin(nx)$

$$\sum y_n(x) = y(x) = \frac{a_0}{2} + \sum_{n=1} a_n\cos(nx) + b_n\sin(nx)$$

Rayleigh Quotient

Consider: $-(p(x)\,y_x)_x + q(x)\,y(x) = \lambda y(x)$

$$-\int_a^b y\,(p(x)\,y_x)_x\,dx + \int_a^b q(x)\,y^2\,dx = \int_a^b \lambda\,y^2\,dx$$

integrate by parts

$$fg = \int f\frac{dg}{dx}\,dx + \int \frac{df}{dx}\,g\,dx$$

$f = y \qquad \frac{dg}{dx} = (p(x)\,y_x)_x \rightarrow g = p(x)\,y_x$

$$\hookrightarrow -y\,p(x)\,y_x\Big|_a^b + \int_a^b p(x)\,y_x^2\,dx + \int_a^b q(x)\,y^2(x)\,dx = \lambda|y|^2$$

$$\hookrightarrow \lambda = \frac{\int_a^b \left(p(x)\,y_x^2 + q(x)\,y^2\right)dx - y\,p(x)\,y_x\Big|_a^b}{|y|^2}$$

(ex) $-y'' + y' = \lambda y \quad 0<x<1, \quad y(0) = y(1) = 0$

[Put in SLP] from the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$, $\xrightarrow{\text{mult by}} \frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)} dx}$

$a_2(x) = -1$
$a_1(x) = 1$ $\rightarrow -\mu(x) = -e^{\int -1\, dx} \rightarrow \mu(x) = e^{-x}$ mult by

$\hookrightarrow -e^{-x}y'' + e^{-x}y' = \lambda e^{-x} y \rightarrow -(e^{-x}y')' = \lambda e^{-x} y$

From: $-(p(x)y')' + q(x)y = \lambda r(x) y$

$q(x) = 0$
$p(x) = e^{-x}, \quad r(x) = e^{-x}$

Solve by expanding derivative; $\rightarrow -y'' + y' - \lambda y = 0$,

$-m^2 + m - \lambda = 0 = m^2 - m + \lambda \Rightarrow m = \frac{1 \pm \sqrt{1 - 4(1)(\lambda)}}{2}$

$\hookrightarrow m = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda} \rightarrow y(x) = Ae^{\frac{1}{2}x} e^{\sqrt{\frac{1}{4}-\lambda}\,x} + Be^{\frac{x}{2}} e^{-\sqrt{\frac{1}{4}-\lambda}\,x}$

Cases

$\lambda = 1/4$: $y(x) = Ae^{\frac{x}{2}} + Be^{\frac{x}{2}} = e^{\frac{x}{2}}(A+B)$

$y(0)=0 \rightarrow A+B=0$ which means $y(x)=0$ trivial soln
$y(1)=0 \rightarrow e^{\frac{1}{2}}(A+B)=0$

$\frac{1}{4}-\lambda > 0$: $y(x) = e^{\frac{x}{2}}\left(Ae^{\sqrt{\frac{1}{4}-\lambda}\,x} + Be^{-\sqrt{\frac{1}{4}-\lambda}\,x}\right)$

$y(0)=0$: $0 = A+B \rightarrow B=-A$
$y(1)=0$ $0 = Ae^{\sqrt{\frac{1}{4}-\lambda}} + Be^{-\sqrt{\frac{1}{4}-\lambda}} \rightarrow A\left(e^{\sqrt{\frac{1}{4}-\lambda}} - e^{-\sqrt{\frac{1}{4}-\lambda}}\right) = 0$

$A \neq 0$: $e^{\sqrt{\frac{1}{4}-\lambda}} = e^{-\sqrt{\frac{1}{4}-\lambda}} \rightarrow \lambda$ is real and $\frac{1}{4}-\lambda>0$

$\hookrightarrow$ only satisfied when $\lambda = \frac{1}{4}$ which is trivial

$\frac{1}{4} - \lambda < 0$:

$y(x) = e^{\frac{x}{2}}\left(A\cos(\sqrt{\tfrac{1}{4}-\lambda}\,x) + B\sin(\sqrt{\tfrac{1}{4}-\lambda}\,x)\right)$

$y(0)=0 \to A=0$, $y(x) = B e^{\frac{x}{2}} \sin(\sqrt{\tfrac{1}{4}-\lambda}\,x)$

$y(1)=0 \to \sin(\sqrt{\tfrac{1}{4}-\lambda}) = 0 \to \sqrt{\tfrac{1}{4}-\lambda} = n\pi$

$\hookrightarrow (n\pi)^2 = \frac{1}{4} - \lambda \to \boxed{\lambda_n = \frac{1}{4} - (n\pi)^2}$

$\boxed{y_n(x) = e^{\frac{x}{2}} \sin(n\pi x), \quad n = 1, 2, 3, \dots}$

orthogonal w.r.t weighting function

Now:

$\langle y_n, y_m \rangle = \int_0^1 (e^{\frac{x}{2}}\sin(n\pi x))(e^{\frac{x}{2}}\sin(m\pi x))\, e^{-x}\,dx = 0$

$f(x) = \sum c_n e^{\frac{x}{2}} \sin(n\pi x)$, $c_n = \dfrac{\langle f | y_n \rangle_w}{\langle y_n | y_n \rangle_w^2}$, $n = 1, 2, 3$

Singular Problems

- bounded
- infinite boundary