Maxwell Stress Tensor

f = force per unit volume

$$\vec{F} = \int_V (\vec{E} + \vec{v}\times\vec{B})\rho\, d\tau = \int_V \overbrace{(\rho \vec{E} + \vec{J}\times\vec{B})}^{f}\, d\tau$$

Express in terms of fields

$$\rho = \varepsilon_0(\nabla\cdot E) \quad \text{and} \quad \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial \vec{E}}{\partial t} = \vec{J}$$

product rule

$$\hookrightarrow \vec{f} = \varepsilon_0(\nabla\cdot E)E + \left(\frac{1}{\mu_0}\nabla\times B - \varepsilon_0\frac{\partial E}{\partial t}\right)\times B \quad \text{and} \quad \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial E}{\partial t}\times B + E\times\underbrace{\frac{\partial B}{\partial t}}$$

$\partial B/\partial t = -\nabla\times E$

$$\hookrightarrow \frac{\partial E}{\partial t}\times B = \frac{\partial}{\partial t}(E\times B) + E\times(\nabla\times E) \quad \text{plug into } \vec{f}$$

$$\hookrightarrow \vec{f} = \varepsilon_0\left[(\vec{\nabla}\cdot E)E - E\times(\nabla\times E)\right] - \frac{1}{\mu_0}\left[\vec{B}\times(\nabla\times B)\right] - \varepsilon_0\frac{\partial}{\partial t}(E\times B)$$

math manipulation: (same for B + $\nabla\cdot B = 0$

$$\nabla(E^2) = 2(E\cdot\nabla)E + 2E\times(\nabla\times E) \quad \text{so} \quad E\times(\nabla\times E) = \frac{1}{2}\nabla(E^2) - (E\cdot\nabla)E$$

0 just to make it look symmetrical

$$\vec{f} = \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right] - \frac{1}{2}\nabla\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

Define: $T_{ij} \equiv \varepsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$

$$\hookrightarrow T_{xx} = \varepsilon_0\left(E_x^2 - \tfrac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_x^2 - \tfrac{1}{2}B^2\right), \quad T_{xy} = \varepsilon_0(E_xE_y) + \frac{1}{\mu_0}(B_xB_y)$$

$$\overleftrightarrow{T} = \begin{array}{ccc} T_{xx} & T_{yx} & T_{zx} \\ \end{array}$$

$$\overleftrightarrow{T} = \begin{bmatrix} \frac{1}{2}\varepsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2) & \varepsilon_0(E_yE_x) + \frac{1}{\mu_0}(B_yB_x) & \varepsilon_0(E_zE_x) + \frac{1}{\mu_0}(B_zB_x) \\ \varepsilon_0(E_xE_y) + \frac{1}{\mu_0}(B_xB_y) & \frac{1}{2}\varepsilon_0(E_y^2 - E_x^2 - E_z^2) + \frac{1}{2\mu_0}(B_y^2 - B_x^2 - B_z^2) & \varepsilon_0(E_zE_y) + \frac{1}{\mu_0}(B_zB_y) \\ \varepsilon_0(E_xE_z) + \frac{1}{\mu_0}(B_xB_z) & \varepsilon_0(E_yE_z) + \frac{1}{\mu_0}(B_yB_z) & \frac{1}{2}\varepsilon_0(E_z^2 - E_x^2 - E_y^2) + \frac{1}{2\mu_0}(B_z^2 - B_x^2 - B_y^2) \end{bmatrix}$$

(Entry labels as written: row 1: T_{xx}, T_{yx}, T_{zx}; row 2: T_{xy}, T_{yy}, T_{zy}; row 3: T_{xz}, T_{yz}, T_{zz}.)

$$(\vec{a}\cdot\overleftrightarrow{T})_j = \sum_i a_iT_{ij} = a_xT_{xj} + a_yT_{yj} + a_zT_{zj}$$

Ok, $T_{xj} = \varepsilon_0\left(E_xE_j - \frac{1}{2}\delta_{xj}E^2\right) + \frac{1}{\mu_0}\left(B_xB_j - \frac{1}{2}\delta_{xj}B^2\right)$

and

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$E^2 = E_x^2 + E_y^2 + E_z^2$

With $\nabla_i T_{ij}$

$(\vec{a}\cdot\overleftrightarrow{T})_j = \sum_i a_i T_{ij} = a_x T_{xj} + a_y T_{yj} + a_z T_{zj}$

and

$T_{xj} = \varepsilon_0 (E_x E_j - \frac{1}{2}\delta_{xj} E^2) + \ldots$ omitting B cuz symetry

For $(\vec{\nabla}\cdot\overleftrightarrow{T})_j$ its really taking derivitive w.r.t i^{th} component

$$(\vec{\nabla}\cdot\overleftrightarrow{T})_j = \frac{\partial}{\partial x} T_{xj} + \frac{\partial}{\partial y} T_{yj} + \frac{\partial}{\partial z} T_{zj}$$

$$= \varepsilon_0 \left(\frac{\partial}{\partial x}(E_x E_j) - \frac{1}{2}\delta_{xj}\frac{\partial}{\partial x}(E^2)\right) +$$
$$\varepsilon_0 \left(\frac{\partial}{\partial y}(E_y E_j) - \frac{1}{2}\delta_{yj}\frac{\partial}{\partial y}(E^2)\right) +$$
$$\varepsilon_0 \left(\frac{\partial}{\partial z}(E_z E_j) - \frac{1}{2}\delta_{zj}\frac{\partial}{\partial z}(E^2)\right)$$

$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{E_z}{\partial z}$

$(\vec{E}\cdot\vec{\nabla})E_j = E_x \frac{\partial}{\partial x} E_j + E_y \frac{\partial}{\partial y} E_j + E_z \frac{\partial}{\partial z} E_j$

$$(\vec{\nabla}\cdot\overleftrightarrow{T})_j = \varepsilon_0\left(\frac{\partial}{\partial x}(E_x E_j) + \frac{\partial}{\partial y}(E_y E_j) + \frac{\partial}{\partial z}(E_z E_j) - \frac{1}{2}\delta_{xj}\frac{\partial}{\partial x}(E^2) - \frac{1}{2}\delta_{yj}\frac{\partial}{\partial y}(E^2) - \frac{1}{2}\delta_{zj}\frac{\partial}{\partial z}(E^2)\right)$$

$E_x \frac{\partial}{\partial x} E_j + E_j \frac{\partial E_x}{\partial x}$ and $(\vec{\nabla}\cdot\vec{E})E_j = \frac{\partial E_x}{\partial x} E_j + \ldots$
$(\vec{E}\cdot\vec{\nabla})E_j = E_x \frac{\partial E_j}{\partial x} + \ldots$

so product rule will combine + make these terms for y, z too

$$(\vec{\nabla}\cdot\overleftrightarrow{T})_j = \varepsilon_0\left((\vec{\nabla}\cdot\vec{E})E_j + (\vec{E}\cdot\vec{\nabla})E_j - \frac{1}{2}\delta_{xj}\frac{\partial}{\partial x}(E^2) - \frac{1}{2}\delta_{yj}\frac{\partial}{\partial y}(E^2) - \frac{1}{2}\delta_{zj}\frac{\partial}{\partial z}(E^2)\right)$$

is $j = x$ then y + z go to zero etc
so its the j^{th} comp of ∇

$$(\vec{\nabla}\cdot\overleftrightarrow{T})_j = \varepsilon_0\left((\vec{\nabla}\cdot\vec{E})E_j + (\vec{E}\cdot\vec{\nabla})E_j - \frac{1}{2}\nabla_j E^2\right) + \frac{1}{\mu_0}\left((\vec{\nabla}\cdot\vec{B})B_j + (\vec{B}\cdot\nabla)B_j - \frac{1}{2}\nabla_j B^2\right)$$

Now for the 3 component of j to get total $(\vec{\nabla}\cdot\overleftrightarrow{T})$, {and the ∇_j would be made to ∇}

you can pull out $\nabla\cdot E$ for each E_j making $\vec{\nabla}\cdot\vec{E}\,\underbrace{(E_x + E_y\ E_z)}_{E}$

Recall:

$$\vec{f} = \underbrace{\varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right] - \frac{1}{2}\nabla\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)}_{\vec{\nabla}\cdot\overleftrightarrow{T}} - \varepsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \varepsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B}), \quad \vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

$$\boxed{\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \varepsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}}$$

Now we have the force per unit volume

total Force

divergence theorem

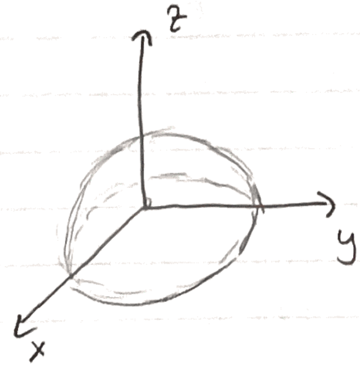
$$\hookrightarrow \vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{a} - \varepsilon_0\mu_0 \frac{d}{dt}\int_V \vec{S}\, d\tau$$

$$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S}\, d\tau$$

$$F_z = \int \sum_j T_{zj}\, da_j$$

Stress Tensor Examples

Example 8.2, net force on the northern hemisphere of a uniformly charged sphere w/ radius R + charge Q

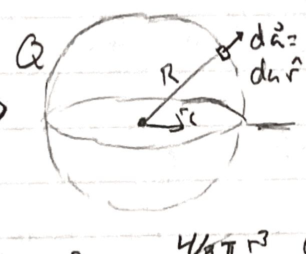


[Has two surfaces the bowl at radius R and the disc] → [Start w/ the bowl]

$d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

and now need to find the field, $\vec{E}$

[E field of a uniformly charged sphere] →



$d\vec{a} = da\,\hat{r}$

$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \rightarrow E\, 4\pi R^2 = \frac{Q}{\varepsilon_0}$

↳ $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^2} \hat{r}$ (outside sphere), $r > R$

↳ for later inside: $E\, 4\pi r^2 = \frac{4/3\pi r^3}{4/3\pi R^3} \frac{Q}{\varepsilon_0} \rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^3} r\, \hat{r}$, $r < R$

Need cartesian, $\hat{r} = \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z}$

↳ Also only looking at z-comp of force due to spherical symmetry x + y cancel

$$\underbrace{F_z = \left(\oint \overleftrightarrow{T} \cdot d\vec{a}\right)}_{\vec{F}} \cdot \hat{z} = \oint T_{zx}\, da_x + T_{zy}\, da_y + T_{zz}\, da_z$$

$$T_{zx} = \varepsilon_0 (E_z E_x) = \varepsilon_0 \Big(\overbrace{\tfrac{1}{4\pi\varepsilon_0} \tfrac{Q}{R^2}}^{\gamma}\Big)^2 \sin\theta\cos\theta\cos\phi$$

$$T_{zy} = \varepsilon_0 (E_z E_y) = \varepsilon_0 \left(\tfrac{1}{4\pi\varepsilon_0} \tfrac{Q}{R^2}\right)^2 \sin\theta\cos\theta\sin\phi$$

$$T_{zz} = \tfrac{1}{2}\varepsilon_0 (E_z^2 - E_x^2 - E_y^2) = \tfrac{1}{2}\varepsilon_0 \left(\tfrac{1}{4\pi\varepsilon_0} \tfrac{Q}{R^2}\right)^2 (\cos^2\theta - \sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi)$$

$-\sin^2\theta$

$-\sin^2\theta(1) = \sin^2\theta$

$$T_{zx}\, da_x = \varepsilon_0 \gamma^2 S_\theta C_\theta C_\phi R^2 S_\theta\, d\theta\, d\phi\, S_\theta C_\phi$$

$$T_{zy}\, da_y = \varepsilon_0 \gamma^2 S_\theta C_\theta S_\phi R^2 S_\theta\, d\theta\, d\phi\, S_\theta S_\phi$$

$$T_{zz}\, da_z = \tfrac{1}{2}\varepsilon_0 \gamma^2 (C_\theta^2 - S_\theta^2) R^2 S_\theta\, d\theta\, d\phi\, C_\theta$$

sum:

$\varepsilon_0 \gamma^2 R^2\, d\theta\, d\phi\, S_\theta C_\theta \left(S_\theta^2 C_\phi^2 + S_\theta^2 S_\phi^2 + \tfrac{1}{2}(C_\theta^2 - S_\theta^2)\right)$

$= \tfrac{1}{2}\varepsilon_0 \gamma^2 R^2 \sin\theta\cos\theta$

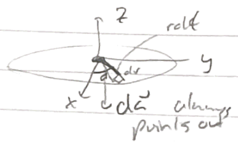
$$F_{bowl} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R}\right)^2 \int_0^{2\pi}\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R}\right)^2 (2\pi)\left(\frac{1}{2}\right) = \boxed{\frac{1}{4\pi\varepsilon_0} \frac{Q^2}{8R^2}}$$

Now for the disc:

E on plane!!

$$\vec{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\, r\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\, r(\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$$

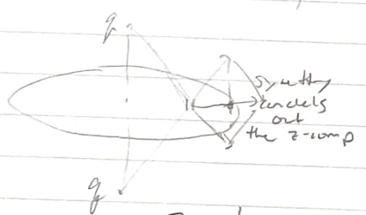
↳ whats $d\vec{a}$?



, θ is constant, $d\phi$ + dr change

$$\boxed{-r\,dr\,d\phi\,\hat{z} = d\vec{a}}$$

$E_z = 0$ (for inside) → just T_{zz} component



$$T_{zz} = \frac{\epsilon_0}{2}\left(\cancel{E_z^2}^{\,0} - E_x^2 - E_y^2\right)$$

$$= -\frac{\epsilon_0}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\right)^2 r^2 \left(\underbrace{\cos^2\phi + \sin^2\phi}_{1}\right)$$

$$T_{zz}\, da_z = +\frac{\epsilon_0}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\right)^2 r^3\, dr\, d\phi$$

$$F_{disk} = \frac{\epsilon_0}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\right)^2 \int_0^{2\pi}\int_0^R r^3\, dr\, d\phi = \frac{\epsilon_0}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\right)^2 2\pi\frac{R^4}{4} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2}}$$

Net force is then: $F_{bowl} + F_{disk} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2} + \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2} = \boxed{\frac{3Q^2}{4\pi\epsilon_0\, 16R^2}}$

Conservation in Electrodynamics

conservation of charge: $Q(t) = \int_V \rho(\vec{r}, t)\, d\tau$

↳ and current flowing out of a boundary $\frac{dQ}{dt} = I = -\oint_S \vec{J} \cdot d\vec{a}$

[using both of the eq] $\int_V \frac{\partial \rho}{\partial t}\, d\tau = -\int_V \nabla \cdot \vec{J}\, d\tau$ ← divergence theorem

Continuity Equation: $\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}}$

Poynting's Theorem

$W_e = \frac{\varepsilon_0}{2} \int E^2\, d\tau$ and $W_B = \frac{1}{2\mu_0} \int B^2\, d\tau$ | Energy per unit volume $u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$

$W = \vec{F} \cdot d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v}\, dt = q\vec{E} \cdot \vec{v}\, dt$

$q \rightarrow \rho\, d\tau$ and $\rho \vec{v} = \vec{J}$

$$\boxed{\frac{dW}{dt} = \int_V (\vec{E} \cdot \vec{J})\, d\tau}$$

Eliminate $\vec{J}$: $\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$

Product Rule: $\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot \underbrace{(\vec{\nabla} \times \vec{E})}_{= -\partial \vec{B}/\partial t} - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$

And $\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t}(B^2)$ and $\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t}(E^2)$

↳ $\vec{E} \cdot \vec{J} = -\frac{1}{2} \frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B})$

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt} \int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau - \frac{1}{\mu_0} \oint_S (\vec{E} \times \vec{B}) \cdot d\vec{a}}$$

total energy in the fields

rate at which energy is transported out of V across the boundary surface

$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

[What if we are in a region of no charge] $\frac{dW}{dt} = 0 \rightarrow \int \frac{\partial u}{\partial t}\, d\tau = -\oint \vec{S} \cdot d\vec{a} = -\int (\vec{\nabla} \cdot \vec{S})\, d\tau$

$$\boxed{\frac{\partial u}{\partial t} = -\vec{\nabla} \cdot \vec{S}}$$

Conservation of Momentum

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt} = -\varepsilon_0\mu_0 \frac{d}{dt}\int_V \vec{S}\, d\tau + \oint_S \overleftrightarrow{T} \cdot d\vec{a}$$

momentum in the fields

$$\vec{P}_{field} = \mu_0 \varepsilon_0 \int_V \vec{S}\, d\tau$$

momentum per unit time flowing through the surface

momentum density: $g = \mu_0 \varepsilon_0 \vec{S} = \varepsilon_0 (\vec{E} \times \vec{B})$

$\overleftrightarrow{T} \cdot d\vec{a}$ is the EM momentum per unit time passing through area $d\vec{a}$

if the total momentum isn't changing $\frac{dp_{mech}}{dt} = 0$ $\rightarrow$ $\int \frac{\partial \vec{g}}{\partial t} d\tau = \oint \overleftrightarrow{T} \cdot d\vec{a} = \int \vec{\nabla} \cdot \overleftrightarrow{T} d\tau$

$$\boxed{\frac{\partial \vec{g}}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T}}$$

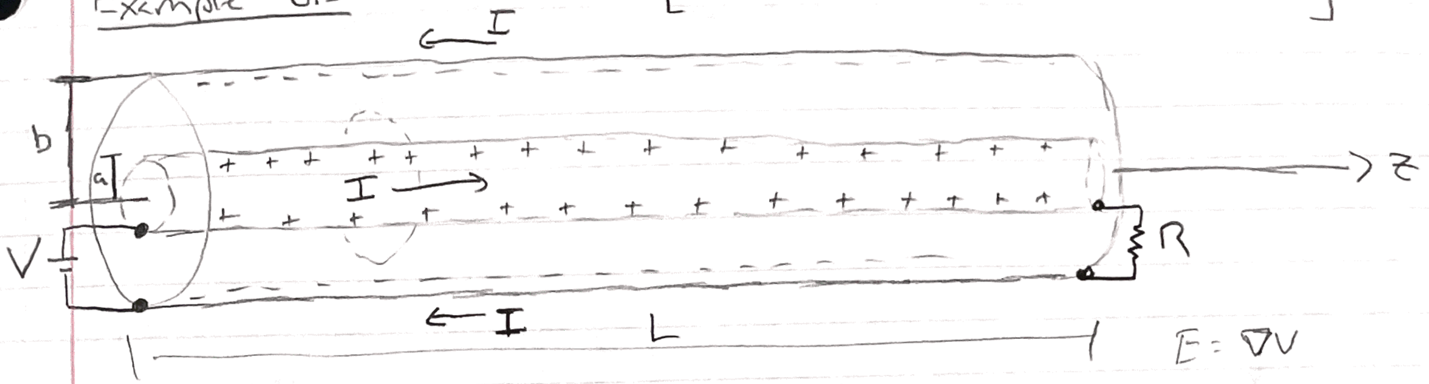
Charges & fields exchange momentum & the momentum between them is conserved

$\vec{S}$ is energy per unit area per unit time by fields

$\mu_0 \varepsilon_0 \vec{S}$ is momentum per unit volume stored in fields

[What is the momentum stored in the fields?]

Example 8.3



$E = \nabla V$

First calculate the E-field & B-field:

$\vec{E}_{out} = 0$, $Q_{encl} = 0$, $\vec{E}_{in} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{s}\hat{s}$ $\Rightarrow$ [inside now] $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \rightarrow 2\pi s L E = \frac{1}{\varepsilon_0}\lambda L$

$\vec{B}_{out} = 0$, $I_{encl} = 0$, $\vec{B}_{in} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$ $\rightarrow$ [Inside now] $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} \rightarrow 2\pi s B = \mu_0 I \;\hat{\phi}$

$\hookrightarrow$ therefore $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\lambda I}{4\pi^2\varepsilon_0 s^2}\hat{z}$

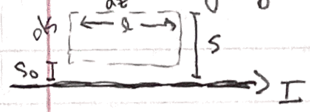
Note: $P = \int \vec{S}\cdot d\vec{a} = \frac{\lambda I}{4\pi^2\varepsilon_0}\int_a^b \frac{1}{s^2}2\pi s\,ds = \frac{\lambda I}{2\pi\varepsilon_0}\ln(b/a) = IV$

[momentum in the field is] $\rightarrow \vec{p} = \mu_0\varepsilon_0\int \vec{S}\,d\tau = \frac{\mu_0\lambda I}{4\pi^2}\hat{z}\int_a^b \frac{1}{s^2}L2\pi s\,ds$

$$\boxed{\vec{p} = \frac{\mu_0\lambda I L}{2\pi}\ln(b/a)\,\hat{z} = \frac{IVL}{c^2}\hat{z}}$$

What if R goes up to decrease the current, and the current changes slowly

$\hookrightarrow$ the changing magnetic field will induce an Electric field.



$\oint \vec{E}\cdot d\vec{l} = E(s_0)L - E(s)L = \frac{-d}{dt}\int \vec{B}\cdot d\vec{a}$, $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$

$d\vec{a} = ds\,dz\,\hat{s}$

$= -\frac{\mu_0 L}{2\pi}\frac{dI}{dt}\int_{s_0}^{s}\frac{1}{s}ds = -\frac{\mu_0 L}{2\pi}\frac{dI}{dt}(\ln s - \ln s_0)$

$\vec{E}(s) = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln(s) + K\right]\hat{z}$

[Field exerts a force on $\pm\lambda$] $\rightarrow \vec{F} = \lambda L\vec{E}\Big|_{s=a} - \lambda L E\Big|_{s=b} = -\frac{\mu_0\lambda L}{2\pi}\frac{dI}{dt}\ln(b/a)\hat{z}$

$\boxed{P_{mech} = \int \vec{F}\,dt = \frac{\mu_0\lambda I L}{2\pi}\ln(b/a)\,\hat{z}}$, total momentum imparted on the cable as the current drops from I to 0

Angular Momentum in the Fields

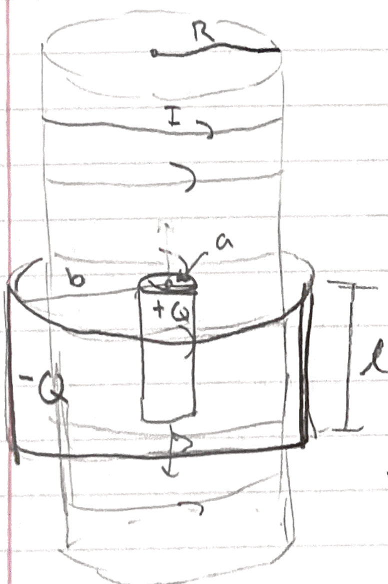
$$\vec{p} = \int \vec{g}\, d\tau$$

So far fields carry energy & momentum $\rightarrow$ $u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$, $\vec{g} = \varepsilon_0 (\vec{E} \times \vec{B})$

Then angular momentum $\rightarrow$ $\vec{\ell} = \vec{r} \times \vec{g} = \varepsilon_0 \left[\vec{r} \times (\vec{E} \times \vec{B})\right]$

Example

As the current is reduced the cylinders begin to rotate



$$\vec{E} = \frac{Q}{2\pi\varepsilon_0 \ell} \frac{1}{s} \hat{s} \qquad a < s < b$$

$$\vec{B} = \mu_0 n I \hat{z} \qquad s < R$$

$$\vec{g} = \varepsilon_0 (\vec{E} \times \vec{B}) = -\frac{\mu_0 n I Q}{2\pi \ell s} \hat{\phi} \quad \text{in } a < s < b$$

$$\vec{r} \times \vec{g} = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ s & 0 & z \\ 0 & g & 0 \end{vmatrix} = (-gz)\hat{s} - 0\hat{\phi} + sg\,\hat{z}$$

$$\vec{L} = \int (-gz\,\hat{s} + sg\,\hat{z})\, d\tau = \int_0^{\ell}\int_a^{R}\int_0^{2\pi} \frac{\mu_0 n I}{2\pi \ell} Q \frac{z}{s} \hat{s}\, s\, d\phi\, ds\, dz \rightarrow 0$$

$$+ \int_0^{\ell}\int_a^{R}\int_0^{2\pi} \left(\frac{\mu_0 n I}{2\pi \ell} Q\, s\, d\phi\, ds\, dz\right)\hat{z} = \frac{Q \mu_0 n I}{2}(R^2 - a^2)\,\hat{z}$$

Let the current turn off

↳ the changing $\vec{B}$ field induces a circumferential E-field

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a}$$

Surface that encloses the flux

$$E\, 2\pi s = -\frac{d}{dt}\left(\pi s^2 B \right) = -\pi s^2 \frac{dB}{dt}$$

outside) that's R^2

$$\begin{cases} \vec{E} = -\frac{1}{2} s \mu_0 n \frac{dI}{dt} \hat{\phi} & s<R \\ \vec{E} = -\frac{1}{2} \mu_0 n \frac{dI}{dt} \frac{R^2}{s} \hat{\phi} & s>R \end{cases}$$

Torque:

outer cylinder $r=b$ ↳ $\vec{\tau}_b = \vec{r} \times \vec{F} = \vec{r} \times Q\vec{E} = \frac{1}{2} \mu_0 n Q R^2 \frac{dI}{dt} \hat{z}$

↳ $\frac{d\vec{L}}{dt} = \tau_{net} \rightarrow \vec{L}_b = \int \tau \, dt = \frac{1}{2} \mu_0 n Q R^2 \int_I^0 dI \, \hat{z}$

$$\boxed{\vec{L}_b = -\frac{1}{2} \mu_0 n I Q R^2}$$

$$\vec{\tau}_a = -\frac{1}{2} \mu_0 n Q a^2 \frac{dI}{dt} \hat{z}$$

$$\boxed{\vec{L}_a = \frac{1}{2} \mu_0 n I Q a^2}$$

yup :)

$$\vec{L}_{em} = -\frac{1}{2} \mu_0 n I Q (R^2 - a^2) \hat{z} = \vec{L}_a + \vec{L}_b$$

$$\begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ 0 & dl & 0 \\ B_s & 0 & B_z \end{vmatrix} = 0\,\hat{s}$$

Reading

8.3 Magnetic forces do no work

- whos doing work on a magnetic crane?

Model $\vec{F} = \int I\,(d\vec{l} \times \vec{B})$, $\vec{I} = \lambda \vec{v}$

$d\vec{l} = dl\,\hat{\phi}$, $\vec{B} = B_s\,\hat{s} + B_z\,\hat{z}$

$\vec{F} = I \int -dl\, B_s\, \hat{z} \Rightarrow F = 2\pi a\, I\, B_s$

$I = \lambda \omega a$

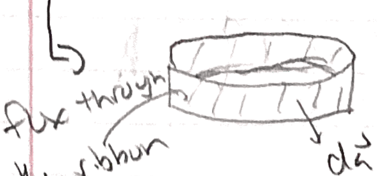
ler →

↳ $dw = 2\pi a^2 \lambda \omega B_s\, dz$ ← if ring rises dz the work done B dW

Who did work? Naively it appears the B field did work
But as the ring rises the magnetic force ⊥ the net velocity of charges in rings

[Since we have a changing flux] → $\mathcal{E} = -\frac{d\phi}{dt}$, back emf that reduces ω

↳



$d\phi = \vec{B} \cdot d\vec{a} = B_s\, 2\pi a\, dz$

$\mathcal{E} \equiv \oint \vec{f} \cdot d\vec{l}$ ← like E, force per unit charge

$\lambda : \frac{dq}{dl}$ $\quad \mathcal{E} = f(2\pi a)$ $\quad -B_s 2\pi a \frac{dz}{dt} = f\, 2\pi a \;\rightarrow\; f = -B_s \frac{dz}{dt}$

F_{dl}: Force on segment dl = $\overset{f\,dq}{f \lambda\, dl} \Rightarrow \overset{\text{total} = F_\ell}{\int \lambda\, 2\pi a} = -B_s \frac{dz}{dt} \lambda 2\pi a$

$|\vec{r} \times \vec{F}| = a F_\ell = a\left(-B_s \frac{dz}{dt}\right) \lambda 2\pi a$

$dw = N\,d\phi = N\omega\, dt = -2\pi a^2 \lambda \omega B_s\, dz$

rotational energy it loses = potential energy gained