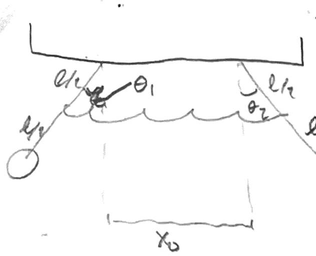
Coupled Oscillations



$$T = \frac{1}{2} m l^2 (\dot\theta_1^2 + \dot\theta_2^2)$$

$$U = mgl(1-\cos\theta_1) + mgl(1-\cos\theta_2)$$

$$+ \frac{1}{2} k x^2$$

x ↑ length of stretched part

→ one will stretch by $\frac{l}{2}\sin(\theta_1)$

other will smoosh by $\frac{l}{2}\sin\theta_2$

$$L = \frac{1}{2} m l^2 (\dot\theta_1^2 + \dot\theta_2^2) - mgl(1-\cos\theta_1) - mgl(1-\cos\theta_2) - \frac{1}{2}k\left(\frac{l}{2}\right)^2(\sin\theta_2 - \sin\theta_1)^2$$

w/ $\left.\begin{array}{l}\sin\theta \approx \theta \\ \cos\theta \approx 1 - \frac{\theta^2}{2}\end{array}\right] \rightarrow L = \frac{m}{2} l^2 (\dot\theta_1^2 + \dot\theta_2^2) - \frac{mgl}{2}(\theta_1^2 + \theta_2^2) - \frac{k}{2}\left(\frac{l}{2}\right)^2(\theta_2 - \theta_1)^2$

Apply E&L E

$$\ddot\theta_1 + \frac{g}{l}\theta_1 - \frac{k}{4m}(\theta_2 - \theta_1) = 0$$

$$\ddot\theta_2 + \frac{g}{l}\theta_2 - \frac{k}{4m}(\theta_2 - \theta_1) = 0$$

define: $\omega_0^2 = \frac{g}{l}$ and $\eta = \frac{kl}{4mg} = \frac{k}{4m}\frac{1}{\omega_0^2}$

$\omega_0^2 \eta = \frac{k}{4m}$

group like θ's and plug in $\omega_0^2 = \frac{g}{l}$ and $\eta = \frac{kl}{4mg}$

$$\ddot\theta_1 + \omega_0^2(1+\eta)\theta_1 - \omega_0^2 \eta \theta_2 = 0$$

$$\ddot\theta_2 + \omega_0^2(1+\eta)\theta_2 - \omega_0^2 \eta \theta_1 = 0$$

now we can put in a matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} + \omega_0^2 \begin{pmatrix}1+\eta & -\eta \\ -\eta & 1+\eta\end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$\left(\begin{array}{l}\text{now solve by} \\ \text{assuming}\end{array}\right] \rightarrow \begin{pmatrix}\theta_1(t) \\ \theta_2(t)\end{pmatrix} = \begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} e^{i\omega t}$

$$\hookrightarrow -\omega^2 \begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} \cancel{e^{i\omega t}} + \omega_0^2 \begin{pmatrix}1+\eta & -\eta \\ -\eta & 1+\eta\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} \cancel{e^{i\omega t}} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

define $\lambda = \frac{\omega^2}{\omega_0^2}$

$$\hookrightarrow -\lambda^2 \begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} + \begin{pmatrix}1+\eta & -\eta \\ -\eta & 1+\eta\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$\hookrightarrow \Phi_1\begin{pmatrix}-\lambda^2 \\ 0\end{pmatrix} + \Phi_2\begin{pmatrix}0 \\ -\lambda^2\end{pmatrix} + \Phi_1\begin{pmatrix}1+\eta \\ -\eta\end{pmatrix} + \Phi_2\begin{pmatrix}-\eta \\ 1+\eta\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \rightarrow \boxed{\begin{pmatrix}-\lambda+1+\eta & -\eta \\ -\eta & -\lambda+1+\eta\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}}$$

Find all non-trivial soln of this, its an eigen value problem.
so take det $= 0$

$$\begin{vmatrix}-\lambda+1+\eta & -\eta \\ -\eta & -\lambda+1+\eta\end{vmatrix} = (1+\eta-\lambda)^2 - \eta^2 = 0 \rightarrow \lambda = 1+\eta \pm \eta \rightarrow \lambda = 1,\ 1+2\eta$$

$\lambda = \frac{\omega^2}{\omega_0^2}$

$$\omega_1^2 = \omega_0^2$$
$$\omega_2^2 = \boxed{(1+2\eta)}\omega_0$$

Eigen vectors

inverse value eq but for the gives ↓ amplitudes

$$\begin{bmatrix} -\lambda+1+\eta & -\eta \\ -\eta & -\lambda+1+\eta \end{bmatrix}\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -\lambda+1+\eta & -\eta & 0 \\ -\eta & -\lambda+1+\eta & 0 \end{array}\right] \rightarrow \Phi_1(-\lambda+1+\eta)$$

$\rightarrow \Phi_1(-\eta) + \Phi_2(-\lambda \dots$

a con... the exponent ... suppose ...

$\hookrightarrow$ [look at eq 2] $\rightarrow$ $\boxed{\dfrac{\Phi_1}{\Phi_2} = \dfrac{1+\eta-\lambda}{\eta}}$ these are different mode eq for different eigenvalues, 1, $1+2\eta$

These values correspond to different modes but what are modes?

[Go back to the eq of motion]

$$\left.\begin{array}{l} \ddot{\theta}_1 + \omega_0^2(1+\eta)\theta_1 - \omega_0^2\eta\,\theta_2 = 0 \\ \ddot{\theta}_2 + \omega_0^2(1+\eta)\theta_2 - \omega_0^2\eta\,\theta_1 = 0 \end{array}\right\} \text{add them together or subtract} \rightarrow \begin{array}{l} \underbrace{(\ddot{\theta}_1+\ddot{\theta}_2)}_{\theta_+} + \omega_0^2(1+\eta)(\theta_1+\theta_2) - \omega_0^2\eta(\theta_1+\theta_2) = 0 \\ \underbrace{(\ddot{\theta}_1-\ddot{\theta}_2)}_{\theta_-} + \omega_0^2(1+\eta)(\theta_1-\theta_2) + \omega_0^2\eta(\theta_1-\theta_2) = 0 \end{array}$$

Let $\overset{\theta_+}{\alpha} = \theta_1 + \theta_2$ and $\overset{\theta_-}{\beta} = \theta_1 - \theta_2$:

these show the eigenvalues as the frequency of oscillation of β and α

$$\begin{bmatrix} \ddot{\alpha} + \omega_0^2\alpha = 0 \\ \ddot{\beta} + (1+2\eta)\omega_0^2\beta = 0 \end{bmatrix}$$

gives 2 <u>modes</u> of oscillation for the sys. one is $\underline{\omega_0^2}$ and the other is $\underline{\omega^2 = \omega_0^2(1+2\eta)}$

Suppose both of these modes are present in a sys and initially these modes are in phase, will they stay in phase?

first relation

$$\begin{bmatrix} \Theta_+ = A\cos(\omega_1 t) \\ \Theta_- = A\cos(\omega_2 t) \end{bmatrix}$$

and

second relation

$$\begin{bmatrix} \Theta_- - \Theta_+ = \omega_2\tau - \omega_1\tau = \Delta\phi \\ \tau \text{ is time},\ \omega \text{ is } \theta/t \end{bmatrix}$$

angle θ_T has gone through; ω: freq, τ: time

$\underbrace{\Delta\omega\tau}_{=\pi} = \Delta\phi$ they are fully out of phase bec $\Delta\phi$ is the phase difference and $\Delta\omega \neq 0$

Back to modes result, if we had N pendulums connected we would have N SHO eq each w/ their own mode. We focus on small vibrations, taylor expansion ensures this.

Consider a conservative sys that consists of coupled oscillators. Suppose further the system can be expressed in terms of generalized coordinates

Suppose a state of stable equilibrium exists, this satisfies the ELE

$$q_k = q_{k,0} \quad , \quad \dot{q}_k = \ddot{q}_k = 0 \quad \text{w/ ELE} \quad \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

[Near equilibrium $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$ must vanish] $\rightarrow$ $\left.\frac{\partial L}{\partial q_k}\right|_0 = \left.\frac{\partial}{\partial q_k}(T-U)\right|_0 = 0$

$\llcorner$ evaluated at equilibrium

[Now Recall position coord] $\rightarrow$ $\dot{X}_\alpha = \sum_k \frac{\partial X_\alpha}{\partial q_k} \underbrace{\frac{\partial q_k}{\partial t}}_{\dot{q}_k} + \cancel{\frac{\partial X_\alpha}{\partial t}}$ assume no time dependence (if it does then it is much more complicated) $\rightarrow \dot{X}_\alpha^2 = \sum_{j,k} \frac{\partial X_\alpha}{\partial q_j}\frac{\partial X_\alpha}{\partial q_k}\dot{q}_j \dot{q}_k$

[look at kinetic Energy] $\rightarrow$ $T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k$ where m_{jk} has to be $\underbrace{\sum_\alpha \overset{\text{mass}}{m_\alpha} \frac{\partial X_\alpha}{\partial q_j}\frac{\partial X_\alpha}{\partial q_k}}_{\text{just evaluate to numbers?}}$, [$m_{jk}$ is the coeff of the $\dot{q}_j$ and $\dot{q}_k$ terms]

[Notice T has no position dependence] $\rightarrow$ $\left.\frac{\partial T}{\partial q_k}\right|_0 = 0$ then from begining eq $\left.\frac{\partial U}{\partial q_k}\right|_0 = 0$

[Now for Potential Taylor expand it about equilibrium] $\rightarrow$ $U(q_j) = \cancel{U_0} + \cancel{\sum_k \left.\frac{\partial U}{\partial q_k}\right|_0 q_k} + \frac{1}{2}\sum_{j,k}\left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_0 q_j q_k + \ldots$

(can bring potential to 0) (sec of this $\rightarrow$ 0)

$\hookrightarrow U \simeq \frac{1}{2}\sum_{jk} A_{jk} q_j q_k$ where $A_{jk} = \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_0$

[Quick justification for m_{jk} being a constant] $\rightarrow m_{jk} = \sum_\alpha m_\alpha \frac{\partial X_\alpha}{\partial q_j}\frac{\partial X_\alpha}{\partial q_k}$

$\simeq m_{jk}(q_{\gamma,0}) + \sum_\gamma \left.\frac{\partial m_{jk}}{\partial q_\gamma}\right|_0 q_\gamma + \ldots$

keeping lowest order ~~term~~ m_{jk} = constant

$$\boxed{\begin{aligned} T &= \frac{1}{2}\sum_{jk} m_{jk}\dot{q}_j\dot{q}_k \\ U &= \frac{1}{2}\sum_{jk} A_{jk} q_j q_k \end{aligned}}$$

$\rightarrow$ Now look at the EOM

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0 \rightarrow \boxed{\frac{\partial U}{\partial q_k} + \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} = 0} \rightarrow \frac{\partial U}{\partial q_k} = \sum_j A_{jk} q_j \ , \quad \frac{\partial T}{\partial \dot{q}_k} = \sum_j m_{jk}\dot{q}_j$$

* could keep $\frac{1}{2}$ but can mult out later *

Substituting!

$$\sum_j \left(A_{jk} q_j + m_{jk}\ddot{q}_j\right) = 0 \rightarrow \text{Now solve ODE} \rightarrow q_j(t) = a_j e^{i(\omega t - \phi)}$$

* mult out $e^{i(\omega t - \phi)}$

$$\hookrightarrow \sum_j \left(A_{jk} - \omega^2 m_{jk}\right) a_j = 0 \longrightarrow$$

for a non trivial soln to exist det = 0, same shape as eigen problem pops out here

$$\left|A_{jk} - \omega^2 m_{jk}\right| = 0$$

* solve eigen value problem

* plug eigen values back in to obtain eigen vectors

q_j is index over generalized coord

k is sum over normal modes

Each ω_r, eigenvalue can be subbed into to obtain ratios $a_1, a_2, a_3 \ldots a_N$ that are components of vector $\vec{a}_r$

* eigenvector

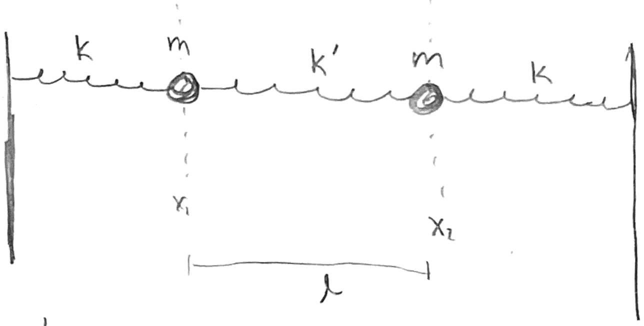
* saying a_{jr} to be the j^th component of the r^th eigenvector?

gen soln is

$$q_j(t) = \mathrm{Re}\left[\sum_r a_{jr} e^{i(\omega_r t - \phi)}\right] = \sum_r a_{jr}\cos(\omega_r t - \phi)$$

Example

0's of x_1, x_2 are when they are in their equillibrium



$(x_2 - x_1 - l)^2$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k' (x_2 - x_1)^2$$

$$U = \frac{1}{2}(k+k') x_1^2 + \frac{1}{2}(k+k') x_2^2 - k' x_1 x_2$$

$$A_{jk} = \left. \frac{\partial^2 U}{\partial q_j \partial q_k} \right|_0 = \begin{cases} A_{11} = \frac{\partial^2 U}{\partial x_1^2} = k + k' \\ A_{22} = \frac{\partial^2 U}{\partial x_2^2} = k + k' \\ A_{12} = \frac{\partial^2 U}{\partial x_1 \partial x_2} = -k' \end{cases}$$

$$A = \begin{bmatrix} k+k' & -k' \\ -k' & k+k' \end{bmatrix}$$

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 = \frac{1}{2} \sum_{j,k} m_{jk} \dot{x}_j \dot{x}_k$$

$$\left. \begin{matrix} m_{11} = m_{22} = m \\ m_{12} = m_{21} = 0 \end{matrix} \right] \rightarrow m \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \rightarrow \text{Now we can } |A_{jk} - \omega^2 m_{jk}| =$$

$$= \begin{vmatrix} k + k' - \omega^2 m & -k' \\ -k' & k + k' - \omega^2 m \end{vmatrix} = 0 \rightarrow (k + k' - m\omega^2)^2 - (-k')^2 = 0$$

$$\omega = \sqrt{\frac{k + k' \pm k'}{m}} \quad \text{so} \quad \overset{\omega_1}{\omega_+} = \sqrt{\frac{k + 2k'}{m}}, \quad \overset{\omega_2}{\omega_-} = \sqrt{\frac{k}{m}}$$

Now eigenvectors: found using this eq...

$$\sum_j (A_{jr} - \omega_r^2 m_{jr}) a_{jr} = 0$$

ω_1/ $$\sum_{j=1}^{2} (A_{j1} - \omega_1^2 m_{j1}) a_{j1} = 0 \rightarrow (A_{11} - \omega_1^2 m_{11}) a_{11} + (A_{21} - \omega_1^2 m_{21}) a_{21} = 0$$

$$\hookrightarrow \left[k + k' - m\left(\frac{k + 2k'}{m}\right) \right] a_{11} - k' a_{21} = 0 \rightarrow \boxed{a_{11} = -a_{21}} \rightarrow a_{j1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ω_2/ $$(A_{12} - \omega_2^2 m_{12}) a_{12} + (A_{22} - \omega_2^2 m_{12}) a_{22} = 0$$

$$-k' a_{12} + \left[(k + k') - \frac{k}{m} m \right] a_{22} = 0 \rightarrow \boxed{a_{12} = a_{22}} \rightarrow a_{j2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus eigenvectors

~~Thus~~ ratio of amplitudes

$$\left. \begin{matrix} \omega_1: \vec{a}_1 = a_{11} \hat{e}_1 + a_{21} \hat{e}_2 = a_{11} (\hat{e}_1 - \hat{e}_2) \\ \omega_2: \vec{a}_2 = a_{12} \hat{e}_1 + a_{22} \hat{e}_2 = a_{22} (\hat{e}_1 + \hat{e}_2) \end{matrix} \right\}$$ general motion later

General motion of ~~spring system~~ coupled oscillations

where: $A_{jk} = \frac{\partial^2 U}{\partial q_j \partial q_k}$ and $m_{jk} = \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k}$

From:

(1) $Aq + m\ddot{q} = 0$ we assumed $q_j = a_j e^{i(\omega t - \delta)}$

↳ [But generally for every eigenvalue] → $q_j(t) = \sum_r a_{jr} e^{i(\omega_r t - \delta_r)}$

j^{th} generalized coordinate

[Lets do some useful manipulation] → $q_j = \sum_r \beta_r a_{jr} e^{i\omega_r t} = \sum_r a_{jr} \eta_r(t)$ then $\dot{q}_j = \sum_r a_{jr} \dot{\eta}_r$

constant

Plug into T now:

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j \dot{q}_k = \frac{1}{2}\sum_{j,k} m_{jk}\left(\sum_r a_{jr}\dot{\eta}_r\right)\left(\sum_s a_{ks}\dot{\eta}_s\right) = \frac{1}{2}\sum_{r,s}\left(\sum_{jk} m_{jk} a_{jr} a_{ks}\right)\dot{\eta}_r \dot{\eta}_s$$

we are showing the eigenvectors, a_r, form an orthonormal set

↳ [Rewriting (1) as] $\sum_j (A_{jk} - \omega^2 m_{jk}) a_j = 0 \rightarrow \omega_s^2 \sum_k m_{jk} a_{ks} = \sum_k A_{jk} a_{ks}$ ← s is the s^{th} root, ~~eigenvalue~~ eigenvalue; ↑ multiply by $\sum_j a_{jr}$

↳ $\omega_s^2 \sum_{jk} m_{jk} a_{jr} a_{ks} = \sum_{jk} A_{jk} a_{jr} a_{ks}$ | → $\omega_r^2 \sum_j m_{jk} a_{jr} = \sum_j A_{jk} a_{jr}$ ← multiply by $\sum_k a_{ks}$

↕ equal

$\omega_r^2 \sum_{jk} m_{jk} a_{jr} a_{ks} = \sum_{jk} A_{jk} a_{jr} a_{ks}$ | *if $\omega_r = \omega_s$ its a degenerate solution, ie $r=s$ ↳ makes sum indeterminite

↳ subtract over → $(\omega_r^2 - \omega_s^2)\left(\sum_{jk} m_{jk} a_{jr} a_{ks}\right) = 0 \rightarrow \sum_{jk} m_{jk} a_{jr} a_{ks} = 0 \quad r \neq s$

Now look at $r=s$ case!

[Relook at T] $T = \frac{1}{2}\sum_{jk} m_{jk}\dot{q}_j \dot{q}_k = \frac{1}{2}\sum_{jk} m_{jk}\left[\sum_r \omega_r a_{jr}\sin(\omega_r t - \delta_r)\right]\left[\sum_s \omega_s a_{ks}\sin(\omega_s t - \delta_s)\right] = \left[\left(\frac{1}{2}\sum_{r,s}\omega_r\omega_s \sin(\omega_r t - \delta_r)\sin(\omega_s t - \delta_s)\right)\cdot\left(\sum_{jk} m_{jk} a_{jr} a_{ks}\right)\right]$ all multiplied together

↳ [apply $r=s$] ⇒ $T = \frac{1}{2}\sum_r \omega_r^2 \sin^2(\omega_r t - \delta_r)\sum_{jk} m_{jk} a_{jr} a_{kr}$, [$\sin^2$ func ≥ 0 so] → $\omega_r^2 \sin^2(\omega_r t - \delta_r) \geq 0$

↳ [Generally for T we know T is positive and become 0 only if all the velocities vanish identically] → $\sum_{jk} m_{jk} a_{jr} a_{kr} \geq 0$ as well

- $\sum m_{jk} a_{jr} a_{ks} = 0$ ($r=s$) means the vectors of a are orthogonal ← opposite vectors $x \cdot y = 0$

- $\sum m_{jk} a_{jr} a_{kr} = 1$ ($r \neq s$) if we force this the we normalized the orthogonal set at the eigenvectors now form a orthonormal set ← same vectors $x \cdot x = 1$

then $\boxed{\sum_{jk} m_{kj} a_{jr} a_{ks} = \delta_{rs}}$

With $\sum_{jk} m_{jk} a_{jr} a_{ks} = \delta_{rs}$; and $T = \frac{1}{2}\sum_{r,s}\left(\sum_{jk} m_{jk} a_{jr} a_{ks}\right)\dot{\eta}_r \dot{\eta}_s$

$\hookrightarrow T = \frac{1}{2}\sum_{r,s} \delta_{rs}\dot{\eta}_r \dot{\eta}_s = \frac{1}{2}\sum_r \dot{\eta}_r^2$

Similarly for U;

$U = \frac{1}{2}\sum_{jk} A_{jk} q_j q_k = \frac{1}{2}\sum_{jk} A_{jk}\left(\sum_r a_{jr}\eta_r\right)\left(\sum_s a_{ks}\eta_s\right)$

$= \frac{1}{2}\sum_{r,s}\underbrace{\left(\sum_{jk} A_{jk} a_{jr} a_{ks}\right)}_{\text{from equation } \omega_s^2 \sum_{jk} m_{jk} a_{jr} a_{ks} = \sum_{jk} A_{jk} a_{jr} a_{ks}}\eta_r \eta_s$

$= \frac{1}{2}\sum_{r,s}\left(\omega_s^2 \sum_{jk} m_{jk} a_{jr} a_{ks}\right)\eta_r \eta_s = \frac{1}{2}\sum_{r,s}\omega_s^2 \delta_{rs}\eta_r \eta_s$

$= \frac{1}{2}\sum_r \omega_r^2 \eta_r^2$

$L = T - U \rightarrow \frac{1}{2}\sum_r\left(\dot{\eta}_r^2 - \omega_r^2 \eta_r^2\right)$

Apply ELE;

$\frac{\partial L}{\partial \eta_r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_r} = 0 \longrightarrow \ddot{\eta}_r + \omega_r^2 \eta_r = 0$] normal coordinates

Recall $q_j(t) = \sum_r a_{jr}\eta_r(t)$, $\hookrightarrow \beta_r e^{i\omega t}$ [So we can use this And to work the general solution]

Recall the 3 spring system we found!

$a_{11} = -a_{21}$ and $a_{12} = a_{22}$

$\omega_1 = \sqrt{\frac{k+2k'}{m}}$ and $\omega_2 = \sqrt{\frac{k}{m}}$

But how do we use this knowledge to solve for ours q's ?

$q_1 = x_1 = a_{11}\eta_1 + a_{12}\eta_2 = a_{11}\eta_1 + a_{22}\eta_2$

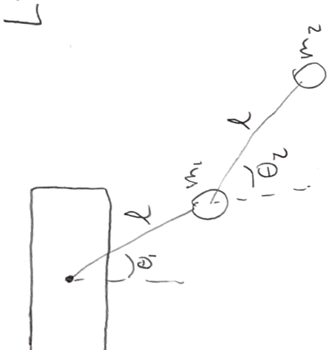
$q_2 = x_2 = a_{21}\eta_1 + a_{22}\eta_2 = -a_{11}\eta_1 + a_{22}\eta_2$

jth gen coord; rth eigenval; ηth eigenvalue

$\hookrightarrow$ solve! $\eta_1 = \frac{1}{2a_{11}}(x_1 - x_2) \leftarrow$ out of phase

$\eta_2 = \frac{1}{2a_{22}}(x_1 + x_2) \leftarrow$ in phase

Double Pendulum



$L = T - V = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2\right) + mg(y_1 + y_2)$

$x_1 = l\sin\theta_1$ and $x_2 = x_1 + l\sin\theta_2$

$y_1 = -l\cos\theta_1$ $\quad y_2 = +y_1 - l\cos\theta_2$ (↑ already has sign)] take derivatives, square and plug in

$L = \frac{m}{2}l^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\left(\cos(\theta_2 - \theta_1)\right)\right) - mgl(2\cos\theta_1 + \cos\theta_2)$

For small displacements $\cos\theta = 1 - \frac{\theta^2}{2} \Rightarrow L = \frac{ml^2}{2}\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\right) - mgl\left(\theta_1^2 + \frac{1}{2}\theta_2^2\right)$

§5 Describing a molecule

- for every deg of freedom we have we have a mode?

Generally:

- → 3 coord about the CM for translational
- → 3 coord to describe rotations

[for each atom in the molecule] $3N-6$ describe vibratory

Consider a molecule

- In the plane each mass has 2 deg of freedom thus → $2N$ d.o.f
- 2 coord to describe translation of CM and 1 coord for a rot

→ generally there are $2N-3$ vibrational modes in the plane.

→ the modes left are out of plane $(3N-6)-(2N-3) = N-3$ modes out,

For a linear molecule we need one less coord for in the plane:

↳ $(3N-5)-(N-1) = 2N-4$ vibrational modes

−1 less

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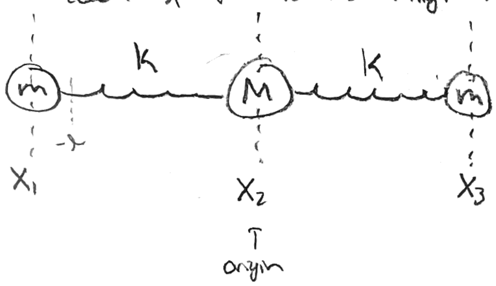
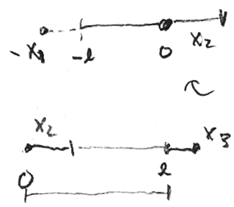
ex each spring has rest length l.



Find potential V:



$$V = \underbrace{\frac{k}{2}\left(X_2 - \underbrace{X_1 - l}_{\text{subtract off whatever } X_1 \text{ and } l \text{ are doing}}\right)^2}_{\text{spring 1}} + \frac{k}{2}\left(X_3 - \underbrace{(X_2 + l)}_{\text{subtract off whatever } X_2 \text{ and } l \text{ are doing}}\right)^2$$

- in equilibrium take derivatives + set = 0

$X_1 - X_2 = -l$ and $X_3 - X_2 = l$ → solve for X_2 → $X_2 = X_1 + l$] displacements about X_2 for X_1; $X_2 = X_3 - l$] displacements about X_2 for X_3

[Lets define a displacement vector] → $\delta x = \begin{pmatrix}\delta x_1 \\ \delta x_2 \\ \delta x_3\end{pmatrix}$, keep $\delta X_2 = X_2$, $\delta X_1 = X_1 + l$, $\delta X_2 = X_3 - l$

Plug in! $V = \frac{k}{2}(\delta X_2 - \delta x_1)^2 + \frac{k}{2}(\delta X_3 - \delta X_2)^2 = \frac{k}{2}(\delta x_1^2 + 2\delta X_2^2 + \delta X_3^2) - k\delta X_2(\delta x_1 + \delta X_3)$

[Now read our matrix ~~off~~] → $A = \frac{\partial^2 V}{\partial x_i x_j} = k\begin{pmatrix}1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1\end{pmatrix}$, $V = \delta\vec{x} \cdot A \cdot \delta\vec{x}$

[Now for T] → $T = \frac{m}{2}(\dot{X}_1^2 + \dot{X}_3^2) + \frac{M}{2}\dot{X}_2^2$ → $m = \frac{\partial^2 T}{\partial \dot{x}_i \partial \dot{x}_j} = \begin{pmatrix}m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m\end{pmatrix}$

Now we can apply our formula

$|A - \omega^2 m| = 0$, to find our eigenvalues, or our modes of frequency

define: $r = \frac{M}{m}$, $\lambda = \frac{\omega^2}{\omega_0^2}$, $\omega_0^2 = \frac{k}{m}$

$$\hookrightarrow \begin{vmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{vmatrix} = 0 ,$$

$$\hookrightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-r\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)\left[(1-\lambda)(2-\overset{r}{\frac{M}{m}}\lambda)-1\right] - (1-\lambda) = 0$$

this yields 3 λ's! or 3 ω's

$$\lambda = 0, 1, 1+2r \rightarrow \omega^2 = 0, \omega_0^2, \omega_0^2(1+2\tfrac{m}{M})$$

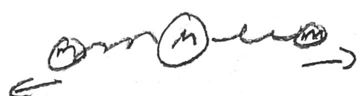
$\omega^2 = 0$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

translation of cm

$\omega^2 = \omega_0^2$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 2-\frac{M}{m} & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



cm doesn't move

$\delta x_{cm} = m(\delta x_1 + \delta x_3) + M(\delta x_2) = 0$

$\delta x_{cm} = 0 = m(1-1)$

$\omega^2 = \omega_0^2(1+2\frac{m}{m})$

$$\begin{pmatrix} -2\frac{m}{m} & -1 & 0 \\ -1 & -\frac{M}{m} & -1 \\ 0 & -1 & -2\frac{m}{m} \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{pmatrix}$$

$\hookrightarrow \delta x_{cm} = m(\delta x_1 + \delta x_3) + M\delta x_2 = m(1+1) + M(-2\frac{m}{M}) = 0$

center of mass still fixed

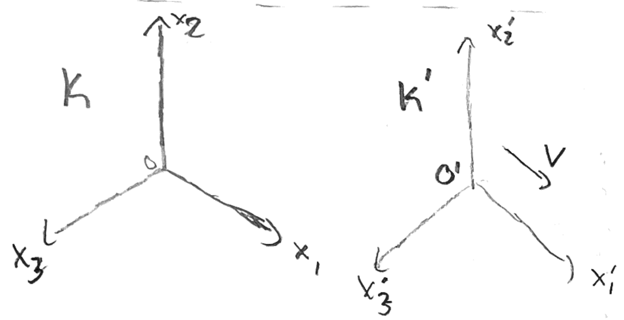
Special Relativity

principle of relativity

- The laws of physical phenomena are the same in all inertial ref frames.
- Only relative motion can be measured, an absolute rest frame is meaningless

↳ The laws of physical phenomena, E&M, show that the <u>speed of light is constant.</u>

$$\nabla^2 \vec{E} = \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \quad v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$



- transformation of coordinates between frames w/ relative motion along x_1, x_1' axis

$$\left.\begin{aligned} x_1' &= x_1 - vt \\ x_2' &= x_2 \\ x_3' &= x_3 \\ t' &= t \end{aligned}\right\} \text{Galilean Transformation}$$

in x_1' frame position x_1 will be whatever it is x_1 minus the speed relative (backwards) at some time t

[Length is the same in both of the frames] → $ds^2 = \sum_i dx_i^2 = \sum_i dx_i'^2 = ds'^2$

*Newtons laws are invariant under a galilean transformation

[Breakdown of the Galilean Transformation]

- two inertial frames k and k'
- moving relative to one another w/ some speed V
- at $t=0$, both have the same origin

*now think about <u>light emitted</u> at the <u>$t=0$</u> and k' moves w/ some <u>speed</u> V

- principle of relativity states it should be <u>c</u> in <u>both frames</u>

↳ $\underbrace{\sum_j x_j^2}_{\text{length in frame } k} - \underbrace{c^2t^2}_{\text{distance light traveled in frame } k} = 0$ in k

↳ $\underbrace{\sum_j x_j'^2}_{\text{length in } k' \text{ frame}} - \underbrace{c^2t'^2}_{\text{distance light traveled in frame } k'} = 0$ in k'

describes a wave of light emitted at the origin but both have same speed c, described in terms of length.

- these both should be the same but if you perform a galilean transformation it doesnt work.

Quick look if it goes along x dir only and v is along x dir it simplifies it:

$\sum x_1^2 = c^2t^2 \rightarrow x_1 = ct$

$x_1'^2 = c^2t'^2 \rightarrow x_1' = ct' \xrightarrow[\text{trans}]{\text{gal}} x_1 - vt = ct \rightarrow x_1 = t(c+v)$ Wrong! <u>Cant be faster than light.</u>

What is the correct transformation then?

• Consider frames k, k' w/ rel velocity v, along x_1, x_1' dir [trivially then] $\rightarrow$ $x_2' = x_2$, $x_3' = x_3$

[Lets find some factor γ to fix our transformation] $\rightarrow$ $x_1' = \gamma(x_1 - vt)$ and $x_1 = \gamma'(x_1' + vt')$ * $\gamma'(v) = \gamma(-v) = \gamma(v)$

$$x_1' = \gamma(x_1 - vt) = \gamma\left(1 - \frac{vt}{x_1}\right)x_1$$

$$x_1 = \gamma\left(1 + \frac{vt}{x_1'}\right)x_1'$$

* the distance the light travels in both should be same, $x_1 = ct$ + $x_1' = ct'$, regardless of v

Plugging in relations:

$$\left.\begin{array}{l} x_1' = \gamma\left(1 - \frac{v}{c}\right)x_1 \\ x_1 = \gamma\left(1 + \frac{v}{c}\right)x_1' \end{array}\right\} \; x_1 = \gamma^2\left(1 - \frac{v^2}{c^2}\right)x_1 \rightarrow 1 = \gamma^2\left(1 - \frac{v^2}{c^2}\right) \rightarrow \boxed{\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

also: $ct' = \gamma\left(1 - \frac{v}{c}\right)x_1 \rightarrow ct' = \gamma ct - \gamma\frac{v}{c}x_1 \rightarrow t' = \gamma\left(t - \frac{v}{c^2}x_1\right)$

Thus our Lorentz transformation is:

$$\boxed{\begin{array}{l} x_1' = \gamma(x_1 - vt) \\ x_2' = x_2 \\ x_3' = x_3 \\ t' = \gamma\left(t - \frac{v}{c^2}x_1\right) \end{array}}$$

Velocities in k and k'

$$u_i = \frac{dx_i}{dt} \quad \text{and} \quad u_i' = \frac{dx_i'}{dt'}$$

$$u_1' = \frac{dx_1'}{dt'} = \frac{(dx_1 - v\,dt)\gamma}{\left(dt - \frac{v}{c^2}dx_1\right)\gamma} = \frac{\left(\frac{dx_1}{dt} - v\right)dt}{\left(1 - \frac{v\frac{dx}{dt}}{c^2}\right)dt} = \boxed{\frac{u_1 - v}{1 - \frac{vu_1}{c^2}} = u_1'}$$

$$u_2' = \frac{dx_2'}{dt'} = \frac{dx_2}{\gamma\left(dt - \frac{v}{c^2}dx_1\right)} = \frac{u_2}{\gamma\left(1 - \frac{uv}{c^2}\right)}$$

$$\frac{\overset{k \text{ frame}}{u} - \overset{\text{rel velocity between them}}{v}}{1 - \frac{vu}{c^2}} = \underset{k' \text{ frame}}{u'}$$

$$\rightarrow u - v = u'\left(1 - \frac{vu}{c^2}\right) = u' - \frac{vuu'}{c^2}$$

$$u\left(1 + \frac{vu'}{c^2}\right) = v + u'$$

$v = 0.8c$

$u' = 0.6c$

$u = ?$

$$u = \frac{v + u'}{1 + \frac{vu'}{c^2}}$$

Length contraction

Lorentz transformation

$x_1' = \gamma(x_1 - vt)$

$x_2' = x_2$

$x_3' = x_3$

$t' = \gamma(t - \frac{v}{c^2}x_1)$

- rod w/ length l lying along x_1 axis in frame k
- an observer in k' moving w/ speed v relative to x_1 measure:

$\rightarrow l' = x_1'(2) - x_1'(1)$

$\rightarrow l' = \gamma(x_1(2) - vt(2)) - \gamma(x_1(1) - vt(1)) = \gamma\left[\underbrace{(x_1(2) - x_1(1))}_{\text{length } l \text{ in frame k}} - v\underbrace{(t(2) - t(1))}_{\Delta t \text{ in frame k}}\right]$

$l' = \gamma(l - v\Delta t)$

*the measurement in frame k' is done like $\Delta t' = 0$

$\Delta t = t'(2) - t'(1) = \gamma(t(2) - \frac{v}{c^2}x_1(2)) - \gamma(t(1) - \frac{v}{c^2}x_1(1))$

$= \gamma\left[\underbrace{(t(2) - t(1))}_{\Delta t} + \frac{v}{c^2}\underbrace{(x_1(2) - x_1(1)}_{l}\right] = 0 \rightarrow \Delta t = \frac{v}{c^2}l$

$l' = \gamma l \overbrace{(1 - \frac{v}{c^2})}^{\gamma^{-2}} = \boxed{\frac{l}{\gamma} = l'}$

distance in rest frame

distance in moving frame

② Time dilation

- consider a clock at position x_1 in frame k w/ $\Delta t = t(2) - t(1)$

↳ [an observer in k' measuring a time interval] $\rightarrow \Delta t' = t'(2) - t'(1) = \gamma\left[(t(2) - \frac{v}{c^2}x_1(2)) - (t(1) - \frac{v}{c^2}x_1(1))\right]$

* $x_1(2) - x_1(1) = 0$ (clock doesn't move)

↳ $\Delta t' = \gamma\Delta t$

time measured in the moving frame

time in the rest frame

Muon Decay

1) Classical Mechanics → $t = \frac{d}{v} = \frac{2000m}{0.98c} \approx 6.8 \mu s$

$N(t) = N(0) e^{-(\ln 2) t/\tau}$ $\sim 45/hr$

2) Relativistic analysis

- An observer on Earth would see the clock of muon run slow

$\Delta t' = \gamma \Delta t$ → $\Delta t = \frac{\Delta t'}{\gamma}$ and $\gamma = \frac{1}{\sqrt{1 - 0.98^2}} \approx 5$

↑ muon frame time = 6.8 μs

↑ Earth frame

*muon has to go 2000m in our frame but we perceive its half life move slowly

$$\boxed{\Delta t = \frac{6.8 \mu s}{5} = 1.36 \mu s}$$

Muon Frame

- An observer w/ the muon in frame k' the distance travell

$l' = \gamma^{-1} l = 400m$, $t = \frac{d}{v} = \frac{400m}{0.98c} = 1.36 \mu s$

↑ moving frame

↑ 2000m

$\& \; N(t) = N(0) e^{-\log 2 \; t/\tau}$ $\sim 536/hr$

which agrees w/ observation

Doppler Shift

Classical:

1) Source is moving to the right at speed V_s

$V = f\lambda \rightarrow \frac{1}{f} = \frac{\lambda}{V}$ where $V = \frac{\lambda}{\Delta t}$ ↑ time it takes for it to travel one distance λ

$\Delta t' = \frac{1}{f'} = \frac{\lambda - V_s \Delta t}{V}$ ← [perceived distance (λ) shortened by the speed of source]

↑ speed of wave

pull out Δt

$= \frac{1}{f}\left(\frac{V - V_s}{V}\right)$

$$f' = f\left(\frac{V}{V - V_s}\right)$$

2) Source is stationary, reciever is moving away

$\Delta t' = \frac{1}{f'} = \frac{\lambda}{V + V_r} = \frac{1}{f}\left(\frac{V}{V + V_r}\right) \rightarrow f' = f\left(\frac{V + V_r}{V}\right)$

3) both moving, so source is moving to the right and reciever is moving away (same dir)

$\frac{1}{f'} = \frac{\lambda - V_s \Delta t}{V + V_r} = \frac{1}{f} \frac{V - V_s}{V + V_r} \rightarrow f' = f \frac{V + V_r}{V - V_s}$

speed of observer relative to the medium

speed of source rel to medium

Relativistic

intuition: if you run w/ light wave you'll see a redshift ($\lambda\uparrow, f\downarrow$), if you run towards it you see blue shift ($\lambda\downarrow, f\uparrow$)

$\underset{\substack{\uparrow \\ \text{total waves} \\ \text{between them}}}{n} \lambda' = \underbrace{c\Delta t'}_{\text{distance light travels in time } \Delta t} - \underbrace{V\Delta t'}_{\text{distance person travels in time } \Delta t} \rightarrow \lambda' = \frac{(c - v)\Delta t'}{n}$, but what is n? $\rightarrow n = \overbrace{f\Delta t}^{\#\text{ of counts after } \Delta t} = f\frac{\Delta t'}{\gamma}$

$f' = \frac{c}{\lambda'} \rightarrow f' = f \underbrace{\frac{\Delta t}{\Delta t'}}_{\gamma} \frac{1}{1 + v/c} = f \frac{\sqrt{1 - u^2/c^2}}{1 + u/c} = f\sqrt{\frac{1 - u/c}{1 + u/c}}$

$$\underset{\substack{\uparrow \\ \text{observed}}}{f'} = \underset{\substack{\uparrow \\ \text{source}}}{f}\sqrt{\frac{1 - u/c}{1 + u/c}}$$

* if u is positive, moving away then you get a $\downarrow f'$ or $\uparrow \lambda'$

* opposite if u is neg

$$f' = f\sqrt{\frac{1-u/c}{1+u/c}}$$

f' ↑ Obser f ↑ Source

(ex)

- distance galaxy moving away from earth

$\lambda_{source} = 434\,nm$

$\lambda_{Earth} = 600\,nm$

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\sqrt{\frac{1-u/c}{1+u/c}}$$

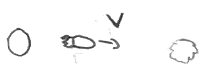
$$\frac{1}{600\,nm} = \frac{1}{434\,nm}\sqrt{\frac{1-u/c}{1+u/c}}$$

(ex) Spacecraft flying to star at speed $V = 0.8c$

Spacecraft sends signal 1 signal/year (measured on spacecraft)

[How often are signals recieved on earth?]

Trip to the star: Spaceship observes earth moving at $-V$



$$f' = f\sqrt{\frac{1-V/c}{1+V/c}} = \frac{1}{3}f$$

f' ↑ observer f ↑ source

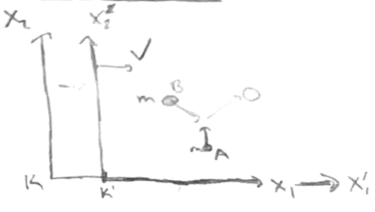
→ observer on Earth sees signals 1/3 the time

Trip back from star

$$f' = f\sqrt{\frac{1- {-V}/c}{1+ {-V}/c}} = 3f$$

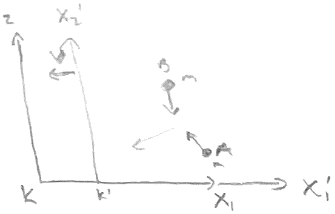
f' ↑ Observer f ↑ Source

Momentum



(collision in frame k)

and



(collision in frame k')

- in k' frame it sees particle B with velocity in one direction is X_2'
- in k frame it sees particle A with velocity in one direction is X_2

} the opposite particle has an extra component of velocity due to the relative frame difference

System k

$$\vec{u}_A = \begin{bmatrix} 0 \\ u_0 \end{bmatrix} \quad \text{then} \quad \vec{p} = \begin{bmatrix} 0 \\ m\,u_0 \end{bmatrix}$$

change in momentum for particle A in frame k? (x_2 direction)

If perfectly elastic then: $P_{f,A} - P_{i,A} = -mu_0 - (mu_0) = -mu_0$

$$\Delta P_{A,X_2} = -2mu_0$$

System k'

$$\vec{u}_B' = \begin{bmatrix} 0 \\ -u_0 \end{bmatrix}$$

, now use lorentz transformation to get u_B' into k frame

(Recall the general equations) $\rightarrow$ $u_1' = \dfrac{u_1 - v}{1 - \frac{v u_1}{c^2}}$ and $u_2' = \dfrac{dx_2'}{dt'} = \dfrac{u_2}{\gamma\left(1 - \frac{u_1 v}{c^2}\right)}$] these take you from the unprimed to primed coord syst

(Reverse these expressions, swap sign on v, and switch primes) $\rightarrow$ $u_1 = \dfrac{u_1' + v}{1 + \frac{v u_1}{c^2}}$ and $u_2 = \dfrac{u_2'}{\gamma\left(1 + \frac{u_1' v}{c^2}\right)}$ u_1 !

$$u_{B,1} = \frac{0 + v}{1 + \frac{v \cdot 0}{c^2}} = v \quad , \quad u_{B,2} = \frac{-u_0}{\sqrt{1 - \frac{v^2}{c^2}}^{\,-1}\left(1 + \frac{0 \cdot v}{c^2}\right)} = -\gamma^{-1} u_0$$

$$P_{B,2} = -\gamma^{-1} m u_0$$

$$P_f - P_i = \gamma^{-1} m u_0 - (-\gamma^{-1} m u_0) = 2\gamma^{-1} m u_0$$

$$\Delta P_{A,2} + \Delta P_{B,2} \neq 0 \;!!$$

We need to modify our definition of momentum

$\longrightarrow$

Recall: $\vec{p} = m \frac{d\vec{x}}{dt}$

Observers dont generally agree on dx/dt, lets instead use the proper time τ

$\vec{p} = m \frac{dx}{dt}\frac{dt}{d\tau}$ $\rightarrow$ $\Delta t = \gamma \Delta \tau_0$ $\rightarrow$ $\frac{dt}{d\tau} = \gamma$

τ time in rest frame

$$\boxed{\vec{p} = \gamma m \frac{d\vec{x}}{dt} \quad \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}}$$

τ is measured in rest frame w.r.t its particle so it moves at speed u in that frame

Restore conservation of momentum?

$P_{A,2} = \frac{mu_0}{\sqrt{1-u_0^2/c^2}}$ and $\Delta P_{A,2} = \frac{-2mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}}$

$P_{B,2} = -\sqrt{1-\frac{v^2}{c^2}}\, m u_0 \gamma$, $\gamma = \frac{1}{\sqrt{1-\frac{u_B^2}{c^2}}}$, u_B is the magnitude of velocity of particle B in frame K

$u_B = \sqrt{u_{B,1}^2 + u_{B,2}^2}$

x component is v

y component is $-\gamma^{-1}u_0$

refers to relative frame velocity

$\rightarrow u_B = \sqrt{v^2 + u_0^2\left(1-\frac{v^2}{c^2}\right)}$

$$P_{B,2} = \frac{-mu_0\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}+\frac{u_0^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}} = \frac{-mu_0\cancel{\sqrt{1-\frac{v^2}{c^2}}}}{\cancel{\sqrt{1-\frac{v^2}{c^2}}}\sqrt{1-\frac{u_0^2}{c^2}}} = \frac{-mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}}$$

$\Delta P_{B,2} = \frac{2mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}}$ $\xrightarrow{\text{So!}}$ $\boxed{\Delta P_{A,2} + \Delta P_{B,2} = 0}$

Energy

$\gamma du + u d\gamma$

work energy theorem: $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = T_2 - T_1$

$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m \vec{u})$ (velocity u) and $d\vec{r} = \vec{u}\,dt$ (velocity u)

$W_{12} = \int \frac{d}{dt}(\gamma m \vec{u}) \cdot \vec{u}\,dt = m\int_0^u u\,d(\gamma u)$ — integrate by parts

$u = u$, $du = du$; $dv = d(\gamma u) \rightarrow v = \gamma u$

$= m\gamma u^2\Big|_0^u - m\int_0^u \frac{u\,du}{\sqrt{1-\frac{u^2}{c^2}}}$ — let $x = \frac{u}{c}$

$= m\gamma u^2 - mc^2\int_0^{u/c} \frac{x\,dx}{\sqrt{1-x^2}} = m\gamma u^2 + mc^2\sqrt{1-(\frac{u}{c})^2} - mc^2$

$\rightarrow 1-(\frac{u}{c})^2$

$= \gamma m u^2 + \gamma^{-1} mc^2 - mc^2 = \gamma m(u^2 + \gamma^{-2}c^2) - mc^2$

$$\boxed{T = \gamma mc^2 - mc^2}$$

Relativistic Kinetic energy

$mc^2 = E_0$

$T = \gamma mc^2 - E_0$

$$\boxed{T + E_0 = \gamma mc^2 = E}$$

Alternate form:

$(\frac{u}{c})^2 = 1 - \gamma^{-2}$ and $\gamma^{-2} = 1 - (\frac{u}{c})^2$

$(pc)^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 (\frac{u^2}{c^2})$

$(pc)^2 = \gamma^2 m^2 c^4 (1 - \gamma^{-2})$

$(pc)^2 = \gamma^2 m^2 c^4 - m^2 c^4$

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$