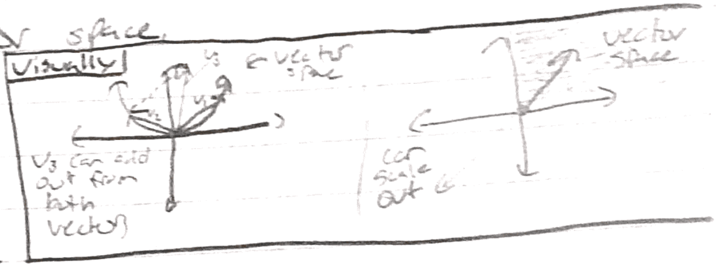
Vector Spaces and Subspaces

→ vector space is using some amount of vectors to scale & and add them to get to all points in that vector space.



Thus it needs:

- to have the zero vector
- scaling keeps it in that subspace
- vector addition keeps it in that subspace

ex

$H = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} a, b \text{ are all } \mathbb{R} \right\} \leq \mathbb{R}^3$ ← H is subspace of $\mathbb{R}^3$

- $a, b = 0 \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓ still in club $\in H$
- $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ 0 \end{bmatrix} \in H$ still in club ✓ and the vector of same form!!
- $c \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \to \begin{bmatrix} ac \\ bc \\ 0 \end{bmatrix}$ still in club ✓

$J = \left\{ \begin{matrix} a \\ b \\ 1 \end{matrix} \; a, b \text{ are all real numbers} \right\} \leq \mathbb{R}^3$ ← $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ can never be in subspace of $\mathbb{R}^3$

Saying you can scale + and add vectors for vector space is the ~~span~~. Has form

set $\{v_1 \cdots v_k\} \to a_1 v_1 + \cdots + a_k v_k \in$ linear combinations.

→ Span $\{v_1 \cdots v_k\}$ in V set of all linear combos of $v_1 \cdots v_k$

- Span of any vectors is a subspace a ~~defined~~ above:
  - → $0 = 0v_1 \cdots 0v_k = 0$
  - → if u, w in span $\{v_1 \cdots v_k\}$; $u = a_1 v_1 + \cdots a_k v_k$ and $w = c_1 v_1 + \cdots + v_k c_k$
  - → $u + w = (a_1 + c_1) v_1 + \cdots + (a_k + c_k) v_k$ ← Span $\{v_1 \cdots v_k\}$ NONE LEAVE Vector Space.
  - → scale → $cu = c(a_1 v_1 + \cdots a_k v_k) \to cu = (ca_1) v_1 + (ca_k) v_k$

Null Spaces | $\text{Nul}\, A = \{ \vec{x} : \vec{x} \text{ is in } \mathbb{R}^n \text{ and } A\vec{x} = \vec{0} \}$

Null A

$A\vec{0} = \vec{0}$, $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$, $A(c\vec{u}) = cA(\vec{u})$

To compute: Take A and do $Ax = 0$ or $[A \mid \vec{0}]$ put into parametric vector form, then those are the soln of x to get 0. Meaning those vector when linear combo'd span the Null space.

$\text{Nul}\, A = \text{Span}\{\text{of parametric vectors}\}$

subspace $\mathbb{R}^n$ ← # of columns tells you dimension

Column Spaces | if columns of $A = [\vec{a}_1 \; \vec{a}_2 \; \vec{a}_3 \cdots \vec{a}_n]$ then

$\text{Col}\, A = \text{Span}\{a_1, a_2, \cdots a_n\}$

if $\vec{b}$ is in Col A, then $\vec{b}$ lin combo of A,

$A \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \vec{b}$

$\text{Col}\, A = \{\vec{b} : \vec{b} = Ax$ for some x in $\mathbb{R}^n\}$

$W = \begin{bmatrix} 6a + 3c \\ 2b \\ b + c + d \\ a + b \end{bmatrix} \to \begin{bmatrix} 6 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ ← if you multiply out your making coeff.

this columns $W = \text{span}\, A$

* better examples on back

subspace of $\mathbb{R}^m$ ← # of entries in columns tells you dimensions

$W = \begin{bmatrix} a \\ 2b \\ 3 \\ a+b \end{bmatrix}$

$\vec{0}$ not in cuz 3 always 3, not subspace of $\mathbb{R}^4$

Linear Independen sets + bases

→ if $c_1v_1 \cdots c_pv_p = 0$ has only then Lin Ind, L.I

→ basis happens if Span{inside} "inside" is L.I + Spans the set. Simplest spanning set.

examples

Let $\{e_1, e_2, e_3\}$ be columns of I_n show $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$

$\boxed{\text{Lin. Ind}}$ $c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, mult out + add → $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

thus L.I.

$\boxed{\text{Span}}$ let $\vec{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ be any vector in $\mathbb{R}^3$ same as $b_1\begin{bmatrix}1\\0\\0\end{bmatrix} + b_2\begin{bmatrix}0\\1\\0\end{bmatrix} + b_3\begin{bmatrix}0\\0\\1\end{bmatrix} = b_1e_1 + b_2e_2 + b_3e_3$

thus $\mathbb{R}^3 = \text{Span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ (each is a unique component to define $\mathbb{R}^3$)

1. $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ check L.I. $\begin{bmatrix}1&1&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$ ~ no free var so L.I

$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ can never have # here

→ not every row has a pivot so it doesnt span $\mathbb{R}^4$ (never reaches that dimension)

2. $\begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}5\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\3\end{bmatrix}$ more vectors than entries so lin dep, and component in each so its in $\mathbb{R}^3$, Span: $\begin{bmatrix}2&5&2&2\\0&-1&2&4\\0&0&1&3\end{bmatrix}$

the span has a pivot in every row so each defines a unique component in $\mathbb{R}^3$

↳ if you removed $\begin{bmatrix}2\\4\\3\end{bmatrix}$ then you would still span $\mathbb{R}^3$, this would be the basis

Basis Example

find basis for $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}5\\-1\\5\end{bmatrix}, \begin{bmatrix}6\\-1\\6\end{bmatrix}, \begin{bmatrix}2\\4\\3\end{bmatrix}$ (v_1, v_2, v_3, v_4), → $H = \text{Span}\{v_1, v_2, v_3, v_4\}$ observe $v_3 = v_1 + v_2$ → $H = \text{Span}\{v_1, v_2, v_4\}$

Still need to check if v_1, v_2, v_4 are L.I (or combos of eachother) → $\begin{bmatrix}1&5&2&0\\0&-1&4&0\\1&5&3&0\end{bmatrix}$ → $\begin{bmatrix}1&5&2&0\\0&1&-4&0\\0&0&1&0\end{bmatrix}$ no free var so only triv soln

Bases for Nul A

→ when you do (A|0) and get parametric vector form these are Ind, heres why

lets say you get $\vec{x} = x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-2\\0\\-2\\1\\0\end{bmatrix} + x_5\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ these ones each define a unique component.

Bases for Col B

In echelon form so

lets say: $B = \begin{bmatrix}1&0&-1&0&-4\\0&1&1&0&5\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}$ (columns b_1, b_2, b_3, b_4, b_5) → $\text{Col } B = \text{Span}\{b_1, b_2, b_3, b_4, b_5\}$

are they L.I though?

$b_3 = b_2 - b_1$ also $b_5 = 5b_2 - 4b_1$ so you can take those out, only

only take columns with pivots for same reason as: thus $\text{Col } B = \text{Span}\{b_1, b_2, b_4\}$

* if not in echelon form, reduce to find pivots / find which b but use original matrix columns bec/ the echelon column space is different

(stays on its own span after its transformation $A\vec{v}$)

Eigenvectors and Eigenvalues

definition $A\vec{v} = \lambda\vec{v}$, for $\vec{v} \neq 0$ (so its interesting) looking for v's and λ such that when you do matrix multiplication it returns a scalar of that vector.

Finding eigen vectors: (given λ)

$A\vec{v} = \lambda\vec{v} \rightarrow \lambda\vec{v} - A\vec{v} = \vec{0} \rightarrow \vec{v}(I_n\lambda - A) = \vec{0}$

(given λ) • now to satisfy $A\vec{v} = \lambda\vec{v}$ you have to find all nontrivial soln of this matrix

call $I_n\lambda - A = G$ then your solving $G\vec{v} = \vec{0}$ → which is the same as finding the Nulspace ($Ax=0$)

ex $\lambda = 3, 2$ w/ $A = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix}$

compute ($\lambda=3$): $I_n\lambda - A$:

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & 2 \\ -1 & -1 & 1 \\ -3 & -6 & 4 \end{bmatrix}$ $(I_n\lambda - A)$

now solve $\vec{v}\begin{bmatrix} -1 & -4 & 2 \\ -1 & -1 & 1 \\ -3 & -6 & 4 \end{bmatrix} = \vec{0} \rightarrow \left[\begin{array}{ccc|c} -1 & -4 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ -3 & -6 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\lambda = 3 \Rightarrow \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2/3 v_3 \\ 1/3 v_3 \\ v_3 \end{bmatrix} = v_3\begin{bmatrix} 2/3 \\ 1/3 \\ 1 \end{bmatrix}$ so $Nul(3I_n - A) = Span\left\{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right\}$

now for $\lambda = 2$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{G\vec{v}=0} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $v = \begin{bmatrix} -2v_2 + v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

• $Nul(2I_3 - A) = Span\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

eigenvectors

Finding eigenvalues using $(A - \lambda I_n)\vec{v} = \vec{0}$, we dont want triv sol for eigenvalues

meaning we dont want a pivot in every column, so look at RREF → if in RREF you get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → triv sol'n; $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ → non triv solut → non triv soln always have 0 determinant, need 0 on diagonal → note: $\lambda = 0$ means $A\vec{v} = \vec{0}\vec{v}$ which isnt very interesting for eigen stuff.

* trivial solution is when $\vec{v} = \vec{0}$, but we dont want that the only way $v \neq 0$ is if there isnt a pivot in every column $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ corresponds to $v = 0$, if you dont want $v = 0$ you need a row of zeros, thus det has to be zero.

$$\boxed{\det(A - \lambda I_n) = 0}$$

this what we will use to find λ's

Diagonalization and Change of Basis

with Eigen values, Eigen vectors, Eigen basis, etc..

that doesn't transform vectors into eigen stuff that

(Gets you change of Basis for Eigen)

For Eigen values + Functions

- get basis for eigen space
  - ↳ get lin ind vectors that are eigenvectors
    - ↳ naturally comes w/ solving for them like we did on back.
- make those vectors into a matrix. (can do bec if they span you can turn into matrix by making it columns from $A = [v_1 \; v_2]$ from the $x_1\vec{v}_1 + x_2\vec{v}_2 = Ax = [v_1 \; v_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, mult out u see

Change of basis recap w/ matrix's:

ex

$B = \{B\text{'s basis vectors}\}$

$C = \{C\text{'s basis vectors}\}$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}}_{\substack{\text{(inverse change of basis vector)} \\ P_{C \leftarrow B}}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{(transformation)}} \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}_{\substack{\text{(change of basis vector)} \\ P_{B \leftarrow C}}} \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\substack{\text{(vector)} \\ [x]_C}}$$

(convert back to her lang "C") ③ (apply trans in our lang) ② (First transform to our language "B") ①

ex we found, $\left\{\lambda_1 = 3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \left\{\lambda_2 = 2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ (v_1, v_2)

corresponding to the matrix → $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

(lin ind & spans full space) → So basis $E = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

*IDK IF RIGHT BUT HAVE TO GO W/ IT *For sake of the

Changing basis so these eigen vectors are your basis vectors → you want them to be lin combo, span, so $E_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so from before you see → $P_{C \leftarrow E} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(so scales are relevant to cartesian) →

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}}_{\substack{\text{inverse of Eigen basis} \\ P_{E \leftarrow C}}} \underbrace{\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}}_{\substack{\text{transformation} \\ A \text{ in } C}} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{change of basis to Eigen basis} \\ P_{C \leftarrow E}}} v = (\lambda I_n) v = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} v$$

Eigen vector is naturally

$$S^{-1} A S = \Lambda$$

diagonal

always is diagonal because the whole point of doing this w/ eigenvectors is the basis vectors (v_1, v_2) get scaled during the transformation to this which corresponds

* transfer to eigenspace
* do transformation
* scaled vectors come out of basis vect
* scaled basis matrix always is diagonal

note not everything can become diagonal

George Baby

Eigenvalues and Eigenvectors

Let A be $n \times n$ throughout

A (complex) scalar, λ, is called an Eigenvalue of A, if there is a non-trivial vector $\vec{v}$ which satisfys $A\vec{v} = \lambda\vec{v}$

non trivial vectr V is Eigen vectr associated w/ λ.

$\{\lambda, \vec{v}\}$ eigen pairs

ex) let $A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$ is $\vec{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors

soln

$$A\vec{v}_1 = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 15+20 \\ 20+8 \end{bmatrix} = \begin{bmatrix} 35 \\ 28 \end{bmatrix} = 7\vec{v}_1$$

eigen pairs $\lambda = 7$, $\vec{v}_1$ = eigenvecor

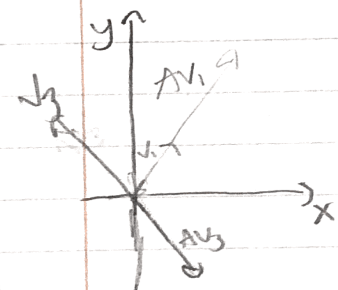
$$A\vec{v}_2 = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 - 5 \\ 16 - 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

not an eigenvecter

$$A\vec{v}_3 = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3+5 \\ -4+2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2V_3$$

yes eigen pairs

Contraction or dilation



let $\vec{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$

let $V_4 = 3V_1 = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$

then $AV_4 = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 15 \\ 12 \end{bmatrix} = \begin{bmatrix} 45+60 \\ 60+24 \end{bmatrix} = \begin{bmatrix} 105 \\ 84 \end{bmatrix} = 7V_4$

that is $\{\lambda = 7, \begin{bmatrix} 15 \\ 12 \end{bmatrix}\}$

scalars of eigen vectors will also be eigenvectors

$x_1 = -2/3$
$x_2 = 1/3$

Basic Fact: Suppose $\{\lambda, \vec{v}\}$ is eigen pair of A

for any scalar (could be complex) α $\{\lambda, \alpha\vec{v}\}$

$$A(\alpha\vec{v}) = \alpha(A\vec{v}) = \alpha(\lambda\vec{v}) = \lambda(\alpha\vec{v})$$

Finding Eigenvectors given, λ of A

if $\vec{v}$ is an Eigenvector

$$A\vec{v} = \lambda\vec{v} \rightarrow \lambda\vec{v} - A\vec{v} = \vec{0} \rightarrow$$

has form $Ax = 0$

$$\lambda I_n\vec{v} - A\vec{v} = \vec{0} \Rightarrow (\lambda I_n - A)\vec{v} = \vec{0}$$

$\therefore$ find all non trivial soln of this

$\Rightarrow$ find Nul Space of $(\lambda I_n - A)$

$\text{Nul}(\lambda I_n - A)$ = eigenspace a subspace $\mathbb{R}^n$

ex $\lambda = 3$, $\lambda = 2$ are eigenvalues

$\lambda = 3$

$$A = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix}, \quad \text{Nul}(3I_3 - A)$$

(because want be Identity m)

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & 2 \\ -1 & -1 & 1 \\ -3 & -6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\text{Nul}(3I_3 - A) = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right\}$

$x_1 = -2x_2 + x_3$
$x_2 = x_2$
$x_3 = x_3$

$$\vec{x} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$

might be 0

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix} = \text{in RREF} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Nul}(2I_3 - A) = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$v_1, \ldots v_k$ be k eigenvects of A which have distinct $\lambda_1, \ldots \lambda_k$ ie, $\lambda_i \neq \lambda_j$, $i \neq j$

then $\{v_1, \ldots v_k\}$ is lin ind

How do you find eigen values?

there's a 0 on diagonal or not a pivot in every column

5.2

$(A - \lambda I_n)\vec{v} = \vec{0}$ from inverse matrix theorem there is a non trivial soln to this is det = 0

det

$|A - \lambda I_n| = 0$ or $|\lambda I_n - A| = 0$

$P_\lambda(A) = |\lambda I_n - A|$

$|\lambda I_n - A| > 0$ will have n zeros, possibly repeated, possibly complex

these zeros, $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n$ will be eigen vals

ex

$A = \begin{bmatrix} 3 & 3 \\ 6 & -4 \end{bmatrix}$ $\rightarrow$ $|\lambda I_n - A| = 0$, $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ 6 & -4 \end{bmatrix}$

$= \begin{vmatrix} \lambda - 3 & -3 \\ -6 & \lambda + 4 \end{vmatrix} = 0$ ad-bc $\rightarrow$ $(\lambda+4)(\lambda-3) - 6 \cdot 3 = 0$

$\lambda^2 + \lambda - 12 - 18$

$\lambda^2 + \lambda - 30 = 0$

$(\lambda+6)(\lambda-5) = 0$

$\lambda_1 = -6$ $\lambda_2 = 5$

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $|\lambda I_2 - A| = 0$ becomes

$\begin{vmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{vmatrix}$ or $(\lambda-2)^2 - 1 = 0$, $\lambda^2 - 4\lambda + 3 = 0$ or $(\lambda-3)(\lambda-1) = 0$ $\lambda_1 = 1$ $\lambda_2 = 3$

for $\lambda = 1$: $Nul(1 \cdot I_n - A)$ $I_2 - A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $Nul = Span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

for $\lambda = 3$ $\Rightarrow Nul(3 I_n - A) \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow Nul = Span\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -2 & 2 \\ 2 & 0 & 0 \end{bmatrix} \Rightarrow |\lambda I_3 - A| = 0 \quad \begin{vmatrix} \lambda-1 & 3 & -3 \\ -2 & \lambda+2 & -2 \\ -2 & 0 & \lambda \end{vmatrix}$$

$$(-1)^{3+1} \cdot (-2)\left(\begin{vmatrix} 3 & -3 \\ \lambda+2 & -2 \end{vmatrix}\right) + 0 + (-1)^{3+3} \cdot \lambda \cdot \begin{vmatrix} \lambda-1 & 3 \\ -2 & \lambda+2 \end{vmatrix} = 0$$

$$-2\left[-6 + 3(\lambda+2)\right] + \lambda\left[(\lambda-1)(\lambda+2) + 6\right] = 0$$

or $\lambda(\lambda+2)(\lambda-1) = 0$, $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = 1$

$$-2 I_3 - A = \begin{bmatrix} -3 & 3 & -3 \\ -2 & 0 & -2 \\ -2 & 0 & -2 \end{bmatrix} \xrightarrow{\det} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}, \left\{-2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Note $\lambda = 0$ is an eigenvalue iff $|A| = 0$
solving $Ax = 0x$ of a square matrix.

if $\det(A) = 0$ that means there's a row of 0's, not $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
eigenvalues look for nontrivial solutions the only way
you can't have trivial soln for $Ax = 0x = \vec{0}$ is when it's not

$\rightarrow$ find eigenvalues, λ, by solving $A\vec{x} = \lambda\vec{x} \Rightarrow \vec{x}\underbrace{(A - \lambda I_n)}_{\text{matrix } P} = 0$
and setting $\det(P) = 0$ meaning since A is square, P is square
if you put it in rref form (or reduce) you'll get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ a row of
zeros: you don't want a pivot in every row bec that's only
trivial solution.

$\rightarrow$ find eigenvectors with associated λ, do $\vec{x}(A - \lambda I_n) = 0$ or $P\vec{x} = 0$
since thats just finding Nul space u call it that.
the vectors you get will be lin ind and span eigenspace

$\boxed{63}$

Given A, $n \times n$, A is diagonalizable if there is a nonsingular matrix S, such that

$$S^{-1}AS = \Lambda \text{, diagonal matrix}$$

S is similarity transformation w.r.t A

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

ex

$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, then $S = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ is similarity transformation of A

$$S^{-1} = \frac{1}{-2+1}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{check}$$

$$= \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}}_{S^{-1}} \underbrace{\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}}_{S^{-1}} \underbrace{\begin{bmatrix} 7-2 & 7-4 \\ -4-1 & -4-2 \end{bmatrix}}_{AS} = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}}_{S^{-1}} \underbrace{\begin{bmatrix} 5 & 3 \\ -5 & -6 \end{bmatrix}}_{AS}$$

$$= \underbrace{\begin{bmatrix} 10-5 & 6-6 \\ -5+5 & -3+6 \end{bmatrix}}_{S^{-1}AS} = \text{ and so } S^{-1}AS = \Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

so eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 3$

$\sigma(A) = \{3, 5\}$

- A is diagonalizable if it has n linearly independent vectors.
- if A has n lin ind eigenvectors then take $S = [\vec{V}_1\ \vec{V}_2, \dots \vec{V}_n]$ since columns of S are lin ind S has inverse.

ex $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ then $\lambda I_n - A = \begin{vmatrix} \lambda-7 & -2 \\ 4 & \lambda-1 \end{vmatrix} = P_\lambda(A) = 0 \rightarrow \begin{vmatrix} \lambda-7 & -2 \\ 4 & \lambda-1 \end{vmatrix} = 0$

then $\lambda^2 - \lambda - 7\lambda + 7 + 8 = \lambda^2 - 8\lambda + 15 = 0 = (\lambda-3)(\lambda-5) = 0$, $\sigma(A) = \{3, 5\}$

$3I_2 - A = \begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix}$ so $Nul(3I_2 - A) = \underbrace{\text{Span}\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}}_{\text{eigenvectors}}$ | $Nul(5I_2 - A) = \underbrace{\text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}_{\text{eigenvectors}}$ $\rightarrow S = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

Remark

$$S=\begin{bmatrix}-1&-1\\1&2\end{bmatrix} \quad \text{then} \quad S^{-1}AS=\overset{\Lambda}{\begin{bmatrix}5&0\\0&3\end{bmatrix}}$$

example: Diagonalize, if possible, the following matrix

$$A=\begin{bmatrix}5&-3&0&9\\0&3&1&-2\\0&0&2&0\\0&0&0&2\end{bmatrix}, \quad P_\lambda(A)=\begin{vmatrix}\lambda-5&-3&0&9\\0&\lambda-3&1&-2\\0&0&\lambda-2&0\\0&0&0&\lambda-2\end{vmatrix}$$

determinant of tri3 is its diagonals

$\lambda=2$

$|\lambda I_4-A|=0$, $(\lambda-5)(\lambda-3)(\lambda-2)(\lambda-2)$, $\lambda_1=5, \lambda_2=3, \lambda_3=2=\lambda_4$ mult of 2

$$2I_2-A=\begin{bmatrix}-3&3&0&-9\\0&-1&-1&2\\0&0&0&0\\0&0&0&0\end{bmatrix}\to\begin{bmatrix}-3&0&-3&-3\\0&-1&-1&-2\\0&0&0&0\\0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&1&1&-2\\0&0&0&0\\0&0&0&0\end{bmatrix}=(2I_2-A)$$

mult of 2 but still diagonalizable bcc $\dim(Nul(2I_2-A))=2 \rightsquigarrow$ Span $\left\{\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\2\\0\\1\end{bmatrix}\right\}$

$\lambda=5$

$$\begin{bmatrix}5&3&0&-9\\0&2&-1&2\\0&0&3&0\\0&0&0&3\end{bmatrix}\sim\begin{bmatrix}0&1&0&0\\0&0&+1&0\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad Nul(5\lambda_2-A)=\text{Span}\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

$\lambda=3$

$$3I_4-A=\begin{bmatrix}-2&3&0&-9\\0&0&-1&2\\0&0&1&0\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&-3/2&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}=(3I_2-A)$$

$$Nul(3I_4-A)=\text{Span}\left\{\begin{bmatrix}3\\2\\0\\0\end{bmatrix}\right\}$$

For $\{5,e_4\}, \left\{2,\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}\right\}, \left\{2,\begin{bmatrix}-1\\2\\0\\1\end{bmatrix}\right\}, \left\{3,\begin{bmatrix}3\\2\\0\\0\end{bmatrix}\right\}$

$$S=\begin{bmatrix}3&-1&0&-1\\2&2&0&-1\\0&0&0&1\\0&1&1&0\end{bmatrix}\to S^{-1}AS=\begin{bmatrix}3&0&0&0\\0&2&0&0\\0&0&5&0\\0&0&0&2\end{bmatrix}$$

to know where to put λ_i look at how we set up S