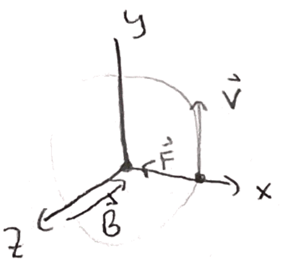
Magnetostatics

- $F_{mag} = Q(\vec{v} \times \vec{B})$ → In the presence of both $\vec{E}$ and $\vec{B}$ → $\vec{F} = Q[\vec{E} + \vec{v} \times \vec{B})$

Example



uniform circular motion

$$ma = \underbrace{\frac{mv^2}{R}}\hat{x} = Q(\vec{v} \times B) = -QvB\,\hat{x} \rightarrow \frac{mv^2}{R} = QvB$$

or $mv = p = QBR$

- Magnetic forces do no work

↳ $dW_{mag} = \vec{F}_{mag} \cdot \underbrace{d\vec{l}}_{\vec{v}\cdot dt} = Q\underbrace{(\vec{v}\times\vec{B})\cdot\vec{v}}_{\text{will be }\perp}dt = 0$

Magnetic forces may alter the direction in which the particle moves but they can't speed it up or slow it down

Current

$I = dq/dt$

line charge density $\lambda = \frac{dq}{dl}$ → $I = \lambda v$ (line charge λ traveling down a wire at speed v)

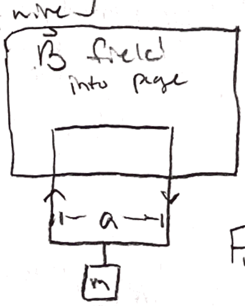
↳ $\vec{I} = \lambda \vec{v}$

$dF_{mag} = dq(\vec{v}\times\vec{B})$ → $\vec{F}_{mag} = \int(\vec{v}\times\vec{B})\,dq = \int(\vec{v}\times\vec{B})\lambda\,dl = \int(\vec{I}\times\vec{B})\,dl$

↳ $\vec{I}$ and $d\vec{l}$ point in the same direction so $\vec{F}_{mag} = \int I\,(d\vec{l}\times\vec{B})$

$\boxed{\text{If } I \text{ is constant along wire}}$ → $\vec{F}_{mag} = I\int(d\vec{l}\times\vec{B})$

ex



$\boxed{\text{For what current } I\text{, would the magnetic force upward cancel gravity}}$

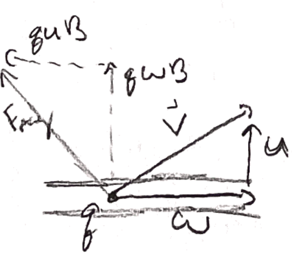
w/ RHR you want the current clockwise to have F_{mag} upwards

angle between I and B is $\perp$ so:

$$F_{mag} = \int_0^a IB\sin(90°)\,dl = IB\int_0^a dl = \underbrace{IBa} = mg \rightarrow I = \frac{mg}{Ba}$$

$W_B = IBah$? no its the battery

what is we increase I does B do work?



- as the loop rises the charge q gains an upward component u (speed of loop)
- it still has its velocity as it goes around the loop (w)
- $\vec{B}$ is always $\perp$ $\vec{v}$ by $\underset{\lambda\vec{v}}{\vec{I}}\times\vec{B}$
- v no longer points straight up, B is $\perp$ to the net displacement of charge (why it does no work)

$\boxed{\text{we still have force components}}$ →

$F_\perp = \lambda w a B$

$F_\parallel = \lambda u a B$

opposes the flow of current / battery has to push it back (work)

→ $W_{battery} = \int F_\parallel\,dl = \int F_\parallel\,w\,dt = \lambda a B\int u w\,dt$ (horz velocity; $I = \lambda w$)

$= IaB\underbrace{\int u\,dt}_{h} = IaBh$

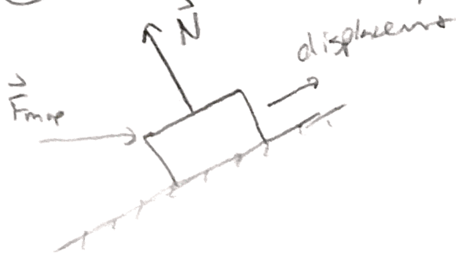
So $W_{batt} = IBah$

- work is done by the battery

(the battery is pushing, B field makes a ⊥ comp., opposes the motion too)

- the magnetic force redirected the horizontal force of the battery the vertical motion of the loop

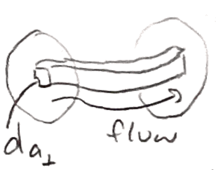
(ex) mechanical analogy



- sliding a trunk up a frictionless ramp
- I push horizontally w/ a mop
- $\vec{N}$ does no work bc it's ⊥ to the displacement
  ↳ it does have a vertical component tho (and a horz one) (like F_{mag} having a ⊥ component)
- you overcome the the horizontal component of the $\vec{N}$
- You are obviously doing the work here, even though your force is purely horizontal

- The normal force redirects your push (the battery) from horizontal to vertical (you need a vertical force component to get it up the wall!)

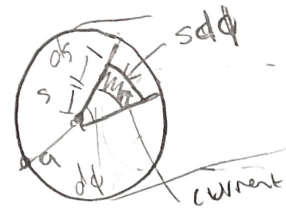
Volume current



($da_\perp$ runs parallel to the flow) $\rightarrow$ $\vec{J} \equiv \frac{d\vec{I}}{da_\perp}$, current per unit area

$I = \frac{dq}{dt}$ $\quad \rho = \frac{dq}{d\tau}$

$$\rightarrow \vec{J} = \frac{d}{da}\left(\frac{dq}{dt}\right) = \frac{d}{da}\left(\frac{d}{dt}(\rho\, d\tau)\right) = \rho \frac{dx}{dt}\frac{da_\perp}{da_\perp} = \rho\vec{v}$$



$$\vec{F}_{mag} = \int(\vec{v}\times\vec{B})\underbrace{\rho\, d\tau}_{dq} = \int(\vec{J}\times\vec{B})\,d\tau$$

Surface is

$\vec{K} = \frac{dI}{dl_\perp}$, $\vec{K} = \sigma\vec{v}$, $F_{mag} = \int(\vec{K}\times\vec{B})\,da$

* the charge density can be in any direction, you just want the length or area element perpendicular so you can integrate over the area or surface

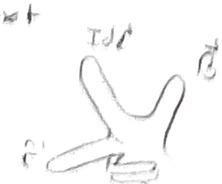
Biot-Savart law

Steady currents produce magnetic fields that are constant in time
(no charge pile ups, that it's been going on forever)

(a moving point charge can't constitute a steady current, if its here in one frame its gone in the next)

↳ $\frac{\partial \rho}{\partial t} = 0$ and $\frac{\partial \vec{J}}{\partial t} = 0$

Then: $\boxed{\vec{\nabla} \cdot \vec{J} = 0}$

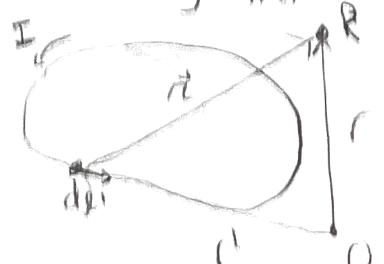


$1T = \frac{N}{A \cdot m}$

Biot-Savart law

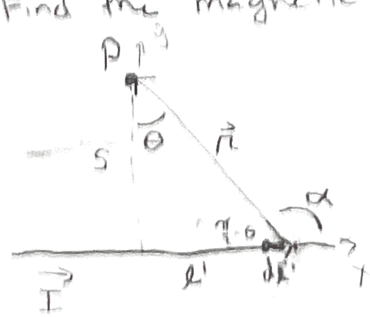
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{r}}{|r|^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{r}}{|r|^2}$$

$, \mu_0 = 4\pi \cdot 10^{-7} \, N/A^2$



example 5.5

• Find the magnetic field a distance s from a long straight wire with current I



$d\vec{l}' = dl' \hat{x}$

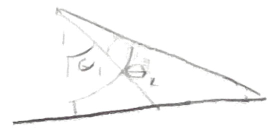
$\vec{r} = r\sin\theta\,\hat{x} + r\cos\theta\,\hat{y}$

$\hat{r} = \frac{\vec{r}}{r} = \sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dl & 0 & 0 \\ \sin\theta & \cos\theta & 0 \end{vmatrix} = 0\hat{x} - 0\hat{y} + dl\cos\theta\,\hat{z}$

$\boxed{d\vec{l}' \times \hat{r} = dl\cos\theta\,\hat{z}}$

$l' = s\tan\theta \rightarrow dl' = \frac{s}{\cos^2\theta} d\theta$ and $s = r\cos\theta \rightarrow \frac{1}{r^2} = \frac{\cos^2\theta}{s^2}$

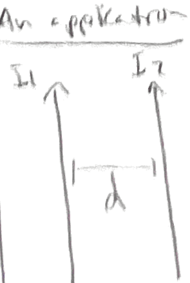


$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\cos^2\theta}{s^2} \frac{s}{\cos^2\theta} \cos\theta \, d\theta = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{1}{s} \cos\theta \, d\theta$$

$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi s}(\sin\theta_1 - \sin\theta_2)$, for the infinite wire let $\theta_1 = -\frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi s}}$$

An application: find the force of attraction between two long parallel wires



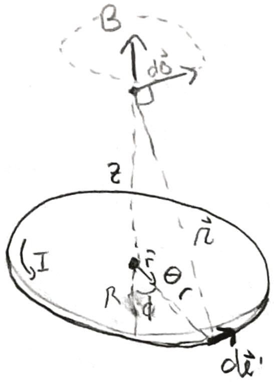
$\vec{F}_{mag} = \int (\vec{I} \times \vec{B}) dl$; $B_{1,2 \text{ found}} = \frac{\mu_0 I_1}{2\pi d} \rightarrow F = I_2 \left(\frac{\mu_0 I_1}{2\pi d}\right) \int dl$

$$\boxed{\frac{\text{Force}}{\text{length}} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}}$$

Example 5.6

- Find the magnetic field a distance z above the center of a ...

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}\times\hat{\mathscr{r}}}{|\mathscr{r}|^2}$$



$d\vec{l}' = dl\hat{\phi}$

and

$\vec{\mathscr{r}} = R\hat{r} + z\hat{z}$ and $|\mathscr{r}|^2 = R^2 + z^2 + \cancel{2Rz(\hat{r}\cdot\hat{z})}^{0}$ $\hat{r}\perp\hat{z}$

$$\hat{\mathscr{r}} = \underbrace{\frac{R}{\sqrt{R^2+z^2}}}_{R_m}\hat{r} + \underbrace{\frac{z}{\sqrt{R^2+z^2}}}_{z_m}\hat{z}$$

$$d\vec{l}\times\hat{\mathscr{r}} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ 0 & dl & 0 \\ R_m & 0 & z_m \end{vmatrix} = \frac{z}{\sqrt{R^2+z^2}}\,dl\,\hat{x} - 0\hat{\phi} - \frac{R}{\sqrt{R^2+z^2}}\,dl\,\hat{z}$$

- $$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{z\hat{r}}{(R^2+z^2)^{3/2}}\underset{\uparrow\, R\,d\phi}{dl} - \int_0^{2\pi}\frac{R}{(R^2+z^2)^{3/2}}\underset{\uparrow\, R\,d\phi}{dl}\,\hat{z} = -\frac{\mu_0 I}{4\pi}\frac{2\pi R^2}{(R^2+z^2)^{3/2}} = \boxed{\frac{-\mu_0 I R^2}{2(R^2+z^2)^{3/2}}}$$

minus sign wrong

$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

↳ $\cancel{\int\cos\phi\,d\phi}^{0} + \cancel{\int\sin\phi\,d\phi}^{0}$

(rest are constants)

Surface and Volume currents

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_l \frac{\vec{I}\times\hat{\mathscr{r}}}{|\mathscr{r}|^2}\,dl \quad , \quad I = \int \vec{J}\,da_\perp$$

$$= \frac{\mu_0}{4\pi}\int_l\int_{a_\perp}\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\underbrace{dl\,da_\perp}_{d\tau'} \rightarrow \boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'}$$

Likewise for $\vec{K}(\vec{r}')$: $\vec{B}(\vec{r}') = \frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,da$

↳ Always remember Biot Savart only holds for steady current

~~Infinite solenoid~~

$$\int\frac{a^2}{(a^2+z^2)^{3/2}}\,dz = \frac{z}{\sqrt{a^2+z^2}} \quad , I = nI$$

gence and Curl of B

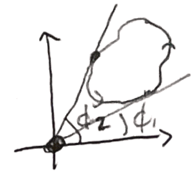
... the example of the B-field for an infinite wire we had → Using Biot Savart Law $\boxed{\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}}$

The field has a curl!

$\int(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$

Curl of a field is related to the line integral around it → $\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s}\, dl = \frac{\mu_0 I}{2\pi s}\underbrace{\oint dl}_{\text{circumference of circle}} = \mu_0 I$

The answer is independent of s, B decreases at the same rate as the circumference increases → notice if it didn't enclose the wire at all



$\int_{\phi_1}^{\phi_1} d\phi = 0$

If we had a bundle of wires each would contribute a $\mu_0 I$, $\underbrace{\oint \vec{B}_1\cdot d\vec{l} + \oint \vec{B}_2\cdot d\vec{l}}_{\text{same loop}} = 2\mu_0 I$ → so we need the number of wires enclosed, the ones that aren't are 0 → $\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}}$

$I_{encl} = \int \vec{J}\cdot d\vec{a}$

Continuing math:

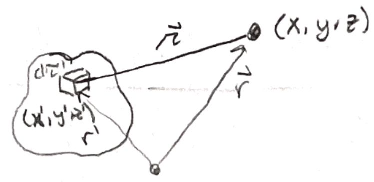
$\int(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0\int \vec{J}\cdot d\vec{a}$ → $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$ (only true for infinite straight wires) next is a formal proof

Divergence and curl of B

B is a function of (x, y, z)

J is a function of (x', y', z')

$\vec{\mathscr{r}} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$

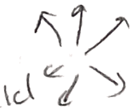


$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$

divergence

$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$ → $\vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\vec{\nabla}\times\vec{J}) - \vec{J}\cdot\underbrace{\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{\text{no curl, radial field}}$

unprimed coord / primed coord



$\vec{\nabla}\times$ whatever looks at the curl of unprimed coord, $\vec{J}$ is independent of that, so $\vec{\nabla}\times\vec{J} = 0$

Thus $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

Curl $\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$ ⇒ $\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{J}\underbrace{\left(\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{4\pi\delta^3(\vec{\mathscr{r}})} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}(\vec{\nabla}\cdot\vec{J})$

$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot|\vec{\nabla}\vec{J}| \to 0$

$\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\vec{\nabla}\right)\vec{J} - \cancel{(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} +$ involves terms like $\frac{\partial J_x'}{\partial x} = 0$

shown to be 0 on page 237

and $\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\, 4\pi\,\delta^3(\vec{r}-\vec{r}')\, d\tau'$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 J(\vec{r})}$$

Ampères Law

$\vec\nabla \times \vec B = \mu_0 \vec J$ (ampères law) $\to$ [integral form] $\to \int(\vec\nabla\times\vec B)\cdot d\vec a = \oint \vec B\cdot d\vec l = \mu_0 \int \vec J \cdot d\vec a = \mu_0 I_{enc}$

$\oint \vec B \cdot d\vec l = \mu_0 I_{encl}$, [RHR, fingers indicate direction of integration around the loop (Amperian loop), thumb indicates positive current]

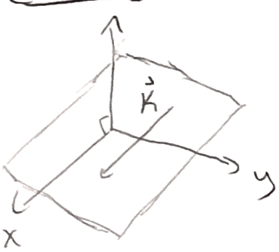
Ex

$$\oint \vec B\cdot d\vec l = B\oint dl = B\,2\pi s = \mu_0 I_{encl} = \mu_0 I$$

by RHR $\hat\phi$ $\hat\phi \leftarrow$ direction of loop

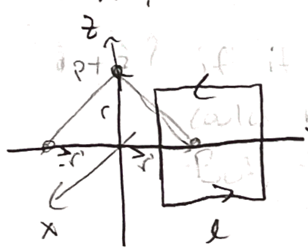
$$\boxed{B = \frac{\mu_0 I}{2\pi s}}$$

example Find the B-field of an infinite uniform surface current $\vec K = K\hat x$



Direction of $\vec B$

- $\hat x$? no $\vec B$ has to be $\perp$ to the current density $\vec K$

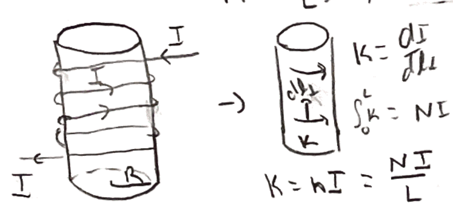


$\vec B$ $\hat z$ component cancels out by RHR for $-\vec r'$ and $+\vec r'$, pg 235

(do thumb way for a wire but when you start to curl your fingers keep it in the plane in that dir)

$$\oint \vec B\cdot d\vec l = 2Bl = \mu_0 I_{encl} = \mu_0 K l \to B = \frac{\mu_0 K}{2} \to \vec B = \begin{cases} \frac{\mu_0 K}{2}\hat y & \text{for } z<0 \\ -\frac{\mu_0 K}{2}\hat y & \text{for } z>0 \end{cases}$$

example consists of $n = \frac{N \leftarrow \text{total number}}{L = \text{length}}$ circular windings



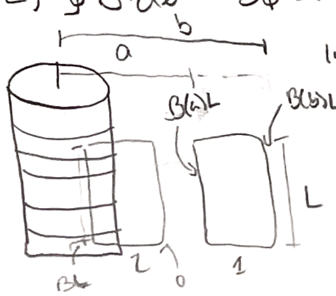
$K = \frac{dI}{dl_\parallel}$, $\int_0^L K = NI$, $K = nI = \frac{NI}{L}$

- Is there a B_s? If B_s is positive and we reverse the current B_s would be negative
  ↳ same as flipping the solenoid, which shouldn't change anything radial

B_ϕ?

- the amperian loop encloses no current
  ↳ current has to go through the loop not run parallel.

↳ $\oint \vec B\cdot d\vec l = B_\phi 2\pi s = \mu_0 I_{encl} = 0$, so $\boxed{B_\phi = 0}$, thus we only have a $\hat z$ component



loop 1: $\oint \vec B\cdot d\vec l = B(a) - B(b) L = 0 \to \boxed{B(a) = B(b)}$

* same everywhere outside the loop, should go to zero infinitely far away, thus its 0 everywhere outside loop,

loop 2: $\oint \vec B\cdot d\vec l = BL = \mu_0 (InL)$

$$\vec B = \begin{cases} \mu_0 n I \hat z & \text{inside solenoid} \\ 0 & \text{outside solenoid} \end{cases}$$

gnetic Vector Potential

$\vec{\nabla}\cdot\vec{B}=0 \rightarrow \vec{B}=\vec{\nabla}\times\vec{A}$ the divergence of a curl must always be 0

$\vec{A}$ vector potential

amperes law

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}=\mu_0\vec{J}$$

- for $\vec{E}$ you can add any function whos gradient is 0 (a constant) without altering the physical quantity $\vec{E}$
- for $\vec{B}$, you can add to $\vec{A}$ any function who's curl is 0

↳ for $\vec{E}$ $\quad \vec{\nabla}\times\vec{E}=0$ so $\rightarrow \vec{E}=-\vec{\nabla}V$, we can exploit this
gradient of scalar

for $\vec{A}$

so $\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$ → call $\boxed{\vec{\nabla}\cdot\vec{A}=0}$

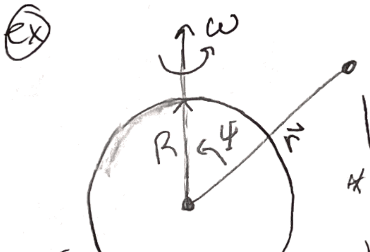
scalar

See pg 245 for line + surface currents

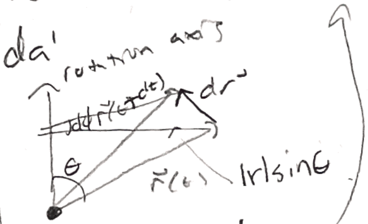
↳ thus $\boxed{\nabla^2\vec{A}=-\mu_0\vec{J}}$ $\xrightarrow{\text{poisson's eq}}$ $\boxed{\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{\mathscr{r}}\,d\tau'}$

Ex A spherical shell of radius R, carries a uniform surface charge σ, spinning at angular velocity $\vec{\omega}$

$d\vec{\phi}=\hat{\phi}\,d\phi$
$d\vec{r}=d\vec{\phi}\times\vec{r}$



$\boxed{\text{Find } \vec{A} \text{ at point } \vec{r}}$ $\quad \vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{K}}{\mathscr{r}}\,da'$



$dr=(r\sin\theta)\,d\phi$
arc length!! cross
then think of it as cross!!

current x-y axis so $\vec{\omega}$ lies in xz plane

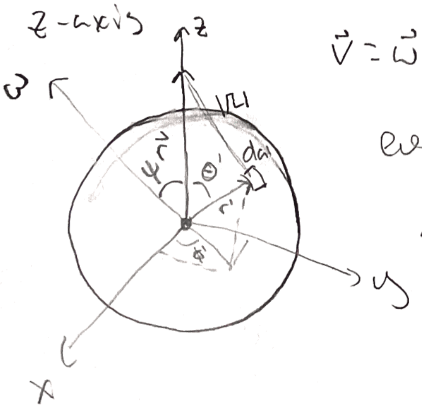
$\vec{K}=\sigma\vec{v} \rightarrow$ whats $\vec{v}$? $\quad \vec{v}=\vec{\omega}\times\vec{r}$

$\vec{r}'=R\sin\theta'\cos\phi'\,\hat{x}+R\sin\theta'\sin\phi'\,\hat{y}+R\cos\theta'\,\hat{z}$

$\vec{\omega}=\omega\sin\psi\,\hat{x}+\omega\cos\psi\,\hat{z}$

$\vec{v}=\vec{\omega}\times\vec{r}'=\begin{vmatrix}\hat{x} & & \\ & & \\ & & \end{vmatrix}=R\omega\left[-\cos\psi\sin\theta'\sin\phi'\,\hat{x}+(\cos\psi\sin\theta'\cos\phi'-\sin\psi\cos\theta')\hat{y}+\sin\psi\sin\theta'\sin\phi'\,\hat{z}\right]$

its easier to let $\vec{r}$ lie along the z-axis



every term w/ $\sin\phi'$ or $\cos\phi$ is zero $\quad \int_0^{2\pi}\sin\phi'\,d\phi'=0$ $\quad$ same for cos

$$A(\vec{r})=\frac{-\mu_0R^3\sigma\omega\sin\psi}{2}\left(\int_0^{\pi}\frac{\cos\theta'\sin\theta'\,d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\right)\hat{y}$$

since $\mathscr{r}=\sqrt{R^2+r^2-2Rr\cos\theta'}$

$da'=R^2\sin\theta'\,d\theta'\,d\phi'$

Integrating $(R+r)^2 = R^2 + r^2 + 2rR$, $(R-r)^2 = R^2 + r^2 - 2rR$

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2+r^2-2Rru}}\,du = -\frac{R^2+r^2+Rru}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Big|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\right]$$

if $R>r$: $\frac{2r}{3R^2}$; if $R<r$: $\frac{2R}{3r^2}$

$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2}\begin{cases} 2r/3R^2 & R>r \\ 2R/3r^2 & R<r \end{cases}\hat{y},$$

note: $\boxed{\vec{\omega}\times\vec{r} = -\omega r \sin\psi\,\hat{y}}$

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}), & \text{inside} \\ \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}), & \text{out} \end{cases}$$

"natural coordinates"

$\vec{\omega} = \omega\hat{z}$ $\quad \vec{\omega}\times\vec{r} = \omega r\sin\theta$ (dir?)

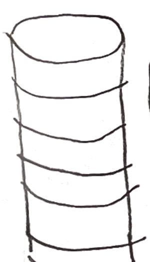
$\vec{r} = r\hat{r}$ $\quad \hat{z}\times\hat{r} = \hat{\phi}$

B inside

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) = \frac{2}{3}\mu_0\sigma R\omega\,\hat{z} = \frac{2}{3}\mu_0\sigma R\,\vec{\omega}$$

Solenoid w/ n turns per unit length, radius R and current I

→ take a look at $\oint \vec{A}\cdot d\vec{\ell} = \underbrace{\int(\underbrace{\vec{\nabla}\times\vec{A}}_{\vec{B}})\cdot d\vec{a}}_{\text{stokes}} = \int\vec{B}\cdot d\vec{a} = \Phi$ = flux of B through the loop



(wire inside)

[lets find the flux] → $\oint\vec{A}\cdot d\vec{\ell} = A(2\pi s) = \int\vec{B}\cdot d\vec{a} = \underbrace{\mu_0 n I}_{\text{field of B for solenoid}}(\pi s^2)$

$$\vec{A} = \frac{\mu_0 n I}{2}s\,\hat{\phi} \quad \text{for } s\le R$$

[Amperian loop outside] → $\int\vec{B}\cdot d\vec{a} = \mu_0 n I(\pi R^2)$ → $\vec{A} = \frac{\mu_0 n I}{2}\frac{R^2}{s}\hat{\phi}$ $\quad s\ge R$

all that needs to happen!)

$\vec{\nabla}\times\vec{A} = \vec{B}$

and

$\vec{\nabla}\cdot\vec{A} = 0$

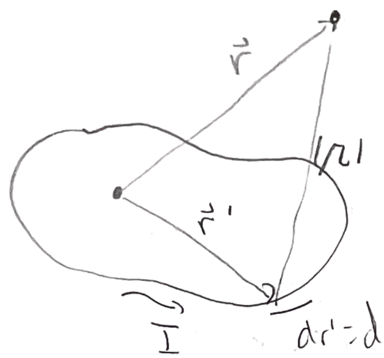
$\vec{A}$ has same dir as $\vec{I}$, or $\vec{J}$, $\vec{K}$

if all the current flows that way $\quad \vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{\kappa}d\tau$ (if it doesn't extend to ∞)

ltipole expansion

Remember → $\frac{1}{\eta} = \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\eta}\,d\vec{l}' = \frac{\mu_0 I}{4\pi}\sum \frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\,d\vec{l}'$$



$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r'\cos\alpha\,d\vec{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)d\vec{l}' + \dots\right]$$

$\oint d\vec{l}' = 0$ total vector displacement around a closed loop

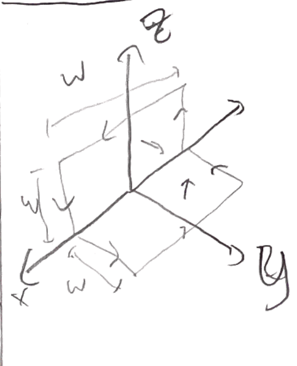
$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha\,d\vec{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}')\,d\vec{l}'$$

$\hat{r}\cdot\vec{r}' = r'\cos\alpha$, where α is the angle between $\vec{r}$ and $\vec{r}'$

↳ $\oint(\hat{r}\cdot\vec{r}')\,d\vec{l}' = -\hat{r}\times\int d\vec{a}'$

[vector area of the loop, direction by the RHR]

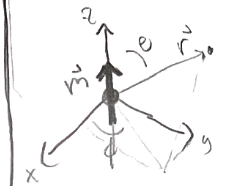
$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\left(-\hat{r}\times \underbrace{I\int d\vec{a}}_{\vec{m}}\right) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}, \quad m = I\int d\vec{a} = I\vec{a}$$



$\vec{m} = I(\vec{a})$, [can be thought of as a superposition of two planes] → $\vec{m}$

$\vec{m} = Iw^2\hat{y} + Iw^2\hat{z}$

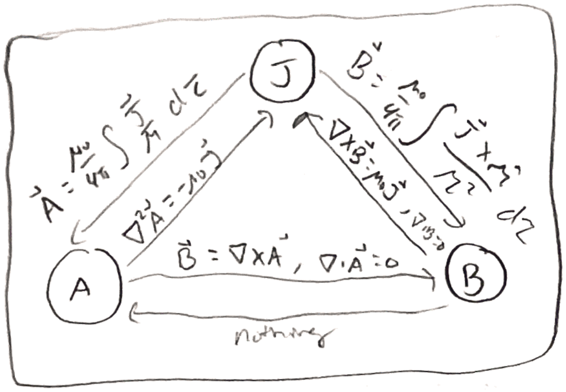
$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$, bc $\hat{z}\times\hat{r} = \hat{\phi}$

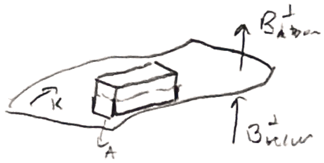


and $\boxed{\vec{\nabla}\times\vec{A} = \vec{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})}$

Boundary Conditions

$\vec{\nabla}\cdot\vec{B}=0 \rightarrow \oint \vec{B}\cdot d\vec{a}=0$





$\hookrightarrow (B^{\perp}_{above}-B^{\perp}_{below})A=0$

$$\boxed{B^{\perp}_{above}=B^{\perp}_{below}}$$

Tangential components

$\oint \vec{B}\cdot d\vec{l}=(B^{\parallel}_{above}-B^{\parallel}_{below})l=\mu_0 I_{encl}=\mu_0 K l$ $\quad : K=\frac{dI}{dl_{\perp}}$ to k

$\hookrightarrow$ $\boxed{B^{\parallel}_{above}-B^{\parallel}_{below}=\mu_0 K}$ (Just as the electric field suffers a discontinuity at a surface charge, B is discontinuous at surface current)

$\hookrightarrow \vec{B}_{above}-\vec{B}_{below}=\mu_0(\vec{K}\times\hat{n})$, $\hat{n}$ is unit vector $\perp$ to surface

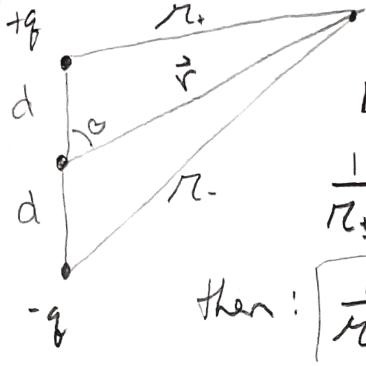
For the vector potential

$\vec{A}_{above}=\vec{A}_{below}$ (perp and $\parallel$ are continuous)

$$\frac{\partial\vec{A}_{above}}{\partial n}-\frac{\partial\vec{A}_{below}}{\partial n}=-\mu_0\vec{K}$$

multipoles

electric dipole : $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$



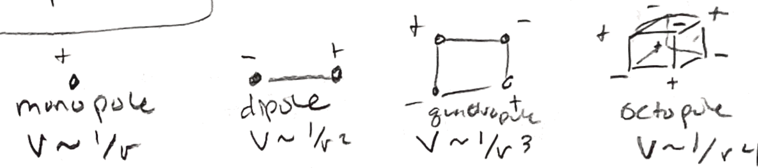
$\mathscr{r}_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp r d\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$

Lets go to the regime $r >> d$:

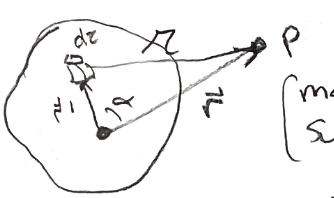
$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$

then : $\boxed{\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \approx \frac{d}{r^2}\cos\theta}$

$\to \boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}}$



Generalize this: w/ $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}} d\tau'$



$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$

(make the substitution) $\to \mathscr{r} = r\sqrt{1+\varepsilon}$, $\varepsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

$\to \frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \dots\right)$ pg 153

* If you sub ε back in and pull out corresponding power of $\left(\frac{r'}{r}\right)^n$ you get Legendre polynomials, so this has the form:

$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta) \to \boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(r')d\tau'}$

r is constant as far as the integration is concerned

physical pole

Potential is the sum of pure poles

The monopole & dipole terms

- the expansion is dominated by $V_{mon} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$ at large r.
- If the total charge is zero, the dominant term is the dipole term

↳ why isn't it always dominated by monopole term

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int \underbrace{r'\cos\alpha}_{\hat{r}\cdot\vec{r}'}\,\rho(\vec{r}')\,d\tau$$

, α is the angle between $\vec{r}$ and $\vec{r}'$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\int \vec{r}'\rho(\vec{r}')\,d\tau \quad , \quad \vec{p} = \int \vec{r}'\rho(\vec{r}')\,d\tau$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

[↳ for a collection of point charges] $\rightarrow \vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i'$

For dipole (q & $-q$): $\vec{p} = q\vec{r}_+' - q\vec{r}_-' = q\vec{d}$ ← vector from the negative charge to the positive charge

---

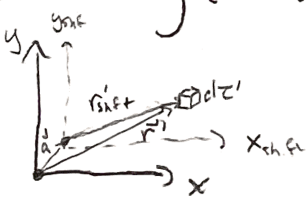
Moving the origin:

- lets say we displace the origin by an amount $\vec{a}$

$$\vec{p}_{shift} = \int \vec{r}_{shift}'\,\rho(\vec{r}')\,d\tau' = \int(\vec{r}' - \vec{a})\rho(\vec{r}')\,d\tau' = \int \vec{r}'\rho(\vec{r}')\,d\tau' - \vec{a}\int\rho(\vec{r}')\,d\tau'$$

↑ why w/ respect to old $\vec{r}'$

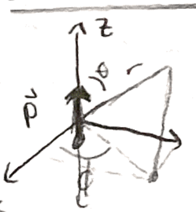
$$\vec{p}_{shift} = \vec{p} - Q\vec{a}$$



If $Q = 0$ then:

$$\vec{p}_{shift} = \vec{p}$$

Electric Field of a dipole



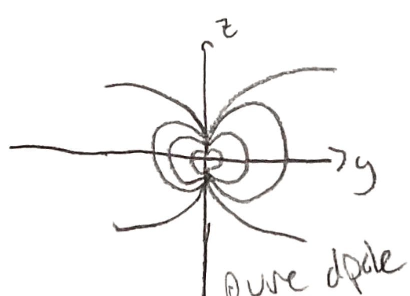
$$V_{dip}(r,\theta) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$ now do $-\vec{\nabla}V_{dip}$ in spherical

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}, \quad E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

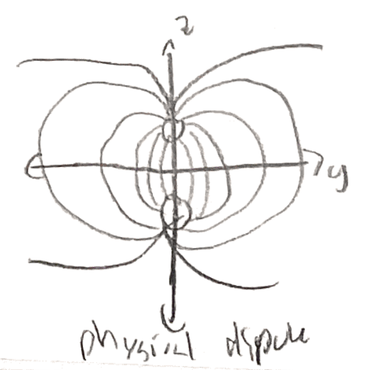
$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

↳ only holds for a physical dipole if $r \gg d$

↑ shortening d or big r



pure dipole



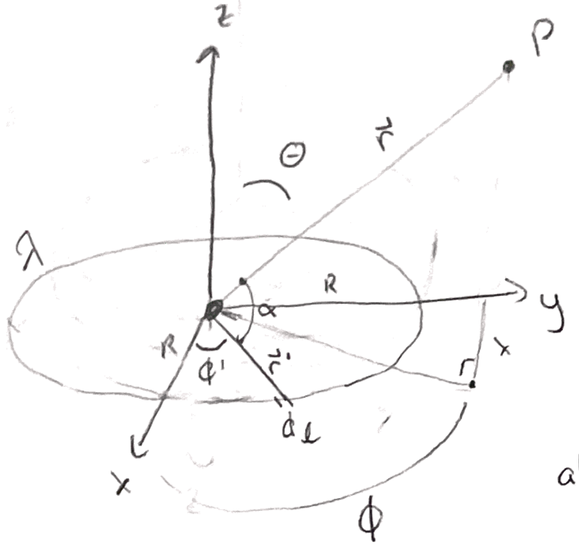
physical dipole

E+M dipole stuff

looking at the first few terms of

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(\vec{r}')\,d\tau'$$

α angle between $\vec{r}$ and $\vec{r}'$



Looking at first 3:

$$V_{mono} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\underbrace{\int \lambda(\vec{r}')\,dl'}_{Q = 2\pi r'\lambda} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^{2\pi} r'\cos\alpha\,\lambda\,\vec{r}'\,d\phi', \quad |r'| = R$$

we know: $\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$

$\vec{r}' = R\cos\phi'\,\hat{x} + R\sin\phi'\,\hat{y}$

also: $\vec{r}\cdot\vec{r}' = r\underset{R}{r'}\cos\alpha$, we can get $\cos\alpha$ in terms of ϕ'

$$\vec{r}\cdot\vec{r}' = rR(\sin\theta\cos\phi\cos\phi' + \sin\theta\sin\phi\sin\phi')$$

$$\vec{r}\cdot\vec{r}' = rR\cos\alpha$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^2}\int_0^{2\pi}(r')^2(\sin\theta\cos\phi\cos\phi' + \sin\theta\sin\phi\sin\phi')\,d\phi' = 0$$

essentially $\underbrace{\int_0^{2\pi}\cos\phi'\,d\phi'}_{=0} + \underbrace{\int_0^{2\pi}\sin\phi'\,d\phi'}_{=0}$

Quadripole:

$$V_{quad} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int (\overset{R}{r'})^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\lambda\,\underset{R\,d\phi'}{dl'}$$

, to compute, plug that expression in for $\cos\alpha$ again

* answer is it has a quadripole term *

Field at a distance d from a uniformly charged sphere

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$$

for $r > a$: $4\pi r^2 E = \frac{Q}{\varepsilon_0} \rightarrow \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

$r < a$: $4\pi r^2 E = \frac{Q_{encl}}{\varepsilon_0}$

$\int \rho\,d\tau$ $\int \rho\,d\tau$

$Q_{encl} = \gamma Q$

$\rho\frac{4\pi}{3}r^3 = \gamma\frac{4\pi}{3}a^3\rho$ $\quad \gamma = \frac{r^3}{a^3}$

$$\boxed{\vec{E} = \frac{Qr}{4\pi\varepsilon_0 a^3}\hat{r}}$$

?

does it match with $\rightarrow \vec{E}_{dip} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

$$= \frac{2p}{4\pi\varepsilon_0 r^3}$$