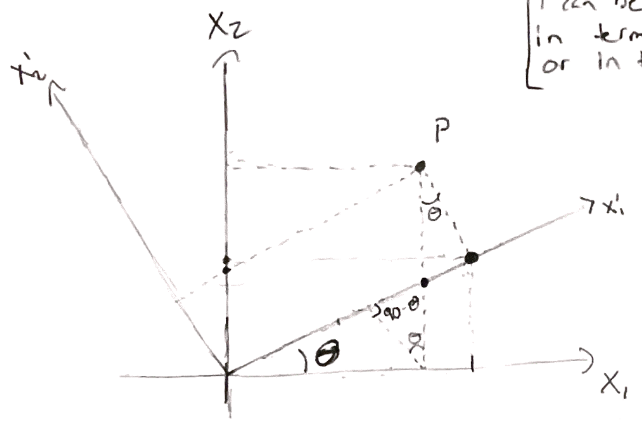
Coordinate Transformations

Class notes



[P can be written in terms of (X_1, X_2) or in terms of (X_1', X_2')]

General Definitions

$$\begin{bmatrix} X_1' = \alpha X_1 + \beta X_2 \rightarrow \vec{X}_1' = \alpha \hat{X}_1 + \beta \hat{X}_2 \\ X_2' = -a X_1 + b X_2 \end{bmatrix}$$

↑[negative sign prob not needed]

(answer)

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} X_1' + \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} X_2'$$

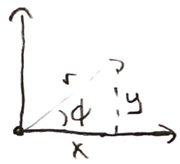
$$\vec{P} = a \cdot \hat{X}_1 + b \hat{X}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$$

also

$$\vec{P} = c \cdot \hat{X}_1' + d \hat{X}_2' = \begin{bmatrix} c \\ d \end{bmatrix}$$

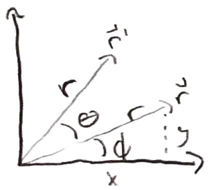
eh

(Deriving the rotation Matrix)



$$\vec{r} = r\cos(\phi)\hat{x} + r\sin\phi\,\hat{y} = \begin{array}{c} \begin{bmatrix} r\cos(\phi) \\ r\sin(\phi) \end{bmatrix} \end{array} \begin{array}{l} \leftarrow x \\ \leftarrow y \end{array}$$

Now rotate it up



$$\vec{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix} = \begin{bmatrix} r\cos(\phi)\cos\theta - r\sin(\phi)\sin\theta \\ r\sin\theta\cos\phi + r\sin\phi\cos(\theta) \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

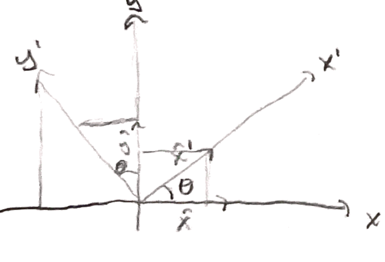
for me: $\vec{r}' = (x\cos\theta - y\sin\theta)\hat{x} + (x\sin\theta + y\cos\theta)\hat{y}$

$$\vec{r}' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Could be wrong

This Makes Sense to Me!

Looking at where $\hat{x}$ and $\hat{y}$ go.



$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow e_1 = \begin{array}{c} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \hat{x} \\ \leftarrow \hat{y} \end{array}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{R} e_2 = \begin{array}{c} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \hat{x} \\ \leftarrow \hat{y} \end{array}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$$

when $R = x\hat{e}_1 + y\hat{e}_2$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

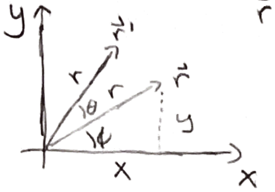
Coordinate Transformations

Rotating a coordinate system

$$\vec{r} = r\cos(\phi)\,\hat{x} + r\sin\phi\,\hat{y} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} \begin{matrix} x \\ y \end{matrix}$$

now rotate it

$$\vec{r}\,' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix} = \begin{bmatrix} \overbrace{r\cos\phi}^{x}\cos\theta - \overbrace{r\sin\phi}^{y}\sin\theta \\ r\sin\theta\cos\phi + r\sin\phi\cos\theta \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} x + \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} y$$

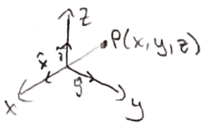


$$\vec{r}\,' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Cartesian

$\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$

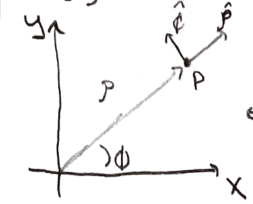
$|\vec{r}| = \sqrt{x^2+y^2+z}$



to →

Polar/Cylindrical

$\vec{r} = \rho\hat{\rho} + z\hat{z}$



z out of page ⊙

$P(\rho, \phi, z)$

$x = \vec{r}\cdot\hat{e}_1$
$y = \vec{r}\cdot\hat{e}_2$
$z = \vec{r}\cdot\hat{e}_3$

$\rho = \sqrt{x^2+y^2}$
$\phi = \tan^{-1}(y/x)$

Unit vectors $\hat{\rho}, \hat{\phi}$ in cartesian Polar

1. $\hat{\rho} = ? \rightarrow \vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$

$\left.\begin{matrix} x = \rho\cos\phi \\ y = \rho\sin\phi \end{matrix}\right]$ $\vec{r} = \underbrace{\rho\cos\phi\,\hat{e}_1 + \rho\sin\phi\,\hat{e}_2}_{\rho\hat{\rho}} + \underbrace{z\hat{e}_3}_{+z\hat{z}}$

$\boxed{\hat{\rho} = \cos\phi\,\hat{e}_1 + \sin\phi\,\hat{e}_2}$

Unit vector for $\hat{\phi}$ → $\hat{\phi} = \alpha\hat{e}_1 + \beta\hat{e}_2$, $\hat{\phi}\cdot\hat{\phi} = 1$, $\hat{\rho}\cdot\hat{\phi} = 0$

$\hat{\phi}\cdot\hat{\phi} = 1 = \alpha^2 + \beta^2$

$\hat{\rho}\cdot\hat{\phi} = 0 = \rho\cos\phi\,\alpha + \rho\sin\phi\,\beta$

→ $\alpha = -\beta\tan\phi$ → $1 = \beta^2\tan^2\phi + \beta^2 \rightarrow 1 = \beta^2\underbrace{(\tan^2\phi + 1)}_{\frac{1}{\cos^2\phi}} \rightarrow$

$\beta = \pm\cos\phi$ (take plus so ccw is positive)

$\alpha = -\sin\phi$

THUS! $\boxed{\hat{\phi} = -\sin\phi\,\hat{e}_1 + \cos\phi\,\hat{e}_2}$

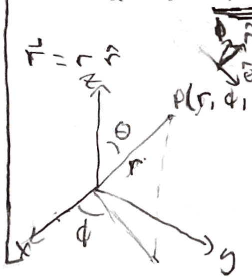
Then!

$$\boxed{\begin{pmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}}$$

← from $\begin{matrix} \hat{\rho} \rightarrow \\ \hat{\phi} \rightarrow \\ \hat{z} \rightarrow \end{matrix} \begin{bmatrix} \cos\phi \\ -\sin\phi \\ 0 \end{bmatrix} \hat{e}_1 + \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \hat{e}_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \hat{e}_3$

Cartesian to Spherical SPHERICAL

$\vec{r} = r\hat{r}$



From geometry

$x = r\sin\theta\cos\phi$ $r = \sqrt{x^2+y^2+z^2}$
$y = r\sin\theta\sin\phi$ $\cos\theta = z/r$
$z = r\cos\theta$ $\tan\phi = y/x$

Looking at Cartesian $\vec{r}$

$\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$ plug in x, y, z

$\vec{r} = r\sin\theta\cos\phi\,\hat{e}_1 + r\sin\theta\sin\phi\,\hat{e}_2 + r\cos\theta\,\hat{e}_3$

1. pull out r and then get $\hat{r}$ by $\vec{r} = r\hat{r}$

$\boxed{\hat{r} = \sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3}$

normal to the $\hat{e}_3, \hat{r}$ plane

$\hat{\phi}$: think about $\hat{e}_3 \times \hat{r} = |\hat{e}_3||\hat{r}|\sin\theta\,\hat{\phi} \rightarrow \hat{\phi} = \dfrac{\hat{e}_3\times\hat{r}}{\sin\theta} = \dfrac{1}{\sin\theta}\begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 0 & 1 \\ S_\theta C_\phi & S_\theta S_\phi & C_\theta \end{vmatrix} = -\sin\phi\,\hat{e}_1 + \cos\phi\,\hat{e}_2$

$\boxed{\hat{\phi} = -\sin\phi\,\hat{e}_1 + \cos\phi\,\hat{e}_2}$

$\hat{\theta} = \hat{\phi}\times\hat{r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -S_\phi & C_\phi & 0 \\ S_\theta C_\phi & S_\theta S_\phi & C_\theta \end{vmatrix} = \boxed{\cos\theta\cos\phi\,\hat{e}_1 + \cos\theta\sin\phi\,\hat{e}_2 - \sin\theta\,\hat{e}_3 = \hat{\theta}}$

(In matrix Form) →

$$\boxed{\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}}$$

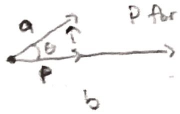
$$\rightarrow \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix}$$

distance formula $= |\vec{r} - \vec{s}|$

dot product intuition $\rightarrow a \cdot b = |a| \cdot |b| \cos\theta$

and $\cos\theta = \frac{|p|}{|a|} \rightarrow a \cdot b = |p| \cdot |b|$

↳ it says how much of a's b component (p) is going into b (if its 0 their ⊥)



if you pull a box 10 meters at an inclined **angle** theres a horizontal and vertical component to Force
- dot product gives amount of force going in the direction of displacement
  ↳ which is what work only cares about ($\vec{F} \cdot d\vec{r}$)

Mechanics Intro Notes

refresher math

Definition of a vector: $r = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{i} + y\hat{j} + z\hat{k}$ (position vector r, components x, y, z)

↳ velocity → V_x, V_y, V_z components, note $e_1 = \hat{x}$, $e_2 = \hat{y}$, $e_3 = \hat{z}$

linear combo: $r = r_1 e_1 + r_2 e_2 + r_3 e_3 = \sum_{i=1}^{3} r_i e_i$ or $r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$, $c\vec{r} = \begin{bmatrix} cr_1 \\ cr_2 \\ cr_3 \end{bmatrix}$

(ex) $F = ma$ says scalar vector, if m is constant

$\boxed{\text{dot product}}$ $\vec{r} \cdot \vec{s} = r_1 s_1 + r_2 s_2 + r_3 s_3$ or $\vec{r} \cdot \vec{s} = rs\cos\theta$

$\vec{r} \cdot \vec{e}$ ← unit vector, $\vec{r} \cdot \vec{s} = \vec{r}^T \vec{s} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = r_1 s_1 + r_2 s_2 + r_3 s_3$

$\|\vec{r}\| = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{r_1^2 + r_2^2 + r_3^2}$, unit vector $\frac{\vec{r}}{\|\vec{r}\|}$, dist between points → $\|r - s\|$

↳ $\cos\theta = \frac{\vec{r} \cdot \vec{s}}{|r||s|}$ (same thing as above)

$\boxed{\text{Cross product}}$ $\vec{r} \times \vec{s} =$ ↳ $\det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix}$ or $\vec{p} = \vec{r} \times \vec{s} = \begin{cases} p_x = r_y s_z - r_z s_y \\ p_y = r_z s_x - r_x s_z \\ p_z = r_x s_y - r_y s_x \end{cases}$

Vector ⊥ to both $\vec{r}$ and $\vec{s}$

$|\vec{r} \times \vec{s}| = rs\sin\theta$, $\frac{d}{dt}(a \times b) = \frac{da}{dt} \times b + a \times \frac{db}{dt}$ (product rule),

define: $F(x) = (f \circ g)(x)$ [$f(g(x))$] the $F'(x) = f'(g(x))\,g'(x)$ (outside, inside fun)

product rule: $\frac{d}{dt}(f(t)\,r(t)) = f\frac{dr}{dt} + \frac{df}{dt}r$, chain rule: if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\boxed{\text{chain rule example}}$, $F(x) = e^{x^2}$, $f(x) = \text{outer}: e^x$, $g(x) = \text{inner}: x^2$, $y = f(u) = e^u$, $u = g(x) = x^2$, $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2xe^{x^2}$

$\boxed{\text{Velocity}}$ if $r = x\hat{x} + y\hat{y} + z\hat{z}$ then $\frac{dr}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} = V = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$

[Unit vectors $\hat{x}, \hat{y}, \hat{z}$ dont depend on time] → [if unit vectors depend on time like in other coordinate systems its not simple] → [polar coordinates this is not true so you use product rule] → NOT just derivative of each of components!

$\boxed{\text{Force}}$ Newton, 1 N is magnitude of force that accelerates a kg w/ an acceleration of $1\frac{m}{s^2}$

↳ direction of Force is the direction of resulting acceleration

pg 30 for reference frames

$\boxed{\text{Newtons Laws}}$ → point masses → bodies w/ rotation, if 2 frames are moving relative to eachother accelerating Newtons laws dont hold

↳ if a second frame is accelerating or rotating relative to an inertial frame, then this second frame is non-inertial

First Law → In the absence of forces particle moves w/ constant V, Second Law → $F = ma = m\dot{v} = m\ddot{r}$

Second Law rephrased → $p = mv$, $\dot{p} = m\dot{v} = ma$ so $F = \dot{p}$

[Newtons Laws for a particle undergoing a constant force] → [F constant in x dir, so $\ddot{x}(t) = \frac{F_0}{m}$] → $\dot{x}(t) = \int \ddot{x}(t)\,dt = V_{initial} + \frac{F_0}{m}t$, with $\dot{x}(0) = V_{initial}$

$x(t) = \int \dot{x}(t)\,dt = x_0 + V_0 t + \frac{F_0}{2m}t^2$

Third Law – [F_{21} is force on 2 by 1, F_{12} is force on 1 by 2]

if object 1 exerts a force on object 2 (F_{21}) then object 2 always exerts a reaction force (F_{12}) given by

$$\boxed{F_{12} = -F_{21}}$$

1 F_{12}

2 F_{21}

[if you push on a wall you dont fall over so there is a reaction force by the wall due to molecular forces]

More dictative look into Newtons 3rd Law on back

* a non inertial RF where first and second law dont hold

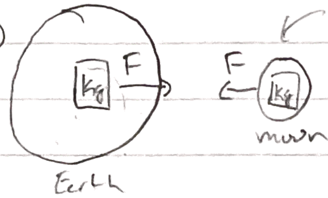
because if the train is moving at constant velocity the ice cube will not move

* if the train accelerates the icecube will fly back even though it experiences no net force! non-inertial RF, same goes for a rotating merry go round

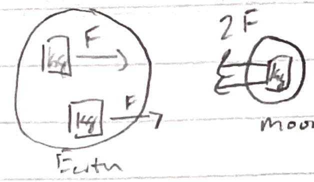
[Earth is a non-inertial RF but can be approximated] on small time + length scales to be inertial

Newton's 3rd law a deeper approach

[a couple examples] gravitational example →

① 

equal and opposite

② 

$-2F = 2F$

$-(2ma) = 2(ma)$

earth?

[in 2 by adding an extra kg the force double on the 1 kg for moon but earth also feels double the force but over 2 kg]

↳ the earth will accelerate less bec it has more mass, more inertia } but feels the same force! Equal and Opposite

[The earth pulls on moons mass but moon pulls on all of earths mass in such a way its equal and opposite, since earths bigger its more spread out + has more inertia causing less acceleration]

Conservation of momentum from third law

(no external forces!)

Quick look

$$F_{12} = -F_{21} \quad \text{and} \quad F = \dot{p} \quad \text{so,} \quad \frac{dp_{12}}{dt} = -\frac{dp_{21}}{dt} \rightarrow \frac{d(p_{12} + p_{21})}{dt} = 0$$

p_1 p_2 for easiness

→ if the rate of change of this sum is zero, which means $p_1 + p_2$ is constant

$p_1 + p_2 = p_{tot} = m_1 v_1 + m_2 v_2$

two particle sys

(net force on 1) ≡ $F_1 = F_{12} + F_1^{ext}$ ← external force is like the sun on earth, moon system or wind by ice skaters

(net force on 2) ≡ $F_2 = F_{21} + F_2^{ext} = \dot{p}_2$ and same for $F_1 = \dot{p}_1$

↳ $p_{tot} = p_1 + p_2$, $\dot{p}_{tot} = \dot{p}_1 + \dot{p}_2 = F_1^{ext} + F_2^{ext}$, $(F_{12} = -F_{21})$,

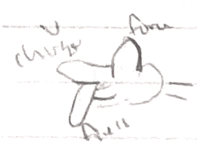
then we can just say $\dot{p}_{tot} = F^{ext}$ (total external force)

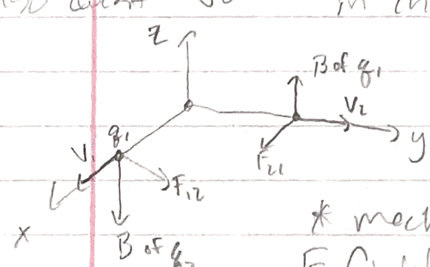
[special case is $F^{ext} = 0$ and P=constant conservation of momentum!]

this is true for multi-particle systems, proof on page 36 of Taylor

*third law is not true in relativity

also doesnt work in charge particles in B field





[Each of the positive charges produces a B field that exerts a force on the another charge]

But arent equal and opposite!

* mechanical momentum isnt only kind of momentum

E fields can also carry momentum (negligible in vacc)

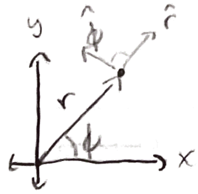
So its ok in low speeds we are good in classical mechanics

back to normality

* solving problems in non-cartesian simplifies things very easily

$$\begin{rmatrix} x = r\cos\phi \\ y = r\sin\phi \end{rmatrix}$$

$$\left.\begin{matrix} x = r\cos\phi \\ y = r\sin\phi \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} r = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \end{matrix}\right.$$



Polar Coordinates

unit vectors → $\hat{r}$ and $\hat{\phi}$

$\hat{r}$: direction moving r while ϕ is fixed

$\hat{\phi}$: direction we move when ϕ increases and r is fixed

$\hat{r} = \frac{r}{|r|}$

ex) twirling a stone in circle, F_r is tension and F_ϕ = air resistance

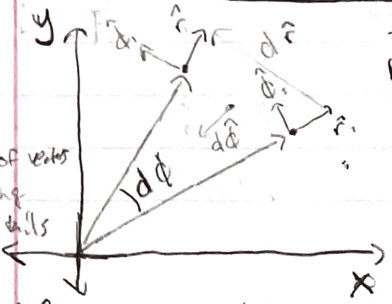
$\hat{r}$ and $\hat{\phi}$ are linearly independent ⟶ $\boxed{F = F_r\hat{r} + F_\phi\hat{\phi}}$

$\hat{r}$ changes as r moves so its not constant like cartesian, find $\frac{d\hat{r}}{dt}$

$\vec{r} = r\hat{r}$

$\frac{d\vec{r}}{dt} \neq \dot{r}\hat{r}$

doing Δ of vectors is connecting heads from tails



to find $\frac{d\hat{r}}{dt}$ algebraically use def of polar coord.

$$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$$
$$\frac{d\hat{r}}{dt} = \hat{x}\frac{d}{dt}(\cos\phi) + \hat{y}\frac{d}{dt}(\sin\phi)$$
$$\frac{d\hat{r}}{dt} = \hat{x}\left[-\sin\phi\frac{d\phi}{dt}\right] + \hat{y}\left[\cos\phi\frac{d\phi}{dt}\right]$$
$$\boxed{\frac{d\hat{r}}{dt} = \frac{d\phi}{dt}\hat{\phi}}$$ wow!

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$
$$\frac{d\hat{\phi}}{dt} = -\hat{x}\cos\phi\frac{d\phi}{dt} - \hat{y}\sin\phi\frac{d\phi}{dt}$$
$$\frac{d\hat{\phi}}{dt} = -\frac{d\phi}{dt}(\hat{x}\cos\phi + \hat{y}\sin\phi)$$
$$\boxed{\frac{d\hat{\phi}}{dt} = -\hat{r}\frac{d\phi}{dt}}$$

triple product rule

$\hat{x}, \hat{y}$ are mag = 1 and dont change direction

not vector $\dot{r}$

Product rule on $\vec{r} = r\hat{r}$: $V = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \Rightarrow \vec{V} = \underbrace{\frac{dr}{dt}}_{V_r}\hat{r} + \underbrace{\frac{d\phi}{dt}r}_{V_\phi}\hat{\phi}$

$$\left\{\begin{matrix} V_r = \dot{r} \\ V_\phi = \dot{\phi} r = \omega r \end{matrix}\right.$$

angular velocity (ω)

$\boxed{\text{To write Second Law: } \frac{d}{dt}V}$ → $a = \dot{\vec{V}} = \ddot{\vec{r}} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$

↳ $\boxed{\text{Product RULE}}$ → $a = \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\phi}\hat{\phi}) = \left(\dot{r}\frac{d\hat{r}}{dt} + \ddot{r}\hat{r}\right) + \left((\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt}\right)$

sub in!

↳ $\boxed{\text{sub in } \frac{d\hat{r}}{dt} \text{ and collect } \hat{r}, \hat{\phi}}$ →

$$\boxed{\vec{a} = (\underbrace{\ddot{r}}_{\text{acceleration as r varies}} - \underbrace{r\omega^2}_{\text{centripetal acceleration}})\hat{r} + (\underbrace{r\alpha}_{\text{tangential acceleration}} + \underbrace{2\dot{r}\omega}_{\text{coriolis acceleration}})\hat{\phi}}$$

angular velocity (ω), angular acceleration (α)

radial stuff; stuff ⊥ r

holy moly thats cool!

← ⊥ r or tangent

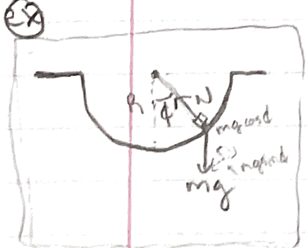
$* \ r\omega^2 = \frac{v^2}{r}$

$\boxed{\begin{matrix} 2^{nd} \text{ Law} \\ F = ma \end{matrix}} \Leftrightarrow \left\{\begin{matrix} F_r = m(\ddot{r} - r\omega^2) \\ F_\phi = m(r\alpha + 2\dot{r}\omega) \end{matrix}\right.$

$* \ \alpha = \ddot{\phi}$

if r is constant like in most problems → $a = -r\omega^2\hat{r} + r\alpha\hat{\phi}$

ex)



• r is constant

↳ $\vec{a} = -r\omega^2\hat{r} + r\alpha\hat{\phi}$

* how long will it take to come back to point of release *

$$\left\{\begin{matrix} F_r = m(-R\omega^2) \rightarrow \sum F_r = mg\cos\phi - N = -mR\omega^2 \\ F_\phi = m(R\alpha) \rightarrow \sum F_\phi = -mg\sin\phi = mR\alpha \end{matrix}\right.$$

or ϕ for small angles

↳ $\alpha = \ddot{\phi} = -\frac{g}{R}\sin\phi$, if you say $\omega = \sqrt{\frac{g}{R}}$ → $\ddot{\phi} = -\omega^2\sin\phi = -\omega^2\phi$

$\ddot{\phi} = -\omega^2\phi$ is a solveable diff eq of form $\phi(t) = A\sin(\omega t) + B\cos(\omega t)$ at t=0 $\phi = B$, $B = \phi_0$ and $\phi(t)$ eq implies $\dot{\phi} = \omega A$, $\dot{\phi} = 0$ so $A = 0$ and $\phi(t) = \phi_0\cos(\omega t)$, returns to ϕ_0 when $\omega t = 2\pi$

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$

[Sorry Newtons 3rd Law Again]

ex 1: If I give you \$10, I am \$10 poorer, and you are \$10 better off. So how does anyone get rich?
→ If we sum +\$10 and −\$10 = \$0, mistake is thinking this applies to the same person. They dont.

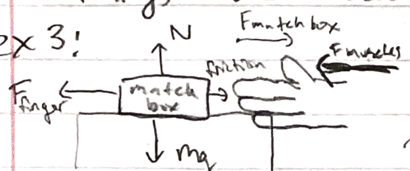
ex 2: sitting on wheelie chair, if you push on wall, wall pushes on you and you roll backwards
↳ or in a swimming pool, you are at the bottom

you push the wall

[By 3rd law you push it equally and it equally pushes on you. The wall/Earth has inertia so it barely moves (reality the wall flexes). But your small so you fly back.]

~ equilibrium can only establish itself when forces act on same object (1st Law)
↳ things move because force is applied to the other object.

ex 3:



$0 = N - mg$ (thats first law thing)

$F_{finger} = -F_{matchbox}$ (3rd Law)

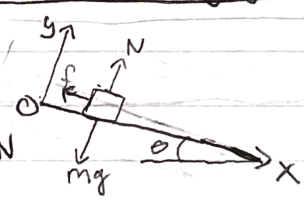
~ person doesnt move bec of friction, if no friction (and ratio of mass to Force) ↔ person would in opposite direction (less bec of inertia) $\frac{F}{m} = a$

F = ma in Coordinates

$\left(F = m\frac{d^2 r}{dt^2}\right)$

• finding components ($F = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$) and ($r = x\hat{x} + y\hat{y} + z\hat{z}$) (constants, constants)
↳ $\ddot{r} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$ so $F = m\ddot{x}\hat{x} + m\ddot{y}\hat{y} + m\ddot{z}\hat{z}$

$F = m\ddot{r} \leftrightarrow \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases}$



RF: where it starts and y normal to the plane, otherwise x and y would vary, also f and N will have a component = 0

ex block sliding on incline plane $f = \mu N$
How far will it travel in time t?

• no motion in y-direction, $\ddot{y} = 0$ therefore $F_y = m\ddot{y} = 0$, y direction is in equilibrium
→ $F_y = N - mg\cos\theta = 0$ → $N = mg\cos\theta$ thus $f = \mu N = \mu mg\cos\theta$

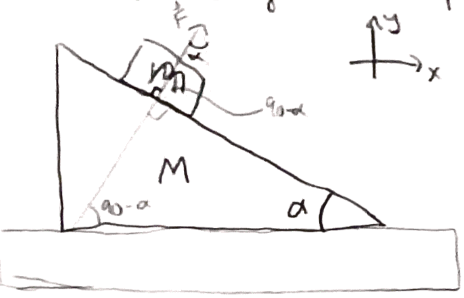
Now: $F_x = m\ddot{x} = \underbrace{mg\sin\theta - \mu mg\cos\theta}_{\text{all these are constants, dont depend on x or t}}$ (weight$_x$ − f_x) → $\ddot{x} = g(\sin\theta - \mu\cos\theta)$ (m's cancel)

$x(t) = \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2$ ↳ for integration and initial cond, $\begin{pmatrix}\dot{x} = 0 \text{ at } t = 0 \\ x = 0 \text{ at } t = 0\end{pmatrix}$

Book notes (1.1)

a block sliding on a plane (Block slides down and Inclined plane can move

and SB = small block



F_1 = normal force of m on inclined plane (IP)

• $\vec{F}_1 = F_1 \sin(\alpha)\hat{\imath} + F_1\cos(\alpha)\hat{\jmath}$

• $\vec{F}_{gravity(SB)} = -mg\hat{\jmath}$

[Let $\vec{A}$ be the acceleration of IP in an inertial ref frame] → [$\vec{A}$ has an $\hat{\imath}$ component $= A_x$]

(need help picking out ref frames)

why?

[Let $\vec{a} + \vec{A}$ be the acceleration of the small block in same ref frame] → [$\vec{a}$ is the accelertn of the block relative to the inclined plane]

I'm guessing related

↳ (now use Newtons laws) → $\vec{F}(SB) = \vec{F}_1 + \vec{F}_{g(SB)} = m(\vec{a} + \vec{A})$ 1.1   * now find $F_{\parallel}$ and $F_{\perp}$

1.2 $F_{\parallel} = mg\sin(\alpha) = ma_{\parallel} + mA_x\cos(\alpha)$ ← [parallel to the top of plane] ← (simple trig verified one)

$F_{\perp} = F_1 - mg\cos(\alpha) = ma_{\perp} + mA_x\sin(\alpha)$ ← [perp to the top of the plane]

$|F_1|$

[$a_{\perp} \equiv 0$ if this is relative to the top of the plane]

1.3 solving for F_1

$$\boxed{F_1 = mg\cos(\alpha) + mA_x\sin(\alpha)}$$

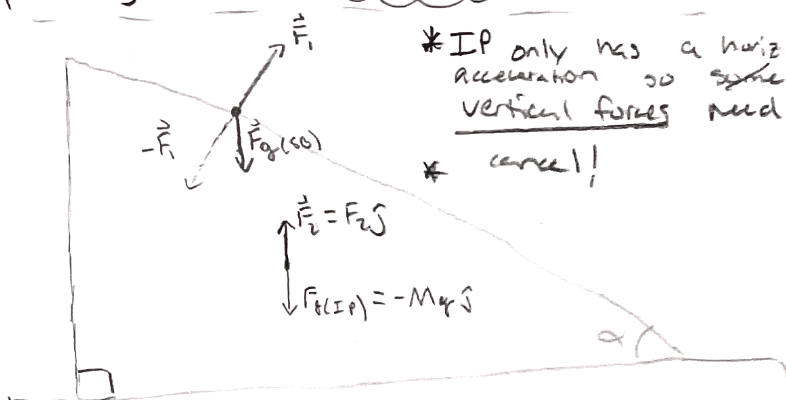
Sum forces on inclined plane!

$\vec{F}_{(IP)} = -\vec{F}_1 + \vec{F}_2 + \vec{F}_{g(IP)} = M\vec{A}$

(3rd law force: IP pushes on m (F_1), m pushes on IP ($-F_1$))

(normal force of IP on ground)

(just vertical so $\vec{F}_2 = F_2\hat{\jmath}$)



* IP only has a horizontal acceleration so ~~some~~ all vertical forces need to cancel!

[Vertical forces on IP] $\underbrace{F_2\hat{\jmath} + (-Mg\hat{\jmath})}_{\text{Forces from IP}} = (-F_1\cos(\alpha))$ → $F_2\hat{\jmath} - Mg\hat{\jmath} = -F_1\cos(\alpha)\hat{\jmath}$

Forces from IP = Forces by m

[horizontal forces] $MA_x = -F_1\sin(\alpha)$

$$\boxed{A_x = -g\left(\frac{\sin(\alpha)\cos(\alpha)}{\sin^2(\alpha) + \frac{M}{m}}\right)}$$ 1.7

(determines motion of the inclined plane)

Now plug F_1 from above into here → $MA_x = -(mg\cos(\alpha) + mA_x\sin(\alpha))\sin(\alpha)$

Motion for small block ⟶

motion for S_{10} is give by $\to (a_{11}) \to$ which we have from eq 1.2 $\to$ $mg\sin(\alpha) = m a_{11} + mA_x \cos(\alpha)$

1.8

$\to a_{11} = g\sin(\alpha) - A_x\cos(\alpha)$

get by plugging in A_x in expression

$\to$ vertical comp of the acceleration $\to a_y = -a_{11}\sin(\alpha) = -g\sin^2(\alpha)\left(\frac{M+m}{M+m\sin^2\alpha}\right)$

$-$ sign bec down

Lets say block starts at height "h" using $y_2 - y_1 = a_y \frac{t^2}{2}$ (final, initial) $\to 0 - h = a_y\frac{t^2}{2} \to t = \sqrt{-2ha_y}$

$\hookrightarrow$ also $a_x = + a_{11}\cos(\alpha)$

Notes to take away

1) • started with 2 objects each having their own coordinates $(x_m, y_m), (X_M, Y_M)$, 4 coordinates, but with our constraints of the problem (m moves ⊥ incline plane and incline plane has $A_y = 0$)

2) • also notice the constraint forces, normal forces are NOT in our final solution, so it should be reasonable to think of a way to solve the problem w/out them in the first place,

$\hookrightarrow$ the constraints restrict the number of degrees of freedom of the motion of whole 1