4.3 Generalized SLP

$$-(p(x)y')' + q(x)y = \lambda r(x) y \quad a<x<b \quad w/ \quad \begin{array}{l}\alpha_1 y(a) + \alpha_2 y(a) = 0 \\ \beta_1 y(b) + \beta_2 y(b) = 0\end{array}$$

↑ we have a positive weight function

Using an integrating factor: consider $a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$

[Note any diff eq of this form can be turned into a SLP] → [SLP of form] → $(p(x)\frac{dy}{dx})' + q(x)y = F(x)$

divide out $a_2(x)$

[Multiply by an integrating factor] → $\mu(x)y'' + \mu(x)\frac{a_1(x)}{a_2(x)}y' + \mu(x)\frac{a_0(x)}{a_2(x)}y = \mu(x)\frac{f(x)}{a_2(x)}$

what does μ need to be to make into one derivative $(\sim)'$ → $\frac{d}{dx}\mu(x)y' = \frac{d\mu}{dx}y' + \mu(x)y''$

$\frac{d\mu}{dx} = \mu(x)\frac{a_1(x)}{a_2(x)}$ to be this diff eq

So! $\frac{d\mu}{dx} = \mu(x)\frac{a_1(x)}{a_2(x)}$ (integrating factor)

↳ $\boxed{\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx}}$ now multiply original by: $\boxed{\frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)}e^{\int\frac{a_1(x)}{a_2(x)}dx}}$

↳ (keep as $\mu(x)$) → $\frac{d}{dx}(\underbrace{\mu(x)}_{p(x)}y') + \underbrace{\mu(x)\frac{a_0(x)}{a_2(x)}}_{q(x)}y = \underbrace{\mu(x)\frac{f(x)}{a_2(x)}}_{F(x)}$

ex $x^2y'' + xy' + 2y = 0$

$\frac{1}{x^2}e^{\int\frac{1}{x}dx} = \frac{1}{x}$ → $0 = xy'' + y' + \frac{2y}{x}$ → $(xy')' + \frac{2}{x}y$ now in SLP

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weighted inner product

$$\langle y_1, y_2\rangle_w = \int_a^b y_1(x)y_2(x)r(x)\,dx \quad \text{and} \quad \|y\|_w = \sqrt{\langle y_1, y_2\rangle_w}$$

$$f(x) = \sum c_n y_n(x) \quad \text{and} \quad c_n = \frac{\langle f | y_n\rangle_w}{\|y\|^2_w}$$

Periodic Boundary Conditions

Consider: $-y'' = \lambda y \quad -\pi < x < \pi$

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi)$$

* not an SLP because of the boundary conditions *

$\lambda = \alpha^2 \rightarrow \quad y'' + \alpha^2 y = 0 \rightarrow y(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$y(\pi) = y(-\pi) \longrightarrow \cancel{A\cos(\alpha\pi)} + B\sin(\alpha\pi) = \cancel{A\cos(\alpha\pi)} - B\sin(\alpha\pi)$

$\hookrightarrow 2B\sin(\alpha\pi) = 0$

$y'(\pi) = y'(-\pi) \rightarrow -A\alpha\sin(\alpha\pi) + \cancel{B\alpha\cos(\alpha\pi)} = A\alpha\sin(\alpha\pi) + \cancel{B\alpha\cos(\alpha\pi)}$

$\hookrightarrow 2A\sin(\alpha\pi) = 0$

$\hookrightarrow$ both satisfied by $\alpha\pi = n\pi \rightarrow \alpha = n$

$y_n(x) = a_n\cos(nx) + b_n\sin(nx)$

$$\sum y_n(x) = y(x) = \frac{a_0}{2} + \sum_{n=1} a_n\cos(nx) + b_n\sin(nx)$$

Rayleigh Quotient

Consider: $-(p(x)\,y_x)_x + q(x)\,y(x) = \lambda\, y(x)$

$$-\int_a^b y\,(p(x)\,y_x)_x\,dx + \int_a^b q(x)\,y^2\,dx = \int_a^b \lambda\, y^2\,dx$$

integrate by parts

$$fg = \int f\frac{dg}{dx}\,dx + \int \frac{df}{dx}\,g\,dx$$

$f = y$, $\quad \frac{dg}{dx} = (p(x)\,y_x)_x \rightarrow g = p(x)\,y_x$

$$\hookrightarrow -y\,p(x)\,y_x\Big|_a^b + \int_a^b p(x)\,y_x^2\,dx + \int_a^b q(x)\,y^2(x)\,dx = \lambda |y|^2$$

$$\hookrightarrow \lambda = \frac{\int_a^b \left(p(x)\,y_x^2 + q(x)\,y^2\right)dx - y\,p(x)\,y_x\Big|_a^b}{|y|^2}$$

ex $-y'' + y' = \lambda y \quad 0<x<1, \quad y(0) = y(1) = 0$

[Put in SLP] from the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$, $\xrightarrow{\text{mult by}} \frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)} dx}$

$a_2(x) = -1$
$a_1(x) = 1$ $\rightarrow -\mu(x) = -e^{\int -1\, dx} \rightarrow \mu(x) = e^{-x}$ mult by

$\hookrightarrow -e^{-x}y'' + e^{-x}y' = \lambda e^{-x} y \rightarrow -(e^{-x}y')' = \lambda e^{-x} y$

From: $-(p(x)y')' + q(x)y = \lambda r(x) y$

$q(x) = 0$
$p(x) = e^{-x}, \quad r(x) = e^{-x}$

Solve by expanding derivative; $\rightarrow -y'' + y' - \lambda y = 0$,

$-m^2 + m - \lambda = 0 = m^2 - m + \lambda \Rightarrow m = \frac{1 \pm \sqrt{1 - 4(1)(\lambda)}}{2}$

$\hookrightarrow m = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda} \rightarrow y(x) = Ae^{\frac{1}{2}x} e^{\sqrt{\frac{1}{4}-\lambda}\,x} + Be^{\frac{x}{2}} e^{-\sqrt{\frac{1}{4}-\lambda}\,x}$

Cases

$\lambda = 1/4$: $y(x) = Ae^{\frac{x}{2}} + Be^{\frac{x}{2}} = e^{\frac{x}{2}}(A+B)$

$y(0) = 0 \rightarrow A + B = 0$
$y(1) = 0 \rightarrow e^{\frac{1}{2}}(A+B) = 0$ which means $y(x) = 0$ trivial sol

$\frac{1}{4} - \lambda > 0$: $y(x) = e^{\frac{x}{2}}\left(Ae^{\sqrt{\frac{1}{4}-\lambda}\,x} + Be^{-\sqrt{\frac{1}{4}-\lambda}\,x}\right)$

$y(0) = 0$: $0 = A + B \rightarrow B = -A$
$y(1) = 0$ $\quad 0 = Ae^{\sqrt{\frac{1}{4}-\lambda}} + Be^{-\sqrt{\frac{1}{4}-\lambda}} \rightarrow A\left(e^{\sqrt{\frac{1}{4}-\lambda}} - e^{-\sqrt{\frac{1}{4}-\lambda}}\right) = 0$

$A \neq 0$: $e^{\sqrt{\frac{1}{4}-\lambda}} = e^{-\sqrt{\frac{1}{4}-\lambda}} \rightarrow \lambda$ is real and $\frac{1}{4} - \lambda > 0$

$\hookrightarrow$ only satisfied when $\lambda = \frac{1}{4}$ which is trivial

$\frac{1}{4} - \lambda < 0$:

$y(x) = e^{\frac{x}{2}} \left(A \cos(\sqrt{\tfrac{1}{4} - \lambda}\, x) + B \sin(\sqrt{\tfrac{1}{4} - \lambda}\, x) \right)$

$y(0) = 0 \to A = 0$, $y(x) = B e^{\frac{x}{2}} \sin(\sqrt{\tfrac{1}{4} - \lambda}\, x)$

$y(1) = 0 \to \sin(\sqrt{\tfrac{1}{4} - \lambda}) = 0 \to \sqrt{\tfrac{1}{4} - \lambda} = n\pi$

$\hookrightarrow (n\pi)^2 = \frac{1}{4} - \lambda \to \boxed{\lambda_n = \frac{1}{4} - (n\pi)^2}$

$\boxed{y_n(x) = e^{\frac{x}{2}} \sin(n\pi x) \text{ , } n = 1, 2, 3, \ldots}$

orthogonal w.r.t weighting function

Now:

$\langle y_n, y_m \rangle = \int_0^1 (e^{\frac{x}{2}} \sin(n\pi x))(e^{\frac{x}{2}} \sin(m\pi x))\, e^{-x}\, dx = 0$

$f(x) = \sum c_n e^{\frac{x}{2}} \sin(n\pi x)$, $c_n = \frac{\langle f | y_n \rangle_w}{\langle y_n | y_n \rangle_w^2}$, $n = 1, 2, 3$

Singular Problems

- bounded
- infinite boundary

Laplaces Equation

[In polar coordinates] $\to u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \qquad 0 < r < R, \ 0 \leq \theta \leq 2\pi$

$u(R,\theta) = f(\theta) \qquad 0 \leq \theta \leq 2\pi,$ [Periodic BC] $u(r,0) = u(r,2\pi)$, $u_\theta(r,0) = u_\theta(r,2\pi)$

Now: $u(r,\theta) = y(r) g(\theta)$

$$y''(r) g(\theta) + \frac{1}{r} y'(r) g(\theta) + \frac{1}{r^2} y(r) g''(\theta) = 0$$

$$\hookrightarrow -\frac{r^2 y''(r) + r y'(r)}{y(r)} = \frac{g''(\theta)}{g(\theta)} = -\lambda \to \boxed{\begin{aligned} r^2 y''(r) + r y'(r) &= \lambda y(r) \\ g''(\theta) &= -\lambda g(\theta) \end{aligned}}$$

Solve g SLP

$g'' = -\lambda g$

$g(0) = g(2\pi)$

$g'(0) = g'(2\pi)$

$\lambda = 0$ is trivial solution $\to g_0(\theta) = 1$

if $\lambda < 0$: then exponential solutions $\to$ doesn't satisfy periodic B.C !!!

So $\lambda > 0$: $\lambda = p^2$, $g(\theta) = a\cos(p\theta) + b\sin(p\theta)$

$g(0) = g(2\pi)$, $g'(0) = g'(2\pi)$ $\begin{cases} (\cos(2\pi p) - 1)a + \sin(2\pi p) b = 0 \\ \sin(2\pi p) a + (1 - \cos(2\pi p)) b = 0 \end{cases}$

Non trivial Soln (det $= 0$):

$(\cos(2\pi p) - 1)(1 - \cos(2\pi p)) + \sin^2(2\pi p) = 0 \to \cos(2\pi p) = 1$

$\hookrightarrow p = \sqrt{\lambda}$, $\boxed{\lambda = n^2, \ n = 1, 2, 3, 4, \ldots}$

take positive root!!

So: $g_0(\theta) = 1$ and $g_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta) \quad n = 1, 2, \ldots$

$\lambda_0 = 0$ eigenvalue

For $\lambda = 0$, $y_0(r) = 1$;

$\lambda_n = n^2$:

$r^2 y''(r) + r y'(r) - n^2 y(r) = 0$ ← Cauchy Euler Eq

$y = r^m \rightarrow y' = m r^{m-1} \rightarrow y'' = m(m-1) r^{m-2}$

$\cancel{r^2} m(m-1) \cancel{r^{m-2}} + \cancel{r} m \cancel{r^{m-1}} - n^2 \cancel{r^m} = 0 \rightarrow \overbrace{m(m-1)}^{m^2 - m} + m - n^2 = 0$

$\hookrightarrow m^2 - n^2 = 0 \rightarrow \boxed{m = \pm n}$

$\hookrightarrow \boxed{y_n(r) = c_n r^{-n} + d_n r^n}$ [now the B.C are] Bounded on $0 \le r \le R$ and $\frac{1}{r^n} \rightarrow \frac{1}{0^n} \rightarrow \infty$

$\hookrightarrow c_n = 0$, let $d_n = 1$

$\hookrightarrow y_n(r) = r^n \quad n = 1, 2, 3, \ldots$

$u_n(r, \theta) = r^n(a_n \cos(n\theta) + b_n \sin(n\theta))$, $u_0(r, \theta) = \frac{a_0}{2}$ ← equivalent constant set to $a/2$

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

$$f(\theta) = u(R, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

[Do fouriers trick to get a_n, b_n] $\rightarrow$

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta$$

Simplifying

$$u(r,\theta) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \frac{r^n}{\pi R^n} \int_0^{2\pi} f(\phi)\big(\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)\big)\,d\phi$$

changed integration variable to ϕ so to not get confused!!

trig identity

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos(n(\theta-\phi))\right)d\phi$$

Recall! $\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$

Solve the sum

$$\rightarrow 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos(n(\theta-\phi)) = 1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{-in(\theta-\phi)}$$

$$= 1 + \sum_{n=1}^{\infty}\Bigg(\underbrace{\frac{re^{i(\theta-\phi)}}{R}}_{<1}\Bigg)^n + \sum_{n=1}^{\infty}\left(\frac{r}{R}e^{-i(\theta-\phi)}\right)^n,$$

$|e^{ix}|$ goes between -1 and 1 and $r<R$

$$\boxed{\text{Recall Sum}}\ \sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \rightarrow \boxed{\text{Compute the sum now}} \rightarrow 1 + \frac{\frac{r}{R}e^{i(\theta-\phi)}}{1-\frac{r}{R}e^{i(\theta-\phi)}} + \frac{\frac{r}{R}e^{-i(\theta-\phi)}}{1-\frac{r}{R}e^{-i(\theta-\phi)}}$$

$$\hookrightarrow 1 + \frac{re^{i(\theta-\phi)}}{R-re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{R-re^{-i(\theta-\phi)}} \overset{\text{chunk of algebra}}{=} \frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-\phi)}$$

Then

$$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,f(\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}\,d\phi}$$

General Results for Laplace's Equation

$$\int_{\Omega} \vec{\nabla}\cdot\vec{\Phi}\, d\bar{V} = \int_{\partial\Omega} \vec{\Phi}\cdot\vec{n}\, dA$$

Deriving Green's Identities, define u a scalar field, $\vec{\Phi}$ vector field

$\vec{\nabla}\cdot(u\vec{\Phi}) = u\,\vec{\nabla}\cdot\vec{\Phi} + \vec{\Phi}\cdot\vec{\nabla}u$ ← just write out component by component

Integrate!) divergence theorem

$$\int_{\partial\Omega} u\vec{\Phi}\cdot\vec{n}\, dA = \int_{\Omega} u\,\vec{\nabla}\cdot\vec{\Phi}\, d\bar{V} + \int_{\Omega} \vec{\Phi}\cdot\vec{\nabla}u\, d\bar{V}, \quad \text{Now let } \vec{\Phi} = \vec{\nabla}v$$

$$\int_{\partial\Omega} u\vec{\nabla}v\cdot\vec{n}\, dA = \int_{\Omega} u\,\vec{\nabla}^2 v\, d\bar{V} + \int_{\Omega} \vec{\nabla}v\cdot\vec{\nabla}u\, d\bar{V}, \quad \text{let } v = u$$

$$\boxed{\int_{\partial\Omega} u\vec{\nabla}u\, d\vec{A} = \int_{\Omega} u\nabla^2 u\, d\bar{V} + \int_{\Omega} \vec{\nabla}u\cdot\vec{\nabla}u\, d\bar{V}}$$

Green's first Identity, in 1D

$$\int_a^b y y''\, dx = -y y'\Big|_a^b - \int_a^b (y')^2\, dx$$

Deriving Greens Identities!

Take)

$$\int_{\partial\Omega} u\vec{\nabla}v\cdot d\vec{A} = \int_{\Omega} u\nabla^2 v\, d\bar{V} + \int_{\Omega} \vec{\nabla}v\cdot\vec{\nabla}u\, d\bar{V}$$

exchanged u's + v's now subtract them

$$\int_{\partial\Omega} v\vec{\nabla}u\cdot d\vec{A} = \int_{\Omega} v\nabla^2 u\, d\bar{V} + \int_{\Omega} \vec{\nabla}u\cdot\vec{\nabla}v\, d\bar{V}$$

$$\hookrightarrow \int_{\partial\Omega} (u\vec{\nabla}v - v\vec{\nabla}u)\cdot d\vec{A} = \int_{\Omega} u\nabla^2 v\, d\bar{V} - \int_{\Omega} v\nabla^2 u\, d\bar{V}$$

$$\hookrightarrow \boxed{\int_{\Omega} u\nabla^2 v\, d\bar{V} = \int_{\Omega} v\nabla^2 u\, d\bar{V} + \int_{\partial\Omega} (u\vec{\nabla}v - v\vec{\nabla}u)\cdot d\vec{A}}$$

Green's Second Identity

Dirichlet problem

$\nabla^2 u = 0 \quad \Omega$

$u = f \quad \text{on } \partial\Omega$ → it is unique

in E and M this is like E^2, so $\frac{1}{2}\varepsilon_0 E^2$ is energy ← when w is potential

[Interesting property of harmonic functions satisfying Dirichlet B.C is that they minimize the energy integral] → $E(w) \equiv \int_\Omega \vec{\nabla} w \cdot \vec{\nabla} w \, dV$

Suppose u satisfies

$\nabla^2 u = 0$ on Ω

$u = f$ on $\partial\Omega$ → $E(u) \le E(w)$ for all w satisfying $w = f$ on $\partial\Omega$

↳ of all the functions that satisfy the B.C, solution to $\nabla^2 u$ is one that minimizes the energy be $E(u)$ is less than all others

Proof let $w = u + v$ where $v = 0$ on boundary $\partial\Omega$

$E(w) = E(u+v) = \int_\Omega \vec{\nabla}(u+v) \cdot \vec{\nabla}(u+v) \, dV$ | $w \equiv u + v$, ~~$\nabla^2 w = 0$ in Ω~~ ~~$w = 0$ on $\partial\Omega$~~ ?

$= \int_\Omega \vec{\nabla} u \cdot \vec{\nabla} u + 2\int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v \, dV + \int_\Omega \vec{\nabla} v \cdot \vec{\nabla} v \, dV$ $v = 0$ on boundary

~~if $v = 0$ on boundary then $\nabla^2 v = 0$ too~~

$= E(u) + E(v) + 2\cancel{\int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v \, dV} = 0$, [Recall Greens 1st Ident] → $\cancel{\int_{\partial\Omega} u \vec{\nabla} v \cdot d\vec{A}} = \int_\Omega u \nabla^2 v \, dV + \int_\Omega \vec{\nabla} v \cdot \vec{\nabla} u \, dV$

$E(w) = E(u) + \underbrace{E(v)}_{\ge 0}$ → $\boxed{E(u) \le E(w)}$ ✓

Cooling of a sphere

- symmetries allows us to reduce dimensions to radius & time
- Given a sphere whose initial temp depends on only the distance from center & boundary thats at constant temp, predict the temp at any point inside the sphere.

[Recall Newtons Law of Cooling] → States the rate at which a body cools is prop. to the difference of T and Tenv → $T'(t) = -h(T - T_e)$ ← constant heat loss coeff.

↳ only applies in the case that the body has a uniform, homogeneous temp.

↳ PDE Problem; consider the temp may vary radially.

$$u_t = k\nabla^2 u \quad , \quad \nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial}{\partial\rho}$$

$$u_t = k\left(u_{\rho\rho} + \frac{2}{\rho}u_\rho\right) \quad 0<\rho<\pi \quad , \quad t>0$$

$$u(\pi,t) = 0 \quad , \quad t>0$$

bounded temp

$$u(\rho,0) = T_0 \quad , \quad 0 \le \rho < \pi$$

Do separation of var;

↳ $-y''(\rho) - \frac{2}{\rho}y'(\rho) = \lambda y(\rho) \qquad 0<\rho<\pi$

↳ to solve introduce $Y(\rho) = \rho y(\rho)$, solve that PDE

↳ $Y = a\cos(\sqrt{\lambda}\rho) + b\sin(\sqrt{\lambda}\rho)$

↳ $y = \frac{1}{\rho}\left(a\cos(\sqrt{\lambda}\rho) + b\sin(\sqrt{\lambda}\rho)\right)$ apply B.C

$$\lambda_n = n^2, \quad y_n = \frac{\sin(n\rho)}{\rho} \quad , \quad g_n = C_n e^{-n^2kt}$$

Apply corner:

$$T_{0p} = \sum c_n \sin(np)$$

$$c_n = \frac{2}{\pi} \int_0^{\pi} T_{0p} \sin(np)\, dp$$

Diffusion in a disk

$T_i(r)$

$\partial\Omega$ at $T=0$

must satisfy

$u_t = k\nabla^2 u \quad , \quad 0<r<R$

↑ Laplacian in polar coord

$0 \le r < R$

$\hookrightarrow u_t = k\left(u_{rr} + \frac{1}{r}u_r\right) \quad , \quad u(R,t)=0, \ t>0, \ u(r,0)=f(r)$

$u(0,t) =$ bounded

$\hookrightarrow$ [separate w/ $u = g(t)y(r)$] $\rightarrow \dfrac{g'(t)}{k\,g(t)} = \dfrac{y'' + \frac{1}{r}y'}{y} = -\lambda$

$\hookrightarrow \boxed{g = e^{-\lambda k t}}$ and $y'' + \frac{1}{r}y' = -\lambda y \rightarrow \boxed{-(ry')' = \lambda r y}$

only $\lambda > 0$! only way to solve this is by assuming solution

$y(r) = \sum_{n=0}^{\infty} a_n r^n \rightarrow$ yields $y(r) = c_1 J_0(\sqrt{\lambda}\, r) + c_2 Y_0(\sqrt{\lambda}\, r)$

Bessel functions!

$Y_0(0) \rightarrow -\infty$ but we want bounded solution so $c_2 = 0$

$\hookrightarrow y(r) = c_1 J_0(\sqrt{\lambda}\, r) \quad , \quad J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^{2n}} z^{2n}$

$y(R) = 0 = c_1 J_0(\sqrt{\lambda}\, r) \rightarrow z_n = \sqrt{\lambda}\, R$, where z_n is zeros of

$\hookrightarrow y_n = J_0\left(\frac{z_n r}{R}\right) \rightarrow u_n = e^{-\left(\frac{z_n}{R}\right)^2 k t} J_0\left(\frac{z_n r}{R}\right)$

$u = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{z_n}{R}\right)^2 k t} J_0\left(\frac{z_n r}{R}\right)$, now do $u(r,0) = f(r)$

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} C_n J_0\left(\frac{z_n r}{R}\right)$$, J_0 functions are orthogonal

$$\hookrightarrow \int_0^R f(r) J_0\left(\frac{z_n r}{R}\right) r\,dr = C_n \left\| J_0\left(\frac{z_n r}{R}\right)\right\|_w^2$$

$\uparrow$ weighting func

$$\boxed{C_n = \frac{\int_0^R f(r) J_0\left(\frac{z_n r}{R}\right) r\,dr}{\left\| J_0\left(\frac{z_n r}{R}\right)\right\|^2}}$$

4.7 Sources on Bounded Domains

Problem: $\begin{cases} u_t - ku_{xx} = f(x,t) & a<x<b, \ t>0 \\ u(a,t) = b_1(t), & u(b,t) = b_2(t) \\ u(x,0) = g(x) \end{cases}$

$\boxed{\text{Split up the problem}} \to \begin{cases} V_t - kV_{xx} = 0 & a<x<b \\ V(a,t) = b_1(t), & V(b,t) = b_2(t) \\ V(x,0) = g(x) \end{cases} + \begin{cases} w_t - kw_{xx} = f(x,t) & a<x<b \\ w(a,t) = 0, & w(b,t) = 0 \\ w(x,0) = 0 \end{cases}$

↑ solve w/ fourier

↑ solve w/ Duhamel

$(u(x,t) = V(x,t) + w(x,t))$

Duhamel's Principle Example

$$\left.\begin{aligned} &u_t - ku_{xx} = f(x,t), \quad 0<x<\pi \\ &u(0,t) = u(\pi,t) = 0 \\ &u(x,0) = 0, \quad 0<x<\pi \end{aligned}\right\}$$

heat flow, u is temperature of a rod whose initial temp is 0 and whose ends are maintained at $0°$
$f(x,t)$ = heat source driving the system

$\boxed{\text{Duhamels Principle}} \to u(x,t) = \int_0^t w(x,t-\tau)\,d\tau$, $\quad \begin{aligned} &w_t - kw_{xx} = 0 \quad 0<x<\pi \\ &w(0,t,\tau) = w(\pi,t,\tau) = 0 \\ &w(x,0,\tau) = f(x,\tau) \quad 0<x<\pi \end{aligned}$

$\boxed{\text{We can solve for } w \text{ in this PDE now, it is}} \to w(x,t,\tau) = \sum_{n=1}^{\infty} c_n e^{-n^2kt} \sin(nx)$

$$c_n = c_n(\tau) = \frac{2}{\pi}\int_0^{\pi} f(x,\tau)\sin(nx)\,dx$$

$\boxed{\text{So the total solution is the following}} \to u(x,t) = \int_0^t \left(\sum_{n=1}^{\infty} c_n(\tau) e^{-n^2k(t-\tau)} \sin(nx) \right) d\tau$

[Can also be solved using eigenvalues & eigenfunctions] → $\lambda_n = n^2$, $y_n = \sin(nx)$

still for duhamel example PDE

↳ there are eigenvalues & functions of homogenous eqn

↳ $u(x,t) = \sum_{n=1}^{\infty} g_n(t) \sin(nx)$,

Assume this solution w/ $f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(nx)$ in a fourier exp,

And, $\boxed{f_n(t) = \frac{2}{\pi} \int_0^{\pi} f(x,t) \sin(nx) dx}$

Now plug into PDE;

$$\frac{\partial}{\partial t}\left(\sum g_n(t) \sin(nx)\right) - k \frac{\partial^2}{\partial x^2}\left(\sum g_n(t) \sin(nx)\right) = \sum_{n=1}^{\infty} f_n(t) \sin(nx)$$

↳ $\sum g_n' \sin(nx) + k \sum n^2 g_n \sin(nx) = \sum f_n(t) \sin(nx)$

↳ $g_n' + n^2 k g_n = f_n$ → first order ODE

↳ $g_n(t) = g_n(0) e^{-n^2 kt} + \int_0^t f_n(\tau) e^{-n^2 k(t-\tau)} d\tau$

get this from $u(x,0) = 0 = \sum g_n(0) \sin(nx)$ or $g_n(0) = 0$

↳ plug $g_n(t)$ into $u(x,t)$ eq

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t f_n(\tau) e^{-n^2 k(t-\tau)} d\tau\right) \sin(nx)}$$

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In order to use seperation of variables you need homogenous B.C.

Reason: lets say v and w solve PDE (u_n but different n's)
they both have to satisfy the B.C, lets say at $x=a$ v and $w = A$
So their sum which also solves pde $S = V + W \rightarrow$ at $x = A$,
$A = A + A$, which is only true for $A = 0$ (homogenous).

Solve (inhomogenous B.C)

$$u_t - k u_{xx} = f(x,t)$$
$$u(a,t) = h_1(t)$$
$$u(b,t) = h_2(t)$$
$$u(x,0) = \phi(x)$$

$$u(x,t) = \underset{\substack{\text{or}\\ w}}{u_{\text{homogenous B.C}}} + u_{\text{steady state}}$$

$u_{\text{homogenous B.C}}$ ↓ dont know PDE yet
$W(a,t) = W(b,t) = 0$
but need homogenous B.C

$u_{\text{steady state}}$ ↓ $\frac{\partial u_{\text{steady}}}{\partial t} = 0$
↓
$u_{xx} = -\frac{1}{k} f(x,t)$
↑ time depend go away?
↳ $u = C_1(t)x + C_2(t)$

$$w(x,t) = u(x,t) - \phi_1(t)x - \phi_2(t)$$
$$\left.\begin{aligned} w(a,t) &= h_1(t) - \phi_1(t)a - \phi_2(t) = 0 \\ w(b,t) &= h_2(t) - \phi_1(t)b - \phi_2(t) = 0 \end{aligned}\right\}$$

Solve $\phi_1(t) = \dfrac{h_2(t) - h_1(t)}{b-a}$ and

$\phi_2(t) = (b h_1(t) - a h_2(t))/(b-a)$

Example

$u_t - k u_{xx} = -x\sin(t) + \sin(\pi x)$ $\quad 0 < x < 1$
$u(0,t) = 1$
$u(1,t) = \cos(t)$
$u(x,0) = \sin(\pi x) + 1$

convert to homogenous B.C
let $w(x,t) = u(x,t) - \underbrace{\phi_1(t)x - \phi_2(t)}_{\text{Find these}}$ require to be homogenous
↳ apply B.C → $w(0,t) = 1 - 0 - \phi_2(t) = 0$
$w(1,t) = \cos(t) - \phi_1(t) - \phi_2(t) = 0$
↳ $\phi_2(t) = 1$, $\phi_1(t) = \cos(t) - 1$

So: $\boxed{w(x,t) = u(x,t) - (\cos(t) - 1)x - 1}$

$w_t = u_t + x\sin(t)$
$\rightarrow w_t - k w_{xx} = u_t + x\sin(t) - k u_{xx} = \overbrace{-x\sin(t) + \sin(\pi x)}^{u_t - k u_{xx}} + \cancel{x\sin(t)}$
$w_{xx} = u_{xx}$

So
$$\begin{cases} w_t - k w_{xx} = \sin(\pi x) \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = \sin(\pi x) \end{cases}$$
→ [Solve by method of duhamels principle or eigenfunction, or if source is time ind. find steady state solution]

Where $w(x,t) = \underset{\substack{\uparrow\\ \text{other part}}}{W} + \underset{\substack{\uparrow\\ \text{steady state}}}{U}$, splitting like this gets rid of source

$\boxed{\text{Solve steady State solution}} \to -kU'' = \sin(\pi x)$, $U(0) = U(1) = 0$

$\hookrightarrow U = \frac{1}{k\pi^2}\sin(\pi x) + c_1 x + c_2$, Apply B.C $c_2 = 0$, $c_1 + c_2 = 0$, $c_1 = 0$

$U = \frac{1}{k\pi^2}\sin(\pi x)$ now $\overline{W} = \omega - U$

$\overline{W}(0,t) = \overline{W}(1,t) = 0$

$\overline{W}(x,0) = \sin(\pi x) - \frac{1}{k\pi^2}\sin(\pi x) = \left(1 - \frac{1}{k\pi^2}\right)\sin(\pi x)$

check $\overline{W}$ gets rid of $\sin(\pi x)$

$\omega = \overline{W} + U \to \overline{W}_t - k\overline{W}_{xx} + \cancel{\sin(\pi x)} = \cancel{\sin}(\pi x) \Rightarrow 0$

$\hookrightarrow \begin{cases} \overline{W}_t - k\overline{W}_{xx} = 0 \\ \overline{W}(0,t) = \overline{W}(1,t) = 0 \\ \overline{W}(x,0) = \sin(\pi x)\left(1 - \frac{1}{k\pi^2}\right) \end{cases}$ w/ $\overline{W} = \omega - \frac{1}{k\pi^2}\sin(\pi x)$

↑ can be solved w/ seperation of varibles

$y_n = \sin(n\pi)$, $g_n = e^{-(n\pi)^2 kt}$

$\lambda_n = \left(\frac{n\pi}{l=1}\right)^2 = (n\pi)^2$ $\to \overline{W}(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 kt}\sin(n\pi x)$

$\hookrightarrow$ initial con $\left(1 - \frac{1}{k\pi^2}\right)\sin(\pi x) = \sum b_n \sin(n\pi x) \to b_1 = \left(1 - \frac{1}{k\pi^2}\right)$

So $\overline{W} = \left(1 - \frac{1}{k\pi^2}\right)e^{-\pi^2 kt}\sin(\pi x)$, now get ω

$\omega = \left(1 - \frac{1}{k\pi^2}\right)e^{-\pi^2 kt}\sin(\pi x) + \frac{1}{k\pi^2}\sin(\pi x)$

and $\omega = u - (\cos(t) - 1)x - 1$

So $\boxed{u(x,t) = \left(1 - \frac{1}{k\pi^2}\right)e^{-\pi^2 kt}\sin(\pi x) + \frac{1}{k\pi^2}\sin(\pi x) + (\cos(t) - 1)x + 1}$