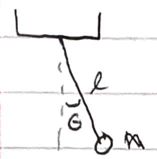
Oscillators

- a system with periodic motion
- equillibrium means remaining in rest state, if potential energy V isnt changing then you can say the system is at equillibrium, ie $\frac{\partial V}{\partial q} \dots \frac{\partial V}{\partial q_k}$ is 0 (for conservative systems only)

ex for a pendulum $V(\theta) = mgl(1-\cos\theta)$ Torque $= F_\theta = \frac{-\partial V}{\partial \theta} = -mgl\sin\theta$, $\sin\theta$ vanishes at 0 and π, two equillibrium points. Obviously 0 and π represent diff kinds of equillibrium. stable unstable.

$\boxed{\text{intro}}$



$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$ lets do taylor expansion of to look at L near $0, \pi$

near 0: $\cos\theta = 1 - \theta^2/2 + \theta^4/24! + \dots \rightarrow L = ml^2\dot{\theta}^2/2 - mgl\theta^2/2$

↳ $L \approx mgl\left(\frac{\left(\sqrt{\frac{l}{g}}\dot{\theta}\right)^2}{2} - \frac{\theta^2}{2}\right)$

↳ scale time by $\tau = t\sqrt{\frac{g}{l}}$, $d\tau = dt\sqrt{\frac{g}{l}} \rightarrow \frac{d\theta}{d\tau} = \frac{d\theta}{dt}\sqrt{\frac{l}{g}} \rightarrow \frac{d\theta}{dt} = \sqrt{\frac{g}{l}}\cdot\frac{d\theta}{d\tau}$

then $L' = \frac{L}{mgl} = \frac{\dot{\theta}^2}{2} - \frac{\theta^2}{2}$ for θ near 0

near π: so $\theta = \pi + \Delta\theta$ so Taylor series of $\cos\theta = \cos(\pi + \Delta\theta) = -1 + \frac{\Delta\theta^2}{2} + \dots$

* $\dot{\theta} \rightarrow \Delta\dot{\theta}$ doing same routine above

$$L_\pi = \frac{\Delta\dot{\theta}^2}{2} + \frac{\Delta\theta^2}{2} \quad \theta \text{ near } \pi$$

- notice the difference in signs. Crucial difference between stable and unstable eq that + or −
- changes force from a restoring one to an anti restoring one when one does the ELE

$\boxed{\text{Generally}}$ how systems behave near equillibrium points → Taylor series $q = q_{eq}$, $\dot{q} = 0$

→ call $q_{eq} = 0$ so about that point

$L(q,\dot{q})$, $L \approx A + Bq + C\dot{q} + Dq^2 + Eq\dot{q} + F\dot{q}^2$ ex: $D = \frac{1}{2}\frac{\partial^2 L}{\partial q^2}$, $F = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}^2}$, $B = \frac{\partial L}{\partial q} \leftarrow$ force on system

so $B = 0$ — constant for this degree of Lagrangian, maybe general idk

$\boxed{\text{Applying ELE}}$ ↳ $\frac{d}{dt}\underbrace{(C + Eq + 2F\dot{q})}_{\partial L/\partial \dot{q}} = \underbrace{2Dq + E\dot{q}}_{\partial L/\partial q}$ $\quad \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$

→ $0 + E\dot{q} + 2F\ddot{q} = 2Dq + E\dot{q}$

or $\boxed{\ddot{q} - \frac{D}{F}q = 0}$ $\quad D = \frac{1}{2}\frac{\partial^2 L}{\partial q^2}$, $F = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}^2}$

you get the same EOM from this Lagrangian $L = \dot{q}^2 + \frac{D}{F}q^2$

↳ back

same lagrangian

$D = \frac{1}{2}\frac{\partial^2 L}{\partial q^2}$ and $F = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}^2}$

So. $L = \dot{q}^2 + \frac{D}{F}q^2$ for taylor expanse of oscillators near $q_{eq} = 0$

- the ω for this is $\omega_0^2 = -\frac{D}{F}$ ~~I dont know if this matters~~ which only makes sense if $\frac{D}{F}$ is negative,
- $F > 0$ because it represents the KE of the Lagrangian (constant) $\dot{q}^2$

(doing it makes the constant term from $\dot{q}^2$ go to numerator of the q term)

also lets scale time by $t = \beta\tau$; $\ddot{q} - \beta^2\frac{D}{F}q = 0$] for after we did ELE

↳ lets choose $\beta = \sqrt{\left|\frac{F}{D}\right|}$, has to be real so we dont have imaginary time

So we have $\ddot{q} - \left|\frac{F}{D}\right| \cdot \frac{D}{F} q$ so depending on D the sign will be ±

↳ then our Lagrangian can be written as $L = \frac{1}{2}(\dot{q}^2 \pm q^2)$

$L = \frac{1}{2}(\dot{q}^2 - q^2)$ is stable and $L = \frac{1}{2}(\dot{q}^2 + q^2)$ is unstable

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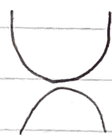
$D = \frac{1}{2}\frac{\partial^2 L}{\partial q^2}$ which tells you about V for conservative, holonomic, scleronomic sys ($L = T - V$)

$\frac{1}{2}\frac{\partial^2 L}{\partial q^2} \sim \frac{\partial^2 V}{\partial q^2}$, second derivative tells you about the curvature of V, concavity.

Looking back at $\ddot{q} - \beta^2\frac{D}{F}q = 0$ if $D > 0$, $\frac{\partial^2 V}{\partial q} > 0$

if $D < 0$, $\frac{\partial^2 V}{\partial q} < 0$

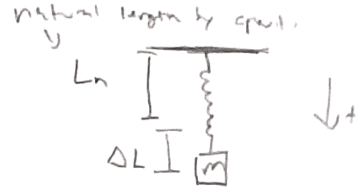


stable

unstable

↳ stable $\ddot{q} - q = 0$ and unstable $\ddot{q} + q = 0$

Lagrangian for simple oscillator can be $L = \frac{\dot{q}^2 - \omega_0^2 q^2}{2}$



Damped and driven oscillators

F_d = damping force, F_s = spring force
F_a = applied force, F_g = gravity

Refresh from ODE: $\overset{mx''}{F_{net}} = F_g + F_s + F_d + F_a$

$mg - kL_n$ cancel so

$mx'' = mg - k(L_n + x) - \beta x' + F_a$, $\beta x'$ bec its a constant change to displacement

$\hookrightarrow \; x'' + \frac{k}{m} + \frac{\beta}{m}x' = F_a/m$ → redefine $\boxed{x'' + 2\lambda x' + \omega^2 x = f(t)}$ $\lambda = \frac{\beta}{2m}$ and $\omega = \sqrt{\frac{k}{m}}$

Now from book: $\ddot{q} + \frac{1}{Q}\dot{q} + \omega_0^2 q = 0$ for just damped

Lets solve this: $q(t) = q_0 e^{\alpha t}$ → $(\alpha^2 + \frac{1}{Q}\alpha + \omega_0^2)e^{\alpha t} = 0$

$$\boxed{\alpha = -\frac{1}{2Q} \pm \frac{1}{2}\sqrt{\left(\frac{1}{Q}\right)^2 - 4\omega_0^2} \;\Rightarrow\; q = Ae^{\alpha t} + Be^{-\alpha t}}$$

you can imagine if α is real then will be exponential something, imaginary you have oscillatory and another type where square root is 0

Underdamped: $\frac{1}{Q^2} - 4\omega_0^2 < 0$ → $Q > \frac{1}{2\omega_0}$

- imaginary roots w/ a negative constant exponential
- solution is thus:

$$\boxed{q(t) = q_0 e^{-t/2Q}\sin(\omega' t - \phi), \quad \omega' = \sqrt{\omega_0^2 - \left(\frac{1}{2Q}\right)^2}}$$

- Both $q(t)$ and $\dot{q}(t)$ are $\sim e^{-t/2Q}$
- $E = \underbrace{T}_{\dot{q}^2} + \underbrace{V}_{q}$ so $E \sim e^{-t/2Q}$ $\quad \frac{dE}{dt} = -\frac{E}{Q}$ ← can repull out, ← this comes out after derivative

Overdamped: $\frac{1}{Q^2} - 4\omega_0^2 > 0$ → $Q < \frac{1}{2\omega_0}$

Using result above w/ imaginary roots: $q(t) = q_0 e^{-t/2Q}\left(e^{i\omega' t} + e^{-i\omega' t}\right)$, $\omega' = \sqrt{\omega_0^2 - \left(\frac{1}{2Q}\right)^2}$

if $Q < \frac{1}{2\omega_0}$ then $\sqrt{\omega_0^2 - \left(\frac{1}{2Q}\right)^2} < 0$ so you get imaginary which cancels i

$$\boxed{q(t) = q_0 e^{-t/2Q}\left(\underbrace{e^{\omega'' t}}_{\text{growth?}} + \underbrace{e^{-\omega'' t}}_{\text{decay}}\right), \quad \omega'' = \sqrt{\left(\frac{1}{2Q}\right)^2 - \omega_0^2}}$$

no growth because if you put $e^{-t/2Q}$ back in then that numerator its negative ← exponential decay still

$\sqrt{\left(\frac{1}{2Q}\right)^2 - \omega_0^2} - \frac{1}{2Q} \overset{?}{<} 0$, rearrange $Q < \frac{1}{2\omega_0}$ → $\omega_0 < \frac{1}{2Q}$, $\frac{1}{2Q} > \omega_0$

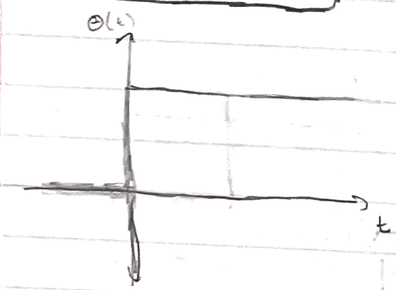
so $\left(\frac{1}{2Q}\right)^2$, greater than ω_0^2 so its not imaginary, being subtracted by ω_0^2, so $\frac{1}{2Q}$ is getting smaller, thus when $\frac{1}{2Q}$ is subtracted from it it has to be negative.

Critical Damping $\quad Q = \frac{1}{2\omega_0} \rightarrow \omega_0 = \frac{1}{2Q}$

makes it decay very fast!

$\alpha = \frac{-1}{2Q} = \overbrace{\omega_0}^{\text{twice!}} \rightarrow q(t) = Ae^{-\omega_0 t} + \underbrace{Bte^{-\omega_0 t}}_{\text{degenerate solution}}$

$\boxed{\text{Forced Oscillator}}$ · consider a force turned on at $t=0$



$$\boxed{\begin{aligned} F(t) &= F_0\,\Theta(t) \\ \Theta(t) &= \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases} \end{aligned}}$$

for $t<0$,

$\ddot{q} + \omega_0^2 q = 0$

$q(t) = q_0 \sin(\omega t + \phi)$

for $t>0$: $\ddot{q} + \omega_0^2 q = F_0$, $q(t) = \frac{F_0}{\omega_0^2}$ (assume q is like a constant)

$q(t) = q_h + q_p = q_0 \sin(\omega t + \phi) + \frac{F_0}{\omega_0^2}$

[at $q(0)=0$ and $\dot{q}(0)=0$, but our function is discontinuous so lets introduce a ε and take the $\lim_{\varepsilon \to 0}$]

$q(t=-\varepsilon) = q(t=\varepsilon) \rightarrow$ when you take lim you get $\rightarrow q(0)=0$

$\dot{q}(t=-\varepsilon) = \dot{q}(t=\varepsilon) \rightarrow \dot{q}(0)=0$

$q(0) = 0 = q_0 \sin(\phi) + \frac{F_0}{\omega_0^2}$

$\dot{q}(0) = 0 = q_0 \omega_0 \cos\phi \rightarrow \phi = \frac{\pi}{2}$ so $0 = -q_0 + \frac{F_0}{\omega_0^2} \rightarrow \boxed{q_0 = \frac{-F_0}{\omega_0^2}}$

$$\boxed{q(t) = \frac{F_0}{\omega_0^2}\left(1 - \cos(\omega_0 t)\right)}$$

Resonance

consider a SHO driven by a sinusoidal force, $F(t) = F_0 \sin(\omega t)$

↳ $G_{SHO}(t-t') = \frac{1}{m\omega_0} \sin(\omega(t-t'))$

$$q(t) = \int_0^t F(t')\,G(t-t')\,dt' = \frac{F_0}{m\omega_0}\int_0^t \sin(\omega t')\sin(\omega_0(t-t'))\,dt'$$

$$q(t) = \frac{F_0}{m}\,\frac{\sin(\omega t) - \left(\frac{\omega}{\omega_0}\right)\sin(\omega_0 t)}{\omega^2-\omega_0^2}$$

what if we drive it at its natural frequency? $\omega \to \omega_0$?

$\lim_{\omega\to\omega_0} q(t) \Rightarrow$ (do derivative since undefined L'Hospital Rule) $\to \left.\frac{dq}{d\omega}\right|_{\omega=\omega_0} = \frac{F_0}{m}\,\frac{t\cos(\omega_0 t) - \frac{1}{\omega_0}\sin(\omega_0 t)}{2\omega_0}$

↳ an interesting thing happens for t, as $t\to\infty$ $\left.\frac{dq}{d\omega}\right|_{\omega=\omega_0} \to \infty$

Damped SHO driven by $F_0\cos(\omega t) = F(t) = \mathrm{Re}(F_0 e^{i\omega t})$

↳ $\ddot{q} + \frac{1}{Q}\dot{q} + \omega_0^2 q = F(t)/m$

guess $q(t) = q_0 e^{i(\omega t)}$ → $q_0 = \frac{F_0/m}{(\omega_0^2-\omega^2) + i\omega/Q}$

$$q(t) = \underbrace{\frac{1}{\omega'} e^{-t/2Q}\sin(\omega' t)}_{\text{transient sol'n}} + \underbrace{\mathrm{Re}\left[\frac{F_0/m\; e^{i\omega t}}{(\omega_0^2-\omega^2) + i\omega/Q}\right]}_{\text{steady-state sol'n}}$$

multiply top and bottom by $(\omega_0^2-\omega^2) - i\ldots$ then take $\mathrm{Re}[\;]$

vanishes after a long time → solution to homogeneous part

↳ $\frac{F_0/m}{(\omega_0^2-\omega^2)^2 + \left(\frac{\omega}{Q}\right)^2}\left[\underbrace{(\omega_0^2-\omega^2)\cos(\omega t)}_{\text{in phase w/ driving force}} + \underbrace{\frac{\omega}{Q}\sin\omega t}_{\text{out of phase w/ driving force}}\right]$

energy put in by driving force

energy taken out by damping

After $t\to\infty$ $q(t) \to \frac{F_0/m}{(\omega_0^2-\omega^2)^2 + \left(\frac{\omega}{Q}\right)^2}\left[\frac{\omega}{Q}\sin(\omega t) + (\omega_0^2-\omega^2)\cos(\omega t)\right]$

↳ look at $\cos(\omega t - \phi) = \cos\phi\cos(\omega t) + \sin\phi\sin(\omega t)$

$\cos\phi = \frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2 + \left(\frac{\omega}{Q}\right)^2}$ and $\sin\phi = \frac{\omega/Q}{\sqrt{(\omega_0^2-\omega^2)^2+\left(\frac{\omega}{Q}\right)^2}}$

$\boxed{\tan\phi = \frac{\omega/Q}{\omega_0^2-\omega^2}}$ ← oscillators are phase shifted w.r.t driving force by this amount

$E = H = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega_0^2 q^2$

Green's Function Golf

want this to simplify model

SHO recieves an impulse, $F = \frac{\Delta p}{\Delta t} = I = F\Delta t = \Delta p = \int F dt$ bec of $\int$

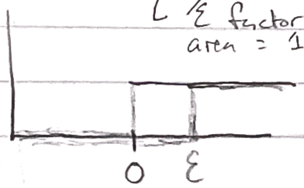
↳ we want impulse force to have infinite strength for infinite short time but finite Δp

Use the Dirac Delta Function $\rightarrow F = F_0 \delta(t-t') \Rightarrow \int F = \int F_0 \delta(t-t')\,dt = F_0(t')$

Forming an impulse from a square wave (Dirac delta in limit)

$$f_{square} = \frac{1}{\varepsilon}\left[f_{step}(t) - f_{step}(t-\varepsilon)\right]$$

↑ $1/\varepsilon$ factor for the limit to keep area = 1, in Dirac Delta notes



Impulse in $\lim_{\varepsilon\to 0}$

$$\boxed{f_{impulse}(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[f_{step}(t) - f_{step}(t-\varepsilon)\right]}$$

ODE Solution to square wave

- $\ddot{q} + \omega_0^2 q = F_{step}$, linear oscillator for $t<\varepsilon$ and $t>\varepsilon$ its a constant (overview)

↳ If we know solution to ODE for $q_{step}(t)$ and $q_{step}(t-\varepsilon)$, you can use linear superposi to combine solutions.

$$\boxed{q_{square}(t) = \frac{1}{\varepsilon}\left(q_{step}(t) - q_{step}(t-\varepsilon)\right)}$$

subtracting away these solutions to get solution of impulse

$q_{impulse}(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[q_{step}(t) - q_{step}(t-\varepsilon)\right] \rightarrow$ definition of derivative! so $\boxed{q_{impulse}(t) = \frac{dq_{step}}{dt}}$

$G = \frac{dq_{step}}{dt}$

look back here for integral at end of notes

$q_{impulse}(t)$ is called the Green's function, G, $q_{impulse} = G$ is a solution to $f_{impulse} = \delta(t-t')$, * $L(q_{impulse}) = f_{impulse}$ *

$$\boxed{L[G] = \delta(t-t')}$$

For a SHO: $\ddot{G} + \omega_0^2 G = \delta(t-t')$! driven by a step function w/ $q(0) = \dot{q}(0) = 0$

we found $q_{step}(t) = 1 - \cos(\omega_0 t)$

using $\omega_0^2 G = \frac{dq_{step}}{dt} = \omega_0 \sin(\omega_0 t)$ ← plugging in $t \to t-t'$ $\rightarrow G(t-t') = \frac{1}{\omega_0}\sin(\omega_0(t-t'))$

Lets look more at Green's function. back →

Our impulse force has infinite force → infinite acceleration, which means the 2nd derivative is discontinous. Lets try to fix that!

[We want a relationship with the derivatives of G to imply continuity] → $L[G] = \delta(t-t')$ → $\int_{t'-\varepsilon}^{t'+\varepsilon} L[G]\,dt = \int_{t'-\varepsilon}^{t'+\varepsilon} \delta(t-t')\,dt = 1$

[The bounds are like so because we can take $\lim \varepsilon \to 0$ to gain continuity] [looking at our SHO this gains continuity for the derivative] → $\int_{t'-\varepsilon}^{t'+\varepsilon} (\ddot{G} + \omega_0^2 G)\,dt = \int_{t'-\varepsilon}^{t'+\varepsilon} \delta(t-t')\,dt$

when this gets integrated we get relationship of derivatives.

[$\ddot{G}$ part] $\int_{t'-\varepsilon}^{t'+\varepsilon} \ddot{G}\,dt = \dot{G}(t'+\varepsilon) - \dot{G}(t'-\varepsilon)$

[G part] $\int_{t'-\varepsilon}^{t'+\varepsilon} G\,dt$, (for small ε you can approximate area) → [can do this because we will take limit of $\varepsilon \to 0$ later]

area = width · height
$= 2\varepsilon \cdot G$
2 bec of $-\varepsilon \to \varepsilon$

$\lim_{\varepsilon\to 0}\left[\dot{G}(t'+\varepsilon) - \dot{G}(t'-\varepsilon) + 2\varepsilon G(t')\right] = 1$ → $\boxed{\dot{G}(t'+\varepsilon) - \dot{G}(t'-\varepsilon) = 1}$ for SHO this is needed for continuity

cute show of what's happening

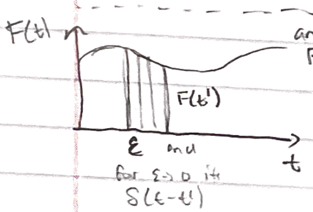
For a SHO we found ① $G(t-t') = \begin{cases} 0, & t \le t' \\ A\sin(\omega_0(t-t') + \phi), & t \ge t' \end{cases}$ could be guess in Greens function for left side is 0 right is harmonic

② [G should be continuous] → $t = t'$: $0 = A\sin(\omega_0(t'-t') + \phi)$ → $0 = \sin\phi$ → $\boxed{\phi = 0}$

③ [$\dot{G}$ is discontinuous use relation from above] → $\underbrace{\dot{G}(t'+\varepsilon)}_{\lim \varepsilon\to 0^+} - \underbrace{\dot{G}(t'-\varepsilon)}_{\lim \varepsilon \to 0^-} = 1$ → $A\omega_0 \cos\phi - 0 = 1$ so $\boxed{A = \frac{1}{\omega_0}}$

Then $G(t-t') = \begin{cases} 0, & t \le t' \\ \frac{1}{\omega_0}\sin(\omega_0(t-t')), & t \ge t' \end{cases}$

Going back to the problem generally [how to use G to get $q(t)$]



• width is infetismal, doing an integral to add up all the $F(t')$ for each $\delta(t-t')$ we use this to grab each $F(t)$ at that time

$\boxed{F(t) = \int^{T} F(t')\,\delta(t-t')\,dt'}$ ← how can relate this arbitrary forcing function to our solution?

• $F(t)$ is a superposition of impulse functions weighted by $F(t')$

• $q(t)$ is a superposition of green's functions weighted by $G(t)$

↳ moreover

$Lq = F(t)$ → $q = L^{-1}F(t)$

$LG = \delta(t-t')$ → $G = L^{-1}\delta(t-t')$

$\underbrace{L^{-1}F(t)}_{q} = \int F(t')\,\underbrace{L^{-1}\delta(t-t')}_{G}\,dt'$

$\boxed{q = \int F(t')\,G(t-t')\,dt'}$

INTRODUCTION

Green's Theorem

Lets say you have some matrix m and vector $\vec{v}$

$m\vec{v} = \vec{f}$ we can do $\vec{v} = m^{-1}\vec{f}$ invert it

Recall matrix operations: ex where: $\vec{V} = \begin{bmatrix} \bar{m}_{11} & \bar{m}_{12} \\ \bar{m}_{21} & \bar{m}_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = (\bar{m}_{11} f_1 + \bar{m}_{12} f_2)\hat{e}_1 + (\bar{m}_{21} f_1 + \bar{m}_{22} f_2)\hat{e}_2$

* bar to indicate inverse

each component so e_1's part, e_2's etc..

$\boxed{V_i = \sum_j \bar{m}_{ij} f_j}$

Differential Equations: $\frac{d^2}{dx^2}y + k^2 y = f(x)$, define $\hat{O} = \frac{d^2}{dx^2} + k^2$

So $\hat{O}y(x) = f(x)$ what if we want inverse? $y(x) = \hat{O}^{-1}f(x)$ — broad idea what Greens function is

*sometimes you can, sometimes you cant, just like how sometimes you cant invert a matrix.

probably gonna be some kind of integral operator: $\frac{dg}{dx} = \int \delta(x-x') \frac{d}{dx'} g \, dx'$ — x' because we do $\frac{d}{dx'}$; when $x = x'$ you get $\frac{dg}{dx}$ back

for $\boxed{y(x) = \int G(x,x') f(x') dx'}$, if you find the green's function you solve the diff eq, how do you solve the greens function

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Main components of Green's function:

If we have $\frac{d^2}{dx^2} f(x) = p(x) \rightarrow f(x) = \int G(x,y) P(x) dy$

and $\hat{O}\, G(x,y) = \delta(x-y)$

w/ this guy

example | derivation on back |, constants of integration, 4, 2nd order and 2 sides ($x<y$ and $x>y$)

$f(0) = f(\pi) = 0$, and the other 2 come from continuity of Green's function,

$[0,y)$ $x<y$ — CASE 1 since $x \neq y$ then $\delta = 0$ so $\frac{d^2}{dx^2}G(x,y) = 0 \rightarrow G(x,y) = Ax + B$ (left side) — A and B can be functions of y

$(y,\pi]$ ex — CASE 2 same thing $\frac{d^2}{dx^2}G(x,y) = 0$ $G(x,y) = Cx + D$ (right side) — *right is G_R and left is G_L*

*note boundary cond $f(\pi)$ doesn't apply for case 1 and likewise $f(0)$ doesn't apply for case 2

Apply $f(0)$ to case 1: $G(0,y) = A \cdot 0 + B = 0$ so $B = 0 \rightarrow \boxed{G(x,y) = Ax}$

Apply $f(\pi)$ to case 2: $G(\pi,y) = C\pi + D = 0$ so $D = -C\pi \rightarrow \boxed{G(x,y) = C(x-\pi)}$

*got 2 constants of integration just need A and C now.

continuity condition

at $x = y$: $Ay = C(y-\pi)$

Then: $Ay = (1+A)(y-\pi) \Rightarrow \boxed{A = \frac{y-\pi}{\pi}} \rightarrow$ then $\boxed{C = 1 + \frac{y-\pi}{\pi} = \frac{y}{\pi}}$

$$\boxed{G = \begin{cases} \frac{y-\pi}{\pi}x, & x<y \\ \frac{y}{\pi}(x-\pi), & x>y \end{cases}}$$

since $f(x) = \int G(x,y) p(x) dx$ (works for any $p(x)$ w/ these initial conditions)

Last one:

$\int_{y-\varepsilon}^{y+\varepsilon} \frac{d^2}{dx^2} G\, dx = \int_{y-\varepsilon}^{y+\varepsilon} \delta(x-y)\, dx = \left.\frac{dG}{dx}\right|_{y-\varepsilon}^{y+\varepsilon}$ now do $\lim_{\varepsilon \to 0^+} \frac{dG}{dx} - \lim_{\varepsilon \to 0^-} \frac{dG}{dx} = 1$, Says: $\frac{dG_R}{dx} - \frac{dG_L}{dx} = 1$ $G_R' = C$, $G_L' = A$ so $\boxed{C - A = 1}$

Derivation of Greens Function (Roughly)

Suppose we have the Linear ODE of form $L(u(x)) = f(x)$

vary along s until you reach $x = s$ then thats your area

[Before I define green's function lets look at a fact] ① $\int_{-\infty}^{\infty} \delta(x-s)\,ds = 1 \rightarrow$ ② $\int_{-\infty}^{\infty} f(s)\,\delta(x-s)\,ds = f(x)$

Now lets define a function G such that when L is applied to it, it equals $\delta(x-s)$.

$$\underbrace{L\,G(x,s)}_{} = \delta(x-s)$$

we can make this equal $f(x)$

we did this because we are essentially integrating an infinite amount of a certain function to obtain $u(x)$

$\hookrightarrow \int f(s)\,L\,G(x,s)\,ds = \int f(s)\,\delta(x-s)\,ds = f(x)$

$\hookrightarrow$ from def of linearity

$$L\underbrace{\int f(s)\,G(x,s)\,ds}_{} = f(x)$$

But wait! this is $u(x)$!

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ex from class

Find solution for a SHO w/ exponentially decaying force,

$\hookrightarrow F(x) = F_0\, e^{-\gamma t}, \quad t > 0$

$\hookrightarrow$ [Solved the greens function for any forced SHO] $G(t-t') = \begin{cases} 0, & t < t' \\ \frac{1}{\omega_0}\sin(\omega_0(t-t')), & t \geq t' \end{cases}$

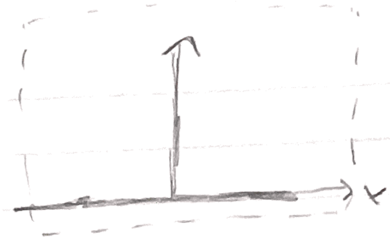
$$q(t) = \int_0^t F(t')\,G(t-t')\,dt' = \frac{F_0}{m\omega_0}\int_0^t \underbrace{\overset{\leftarrow F(t')}{e^{-\gamma t'}}\sin(\omega_0(t-t'))\,dt'}_{}$$

solve by converting to imaginary exponentials

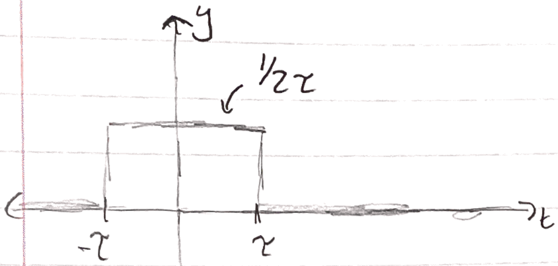
$$= \frac{F/m}{\gamma^2 + \omega_0^2}\left[e^{-\gamma t} - \left(\cos\omega_0 t - \frac{\gamma}{\omega_0}\sin\omega_0 t\right)\right]$$

Dirac Delta Function

Definition $\delta(x) = \begin{cases} \infty, & x=0 \\ 0, & x\neq 0 \end{cases}$ and $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

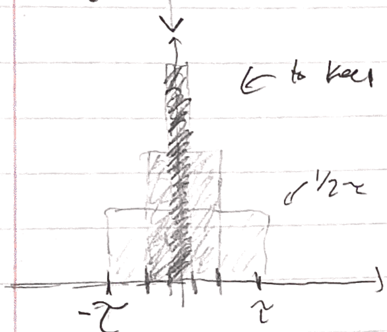


More intuition:



$$\partial(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{everywhere else} \end{cases}$$

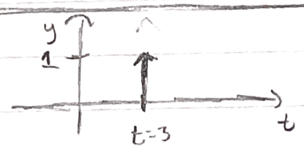
$$\int_{-\infty}^{\infty} \partial(t)\,dt = 2\tau \cdot \frac{1}{2\tau} = 1$$



← to keep area of 1 as $\tau \to 0$ the column gets infinitely high

$$\lim_{\tau \to 0} \partial(t) = \begin{cases} \infty & t=0 \\ 0 & t>0,\ t<0 \end{cases} = \delta(t)$$

Shifting it $\delta(t-3)$ →



↑ where you have spike

$$\int_{-\infty}^{\infty} \delta(t-3)\,dt = 1$$

↳ $\delta(t) = \begin{cases} \infty & t=0 \\ 0 & t\neq 0 \end{cases}$

$$\delta(t-3) = \begin{cases} \infty & t-3=0 \\ 0 & t\neq 0 \end{cases} = \begin{cases} \infty & t=3 \\ 0 & t\neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t')\,dt = f(t')$$