# Power Radiated By a Point Charge

[The Fields of point charge] → $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{R}}{(\vec{\mathscr{R}}\cdot\vec{u})^3}\left((c^2-v^2)\vec{u} + \vec{\mathscr{R}}\times(\vec{u}\times\vec{a})\right)$

$\vec{B} = \frac{1}{c}\hat{\mathscr{R}}\times\vec{E}$

use $A\times B\times C = B(A\cdot C) - C(A\cdot B)$

[Poynting Vector] → $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0 c}\underbrace{\vec{E}\times\hat{\mathscr{R}}\times\vec{E}}$

$\vec{S} = \frac{1}{\mu_0 c}\left(E^2\hat{\mathscr{R}} - (\hat{\mathscr{R}}\cdot\vec{E})\vec{E}\right)$

* Energy flux of all the fields. Fields from radiation, field energy from the particle as it moves etc...
* radiated energy is the stuff that in effect, detaches itself from the charge and propagates to infinity.

[Calculate Total Power] • the power radiated by the particle at $t_r$

* takes $\frac{\mathscr{R}}{c}$ time for it to reach sphere which is $\boxed{t - t_r = \frac{\mathscr{R}}{c}}$
* Retarded time for all points on the sphere
* $\mathscr{R}$ is from the location of the charge, all points on the sphere get the light at $\frac{\mathscr{R}}{c}$

• Since the radiation goes out to infinity, we take limit to $\mathscr{R}\to\infty$

• for $\int\vec{S}\cdot d\vec{a} = \text{Power}$, $da \propto \mathscr{R}^2$, any $\vec{S}$ term like $\frac{1}{\mathscr{R}^3}, \frac{1}{\mathscr{R}^4}$ doesn't contribute

$\mathscr{R}^2 + 1 + 2\mathscr{R}$, → still drops out

• $\frac{1}{\mathscr{R}^2}$ will give answer

$E^2 \sim \frac{\mathscr{R}^2}{\mathscr{R}^6}(\underbrace{1}_{\text{velocity term}} + \underbrace{\mathscr{R}}_{\text{acceleration term}})^2$ → • velocity term $\frac{1}{\mathscr{R}^4}$ which will drop out
• acceleration term is $\frac{1}{\mathscr{R}^2}$ which stays

$\vec{E}\perp\vec{\mathscr{R}}$

↳ $\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{R}}{(\vec{\mathscr{R}}\cdot\vec{u})^3}(\vec{\mathscr{R}}\times\vec{u}\times\vec{a})$ plug $\vec{E}_{rad}$ into $\vec{S} = \frac{1}{\mu_0 c}\left(E_{rad}^2\hat{\mathscr{R}} - \cancel{(\vec{\mathscr{R}}\cdot\vec{E})\vec{E}}\right)$

$$\boxed{\vec{S}_{rad} = \frac{1}{\mu_0 c}E_{rad}^2\hat{\mathscr{R}}}$$

At time $t_r$ the signal propagates out:

$\vec{u} = c\hat{r} - \vec{v} \Rightarrow \vec{u} = c\hat{r}$, saying at $t_r$, $\vec{v} = 0$

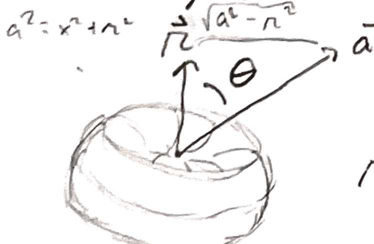
$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0 c^2}\frac{1}{r}\left(\hat{r}\times(\hat{r}\times\vec{a})\right) = \frac{\mu_0 q}{4\pi r}\left((\hat{r}\cdot\vec{a})\hat{r} - \vec{a}\right)$$

$\left(\frac{\mu_0 q}{4\pi r}\right)^2\left[\left((\hat{r}\cdot\vec{a})\hat{r} - \vec{a}\right)\right.$
$\left.\left((\hat{r}\cdot\vec{a})\hat{r} - \vec{a}\right)\right]$

$= \left(\frac{\mu_0 q}{4\pi r}\right)^2\left((\hat{r}\cdot\vec{a})^2 + a^2 - 2(\hat{r}\cdot\vec{a})(\hat{r}\cdot\vec{a})\right)$

$= \left(\frac{\mu_0 q}{4\pi r}\right)^2\left(a^2 - (\hat{r}\cdot\vec{a})^2\right)$

$$\vec{S}_{rad} = \frac{1}{\mu_0 c}E_{rad}^2\hat{r} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi r}\right)^2\left(a^2 - (\hat{r}\cdot\vec{a})^2\right)\hat{r}$$




$$\vec{S}_{rad} = \frac{\mu_0 q^2}{16\pi^2 c^2}\frac{1}{r^2}\left(a^2 - (\hat{r}\cdot\vec{a})^2\right)\hat{r}$$

$$\hat{r}\cdot\vec{a} = (a\cos\theta)^2 \rightarrow a^2 - a^2\cos^2\theta = a^2\overbrace{(1-\cos^2\theta)}^{\sin^2\theta}$$

$$\boxed{\vec{S}_{rad} = \frac{\mu_0 q^2}{16\pi^2 c^2}\frac{a^2}{r^2}\sin^2\theta\,\hat{r}}$$

, $\begin{bmatrix}\text{total}\\ \text{power}\end{bmatrix} \rightarrow P = \oint \vec{S}_{rad}\cdot d\vec{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c^2}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta\, d\theta\, d\phi$

$\int \sin^2\theta = \frac{8\pi}{3}$ or something

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}}$$

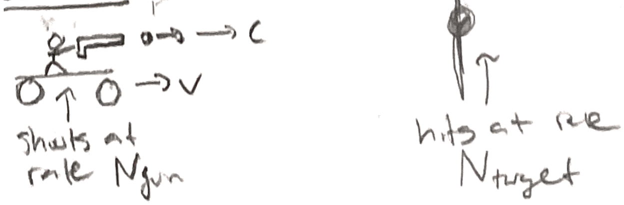
Larmor Formula

*Derived for $V=0$, holds well for $V \ll c$. $V \neq 0$ is harder.

1. $E_{rad}$ becomes more complicated w/ $v$ term

2. Also, the rate at which energy passes through the sphere, $\neq$ as the rate at which energy left the particle.

Because:




↳ $N_{gun} = (1 - \frac{v}{c})N_{target}$ or generally $N_{gun} = \left(1 - \frac{\hat{r}\cdot\vec{v}}{c}\right)N_{target}$

* Shoots at constant rate $N_{gun}$, as it moves toward target more bullets hit it same time interval


frequency

$N_{gun} = \frac{v}{L'}$

$N_{target} = \frac{v}{L}$

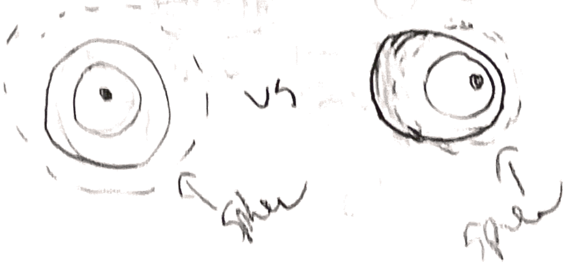
$N = f =$ hits/sec

$N_{gun} L = N_{target} L'$

$N_{gun} = N_{target}\frac{L}{L'} = (1 - \frac{v}{c})N_{target}$

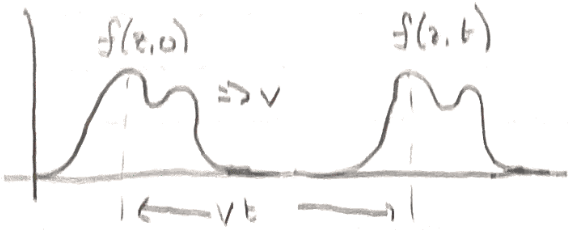
From before:

$$\frac{L'}{L} = \frac{1}{1 - \frac{v}{c}} \rightarrow$$



# Electromagnetic Waves

Describing a wave mathematically



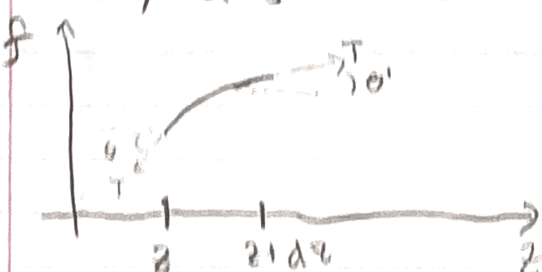

* given initial shape $g(z) = f(z,0)$

• displacement at point $z$ at later time $t$ is the same as the displacement a distance $vt$ to the left $(z - vt)$

$$f(z,t) = f(z - vt) = g(z - vt)$$

## Stretch String Supporting Wave Motion

• Why does a stretched string support wave motion?




*imagine a very long string under tension $T$

* Now displace it from equilibrium *

for small distortions

Net vertical force: $\Delta F = T\sin\theta' - T\sin\theta \approx T(\underbrace{\tan\theta' - \tan\theta}_{\text{slopes}})$

$$\Delta F = T\left(\frac{\partial f}{\partial z}\Big|_{z+\Delta z} - \frac{\partial f}{\partial z}\Big|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z$$

Newtons second law: $\Delta F = ma = (\underbrace{\mu}_{\text{mass per unit length}}\Delta z)\frac{\partial^2 f}{\partial t^2}$

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}}\quad , \quad v = \sqrt{\frac{T}{\mu}} = \text{speed of propagation}$$

↓

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

(admits all solutions of the form) $f(z,t) = g(z - vt)$

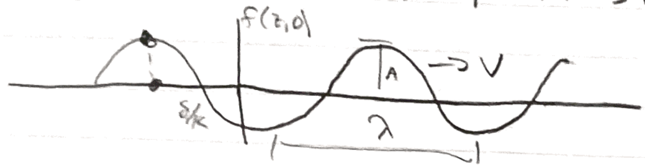
Little Proof $u \equiv z - vt$, $f(z,t) = g(z - vt)$

$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du} \rightarrow \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2}$ solves $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}$

$\frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du} \rightarrow \frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2}$ ✓

Another Class of solutions is: $f(z,t) = h(z + vt)$
Most general Solution is: $f(z,t) = g(z - vt) + h(z + vt)$

Sinusoidal Waves $f(z,t) = A\cos(k(z - vt) + \delta)$, $\lambda = \frac{2\pi}{k}$




time for one full cycle (period) $T = \frac{\lambda}{v} = \frac{2\pi}{kv}$ and (frequency) # of oscillations per unit time (cycles) $\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$.

(angular freq) # of radians swept out per unit time: $\omega = 2\pi\nu = kv$

$\rightarrow f(z,t) = A\cos(kz - \omega t + \delta)$ $\frac{\delta}{k}$ meant delay & swapped signs bec delay would be to the right

Compare!

$f(z,t) = A\cos(kz + \omega t - \delta)$ (wave traveling to left)

$\rightarrow f(z,t) = A\cos(-kz - \omega t + \delta)$

*Change wave direction by changing $k$ ✓

$\boxed{\text{Complex Notation}}$ $e^{i\theta} = \cos\theta + i\sin\theta$ so $f(z,t) = \mathrm{Re}\left[Ae^{i(kz - \omega t + \delta)}\right]$

$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz - \omega t)} \rightarrow f(z,t) = \mathrm{Re}\left[\tilde{f}(z,t)\right]$ and $\tilde{A} \equiv Ae^{i\delta}$

$\boxed{\text{Any wave can be expressed as a linear combination of sin}}$ $\rightarrow \tilde{f}(z,t) = \int_{-\infty}^{\infty}\tilde{A}(k)\,e^{i(kz - \omega t)}\,dk$

# Boundary Conditions

- Two strings attached to one another but different mass per unit lengths, $\mu$ so different speed $V$'s bec $V=\sqrt{T/\mu}$

- Two different strings meet at $z=0$

$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)}$ $z<0$ (incident wave)

$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{V_1}{V_2}$

↳ Reflected wave: $\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}$ $z<0$

Same speed $V$ bec $T$ and $\mu$ don't change just energy split

↳ Transmitted wave: $\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)}$ $z>0$

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)} \\ \tilde{A}_T e^{i(k_2 z - \omega t)} \end{cases}$$

At the joint $\underbrace{f(0^-,t) = f(0^+,t)}_{1}$ and $\underbrace{\frac{\partial f}{\partial z}\Big|_{0^-} = \frac{\partial f}{\partial z}\Big|_{0^+}}_{2}$

1 $\tilde{A}_I + \tilde{A}_R = \tilde{A}_T$ and 2 $k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T$

↳ $\tilde{A}_R = \frac{k_1 - k_2}{k_1 + k_2}\tilde{A}_I$ , $\tilde{A}_T = \frac{2k_1}{k_1+k_2}\tilde{A}_I$

↓ velocities ↓ velocities

$\tilde{A}_R = \frac{V_2 - V_1}{V_2 + V_1}\tilde{A}_I$ $\tilde{A}_T = \frac{2V_2}{V_2+V_1}\tilde{A}_I$ , remember $\tilde{A} = A e^{i\delta}$

- if second string is lighter than first, $\mu_2 < \mu_1$, $V_2 > V_1$ all three have same phase angle $\delta_R = \delta_T = \delta_I$
- if second string is heavier than first $V_2 < V_1$ reflected is out of phase by $180°$ → $\delta_R + \pi = \delta_T = \delta_I$ makes wave upside down

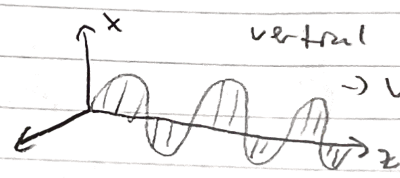
# Polarization

- waves down a string when you shake it are transverse bc/ displacement is $\perp$ to direction of propagation
- longitudinal waves are like compressions in a slinky

Two kinds of polarization: up and down + left and Right
vertical horizontal

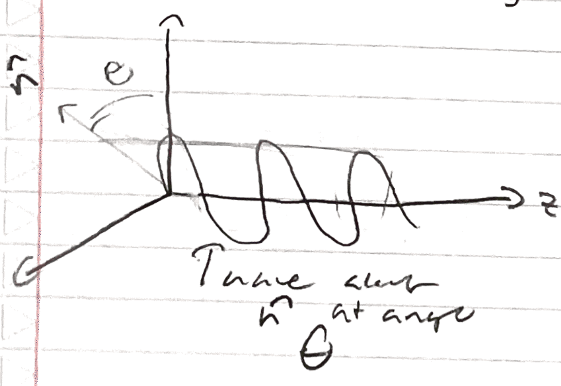
$\tilde{f}_v(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{x}$

$\tilde{f}_h(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{y}$




Generally: $\tilde{f}(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{n}$ and $\hat{n}\cdot\hat{z} = 0$

$\hookrightarrow \hat{n} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$




$\tilde{f}(z,t) = \tilde{A}\cos\theta\, e^{i(kz-\omega t)} \hat{x} + \tilde{A}\sin\theta\, e^{i(kz-\omega t)} \hat{y}$

Superposition of two waves

$E = pc$ $\qquad$ $\hat{z} \times \vec{E}_0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ E_x & E_y & 0 \end{vmatrix} = -E_y + E_x$

## Wave equation for E and B

Regions of no charge or current:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t \qquad \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

} set of coupled first order PDE, decouple

↑ apply curl to this

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$
$$\nabla(\cancel{\nabla \cdot E}^{\,0}) - \nabla^2 E = \nabla \times \left(-\frac{\partial B}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times B) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

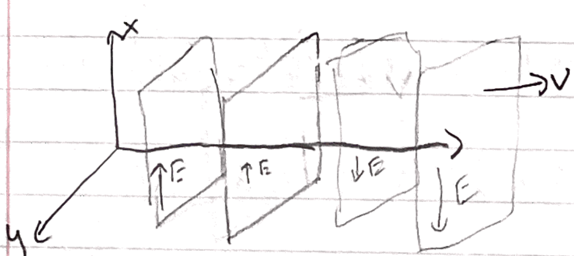
$$\nabla \times (\nabla \times B) = \nabla(\cancel{\nabla \cdot B}^{\,0}) - \nabla^2 B = \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times E) = -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Thus $\boxed{\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \text{and} \quad \nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}}$ Wave eq!

$$V = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \,!!$$

## Monochromatic Plane Waves

* light from distance stars are plane waves, all rays of light come in parallel




$$\boxed{\begin{aligned} \tilde{\vec{E}}(z,t) &= \tilde{\vec{E}}_0 e^{i(kz - \omega t)} \\ \tilde{\vec{B}}(z,t) &= \tilde{\vec{B}}_0 e^{i(kz - \omega t)} \end{aligned}}, \quad \omega = ck$$

## Further Restrictions on Wave equation Solutions

EM waves are transverse

z diff being 0 is interesting bec no change in dir of V!

$$\vec{\nabla} \cdot \tilde{\vec{E}} = 0 \rightarrow 0 = \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y} + \frac{\partial \tilde{E}_z}{\partial z} \rightarrow \frac{\partial}{\partial x}(\cancel{E_x - B_x}) + \frac{\partial}{\partial y}(\cancel{E_y - B_y}) + \underbrace{\frac{\partial}{\partial z}(E_z - B_z)}_{0} = 0$$
$$0 = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

cross outs are 0 bec wave has form $\cos(kz - \omega t)$ only z-dep!

$$\nabla \times E = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\cancel{\frac{\partial E_y}{\partial x}} - \cancel{\frac{\partial E_x}{\partial y}}\right)\hat{z}$$

no x,y dep $\qquad$ no x,y dep $\qquad$ no x,y dep

$$(\vec{\nabla} \times \vec{E}) \cdot \hat{x} = -\frac{\partial \vec{B}}{\partial t} \cdot \hat{x}$$

same $(\vec{\nabla} \times \vec{E}) \cdot \hat{y} = -\frac{\partial \vec{B}}{\partial t} \cdot \hat{y}$

$$-k(E_0)_y = \omega(B_0)_x \quad \text{and} \quad k(E_0)_x = \omega(B_0)_y$$

$$(\tilde{B}_0)_x = -\frac{k}{\omega}(\tilde{E}_0)_y \quad \text{and}$$
$$(\tilde{B}_0)_y = \frac{k}{\omega}(E_0)_x \qquad \hat{z} \times \vec{E} = -E_y \hat{x} + E_x \hat{y}$$

$$\rightarrow \boxed{\tilde{B}_0 = \underbrace{\frac{k}{\omega}}_{1/c}(\hat{z} \times \tilde{\vec{E}}_0)}$$

in phase and perpendicular


Energy in volume $= \underbrace{u A c \Delta t}_{\text{volume}}$ → $\underbrace{\frac{\text{energy}}{\text{time}}}_{\text{power}} \frac{1}{\text{Area}} = uc$

Amplitudes: $B_0 = \frac{E_0}{c}$, polarization dir in dir of $E$

## Generalized direction

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{n} \quad \text{and} \quad \tilde{\vec{B}}(\vec{r},t) = \frac{1}{c}\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n})$$

$$= \frac{1}{c}(\hat{k}\times\vec{E})$$

* $\hat{k}$ points in direction of propagation * magnitude is $k$
  ↳ $\vec{k}\cdot\vec{r} \Rightarrow kz$

* $\hat{n}$ is polarization vector, $\hat{n}\cdot\hat{k} = 0$ *, direction of $E$

Or

$$\vec{E}(\vec{r},t) = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\,\hat{n}$$
$$\vec{B}(\vec{r},t) = \frac{1}{c} E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)(\hat{k}\times\hat{n})$$

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Energy + Momentum in waves

$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$, $B^2 = \frac{E^2}{c^2} = \mu_0\epsilon_0 E^2$ → $u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$

↳ Direction of Energy Travel: $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = cu\,\hat{z}$ (for monochromatic plane wave)

Momentum: $\vec{g} = \frac{1}{c^2}\vec{S} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{z} = \frac{u}{c}\hat{z}$

Averages, $\langle \cos^2(\theta)\rangle = 1/2$ so

$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2$, $\underbrace{\langle \vec{S}\rangle = \frac{1}{2}c\,\epsilon_0 E_0^2\,\hat{z}}_{\text{average power per unit area is Intensity}}$, $\langle \vec{g}\rangle = \frac{1}{2c}\epsilon_0 E_0^2\,\hat{z}$

Radiation Pressure $= P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}$

$\Delta p = \langle g \rangle A c \Delta t$

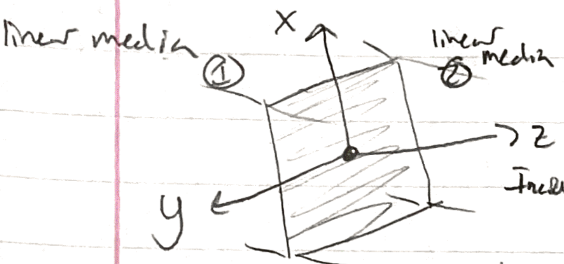
$\boxed{\text{Boundary Conditions}}$ i) $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$ iii) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$

ii) $B_1^{\perp} = B_2^{\perp}$ iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$

$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$

$v = c/n$

## Reflection and Transmission at Normal Incidence




A wave polarized in the x-dir approaches from the left:

Incident $\begin{cases} \tilde{\vec{E}}_I(z,t) = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\,\hat{x} \\ \tilde{\vec{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\,\hat{y} \end{cases}$

* what is light polarized differently instead

Reflection $\begin{cases} \tilde{\vec{E}}_R = \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\,\hat{x} \\ \tilde{\vec{B}}_R = \frac{1}{v_1}\tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\,(-\hat{y}) \end{cases}$

direction by pointing vector

* how can we say at $z=0$ it's a maximum at the boundary?

Transmitted $\begin{cases} \tilde{\vec{E}}_T = \tilde{E}_{0T}\, e^{i(k_2 z - \omega t)}\,\hat{x} \\ \tilde{\vec{B}}_T = \frac{1}{v_2}\tilde{E}_{0T}\, e^{i(k_2 z - \omega t)}\,\hat{y} \end{cases}$

These equations have to satisfy the boundary cond. at $z=0$

Total E field on left = $\tilde{\vec{E}}_I + \tilde{\vec{E}}_R$, B field = $\tilde{\vec{B}}_I + \tilde{\vec{B}}_R$

Total E + B on Right = $\tilde{\vec{E}}_T$, $\tilde{\vec{B}}_T$

No z component of E, and B so no E field, B field $\perp$ to surface so i) = 0, ii) = 0

iii) $(\tilde{\vec{E}}_I + \tilde{\vec{E}}_R)^{\parallel} = (\tilde{\vec{E}}_T)^{\parallel} \rightarrow \left(\tilde{E}_{0I}\,\cancel{e^{i(k_1 z - \omega t)}}^{\,z=0}\,\hat{x} + \tilde{E}_{0R}\,\cancel{e^{i(-k_1 z - \omega t)}}^{\,z=0}\,\hat{x}\right)^{\parallel} = \left(\tilde{E}_{0T}\,\cancel{e^{i(k_2 z - \omega t)}}^{\,z=0}\,\hat{x}\right)^{\parallel}$

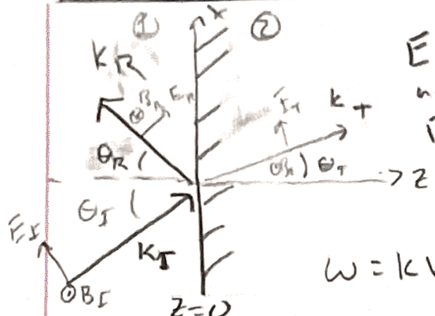
* the component $\parallel$ to surface is just $E\,\hat{x}$, dot w/ $\hat{x}$

↳ $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

$\boxed{\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_1}{\mu_2 n_2}}$

iv) $\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0I} - \frac{1}{v_1}\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\frac{1}{v_2}\tilde{E}_{0T} \rightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \beta\,\tilde{E}_{0T}$

↳ Solve $\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I}$, $\tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I}$




E and B

$$\tilde{\vec E}_I(\vec r,t)=\tilde E_{0I}\, e^{i(\vec k_I\cdot\vec r-\omega t)}, \quad \tilde{\vec B}_I(\vec r,t)=\frac{1}{v_1}(\hat k_I\times\tilde{\vec E}_I)$$

$$\tilde{\vec E}_R(\vec r,t)=\tilde E_{0R}\, e^{i(\vec k_R\cdot\vec r-\omega t)}, \quad \tilde{\vec B}_R(\vec r,t)=\frac{1}{v_1}(\hat k_R\times\tilde{\vec E}_R)$$

$$\tilde{\vec E}_T(\vec r,t)=\tilde E_{0T}\, e^{i(\vec k_T\cdot\vec r-\omega t)}, \quad \tilde{\vec B}_T(\vec r,t)=\frac{1}{v_2}(\hat k_T\times\tilde{\vec E}_T)$$

$\boxed{k_I=k_R=\frac{v_2}{v_1}k_T}$

$\omega=kv$, $\omega$ is same for all, $\omega=k_Iv_1=k_Rv_1=k_Tv_T$

for all $x$ → quick proof: $Ae^{iax}+Be^{ibx}=Ce^{icx}$, $a=b=c\to A+B=C$

[at $x=0$: $A+B=C$ done ✓]

[at $x=0$, 1st der] → $aA+bB=cC$

[at $x=0$, 2nd der] → $a^2A+b^2B=c(cC)$ → $a^2A+b^2B=c(aA+bB)$

$(A+B)(a^2A+b^2B)=(A+B)c(aA+bB)=$ … $cC$ plug in …

$a^2A^2+b^2AB+a^2AB+b^2B^2=(aA+bB)^2=a^2A^2+2abAB+b^2B^2$ → $AB(a-b)^2=0$, $A,B$ non zero

so $a=b$ → $(A+B)e^{iax}=Ce^{icx}\xrightarrow{\text{diff}}a(A+B)=cC$ → $aC=cC$, $a=b=c$ ✓

[All boundary cond have the form] → $(\ )e^{i(\vec k_I\cdot\vec r-\omega t)}+(\ )e^{i(\vec k_R\cdot\vec r-\omega t)}=(\ )e^{i(\vec k_T\cdot\vec r-\omega t)}$ at $z=0$

something from bound conditions

[Now use proof above $a=b=c$] → [$\vec k_I\cdot\vec r=$, $\vec k_R\cdot\vec r=$, $\vec k_T\cdot\vec r$]

$x(k_I)_x+y(k_I)_y=x(k_R)_x+y(k_R)_y=x(k_T)_x+y(k_T)_y$

→ [$x,y$ independently change from one another so →] $(k_I)_y=(k_R)_y=(k_T)_y$

$[(k_I)_x=(k_R)_x=(k_T)_x]$

First law

* $\vec k$ points in the dir of the wave, for any wave, its incident, reflected, transmitted components will all lie in the same plane, plane of incidence. [So we orient our axis so $k_I$ lies in $xz$ plane]

Second law: $(k_I)_x=(k_R)_x=(k_T)_x\Rightarrow k_I\sin(\theta_I)=k_R\sin(\theta_R)=k_T\sin(\theta_T)$

from constant $\omega$: $k_I=k_R$ → $\boxed{\theta_I=\theta_R}$ law of reflection

Third law: $\boxed{\frac{\sin(\theta_T)}{\sin(\theta_I)}=\frac{k_T}{k_I}=\frac{v_2}{v_1}=\frac{n_1}{n_2}}$ Snell's Law, $\theta_T$ = angle of transmission or angle of refraction

**BOUNDARY CONDITIONS**, forget $e^{\dots}$ since we proved they all have to be equal

* say polarization is in plane, $E_I$ in $xz$ plane

i) $\varepsilon_1(\tilde{\vec E}_{0I}+\tilde{\vec E}_{0R})_z^{\perp}=\varepsilon_2(\tilde{\vec E}_{0T})_z$ → $\varepsilon_1(-\tilde E_{0I}\sin(\theta_I)+\tilde E_{0R}\sin(\theta_R))=\varepsilon_2(-\tilde E_{0T}\sin(\theta_T))$

ii) $(\tilde{\vec B}_{0I}+\tilde{\vec B}_{0R})_z=(\tilde{\vec B}_{0T})_z$ → $0=0$ no z-comp

iii) $(\tilde{\vec E}_{0I}+\tilde{\vec E}_{0R})_{x,y}=(\tilde{\vec E}_{0T})_{x,y}$ — [pair of eqs, one for x one for y] → no y comp → $\tilde E_{0I}\cos(\theta_I)+\tilde E_{0R}\cos(\theta_R)=\tilde E_{0T}\cos(\theta_T)$

iv) $\frac{1}{\mu_1}(\tilde{\vec B}_{0I}+\tilde{\vec B}_{0R})_{x,y}=\frac{1}{\mu_2}(\tilde{\vec B}_{0T})_{x,y}$, B in y dir → $\frac{1}{\mu_1v_1}(\tilde E_{0I}-\tilde E_{0R})=\frac{1}{\mu_2v_2}\tilde E_{0T}$

use Snells law & law of reflection

$\beta$ is same if you work out

[i) and iv)] $E_{0I}-E_{0R}=\beta E_{0T}$ and [iii)] $E_{0I}+E_{0R}=\alpha E_{0T}$, $\alpha=\frac{\cos(\theta_T)}{\cos(\theta_I)}$

$$\boxed{\tilde E_{0R}=\frac{\alpha-\beta}{\alpha+\beta}E_{0I}\quad\text{and}\quad\tilde E_{0T}=\left(\frac{2}{\alpha+\beta}\right)\tilde E_{0I}}$$

for polarization in the plane of incidence

$$\alpha=\frac{\sqrt{1-\sin^2(\theta_T)}}{\cos(\theta_I)}=\frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin(\theta_I)\right)^2}}{\cos(\theta_I)}$$

Linear Media magnetism: $M = \chi_m H$

$\hookrightarrow \vec{B} = \mu_0(\vec{H} + \vec{M}) = \underbrace{\mu_0(1+\chi_m)}_{\mu}\vec{H} = \mu\vec{H}$

# EM Waves in Matter

Recall:
$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$$

If the medium linear, $\vec{D} = \varepsilon\vec{E}$, $\vec{H} = \frac{1}{\mu}\vec{B}$ and homogenous

$$\left.\begin{array}{ll} \nabla\cdot E = 0 & \nabla\times E = -\frac{\partial B}{\partial t} \\ \nabla\cdot B = 0 & \nabla\times B = \mu\varepsilon\frac{\partial E}{\partial t} \end{array}\right\}$$ only difference w/ vacuum analogs is $\mu_0\varepsilon_0$ by $\mu\varepsilon$

Wave speed: $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$ where $n \equiv \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$ = index of refraction

$n \cong \sqrt{\varepsilon_r}$ ← may depend on frequency

Everything carries over! $u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right)$, $\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B})$

Intensity $= I = \frac{1}{2}\varepsilon v E_0^2$, $\omega = kv$, $\frac{B_0}{v} = E_0$

## Review of Boundary conditions in Matter

$V = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{r}\,da' + \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_b}{r}\,d\tau'$

[Quick review of polarization] → Just by saying an object is polarized, lots of dipoles lined up → work out potential to get → $\sigma_b \equiv \vec{P}\cdot\hat{n}$, $\rho_b = -\vec{\nabla}\cdot\vec{P}$

↳ By looking at the total charge density, $\rho = \rho_b + \rho_f$ ($\varepsilon_0\nabla\cdot E = \rho$) → $\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P}$, $\vec{\nabla}\cdot\vec{D} = \rho_f$

↳ linear medium: $\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0 E + \varepsilon_0\chi_e E = \varepsilon\vec{E}$, $\vec{D} = \varepsilon\vec{E}$

[Quick review of magnetization] → $\vec{m} = \text{dipole moment} = I\vec{a}$, object can have many little m's → $\vec{M} = \vec{m}/\text{volume}$

like above looking at the vector potential of magnetized object!

$\vec{J}_b = \vec{\nabla}\times\vec{M}$
$\vec{K}_b = \vec{M}\times\hat{n}$
→ Say there can be two currents $\vec{J} = \vec{J}_b + \vec{J}_f$ → rewrite Amperes law → $\frac{1}{\mu_0}(\nabla\times B) = J$ → $\nabla\times\underbrace{\left(\frac{1}{\mu_0}B - M\right)}_{H} = J_f$

## Maxwell Eqs in Matter

$da = \sigma_b\,da_\perp$

$I = \frac{d\sigma_b}{dt}da_\perp = \frac{\partial P}{\partial t}da_\perp$

$\sigma_b = \vec{P}\cdot\hat{n} = P$ → $(J = \frac{\partial P}{\partial t})$

$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0\nabla\cdot E)$

$= -\nabla\cdot\left(\varepsilon_0\frac{\partial E}{\partial t}\right)$ this correction fixes Ampere's eq

$\boxed{\begin{array}{l}\nabla\cdot B = 0 \\ \nabla\times E = -\frac{\partial B}{\partial t}\end{array}}$ Don't involve P or J so don't change → Gauss: $\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$

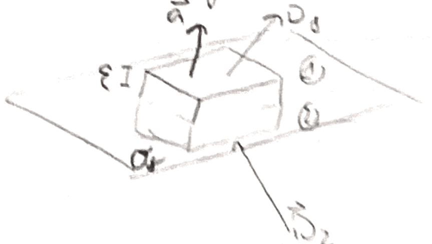
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ → $\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$

w/ $D = \varepsilon_0 E + P$ & $H = \frac{1}{\mu_0}B - M$ → $\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial D}{\partial t}}$

# Boundary Conditions in Matter

Rewrite Maxwell eq in Matter in Integral form:

$$\boxed{\begin{aligned} &\oint \vec{D}\cdot d\vec{a} = Q_{free} \\ &\oint_S \vec{B}\cdot d\vec{a} = 0 \\ &\oint_P \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} \\ &\oint_P \vec{H}\cdot d\vec{l} = I_{free} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{a} \end{aligned}}$$




Gaussian pill box we obtain:

$$\vec{D}_1\cdot\vec{a} - \vec{D}_2\cdot\vec{a} = \sigma_f a$$

Take limit as $\varepsilon \to 0$ so edge sides contribute nothing as well as and $\rho$ in the region. So just on the surface:

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f}$$ Same w/ $\oint \vec{B}\cdot d\vec{a} = 0 \to \boxed{B_1^{\perp} - B_2^{\perp} = 0}$

Now make a thin amperian loop: $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$


Take $\varepsilon \to 0$ so ends are gone: $\underbrace{\vec{E}_1\cdot\vec{l} - \vec{E}_2\cdot\vec{l}}_{\text{in } \parallel \text{ dir}} = \cancel{-\frac{d}{dt}\int \vec{B}\cdot d\vec{a}} \to 0 \to \boxed{E_1^{\parallel} - E_2^{\parallel} = 0}$
And the flux vanishes

Same logic for $\oint \vec{H}\cdot d\vec{l} = I_{free} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{a} \to 0$ as $\varepsilon \to 0$ $\to \vec{H}_1\cdot\vec{l} - \vec{H}_2\cdot\vec{l} = I_{free}$
just ∥ amp

no volume current density as $\varepsilon \to 0$

$* \; I_{free} = \vec{K}_f\cdot(\underbrace{\hat{n}\times\vec{l}}_{\text{normal to amperian loop}}) = (\vec{K}_f \times \hat{n})\cdot\vec{l} \to \boxed{H_1^{\parallel} - H_2^{\parallel} = \vec{K}_f \times \hat{n}}$

In linear Media where: $\vec{D} = \varepsilon\vec{E}$, $\vec{B} = \mu\vec{H}$ and no free charge no free current

$$\boxed{\begin{aligned} &\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0 \qquad && E_1^{\parallel} - E_2^{\parallel} = 0 \\ &B_1^{\perp} - B_2^{\perp} = 0 && \frac{1}{\mu_1}B_1^{\parallel} - \frac{1}{\mu_2}B_2^{\parallel} = 0 \end{aligned}}$$

# Guided Waves

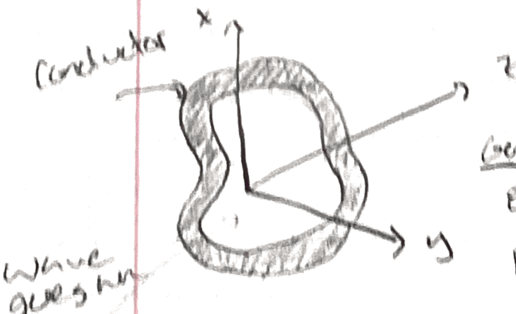
- started w/ plane waves w/ infinite extent
- now we consider EM waves confined to a hollow pipe, perfect conductor

net charge just as much + as minus

↳ then $\vec{E} = 0$ and $\vec{B} = 0$ inside the material

↳ if $E = 0$ then by $\nabla \cdot E = \rho/\epsilon_0$, $\rho = 0$ too

↳ only charge that's left is on the surface




General Boundary Conditions

$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$ and $E_2^{\parallel} - E_1^{\parallel} = 0$

$B_1^{\perp} - B_2^{\perp} = 0$ and $H_1^{\parallel} - H_2^{\parallel} = \vec{K}_f \times \hat{n}$

$B^{\perp} = 0$, $\frac{\partial B}{\partial t} = 0$

$E_1^{\parallel}, B_1^{\perp}$

inner wall

$E_2^{\parallel}, B_2^{\perp} = 0$

$\left[\begin{array}{l}\text{If } \vec{E} = 0 \text{ inside the} \\ \text{the conductor then}\end{array}\right] \rightarrow \vec{E}^{\parallel} = 0$ and $B^{\perp} = 0$

(propagating down wave guide)

General Waves: $\vec{E}(x,y,z,t) = \vec{E}_0(x,y)\, e^{i(kz - \omega t)}$ and $\vec{B} = \vec{B}_0\, e^{i(kz - \omega t)}$

* Need the wave above to satisfy Maxwell eq

* For the above Boundary Conditions

$\left[\begin{array}{l}\text{Must Satisfy} \\ \text{Maxwell Eq.}\end{array}\right]$ $\vec{\nabla} \cdot \vec{E} = 0$ $\quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0$ $\quad \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Notation: $\longrightarrow \vec{E}_0 = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$, $\vec{B}_0 = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z$

$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x$

$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y$

Plug in $\vec{E} = \vec{E}_0(x,y)\, e^{i(kz - \omega t)}$

taking derivative of $z$, all others are zero, we know $z$-dependence

$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \rightarrow \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z$

$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2} E_x$

$ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2} E_y$

if $k = \frac{\omega}{c}$ then these eq are indeterminate

Solve for $E_x$, $E_y$ $B_x$, $B_y$

$$E_x = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$E_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_x = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

Need to determine longitudinal components $E_z$ and $B_z$

We have the remaining maxwell eq.

$\vec{\nabla}\cdot\vec{E} = 0$ and

$\vec{\nabla}\cdot\vec{B} = 0$

Put into div ! the $E_x$, $E_y$, $B_x$, $B_y$ (So only $E_z$ & $B_z$ left)

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right)E_z = 0\\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right)B_z = 0\end{aligned}\right\}$$ uncoupled eq in $E_z$ & $B_z$

$E_z = 0 \rightarrow$ transverse electric waves TE waves

$B_z = 0 \rightarrow$ transverse magnetic waves TM waves

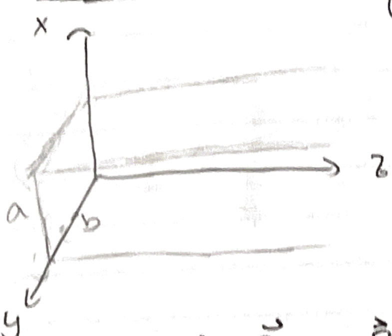
Both $0 \rightarrow$ TEM waves

(TEM waves can't be in hollow wave guide) $\rightarrow$ $E_z = 0$ $\hookrightarrow$ gauss $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$

$B_z = 0$

$\nabla\times E = -\frac{\partial B}{\partial t} \rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$

## Guided Waves




Wave propagates: $\tilde{\vec{E}}(x,y,z,t) = \tilde{\vec{E}}_0(x,y)\, e^{i(kz-\omega t)}$

$\tilde{\vec{B}}(x,y,z,t) = \tilde{\vec{B}}_0(x,y)\, e^{i(kz-\omega t)}$

$\boxed{\text{Maxwell Equations}}$ $\vec{\nabla}\cdot\vec{E}=0$ $\quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla}\cdot\vec{B}=0$ $\quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

$\boxed{\text{Boundary conditions}}$ E is zero in conductor then by

$E_1^{\parallel} - E_2^{\parallel} = 0 \to E^{\parallel} = 0$ on surface

$B^{\perp} = 0$ as well on surface

Components of $\vec{E}$ and $\vec{B}$:

Pull out and cancel

$\tilde{\vec{E}} = (E_x\hat{x} + E_y\hat{y} + E_z\hat{z})\, e^{i(kz-\omega t)}$ and $\tilde{\vec{B}} = (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})\, e^{i(kz-\omega t)}$

$-\partial B/\partial t$ ?

$e^{i}\sim$ $\vec{\nabla}\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - ikE_y\right)\hat{x} + \left(ikE_x - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z} = +i\omega B_x\hat{x} + i\omega B_y\hat{y} + i\omega B_z\hat{z}$

Component by component have to equal!

$$-ikE_y + \frac{\partial E_z}{\partial y} = i\omega B_x, \quad -\frac{\partial E_z}{\partial x} + ikE_x = i\omega B_y, \quad -\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} = i\omega B_z$$

For $\nabla\times B$ same but flip E & B and $1/c^2$!

$$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x, \quad ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y, \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$$

Solve for $E_x$, $E_y$, $B_x$, $B_y$ in terms of z-components

solve other var

$-ikE_y + \frac{\partial E_z}{\partial y} = i\omega B_x$

$ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$

$\to -ikE_y + \frac{\partial E_z}{\partial y} = \omega\left(\frac{-i\omega}{kc^2}E_y + \frac{1}{k}\frac{\partial B_z}{\partial x}\right)$

$\hookrightarrow -ikE_y + \frac{\omega^2}{kc^2}E_y = \frac{\omega}{k}\frac{\partial B_z}{\partial x} - \frac{\partial E_z}{\partial y}$

$$\boxed{E_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)}$$

$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x$

$-\frac{\partial E_z}{\partial x} + ikE_x = i\omega B_y$

math

$$\boxed{E_x = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)}$$

$$\boxed{B_x = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)} \quad \text{and} \quad \boxed{B_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)}$$

Finally Use $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{B} = 0$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0 \to \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial^2 E_z}{\partial x^2} + \omega\frac{\partial^2 B_z}{\partial x\partial y} + k\frac{\partial^2 E_z}{\partial y^2} - \omega\frac{\partial^2 B_z}{\partial x\partial y}\right) + ikE_z = 0$$

$$\frac{1}{(\frac{\omega}{c})^2 - k^2}\left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2}\right) + E_z = 0 \to \boxed{\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(\left(\frac{\omega}{c}\right)^2 - k^2\right)E_z = 0}$$

$$\frac{\partial^2}{\partial x^2}E_z + \frac{\partial^2}{\partial y^2}E_z + \left(\frac{\omega^2}{c^2} - k^2\right)E_z = 0$$

$$\frac{\partial^2}{\partial x^2}B_z + \frac{\partial^2 B_z}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right)B_z = 0$$

Boundary Conditions

$\vec{E}^{\parallel} = 0 \,|_{surface}$

$\vec{B}_{\perp} = 0 \,|_{surface}$

w/ $E_x = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$ $\quad B_x = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$

$E_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$ $\quad B_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$

---

TM Waves, $B_z = 0$, just need to solve $E_z$ eq

have to be minus bec 0 on walls!!!

$E_z = X(x)Y(y) \rightarrow \frac{\partial^2}{\partial x^2}XY + \frac{\partial^2}{\partial y^2}XY + \left(\frac{\omega^2}{c^2} - k^2\right)XY = 0$

$\hookrightarrow \underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{\text{constants}} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{\text{constants}} + \left(\frac{\omega^2}{c^2} - k^2\right) = 0 \rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$

[General Solution for $X(x)$] $\rightarrow X(x) = A\sin(k_x x) + B\cos(k_x x)$

Now for Boundary conditions $E^{\parallel} = 0$

$E_z(x=0) = 0$ and $E_z(x=a) = 0$

$\hookrightarrow 0 = A\cdot 0 + B \rightarrow B = 0 \rightarrow 0 = A\sin(k_x a) \rightarrow \boxed{k_x = \frac{m\pi}{a}}$

[General Solution for $Y(y)$] $\rightarrow Y(y) = A\sin(k_y y) + B\cos(k_y y) \rightarrow E_z(y=0) = 0$ and $E_z(y=b) = 0$

$\boxed{k_y = \frac{n\pi}{a}}$

Total Solution $E_z(x,y) = E_0 \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)$, $k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$

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TE Waves $E_z = 0$, Solve $B_z$ eq, same general solu

$B_z = X(x)Y(y) \rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial y^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right) = 0 \rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$

$\hookrightarrow$ [General Solution for $X(x)$] $\rightarrow X(x) = A\sin(k_x x) + B\cos(k_x x)$ | Boundary cond $B_x(x=0)$ and $B_x(x=a) = 0$

But $B_x = 0 = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c}\cdot 0\right) \rightarrow \frac{\partial B_z}{\partial x} = \frac{\partial X(x)}{\partial(x)}Y(y) = 0, \quad \frac{\partial X(x)}{\partial x} = 0$

$0 = Ak_x\cos(k_x\cdot 0) - Bk_x\sin(k_x\cdot 0) \rightarrow A = 0$

$0 = -Bk_x\sin(k_x a) \rightarrow k_x = \frac{n\pi}{A}$

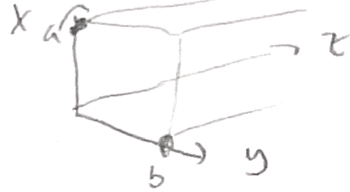
$\Big\}\; X(x) = B\cos(\frac{m\pi}{a}x)$

[General solution for $Y(y)$] $\rightarrow Y(y) = A\sin(k_y y) + B\cos(k_y y)$ | $B_y = 0 \rightarrow \frac{\partial Y(y)}{\partial y} = 0$

$\hookrightarrow Y(y) = B\cos(\frac{n\pi}{b}y)$

$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$

Total Solution $B_z(x,y) = B_0\cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)$




$E_z = 0$

## TE Waves in a Rectangular Guide

Need to solve $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right) B_z = 0$ (b/c $E_z = 0$ so just B)

↳ By separation of Variables: $B_z = X(x)Y(y)$

↳ $Y \frac{d^2X}{dx^2} + X \frac{d^2Y}{dy^2} + \left(\left(\frac{\omega}{c}\right)^2 - k^2\right) XY = 0$

↳ $\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{must be a constant} \, = \, -k_x^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{must be a constant} \, = \, -k_y^2} + \left(\left(\frac{\omega}{c}\right)^2 - k^2\right) = 0$ → $-k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$

$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2$ and $\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$

↳ $X(x) = A\sin(k_x x) + B\cos(k_x x)$

Boundary Conditions

↳ $E_z = 0$ → $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$ (gauss's law w/ $E_z = 0$)

$\vec{E}^{\parallel} = 0$
$B^{\perp} = 0$ } $B_x(x=0) = 0$
$B_x(x=a) = 0$

↳ $B_x = 0 = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - 0\right)$ → $\frac{\partial B_z}{\partial x} = 0$ so $\frac{\partial B_z}{\partial x}\Big|_{x=0}^{x=a} = 0$ → $\frac{\partial X(x)}{\partial x} = 0$

↳ $\frac{dX(x)}{dx} = Ak_x\cos(k_x x) - Bk_x\sin(k_x x)$ → at $x=0$ → $A = 0$
at $x = a$ → $\sin(k_x a) = 0$

↳ $k_x a = m\pi$ and same for Y $k_y b = n\pi$

↳ $$\boxed{\begin{gathered} B_z = B_0\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \\ k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} \end{gathered}}$$

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right)}$$

cut off frequency

if $\omega < c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \equiv \omega_{mn}$ then $k$ is imaginary which leads to exponentially decaying fields

↳ lowest cut off is when $m=1, n=0$, $\boxed{\omega_{10} = \frac{c\pi}{a}}$ frequencies less than this won't propagate at all

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

↳ velocity $= v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} > c$

↳ $v_g = \frac{1}{dk/d\omega} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c$ (speed at which energy is carried