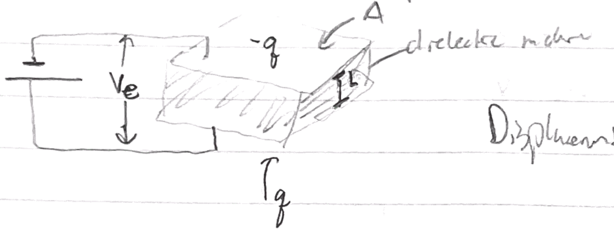
8

"Static"

Thermodynamics in electric & magnetic fields

$dU = \delta Q - \delta W$ * So far we looked at $\delta W = P\,dV$, originates from mechanical work

① dielectric materials, capacitor



$E = \frac{V_e}{L}$

Displacement field $D = \frac{q}{A}$, $\vec{\nabla}\vec{D} = \rho$

$$\int_{vol} \vec{\nabla}\vec{D}\, d^3r = \oint_{surface} \vec{D}\, d^2r = DA \to \int \rho\, d^3\vec{r} = q$$

charge density

• Energy contain in capacitor reduced which means $W_{cap} > 0$ done by capacitor] reduce the amount of charge on capacitor

$W_{cap} = -\int V_e\, dq$ [if $dq < 0$, $V_e > 0$, $W_{cap} > 0$]

$\hookrightarrow V_e\, dq = E L A\, dD$ ~> $D = \frac{q}{A}$ so $dD = \frac{dq}{A}$

$$V_e\, dq = V_{ol}\, E\, dD \to W_{cap} = -V_{ol}\int E\, dD$$

Use $D = \varepsilon_0 E$, if no material is present, work still done by changing the field energy in the capacitor

(for no material) (vacuum)

$$W_{empty\ cap} = -V_{ol}\,\varepsilon_0 \int E\, dE$$

$P = \gamma E$

$D = \varepsilon_0 E + \gamma E$

$D = \underbrace{(\varepsilon_0 + \gamma)}_{\varepsilon} E$ and $\varepsilon = \varepsilon_0\varepsilon_r$

do: $\boxed{W_{sys} = W_{cap} - W_{empty\ cap}} = -V_{ol}\int E\, dD + V_{ol}\int \varepsilon_0 E\, dE$;

$$W_{sys} = -V\int E(t)\left(\frac{dD(t)}{dt} - \varepsilon_0 \frac{dE(t)}{dt}\right) dt$$; use now $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ polarization = total dipole moment per volume

$$\boxed{W_{sys} = -V_{ol}\int E(t)\, \frac{dP(t)}{dt}\, dt \quad , \quad \delta W = -V_{ol}\, E\, dP}$$

V = constant, $PdV = 0$

$$dU = \delta Q - \delta W \Rightarrow dU = \delta Q + E\, dP$$

w/ $V_n P = P_e$

$\hookrightarrow$ $dU = \delta Q + E\, dP_e$ Correspond

compare $\updownarrow$ $E \leftrightarrow -P$

$dU = \delta Q - P\, dV$ $P_e \hookrightarrow V$

$$dU = TdS + E\, dP_e$$ do Legendre transform

$$dU = TdS + d(EP_e) + P_e\, dE$$

$$d(\underbrace{U - EP_e}) = TdS - P_e\, dE$$

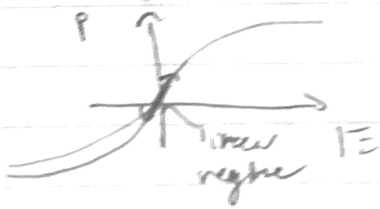
$$H = H(S, E)$$

$$dH = TdS - P_e\, dE$$

$$dH = d(TS) - SdT - P_e\, dE \Rightarrow d(\underbrace{H - TS}) = -SdT - P_e\, dE$$

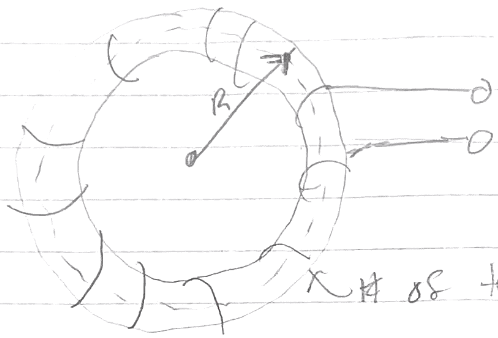
$G = G(T, E)$

$$\boxed{\begin{gathered} dG = -SdT - P_e\, dE \\ S = -\left(\frac{\partial G}{\partial T}\right)_E, \quad P_e = -\left(\frac{\partial G}{\partial E}\right)_T \end{gathered}}$$

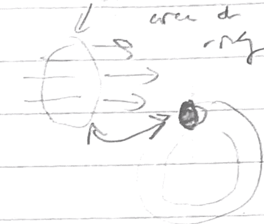


$\uparrow$ dipole moment or polarization

Magnetic Materials



# of turns of wire



Faradays law: $\oint_{\text{circle cross section}} \vec{E}(\vec{r})\, d\vec{r} = -\dfrac{d\phi_B}{dt}$, $\phi_B = \int \vec{B}\, d^2\vec{r}$

$\phi = A \cdot B$ (area, constant B field)

Amperes law:

$$\int_{\text{red circle}} \vec{H}\, d\vec{r} = I_{tot} \quad \text{where} \quad I_{tot} = NI$$

Reduction of B field → work done by Ring → $V_{ind} = -\dfrac{d\phi}{dt} = -A\dfrac{dB}{dt}$

Work done by the ring per time $\dfrac{dW_{ring}}{dt} = N V_{ind} I$ → $\oint H dr = 2\pi R H = NI$

$I = \dfrac{2\pi R H}{N}$

$$\frac{dW_{ring}}{dt} = -A 2\pi R \frac{dB}{dt} H = -V_{ring} H \frac{dB}{dt}$$

(V_{ring}: volume)

if $\frac{dB}{dt} < 0$ system does work for us

$\vec{B} = \mu_0(\vec{H} + \vec{M})$ $\quad \vec{M}$ is the magnetization = magnetic dipole moment per volume

no material $\vec{M} = 0$, $\vec{B} = \mu_0 \vec{H}$, $\dfrac{dW_{mm}}{dt} = \dfrac{dW_{ring}}{dt} - \left(-V_{ring} H \mu_0 \dfrac{dH}{dt}\right)$

$$\frac{dW_{mm}}{dt} = -V_{ring} \mu_0 H \frac{dM}{dt} \qquad = -V_{ring} H\left(\frac{dB}{dt} - \mu_0 \frac{dH}{dt}\right)$$

$\delta W = -V\mu_0 H\, dM$

$$d(HM)$$

$$dU = TdS + \mu_0 VHdM$$

$$d(\underbrace{U - \mu_0 VMH}_{H_{em} = H_{em}(S, H)}) = TdS - \mu_0 VMdH$$

$$d(\underbrace{H_{em} - TS}_{G(T,H)}) = -SdT - \mu_0 VMdH$$

$$dG = -SdT - \mu_0 VMdH$$

$$S = -\left(\frac{\partial G}{\partial T}\right)_H \quad \text{and} \quad M = -\frac{1}{\mu_0 V}\left(\frac{\partial G}{\partial H}\right)_T$$

Basic Concepts of Thermochemistry

- Use minimal gibbs free energy for chemical processes at constant T&P to derive mass action

↳ $G = \mu N$ and $dG = -SdT + Vdp + \underbrace{\mu dN}$

We want to go from multicomponent systems to single component

[We need to generalize to single component systems]

* We also only do single phase systems for simplicity

↳ Specifically looking at reactions

ex) $O_2 + 2CO \rightleftharpoons 2CO_2$

- Gibbs of a multicomponent system is $G = G(N_1, \cdots, N_j, T, P)$ ↳[w/ N_j particles]

↳ $G = \frac{\partial G}{\partial N_1}N_1 + \cdots + \underbrace{\frac{\partial G}{\partial N_j}}_{\mu_j}N_j + \cdots \frac{\partial}{\partial N} = \sum_j \mu_j N_j$ ← what you'd expect as a generalization of $G = \mu N$

↓ $dG = d\sum_j \mu_j N_j$

↳ $dG = -SdT + VdP + \sum_j \mu_j dN_j$ w/ [we can make a relation]

$dG - dG = -SdT + Vdp + \sum_j \mu_j dN_j - \sum_j d\mu_j N_j - \sum_j \mu_j dN_j$

↳ $\boxed{0 = -SdT + VdP - \sum_j d\mu_j N_j}$ *Gibbs-Duhem eq*

product rule

if P, T constant → $\sum_j d\mu_j N_j = 0$ ← chemical potentials of components are related

↳ ex) 2 component mixture $N_1 d\mu_1 + N_2 d\mu_2 = 0$

$\boxed{\text{Chemical Reactions}}$ → $3H_2 + N_2 \rightleftharpoons 2NH_3$ $\xrightarrow{\text{convert}}$ $\underbrace{-3H_2 - N_2}_{\text{reactants}} + \underbrace{2NH_3}_{\text{product}} = 0$

generalize! → $\gamma_1 A_1 + \gamma_2 A_2 + \cdots + \gamma_\sigma A_\sigma = 0$ [Product if γ is positive, reactant if γ is negative]

↳ $G = G(N_1, \ldots, N_j, T, P)$ → min $dG = 0$ w/ T, P constant, $\sum_j \mu_j dN_j = 0$ (in equilibrium)

dN_j is a single particle element $= \gamma_j$ → $\boxed{\sum_j \mu_j \gamma_j = 0}$

$\mu = \int \left(\frac{\partial \mu}{\partial P}\right)_T dP + f(T)$ can just call $f(T)$ this

[Chemical Potential for an ideal gas] → $G = \mu N$ → $\underbrace{\left(\frac{\partial G}{\partial P}\right)_T}_{V} = N\left(\frac{\partial \mu}{\partial P}\right)_T$ → $\frac{V}{N} = \left(\frac{\partial \mu}{\partial P}\right)_T = \frac{k_B T}{P}$ (ideal gas relation, $V = T/P$) $\boxed{\mu = k_B T \ln \frac{P}{P_r} + k_B T \phi(T)}$

L For each reactant + product:

*also need law of partial pressures $\frac{P_j}{N_j} = \frac{P}{N}$ → $P_j = P\frac{N_j}{N} = P[A_j]$ ($P = \sum_j P_j$)

$\sum_j \mu_j \gamma_j = 0$, where $\mu_j \overset{\text{(for ideal gas)}}{=} k_B T \ln\left(\frac{P_j}{P_r}\right) + k_B T \phi_j(T)$, $[A_j]$ is the concentration of component j

↳ $k_B T \sum_j \gamma_j \left(\ln P\frac{[A_j]}{P_r} + \phi_j(T)\right) = 0$

↳ $\sum_j \gamma_j \left[\ln \frac{P}{P_r} + \ln[A_j] + \phi_j(T)\right] = 0$

↳ $\sum_j \ln[A_j]^{\gamma_j} = -\sum_j \gamma_j \left[\ln\left(\frac{P}{P_r} + \phi_j(T)\right)\right]$

↳ $e^{\sum_j \ln[A_j]^{\gamma_j}} = \underbrace{e^{-\sum \gamma_j [\ln(P/P_r) + \phi_j(T)]}}_{K(P,T)}$

($e^{\ln(x) + \ln(y)} = x \cdot y$ ↑ but we have j of these)

↳ $\boxed{\prod_j [A_j]^{\gamma_j} = K(P,T)}$ law of mass action →

ex) $K = \frac{[NH_3]^2}{[H_2]^3[N_2]}$ of $3H_2 + N_2 \rightleftharpoons 2NH_3$

if K is big: $[A_j]$ large w/ $\gamma_j > 0$; product concentration large

if K is big: $[A_j]$ small w/ $\gamma_j < 0$; reactant conc low

if K is small: $[A_j]$ small w/ $\gamma_j > 0$; product conc low

if K is small: $[A_j]$ big w/ $\gamma_j < 0$; reactant conc large

$K = \frac{\text{product}}{\text{reactant}}$

$\mathcal{K}$ is a function of P, thus we can use it to change the amount of product

$$\mathcal{K} = e^{-\sum_j \gamma_j [\ln \frac{P}{P_r} + \phi_j(T)]} = e^{-\ln(\frac{P}{P_r}\sum_j \gamma_j - \sum_j \gamma_j \phi_j(T))} = \boxed{\left(\frac{P}{P_r}\right)^{-\sum_j \gamma_j} e^{-\sum_j \gamma_j \phi_j(T)} = \mathcal{K}}$$

if $\sum_j \gamma_j > 0$ $\uparrow P$ (in denominator now) decreases $\mathcal{K}$, equilibrium shifts towards reactants

if $\sum_j \gamma_j < 0$ $\uparrow P$ (in numerator now) increases $\mathcal{K}$, equilibrium shifts towards products

$\mathcal{K}$ is a function of T, thus we can use it to change the amount of product

$$\ln(\mathcal{K}) = \ln\left(\frac{P}{P_r}\right)^{-\sum_j \gamma_j} - \sum_j \gamma_j \phi_j(T) \rightarrow \left(\frac{\partial(\ln(\mathcal{K}))}{\partial T}\right)_P = -\sum_j \gamma_j \frac{d\phi_j}{dT}$$

Go back to:

$$\mu_j = k_B T \ln\left(\frac{P_j}{P_r}\right) + k_B T \phi_j(T) \rightarrow \left(\frac{\partial \mu_j}{\partial T}\right)_P = k_B \ln\left(\frac{P_j}{P_r}\right) + k_B \phi_j(T) + k_B T \frac{d\phi_j}{dT} = \mu_j + k_B T \frac{d\phi_j}{dT}$$

Now look at Gibbs:

$$G = H - TS \rightarrow \mu_j = h_j - Ts_j \quad \text{and} \quad dG = -SdT + VdP + \mu dN, \left(\frac{\partial \mu_j}{\partial T}\right)_P = -s_j$$

$$\hookrightarrow \mu_j = h_j + T\left(\frac{\partial \mu_j}{\partial T}\right)_P \xrightarrow{\text{combine}} \boxed{h_j = -k_B T^2 \frac{d\phi_j}{dT}}$$

Now look at $\left(\frac{\partial \ln \mathcal{K}}{\partial T}\right)_P = -\sum \gamma_j \frac{d\phi_j}{dT} \rightarrow \left(\frac{\partial \ln \mathcal{K}}{\partial T}\right)_P = \frac{\sum_j \gamma_j h_j]}{k_B T^2} = \Delta h = \frac{\Delta h}{k_B T^2}$

(we can integrate)

$$\ln \frac{\mathcal{K}(T_2)}{\mathcal{K}(T_1)} = \int_{T_1}^{T_2} \frac{\Delta h}{k_B T^2} dT \Rightarrow \mathcal{K}(T_2) = \mathcal{K}(T_1) e^{\frac{-\Delta h}{k_B}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)}$$

if $\Delta h < 0$: heat leaves the system, reaction is exothermic, $\mathcal{K}$ decreases w/ increasing T, yield goes down

if $\Delta h > 0$: the reaction is endothermic, $\mathcal{K}$ increases w/ ~~increasing~~ increasing T, yield goes up

Haber Bosch Process

- industrial route to ammonia (Nobel Prize)

↳ $3H_2 + N_2 \rightleftharpoons 2NH_3 \quad \Delta h = -92.5\ kJ/mol$

- at room temperature reaction is slow
- good idea to increase T to increase reaction speed

↳ even though its a good idea to increase T in a reaction (from intuition of molecules) you better not over do it. We know this from our law of mass action.

Since $\Delta h < 0$, reaction is exothermic, K decreases w/ increasing T, yield goes down

$$\left(K(T_2) = K(T_1)\, e^{\frac{-\Delta h}{k_B}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)} \right)$$

Cant go further w/ Temperature, but we can use pressure:

$-3H_2 - N_2 + 2NH_3 = 0$

$\gamma(NH_3) = 2,\ \gamma(H_2) = -3,\ \gamma(N_2) = -1$

$$K = \left(\frac{P}{P_r}\right)^{-\sum_j \gamma_j} e^{-\sum_j \gamma_j \phi_j(T)}$$

$\rightarrow K = \left(\frac{P}{P_r}\right)^2 \cdot \text{constant}$

sum it up, $-3-1+2 = -2 < 0$

↳ increase pressure to get more yield.

- you can also remove ammonia so more stuff reacts to try to get it to equillibrium,

$K = \dfrac{[NH_3]^2}{[H_2]^3[N_2]}$ ← removing $[NH_3]$, increases yield

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$\boxed{3^{rd}\ \text{Law of Thermodynamics}}$

Nernst Heat Theorem: In the neighborhood of abs 0, all reactions in a liquid or solid in internal equillibrium take place w/ no change in entropy

↳ $\lim_{T\to 0}(S_1 - S_2) = 0$

We know at P, T = constant equillibrium is determined by $G \Rightarrow \min$, $\Delta G = G_2 - G_1$

$G = H - TS$ and $dG = -SdT + VdP$

↓

$G = H + T\left(\frac{\partial G}{\partial T}\right)_P \longrightarrow \Delta G = \Delta H + T\left(\frac{\partial \Delta G}{\partial T}\right)_P$

as $T \to 0$, $\Delta G \to \Delta H$, $\left(\frac{\partial \Delta G}{\partial T}\right)_P \to 0$

From $\lim_{T\to 0}\left(\frac{\partial \Delta G}{\partial T}\right)_P = 0 \rightarrow \lim_{T\to 0}(S_1 - S_2) = 0$

↑ ΔS

↳ Plancks Further generalization of 3^{rd} Laws

3rd Law

[Entropy of every solid or liquid substance in internal equillibrium at absolute 0 is itself 0]

$$\lim_{T \to 0} S = 0$$

$dS = \frac{\delta Q}{T} \rightarrow dS(V = const) = \int_0^T C_v \frac{dT}{T}, \quad dS(P = const) = \int_0^T C_p \frac{dT}{T}$

↳ you cant have sol'n at $T=0$ thus C_v and $C_p \to 0$ at $T=0$.

$\lim_{T \to 0} C_v = \lim_{T \to 0} C_p = 0$ * requires QM to derive in terms of statistical mechanics

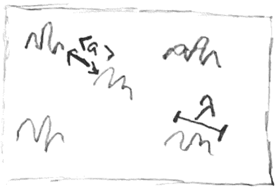
↳ Nernst! $\lim_{T \to 0} (S_1 - S_2) = 0 \rightarrow \lim_{T \to 0} \left(\frac{\partial S}{\partial P}\right)_T = \lim_{T \to 0} \left(\frac{\partial S}{\partial V}\right)_T = 0$

Use maxwell relation → $\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P \rightarrow \lim_{T \to 0} \left(\frac{\partial V}{\partial T}\right)_P = 0, \quad \boxed{\alpha_V = \lim_{T \to 0} \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = 0}$

Kinetic Theory

$E_k = \frac{p^2}{2m}$

- consider a dilute gas of N atoms in a box



Classical limit → $\lambda \ll \langle a \rangle$

$p = \frac{h}{\lambda} \rightarrow \lambda = \frac{h}{p} = \frac{h}{\sqrt{2m E_{kinetic}}} \approx \frac{h}{\sqrt{2m k_B T}}$ ✓ derived in bulk

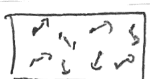
↳ $\frac{h}{\sqrt{2mk_BT}} \ll \left(\frac{L}{N^{1/3}} = \langle a \rangle \rightarrow \right)$

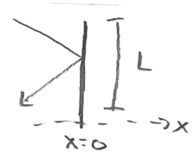
and w/ density $n = \frac{N}{L^3}$

$$\boxed{\frac{h\, n^{1/3}}{\sqrt{2m k_B T}} \ll 1 \quad \text{classical limit}}$$

assumes on average evenly spread out system, think of a lattice $N^{1/3}$ are on one side times their average distance is L

Ideal gas





$p_i = m v_x$

$p_f = m v_x(t + \Delta t)$

momentum change when hitting the wall!

$\Delta \vec{p} = \int_t^{t+\Delta t} \frac{dp}{dt'} dt' = \int_t^{t+\Delta t} \vec{F}(t') dt'$ $\xrightarrow{\text{divide each side by } \Delta t}$ $F^{av}_{x,\,atom} = \frac{1}{\Delta t} \int_t^{t+\Delta t} F_x(t') dt'$

(x component of the momentum change) $\Delta p_x =$

$\underbrace{m v_x(t+\Delta t)}_{\text{after collision}} - \underbrace{m v_x(t)}_{\text{before collision}} = \underbrace{-2 m v_x(t)}_{\text{all that happens before and after is change of sign but magnitude stays the same}} = F^{av}_{x,\,atom} \Delta t$ & 3rd Law: $F^{av}_{x,\,atom} = F^{av}_{x,\,wall}$

what's Δt?

[We can't choose when the force is turned off & when its on, so we chose Δt to be time between 2 successive collisions]

ex [hits wall hits opposite wall & comes back & hits wall, the time it takes to do that is the time we average over]

* $F^{av}_{x,\,wall} \Delta t = 2 m v_x(t)$, w/ $\Delta t = \frac{2 L_x}{v_x}$, twice the time it takes for it to reach the wall at speed v_x

↳ $F^{av}_{x,\,wall} = \frac{m v_x^2}{L_x}$

total force on Back

$$F^{x}_{av/wall} = \frac{m v_x^2}{L_x}$$

(Total Force by N particles) $\rightarrow$ $F_x^{tot} = \sum_{i=1}^{N} \frac{m_i v_{x,i}^2}{L_x}$ and $A = L_y L_z$ $P = \frac{F}{A}$ so;

$$P = \frac{F_x^{tot}}{L_y L_z} = \frac{1}{L_y L_z L_x} \sum \sim = \frac{1}{V} \sum_{i=1}^{N} m_i v_{x,i}^2$$ now do for each direction

$$3P = \frac{1}{V} \sum_{i=1}^{N} m_i (v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2) \Rightarrow \boxed{P = \frac{1}{3V} \sum_{i=1}^{N} m_i v_i^2}$$

To further simplify look at the time average:

$$\frac{K^{tot}}{N} = K^{av} \rightarrow \underbrace{\frac{\frac{1}{2}\sum_{i=1}^{N} m_i v_i^2}{N}}_{\text{(adding up + dividing by how many there are)}} = \frac{1}{2} \overline{m_i v_i^2}^{\text{average}} \xrightarrow{\text{for all mass the same}} \frac{1}{2} \overline{m v^2} = \frac{1}{2} m \overline{v^2}$$

$\hookrightarrow$ $PV = N \frac{2}{3} \left(\frac{1}{2} m \overline{v^2}\right)$

$\hookrightarrow$ (compare w/ ideal gas equation) $\rightarrow$ $PV = N k_B T$

$$\frac{1}{2} m \overline{v^2} = \frac{3}{2} k_B T$$

$$* \; v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3 k_B T}{m}}$$

Calculating the statistical average of $\overline{v^2}$;

To do so we use a distribution function.

Specifically, we use one of the form $f(\vec{r}, \vec{p}, t)\, d^3\vec{r}\, d^3\vec{p} \propto$ [# of molecules at a given time t within a volume element $d^3 r$ located at r and momentum element $d^3\vec{p}$ around $\vec{p} = m\vec{v}$]

To do an average:

$$\overline{v^2} = \frac{\int \overbrace{v^2 f(\vec{r}, \vec{p}, t)}^{\text{function averaging over}}\, d^3\vec{r}\, d^3\vec{p}}{\int f(\vec{r}, \vec{p}, t)\, d^3\vec{r}\, d^3\vec{p}}$$

$\rightarrow$ kinda like $\frac{1}{T}\int_0^T f(t)\,dt$

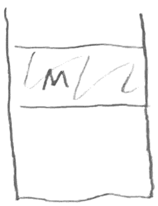
Next tricks $\rightarrow$ if $f(\vec{r}, \vec{p}, t) = f(\vec{r}, \vec{p}) = f_1(\vec{p})\, f_2(\vec{r})$ then

$$\overline{v^2} = \frac{\int v^2 f_1(\vec{p})\, d^3\vec{p}}{\int f_1(\vec{p})\, d^3\vec{p}} \cdot \cancel{\frac{\int f_2(\vec{r})\, d^3\vec{r}}{\int f_2(\vec{r})\, d^3\vec{r}}} = \frac{\int v^2 f_1(\vec{p})\, d^3\vec{p}}{\int f_1(\vec{p})\, d^3\vec{p}}$$

this will be important once we show that we can seperate it like this.

How do we get f? → Turns out to be the Maxwell-Boltzmann dist.

To get a feel for it recall the following example.



$P(h+\Delta h)$

→ $P(h) = P(h+\Delta h) + \frac{Mg}{A} = P(h+\Delta h) + \frac{\rho(h) A \Delta h g}{A} = P(h+\Delta h) + \rho(h)\Delta h g$

[Air is ideal gas] → $PV = nRT \to P\underbrace{\frac{V}{M}}_{1/\rho} = \frac{n}{M}RT \to \rho = \frac{M}{n}\frac{P}{RT}$

↳ $\frac{P(h+\Delta h) - P(h)}{\Delta h} = -\frac{M}{n}\frac{g P(h)}{RT(h)} \to \frac{dP}{dh} = -\frac{M}{n}\frac{g P(h)}{nRT(h)}$

In case T = const. ($M = n N_A m$ and $R = N_A k_B$)

$\frac{dP}{dh} = \frac{-mg P(h)}{k_B T} \to \int_{P_0}^{P}\frac{dP}{P} = \frac{-mg}{k_B T}\int_0^h dh' \to$

$$\boxed{\begin{aligned} P(h) &= P_0 e^{-\frac{mgh}{k_BT}} \\ \rho(h) &= \rho_0 e^{-\frac{mgh}{k_BT}} \end{aligned}}$$

Energy

Energy

$\rho(\vec r)\, d^3\vec r$ = # of particles in $d^3\vec r$ ∝ probability of finding a particle in $d^3\vec r$

$\int f(\vec r,\vec p)\, d^3\vec p = \rho(r) = \rho(x) = \rho_0 e^{-\frac{mgx}{k_BT}} = \rho_0 e^{-\frac{E_{pot}(x)}{k_BT}}$

we can say $\rho(\vec r) = \int f(\vec r,\vec p)$ bec if we integrate $\rho(\vec r)\, d^3\vec r$ we get # of particles which is what happens if you integrate the whole prob density function

[Assume r, p contribute to the energy] → $f(\vec r,\vec p) = f\left(E = \frac{p^2}{2m} + E_{pot}(\vec r)\right) \to \int d^3\vec p\, f(E) = \rho_0 e^{-\frac{E_{pot}}{k_BT}}$

[Take derivative wrt dE_{pot} of both sides] $\frac{d}{dE_{pot}}\left(\int f(E)\, d^3\vec p\right) = \frac{d}{dE_{pot}}\left(\rho(\vec r)\right) = \int d^3\vec p\, \frac{df(E)}{dE}\underbrace{\frac{dE}{dE_{pot}}}_{1} = -\frac{1}{k_BT}\rho_0 e^{-\frac{E_{pot}}{k_BT}}$

also = integral part of f(E) earlier

↳ $\int d^3\vec p\, \frac{df(E)}{dE} = -\frac{1}{k_BT}\int d^3\vec p\, f(E)$

↳ $\int d^3\vec p\left(\frac{df(E)}{dE} + \frac{1}{k_BT}f(E)\right) = 0 \to \frac{df(E)}{dE} + \frac{1}{k_BT}f(E) = 0 \to \int \frac{1}{f}df \propto \frac{-E}{k_BT}$

↳ $f \propto e^{-E/k_BT} \propto e^{-\frac{1}{k_BT}\left(\frac{p^2}{2m} + E(\vec r)\right)} = e^{-\frac{1}{k_BT}\left(\frac{p^2}{2m}\right)} e^{-\frac{1}{k_BT}E(\vec r)} = f_1(\vec p^2)\, f_2(\vec r)$

↳ $N = C\int_{-\infty}^{\infty} d^3\vec p\, e^{-\frac{1}{k_BT}\frac{p^2}{2m}}$