Fine Structure of Hydrogen

The hamiltonian to originally solve this → $H = \frac{-\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$ (coulomb potential)

Fine Structure → relativistic correction → spin orbit coupling

Spin Orbit Coupling → from the electrons point of view, the proton is circling it. Positive charge sets up a B field

↳ This B-field exerts a torque on the spinning electron

↳ (tending to align its magnetic moment μ along the direction of the field)

$\boxed{H = -\vec{\mu}\cdot\vec{B}}$, need to find $\vec{B}$ and $\vec{\mu}$ now

Calculating $\vec{B}$

$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 e}{2rT}$ and $L = rmv = rm(2\pi r/T)$ so $\frac{1}{T} = \frac{L}{2\pi r^2 m}$

↳ $\vec{B} = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{e}{mc^2r^3}\right)\vec{L}$, $\vec{L}$ and $\vec{B}$ point in the same direction

Calculating μ

$\mu = I\cdot\text{area} = \pi r^2 \cdot q/T$, $S = I\omega = mr^2\cdot\frac{2\pi}{T}$ → $\vec{\mu} = \left(\frac{q}{2m}\right)\vec{S}$

↳ In reality! $\boxed{\vec{\mu}_e = -\frac{e}{m}\vec{S}}$

ALL Together:

$$\hat{H} = \left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L} \xrightarrow{\text{Thomas Precession}} \boxed{H'_{\text{Spin Orbit}} = \frac{e^2}{8\pi\varepsilon_0}\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L}}$$

What states can we use!

$[H'_{so}, L^2] = 0$

$[H'_{so}, S^2] = 0$

$[H'_{so}, \vec{J}] = 0$, $\vec{J} = \vec{L} + \vec{S}$

Eigenstates of L_z & S_z aren't eigenstates of H'

Proof that good states are ones ~~that commute w/ the~~ a third operator commutes w/ H' w/ distinct eigenvals (291)

J^2: $\rightarrow (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S}) = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$

$\hookrightarrow \vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2)$ w/ eigenvalues

$$\frac{\hbar^2}{2}\big(\underbrace{j(j+1)}_{\text{same as } l \text{ or } s \text{ since just a sum}} - l(l+1) - s(s+1)\big)$$

For $\psi^0_{n,l,m}$ the wave function of hydrogen atom

$\boxed{\text{The first order correction to energy}} \rightarrow E^1_{\substack{\text{spin}\\\text{orbit}}} = \langle \psi^0_{n,l,m} | H'_{\substack{\text{spin}\\\text{orbit}}} | \psi_{n',l',m'} \rangle$

$$E^1_{\substack{\text{spin}\\\text{orbit}}} = \langle \psi_{n,l,m} | \frac{e^2}{8\pi\varepsilon_0}\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L} | \psi_{n',l',m'} \rangle = \frac{e^2}{8\pi\varepsilon_0}\frac{1}{m^2c^2}\langle \psi_{n,l,m} | \frac{S\cdot L}{r^3} | \psi_{n',l',m'} \rangle$$

$$= \frac{e^2}{8\pi\varepsilon_0}\frac{1}{m^2c^2}\frac{\hbar^2/2\,(j(j+1) - l(l+1) - 3/4)}{\langle r^3 \rangle}, \quad \underbrace{\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{l(l+1/2)(l+1)n^3a^3}}_{\text{hw problem?}}$$

$$E^1_{SO} = \underbrace{\frac{e^2}{8\pi\varepsilon_0}\frac{\hbar^2}{2m}\frac{1}{n^4}\frac{1}{a^3}}_{(E^0_n)^2}\left(n\frac{(j(j+1) - l(l+1) - 3/4)}{l(l+1/2)(l+1)}\right)\frac{1}{mc^2}, \quad E_n = -\frac{m_e}{2\hbar^2}\left[\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n^2}$$

$$\frac{e^2}{4\pi\varepsilon_0} = \frac{\hbar^2}{m_e a} \rightarrow E_n = \frac{m_e}{2\hbar^2}\frac{\hbar^4}{m_e^2a^2}\frac{1}{n^2} = \frac{\hbar^2}{2m_e a^2}\frac{1}{n^2}$$

$$E^1_{SO} = \frac{(E_n)^2}{mc^2}\left(\frac{n(j(j+1) - l(l+1) - 3/4)}{l(l+1/2)(l+1)}\right)$$

The quantum numbers can't be used, we use now n, l, s, j, m_j

$$l^2 + \frac{1}{2} + \frac{3}{4}l = 0$$

$$l = \frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4} - 4\cdot\frac{1}{2}}}{2} = \frac{-\frac{3}{2} \pm \sqrt{\frac{1}{4}}}{2} = -\frac{3}{4} \pm \frac{1}{4}, \quad \boxed{l = -1}, \quad l = -1/2$$

can't have negative l

Zeeman Effect

- atom placed in a B-field, energy levels shift

$$H_Z' = -(\vec{\mu}_\ell + \vec{\mu}_s) \cdot \vec{B}_{ext} \quad , \vec{\mu}_s = -\frac{e}{m}\vec{S}, \quad \vec{\mu}_\ell = -\frac{e}{2m}\vec{L}$$

$$H_Z' = \frac{e}{2m}(\vec{L} + 2\vec{S}) \cdot \vec{B}_{ext}$$, if $B_{ext} \ll B_{int}$ it can be treated as a small perturbation

- If $B_{ext} \gg B_{int}$ then zeeman dominates + fine structure is a small perturbation

Unperturbed Hamiltonian: $H_{bohr} + H_{fine\ struct}$, w/ $|n \ell j m_j\rangle$, E_{nj}

* energy doesn't depend on ℓ, m_j while $|n \ell j m_j\rangle$ does so there are degenerate energies

↳ But H_z' commutes w/ J_z ($\vec{B}_{ext}$ in z-dir) and L^2 and each of the degenerate states is uniquely labeled by m_j and ℓ

Thus! $E_z' = \langle n \ell j m_j | H_z' | n \ell j m_j \rangle = \frac{e}{2m} B_{ext}\, \hat{k} \cdot \langle \vec{L} + 2\vec{S} \rangle$

and $\vec{L} + 2\vec{S} = \vec{J} + \vec{S}$

$\vec{J} = \vec{L} + \vec{S}$ = constant, $\vec{L}$ and $\vec{S}$ precess rapidly
(B-field does no work)

proj $\vec{A}$ onto $\vec{B} = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|^2}\vec{B}$ → proj $\vec{S}$ onto $\vec{J} = \frac{(\vec{S} \cdot \vec{J})}{|\vec{J}|^2}\vec{J}$

$\vec{S}_{ave} = \left(\frac{\vec{S} \cdot \vec{J}}{|\vec{J}|^2}\right)\vec{J}$ bec all horizontal components cancel out ⟷

$$\vec{S}_{ave} = \frac{\vec{S}\cdot\vec{J}}{J^2}\vec{J}$$

$$\vec{L} = \vec{J} - \vec{S} \Rightarrow L^2 = J^2 + S^2 - 2\vec{J}\cdot\vec{S} \Rightarrow \vec{S}\cdot\vec{J} = \frac{1}{2}\left(J^2 + S^2 - L^2\right)$$

$$\vec{S}\cdot\vec{J} = \frac{\hbar^2}{2}\left(j(j+1) + s(s+1) - \ell(\ell+1)\right)$$

$\langle \vec{L} + 2\vec{S} \rangle = \langle \vec{J} + \vec{S} \rangle$ replace $\vec{S}$ by $\vec{S}_{avg}$

$$= \langle \underbrace{\left(1 + \frac{\vec{S}\cdot\vec{J}}{J^2}\right)}_{\text{scalar}} \vec{J} \rangle = \underbrace{\left[1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)} \right]}_{g_J \text{ factor}} \langle \vec{J} \rangle$$

So: $E_Z^1 = \frac{e}{2m} B_{ext}\, \hat{k}\cdot\left(g_J \langle \vec{J} \rangle\right)$, $\hat{k}\cdot\langle\vec{J}\rangle = m_j \hbar$

$$= \frac{e\hbar}{2m} B_{ext}\, g_J\, m_j = \boxed{\mu_B\, g_J\, m_j\, B_{ext}}$$

m_j is $\pm j$

$$E_{total} = E_{nj} + E_Z^1$$

• Checking off diagonals of Zeeman Hamiltonian

When expanding hamiltonian you get 2 eq!

$$\alpha \overbrace{\langle \psi_a^0 | H' | \psi_a^0 \rangle}^{W_{aa}} + \beta \overbrace{\langle \psi_a^0 | H' | \psi_b^0 \rangle}^{W_{ab}} = \alpha E^1 \qquad \text{w/ } \psi^0 = \alpha \psi_a^0 + \beta \psi_b^0$$

$$\alpha \overbrace{\langle \psi_b^0 | H' | \psi_a^0 \rangle}^{W_{ba}} + \beta \overbrace{\langle \psi_b^0 | H' | \psi_b^0 \rangle}^{W_{bb}} = \beta E^1$$

<u>For B in z-direction:</u> $H_z' = \frac{e}{2m}(L_z + 2S_z) B_z$

on the off diagonal terms you just get ψ_a back (w/ factors of $\hbar$ and stuff) and $\langle \psi_b^0 | \psi_a^0 \rangle = 0$ so off diagonal terms are 0.

EPR Paradox

* in quantum mechanics we cannot say that a particle whose spin is measured up had that spin prior to measurement
  ↳ could of been in a superposition state
  ↳ we can only know one spin component of the particle because measurement of one component disturbs our knowledge of all other components. Because of these deficiencies, einstein believed QM was incomplete

1935, Einstein, Boris, Podolsky, Rosen published a paper to expose the shortcomings of QM
↳ essence of argument is that if you believe that a measurement on two widely separated particles can't influence one another then QM of two prepared system leads you to conclude the physical properties are really there!

EPR Experiment

* unstable pion decays into two spin 1/2 particles
* due to conservation laws, their momentum + spin must be in opposite dir

Observers A and B are on opposite sides w/ stern gerlachs to measure the spin

* whenever one observer measures spin up the other measures spin down, along the same direction *

Quantum state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right) \quad \text{(two particle system!)}$$

↑ entangled state

* if A measures spin up then B is spin down, and thats known instantly. * → if A is closer than B so it measures it first, then the state $|\psi\rangle$ instantly collapses, even if B hasn't reached the stern gerlach yet.

* still all random *

$\boxed{\text{So}}$ lets say observer measures $S_{1z} = +\frac{\hbar}{2}$ → we know instantly that $S_{2z} = -\frac{\hbar}{2}$ → ψ collapsed to $|\uparrow\rangle_1 |-\rangle_2$

so measurement of A somehow determines 2

EPR states that bc/ we can say w/ 100% certainty the second particle state it was a "real" property all along
↳ and is independent of what observer A does! (Einstein's locality principle)

* correlation of spin measurements is independent of the choice of measurement direction. If A meas $S_{1x} = +\hbar/2$ then $S_{2x} = -\hbar/2$
A is free to choose S_{1x}, S_{1y}, S_{1z} so S_{2x}, S_{2y}, S_{2z} are all elements of reality
↳ in quantum we can only know one component at a time so
EPR concludes QM is incomplete!!

John Bell
- 1964 he showed that hidden variables we cannot know can't be there!

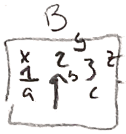
Bell's Argument
- observers A and B make measurements along different set of dir
- consider 3 dir, $\hat{a}, \hat{b}, \hat{c}$ each 120° from one another

* Each observer makes measurements along one of these directions chosen randomly

* Any single observer can only be spin up or down in that direction
↳ record the results independent of the direction

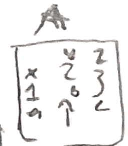
(Note! this always happen no matter what your apparatus is) →

B



no matter what you flip this to you know the other has to be opposite!

A



Now make instruction sets for the particles (hidden variable theory) → There are 3 directions we can measure $\hat{a}, \hat{b}, \hat{c}$ → $\hat{a}\hat{a}$, $\hat{a}\hat{b}$, $\hat{a}\hat{c}$, $\hat{b}\hat{a}$, $\hat{b}\hat{b}$, $\hat{b}\hat{c}$, $\hat{c}\hat{a}$, $\hat{c}\hat{b}$, $\hat{c}\hat{c}$ ← can measure in same direction

measure particle B in $\hat{c}$ direction and particle A in $\hat{a}$ direction

Of all the combinations of $a \times B(\hat{a}, \hat{b}, \hat{c})$ to measure along what are probabilities?

if instruction set is $(\hat{a}+, \hat{b}+, \hat{c}+)$ then any measurement along any of the axis yields spin up, $|+\rangle$, so the other has to be opposite

Populations	Particle 1	Particle 2
N_1	$(\hat{a}+, \hat{b}+, \hat{c}+)$	$(\hat{a}-, \hat{b}-, \hat{c}-)$
N_2	$(\hat{a}+, \hat{b}+, \hat{c}-)$	$(\hat{a}-, \hat{b}-, \hat{c}+)$
N_3	$(\hat{a}+, \hat{b}-, \hat{c}+)$	$(\hat{a}-, \hat{b}+, \hat{c}-)$
N_4	$(a+, b-, c-)$	$(a-, b+, c+)$
N_5	$(a-, b+, c+)$	$(a+, b-, c-)$
N_6	$(a-, b+, c-)$	$(a+, b-, c+)$
N_7	$(\hat{a}-, \hat{b}-, \hat{c}+)$	$(\hat{a}+, \hat{b}+, \hat{c}-)$
N_8	$a-, b-, c-$	$a+, b+, c+$

Now calculate probabilities that the particles are the same or opp

For populations w/ all axis measuring the same direction, N_1, N_2 have 0.1 chance of measuring spin in same direction

So

$P_{same} = 1$, $P_{opp} = 0$ } for types 1 and 8

For N_2 through N_7 its a little more complicated

Denote one observers measurement as + or −, regardless of axis measurement.

For type 2 | Particle 1 $(a+, b+, c-)$, Particle 2 $(a-, b-, c+)$ * for measuring particles theres 9 results

1 + − → $a+, a-$ so Particle 1 is spin up, 2 is down (a possibility), these are all possible combos
2 + − → $a+, b-$ so 1 is up, 2 is down in b dir
3 + + → $a+, c+$ so 1 is up, 2 is up in c dir
4 + − → $b+, a-$ so 1 is up in b dir, 2 is down in a dir
5 + − → $b+, b-$ so 1 is up in b, 2 is down in b
6 + +
7 − −
8 − −
9 − +

5 are opposite → $P_{opp} = 5/9$

4 are same → $P_{same} = 4/9$

To recap

In a local hidden variable theory needs an instruction set to tell you what the result of measurement on $\hat{a}, \hat{b}, \hat{c}$ will be

$(a+, b+, c+) \rightarrow$ any measurement along these dir will give spin up

$\rightarrow (a+, b+, c+)$ must be paired w/ $(a-, b+, c-)$ bec a measurement along $a+$ will give spin up so second has to be spin down

Now we can measure in any direction so do all combos $\rightarrow$ this is the table I drew.

9 possible combos for a, b, c for each population

Particle 1, Particle 2

$\rightarrow \hat{a}\hat{a}, \hat{a}\hat{b}, \hat{a}\hat{c}, \hat{b}\hat{a}, \hat{b}\hat{b}, \hat{b}\hat{c}, \hat{c}\hat{a}, \hat{c}\hat{b}, \hat{c}\hat{c}$

For population 2

$(a+, b+, c-)$ Particle 1 $\quad (a-, b-, c+)$ Particle 2

$\rightarrow +-, +-, ++, +-, +-, ++, --, --, -+$

so 5 opposite, 4 same

In total

$P_{opp} = 1, \; P_{same} = 0 \;\}$ 1 and 8 and $P_{opp} = 5/9, \; P_{same} = 4/9 \;\} \; 2 \rightarrow 7$

Now do for all measurements

$$P_{same} = \frac{1}{\sum N_i} \frac{4}{9}(N_2 + N_3 + N_4 + N_5 + N_6 + N_7) \leq 4/9$$

$$P_{odd} : \frac{1}{\sum N_i} \left(N_1 + N_8 + \frac{5}{9}(N_2 + N_3 + N_4 + N_5 + N_6 + N_7) \right) \geq 5/9$$

- for EPR

$P_{same} \leq \frac{4}{9}$ in

$P_{opp} \geq \frac{5}{9}$

Quantum we found | $P_{same} = \sin^2(\theta/2)$

$P_{opp} = \cos^2(\theta/2)$

What are the angles between $\hat{a}, \hat{b}, \hat{c}$?

1/3 of time you measure same $(\hat{a}, \hat{a})$ or $\hat{b}\hat{b}$ or $\hat{c}\hat{c}$ $\rightarrow 3/9 = 1/3$

2/3 of time you measuser a mix ab, ac, ba, bc, ca, cb $\rightarrow 6/9 = 2/3$

- the angle between A + B is $0°$ 1/3 of time
- the angle between A + B is $120°$ 2/3 of time

$$P_{same} = \frac{1}{3}\Big(\underbrace{\sin^2\left(\tfrac{0}{2}\right)}_{=0 \text{ as it should be!}}\Big) + \frac{2}{3}\left(\sin^2\left(\tfrac{120}{2}\right)\right) = \frac{1}{2}$$ = probability of both measure spin up or spin dn

$$P_{opp} = \frac{1}{3}\Big(\underbrace{\cos^2\left(\tfrac{0}{2}\right)}_{1 \text{ as it should be!}}\Big) + \frac{2}{3}\left(\cos^2\left(\tfrac{120}{2}\right)\right) = 1/2$$ = probability of measure opposite

Not consistent w/ hidden variable theory!

What does quantum mechanics say the probabilities should be

- spin 1/2 system
- assume that observer A records $|+\rangle$ along some direction $(\hat{a}, \hat{b}, \hat{c})$
  ↳ define that direction as z-axis
- observer B measures along $\hat{n}$ direction at some angle θ w.r.t z-axis

[Probability that observer A records a + along the z-axis and that observer B records a + along $\hat{n}$-direction is]

Recall $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right)$

* for one particle sys, to get probability we then
* $\langle\psi_1|\psi\rangle = c_1$ then square

$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$ *

*

For two particle system, dot w/

$\underset{1}{\langle ?|}\ \underset{2}{\langle ?|}$ → [we are trying to measure particle 1 in $\underset{1z}{\langle +|}$ and 2 in $\underset{2\hat{n}}{\langle +|}$ state]

↳ $\left|\left(\underset{1z}{\langle +|}\ \underset{2\hat{n}}{\langle +|}\right)|\psi\rangle\right|^2 = P_{++}$ = probability that both are spin up in those directions

$$P_{++} = \left|\ \underset{1}{\langle +|}\left(\cos\left(\tfrac{\theta}{2}\right)\underset{2}{\langle +|} + e^{-i\phi}\sin\left(\tfrac{\theta}{2}\right)\underset{2}{\langle -|}\right)\frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right)\right|^2$$

$$= \left|\left(\cos\tfrac{\theta}{2}\underset{2}{\langle +|} + e^{-i\phi}\sin\tfrac{\theta}{2}\underset{2}{\langle -|}\right)\frac{1}{\sqrt{2}}\left(|-\rangle_2\right)\right|^2 = \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)$$

P_{--} = the same thing (both z, and $\hat{n}$ measure spin down

↳ $P_{same} = \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right) + \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right) = \boxed{\sin^2\left(\frac{\theta}{2}\right)}$

P_{+-} = probability that A records + and B records − along $\hat{n}$ is

$P_{+-} = \left|\left(\underset{1}{\langle +|}\ \underset{\hat{n}2}{\langle -|}\right)|\psi\rangle\right|^2 = \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right)$, P_{-+} = same

$$\boxed{P_{opp} = \cos^2\left(\frac{\theta}{2}\right)}$$

Quantum Information Processing

[Feynman asked: Can a classical computer model a quantum system?] → Say we want to model the quantum mechanical time evolution of → a 50 particle system of spin 1/2 particles, ↳ need 2^{50} χ's to describe this or 10^{15} coefficients

100 particles → 10^{30} coeff
300 particles → 10^{90} coeff
→ [A classical computer would have to keep track of all the coef] → $\boxed{\text{nature has no trouble w/ this so use nature as computer,}}$

Qubits

- classically we use binary digits or bits to store information
  - ↳ each bit has value 0 or 1, strung together to represent numbers (00110 0101)
  - ↳ job of classical computer is to store bits
  - ↳ since its either 1 or 0, classical computer uses on/off switches
- in quantum info processing, info is stored in quantum bits or qubits
  - ↳ **qubit**: quantum system w/ two possible states (analogous to 0 and 1)
  - ↳ canonical qubit: spin 1/2 system: spin up χ_+, $|+\rangle$, or spin down χ_-, $|-\rangle$
  - (other types are hyperfine levels in atoms + polarization states of photons)

* [Refer to qubit states as the following] → $\boxed{|0\rangle = |+\rangle \qquad \underbrace{|1\rangle}_{\text{qubit state}} = \underbrace{|-\rangle}_{\text{spin 1/2 state}}}$ [Defining our qubit states as spin 1/2 states]

Superposition States

* key difference between bits and qubits is qubits can exist in a superposition of states

[General qubit superposition] → $|\psi\rangle = c_0|0\rangle + c_1|1\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$

[With probability to be in each state] → $P_0 = |\langle 0|\psi\rangle|^2 = |c_0|^2$
$P_1 = |\langle 1|\psi\rangle|^2 = |c_1|^2$

Very different from classical computer where its 0 or 1 w/ 100% certainty!

[But, quantum superposition states are more than simple probability mixtures of different possibilities]

ex) $|\psi\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

- 50/50 chance of being in z-dir
- 100% chance of being in x-dir

so there is some certainty

Superposition states are at the heart of power of quantum info processing

↓ because ↓

The amount of information grows exponentially w/ # of qubits

Example ; building a 2 qubit system

4 comps

$|\downarrow\downarrow\rangle = |\downarrow\rangle_A |\downarrow\rangle_B = |00\rangle$

$|\downarrow\uparrow\rangle = |\downarrow\rangle_A |\uparrow\rangle_B = |10\rangle$

$|\uparrow\downarrow\rangle = |\uparrow\rangle_A |\downarrow\rangle_B = |01\rangle$

$|\uparrow\uparrow\rangle = |\uparrow\rangle_A |\uparrow\rangle_B = |11\rangle$

General superposition

$$|\psi\rangle = \underset{\text{off}}{c_{00}|00\rangle} + c_{01}|01\rangle + c_{10}|10\rangle + \underset{\text{on}}{c_{11}|11\rangle} \leftarrow \text{4 pieces of information!}$$

mixed state but more info!

↳ contains 2^2 pieces of info (2: spin up or spin down; exponent: number of qubits)

so generally 2^N ← number of qubits

← classical bit 0 1 just has 2 pieces of information

* turns out even though there are 2^N pieces of info, we can only extract N pieces of classical information through measurements

↳ trick is to harness all this info in superposition of states (but hidden from direct measurement)

↳ To get more, they <u>perform operations that effect all the qubits at once</u>

↳ <u>performing operations all at once we achieve quantum parallelism</u>

↳ can do this classically but need two computers

Quantum NOT-Gate

- Show that spin precession of a general state at π rotation about the x-axis is equivalent to quantum NOT gate

↳ we want spin to precess about x-axis so we put B-field in x-direction

Recall $\hat{H} = -\vec{\mu}\cdot\vec{B}$, $\vec{\mu} = \gamma\hat{S}$, $\gamma = \frac{-e}{m}$

↳ $\hat{H} = \frac{e}{m} B_0 \hat{S}_x$, $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ → $\hat{H} = \frac{e\hbar}{2m} B_0 \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$\left[\begin{array}{l}\text{General Solution}\\ \text{To the S.E.}\end{array}\right]$ → $|\chi(t)\rangle = c_+'|\chi_x\rangle_+ e^{-iE_+t/\hbar} + c_-'|\chi_x\rangle_- e^{iE_-t/\hbar}$

time dependent part

eigenvectors of hamiltonian, which are eigenvectors of S_x

Get E_+, E_-: $H|\chi_x\rangle_+ = E_+|\chi_x\rangle_+$ → $\frac{e}{m} B_0 \underbrace{\hat{S}_x|\chi_x\rangle_+}_{\frac{\hbar}{2}|\chi_x\rangle_+} = E_+|\chi_x\rangle_+$

↳ $\boxed{E_{+/-} = \pm\frac{e\hbar}{2m}B_0}$, $\omega_0 = \frac{eB_0}{m}$, $E_{+/-} = \pm\frac{\hbar}{2}\omega_0$

Initial cond: $|\chi(0)\rangle = c_+|+\rangle + c_-|-\rangle = c_+|\chi_z\rangle_+ + c_-|\chi_z\rangle_-$

also equals $|\chi(0)\rangle = c_+'|\chi_x\rangle_+ + c_-'|\chi_x\rangle_-$

$|\chi_x\rangle_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$|\chi_x\rangle_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

↳ $c_+'|\chi_x\rangle_+ + c_-'|\chi_x\rangle_- = c_+|\chi_z\rangle_+ + c_-|\chi_z\rangle_-$

↳ $c_+' = c_+ \,{}_+\langle\chi_x|\chi_z\rangle_+ + c_- \,{}_+\langle\chi_x|\chi_z\rangle_- = \frac{1}{\sqrt{2}}(c_+ + c_-)$

↳ $c_-' = c_+ \,{}_-\langle\chi_x|\chi_z\rangle_+ + c_- \,{}_-\langle\chi_x|\chi_z\rangle_- = \frac{1}{\sqrt{2}}(c_+ - c_-)$

So $\boxed{|\chi(t)\rangle = \frac{1}{\sqrt{2}}(c_+ + c_-)e^{-i\omega_0 t/2}|\chi_x\rangle_+ + \frac{1}{\sqrt{2}}(c_+ - c_-)e^{i\omega_0 t/2}|\chi_x\rangle_-}$

↳ apply field till $\omega_0 t = \pi$ → $|\chi(t)\rangle = \frac{1}{\sqrt{2}}(c_+ + c_-)e^{-i\pi/2}|\chi_x\rangle_+ + \frac{1}{\sqrt{2}}(c_+ - c_-)e^{i\pi/2}|\chi_x\rangle_-$

Put back in z-basis: $|\chi_x\rangle_+ = \frac{1}{\sqrt{2}}(|\chi_z\rangle_+ + |\chi_z\rangle_-)$, $|\chi_x\rangle_- = \frac{1}{\sqrt{2}}(|\chi_z\rangle_+ - |\chi_z\rangle_-)$

Algebra: $|\psi(t = \frac{\pi}{\omega_0})\rangle = -i(c_-|\chi_z\rangle_+ + c_+|\chi_z\rangle_-)$ ← same as original $|\psi\rangle$ but c's are flipped

$$\boxed{\begin{pmatrix}c_{+,new}\\ c_{-,new}\end{pmatrix} = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}c_+\\ c_-\end{pmatrix}}$$

Quantum Gates

- classical computer uses gates on bits to perform tasks
  - ↳ NOT gate is 1 bit input & output
  - ↳ AND / OR is 2 bit input + 1 bit output

Stern Gerlacs aren't quantum gates
↳ quantum gates are devices that alter the relative coefficients in a qubit superposition w/out destroying coherence

1 qubit Gate: $|\psi_{in}\rangle = c_0|0\rangle + c_1|1\rangle \rightarrow |\psi_{out}\rangle = c_0'|0\rangle + c_1'|1\rangle$

[For any general 1 qubit gate we can write a matrix to transform the input to output coefficients] $\rightarrow \begin{pmatrix} c_0' \\ c_1' \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$, $UU^\dagger = 1$

to preserve the coherence of the qubit

Quantum NOT Gate → $|0\rangle \rightarrow |1\rangle$ and also $|1\rangle \rightarrow |0\rangle$
↳ also changes superposition $a|0\rangle + b|1\rangle \rightarrow b|0\rangle + a|1\rangle$

$U_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} c_0' \\ c_1' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_0 \end{pmatrix}$

Ex 16.1

- Show that spin precession transformation of a general spin state for a π rotation about x-axis is equivalent to a quantum NOT gate

[initial general State] $\rightarrow |\psi(0)\rangle = c_+|+\rangle + c_-|-\rangle$

Rewrite in x-basis:

Quantum Gates

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$U_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, flips the qubit, $\boxed{\otimes}$

$\boxed{\text{Hadamard Gate}} \rightarrow U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, [made by π rotation around z-axis followed by a $\pi/2$ rotation around y-axis]

$$\left.\begin{aligned} U_H|0\rangle &= \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ U_H|1\rangle &= \tfrac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \end{aligned}\right\}$$ creates a superposition of states!

b) $q_{in} \xrightarrow{\text{cloned qubit}} \boxed{Z} — \boxed{X} — q_{out}$ ← read qubit

CNOT Gate

$U_{CNot}|00\rangle = |00\rangle$

$U_{CNot}|01\rangle = |01\rangle$

$U_{CNot}|10\rangle = |11\rangle$

$U_{CNot}|11\rangle = |10\rangle$

$|0\,0\rangle = \underbrace{|0\rangle_c}_{\text{control qubit}}\underbrace{|0\rangle_T}_{\text{target qubit}}$

if $|1\rangle_c$ then it switches target qubit, otherwise it does nothing

$U_{CNot} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $U_{CNot}|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix} = \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix}$

general superposition of states

Creating a Bell state

Ex

call one control ; call one target

Remember, physically these are represented by spin up + down electrons

(Take two qubits) → $|1\rangle_c |0\rangle_T$

$$\left(H_{ad}|1\rangle_c\right)|0\rangle_T = \frac{1}{\sqrt{2}}\left(|0\rangle_c - |1\rangle_c\right)|0\rangle_T = \frac{1}{\sqrt{2}}\left(|0\rangle_c|0\rangle_T - |1\rangle_c|0\rangle_T\right)$$

(Now apply the U_{CNot} Gate) → $\frac{1}{\sqrt{2}}\left(U_{CNot}\left(|00\rangle - |10\rangle\right)\right) = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$

[This is one of the Bell states] → Now they are entangled, question, why is U_{CNot} gate needed?

Another Example

$|\psi_1\rangle = |11\rangle = |1\rangle_c |1\rangle_T$, [Act w/ the Hadamard gate on control qubit] $\left(U_{Had}|1\rangle_c\right)|1\rangle_T$

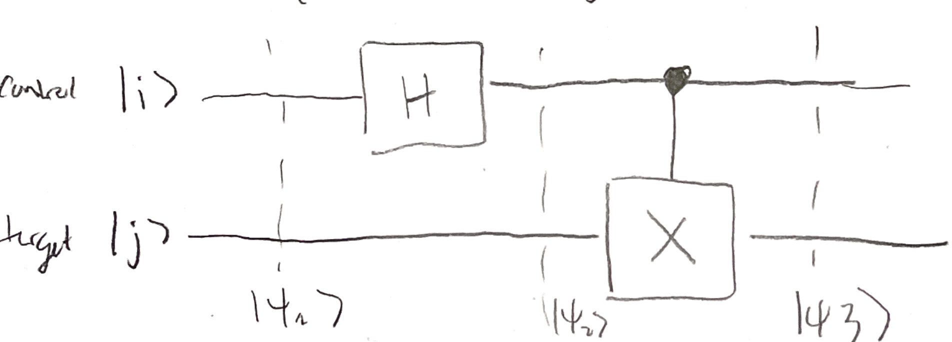
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle_c - |1\rangle_c\right)$$

Thus: $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_c - |1\rangle_c\right)|1\rangle_T = \frac{1}{\sqrt{2}}\left(|01\rangle - |11\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ ← not a bell state!

$$|\psi_3\rangle = U_{CNot}|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$ ← which is a bell state $|\beta_{11}\rangle$

labeled like this because we started w/ $|11\rangle$



$|\beta_{ij}\rangle$