A Proof of Induction

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The standard technique to prove facts about the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers is induction: if P is a predicate on $\mathbb{N}$, it suffices to prove $P(0)$ and that $P(n) \implies P(n+1)$ for all $n \in \mathbb{N}$ to deduce that $P(n)$ holds for all $n \in \mathbb{N}$. Intuitively, this technique can be justified by noting that $P(1)$ holds since $P(0)$ holds, $P(2)$ holds since $P(1)$ holds, and so on.

However, this does not constitute a rigorous proof. In order to prove that induction is a valid proof technique, we will need the following property of $\mathbb{N}$: for every nonempty subset $B \subseteq \mathbb{N}$, there exists a "first element" of B, i.e, there exists a $b_0 \in B$ such that $b_0 \leq b$ for all $b \in B$. We will also need the following three basic facts about $\mathbb{N}$: if $a \in \mathbb{N}$ is nonzero, then $a = n + 1$ for some $n \in \mathbb{N}$, $n < n+1$ for each $n \in \mathbb{N}$, and it cannot be that $n < m$ and $m \leq n$ for $n, m \in \mathbb{N}$.[1]

We can represent a predicate P as a subset $A \subseteq \mathbb{N}$ of the elements for which the predicate holds, i.e,

$$A = \{n \in \mathbb{N} : P(n) \text{ is true}\}. \tag{1}$$

With this representation, we can prove that induction does indeed work.

Theorem (Induction)**.** *Let $A \subseteq \mathbb{N}$. Suppose that $0 \in A$ and that for all $n \in A$, $n + 1 \in A$. Then $A = \mathbb{N}$.*

Proof. Suppose for contradiction that $A \neq \mathbb{N}$. Then, $B = \mathbb{N} \setminus A \neq \varnothing$, and so by the property of $\mathbb{N}$ mentioned above, there exists a $b_0 \in B$ such that $b_0 \leq b$ for all $b \in B$. Since $0 \in A$, $b_0 \neq 0$, and so $b_0 = n + 1$ for some $n \in \mathbb{N}$. Now, it cannot be that $n \in B$, since then $b_0 = n + 1 \leq n$, contradicting that $n < n + 1$. Therefore $n \in A$, and so by assumption $b_0 = n + 1 \in A$. This contradicts that $b_0 \in B$, and so it must be that $A = \mathbb{N}$. $\square$

We can also prove that strong induction is valid.

Theorem (Strong induction)**.** *Let $A \subseteq \mathbb{N}$. Suppose that $0 \in A$ and that for all $n \in \mathbb{N}$,*

$$\{k \in \mathbb{N} : k < n\} \subseteq A \implies n \in A. \tag{2}$$

Then $A = \mathbb{N}$.

[1] Proving that $\mathbb{N}$ satisfies these properties usually requires us to know the set-theoretic details of how $\mathbb{N}$ is constructed and how $+$ and $<$ are defined.

Proof. Suppose for contradiction that $A \neq \mathbb{N}$. Then, $B = \mathbb{N} \setminus A \neq \varnothing$, and so there exists a $b_0 \in B$ such that $b_0 \leq b$ for all $b \in B$. Let $n \in \mathbb{N}$ be such that $n < b_0$. It cannot be that $n \in B$, since then $b_0 \leq n$, a contradiction. Hence, $n \in A$, and since n was arbitrary,

$$\{n \in \mathbb{N} : n < b_0\} \subseteq A. \tag{3}$$

By assumption, we then have that $b_0 \in A$, a contradiction. Therefore it must be that $A = \mathbb{N}$. $\square$

It is also possible to prove that normal or strong induction with an arbitrary starting point are also valid. In addition, we could prove strong induction and other forms of induction just using normal induction.

The main property that we used about $\mathbb{N}$, that nonempty subsets have "first elements", generalizes to the concept of a well-ordered set. For well-ordered sets, we have a corresponding induction principle called transfinite induction. The validity of this induction principle can be proved in a similar way as strong induction, by assuming a subset is nonempty and therefore has a first element and then deriving a contradiction. For $\mathbb{N}$, transfinite induction reduces to strong induction.