Using Connectedness to Show Constancy

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Connectivity is one of the main topological properties that we utilize when working with a space. A particularly useful property of a connected space is that we can often extend "local" results to "global" results. In this note, we state and prove one of these "extension" results and use it in a concrete setting.

The question we consider is whether a function on a space is constant. This is a "global" assertion about a function, and we have a corresponding "local" assertion that we want to extend.

Definition 1. Let X be a space and S be a set. We say that $f : X \to S$ is locally constant if for each $x \in X$ there is an open neighborhood U of x such that f is constant on U.

In particular, every constant function is locally constant. When the domain is connected, the converse holds as well.

Theorem 2. *Let X be a connected space and S be a set. Then, if $f : X \to S$ is locally constant, it is constant.*

Proof. If X is empty, then the assertion that f is constant is vacuously true, and we are done.

Otherwise, assume that X is nonempty. Then, there is an $x_0 \in X$. Define

$$A = \{x \in X : f(x) = f(x_0)\}.$$

We first claim that A is open. Indeed, for each $a \in A$, since f is locally constant, there is an open neighborhood U of a such that f is constant on U. Hence, for each $x \in U$, $f(x) = f(a) = f(x_0)$, and so $x \in A$. Therefore $U \subseteq A$, and so a is an interior point of A. As a was arbitrary, A is open.

We now claim that A is closed. For each $b \in X \setminus A$, since f is locally constant, there is an open neighborhood V of b such that f is constant on V. Thus, for each $x \in V$, $f(x) = f(b) \neq f(x_0)$, so $x \in X \setminus A$. Hence, $V \subseteq X \setminus A$, and so b is an interior point of $X \setminus A$. Since b was arbitrary, $X \setminus A$ is open and therefore A is closed.

We thus have that A is clopen, and since $x_0 \in A$ we also have that A is nonempty. By connectedness of X, the only nonempty clopen subset of X is X itself, so

$A = X$. Thus, for all $x, y \in X$, $x, y \in A$, and so $f(x) = f(x_0) = f(y)$. We conclude that f is constant. $\square$

We now take a look at a concrete example of this result. In calculus, we usually assert (inaccurately) that if a function's derivative is always zero, then it is constant. A more precise (and correct) statement is as follows.

Theorem 3. *Let $U \subseteq \mathbb{R}$ be open and $f : U \to \mathbb{R}$ be differentiable. Suppose that U is connected and that $f'(x) = 0$ for all $x \in U$. Then f is constant.*

Proof. Note that since f is differentiable, it is also continuous.

We wish to show that f is locally constant. Let $x_0 \in U$. Since U is open, we can find a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq U$. Let $x \in (x_0 - \delta, x_0 + \delta)$. There are three cases.

First, if $x = x_0$, then $f(x) = f(x_0)$.

Second, if $x < x_0$, then $[x, x_0] \subseteq U$. Since f is continuous on $[x, x_0]$ and differentiable on (x, x_0), by the mean value theorem there exists a $c \in (x, x_0)$ such that

$$f(x_0) - f(x) = f'(c)(x_0 - x) = 0 \cdot (x_0 - x) = 0$$

since $f'(y) = 0$ for all $y \in U$. Hence, $f(x) = f(x_0)$.

Third, if $x_0 < x$, then $[x_0, x] \subseteq U$, and a similar argument to the case when $x < x_0$ shows that $f(x) = f(x_0)$.

In all three cases, $f(x) = f(x_0)$, so for any $x, y \in (x_0 - \delta, x_0 + \delta)$, $f(x) = f(x_0) = f(y)$. Therefore f is constant on $(x_0 - \delta, x_0 + \delta)$, which is an open neighborhood of x_0.

We conclude that f is locally constant. Since U is connected, by Theorem 2 we have that f is constant. $\square$

A similar result holds in higher dimensions.

Theorem 4. *Let n be a positive integer, $U \subseteq \mathbb{R}^n$ be open and connected, and $f : U \to \mathbb{R}$. Suppose that f is differentiable and that $\nabla f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in U$. Then f is constant.*

The proof is similar to the $n = 1$ case: we show that f is locally constant using a generalization of the mean value theorem, and then invoke Theorem 2.

This last result is interesting in that it allows us to detect a topological property using differential methods: if we can find a differentiable nonconstant function on U whose derivative is always zero, then U must be disconnected.

More generally, we can investigate topological properties of differentiable manifolds by studying differentiable functions on them. This leads us naturally to the field of Morse theory.