A Proof of the Fundamental Theorem of Calculus

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Despite being such an important theorem for the theory of integration, the fundamental theorem of calculus (FTC) is actually fairly straightforward to prove. (The theorem we prove here is usually referred to as the second part of the FTC, but it is just as fundamental as the first part.) The main tool is the mean value theorem.

We use the Darboux integral, which is equivalent to the usual Riemann integral.

Theorem (FTC)**.** *Let a and b be real numbers with $a < b$, and let f and F be functions $[a, b] \to \mathbb{R}$. Suppose that f is bounded and integrable and that F is continuous on $[a, b]$ and differentiable on (a, b). If $F'(x) = f(x)$ for all $x \in (a, b)$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{1}$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, where n is a positive integer and

$$a = x_0 < x_1 < \cdots < x_n = b. \tag{2}$$

For each $i \in \{1, \ldots, n\}$, since F is continuous on $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i), by the mean value theorem there exists a $c_i \in (x_{i-1}, x_i) \subseteq (a, b)$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}). \tag{3}$$

We have that $F'(c_i) = f(c_i)$ since $c_i \in (a, b)$, and so

$$F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1}). \tag{4}$$

Therefore

$$\sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \geq f(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1}) \tag{5}$$

and

$$\inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \leq f(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1}). \tag{6}$$

Hence,

$$U(f, P) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \geq \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) \tag{7}$$

and

$$L(f,P) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \leq \sum_{i=1}^{n} F(x_i) - F(x_{i-1}). \tag{8}$$

This sum telescopes,[1] and so

$$\sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = F(x_n) - F(x_0) = F(b) - F(a) \tag{9}$$

and

$$L(f,P) \leq F(b) - F(a) \leq U(f,P). \tag{10}$$

Since P was arbitrary,

$$\underline{\int_a^b} f(x)\, dx = \sup_P L(f,P) \leq F(b) - F(a) \leq \inf_P U(f,P) = \overline{\int_a^b} f(x)\, dx. \tag{11}$$

By assumption, f is integrable, and so

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx \tag{12}$$

and therefore

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{13}$$

$\square$

[1] It is interesting to note that the telescoping of a sum is almost like a discrete FTC, where we replace integrals with sums and derivatives with differences. This "discrete FTC" then shows up in the proof of the "continuous" FTC.