Approximating the Uniform Norm via Discrete Sampling

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Abstract

Given a polynomial on the complex unit circle, we can approximate its uniform norm by sampling at a finite number of points. We discuss here some of the known results and open problems around estimating the relative error in this approximation. We also prove some minor generalizations of these results.

Contents

1 Introduction

Suppose that we have some periodic signal that we wish to process in some way. An important feature of the signal that is relevant for several real-world applications is the peak amplitude, i.e, the maximum absolute value of the signal.

It might be infeasible or unnecessary to determine the exact peak amplitude of a signal, and so instead one can approximate it. One way to approximate the peak amplitude is to sample the signal at several points and take the maximum of the absolute values of the signal at these points. Two natural questions arise: how well does this approximate the peak amplitude, and which choices of sampling points are best?

To begin to answer these questions, we first translate the problem into more mathematical language. We can represent a signal as a periodic real-valued function f of a real variable, say with period $T > 0$. The peak amplitude of f is then

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in[0,T]} |f(x)|, \tag{1}$$

using the periodicity of f.

One important class of periodic functions are the real trigonometric polynomials (of period T), which are of the form

$$s(x) = \sum_{n=0}^{N} a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right), \quad x \in \mathbb{R}, \tag{2}$$

where the a_n and b_n are real coefficients. If we assume that the signal f is continuous, then it turns out that f can be approximated arbitrarily well by a real trigonometric polynomial. We therefore focus on trying to approximate the peak amplitude of a real trigonometric polynomial.

Using Euler's formula, we can also write the real trigonometric polynomial s in the form

$$s(x) = \sum_{n=-N}^{N} c_n e^{2\pi nix/T} = \sum_{n=-N}^{N} c_n \left(e^{2\pi ix/T}\right)^n, \quad x \in \mathbb{R}, \tag{3}$$

where the c_n are complex coefficients. By periodicity, we only need to consider $x \in [0, T]$ in order to find the peak amplitude, and as x goes from 0 to T, $e^{2\pi ix/T}$ traces out the unit circle. Hence, to find the peak amplitude of s, it suffices to find the maximum absolute value of the Laurent polynomial

$$L(z) = \sum_{n=-N}^{N} c_n z^n, \quad |z| = 1, \tag{4}$$

where z is a complex variable. Now, as $|z^N| = 1$ when $|z| = 1$, we have that

$$|L(z)| = |z^N L(z)| = \left|\sum_{n=-N}^{N} c_n z^{N+n}\right| = \left|\sum_{n=0}^{2N} c_{n-N} z^n\right|. \tag{5}$$

We therefore need to approximate the maximum absolute value of the complex polynomial

$$p(z) = \sum_{n=0}^{2N} c_{n-N} z^n, \quad |z| = 1. \tag{6}$$

In particular, we wish to investigate how sampling the values of p at finitely many points allows us to approximate the maximum absolute value of p. This is the task which we discuss in this paper.

Paper Outline: First, in Section 2, we formalize the problem statement and state some classical results that we will use later. In Section 3, we discuss some of the existing results for a particular case of the problem. We then prove some generalizations of these results in Section 4 and apply these generalizations to some open problems in Section 5.

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2 Background

2.1 Formalizing the Problem

Fix nonnegative integers N and n, and assume that $N > n$. We denote the complex vector space of all polynomials of degree at most n with complex coefficients by $\mathbb{P}_n$. This vector space is finite-dimensional, with dimension $n+1$.

We denote the complex unit circle by

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \tag{7}$$

and the Nth roots of unity by

$$\mathbb{T}_N = \left\{z \in \mathbb{C} : z^N = 1\right\} = \left\{e^{2\pi ij/N} : j = 0, \dots, N-1\right\}. \tag{8}$$

Note that $\mathbb{T}_N \subseteq \mathbb{T}$ and that $\#\mathbb{T}_N = N$.

We can now define two norms on $\mathbb{P}_n$. First, we have the uniform norm $\|\cdot\|$ defined by

$$\|p\| = \sup_{z\in\mathbb{T}} |p(z)|, \quad p \in \mathbb{P}_n, \tag{9}$$

which is always finite since polynomials are continuous functions and $\mathbb{T}$ is compact. We also have the sampling norm $\|\cdot\|_N$ defined by

$$\|p\|_N = \sup_{z\in\mathbb{T}_N} |p(z)| = \max_{0\le j\le N-1} \left|p\left(e^{2\pi ij/N}\right)\right|, \quad p \in \mathbb{P}_n. \tag{10}$$

This defines a norm on $\mathbb{P}_n$ since a polynomial of degree at most n that has $N > n$ roots is necessarily the zero polynomial.

More generally, if $F \subseteq \mathbb{T}$ is any subset such that $\#F = N$, we can define the sampling norm $\|\cdot\|_F$ given by

$$\|p\|_F = \sup_{z \in F} |p(z)|, \quad p \in \mathbb{P}_n. \tag{11}$$

With this notation, $\|\cdot\|_N = \|\cdot\|_{\mathbb{T}_N}$.

Now, fix a subset $F \subseteq \mathbb{T}$ such that $\#F = N$. We wish to see how good of an approximation $\|\cdot\|_F$ is to $\|\cdot\|$. In particular, we want to investigate the relative error

$$\left| \frac{\|p\| - \|p\|_F}{\|p\|} \right| = \left| 1 - \frac{\|p\|_F}{\|p\|} \right|, \quad p \in \mathbb{P}_n^*, \tag{12}$$

where $\mathbb{P}_n^* \subseteq \mathbb{P}_n$ is the set of all nonzero polynomials in $\mathbb{P}_n$. Since $\|p\|_F \leq \|p\|$ already holds for all $p \in \mathbb{P}_n$, the relative error simply becomes

$$1 - \frac{\|p\|_F}{\|p\|}, \quad p \in \mathbb{P}_n^*. \tag{13}$$

Therefore one way to quantify how well $\|\cdot\|_F$ approximates $\|\cdot\|$ is to investigate the quantity

$$\sup_{p \in \mathbb{P}_n^*} \left(1 - \frac{\|p\|_F}{\|p\|} \right) = 1 - \left(\sup_{p \in \mathbb{P}_n^*} \frac{\|p\|}{\|p\|_F} \right)^{-1}. \tag{14}$$

We therefore define

$$K(F, n) = \sup_{p \in \mathbb{P}_n^*} \frac{\|p\|}{\|p\|_F}. \tag{15}$$

As $K(F, n)$ gets closer to 1, the approximation of $\|\cdot\|$ via discrete sampling on F gets better. Estimating this quantity for various choices of F, N, and n is the problem we will focus on in this paper.

As remarked earlier, $\|p\|_F \leq \|p\|$ for all $p \in \mathbb{P}_n$, and so $1 \leq \|p\|/\|p\|_F$ for all $p \in \mathbb{P}_n^*$. Therefore $1 \leq K(F, n)$, and in particular $K(F, n)$ is always positive.

In addition, we can show that $K(F, n)$ is always finite. Since $\|\cdot\|$ and $\|\cdot\|_F$ are both norms on the finite-dimensional vector space $\mathbb{P}_n$, they must be equivalent norms, and so there exist positive constants c and C such that

$$c\|p\|_F \leq \|p\| \leq C\|p\|_F, \quad p \in \mathbb{P}_n. \tag{16}$$

Therefore

$$c \leq \frac{\|p\|}{\|p\|_F} \leq C, \quad p \in \mathbb{P}_n^*, \tag{17}$$

and so $c \leq K(F, n) \leq C$. In particular, $K(F, n) < \infty$, and we can reinterpret $K(F, n)$ as the optimal value of the constant C.

2.2 Auxiliary Results

We will need a number of classical results for the proofs in Section 4, so for convenience we will state them here. We use the notation of Section 2.1.

The first result is a special case of a theorem of Bernstein [1].

Theorem 2.1. *Let n be a nonnegative integer and $p \in \mathbb{P}_n$. Then $\|p'\| \leq n\|p\|$.*

The next two results are standard theorems from complex analysis. They can be found in the reference [11] as Proposition 3.1 (iii) and Proposition 3.2, respectively, in Chapter 1.

Theorem 2.2. *Let γ be a smooth contour in the complex plane and f a continuous function $\mathbb{C} \to \mathbb{C}$. Then*

$$\left| \int_\gamma f(z)\, dz \right| \leq \sup_\gamma |f(z)| \cdot \ell(\gamma), \tag{18}$$

where $\ell(\gamma)$ is the length of the contour γ and the supremum is taken over the points on the contour.

Theorem 2.3. *Let z_0 and z_1 be points in the complex plane and γ be a smooth contour in the complex plane from z_0 to z_1. Suppose that F is a holomorphic function $\mathbb{C} \to \mathbb{C}$. Then,*

$$\int_\gamma F'(z)\, dz = F(z_1) - F(z_0). \tag{19}$$

3 Known Results

A number of estimates of $K(F, n)$ are known in the special case when $F = \mathbb{T}_N$. We leverage these estimates in Section 4.2 to prove estimates for more general F.

The first result, in the special case when $N = n + 1$, is due to Marcinkiewicz [7].

Theorem 3.1 (Marcinkiewicz, 1937)**.** *Let n be an integer such that $n \geq 2$. Then, there exist positive constants C_1 and C_2, independent of n, such that*

$$C_1 \log(n) \leq K(\mathbb{T}_{n+1}, n) \leq C_2 \log(n). \tag{20}$$

This result was then generalized to arbitrary $N > n$ by Rakhmanov and Shekhtman [9].

Theorem 3.2 (Rakhmanov and Shekhtman, 2006)**.** *Let N and n be nonnegative integers with $N > n$. Then, there exist positive constants C_1 and C_2, independent of N and n, such that*

$$1 + C_1 \log\left(\frac{N}{N-n}\right) \leq K(\mathbb{T}_N, n) \leq 1 + C_2 \log\left(\frac{N}{N-n}\right). \tag{21}$$

This result implies in particular that $K(\mathbb{T}_N, n) \to 1$ as $N \to \infty$, as expected. Therefore we can approximate $\|\cdot\|$ arbitrarily well by sampling on $\mathbb{T}_N$ for very large N.

There is a better upper bound when N is much larger than n, due to Sheil-Small [10].

Theorem 3.3 (Sheil-Small, 2008). *Let N and n be nonnegative integers with $N > n$. Then,*

$$K(\mathbb{T}_N, n) \leq \sqrt{\frac{N}{N-n}}. \tag{22}$$

Furthermore, both the upper bound due to Rakhmanov and Skekhtman and the upper bound due to Sheil-Small can be improved when $N \geq 2n$ using a bound due to Dubinin [4].

Theorem 3.4 (Dubinin, 2011). *Let N and n be nonnegative integers with $N > n$. Then,*

$$K(\mathbb{T}_N, n) \leq \sec\left(\frac{\pi n}{2N}\right). \tag{23}$$

This upper bound is obtained if N is a multiple of n.

This result was generalized by Kalmykov to when the uniform norm is taken over an arc of the unit circle instead [6].

For the analogous problem of comparing the uniform and sampling norms when the polynomials are real, results have been obtained by Coppersmith and Rivlin [3], Kalmykov [6], and Rakhmanov [8].

4 New Results

The known results in Section 3 are only for the special case when $F = \mathbb{T}_N$. In this section we develop some results for more general F. First, in Section 4.1 we establish some independent results relating $K(F, n)$ and $K(\mathbb{T}_N, n)$ when F is "close" to $\mathbb{T}_N$. Then, in Section 4.2, we use the results of Section 4.1 and the existing results from Section 3 to derive some more explicit bounds on $K(F, n)$.

4.1 Independent Results

Fix nonnegative integers N and n, and assume that $N > n$. The points of $\mathbb{T}_N$ are then given by $e^{i\phi_j}$, $j \in \{0, \ldots, N-1\}$, where $\phi_j = 2\pi j/N$. For each $j \in \{0, \ldots, N-1\}$, we perturb the angle ϕ_j by an angle $\theta_j \in [-\pi, \pi]$. We then have the perturbed sample points

$$F = \left\{e^{i(\phi_j + \theta_j)} : j \in \{0, \ldots, N-1\}\right\}. \tag{24}$$

We assume that the perturbations do not cause any of the perturbed points to align, so that $\#F = N$. Furthermore, any subset $F \subseteq \mathbb{T}$ with $\#F = N$ can be obtained in this way for particular choices of the θ_j's.

We can measure the extent to which F is a perturbation of $\mathbb{T}_N$ by the quantity

$$\theta = \max_{0 \le j \le N-1} |\theta_j|. \tag{25}$$

Our main result here is that when θ is sufficiently small (in a way that depends on N and n) we can bound $K(F,n)$ using $K(\mathbb{T}_N,n)$.

Theorem 4.1. *Suppose that*

$$n\theta < \frac{1}{K(\mathbb{T}_N,n)}. \tag{26}$$

Then,

$$\left(\frac{1}{K(\mathbb{T}_N,n)} + n\theta\right)^{-1} \le K(F,n) \le \left(\frac{1}{K(\mathbb{T}_N,n)} - n\theta\right)^{-1}. \tag{27}$$

For the proof, we will need the following lemma.

Lemma 4.1. *Suppose that* $j \in \{0,\ldots,N-1\}$ *and that* $p \in \mathbb{P}_n$. *Then,*

$$\left|p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j+\theta_j)}\right)\right| \le n\theta\|p\|. \tag{28}$$

Proof of Lemma 4.1. Let γ be the contour lying on the unit circle from $e^{i\phi_j}$ to $e^{i(\phi_j+\theta_j)}$, i.e, the arc of the unit circle between the points with the angles ϕ_j and $\phi_j+\theta_j$. By Theorem 2.3,

$$\left|p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j+\theta_j)}\right)\right| = \left|\int_\gamma p'(z)\, dz\right|. \tag{29}$$

Now, the length of the contour γ is $\ell(\gamma) = |\theta_j|$, and so by Theorem 2.2,

$$\left|\int_\gamma p'(z)\, dz\right| \le \sup_\gamma |p'(z)| \cdot \ell(\gamma) \le \sup_{z\in\mathbb{T}} |p'(z)|\,|\theta_j| = \|p'\|\,|\theta_j| \le \|p'\|\theta. \tag{30}$$

Finally, by Theorem 2.1, $\|p'\| \le n\|p\|$, and so

$$\left|p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j+\theta_j)}\right)\right| \le \|p'\|\theta \le n\theta\|p\|. \tag{31}$$

□

Proof of Theorem 4.1. We first prove the upper bound. Let $p \in \mathbb{P}_n^*$. Then,

$$\begin{aligned}
\|p\|_F &= \max_{0\le j\le N-1} \left|p\left(e^{i(\phi_j+\theta_j)}\right)\right| && (32)\\
&\ge \max_{0\le j\le N-1} \left(\left|p\left(e^{i\phi_j}\right)\right| - \left|p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j+\theta_j)}\right)\right|\right) && (33)\\
&\ge \max_{0\le j\le N-1} \left|p\left(e^{i\phi_j}\right)\right| - \max_{0\le j\le N-1} \left|p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j+\theta_j)}\right)\right| && (34)\\
&= \|p\|_N - \max_{0\le j\le N-1} \left|p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j+\theta_j)}\right)\right|. && (35)
\end{aligned}$$

By Lemma 4.1, we then have that

$$\|p\|_F \geq \|p\|_N - n\theta\|p\|. \tag{36}$$

Since

$$\frac{\|p\|}{\|p\|_N} \leq K(\mathbb{T}_N, n), \tag{37}$$

we therefore have that

$$\|p\|_F \geq \frac{\|p\|}{K(\mathbb{T}_N, n)} - n\theta\|p\| = \|p\| \left(\frac{1}{K(\mathbb{T}_N, n)} - n\theta \right). \tag{38}$$

By assumption,

$$\frac{1}{K(\mathbb{T}_N, n)} - n\theta > 0, \tag{39}$$

and so we can rearrange to obtain that

$$\frac{\|p\|}{\|p\|_F} \leq \left(\frac{1}{K(\mathbb{T}_N, n)} - n\theta \right)^{-1}. \tag{40}$$

Since p was arbitrary,

$$K(F, n) \leq \left(\frac{1}{K(\mathbb{T}_N, n)} - n\theta \right)^{-1}. \tag{41}$$

We now prove the lower bound. Let $0 < \varepsilon < K(\mathbb{T}_N, n)$. By the definition of $K(\mathbb{T}_N, n)$, there exists a $p \in \mathbb{P}_n^*$ such that

$$\frac{\|p\|}{\|p\|_N} > K(\mathbb{T}_N, n) - \varepsilon. \tag{42}$$

Therefore

$$\|p\|_N < \frac{\|p\|}{K(\mathbb{T}_N, n) - \varepsilon}. \tag{43}$$

Next, we have that

$$\begin{aligned}
\|p\|_F &= \max_{0 \leq j \leq N-1} \left| p\left(e^{i(\phi_j + \theta_j)}\right) \right| && (44)\\
&\leq \max_{0 \leq j \leq N-1} \left(\left| p\left(e^{i\phi_j}\right) \right| + \left| p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j + \theta_j)}\right) \right| \right) && (45)\\
&\leq \max_{0 \leq j \leq N-1} \left| p\left(e^{i\phi_j}\right) \right| + \max_{0 \leq j \leq N-1} \left| p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j + \theta_j)}\right) \right| && (46)\\
&= \|p\|_N + \max_{0 \leq j \leq N-1} \left| p\left(e^{i\phi_j}\right) - p\left(e^{i(\phi_j + \theta_j)}\right) \right|. && (47)
\end{aligned}$$

By Lemma 4.1, we then have that

$$\|p\|_F \leq \|p\|_N + n\theta\|p\|. \tag{48}$$

Therefore

$$\|p\|_F \leq \frac{\|p\|}{K(\mathbb{T}_N,n)-\varepsilon} + n\theta\|p\| = \|p\|\left(\frac{1}{K(\mathbb{T}_N,n)-\varepsilon} + n\theta\right). \tag{49}$$

We can then rearrange this to yield that

$$K(F,n) \geq \frac{\|p\|}{\|p\|_F} \geq \left(\frac{1}{K(\mathbb{T}_N,n)-\varepsilon} + n\theta\right)^{-1}. \tag{50}$$

Since ε was arbitrary,

$$K(F,n) \geq \left(\frac{1}{K(\mathbb{T}_N,n)} + n\theta\right)^{-1}. \tag{51}$$

□

Letting $\theta \to 0^+$, the upper and lower bounds collapse to the equality $K(F,n) = K(\mathbb{T}_N,n)$, as expected. Therefore we can make the ratio $K(F,n)/K(\mathbb{T}_N,n)$ as close to 1 as we desire by restricting the maximum size of the perturbations. In particular, small perturbations to the sample points of $\mathbb{T}_N$ will yield small deviations in the accuracy of the approximation made by sampling via $\mathbb{T}_N$.

The assumed bounds on θ and the derived bounds on $K(F,n)$ in Theorem 4.1 can be determined using knowledge of $K(\mathbb{T}_N,n)$ and θ. Therefore results about $K(\mathbb{T}_N,n)$ can be translated into results about small perturbations of $\mathbb{T}_N$ which involve more explicit bounds on θ and $K(F,n)$. We derive some of these results in Section 4.2 using the results of Section 3.

We can also derive a corollary of Theorem 4.1 that makes the bounds more straightforward.

Corollary 4.1. *Let $\delta \in [0,1)$ and suppose that*

$$n\theta \leq \frac{\delta}{K(\mathbb{T}_N,n)}. \tag{52}$$

Then,

$$\frac{1}{1+\delta}K(\mathbb{T}_N,n) \leq K(F,n) \leq \frac{1}{1-\delta}K(\mathbb{T}_N,n). \tag{53}$$

Proof. Since

$$n\theta \leq \frac{\delta}{K(\mathbb{T}_N,n)} < \frac{1}{K(\mathbb{T}_N,n)}, \tag{54}$$

by Theorem 4.1,

$$\begin{aligned}
K(F,n) &\leq \left(\frac{1}{K(\mathbb{T}_N,n)} - n\theta\right)^{-1} &(55)\\
&\leq \left(\frac{1}{K(\mathbb{T}_N,n)} - \frac{\delta}{K(\mathbb{T}_N,n)}\right)^{-1} &(56)\\
&= \frac{1}{1-\delta}K(\mathbb{T}_N,n). &(57)
\end{aligned}$$

and

$$K(F,n) \geq \left(\frac{1}{K(\mathbb{T}_N,n)} + n\theta \right)^{-1} \tag{58}$$

$$\geq \left(\frac{1}{K(\mathbb{T}_N,n)} + \frac{\delta}{K(\mathbb{T}_N,n)} \right)^{-1} \tag{59}$$

$$= \frac{1}{1+\delta} K(\mathbb{T}_N,n). \tag{60}$$

□

Substituting explicit values for δ, we can find ranges of perturbations of $\mathbb{T}_N$ for which we can guarantee that $K(F,n)$ is within a certain proportion of $K(\mathbb{T}_N,n)$. Taking $\delta = 1/4$, for example, we get that if

$$n\theta \leq \frac{1}{4K(\mathbb{T}_N,n)}, \tag{61}$$

then we have the estimate

$$\frac{4}{5} K(\mathbb{T}_N,n) \leq K(F,n) \leq \frac{4}{3} K(\mathbb{T}_N,n). \tag{62}$$

4.2 Using Known Results

Combining the known results in Section 3 and the results of Section 4.1, we can derive more explicit results on bounding $K(F,n)$. We keep the notation from Section 4.1.

First, we can draw the same conclusion of Corollary 4.1 with a stronger but more simple bound on θ.

Corollary 4.2. *Let $\delta \in [0,1)$ and suppose that*

$$\theta \leq \frac{\delta}{N^{3/2}}. \tag{63}$$

Then,

$$\frac{1}{1+\delta} K(\mathbb{T}_N,n) \leq K(F,n) \leq \frac{1}{1-\delta} K(\mathbb{T}_N,n) \tag{64}$$

and

$$K(F,n) \leq \frac{1}{1-\delta} \sqrt{\frac{N}{N-n}}. \tag{65}$$

Proof. We have that

$$n\theta \leq \frac{\delta n}{N^{3/2}} \leq \frac{\delta N}{N^{3/2}} = \delta \sqrt{\frac{1}{N}} \leq \delta \sqrt{\frac{N-n}{N}}. \tag{66}$$

By Theorem 3.3,

$$K(\mathbb{T}_N, n) \leq \sqrt{\frac{N}{N-n}}, \tag{67}$$

and so

$$n\theta \leq \delta\sqrt{\frac{N-n}{N}} \leq \frac{\delta}{K(\mathbb{T}_N, n)}. \tag{68}$$

Thus, by Corollary 4.1,

$$\frac{1}{1+\delta}K(\mathbb{T}_N, n) \leq K(F, n) \leq \frac{1}{1-\delta}K(\mathbb{T}_N, n). \tag{69}$$

Furthermore,

$$K(F, n) \leq \frac{1}{1-\delta}K(\mathbb{T}_N, n) \leq \frac{1}{1-\delta}\sqrt{\frac{N}{N-n}}. \tag{70}$$

□

We can derive the same conclusion with a weaker bound on θ as long as we have some guarantees on how large N is compared to n. Informally, we can afford larger perturbations of the sample points if N is sufficiently larger than n.

Corollary 4.3. *Let $\delta \in [0, 1)$ and $\alpha \in [1, \infty)$ and suppose that $N \geq \alpha n$ and*

$$\theta \leq \frac{\delta\sqrt{\alpha - 1}}{N}. \tag{71}$$

Then,

$$\frac{1}{1+\delta}K(\mathbb{T}_N, n) \leq K(F, n) \leq \frac{1}{1-\delta}K(\mathbb{T}_N, n). \tag{72}$$

Proof. We have that

$$n\theta \leq \frac{\delta n\sqrt{\alpha - 1}}{N} = \frac{\delta\sqrt{n}\sqrt{\alpha n - n}}{\sqrt{N}\sqrt{N}} \leq \delta\sqrt{\frac{N}{N}}\sqrt{\frac{N-n}{N}} = \delta\sqrt{\frac{N-n}{N}}. \tag{73}$$

By Theorem 3.3,

$$K(\mathbb{T}_N, n) \leq \sqrt{\frac{N}{N-n}}, \tag{74}$$

and so

$$n\theta \leq \delta\sqrt{\frac{N-n}{N}} \leq \frac{\delta}{K(\mathbb{T}_N, n)}. \tag{75}$$

Thus, by Corollary 4.1,

$$\frac{1}{1+\delta}K(\mathbb{T}_N, n) \leq K(F, n) \leq \frac{1}{1-\delta}K(\mathbb{T}_N, n). \tag{76}$$

□

In particular, setting $\alpha = 2$, the conclusion holds if $N \geq 2n$ and $\theta \leq \delta/N$.

Similar corollaries can be derived using Theorem 3.2 and Theorem 3.4, where we can get the same conclusion of Corollary 4.1 with a bound on θ that does not depend on $K(\mathbb{T}_N, n)$.

5 Open Problems

There are a number of conjectures around the quantity $K(F, n)$. We discuss a few of them in Section 5.1 and prove a special case of one of them in Section 5.2.

5.1 Conjecture Statements

The first conjecture is due to Erdős [5].

Conjecture 5.1. *Let N and n be nonnegative integers with $N > n$ and $F \subseteq \mathbb{T}$ be such that $\#F = N$. Then, there exists a positive constant C, independent of N, n, and F, such that*

$$K(F, n) \geq C \log \left(\frac{N}{N - n} \right). \tag{77}$$

This lower bound is similar to the bound due to Rakhmanov and Shekhtman (Theorem 3.2), but with a general F instead of $\mathbb{T}_N$. This conjecture would follow from Theorem 3.2 and the following conjecture, which is stated in [9].

Conjecture 5.2. *Let N and n be nonnegative integers with $N > n$ and $F \subseteq \mathbb{T}$ be such that $\#F = N$. Then,*

$$K(F, n) \geq K(\mathbb{T}_N, n). \tag{78}$$

Informally, this conjecture states that $\mathbb{T}_N$ is the optimal choice of N sampling points for a fixed N, since we always have that $K(\mathbb{T}_N, n) \geq 1$ and the accuracy improves the closer $K(F, n)$ is to 1. This conjecture was proven in the special case of $N = n + 1$ by de Boor and Pinkus [2].

5.2 Partial Solutions

Using the results of Section 4, we can prove Conjecture 5.1 in the special case when F is a sufficiently small perturbation of $\mathbb{T}_N$. We use the notation of Section 4 for the following result.

Theorem 5.1. *Suppose that either*

$$\theta \leq \frac{1}{2N^{3/2}} \tag{79}$$

or

$$N \geq 2n \quad \textit{and} \quad \theta \leq \frac{1}{2N}. \tag{80}$$

Then, there exists a positive constant C, independent of N and n, such that

$$K(F,n) \geq C \log\left(\frac{N}{N-n}\right). \tag{81}$$

Proof. If $\theta \leq \delta/N^{3/2}$, then by Corollary 4.2 with $\delta = 1/2$,

$$K(F,n) \geq \frac{2}{3} K(\mathbb{T}_N, n). \tag{82}$$

If $N \geq 2n$ and $\theta \leq 1/(2N)$, then by Corollary 4.3 with $\delta = 1/2$ and $\alpha = 2$,

$$K(F,n) \geq \frac{2}{3} K(\mathbb{T}_N, n). \tag{83}$$

Hence, by Theorem 3.2, in both cases we have that

$$K(F,n) \geq \frac{2}{3}\left(1 + C_1 \log\left(\frac{N}{N-n}\right)\right) \geq \frac{2}{3} C_1 \log\left(\frac{N}{N-n}\right), \tag{84}$$

where C_1 is a positive constant independent of N and n. Setting $C = 2C_1/3$, C is a positive constant independent of N and n, and

$$K(F,n) \geq C \log\left(\frac{N}{N-n}\right). \tag{85}$$

□

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