Supplementary: Kinetic Plasma Dispersion Relation for Arbitrary Distributions under Expansions

Hua-sheng XIE, huashengxie@gmail.com, ENN

Saturday 11$^{\text{th}}$ January, 2025

Abstract

This document provides the equations to solve the electromagnetic magnetized non-relativistic kinetic dispersion relation for arbitrary gyrotropic distributions. The distribution functions in the parallel and perpendicular directions are expanded using three types of expansions: Hermite-Hermite (HH), GPDF-Hermite (GH), and GPDF-GPDF (GG) bases. All of these expansions yield rational dispersion relations. For the HH model, by applying rational approximation to the parallel integral plasma dispersion function, we also present the final eigenvalue matrix, which allows for solving all solutions without requiring an initial guess.

1 The Kinetic Dispersion Relation

We consider only the electromagnetic magnetized model. In this work, we limit our discussion to linear plasma waves in an infinite, uniform, and homogeneous system. The background magnetic field is assumed to be $\boldsymbol{B}_0 = (0, 0, B_0)$, and the wave vector is given by $\boldsymbol{k} = (k_x, 0, k_z) = (k\sin\theta, 0, k\cos\theta)$, where $k_\parallel = k_z$ and $k_\perp = k_x$. The plasma consists of S species, indexed by $s = 1, 2, \cdots, S$. The electric charge, mass, and density of each species are denoted by q_s, m_s, and n_{s0}, respectively. The dispersion relation is given by [6, 3, 12]

$$\bar{D}(\omega, \boldsymbol{k}) \quad = \quad |\boldsymbol{D}(\omega, \boldsymbol{k})| = |\boldsymbol{K}(\omega, \boldsymbol{k}) + (\boldsymbol{kk} - k^2\boldsymbol{I})\frac{c^2}{\omega^2}| = 0,$$

i.e.,

$$\bar{D}(\omega, \boldsymbol{k}) = \begin{vmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{vmatrix} = \begin{vmatrix} K_{xx} - \frac{k_z^2c^2}{\omega^2} & K_{xy} & K_{xz} + \frac{k_zk_xc^2}{\omega^2} \\ K_{yx} & K_{yy} - \frac{k^2c^2}{\omega^2} & K_{yz} \\ K_{zx} + \frac{k_zk_xc^2}{\omega^2} & K_{zy} & K_{zz} - \frac{k_x^2c^2}{\omega^2} \end{vmatrix} = 0, \tag{1}$$

where

$$\boldsymbol{K} = \boldsymbol{I} + \boldsymbol{Q} = \boldsymbol{I} - \frac{\boldsymbol{\sigma}}{i\omega\epsilon_0}, \quad \boldsymbol{Q} = -\frac{\boldsymbol{\sigma}}{i\omega\epsilon_0}. \tag{2}$$

with

$$\boldsymbol{\sigma} = -i\sum_s \frac{q_s^2 n_{s0}}{m_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi v_\perp dv_\perp dv_\parallel}{\omega - n\omega_{cs} - k_\parallel v_\parallel} \times \begin{bmatrix} A_s\frac{n^2v_\perp}{\mu_s^2}J_n^2 & iA_s\frac{nv_\perp}{\mu_s}J_nJ_n' & B_s\frac{nv_\perp}{\mu_s}J_n^2 \\ -iA_s\frac{nv_\perp}{\mu_s}J_nJ_n' & A_sv_\perp {J_n'}^2 & -iB_sv_\perp J_nJ_n' \\ A_s\frac{nv_\parallel}{\mu_s}J_n^2 & iA_sv_\parallel J_nJ_n' & B_sv_\parallel J_n^2 \end{bmatrix}, \tag{3}$$

and

$$\mu_s = \frac{k_\perp v_\perp}{\omega_{cs}}, \quad A_s = \Big(1 - \frac{k_\parallel v_\parallel}{\omega}\Big)\frac{\partial f_{s0}}{\partial v_\perp} + \frac{k_\parallel v_\perp}{\omega}\frac{\partial f_{s0}}{\partial v_\parallel}, \quad B_s = \frac{n\omega_{cs}v_\parallel}{\omega v_\perp}\frac{\partial f_{s0}}{\partial v_\perp} + \Big(1 - \frac{n\omega_{cs}}{\omega}\Big)\frac{\partial f_{s0}}{\partial v_\parallel}. \tag{4}$$

Eq.(3) is valid for non-relativistic arbitrary gyrotropic distributions. Here, $J_n = J_n(\mu_s)$ is the Bessel function of the first kind of order n, and its derivative $J_n' = dJ_n(\mu_s)/d\mu_s$, $J_{n+1}(\mu_s) + J_{n-1}(\mu_s) = \frac{2n}{\mu_s}J_n(\mu_s)$ and $J_{n+1}(\mu_s) -$

$J_{n-1}(\mu_s) = -2J'_n(\mu_s)$, $J_{-n} = (-1)^n J_n$. And, $\boldsymbol{n} = \frac{\boldsymbol{k}c}{\omega}$, $\omega_{cs} = \frac{q_s B_0}{m_s}$, $\omega_{ps} = \sqrt{\frac{n_{s0} q_s^2}{\epsilon_0 m_s}}$, $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ where ω_{ps} and ω_{cs} are the plasma frequency and cyclotron frequency of each species (note that for electron, $q_e < 0$ and thus $\omega_{ce} < 0$), c is the speed of light, ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space, and $\boldsymbol{n}$ is the refractive index vector.

In this work, we write the normalized distribution function of species s to orthogonal basis functions expansion form

$$f_{s0}(v_\parallel, v_\perp) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{s0,lm} W_{sz}(v_\parallel) W_{sx}(v_\perp) \rho_{sz,l}(v_\parallel) u_{sx,m}(v_\perp), \quad -\infty < v_\parallel < \infty, \quad 0 \le v_\perp < \infty, \tag{5}$$

where $\rho_{sz,l}$ and $u_{sx,m}$ are basis functions for parallel and perpendicular directions, with W_{sz} and W_{sx} be corresponding weight functions, which satisfy

$$\int_{-\infty}^{\infty} W_{sz}(v) \rho_n(v) \rho^*_{n'}(v) dv = A_{sz} \delta_{n,n'}, \quad \int_{-\infty}^{\infty} W_{sx}(v) u_n(v) u^*_{n'}(v) dv = A_{sx} \delta_{n,n'}. \tag{6}$$

Thus, we have

$$a_{s0,lm} = \frac{1}{A_{sx} A_{sz}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{s0}(v_\parallel, v_\perp) \rho^*_{sz,l}((v_\parallel) h^*_{sx,m}(v_\perp) dv_\parallel dv_\perp. \tag{7}$$

Note also that $\int f_{s0} d\boldsymbol{v}^3 = \int_{-\infty}^{\infty} \int_0^{\infty} 2\pi v_\perp f_{s0} dv_\perp dv_\parallel = 1$.

The advantage of Eq. (5) is that the distribution function f_{s0} is decoupled into parallel and perpendicular velocity components, allowing the integral in Eq. (3) to be separated for $v_\parallel$ and $v_\perp$, i.e., $f_{s0}(v_\parallel, v_\perp) = \sum f_l(v_\parallel) \cdot f_m(v_\perp)$. The performance is largely dependent on the choice of basis functions. Early attempts utilized orthogonal basis functions such as Hermite, Legendre, or Chebyshev polynomials [15]. In practical tests, these basis functions for parallel integration often require high orders for general cases, leading to slow convergence and high computational cost [5].

Fortunately, the Fourier-related rational basis functions introduced by Weideman [11, 16] for parallel integration are found to be both fast and highly accurate for the generalized plasma dispersion function (GPDF) [1]. For this reason, we refer to them as the GPDF basis. Another expansion method involves fitting or interpolating the parallel direction using spline functions and employing Legendre basis functions for the perpendicular direction. This approach has been used for studying wave heating in the ion cyclotron range of frequencies (ICRF) [18, 19].

Our goal is to solve the dispersion relation for arbitrary distributions with high accuracy and to obtain all important solutions without requiring an initial guess. Therefore, it is essential to express the dispersion relation in a rational form with respect to the wave frequency ω [2, 12, 17]. Direct grid integration [7, 8, 5] or spline-based fitting [9, 10] cannot yield a rational dispersion relation for ω.

By decoupling the distribution function, the solution for the unknown ω—given the input parameters (k, θ, etc.)—requires computing the parallel integral iteratively for each ω. In contrast, the perpendicular integral only needs to be calculated once. As such, for the perpendicular integral, high-order expansion has little impact on computational speed, whereas for the parallel integral, it is preferable to choose an expansion method that is both accurate and efficient.

To our knowledge, two types of expansions satisfy these requirements: being general for arbitrary distributions and transformable into a matrix method to solve for all solutions. The first type is the GPDF basis [1], which is related to the Fourier basis and exhibits fast, exponentially convergent behavior. It can be transformed into a matrix method similar to the case for κ distributions [14, 13]. The second type is the Hermite basis, which is closely related to the Maxwellian distribution and provides a form suitable for J-pole expansion. This form enables a matrix-based approach [2, 12] with a smaller matrix dimension, allowing for the computation of all solutions without requiring initial guesses.

It is noted that the default GPDF basis may not be suitable for the perpendicular direction because the integral $\int_0^\infty \frac{x^p}{1+x^2} dx$ diverges for $p \ge 2$, requiring modifications. Furthermore, in double-precision floating-point arithmetic, the Hermite basis may not perform well for highly accurate calculations of the parallel direction. However, for most applications, it is advantageous over the GPDF basis due to its smaller required matrix dimension. Quadruple precision can be used for more accurate computations [19].

In this work, we primarily demonstrate the calculation of the dispersion relation and its transformation into a matrix eigenvalue problem to solve for all solutions using Hermite-Hermite (HH) basis functions for the parallel and perpendicular directions, respectively. For the GPDF-Hermite (GH) and GPDF-GPDF (GG) bases, we provide the final dispersion relations but omit the details of the matrix transformation.

2 Hermite-Hermite Basis Function Expansion

As previously mentioned, while arbitrary bases can be used for the expansion, selecting an appropriate basis can significantly enhance performance.

2.1 Choose the basis function

For simplication, we choose Hermite basis for both parallel and perpendicular direction in this section to demonstrate the performance of our approach, with

$$f_{s0}(v_\parallel, v_\perp) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{s0,lm}\rho_{sz,l}(v_\parallel)u_{sx,m}(v_\perp), \quad -\infty < v_\parallel < \infty, \quad 0 \le v_\perp < \infty, \tag{8}$$

where $\rho_{sz,l}$ and $u_{sx,m}$ are basis functions for parallel and perpendicular directions, with corresponding weight functions $W_{sz}(v_\parallel) = W_{sx}(v_\perp) = 1$ are omitted in the above form, which satisfy

$$\int_{-\infty}^{\infty} \rho_{sz,n}(v)\rho^*_{sz,n'}(v)dv = A_{sz}\delta_{n,n'}, \quad \int_{-\infty}^{\infty} u_{sx,n}(v)u^*_{sx,n'}(v)dv = A_{sx}\delta_{n,n'}. \tag{9}$$

Here, the asterisk denotes complex conjugation and $\delta_{n,n'}$ is the Kronechker delta. Thus, we have

$$a_{s0,lm} = \frac{1}{A_{sx}A_{sz}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{s0}(v_\parallel, v_\perp)\rho^*_{sz,l}((v_\parallel)u^*_{sx,m}(v_\perp)dv_\parallel dv_\perp. \tag{10}$$

We choose Hermite basis for $v_\parallel$ direction

$$\rho_{sz,n}(z) = h_n\Big(\frac{\sqrt{2}(z-d_{sz})}{L_{sz}}\Big), \quad -\infty < z < \infty, \tag{11}$$

with corresponding velocity widths be L_{sz} and drift velocity d_{sz}. Here, $\sqrt{2}$ is to convinient normalized to Maxwellian case as in BO code[12]. The basis function for perpendicular $v_\perp$ direction is choosen as

$$u_{sx,n}(x) = h_n\Big(\frac{\sqrt{2}(x-d_{sx})}{L_{sx}}\Big), \quad -\infty < x < \infty, \tag{12}$$

with corresponding velocity widths be L_{sx} and ring beam velocity d_{sx}. For most studies, we can set $d_{sx} = 0$. Note that we have extend $0 \le v_\perp < \infty$ to $-\infty < v_\perp < \infty$, with $f_{s0}(v_\parallel, v_\perp) = f_{s0}(v_\parallel, -v_\perp)$. Here h_n is Hermite function

$$h_n(y) = \sqrt{\frac{1}{2^n n!\sqrt{\pi}}}H_n(y)e^{-y^2/2}, \quad \int_{-\infty}^{\infty} h_n(y)h_{n'}(y)dy = \delta_{nn'}, \tag{13}$$

$$H_0(y) = 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2 - 2, \quad H_3(y) = 8y^3 - 12y, \quad \cdots \tag{14}$$

$$H'_n(y) = 2nH_{n-1}(y), \quad H_n(-y) = (-1)^n H_n(y), \quad H_{n+1}(y) = 2yH_n((y) - 2nH_{n-1}(y). \tag{15}$$

We have $A_{sz} = L_{sz}/\sqrt{2}$ and $A_{sx} = L_{sx}/\sqrt{2}$. For later more convenient of calculations, we can rewrite the above expansion to the follow form

$$f_{s0}(v_\parallel, v_\perp) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{s0,lm}\rho_{sz,l}(v_\parallel)u_{sx,m}(v_\perp) = c_{s0}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{s,lm}\cdot g_{sz,l}(v_\parallel)\cdot g_{sx,m}(v_\perp), \tag{16}$$

with

$$g_{sz,l}(v_\parallel) = (\frac{v_\parallel - d_{sz}}{L_{sz}})^l e^{-(\frac{v_\parallel - d_{sz}}{L_{sz}})^2}, \quad g_{sx,m}(v_\perp) = (\frac{v_\perp - d_{sx}}{L_{sx}})^m e^{-(\frac{v_\perp - d_{sx}}{L_{sx}})^2}, \tag{17}$$

and normalized coefficient

$$c_{s0} = \frac{1}{\pi^{3/2} L_{sz} L_{sx}^2 R_s}, \quad R_s = \exp\Big(-\frac{d_{sx}^2}{L_{sx}^2}\Big) + \frac{\sqrt{\pi} d_{sx}}{L_{sx}} \mathrm{erfc}\Big(-\frac{d_{sx}}{L_{sx}}\Big), \tag{18}$$

and $\mathrm{erfc}(-x) = 1-\mathrm{erf}(-x) = 1+\mathrm{erf}(x)$ is the complementary error function. After obtaining the coefficients $a_{s0,lm}$, it is readily to calculate the coefficients $a_{s,lm}$. Note that this new $g_{sx,m}$ is not orthogonal.

The benefits of the above choice of basis functions include: (1) It naturally reduces to the drift bi-Maxwellian ring beam case [12] by retaining only the lowest-order term with $a_{s,00} = 1 \neq 0$. (2) The parallel integral can be expressed with only one Maxwellian Z function, and the perpendicular integral can also be reduced to a single term at the outset. The second benefit allows the J-pole matrix method in BO [12] to be directly applied here, while maintaining the same matrix dimension, even if a larger J is required for accurate computation. This means that a small modification of the Maxwellian-based BO code is sufficient to handle generalized arbitrary distributions.

2.2 Calculate the dispersion relation

We define

$$f_{s0,lm}(v_\parallel, v_\perp) \equiv a_{s,lm} g_{sz,l}(v_\parallel) g_{sx,m}(v_\perp) = a_{s,lm} f_{s0z,l}(v_\parallel) f_{s0x,m}(v_\perp), \tag{19}$$

$$f_{s0}(v_\parallel, v_\perp) = c_{s0} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} f_{s0,lm}(v_\parallel, v_\perp), \quad f_{s0z,l}(v_\parallel) \equiv g_{sz,l}(v_\parallel), \quad f_{s0x,m}(v_\perp) \equiv g_{sx,m}(v_\perp). \tag{20}$$

Hence, we have

$$\frac{\partial f_{s0,lm}}{\partial v_\parallel} = a_{s,lm} f_{s0x,m}(v_\perp) \frac{\partial f_{s0z,l}(v_\parallel)}{\partial v_\parallel}, \quad \frac{\partial f_{s0,lm}}{\partial v_\perp} = a_{s,lm} f_{s0z,l}(v_\parallel) \frac{\partial f_{s0x,m}(v_\perp)}{\partial v_\perp}, \tag{21}$$

$$\frac{\partial f_{s0z,l}(v_\parallel)}{\partial v_\parallel} = -\frac{1}{L_{sz}} \Big[2 f_{s0z,l+1} - l f_{s0z,l-1} \Big], \tag{22}$$

$$\frac{\partial f_{s0x,m}(v_\perp)}{\partial v_\perp} = -\frac{1}{L_{sx}} \Big[2 f_{s0x,m+1} - m f_{s0x,m-1} \Big]. \tag{23}$$

Here, we set $f_{s0z,l} = f_{s0x,m} = 0$ for all $l, m < 0$. The above derivatives equations are valid for all $l, m = 0, 1, 2, \cdots$. We have also numerically checked the valid of the above two derivatives.

For parallel integral $[\mathrm{Im}(\zeta_{sn}) > 0]$, define

$$Z_{l,p}(\zeta_{sn}) \equiv -\frac{k_\parallel}{L_{sz}^p \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{v_\parallel^p}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \Big(\frac{v_\parallel - d_{sz}}{L_{sz}}\Big)^l e^{-(\frac{v_\parallel - d_{sz}}{L_{sz}})^2} dv_\parallel = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{g_{l,p}(z)}{z - \zeta_{sn}} dz, \quad g_{l,p}(z) \equiv (z+d)^p z^l e^{-z^2}, \tag{24}$$

we have[1]

$$Z_{0,0}(\zeta) = Z(\zeta), \quad Z_{1,0}(\zeta) = [1 + \zeta Z(\zeta)], \quad Z_{2,0}(\zeta) = \zeta[1 + \zeta Z(\zeta)], \tag{25}$$

$$Z_{3,0}(\zeta) = \frac{1}{2} + \zeta^2 [1 + \zeta Z(\zeta)], \quad Z_{l+1,0}(\zeta) = \zeta Z_{l,0} + I_l, \tag{26}$$

$$Z_{l,1} = d Z_{l,0} + Z_{l+1,0}, \quad Z_{l,2} = d^2 Z_{l,0} + 2d Z_{l+1,0} + Z_{l+2,0}, \tag{27}$$

with $\zeta_{sn} = \frac{\omega - k_\parallel d_{sz} - n\omega_{cs}}{k_\parallel L_{sz}}$, $d = \frac{d_{sz}}{L_{sz}}$, $p = 0, 1, 2$. And we redefine $Z_{l,0} \equiv Z_l$ for simplify the notation. Here, $I_{2l+1} \equiv \pi^{-1/2} \int_{-\infty}^{\infty} x^{2l+1} e^{-x^2} dx = 0$ and $I_{2l} \equiv \pi^{-1/2} \int_{-\infty}^{\infty} x^{2l} e^{-x^2} dx = \Gamma(l + \frac{1}{2})/\sqrt{\pi}$, where Γ is Euler Gamma function, and $I_0 = 1$, $I_{2l} = \frac{2l-1}{2l} I_{2l-2}$. Here, $Z_0^{(l)}(\zeta)$ means l-th derivatites of $Z_0(\zeta)$. The above functions can be readily[1] calculated to high accuracy used the standard Maxwellian plasma dispersion functions $Z(\zeta)$. Also, it can be rewritten to J-pole expansion form[17, 12].

For perpendicular integral

$$\Gamma_{an,m,p}(a_s, d_s) \equiv \frac{1}{L_{sx}^{p+1}} \int_0^{\infty} v_\perp^p J_n^2\Big(\frac{k_\perp v_\perp}{\omega_{cs}}\Big)\Big(\frac{v_\perp - d_{sx}}{L_{sx}}\Big)^m e^{-(\frac{v_\perp - d_{sx}}{L_{sx}})^2} dv_\perp = \int_0^{\infty} x^p J_n^2(a_s x)(x - d_s)^m e^{-(x-d_s)^2} dx,$$

$$\Gamma_{bn,m,p}(a_s, d_s) \equiv \frac{1}{L_{sx}^{p+1}} \int_0^{\infty} v_\perp^p J_n\Big(\frac{k_\perp v_\perp}{\omega_{cs}}\Big) J_n'\Big(\frac{k_\perp v_\perp}{\omega_{cs}}\Big)\Big(\frac{v_\perp - d_{sx}}{L_{sx}}\Big)^m e^{-(\frac{v_\perp - d_{sx}}{L_{sx}})^2} dv_\perp = \int_0^{\infty} x^p J_n(a_s x) J_n'(a_s x)(x - d_s)^m e^{-(x-d_s)^2} dx,$$

$$\Gamma_{cn,m,p}(a_s, d_s) \equiv \frac{1}{L_{sx}^{p+1}} \int_0^{\infty} v_\perp^p {J_n'}^2\Big(\frac{k_\perp v_\perp}{\omega_{cs}}\Big)\Big(\frac{v_\perp - d_{sx}}{L_{sx}}\Big)^m e^{-(\frac{v_\perp - d_{sx}}{L_{sx}})^2} dv_\perp = \int_0^{\infty} x^p {J_n'}^2(a_s x)(x - d_s)^m e^{-(x-d_s)^2} dx.$$

with $a_s = k_\perp \rho_{cs}$, $d_s = d_{sx}/L_{sx}$, $\rho_{cs} = L_{sx}/\omega_{cs}$, $p = 0,1,2,3$. These integrals can be calculated numerically[12]. For $d_{sx} = 0$, they are relavant to the modified Bessel functions of the first kind as in the Maxwellian case.

Note that we have the normalized condition: $1 = \int_{-\infty}^{\infty}\int_0^{\infty} 2\pi v_\perp f_{s0} dv_\perp dv_\parallel = 2\pi c_{s0}\sum_{l,m} a_{s,lm}\int_{-\infty}^{\infty}\int_0^{\infty} v_\perp g_{sz,l} g_{sx,m} dv_\perp dv_\parallel = 2\pi^{3/2} L_{sz} L_{sx}^2 c_{s0}\sum_{l,m} a_{s,lm} I_l \int_0^\infty x(x-d_s)^m e^{-(x-d_s)^2} dx$.

Hence, we have

$$A_s = -c_{s0}\sum_l\sum_m a_{s,lm}\Big\{\frac{\omega - k_\parallel v_\parallel}{\omega} f_{s0z,l}\frac{1}{L_{sx}}[2f_{s0x,m+1} - m f_{s0x,m-1}] + \frac{k_\parallel v_\perp}{\omega} f_{s0x,m}\frac{1}{L_{sz}}[2f_{s0z,l+1} - l f_{s0z,l-1}]\Big\},$$
$$B_s = -c_{s0}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs} v_\parallel}{\omega v_\perp} f_{s0z,l}\frac{1}{L_{sx}}[2f_{s0x,m+1} - m f_{s0x,m-1}] + \frac{\omega - n\omega_{cs}}{\omega} f_{s0x,m}\frac{1}{L_{sz}}[2f_{s0z,l+1} - l f_{s0z,l-1}]\Big\}.$$

Define

$$\boldsymbol{X}_{sn} = \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}}\times\begin{bmatrix} A_s\frac{n^2 v_\perp}{\mu_s^2}J_n^2 & iA_s\frac{n v_\perp}{\mu_s}J_n J_n' & B_s\frac{n v_\perp}{\mu_s}J_n^2 \\ -iA_s\frac{n v_\perp}{\mu_s}J_n J_n' & A_s v_\perp {J_n'}^2 & -iB_s v_\perp J_n J_n' \\ A_s\frac{n v_\parallel}{\mu_s}J_n^2 & iA_s v_\parallel J_n J_n' & B_s v_\parallel J_n^2\end{bmatrix}. \tag{31}$$

We have

$$\begin{aligned}
X_{sn11} &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}}\times A_s\frac{n^2 v_\perp}{\mu_s^2}J_n^2 = \frac{n^2\omega_{cs}^2}{k_\perp^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}}\times A_s J_n^2 \\
&= -c_{s0}\frac{n^2\omega_{cs}^2}{k_\perp^2}\sum_l\sum_m a_{s,lm}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}}\times\Big\{\frac{\omega - k_\parallel v_\parallel}{\omega} f_{s0z,l}\frac{1}{L_{sx}}[2f_{s0x,m+1} - m f_{s0x,m-1}] \\
&\quad + \frac{k_\parallel v_\perp}{\omega} f_{s0x,m}\frac{1}{L_{sz}}[2f_{s0z,l+1} - l f_{s0z,l-1}]\Big\}J_n^2 \\
&= 2\frac{1}{L_{sz}L_{sx}^2 R_s}\frac{n^2\omega_{cs}^2}{k_\perp^2}\sum_l\sum_m a_{s,lm}\Big\{(\frac{\omega}{k_\parallel}Z_{l,0} - L_{sz}Z_{l,1})[2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}] + \Gamma_{an,m,1}\frac{L_{sx}^2}{L_{sz}}[2Z_{l+1,0} - lZ_{l-1,0}]\Big\} \\
&= \frac{2}{R_s}\frac{n^2\omega_{cs}^2}{k_\perp^2 L_{sx}^2}\sum_l\sum_m a_{s,lm}\Big\{(\zeta_{sn}Z_{l,0} + \frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0} - Z_{l+1,0})[2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}] + \Gamma_{an,m,1}\frac{L_{sx}^2}{L_{sz}^2}[2Z_{l+1,0} - lZ_{l-1,0}]\Big\} \\
&= \frac{2}{R_s}\frac{n^2\omega_{cs}^2}{k_\perp^2 L_{sx}^2}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_l - I_l)(2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}) + \Gamma_{an,m,1}\frac{L_{sx}^2}{L_{sz}^2}(2Z_{l+1} - lZ_{l-1})\Big\},
\end{aligned}$$

$$\begin{aligned}
X_{sn12} &= -X_{sn21} = \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}}\times iA_s\frac{n v_\perp}{\mu_s}J_n J_n' = i\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}}\times A_s J_n J_n' \\
&= \frac{2i}{L_{sz}L_{sx}^2 R_s}\frac{n\omega_{cs}}{k_\perp}L_{sx}\sum_l\sum_m a_{s,lm}\Big\{(\frac{\omega}{k_\parallel}Z_{l,0} - L_{sz}Z_{l,1})[2\Gamma_{bn,m+1,1} - m\Gamma_{bn,m-1,1}] + \Gamma_{bn,m,2}\frac{L_{sx}^2}{L_{sz}}[2Z_{l+1,0} - lZ_{l-1,0}]\Big\} \\
&= \frac{2i}{R_s}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_l - I_l)[2\Gamma_{bn,m+1,1} - m\Gamma_{bn,m-1,1}] + \Gamma_{bn,m,2}\frac{L_{sx}^2}{L_{sz}^2}[2Z_{l+1} - lZ_{l-1}]\Big\} \\
&\to \frac{2i}{R_s}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_l(2\Gamma_{bn,m+1,1} - m\Gamma_{bn,m-1,1}) + \Gamma_{bn,m,2}\frac{L_{sx}^2}{L_{sz}^2}(2Z_{l+1} - lZ_{l-1})\Big\},
\end{aligned}$$

$$\begin{aligned}
X_{sn22} &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}}\times A_s v_\perp {J_n'}^2 = \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}}\times A_s v_\perp {J_n'}^2 \\
&= 2\frac{1}{L_{sz}R_s}\sum_l\sum_m a_{s,lm}\Big\{(\frac{\omega}{k_\parallel}Z_{l,0} - L_{sz}Z_{l,1})[2\Gamma_{cn,m+1,2} - m\Gamma_{cn,m-1,2}] + \Gamma_{cn,m,3}\frac{L_{sx}^2}{L_{sz}}[2Z_{l+1,0} - lZ_{l-1,0}]\Big\} \\
&= \frac{2}{R_s}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_l - I_l)(2\Gamma_{cn,m+1,2} - m\Gamma_{cn,m-1,2}) + \Gamma_{cn,m,3}\frac{L_{sx}^2}{L_{sz}^2}(2Z_{l+1} - lZ_{l-1})\Big\},
\end{aligned}$$

$$
\begin{aligned}
X_{sn31} &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_s\frac{nv_\parallel}{\mu_s}J_n^2=\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\parallel dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_sJ_n^2\\
&= 2\frac{1}{L_{sx}^2R_s}\frac{n\omega_{cs}}{k_\perp}\sum_l\sum_m a_{s,lm}\Big\{(\frac{\omega}{k_\parallel}Z_{l,1}-L_{sz}Z_{l,2})[2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0}]+\Gamma_{an,m,1}\frac{L_{sx}^2}{L_{sz}}[2Z_{l+1,1}-lZ_{l-1,1}]\Big\}\\
&= 2\frac{1}{L_{sx}^2R_s}\frac{n\omega_{cs}}{k_\perp}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel}(\frac{d_{sz}}{L_{sz}}Z_l+Z_{l+1})-d_{sz}I_l-L_{sz}I_{l+1})[2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0}]\\
&\quad+\Gamma_{an,m,1}\frac{L_{sx}^2}{L_{sz}}[2(\frac{d_{sz}}{L_{sz}}Z_{l+1}+Z_{l+2})-l(\frac{d_{sz}}{L_{sz}}Z_{l-1}+Z_l)]\Big\}\\
&\to 2\frac{L_{sz}}{L_{sx}R_s}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sz}}(\frac{d_{sz}}{L_{sz}}Z_l+Z_{l+1})(2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0})\\
&\quad+\Gamma_{an,m,1}\frac{L_{sx}^2}{L_{sz}^2}[2(\frac{d_{sz}}{L_{sz}}Z_{l+1}+Z_{l+2})-l(\frac{d_{sz}}{L_{sz}}Z_{l-1}+Z_l)]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
X_{sn32} &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times iA_sv_\parallel J_nJ_n'=i\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_sJ_nJ_n'\\
&= \frac{2i}{L_{sz}L_{sx}^2R_s}L_{sz}L_{sx}\sum_l\sum_m a_{s,lm}\Big\{(\frac{\omega}{k_\parallel}Z_{l,1}-L_{sz}Z_{l,2})[2\Gamma_{bn,m+1,1}-m\Gamma_{bn,m-1,1}]+\Gamma_{bn,m,2}\frac{L_{sx}^2}{L_{sz}}[2Z_{l+1,1}-lZ_{l-1,1}]\Big\}\\
&= \frac{2iL_{sz}}{L_{sx}R_s}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}(\frac{d_{sz}}{L_{sz}}Z_l+Z_{l+1})-d_{sz}I_l-L_{sz}I_{l+1})[2\Gamma_{bn,m+1,1}-m\Gamma_{bn,m-1,1}]\\
&\quad+\Gamma_{bn,m,2}\frac{L_{sx}^2}{L_{sz}^2}[2(\frac{d_{sz}}{L_{sz}}Z_{l+1}+Z_{l+2})-l(\frac{d_{sz}}{L_{sz}}Z_{l-1}+Z_l)]\Big\}\\
&\to \frac{2iL_{sz}}{L_{sx}R_s}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sz}}(\frac{d_{sz}}{L_{sz}}Z_l+Z_{l+1})(2\Gamma_{bn,m+1,1}-m\Gamma_{bn,m-1,1})\\
&\quad+\Gamma_{bn,m,2}\frac{L_{sx}^2}{L_{sz}^2}[2(\frac{d_{sz}}{L_{sz}}Z_{l+1}+Z_{l+2})-l(\frac{d_{sz}}{L_{sz}}Z_{l-1}+Z_l)]\Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
X_{sn13} &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times B_s\frac{nv_\perp}{\mu_s}J_n^2=\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times B_sJ_n^2\\
&= -2\pi\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times\sum_l\sum_m a_{s,lm}\Big\{(n\omega_{cs}v_\parallel)f_{s0z,l}\frac{1}{L_{sx}}[2f_{s0x,m+1}-mf_{s0x,m-1}]\\
&\quad+(\omega-n\omega_{cs})v_\perp f_{s0x,m}\frac{1}{L_{sz}}[2f_{s0z,l+1}-lf_{s0z,l-1}]\Big\}J_n^2\\
&= \frac{2}{L_{sz}L_{sx}^2R_s}\frac{n\omega_{cs}}{k_\perp k_\parallel}L_{sz}\sum_l\sum_m a_{s,lm}\Big\{n\omega_{cs}Z_{l,1}[2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0}]+(\omega-n\omega_{cs})\frac{L_{sx}^2}{L_{sz}^2}\Gamma_{an,m,1}[2Z_{l+1,0}-lZ_{l-1,0}]\Big\}\\
&= \frac{2}{L_{sx}^2R_s}\frac{n\omega_{cs}}{k_\perp k_\parallel}\sum_l\sum_m a_{s,lm}\Big\{n\omega_{cs}(\frac{d_{sz}}{L_{sz}}Z_l+Z_{l+1})[2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0}]\\
&\quad+\frac{L_{sx}^2}{L_{sz}^2}\Gamma_{an,m,1}[k_zL_{sz}(2(Z_{l+2}-I_{l+1})-l(Z_l-I_{l-1}))+k_zd_{sz}(2Z_{l+1}-lZ_{l-1})]\Big\}\\
&\to \frac{2}{R_s}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_zL_{sx}}(\frac{d_{sz}}{L_{sz}}Z_l+Z_{l+1})(2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0})\\
&\quad+\frac{L_{sx}}{L_{sz}}\Gamma_{an,m,1}[(2Z_{l+2}-lZ_l)+\frac{d_{sz}}{L_{sz}}(2Z_{l+1}-lZ_{l-1})]\Big\}\to X_{sn31},
\end{aligned}
$$

$$
\begin{aligned}
X_{sn23} &= -\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times(-iB_s v_\perp J_n J_n') = -i\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{2\pi\omega v_\perp^2 dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times B_s J_n J_n' \\
&= \frac{-2i}{L_{sz}L_{sx}^2R_s}\frac{L_{sx}}{k_\parallel}L_{sz}\sum_l\sum_m a_{s,lm}\Big\{n\omega_{cs}Z_{l,1}[2\Gamma_{bn,m+1,1}-m\Gamma_{bn,m-1,1}]+(\omega-n\omega_{cs})\frac{L_{sx}^2}{L_{sz}^2}\Gamma_{bn,m,2}[2Z_{l+1,0}-lZ_{l-1,0}]\Big\} \\
&= \frac{-2i}{L_{sz}L_{sx}^2R_s}\frac{L_{sx}}{k_\parallel}L_{sz}\sum_l\sum_m a_{s,lm}\Big\{n\omega_{cs}(\frac{d_{sz}}{L_{sz}}Z_l+Z_{l+1})[2\Gamma_{bn,m+1,1}-m\Gamma_{bn,m-1,1}] \\
&\quad+\frac{L_{sx}^2}{L_{sz}^2}\Gamma_{bn,m,2}[k_zL_{sz}(2(Z_{l+2}-I_{l+1})-l(Z_l-I_{l-1}))+k_zd_{sz}(2Z_{l+1}-lZ_{l-1}-)]\Big\} \\
&\to \frac{-2i}{R_s}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_zL_{sx}}(\frac{d_{sz}}{L_{sz}}Z_l+Z_{l+1})(2\Gamma_{bn,m+1,1}-m\Gamma_{bn,m-1,1}) \\
&\quad+\frac{L_{sx}}{L_{sz}}\Gamma_{bn,m,2}[(2Z_{l+2}-lZ_l)+\frac{d_{sz}}{L_{sz}}(2Z_{l+1}-lZ_{l-1})]\Big\}\to -X_{sn32},
\end{aligned}
$$

$$
\begin{aligned}
X_{sn33} &= \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times B_s v_\parallel J_n^2 = \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times B_s J_n^2 \\
&= \frac{2}{L_{sz}L_{sx}^2R_s}\frac{L_{sz}^2}{k_\parallel}\sum_l\sum_m a_{s,lm}\Big\{n\omega_{cs}Z_{l,2}[2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0}]+(\omega-n\omega_{cs})\frac{L_{sx}^2}{L_{sz}^2}\Gamma_{an,m,1}[2Z_{l+1,1}-lZ_{l-1,1}]\Big\} \\
&= \frac{2}{R_s}\frac{L_{sz}^2}{L_{sx}^2}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_zL_{sz}}(\frac{d_{sz}^2}{L_{sz}^2}Z_l+2\frac{d_{sz}}{L_{sz}}Z_{l+1}+Z_{l+2})(2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0}) \\
&\quad+\frac{L_{sx}^2}{L_{sz}^2}\Gamma_{an,m,1}[(2\frac{d_{sz}}{L_{sz}}(Z_{l+2}-I_{l+1})+2(Z_{l+3}-I_{l+2})-l\frac{d_{sz}}{L_{sz}}(Z_l-I_{l-1})-l(Z_{l+1}-I_l)) \\
&\quad+\frac{d_{sz}}{L_{sz}}(2\frac{d_{sz}}{L_{sz}}Z_{l+1}+2Z_{l+2}-l\frac{d_{sz}}{L_{sz}}Z_{l-1}-lZ_l)]\Big\},
\end{aligned}
$$

where

$$
\boldsymbol{Q}=\sum_s\boldsymbol{Q}_s,\quad \boldsymbol{Q}_s=\frac{\omega_{ps}^2}{\omega^2}\boldsymbol{P}_s,\quad \boldsymbol{P}_s=\sum_n\boldsymbol{P}_{sn},\quad \boldsymbol{P}_{sn}=c_{s0}\boldsymbol{X}_{sn}. \tag{32}
$$

Here, '$\to$' means the constant terms I_l has been summation out for n. We have used $\sum_{n=-\infty}^{\infty}J_nJ_n'=0$, $\sum_{n=-\infty}^{\infty}nJ_n^2=0$, $\sum_{n=-\infty}^{\infty}nJ_nJ_n'=0$, $\sum_{n=-\infty}^{\infty}J_n^2=1$, $\sum_{n=-\infty}^{\infty}(J_n')^2=\frac{1}{2}$ and $\sum_{n=-\infty}^{\infty}\frac{n^2J_n^2}{x^2}=\frac{1}{2}$.

Keep only the $l=0,m=0$ terms, the above dispersion relation can reduce to the drift bi-Maxwellian ring beam case[12]. By further set $d_{sx}=0$, it can reduce to the drift bi-Maxwellian case[2].

After the above expansion, the dispersion relation Eq. (1) can be readily solved using conventional iterative root-finding methods. Since the perpendicular integral $\Gamma(a_s)$ needs to be calculated only once, and the parallel integrals $Z(\zeta_{sn})$ are related to the Maxwellian Z function and can be computed efficiently [1, 17], this approach can achieve high-accuracy solutions with reduced computational cost. In principle, this new approach can surpass previous methods [7, 8, 9, 5, 10] in both accuracy and speed. Beyond these advantages, we aim to go further—eliminating the requirement for an initial guess, as implemented in the BO code [2, 12, 14, 13].

2.3 Transfrom to Matrix Eigenvalue Problem

We only consider $k_z\geq 0$ here, for Maxwellian Z function J-pole expansion, we have[12]

$$
Z_l(\zeta)\simeq\sum_{j=1}^{J}\frac{b_jc_j^l}{\zeta-c_j}, \tag{33}
$$

Here, we have used[17] $\sum_j b_j=-1$, $\sum_j b_jc_j=0$, $\sum_j b_jc_j^2=-1/2$, $\sum_j b_jc_j^3=0$, $\cdots$. This also means that if we need high order Hermite fitting of distribution function, the corresponding coefficients should also keep to similar order, usually we need set $J \geq l_{max}+4$. To double precision, we have calculated to $J=24$. Hence, the present solver can support to $l_{max} \simeq 20$. Though higer l_{max} can also calculate, the accuracy would reduce.

The electromagnetic case is much complicated. However, the linear transformation for $\boldsymbol{Q}(\omega, \boldsymbol{k})$ is still similar to the original BO-K/PDRK derivation.

To seek an equivalent linear system, the Maxwell's equations

$$\partial_t \boldsymbol{E} = c^2 \nabla \times \boldsymbol{B} - \boldsymbol{J}/\epsilon_0, \tag{34a}$$

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{34b}$$

do not need to be changed. We only need to seek a new linear system for $\boldsymbol{J} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$.

Considering the defination $\boldsymbol{\sigma}_s = -i\epsilon_0 \omega \boldsymbol{Q}_s = -i\epsilon_0 \frac{\omega_{ps}^2}{\omega} \boldsymbol{P}_s$, after J-pole expansion, we have

- $P_{s11} = \sum_n \frac{2}{R_s} \frac{n^2\omega_{cs}^2}{k_\perp^2 L_{sx}^2} \sum_{l,m} a_{s,lm} \Big\{ (\frac{n\omega_{cs}}{k_\parallel L_{sz}} Z_l - I_l)(2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}) + \Gamma_{an,m,1} \frac{L_{sx}^2}{L_{sz}^2}(2Z_{l+1} - lZ_{l-1}) \Big\}$
$\simeq \sum_{n,j} \frac{2}{R_s} \frac{n^2\omega_{cs}^2}{k_\perp^2 L_{sx}^2} \frac{k_z L_{sz} b_j}{\omega - c_{snj}} \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_\parallel L_{sz}} c_j^l (2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}) + \Gamma_{an,m,1} \frac{L_{sx}^2}{L_{sz}^2} [2c_j^{l+1} - lc_j^{l-1}] \Big\}$
$- \frac{2}{R_s} \frac{\omega_{cs}^2}{k_\perp^2 L_{sx}^2} \sum_{l,m} a_{s,lm} \Big\{ I_l \sum_n n^2 (2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}) \Big\} = \sum_{n,j} \frac{p_{11snj}}{\omega - c_{snj}} - 1$
- $P_{s12} \simeq \sum_{n,j} \frac{2i}{R_s} \frac{n\omega_{cs}}{k_\perp L_{sx}} \frac{k_z L_{sz} b_j}{\omega - c_{snj}} \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_\parallel L_{sz}} c_j^l (2\Gamma_{bn,m+1,1} - m\Gamma_{bn,m-1,1}) + \Gamma_{bn,m,2} \frac{L_{sx}^2}{L_{sz}^2} (2c_j^{l+1} - lc_j^{l-1}) \Big\} = \sum_{n,j} \frac{p_{12snj}}{\omega - c_{snj}}$.
- $P_{s21} \simeq \sum_{n,j} \frac{p_{21snj}}{\omega - c_{snj}} = \sum_{n,j} \frac{-p_{12snj}}{\omega - c_{snj}}$.
- $P_{s22} \simeq -\sum_n \frac{2}{R_s} \sum_{l,m} a_{s,lm} \Big\{ I_l (2\Gamma_{cn,m+1,2} - m\Gamma_{cn,m-1,2}) \Big\} + \sum_{n,j} \frac{2}{R_s} \frac{k_z L_{sz} b_j}{\omega - c_{snj}} \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_\parallel L_{sz}} c_j^l (2\Gamma_{cn,m+1,2} - m\Gamma_{cn,m-1,2}) + \Gamma_{cn,m,3} \frac{L_{sx}^2}{L_{sz}^2} (2c_j^{l+1} - lc_j^{l-1}) \Big\} = -1 + \sum_{n,j} \frac{p_{22snj}}{\omega - c_{snj}}$.
- $P_{s13} \simeq \sum_{n,j} \frac{2}{R_s} \frac{n\omega_{cs}}{k_\perp L_{sx}} \frac{k_z L_{sz} b_j}{\omega - c_{snj}} \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_z L_{sx}} (\frac{d_{sz}}{L_{sz}} c_j^l + c_j^{l+1})(2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}) + \frac{L_{sx}}{L_{sz}} \Gamma_{an,m,1} [(2c_j^{l+2} - lc_j^l) + \frac{d_{sz}}{L_{sz}} (2c_j^{l+1} - lc_j^{l-1})] \Big\} = \sum_{n,j} \frac{p_{13snj}}{\omega - c_{snj}}$.
- $P_{s31} \simeq \sum_{n,j} \frac{2}{R_s} \frac{n\omega_{cs}}{k_\perp L_{sx}} \frac{k_z L_{sz} b_j}{\omega - c_{snj}} \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_\parallel L_{sx}} (\frac{d_{sz}}{L_{sz}} c_j^l + c_j^{l+1})(2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}) + \Gamma_{an,m,1} \frac{L_{sx}}{L_{sz}} [2(\frac{d_{sz}}{L_{sz}} c_j^{l+1} + c_j^{l+2}) - l(\frac{d_{sz}}{L_{sz}} c_j^{l-1} + c_j^l)] \Big\} = \sum_{n,j} \frac{p_{31snj}}{\omega - c_{snj}}$.
- $P_{s23} \simeq -\sum_{n,j} \frac{2i}{R_s} \frac{k_z L_{sz} b_j}{\omega - c_{snj}} \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_z L_{sx}} (\frac{d_{sz}}{L_{sz}} c_j^l + c_j^{l+1})(2\Gamma_{bn,m+1,1} - m\Gamma_{bn,m-1,1}) + \frac{L_{sx}}{L_{sz}} \Gamma_{bn,m,2} [(2c_j^{l+2} - lc_j^l) + \frac{d_{sz}}{L_{sz}} (2c_j^{l+1} - lc_j^{l-1})] \Big\} = \sum_{n,j} \frac{p_{23snj}}{\omega - c_{snj}}$.
- $P_{s32} \simeq \sum_{n,j} \frac{2i}{R_s} \frac{k_z L_{sz} b_j}{\omega - c_{snj}} \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_\parallel L_{sx}} (\frac{d_{sz}}{L_{sz}} c_j^l + c_j^{l+1})(2\Gamma_{bn,m+1,1} - m\Gamma_{bn,m-1,1}) + \Gamma_{bn,m,2} \frac{L_{sx}}{L_{sz}} [2(\frac{d_{sz}}{L_{sz}} c_j^{l+1} + c_j^{l+2}) - l(\frac{d_{sz}}{L_{sz}} c_j^{l-1} + c_j^l)] \Big\} = \sum_{n,j} \frac{p_{32snj}}{\omega - c_{snj}}$.
- $P_{s33} \simeq \sum_{n,j} \frac{2}{R_s} \frac{L_{sz}}{L_{sx}} \frac{k_z L_{sz} b_j}{\omega - c_{snj}} \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_z L_{sx}} (\frac{d_{sz}^2}{L_{sz}^2} c_j^l + 2\frac{d_{sz}}{L_{sz}} c_j^{l+1} + c_j^{l+2})(2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}) + \frac{L_{sx}}{L_{sz}} \Gamma_{an,m,1} [(2\frac{d_{sz}}{L_{sz}} c_j^{l+2} + 2c_j^{l+3} - l\frac{d_{sz}}{L_{sz}} c_j^l - lc_j^{l+1}) + \frac{d_{sz}}{L_{sz}} (2\frac{d_{sz}}{L_{sz}} c_j^{l+1} + 2c_j^{l+2} - l\frac{d_{sz}}{L_{sz}} c_j^{l-1} - lc_j^l)] \Big\} - \sum_n \frac{2}{R_s} \sum_{l,m} a_{s,lm} \Big\{ \Gamma_{an,m,1} (2\frac{d_{sz}}{L_{sz}} I_{l+1} + 2I_{l+2} - l\frac{d_{sz}}{L_{sz}} I_{l-1} - lI_l) \Big\} = -1 + \sum_{n,j} \frac{p_{33snj}}{\omega - c_{snj}}$.

Here, $\sum_{n,j} = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{J}$ and $\sum_{l,m} = \sum_l \sum_m$. In the above, say, $p_{11snj} = \frac{2}{R_s} \frac{n^2\omega_{cs}^2}{k_\perp^2 L_{sx}^2} k_z L_{sz} b_j \sum_{l,m} a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_\parallel L_{sz}} c_j^l (2\Gamma_{an,m+1,0} - m\Gamma_{an,m-1,0}) + \Gamma_{an,m,1} \frac{L_{sx}^2}{L_{sz}^2} [2c_j^{l+1} - lc_j^{l-1}] \Big\}$, and other terms are similar.

It is thus easy to find that after J-pole expansion, the relations between $\boldsymbol{J}$ and $\boldsymbol{E}$ has the following form

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = -i\epsilon_0 \begin{pmatrix} \frac{b_{11}}{\omega} + \sum_{snj} \frac{b_{snj11}}{\omega - c_{snj}} & \frac{b_{12}}{\omega} + \sum_{snj} \frac{b_{snj12}}{\omega - c_{snj}} & \frac{b_{13}}{\omega} + \sum_{snj} \frac{b_{snj13}}{\omega - c_{snj}} \\ \frac{b_{21}}{\omega} + \sum_{snj} \frac{b_{snj21}}{\omega - c_{snj}} & \frac{b_{22}}{\omega} + \sum_{snj} \frac{b_{snj22}}{\omega - c_{snj}} & \frac{b_{23}}{\omega} + \sum_{snj} \frac{b_{snj23}}{\omega - c_{snj}} \\ \frac{b_{31}}{\omega} + \sum_{snj} \frac{b_{snj31}}{\omega - c_{snj}} & \frac{b_{32}}{\omega} + \sum_{snj} \frac{b_{snj32}}{\omega - c_{snj}} & \frac{b_{33}}{\omega} + \sum_{snj} \frac{b_{snj33}}{\omega - c_{snj}} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{35}$$

with the coefficients

$$\left\{\begin{array}{c} b_{snj11}=\omega_{ps}^2p_{11snj}/c_{snj},\ b_{11}=-\sum_s\omega_{ps}^2[\sum_n\frac{2}{R_s}\frac{\omega_{cs}^2}{k_\perp^2L_{sx}^2}\sum_{l,m}a_{s,lm}\Big\{I_l\sum_n n^2(2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0})\Big\}+\sum_{nj}p_{11snj}/c_{snj}],\\ b_{snj12}=\omega_{ps}^2p_{12snj}/c_{snj},\ b_{12}=-\sum_s\omega_{ps}^2[\sum_{nj}p_{12snj}/c_{snj}],\\ b_{snj21}=\omega_{ps}^2p_{21snj}/c_{snj},\ b_{21}=-\sum_s\omega_{ps}^2[\sum_{nj}p_{21snj}/c_{snj}],\\ b_{snj22}=\omega_{ps}^2p_{22snj}/c_{snj},\ b_{22}=-\sum_s\omega_{ps}^2[\sum_n\frac{2}{R_s}\sum_{l,m}a_{s,lm}\Big\{I_l(2\Gamma_{cn,m+1,2}-m\Gamma_{cn,m-1,2})\Big\}+\sum_{nj}p_{22snj}/c_{snj}],\\ b_{snj13}=\omega_{ps}^2p_{13snj}/c_{snj},\ b_{13}=-\sum_s\omega_{ps}^2[\sum_{nj}p_{13snj}/c_{snj}],\\ b_{snj31}=\omega_{ps}^2p_{31snj}/c_{snj},\ b_{31}=-\sum_s\omega_{ps}^2[\sum_{nj}p_{31snj}/c_{snj}],\\ b_{snj23}=\omega_{ps}^2p_{23snj}/c_{snj},\ b_{23}=-\sum_s\omega_{ps}^2[\sum_{nj}p_{23snj}/c_{snj}],\\ b_{snj32}=\omega_{ps}^2p_{32snj}/c_{snj},\ b_{32}=-\sum_s\omega_{ps}^2[\sum_{nj}p_{32snj}/c_{snj}],\\ b_{snj33}=\omega_{ps}^2p_{33snj}/c_{snj},\ b_{33}=-\sum_s\omega_{ps}^2[\sum_n\frac{2}{R_s}\sum_{l,m}a_{s,lm}\Big\{\Gamma_{an,m,1}(2\frac{d_{sz}}{L_{sz}}I_{l+1}+2I_{l+2}-l\frac{d_{sz}}{L_{sz}}I_{l-1}-lI_l)\Big\}+\sum_{nj}p_{33snj}/c_{snj}],\\ c_{snj}=k_zv_{dsz}+n\omega_{cs}+k_zv_{zts}c_j. \end{array}\right. \tag{36}$$

In numerical test, considering the cutoff of summation n, i.e., $\sum_n=\sum_{n=-N}^{N}$ with $N\neq\infty$, we find the above original form of b_{11}, b_{22} and b_{33}, converges faster (i.e., can use smaller N to obtain convergent results) than the below form

$$\left\{\begin{array}{l} b_{11}=-\sum_s\omega_{ps}^2[1+\sum_{nj}p_{11snj}/c_{snj}],\\ b_{22}=-\sum_s\omega_{ps}^2[1+\sum_{nj}p_{22snj}/c_{snj}],\\ b_{33}=-\sum_s\omega_{ps}^2[1+\sum_{nj}p_{33snj}/c_{snj}]. \end{array}\right. \tag{37}$$

It is readily to see that all the singularities from $\frac{1}{k_z}$ in $\boldsymbol{P}_s$ are removable. The $\frac{n\omega_{cs}}{k_x}$ singularities at $k_x=0$ in P_{s11}, P_{s12}, P_{s21}, P_{s13}, P_{s31} are also removable. Thus, the overall equations have no singularity and will not meet numerical difficulty. In the solver, to short the code, if $k_x\rho_s<k_\delta$ we set $k_x\rho_s=k_\delta$ for magnetized species in EM version. For example, we can set $k_\delta=10^{-30}$.

Combining Eqs. (34) and (35), the equivalent linear system for electromagnetic dispersion relation can be obtained as

$$\left\{\begin{array}{ccc} \omega v_{snjx} &=& c_{snj}v_{snjx}+b_{snj11}E_x+b_{snj12}E_y+b_{snj13}E_z,\\ \omega j_x &=& b_{11}E_x+b_{12}E_y+b_{13}E_z,\\ iJ_x\epsilon_0 &=& j_x+\sum_{snj}v_{snjx},\\ \omega v_{snjy} &=& c_{snj}v_{snjy}+b_{snj21}E_x+b_{snj22}E_y+b_{snj23}E_z,\\ \omega j_y &=& b_{21}E_x+b_{22}E_y+b_{23}E_z,\\ iJ_y/\epsilon_0 &=& j_y+\sum_{snj}v_{snjy},\\ \omega v_{snjz} &=& c_{snj}v_{snjz}+b_{snj31}E_x+b_{snj32}E_y+b_{snj33}E_z,\\ \omega j_z &=& b_{31}E_x+b_{32}E_y+b_{33}E_z,\\ iJ_z/\epsilon_0 &=& j_z+\sum_{snj}v_{snjz},\\ \omega E_x &=& c^2k_zB_y-iJ_x/\epsilon_0,\\ \omega E_y &=& -c^2k_zB_x+c^2k_xB_z-iJ_y/\epsilon_0,\\ \omega E_z &=& -c^2k_xB_y-iJ_z/\epsilon_0,\\ \omega B_x &=& -k_zE_y,\\ \omega B_y &=& k_zE_x-k_xE_z,\\ \omega B_z &=& k_xE_y, \end{array}\right. \tag{38}$$

which yields a sparse matrix eigenvalue problem. The symbols such as v_{snjx}, $j_{x,y,z}$ and $J_{x,y,z}$ used here do not have direct physical meanings but are analogy to the perturbed velocity and current density in the fluid derivations of plasma waves. The elements of the eigenvector $(E_x,E_y,E_z,B_x,B_y,B_z)$ still represent the original perturbed electric and magnetic fields. Thus, the polarization of the solutions can also be obtained in a straightforward manner. The dimension of the matrix is $N_N=3\times(N_{SNJ}+1)+6=3\times\{[S\times(2\times N+1)]\times J+1\}+6$, N is the harmonic numbers kept for magnetized species, J is the number of the order of J-pole expansion for Z function.

We can also separate the contribution from each species of $\delta\boldsymbol{J}=\sum_s\delta\boldsymbol{J}_s=\sum_s\boldsymbol{\sigma}_s\cdot\delta\boldsymbol{E}$. To do this, we use a relation of $\frac{b_{11}}{\omega}=\sum_s\frac{b_{11s}}{\omega}$, and then we rewrite Eqs. (35) and (37) as

$$
\begin{aligned}
\begin{pmatrix}\delta J_x\\ \delta J_y\\ \delta J_z\end{pmatrix} &= -i\epsilon_0\begin{pmatrix}
\sum_s\frac{b_{s11}}{\omega}+\sum_{snj}\frac{b_{snj11}}{\omega-c_{snj}} & \sum_s\frac{b_{s12}}{\omega}+\sum_{snj}\frac{b_{snj12}}{\omega-c_{snj}} & \sum_s\frac{b_{s13}}{\omega}+\sum_{snj}\frac{b_{snj13}}{\omega-c_{snj}}\\
\sum_s\frac{b_{s21}}{\omega}+\sum_{snj}\frac{b_{snj21}}{\omega-c_{snj}} & \sum_s\frac{b_{s22}}{\omega}+\sum_{snj}\frac{b_{snj22}}{\omega-c_{snj}} & \sum_s\frac{b_{s23}}{\omega}+\sum_{snj}\frac{b_{snj23}}{\omega-c_{snj}}\\
\sum_s\frac{b_{s31}}{\omega}+\sum_{snj}\frac{b_{snj31}}{\omega-c_{snj}} & \sum_s\frac{b_{s32}}{\omega}+\sum_{snj}\frac{b_{snj32}}{\omega-c_{snj}} & \sum_s\frac{b_{s33}}{\omega}+\sum_{snj}\frac{b_{snj33}}{\omega-c_{snj}}
\end{pmatrix}\begin{pmatrix}\delta E_x\\ \delta E_y\\ \delta E_z\end{pmatrix}\\
&= \sum_s \boldsymbol{\sigma}_s\cdot\begin{pmatrix}\delta E_x\\ \delta E_y\\ \delta E_z\end{pmatrix}
\end{aligned}
\tag{39}
$$

with the coefficients

$$
\begin{cases}
b_{snj11}=\omega_{ps}^2p_{11snj}/c_{snj},\ b_{s11}=-\omega_{ps}^2[\sum_n\frac{2}{R_s}\frac{\omega_{cs}^2}{k_\perp^2L_{sx}^2}\sum_{l,m}a_{s,lm}\Big\{I_l\sum_n n^2(2\Gamma_{an,m+1,0}-m\Gamma_{an,m-1,0})\Big\}+\sum_{nj}p_{11snj}/c_{snj}],\\
b_{snj12}=\omega_{ps}^2p_{12snj}/c_{snj},\ b_{s12}=-\omega_{ps}^2[\sum_{nj}p_{12snj}/c_{snj}],\\
b_{snj21}=\omega_{ps}^2p_{21snj}/c_{snj},\ b_{s21}=-\omega_{ps}^2[\sum_{nj}p_{21snj}/c_{snj}],\\
b_{snj22}=\omega_{ps}^2p_{22snj}/c_{snj},\ b_{s22}=-\omega_{ps}^2[\sum_n\frac{2}{R_s}\sum_{l,m}a_{s,lm}\Big\{I_l(2\Gamma_{cn,m+1,2}-m\Gamma_{cn,m-1,2})\Big\}+\sum_{nj}p_{22snj}/c_{snj}],\\
b_{snj13}=\omega_{ps}^2p_{13snj}/c_{snj},\ b_{s13}=-\omega_{ps}^2[\sum_{nj}p_{13snj}/c_{snj}],\\
b_{snj31}=\omega_{ps}^2p_{31snj}/c_{snj},\ b_{s31}=-\omega_{ps}^2[\sum_{nj}p_{31snj}/c_{snj}],\\
b_{snj23}=\omega_{ps}^2p_{23snj}/c_{snj},\ b_{s23}=-\omega_{ps}^2[\sum_{nj}p_{23snj}/c_{snj}],\\
b_{snj32}=\omega_{ps}^2p_{32snj}/c_{snj},\ b_{s32}=-\omega_{ps}^2[\sum_{nj}p_{32snj}/c_{snj}],\\
b_{snj33}=\omega_{ps}^2p_{33snj}/c_{snj},\ b_{s33}=-\omega_{ps}^2[\sum_n\frac{2}{R_s}\sum_{l,m}a_{s,lm}\Big\{\Gamma_{an,m,1}(2\frac{d_{sz}}{L_{sz}}I_{l+1}+2I_{l+2}-l\frac{d_{sz}}{L_{sz}}I_{l-1}-lI_l)\Big\}+\sum_{nj}p_{33snj}/c_{snj}],\\
c_{snj}=k_zv_{dsz}+n\omega_{cs}+k_zv_{zts}c_j.
\end{cases}
\tag{40}
$$

Consequently, we have

$$
\delta\boldsymbol{J}_s=\boldsymbol{\sigma}_s\cdot\delta\boldsymbol{E},\quad \boldsymbol{\sigma}_s=-i\epsilon_0\begin{pmatrix}
\frac{b_{s11}}{\omega}+\sum_{nj}\frac{b_{snj11}}{\omega-c_{snj}} & \frac{b_{s12}}{\omega}+\sum_{nj}\frac{b_{snj12}}{\omega-c_{snj}} & \frac{b_{s13}}{\omega}+\sum_{nj}\frac{b_{snj13}}{\omega-c_{snj}}\\
\frac{b_{s21}}{\omega}+\sum_{nj}\frac{b_{snj21}}{\omega-c_{snj}} & \frac{b_{s22}}{\omega}+\sum_{nj}\frac{b_{snj22}}{\omega-c_{snj}} & \frac{b_{s23}}{\omega}+\sum_{nj}\frac{b_{snj23}}{\omega-c_{snj}}\\
\frac{b_{s31}}{\omega}+\sum_{nj}\frac{b_{snj31}}{\omega-c_{snj}} & \frac{b_{s32}}{\omega}+\sum_{nj}\frac{b_{snj32}}{\omega-c_{snj}} & \frac{b_{s33}}{\omega}+\sum_{nj}\frac{b_{snj33}}{\omega-c_{snj}}
\end{pmatrix}. \tag{41}
$$

We can use Eq. (41) to obtain $\delta\boldsymbol{J}_s$. Since $\delta\boldsymbol{E}$ is known, the first approach requires solving the above 3-by-3 tensor for each species.

The second approach would be more convenient for obtaining $\delta\boldsymbol{J}_s$ based on the fact that a matrix eigenvalue method is used in BO/PDRK. For example, as given in Eq. (132) of Ref. [12], the perturbed current in x direction is

$$
i\delta J_x\epsilon_0=j_x+\sum_{snj}v_{snjx}, \tag{42}
$$

where j_x and v_{snjx} have been solved along with $\delta\boldsymbol{E}$ and $\delta\boldsymbol{B}$. δJ_{sx} can be directly obtained once j_x and v_{snjx} for each species s are known. Similarly, we can obtain δJ_{sy} and δJ_{sz}.

The quantities v_{snjx} are species quantities, but j_x was not in the original version of BO/PDRK. With the addition of only $S-1$ matrix elements, we can replace the matrix element j_x by a sum over S matrix elements j_{sx}, where S is the number of species. To do this, we modified the BO/PDRK matrix equations in Eq. (39), i.e.,

$$
\begin{cases}
\omega j_x=b_{11}\delta E_x+b_{12}\delta E_y+b_{13}\delta E_z,\\
\omega j_y=b_{21}\delta E_x+b_{22}\delta E_y+b_{23}\delta E_z,\\
\omega j_z=b_{31}\delta E_x+b_{32}\delta E_y+b_{33}\delta E_z,
\end{cases}
\tag{43}
$$

as

$$
\begin{cases}
\omega j_{sx}=b_{s11}\delta E_x+b_{s12}\delta E_y+b_{s13}\delta E_z,\\
\omega j_{sy}=b_{s21}\delta E_x+b_{s22}\delta E_y+b_{s23}\delta E_z,\\
\omega j_{sz}=b_{s31}\delta E_x+b_{s32}\delta E_y+b_{s33}\delta E_z.
\end{cases}
\tag{44}
$$

This separation can directly give $j_{sx,y,z}$ from the BO/PDRK matrix, which then yields $\delta J_{sx,y,z}$ through the following equations

$$
\begin{cases}
\delta J_{sx} = (j_{sx}+\sum_{nj}v_{snjx})/(i\epsilon_0),\\
\delta J_{sy} = (j_{sy}+\sum_{nj}v_{snjy})/(i\epsilon_0),\\
\delta J_{sz} = (j_{sz}+\sum_{nj}v_{snjz})/(i\epsilon_0).
\end{cases}
\tag{45}
$$

The updated matrix equations of BO, i.e., Eq.(132) of Ref.[12], become

$$
\left\{\begin{array}{lcc}
\omega v_{snjx} & = & c_{snj}v_{snjx}+b_{snj11}\delta E_x+b_{snj12}\delta E_y+b_{snj13}\delta E_z,\\
\omega j_{sx} & = & b_{s11}\delta E_x+b_{s12}\delta E_y+b_{s13}\delta E_z,\\
i\delta J_x\epsilon_0 & = & j_x+\sum_{snj}v_{snjx},\\
\omega v_{snjy} & = & c_{snj}v_{snjy}+b_{snj21}\delta E_x+b_{snj22}\delta E_y+b_{snj23}\delta E_z,\\
\omega j_{sy} & = & b_{s21}\delta E_x+b_{s22}\delta E_y+b_{s23}\delta E_z,\\
i\delta J_y/\epsilon_0 & = & j_y+\sum_{snj}v_{snjy},\\
\omega v_{snjz} & = & c_{snj}v_{snjz}+b_{snj31}\delta E_x+b_{snj32}\delta E_y+b_{snj33}\delta E_z,\\
\omega j_{sz} & = & b_{s31}\delta E_x+b_{s32}\delta E_y+b_{s33}\delta E_z,\\
i\delta J_z/\epsilon_0 & = & j_z+\sum_{snj}v_{snjz},\\
\omega\delta E_x & = & c^2k_z\delta B_y-i\delta J_x/\epsilon_0,\\
\omega\delta E_y & = & -c^2k_z\delta B_x+c^2k_x\delta B_z-i\delta J_y/\epsilon_0,\\
\omega\delta E_z & = & -c^2k_x\delta B_y-i\delta J_z/\epsilon_0,\\
\omega\delta B_x & = & -k_z\delta E_y,\\
\omega\delta B_y & = & k_z\delta E_x-k_x\delta E_z,\\
\omega\delta B_z & = & k_x\delta E_y,
\end{array}\right. \tag{46}
$$

which yields a sparse matrix eigenvalue problem $\omega\boldsymbol{X}=\boldsymbol{M}(\boldsymbol{k})\cdot\boldsymbol{X}$. The symbols such as v_{snjx}, $j_{sx,y,z}$ and $\delta J_{x,y,z}$ used here are analogous to the perturbed velocity and current density in fluid derivations of plasma waves. The elements of the eigenvector $(\delta E_x,\delta E_y,\delta E_z,\delta B_x,\delta B_y,\delta B_z)$ represent the perturbed electric and magnetic fields. Thus, all variables of one plasma wave mode can be obtained in a straightforward manner. In addition, the dimension of the matrix is $N_N=3\times(N_{SNJ}+S)+6=3\times\{[S\times(2\times N+1)]\times J+S\}+6$, N is the number of harmonics retained for magnetized species, and J is the order of the J-pole expansion used for calculation of the plasma dispersion Z function.

3 GPDF-Hermite Basis Function Expansion

As previously mentioned, while arbitrary bases can be used for the expansion, selecting an appropriate basis can significantly enhance performance.

3.1 Choose the basis function

We choose GPDF basis for parallel direction and Hermite basis for perpendicular direction in this section, with

$$
f_{s0}(v_\parallel,v_\perp)=W_{sz}(v_\parallel)\sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty}a_{s0,lm}\rho_{sz,l}(v_\parallel)u_{sx,m}(v_\perp),\quad -\infty<v_\parallel<\infty,\quad 0\le v_\perp<\infty, \tag{47}
$$

where $\rho_{sz,l}$ and $u_{sx,m}$ are basis functions for parallel and perpendicular directions, with W_{sz} be corresponding weight functions, which satisfy

$$
\int_{-\infty}^{\infty}W_{sz}(v)\rho_{sz,n}(v)\rho^*_{sz,n'}(v)dv=A_{sz}\delta_{n,n'},\quad \int_{-\infty}^{\infty}u_{sx,n}(v)u^*_{sx,n'}(v)dv=A_{sx}\delta_{n,n'}. \tag{48}
$$

Here, the asterisk denotes complex conjugation and $\delta_{n,n'}$ is the Kronechker delta. The weight $W_{sx}=1$ are omitted in the above form. Thus, we have

$$
a_{s0,lm}=\frac{1}{A_{sx}A_{sz}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{s0}(v_\parallel,v_\perp)\rho^*_{sz,l}((v_\parallel)u^*_{sx,m}(v_\perp)dv_\parallel dv_\perp. \tag{49}
$$

We choose GPDF basis [11, 1] for $v_\parallel$ direction

$$
\rho_{sz,n}(z)=\frac{(L_{sz}+iz)^n}{(L_{sz}-iz)^n},\quad W_{sz}(z)=\frac{L_{sz}^2}{L_{sz}^2+z^2},\quad -\infty<z<\infty, \tag{50}
$$

with corresponding velocity widths be L_{sz}. Use

$$e^{i\phi} = \frac{(L_{sz}+iz)}{(L_{sz}-iz)}, \quad z = L_{sz}\cdot\tan\Big(\frac{\phi}{2}\Big), \quad \frac{dz}{d\phi} = \frac{L_{sz}^2+z^2}{2L_{sz}}, \tag{51}$$

and

$$\int_{-\pi}^{\pi} e^{in\phi}e^{-im\phi}d\phi = 2\pi\delta_{m,n}, \tag{52}$$

it is readily to show that $A_{sz} = \pi L_{sz}$. Since the above basis function is relevant to Fourier basis, which would be convergent exponentially and $a_{s,lm}$ can be calculated fastly by fast Fourier transformation (FFT). Another benefit is that the parallel velocity integral Z function can be calculated analytically, thus be also fast and accurate. In practical test, we find the truncation $l = -M, -(M-1), \cdots, -1, 0, 1, \cdots, (M-1), M$ are usually sufficiently accurate for $M \le 16$. Due to the even symmetry of $f_{s0}(v_\perp)$, we have $a_{s0,lm} = 0$ for odd $m = 1, 3, 5, \cdots$.

The basis function for perpendicular $v_\perp$ direction is choosen as

$$u_{sx,n}(x) = h_n\Big(\frac{\sqrt{2}(x-d_{sx})}{L_{sx}}\Big), \quad -\infty < x < \infty, \tag{53}$$

with corresponding velocity widths be L_{sx}. Note that we have extend $0 \le v_\perp < \infty$ to $-\infty < v_\perp < \infty$, with $f_{s0}(v_\parallel, v_\perp) = f_{s0}(v_\parallel, -v_\perp)$. Here h_n is Hermite function

$$h_n(y) = \sqrt{\frac{1}{2^n n!\sqrt{\pi}}}H_n(y)e^{-y^2/2}, \quad \int_{-\infty}^{\infty} h_n(y)h_{n'}(y)dy = \delta_{nn'}, \tag{54}$$

$$H_0(y) = 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2-2, \quad H_3(y) = 8y^3-12y, \quad \cdots \tag{55}$$

$$H_n'(y) = 2nH_{n-1}(y), \quad H_n(-y) = (-1)^n H_n(y), \quad H_{n+1}(y) = 2yH_n((y) - 2nH_{n-1}(y). \tag{56}$$

We have $A_{sx} = L_{sx}/\sqrt{2}$.

For later more convenient of calculations, we can rewrite the above expansion to the follow form

$$f_{s0}(v_\parallel, v_\perp) = W_{sz}(v_\parallel)\sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty} a_{s0,lm}\rho_{sz,l}(v_\parallel)u_{sx,m}(v_\perp) = c_{s0}W_{sz}(v_\parallel)\sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty} a_{s,lm}\cdot\rho_{sz,l}(v_\parallel)\cdot g_{sx,m}(v_\perp), \tag{57}$$

with

$$g_{sx,m}(v_\perp) = (\frac{v_\perp - d_{sx}}{L_{sx}})^m e^{-(\frac{v_\perp-d_{sx}}{L_{sx}})^2}, \tag{58}$$

and normalized coefficient

$$c_{s0} = \frac{1}{\pi^2 L_{sz}L_{sx}^2 R_s}, \quad R_s = \exp\Big(-\frac{d_{sx}^2}{L_{sx}^2}\Big) + \frac{\sqrt{\pi}d_{sx}}{L_{sx}}\mathrm{erfc}\Big(-\frac{d_{sx}}{L_{sx}}\Big), \tag{59}$$

and $\mathrm{erfc}(-x) = 1-\mathrm{erf}(-x) = 1+\mathrm{erf}(x)$ is the complementary error function. After obtaining the coefficients $a_{s0,lm}$, it is readily to calculate the coefficients $a_{s,lm}$. Note that this new $g_{sx,m}$ is not orthogonal.

3.2 Calculate the dispersion relation

We define

$$f_{s0,lm}(v_\parallel, v_\perp) \equiv a_{s,lm}W_{sz}(v_\parallel)\rho_{sz,l}(v_\parallel)u_{sx,m}(v_\perp) = a_{s,lm}f_{s0z,l}(v_\parallel)f_{s0x,m}(v_\perp), \tag{60}$$

$$f_{s0}(v_\parallel, v_\perp) = c_{s0}\sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty} f_{s0,lm}(v_\parallel, v_\perp), \quad f_{s0z,l}(v_\parallel) \equiv W_{sz}(v_\parallel)\rho_{sz,l}(v_\parallel), \quad f_{s0x,m}(v_\perp) \equiv u_{sx,m}(v_\perp). \tag{61}$$

Hence, we have

$$\frac{\partial f_{s0,lm}}{\partial v_\parallel} = a_{s,lm}f_{s0x,m}(v_\perp)\frac{\partial f_{s0z,l}(v_\parallel)}{\partial v_\parallel}, \quad \frac{\partial f_{s0,lm}}{\partial v_\perp} = a_{s,lm}f_{s0z,l}(v_\parallel)\frac{\partial f_{s0x,m}(v_\perp)}{\partial v_\perp}, \tag{62}$$

$$\frac{\partial f_{s0z,l}(v_\parallel)}{\partial v_\parallel} = \frac{\partial}{\partial v_\parallel}\Big[\frac{(L_{sz}+iv_\parallel)^l}{(L_{sz}-iv_\parallel)^l}\frac{L_{sz}^2}{L_{sz}^2+v_\parallel^2}\Big] = \frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}], \tag{63}$$

$$\frac{\partial f_{s0x,m}(v_\perp)}{\partial v_\perp} = \frac{1}{L_{sx}}\Big[mf_{s0x,m-1}-2f_{s0x,m+1}\Big]. \tag{64}$$

Here, we set $f_{s0x,m}=0$ for all $m<0$. The above derivatives equations are valid for all $l=0,\pm1,\pm2,\cdots$ and $m=0,1,2,\cdots$. We have also numerically checked the valid of the above two derivatives.

For parallel integral [$\mathrm{Im}(\zeta_{sn})>0$], define

$$Z_{l,p}(\zeta_{sn})\equiv\frac{k_\parallel}{L_{sz}^p\pi}\int_{-\infty}^{\infty}\frac{v_\parallel^p f_{s0z,l}}{\omega-k_\parallel v_\parallel-n\omega_{cs}}dv_\parallel=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g_{l,p}(z)}{\zeta_{sn}-z}dz, \tag{65}$$

$$g_{l,p}(z)\equiv\frac{(1+iz)^l}{(1-iz)^l}\frac{z^p}{1+z^2}, \tag{66}$$

we have[1]

$$Z_{l,0}=\begin{cases}-\frac{i}{(1-i\zeta_{sn})}, & l=0,\\ -\frac{2i}{1+\zeta_{sn}^2}\frac{(1+i\zeta_{sn})^l}{(1-i\zeta_{sn})^l}, & l>0,\\ 0, & l<0,\end{cases} \tag{67}$$

$$Z_{l,1}=\zeta_{sn}Z_{l,0}-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+iz)^l}{(1-iz)^l}\frac{1}{1+z^2}dz=\zeta_{sn}Z_{l,0}-\delta_{l,0}, \tag{68}$$

$$Z_{l,2}=\zeta_{sn}Z_{l,1}-\frac{1}{\pi}\int_{-\infty}^{\infty}z\frac{(1+iz)^l}{(1-iz)^l}\frac{1}{1+z^2}dz=\zeta_{sn}Z_{l,1}+\mathrm{sgn}(l)\cdot(-1)^l\cdot i, \tag{69}$$

with $\zeta_{sn}=\frac{\omega-n\omega_{cs}}{k_\parallel L_{sz}}$, $p=0,1,2$. And we redefine $Z_{l,0}\equiv Z_l$ for simplify the notation. We have also numerically checked the valid of the above $Z_{l,p}$ equations for different l. Note also that the final right hand side form is also valid for weak damped $\mathrm{Im}(\zeta_{sn})\simeq 0$ mode, since which is naturally analytic continuous for both $\mathrm{Im}(\zeta_{sn})>0$ and $\mathrm{Im}(\zeta_{sn})\leq 0$. They may be not valid for strong damped mode, which is due to that the analytic form of the f_{0s} is not exactly as the one give by the expansion form. In this work, we hope to solve all the solutions of the dispersion relations, hence we mainly use the above rational form of $Z_{l,p}$. It is shown[17] that the rational form is sufficient for weak damped mode, say $\mathrm{Im}(\zeta)>-2$. Note also that $I_{l,0}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+iz)^l}{(1-iz)^l}\frac{1}{1+z^2}dz=\delta_{l,0}$ due to the orthogonal, and $I_{l,1}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+iz)^l}{(1-iz)^l}\frac{z}{1+z^2}dz$.

For perpendicular integral

$$\Gamma_{an,m,p}(a_s,d_s)\equiv\frac{1}{L_{sx}^{p+1}}\int_0^{\infty}v_\perp^p J_n^2(\frac{k_\perp v_\perp}{\omega_{cs}})(\frac{v_\perp-d_{sx}}{L_{sx}})^m e^{-(\frac{v_\perp-d_{sx}}{L_{sx}})^2}dv_\perp=\int_0^{\infty}x^pJ_n^2(a_sx)(x-d_s)^m e^{-(x-d_s)^2}dx,$$

$$\Gamma_{bn,m,p}(a_s,d_s)\equiv\frac{1}{L_{sx}^{p+1}}\int_0^{\infty}v_\perp^p J_n(\frac{k_\perp v_\perp}{\omega_{cs}})J_n'(\frac{k_\perp v_\perp}{\omega_{cs}})(\frac{v_\perp-d_{sx}}{L_{sx}})^m e^{-(\frac{v_\perp-d_{sx}}{L_{sx}})^2}dv_\perp=\int_0^{\infty}x^pJ_n(a_sx)J_n'(a_sx)(x-d_s)^m e^{-(x-d_s)^2}dx,$$

$$\Gamma_{cn,m,p}(a_s,d_s)\equiv\frac{1}{L_{sx}^{p+1}}\int_0^{\infty}v_\perp^p {J_n'}^2(\frac{k_\perp v_\perp}{\omega_{cs}})(\frac{v_\perp-d_{sx}}{L_{sx}})^m e^{-(\frac{v_\perp-d_{sx}}{L_{sx}})^2}dv_\perp=\int_0^{\infty}x^p{J_n'}^2(a_sx)(x-d_s)^m e^{-(x-d_s)^2}dx,$$

with $a_s=k_\perp\rho_{cs}$, $d_s=d_{sx}/L_{sx}$, $\rho_{cs}=L_{sx}/\omega_{cs}$, $p=0,1,2,3$. These integrals can be calculated numerically[12]. For $d_{sx}=0$, they are relavant to the modified Bessel functions of the first kind as in the Maxwellian case.

Note that we have the normalized condition: $1=\int_{-\infty}^{\infty}\int_0^{\infty}2\pi v_\perp f_{s0}dv_\perp dv_\parallel$.

Hence, we have

$$A_s=c_{s0}\sum_l\sum_m a_{s,lm}\Big\{\frac{\omega-k_\parallel v_\parallel}{\omega}f_{s0z,l}\frac{1}{L_{sx}}[mf_{s0x,m-1}-2f_{s0x,m+1}]+ \\ \frac{k_\parallel v_\perp}{\omega}f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}, \tag{73}$$

$$B_s=c_{s0}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}v_\parallel}{\omega v_\perp}f_{s0z,l}\frac{1}{L_{sx}}[mf_{s0x,m-1}-2f_{s0x,m+1}]+ \\ \frac{\omega-n\omega_{cs}}{\omega}f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}. \tag{74}$$

Define

$$\boldsymbol{X}_{sn} = \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times\left[\begin{array}{ccc} A_s\frac{n^2v_\perp}{\mu_s^2}J_n^2 & iA_s\frac{nv_\perp}{\mu_s}J_nJ_n' & B_s\frac{nv_\perp}{\mu_s}J_n^2 \\ -iA_s\frac{nv_\perp}{\mu_s}J_nJ_n' & A_sv_\perp {J_n'}^2 & -iB_sv_\perp J_nJ_n' \\ A_s\frac{nv_\parallel}{\mu_s}J_n^2 & iA_sv_\parallel J_nJ_n' & B_sv_\parallel J_n^2 \end{array}\right]. \quad (75)$$

We have

$$\begin{aligned} X_{sn11} &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_s\frac{n^2v_\perp}{\mu_s^2}J_n^2=\frac{n^2\omega_{cs}^2}{k_\perp^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_sJ_n^2 \\ &= c_{s0}\frac{n^2\omega_{cs}^2}{k_\perp^2}\sum_l\sum_m a_{s,lm}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\pi\omega dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times\Big\{\frac{\omega-k_\parallel v_\parallel}{\omega}f_{s0z,l}\frac{2}{L_{sx}}[mf_{s0x,m-1}-2f_{s0x,m+1}] \\ &\quad +\frac{k_\parallel v_\perp}{\omega}f_{s0x,m}\frac{i}{L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}J_n^2 \\ &= \frac{1}{\pi L_{sz}L_{sx}^2R_s}\frac{n^2\omega_{cs}^2}{k_\perp^2}\sum_l\sum_m a_{s,lm}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times\Big\{(\omega-k_\parallel v_\parallel)f_{s0z,l}\frac{2}{L_{sx}}[mf_{s0x,m-1}-2f_{s0x,m+1}] \\ &\quad +k_\parallel v_\perp f_{s0x,m}\frac{i}{L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}J_n^2 \\ &= \frac{1}{R_s}\frac{n^2\omega_{cs}^2}{k_\perp^2L_{sx}^2}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0}+I_{l,0})(2m\Gamma_{an,m-1,0}-4\Gamma_{an,m+1,0}) \\ &\quad +\Gamma_{an,m,1}\frac{iL_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,0}+2lZ_{l,0}+(l+1)Z_{l+1,0}]\Big\}, \end{aligned}$$

$$\begin{aligned} X_{sn12} &= -X_{sn21}=\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times iA_s\frac{nv_\perp}{\mu_s}J_nJ_n'=i\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_sJ_nJ_n' \\ &= i\frac{1}{R_s}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0}+I_{l,0})(2m\Gamma_{bn,m-1,1}-4\Gamma_{2n,m+1,1}) \\ &\quad +\Gamma_{bn,m,2}\frac{iL_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,0}+2lZ_{l,0}+(l+1)Z_{l+1,0}]\Big\} \\ &\to i\frac{1}{R_s}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0}(2m\Gamma_{bn,m-1,1}-4\Gamma_{2n,m+1,1}) \\ &\quad +\Gamma_{bn,m,2}\frac{iL_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,0}+2lZ_{l,0}+(l+1)Z_{l+1,0}]\Big\}, \end{aligned}$$

$$\begin{aligned} X_{sn22} &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_sv_\perp {J_n'}^2=\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_sv_\perp {J_n'}^2 \\ &= \frac{1}{R_s}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0}+I_{l,0})(2m\Gamma_{cn,m-1,2}-4\Gamma_{cn,m+1,2}) \\ &\quad +\Gamma_{cn,m,3}\frac{iL_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,0}+2lZ_{l,0}+(l+1)Z_{l+1,0}]\Big\}, \end{aligned}$$

$$
\begin{aligned}
X_{sn31} &= X_{sn13} = \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_s\frac{nv_\parallel}{\mu_s}J_n^2 = \frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\parallel dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_s J_n^2 \\
&= c_{s0}\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\parallel dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\sum_l\sum_m a_{s,lm}\Big\{\frac{\omega-k_\parallel v_\parallel}{\omega}f_{s0z,l}\frac{1}{L_{sx}}[mf_{s0x,m-1}-2f_{s0x,m+1}]+ \\
&\quad \frac{k_\parallel v_\perp}{\omega}f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}J_n^2 \\
&= \frac{1}{\pi L_{sz}L_{sx}^2R_s}\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{v_\parallel dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\sum_l\sum_m a_{s,lm}\Big\{(\omega-k_\parallel v_\parallel)f_{s0z,l}\frac{1}{L_{sx}}[2mf_{s0x,m-1}-4f_{s0x,m+1}]+ \\
&\quad k_\parallel v_\perp f_{s0x,m}\frac{i}{L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}J_n^2 \\
&= \frac{L_{sz}}{L_{sx}R_s}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,1}+I_{l,1})(2m\Gamma_{an,m-1,0}-4\Gamma_{an,m+1,0}) \\
&\quad +\Gamma_{an,m,1}\frac{iL_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,1}+2lZ_{l,1}+(l+1)Z_{l+1,1}]\Big\} \\
&\to \frac{1}{R_s}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sx}}Z_{l,1}(2m\Gamma_{an,m-1,0}-4\Gamma_{an,m+1,0}) \\
&\quad +\Gamma_{an,m,1}\frac{iL_{sx}}{L_{sz}}[(l-1)Z_{l-1,1}+2lZ_{l,1}+(l+1)Z_{l+1,1}]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
X_{sn32} &= -X_{sn23} = \int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times iA_s v_\parallel J_n J_n' = i\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times A_s J_n J_n' \\
&= i\int_{-\infty}^{\infty}\int_0^{\infty}\frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}c_{s0}\sum_l\sum_m a_{s,lm}\Big\{\frac{\omega-k_\parallel v_\parallel}{\omega}f_{s0z,l}\frac{1}{L_{sx}}[mf_{s0x,m-1}-2f_{s0x,m+1}] \\
&\quad +\frac{k_\parallel v_\perp}{\omega}f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}J_n J_n' \\
&= \frac{i}{\pi L_{sz}L_{sx}^2R_s}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{v_\parallel v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\sum_l\sum_m a_{s,lm}\Big\{(\omega-k_\parallel v_\parallel)f_{s0z,l}\frac{1}{L_{sx}}[2mf_{s0x,m-1}-4f_{s0x,m+1}] \\
&\quad +k_\parallel v_\perp f_{s0x,m}\frac{i}{L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}J_n J_n' \\
&= i\frac{L_{sz}}{L_{sx}R_s}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,1}+I_{l,1})(2m\Gamma_{bn,m-1,1}-4\Gamma_{bn,m+1,1}) \\
&\quad +\Gamma_{bn,m,2}\frac{iL_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,1}+2lZ_{l,1}+(l+1)Z_{l+1,1}]\Big\} \\
&\to i\frac{1}{R_s}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sx}}Z_{l,1}(2m\Gamma_{bn,m-1,1}-4\Gamma_{bn,m+1,1}) \\
&\quad +\Gamma_{bn,m,2}\frac{iL_{sx}}{L_{sz}}[(l-1)Z_{l-1,1}+2lZ_{l,1}+(l+1)Z_{l+1,1}]\Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
X_{sn33} &= \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times B_s v_\parallel J_n^2=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\times B_s J_n^2\\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}c_{s0}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}v_\parallel}{\omega v_\perp}f_{s0z,l}\frac{1}{L_{sx}}[mf_{s0x,m-1}-2f_{s0x,m+1}]\\
&\quad+\frac{\omega-n\omega_{cs}}{\omega}f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}J_n^2\\
&= \frac{1}{\pi L_{sz}L_{sx}^2R_s}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{dv_\perp dv_\parallel}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\sum_l\sum_m a_{s,lm}\Big\{n\omega_{cs}v_\parallel^2 f_{s0z,l}\frac{1}{L_{sx}}[2mf_{s0x,m-1}-4f_{s0x,m+1}]\\
&\quad+v_\parallel v_\perp(\omega-n\omega_{cs})f_{s0x,m}\frac{i}{L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}J_n^2\\
&= \frac{L_{sz}^2}{L_{sx}^2R_s}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,2}(2m\Gamma_{an,m-1,0}-4\Gamma_{1n,m+1,0})\\
&\quad+\Gamma_{an,m,1}i\frac{L_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,2}+2lZ_{l,2}+(l+1)Z_{l+1,2}+(l-1)I_{l-1,1}+2lI_{l,1}+(l+1)I_{l+1,1}]\Big\}\\
&= \frac{L_{sz}^2}{L_{sx}^2R_s}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,2}(2m\Gamma_{an,m-1,0}-4\Gamma_{1n,m+1,0})\\
&\quad+\Gamma_{an,m,1}i\frac{L_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,2}+2lZ_{l,2}+(l+1)Z_{l+1,2}+2i\delta_{l,0}]\Big\},
\end{aligned}
$$

where

$$
\boldsymbol{Q}=\sum_s\boldsymbol{Q}_s,\quad \boldsymbol{Q}_s=\frac{\omega_{ps}^2}{\omega^2}\boldsymbol{P}_s,\quad \boldsymbol{P}_s=\sum_n\boldsymbol{P}_{sn},\quad \boldsymbol{P}_{sn}=c_{s0}\boldsymbol{X}_{sn}. \tag{76}
$$

We have used $(l-1)I_{l-1,1}+2lI_{l,1}+(l+1)I_{l+1,1}=2i\delta_{l,0}$.

Here, '$\to$' means the constant terms I_l has been summation out for n. We have used $\sum_{n=-\infty}^{\infty}J_nJ_n'=0$, $\sum_{n=-\infty}^{\infty}nJ_n^2=0$, $\sum_{n=-\infty}^{\infty}nJ_nJ_n'=0$, $\sum_{n=-\infty}^{\infty}J_n^2=1$, $\sum_{n=-\infty}^{\infty}(J_n')^2=\frac{1}{2}$ and $\sum_{n=-\infty}^{\infty}\frac{n^2J_n^2}{x^2}=\frac{1}{2}$.

After the above expansion, the dispersion relation Eq.(1) can be readily solved use conventional iterative root finding method. Since the perpendicular integral $\Gamma(a_s)$ need only calculate once, and the parallel integrals $Z(\zeta_{sn})$ are finite rational form, this approach can obtain high accurate solution with less computation cost. The above dispersion relation can also be transformed to matrix method similar to the κ distribution case[14, 13]. We omit the transformation here since it is slightly complicated.

4 GPDF-GPDF Basis Function Expansion

As previously mentioned, while arbitrary bases can be used for the expansion, selecting an appropriate basis can significantly enhance performance.

4.1 Choose the basis function

We choose GPDF basis for parallel direction and modified GPDF basis for perpendicular direction in this section, with

$$
f_{s0}(v_\parallel,v_\perp)=W_{sz}(v_\parallel)W_{sx}^2(v_\perp)\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_{s0,lm}\rho_{sz,l}(v_\parallel)u_{sx,m}(v_\perp),\quad -\infty<v_\parallel<\infty,\quad 0\le v_\perp<\infty, \tag{77}
$$

where $\rho_{sz,l}$ and $u_{sx,m}$ are basis functions for parallel and perpendicular directions, with W_{sz} and W_{sx} be corresponding weight functions, which satisfy

$$
\int_{-\infty}^{\infty}W_{sz}(v)\rho_{sz,n}(v)\rho_{sz,n'}^*(v)dv=A_{sz}\delta_{n,n'},\quad \int_{-\infty}^{\infty}W_{sx}(v)u_{sx,n}(v)u_{sx,n'}^*(v)dv=A_{sx}\delta_{n,n'}. \tag{78}
$$

Here, the asterisk denotes complex conjugation and $\delta_{n,n'}$ is the Kronechker delta. Note for perpendicular direction, we use $W_{sx}^2(v_\perp)$ instead of $W_{sx}(v_\perp)$ is to avoid the divergence of later perpendicular integral. Thus, we have

$$a_{s0,lm} = \frac{1}{A_{sx}A_{sz}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f_{s0}(v_\parallel, v_\perp)}{W_{sx}(v_\perp)} \rho_{sz,l}^*((v_\parallel) u_{sx,m}^*(v_\perp) dv_\parallel dv_\perp. \tag{79}$$

We choose both GPDF basis [11, 1] for $v_\parallel$ and $v_\perp$ direction

$$\rho_{sz,n}(z) = \frac{(L_{sz}+iz)^n}{(L_{sz}-iz)^n}, \quad W_{sz}(z) = \frac{L_{sz}^2}{L_{sz}^2+z^2}, \quad -\infty < z < \infty, \tag{80}$$

$$u_{sx,n}(x) = \frac{(L_{sx}+ix)^n}{(L_{sx}-ix)^n}, \quad W_{sx}(x) = \frac{L_{sx}^2}{L_{sx}^2+x^2}, \quad 0 < x < \infty, \tag{81}$$

with corresponding velocity widths be L_{sz} and L_{sx}. Use

$$e^{i\phi} = \frac{(L+iz)}{(L-iz)}, \quad z = L\cdot\tan\Big(\frac{\phi}{2}\Big), \quad \frac{dz}{d\phi} = \frac{L^2+z^2}{2L}, \tag{82}$$

and

$$\int_{-\pi}^{\pi} e^{in\phi}e^{-im\phi}d\phi = 2\pi\delta_{m,n}, \tag{83}$$

it is readily to show that $A_{sz} = \pi L_{sz}$ and $A_{sx} = \pi L_{sx}$. Since the above basis function is relevant to Fourier basis, which would be convergent exponentially and $a_{s,lm}$ can be calculated fastly by fast Fourier transformation (FFT). Another benefit is that the parallel velocity integral Z function can be calculated analytically, thus be also fast and accurate. In practical test, we find the truncation $l = -M, -(M-1), \cdots, -1, 0, 1, \cdots, (M-1), M$ are usually sufficiently accurate for $M \le 16$. Note that we have extend $0 \le v_\perp < \infty$ to $-\infty < v_\perp < \infty$, with $f_{s0}(v_\parallel, v_\perp) = f_{s0}(v_\parallel, -v_\perp)$. Due to the even symmetry of $f_{s0}(v_\perp)$, we have $a_{s,l,m} = a_{s,l,-m}$. The benifit of GPDF-GPDF expansion is that the coefficients can be calculate fastly use FFT.

For later more convenient of calculations, we can rewrite the above expansion to the follow form

$$\begin{aligned} f_{s0}(v_\parallel, v_\perp) &= W_{sz}(v_\parallel)W_{sx}^2(v_\perp) \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_{s0,lm}\rho_{sz,l}(v_\parallel)u_{sx,m}(v_\perp) \\ &= c_{s0}W_{sz}(v_\parallel)W_{sx}^2(v_\perp) \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_{s,lm}\cdot\rho_{sz,l}(v_\parallel)\cdot u_{sx,m}(v_\perp), \end{aligned} \tag{84}$$

with the normalized coefficient

$$c_{s0} = \frac{1}{\pi^2 L_{sz}L_{sx}^2}. \tag{85}$$

After obtaining the coefficients $a_{s0,lm}$, it is readily to calculate the coefficients $a_{s,lm}$.

4.2 Calculate the dispersion relation

We define

$$f_{s0,lm}(v_\parallel, v_\perp) \equiv a_{s,lm}W_{sz}(v_\parallel)W_{sx}^2(v_\perp)\rho_{sz,l}(v_\parallel)u_{sx,m}(v_\perp) = a_{s,lm}f_{s0z,l}(v_\parallel)f_{s0x,m}(v_\perp), \tag{86}$$

$$f_{s0}(v_\parallel, v_\perp) = c_{s0}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f_{s0,lm}(v_\parallel, v_\perp), \quad f_{s0z,l}(v_\parallel) \equiv W_{sz}(v_\parallel)\rho_{sz,l}(v_\parallel), \quad f_{s0x,m}(v_\perp) \equiv W_{sx}^2(v_\perp)u_{sx,m}(v_\perp). \tag{87}$$

Hence, we have

$$\frac{\partial f_{s0,lm}}{\partial v_\parallel} = a_{s,lm}f_{s0x,m}(v_\perp)\frac{\partial f_{s0z,l}(v_\parallel)}{\partial v_\parallel}, \quad \frac{\partial f_{s0,lm}}{\partial v_\perp} = a_{s,lm}f_{s0z,l}(v_\parallel)\frac{\partial f_{s0x,m}(v_\perp)}{\partial v_\perp}, \tag{88}$$

$$\frac{\partial f_{s0z,l}(v_\parallel)}{\partial v_\parallel} = \frac{\partial}{\partial v_\parallel}\Big[\frac{(L_{sz}+iv_\parallel)^l}{(L_{sz}-iv_\parallel)^l}\frac{L_{sz}^2}{L_{sz}^2+v_\parallel^2}\Big] = \frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1} + 2lf_{s0z,l} + (l+1)f_{s0z,l+1}], \tag{89}$$

$$\frac{\partial f_{s0x,m}(v_\perp)}{\partial v_\perp} = \frac{\partial}{\partial v_\perp}\Big[\frac{(L_{sx}+iv_\perp)^m}{(L_{sx}-iv_\perp)^m}\Big(\frac{L_{sx}^2}{L_{sx}^2+v_\perp^2}\Big)^2\Big] = \frac{i}{2L_{sx}}\Big[(m-2)f_{s0x,m-1} + 2mf_{s0x,m} + (m+2)f_{s0x,m+1}\Big]. \tag{90}$$

The above derivatives equations are valid for all $l,m=0,\pm1,\pm2,\cdots$. We have also numerically checked the valid of the above two derivatives.

For parallel integral [$\mathrm{Im}(\zeta_{sn})>0$], define

$$Z_{l,p}(\zeta_{sn})\equiv\frac{k_\parallel}{L_{sz}^p\pi}\int_{-\infty}^{\infty}\frac{v_\parallel^p f_{s0z,l}}{\omega-k_\parallel v_\parallel-n\omega_{cs}}dv_\parallel=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g_{l,p}(z)}{\zeta_{sn}-z}dz, \tag{91}$$

$$g_{l,p}(z)\equiv\frac{(1+iz)^l}{(1-iz)^l}\frac{z^p}{1+z^2}, \tag{92}$$

we have[1]

$$Z_{l,0}=\begin{cases}-\frac{i}{(1-i\zeta_{sn})}, & l=0,\\ -\frac{2i}{1+\zeta_{sn}^2}\frac{(1+i\zeta_{sn})^l}{(1-i\zeta_{sn})^l}, & l>0,\\ 0, & l<0,\end{cases} \tag{93}$$

$$Z_{l,1}=\zeta_{sn}Z_{l,0}-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+iz)^l}{(1-iz)^l}\frac{1}{1+z^2}dz=\zeta_{sn}Z_{l,0}-\delta_{l,0}, \tag{94}$$

$$Z_{l,2}=\zeta_{sn}Z_{l,1}-\frac{1}{\pi}\int_{-\infty}^{\infty}z\frac{(1+iz)^l}{(1-iz)^l}\frac{1}{1+z^2}dz=\zeta_{sn}Z_{l,1}+\mathrm{sgn}(l)\cdot(-1)^l\cdot i, \tag{95}$$

with $\zeta_{sn}=\frac{\omega-n\omega_{cs}}{k_\parallel L_{sz}}$, $p=0,1,2$. And we redefine $Z_{l,0}\equiv Z_l$ for simplify the notation. We have also numerically checked the valid of the above $Z_{l,p}$ equations for different l. Note also that the final right hand side form is also valid for weak damped $\mathrm{Im}(\zeta_{sn})\simeq0$ mode, since which is naturally analytic continuous for both $\mathrm{Im}(\zeta_{sn})>0$ and $\mathrm{Im}(\zeta_{sn})\leq0$. They may be not valid for strong damped mode, which is due to that the analytic form of the f_{0s} is not exactly as the one give by the expansion form. In this work, we hope to solve all the solutions of the dispersion relations, hence we mainly use the above rational form of $Z_{l,p}$. It is shown[17] that the rational form is sufficient for weak damped mode, say $\mathrm{Im}(\zeta)>-2$. Note also that $I_{l,0}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+iz)^l}{(1-iz)^l}\frac{1}{1+z^2}dz=\delta_{l,0}$ due to the orthogonal, and $I_{l,1}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+iz)^l}{(1-iz)^l}\frac{z}{1+z^2}dz$.

For perpendicular integral

$$\Gamma_{an,m,p}(a_s)\equiv\frac{1}{L_{sx}^{p+1}}\int_0^{\infty}v_\perp^pJ_n^2(\frac{k_\perp v_\perp}{\omega_{cs}})f_{s0x,m}dv_\perp=\int_0^{\infty}x^pJ_n^2(a_sx)\frac{(1+ix)^m}{(1-ix)^m}\frac{1}{(1+x^2)^2}dx, \tag{96}$$

$$\Gamma_{bn,m,p}(a_s)\equiv\frac{1}{L_{sx}^{p+1}}\int_0^{\infty}v_\perp^pJ_n(\frac{k_\perp v_\perp}{\omega_{cs}})J_n'(\frac{k_\perp v_\perp}{\omega_{cs}})f_{s0x,m}dv_\perp=\int_0^{\infty}x^pJ_n(a_sx)J_n'(a_sx)\frac{(1+ix)^m}{(1-ix)^m}\frac{1}{(1+x^2)^2}dx, \tag{97}$$

$$\Gamma_{cn,m,p}(a_s)\equiv\frac{1}{L_{sx}^{p+1}}\int_0^{\infty}v_\perp^p{J_n'}^2(\frac{k_\perp v_\perp}{\omega_{cs}})f_{s0x,m}dv_\perp=\int_0^{\infty}x^p{J_n'}^2(a_sx)\frac{(1+ix)^m}{(1-ix)^m}\frac{1}{(1+x^2)^2}dx. \tag{98}$$

with $a_s=k_\perp\rho_{cs}$, $\rho_{cs}=L_{sx}/\omega_{cs}$, $p=0,1,2,3$. These integral can be calculated numerically. Due to the $\frac{1}{(1+x^2)^2}$ term, the integral is convergent for even $p=3$. Note also $J_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}$ for small x.

Hence, we have

$$A_s=c_{s0}\sum_l\sum_m a_{s,lm}\Big\{\frac{\omega-k_\parallel v_\parallel}{\omega}f_{s0z,l}\frac{i}{2L_{sx}}[(m-2)f_{s0x,m-1}+2mf_{s0x,m}+(m+2)f_{s0x,m+1}]+\frac{k_\parallel v_\perp}{\omega}f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}, \tag{99}$$

$$B_s=c_{s0}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}v_\parallel}{\omega v_\perp}f_{s0z,l}\frac{i}{2L_{sx}}[(m-2)f_{s0x,m-1}+2mf_{s0x,m}+(m+2)f_{s0x,m+1}]+\frac{\omega-n\omega_{cs}}{\omega}f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1}+2lf_{s0z,l}+(l+1)f_{s0z,l+1}]\Big\}. \tag{100}$$

Note that we have the normalized condition: $1=\int_{-\infty}^{\infty}\int_0^{\infty}2\pi v_\perp f_{s0}dv_\perp dv_\parallel$.

Define

$$\boldsymbol{X}_{sn} = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times \begin{bmatrix} A_s\frac{n^2 v_\perp}{\mu_s^2}J_n^2 & iA_s\frac{nv_\perp}{\mu_s}J_nJ_n' & B_s\frac{nv_\perp}{\mu_s}J_n^2 \\ -iA_s\frac{nv_\perp}{\mu_s}J_nJ_n' & A_s v_\perp {J_n'}^2 & -iB_s v_\perp J_nJ_n' \\ A_s\frac{nv_\parallel}{\mu_s}J_n^2 & iA_s v_\parallel J_nJ_n' & B_s v_\parallel J_n^2 \end{bmatrix}. \tag{101}$$

We have

$$\begin{aligned}
X_{sn11} &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times A_s\frac{n^2 v_\perp}{\mu_s^2}J_n^2 = \frac{n^2\omega_{cs}^2}{k_\perp^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times A_sJ_n^2 \\
&= c_{s0}\frac{n^2\omega_{cs}^2}{k_\perp^2}\sum_l\sum_m a_{s,lm}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\pi\omega dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times \Big\{ \\
&\quad \frac{\omega - k_\parallel v_\parallel}{\omega} f_{s0z,l}\frac{i}{L_{sx}}[(m-2)f_{s0x,m-1} + 2mf_{s0x,m} + (m+2)f_{s0x,m+1}] \\
&\quad + \frac{k_\parallel v_\perp}{\omega} f_{s0x,m}\frac{i}{L_{sz}}[(l-1)f_{s0z,l-1} + 2lf_{s0z,l} + (l+1)f_{s0z,l+1}]\Big\}J_n^2 \\
&= \frac{i}{\pi L_{sz}L_{sx}^2}\frac{n^2\omega_{cs}^2}{k_\perp^2}\sum_l\sum_m a_{s,lm}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times \Big\{ \\
&\quad (\omega - k_\parallel v_\parallel) f_{s0z,l}\frac{2}{L_{sx}}[(m-2)f_{s0x,m-1} + 2mf_{s0x,m} + (m+2)f_{s0x,m+1}] \\
&\quad + k_\parallel v_\perp f_{s0x,m}\frac{1}{L_{sz}}[(l-1)f_{s0z,l-1} + 2lf_{s0z,l} + (l+1)f_{s0z,l+1}]\Big\}J_n^2 \\
&= i\frac{n^2\omega_{cs}^2}{k_\perp^2 L_{sx}^2}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0} + I_{l,0})[(m-2)\Gamma_{an,m-1,0} + 2m\Gamma_{an,m,0} + (m+2)\Gamma_{an,m+1,0}] \\
&\quad + \Gamma_{an,m,1}\frac{L_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,0} + 2lZ_{l,0} + (l+1)Z_{l+1,0}]\Big\},
\end{aligned}$$

$$\begin{aligned}
X_{sn12} &= -X_{sn21} = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times iA_s\frac{nv_\perp}{\mu_s}J_nJ_n' = i\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times A_sJ_nJ_n' \\
&= -\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0} + I_{l,0})[(m-2)\Gamma_{bn,m-1,1} + 2m\Gamma_{2n,m,1} + (m+2)\Gamma_{2n,m+1,1}] \\
&\quad + \Gamma_{bn,m,2}\frac{iL_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,0} + 2lZ_{l,0} + (l+1)Z_{l+1,0}]\Big\} \\
&\to -\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0}[(m-2)\Gamma_{bn,m-1,1} + 2m\Gamma_{2n,m,1} + (m+2)\Gamma_{2n,m+1,1}] \\
&\quad + \Gamma_{bn,m,2}\frac{L_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,0} + 2lZ_{l,0} + (l+1)Z_{l+1,0}]\Big\},
\end{aligned}$$

$$\begin{aligned}
X_{sn22} &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times A_s v_\perp {J_n'}^2 = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times A_s v_\perp {J_n'}^2 \\
&= i\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}}Z_{l,0} + I_{l,0})[(m-2)\Gamma_{cn,m-1,2} + 2m\Gamma_{cn,m,2} + (m+2)\Gamma_{cn,m+1,2}] \\
&\quad + \Gamma_{cn,m,3}\frac{L_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,0} + 2lZ_{l,0} + (l+1)Z_{l+1,0}]\Big\},
\end{aligned}$$

$$
\begin{aligned}
X_{sn31} &= X_{sn13} = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times A_s \frac{n v_\parallel}{\mu_s} J_n^2 = \frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\pi\omega v_\parallel dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times A_s J_n^2 \\
&= c_{s0}\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\pi\omega v_\parallel dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \sum_l\sum_m a_{s,lm}\Big\{ \\
&\quad \frac{\omega - k_\parallel v_\parallel}{\omega} f_{s0z,l}\frac{i}{2L_{sx}}[(m-2)f_{s0x,m-1} + 2m f_{s0x,m} + (m+2)f_{s0x,m+1}] + \\
&\quad \frac{k_\parallel v_\perp}{\omega} f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1} + 2l f_{s0z,l} + (l+1)f_{s0z,l+1}]\Big\} J_n^2 \\
&= \frac{i}{\pi L_{sz}L_{sx}^2}\frac{n\omega_{cs}}{k_\perp}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{v_\parallel dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \sum_l\sum_m a_{s,lm}\Big\{ \\
&\quad (\omega - k_\parallel v_\parallel) f_{s0z,l}\frac{1}{L_{sx}}[(m-2)f_{s0x,m-1} + 2m f_{s0x,m} + (m+2)f_{s0x,m+1}] + \\
&\quad k_\parallel v_\perp f_{s0x,m}\frac{1}{L_{sz}}[(l-1)f_{s0z,l-1} + 2l f_{s0z,l} + (l+1)f_{s0z,l+1}]\Big\} J_n^2 \\
&= i\frac{L_{sz}}{L_{sx}}\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}} Z_{l,1} + I_{l,1})[(m-2)\Gamma_{an,m-1,0} + 2m\Gamma_{an,m,0} + (m+2)\Gamma_{an,m+1,0}] \\
&\quad + \Gamma_{an,m,1}\frac{L_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,1} + 2l Z_{l,1} + (l+1)Z_{l+1,1}]\Big\} \\
&\to i\frac{n\omega_{cs}}{k_\perp L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sx}} Z_{l,1}[(m-2)\Gamma_{an,m-1,0} + 2m\Gamma_{an,m,0} + (m+2)\Gamma_{an,m+1,0}] \\
&\quad + \Gamma_{an,m,1}\frac{L_{sx}}{L_{sz}}[(l-1)Z_{l-1,1} + 2l Z_{l,1} + (l+1)Z_{l+1,1}]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
X_{sn32} &= -X_{sn23} = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times i A_s v_\parallel J_n J_n' = i\int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times A_s J_n J_n' \\
&= i\int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} c_{s0}\sum_l\sum_m a_{s,lm}\Big\{ \\
&\quad \frac{\omega - k_\parallel v_\parallel}{\omega} f_{s0z,l}\frac{i}{2L_{sx}}[(m-2)f_{s0x,m-1} + 2m f_{s0x,m} + (m+2)f_{s0x,m+1}] \\
&\quad + \frac{k_\parallel v_\perp}{\omega} f_{s0x,m}\frac{i}{2L_{sz}}[(l-1)f_{s0z,l-1} + 2l f_{s0z,l} + (l+1)f_{s0z,l+1}]\Big\} J_n J_n' \\
&= -\frac{1}{\pi L_{sz}L_{sx}^2}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{v_\parallel v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \sum_l\sum_m a_{s,lm}\Big\{ \\
&\quad (\omega - k_\parallel v_\parallel) f_{s0z,l}\frac{1}{L_{sx}}[(m-2)f_{s0x,m-1} + 2m f_{s0x,m} + (m+2)f_{s0x,m+1}] \\
&\quad + k_\parallel v_\perp f_{s0x,m}\frac{1}{L_{sz}}[(l-1)f_{s0z,l-1} + 2l f_{s0z,l} + (l+1)f_{s0z,l+1}]\Big\} J_n J_n' \\
&= -\frac{L_{sz}}{L_{sx}}\sum_l\sum_m a_{s,lm}\Big\{(\frac{n\omega_{cs}}{k_\parallel L_{sz}} Z_{l,1} + I_{l,1})[(m-2)\Gamma_{bn,m-1,1} + 2m\Gamma_{bn,m,1} + (m+2)\Gamma_{bn,m+1,1}] \\
&\quad + \Gamma_{bn,m,2}\frac{L_{sx}^2}{L_{sz}^2}[(l-1)Z_{l-1,1} + 2l Z_{l,1} + (l+1)Z_{l+1,1}]\Big\} \\
&\to -\sum_l\sum_m a_{s,lm}\Big\{\frac{n\omega_{cs}}{k_\parallel L_{sx}} Z_{l,1}[(m-2)\Gamma_{bn,m-1,1} + 2m\Gamma_{bn,m,1} + (m+2)\Gamma_{bn,m+1,1}] \\
&\quad + \Gamma_{bn,m,2}\frac{L_{sx}}{L_{sz}}[(l-1)Z_{l-1,1} + 2l Z_{l,1} + (l+1)Z_{l+1,1}]\Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
X_{sn33} &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times B_s v_\parallel J_n^2 = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \times B_s J_n^2 \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{2\pi\omega v_\parallel v_\perp dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} c_{s0} \sum_l \sum_m a_{s,lm} \Big\{ \\
&\quad \frac{n\omega_{cs} v_\parallel}{\omega v_\perp} f_{s0z,l} \frac{i}{2L_{sx}} [(m-2) f_{s0x,m-1} + 2m f_{s0x,m} + (m+2) f_{s0x,m+1}] \\
&\quad + \frac{\omega - n\omega_{cs}}{\omega} f_{s0x,m} \frac{i}{2L_{sz}} [(l-1) f_{s0z,l-1} + 2l f_{s0z,l} + (l+1) f_{s0z,l+1}] \Big\} J_n^2 \\
&= \frac{i}{\pi L_{sz} L_{sx}^2} \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{dv_\perp dv_\parallel}{\omega - k_\parallel v_\parallel - n\omega_{cs}} \sum_l \sum_m a_{s,lm} \Big\{ \\
&\quad n\omega_{cs} v_\parallel^2 f_{s0z,l} \frac{1}{L_{sx}} [(m-2) f_{s0x,m-1} + 2m f_{s0x,m} + (m+2) f_{s0x,m+1}] \\
&\quad + v_\parallel v_\perp (\omega - n\omega_{cs}) f_{s0x,m} \frac{1}{L_{sz}} [(l-1) f_{s0z,l-1} + 2l f_{s0z,l} + (l+1) f_{s0z,l+1}] \Big\} J_n^2 \\
&= i\frac{L_{sz}^2}{L_{sx}^2} \sum_l \sum_m a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_\parallel L_{sz}} Z_{l,2} [(m-2)\Gamma_{an,m-1,0} + 2m\Gamma_{1n,m,0} + (m+2)\Gamma_{1n,m+1,0}] \\
&\quad + \Gamma_{an,m,1} \frac{L_{sx}^2}{L_{sz}^2} [(l-1) Z_{l-1,2} + 2l Z_{l,2} + (l+1) Z_{l+1,2} + (l-1) I_{l-1,1} + 2l I_{l,1} + (l+1) I_{l+1,1}] \Big\} \\
&= i\frac{L_{sz}^2}{L_{sx}^2} \sum_l \sum_m a_{s,lm} \Big\{ \frac{n\omega_{cs}}{k_\parallel L_{sz}} Z_{l,2} [(m-2)\Gamma_{an,m-1,0} + 2m\Gamma_{1n,m,0} + (m+2)\Gamma_{1n,m+1,0}] \\
&\quad + \Gamma_{an,m,1} \frac{L_{sx}^2}{L_{sz}^2} [(l-1) Z_{l-1,2} + 2l Z_{l,2} + (l+1) Z_{l+1,2} + 2i\delta_{l,0}] \Big\},
\end{aligned}
$$

where

$$
\boldsymbol{Q} = \sum_s \boldsymbol{Q}_s, \quad \boldsymbol{Q}_s = \frac{\omega_{ps}^2}{\omega^2} \boldsymbol{P}_s, \quad \boldsymbol{P}_s = \sum_n \boldsymbol{P}_{sn}, \quad \boldsymbol{P}_{sn} = c_{s0} \boldsymbol{X}_{sn}. \tag{102}
$$

We have used $(l-1)I_{l-1,1} + 2lI_{l,1} + (l+1)I_{l+1,1} = 2i\delta_{l,0}$.

Here, '$\rightarrow$' means the constant terms I_l has been summation out for n. We have used $\sum_{n=-\infty}^{\infty} J_n J_n' = 0$, $\sum_{n=-\infty}^{\infty} n J_n^2 = 0$, $\sum_{n=-\infty}^{\infty} n J_n J_n' = 0$, $\sum_{n=-\infty}^{\infty} J_n^2 = 1$, $\sum_{n=-\infty}^{\infty} (J_n')^2 = \frac{1}{2}$ and $\sum_{n=-\infty}^{\infty} \frac{n^2 J_n^2}{x^2} = \frac{1}{2}$.

After the above expansion, the dispersion relation Eq.(1) can be readily solved use conventional iterative root finding method. Since the perpendicular integral $\Gamma(a_s)$ need only calculate once, and the parallel integrals $Z(\zeta_{sn})$ are finite rational form, this approach can obtain high accurate solution with less computation cost. The above dispersion relation can also be transformed to matrix method similar to the κ distribution case[14, 13]. We omit the transformation here since it is slightly complicated.

Last update: Saturday 11th January, 2025 15:39.

References

[1] H. S. Xie, Generalized plasma dispersion function: One-solve-all treatment, visualizations, and application to Landau damping, Phys. Plasmas 20, 092125 (2013).

[2] H. S. Xie and Y. Xiao, PDRK: A General Kinetic Dispersion Relation Solver for Magnetized Plasma, Plasma Science and Technology, 18, 2, 97 (2016). DOI: 10.1088/1009-0630/18/2/01. Update/bugs fixed at http://hsxie.me/codes/pdrk/ or https://github.com/hsxie/pdrk/.

[3] D. A. Gurnett and A. Bhattacharjee, Introduction to plasma physics: with space and laboratory applications, Cambridge, 2005.

[4] H. S. Xie, PDRF: A general dispersion relation solver for magnetized multi-fluid plasma, Computer Physics Communications, 185, 670–675 (2014).

[5] Daniel Verscharen, Kristopher G. Klein, Benjamin D. G. Chandran, Michael L. Stevens, Chadi S. Salem and Stuart D. Bale, ALPS: The Arbitrary Linear Plasma Solver, J. Plasma Phys., 84 (2018) 905840403.

[6] D. Summers, S.Xue and Richard M. Thorne, Calculation of the dielectric tensor for a generalized Lorentzian (kappa) distribution function, Phys. Plasmas, 1, 2012 (1994).

[7] Y. Matsuda and G. R. Smith, A microinstability code for a uniform magnetized plasma with an arbitrary distribution function, Journal of Computational Physics, 1992, 100, 229-235.

[8] P. Hellinger and P. M. Travnicek, Proton core-beam system in the expanding solar wind: Hybrid simulations, Journal of Geophysical Research: Space Physics, 2011, 116, A11101 .

[9] P. Astfalk and F. Jenko, LEOPARD: A grid-based dispersion relation solver for arbitrary gyrotropic distributions, J. Geophys. Res. Space Physics, 122, 89-101 (2017).

[10] S. W. A. Irvine, Collective instability and physics of the anomalous Doppler resonance in fusion plasmas, University of Warwick, PhD thesis, 2018.

[11] J. A. C. Weideman, Computation of the Complex Error Function, SIAM Journal on Numerical Analysis, 1994, 31, 1497-1518.

[12] H.S. Xie, BO: A unified tool for plasma waves and instabilities analysis, Comput. Phys. Comm. 244 (2019) 343-371; Xie, H. S., Denton, R., Zhao, J. S. and Liu, W, BO 2.0: Plasma Wave and Instability Analysis with Enhanced Polarization Calculations arXiv:2103.16014, 2021. https://github.com/hsxie/bo/.

[13] W. Bai, H. S. Xie, C. C. Wu, Y.X. Pu, P. C. Yu, BO-KM: A comprehensive solver for dispersion relation of obliquely propagating waves in magnetized multi-species plasma with anisotropic drift kappa-Maxwellian distribution, Computer Physics Communications 307 (2025) 109434.

[14] W. Bai and H. S. Xie, Toward developing a comprehensive algorithm for solving kinetic plasma dispersion relations for parallel propagation with a kappa distribution, Phys. Plasmas 30, 043903 (2023).

[15] P. A. Robinson, Systematic methods for calculation of the dielectric properties of arbitrary plasmas. J. Comput. Phys. 88, 381–392 (1990).

[16] J. A. C. Weideman, Computing the Hilbert transform on the real line. Math. Comput. 64, 745–762 (1995).

[17] H. S. Xie, Rapid computation of the plasma dispersion function: Rational and multi-pole approximation, and improved accuracy, AIP Advances 14, 075007 (2024).

[18] M. Brambilla and R. Bilato, An efficient interface between the solvers of Maxwell's equations and of the quasilinear kinetic equation for simulations of ion cyclotron heating in tokamak plasmas, Computer Physics Communications 184 (2013) 2053–2060.

[19] R. Bilato, M. Brambilla and Z. Jiang, Implementing zero-banana-width quasilinear operator for fast ICRF simulations, Journal of Physics: Conference Series 401 (2012) 012001.