Chapter 2. Transformations & Expectations

2.1 Distributions of functions of a rand. variable

Given $X \sim F_X(\cdot)$ and $g: \mathcal{X} \mapsto \mathcal{Y}$, $Y = g(X) \sim ?$

$$\boxed{\begin{aligned} P_Y(A) = P(Y \in A) &= P(\{s \mid Y(s) \in A\}) = P(\{s \mid g(X(s)) \in A\}) \\ &= P(\{s \mid X(s) \in g^{-1}(A)\}) \end{aligned}}$$

(e.g.) $X \sim B(n, p)$

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n$$

$$g(x) = n - X$$

$Y = g(X) = n - X$ (= # of failures) $\sim ?$

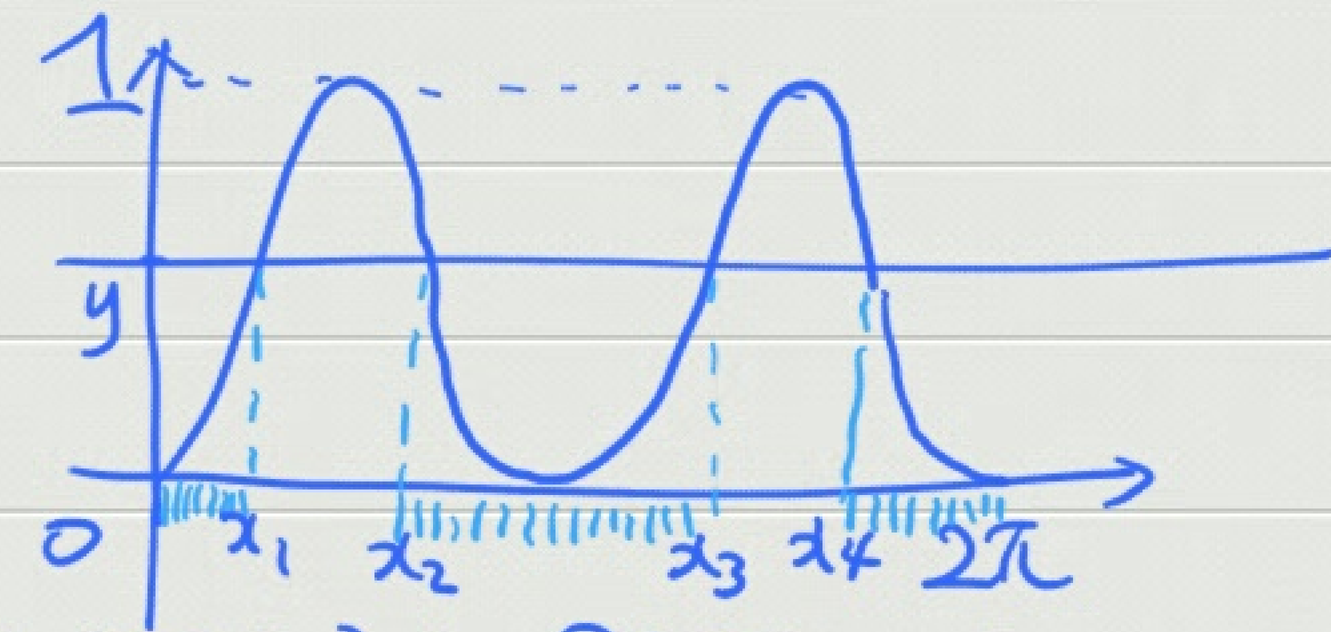
<u>sol.</u> $\mathcal{X} = \{0, 1, 2, \cdots, n\} \xrightarrow{g} \mathcal{Y} = \{0, 1, 2, \cdots, n\}$

$$f_Y(y) = P(Y = y) = P(n - X = y) = P(X = n - y) = \binom{n}{n-y} p^{n-y} (1-p)^y : B(n, p')$$

$= \binom{n}{y}$, p', $1-p$

(e.g.) $X \sim \text{Uniform}(0, 2\pi)$

$Y = \sin^2 X \sim ?$



sol. $\mathcal{X} = [0, 2\pi]$, $\mathcal{Y} = [0, 1]$

$$P(Y \le y) = P(X \le x_1) + P(x_2 \le X \le x_3) + P(X \ge x_4)$$

where x_1, x_2, x_3 and x_4 are the solutions to $\sin^2 x = y$. //

Thm: $X \sim F_X(\cdot)$, $Y = g(X)$ with $g : \mathcal{X} \mapsto \mathcal{Y}$

a. If g is $\uparrow$ on $\mathcal{X}$, then $F_Y(y) = F_X(g^{-1}(y))$, $y \in \mathcal{Y}$

b. If g is $\downarrow$ on $\mathcal{X}$, then $F_Y(y) = 1 - F_X(g^{-1}(y))$, $y \in \mathcal{Y}$

(e.g.) $X \sim \mathcal{U}niform(0,1)$. Then $F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$.

$Y = g(X) = -\log X \sim ?$

sol. $\mathcal{X} = [0,1]$, $\mathcal{Y} = (0, \infty)$.

$F_X(x) = x$, $0 \le x \le 1$

$\forall y \in \mathcal{Y}$, $F_Y(y) \underset{(\because\, g : \downarrow\ f^{tn})}{=} 1 - F_X(g^{-1}(y)) = 1 - F_X(e^{-y}) = \overbrace{1 - e^{-y}}^{\text{Exponential}}$. //

(or $y > 0$)

$\underline{\text{Thm}}$: $X \sim f_X(\cdot)$, $Y = g(X)$ with g : monotone, $\mathcal{X} \mapsto \mathcal{Y}$.

then $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$, $y \in \mathcal{Y}$.

pf By the chain rule... //

(e.g.) $X \sim \text{Gamma}(\alpha, \beta)$, $Y = g(X) = 1/X \sim$?

$\underline{\text{sol}}$. $g^{-1}(y) = 1/y$. $\frac{d}{dy} g^{-1}(y) = -1/y^2$.

$$f_Y(y) = f_X(1/y) \left| -1/y^2 \right|$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{1}{y}\right)^{\alpha-1} e^{-1/(\beta y)} \cdot \frac{1}{y^2} = \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{1}{y}\right)^{\alpha+1} e^{-1/(\beta y)}, \quad y > 0 \text{ //}$$

(e.g.) $X \sim N(0,1)$, $Y = X^2 \sim$?

sol. $g(x) = x^2$ is not monotone on $\mathbb{R}$, but monotone on $(-\infty, 0)$ and $(0, \infty)$.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2} \left| -\frac{1}{2\sqrt{y}} \right| + \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \left| \frac{1}{2\sqrt{y}} \right|$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y}} e^{-y/2}, \quad y > 0. \quad //$$

χ_1^2

$\underline{\text{Th}^{m}}$: $X \sim F_X(\cdot)$, conti.

$\Rightarrow Y = F_X(X) \sim \text{Uniform}(0,1)$.

pf: $P(Y \le y) = P(F_X(X) \le y) = P(X \le F_X^{-1}(y)) = F_X(F_X^{-1}(y))$

$= y.$ //

(e.g.) $X \sim \mathcal{E}xp(1)$. $\quad f_X(x) = e^{-x}$, $x > 0$.

$F_X(x) = 1 - e^{-x}$

$Y = 1 - F_X(X) = 1 - e^{-X} \sim \text{Uniform}(0,1)$. //

2.2 Expected values

Def $E\,g(X) = \begin{cases} \sum_{x \in \mathcal{X}} g(x) f(x) \\ \int_{\mathcal{X}} g(x) f(x)\, dx \end{cases}$ (the expectation or the mean)

(e.g.) $X \sim$ Exponential (λ)

$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x > 0$

$E(X) = \int_0^\infty x \cdot \frac{1}{\lambda} e^{-x/\lambda}\, dx = \lambda.$ //

(e.g.) $X \sim B(n, p)$

$E(X) = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \times \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$

$= \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-1-(x-1))!} p^{x-1} (1-p)^{n-1-(x-1)} \times np = \sum_{y=0}^{n-1} B(n-1, p) \times np = np$ //

(e.g.) $X \sim \text{Cauchy}(0,1)$

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi(1+x^2)}\, dx = \infty. \quad //$$

$\underline{\text{Th}^m}$: (see Th^m 2.2.5 in page 57).

pf: (HW)

(e.g.) $\min_a E(X-a)^2$

$\underline{\text{sol}}$. $E(X-a)^2 = EX^2 - 2aE(X) + a^2$

$$\frac{d}{da} E(X-a)^2 = -2E(X) + 2a = 0 \qquad \therefore\ a^* = \arg\min_a E(X-a)^2 = E(X). \quad //$$

2.3 Moments and moment generating functions

Defn. $\mu_n' = EX^n$ (n-th moment of a r.v. X)

$\mu_n = E(X-\mu)^n$ (n-th central moment of a r.v. X)

where $\mu = \mu_1' = E(X)$.

Defn $\mathrm{Var}(X) = E(X - EX)^2$. (variance of a r.v. X)

Thm: $\mathrm{Var}(aX+b) = a\,\mathrm{Var}(X)$.

Pf (HW).

Defn: $M_X(t) = E e^{tX}$ (the moment generating function of a r.v. X).
provided that the expectation exists for t in some nbhd. of 0.
"neighborhood"

Thm: $EX^n = M_X^{(n)}(0)$?

Pf: $M_X'(t) = \frac{d}{dt}\int_x e^{tx} f_X(x)\,dx = \int_x \frac{d}{dt} e^{tx} f_X(x)\,dx = \int_x e^{tx} \times x f_X(x)\,dx$

$M_X'(0) = \int_x e^{0\times x} \times x f_X(x)\,dx = \int_x x f_X(x)\,dx = \mu_1'$.

$M_X^{(n)}(t) = \frac{d^n}{dt^n}\int_x e^{tx} f_X(x)\,dx = \int_x \frac{d^n}{dt^n} e^{tx} f_X(x)\,dx = \int_x e^{tx} \times x^n f_X(x)\,dx$

$M_X^{(n)}(0) = \int_x e^{0\times x} \times x^n f_X(x)\,dx = \int_x x^n f_X(x)\,dx = \mu_n'$.

(e.g.) $X \sim f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$, $x > 0$

- $M_X(t) = Ee^{tX} = \int_0^\infty e^{tx} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-\underbrace{(1/\beta - t)}_{>0} x}\, dx \quad \text{i.e. } t < 1/\beta$$

$$= \frac{\Gamma(\alpha)(\frac{1}{\beta} - t)^{-\alpha}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \text{Gamma}(\alpha, (1/\beta - t)^{-1})\, dx$$

$$= (1 - \beta t)^{-\alpha}, \quad t < 1/\beta.$$

- $E(X) = M_X'(t)\Big|_{t=0} = -\alpha(1-\beta t)^{-\alpha-1} \cdot (-\beta)\Big|_{t=0} = \alpha\beta$ //

(e.g.) $B(n,p)$: $M_X(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$

$$= (pe^t + 1-p)^n \,//$$

$\boxed{\text{· nonuniqueness of moments : (see Example 2.3.10 in page 64)}}$

(i) Bounded support $\longrightarrow$ the moments uniquely determines the distribution.

(ii) mgf exists in a nbhd. of 0 $\longrightarrow$ the moments uniquely determines the distn
no matter what its support.

$\underline{Th^m}$: $X_1, X_2, \cdots$, a seq. of r.v.'s, each with mgf $M_1(t), M_2(t), \cdots$

$\lim_{i\to\infty} M_i(t) = M(t)$ for all $t \in$ nbhd of 0 and $M(t)$ is an mgf

$\Rightarrow \exists 1$ cdf F_X corresponding to $M(\cdot)$ s.t. $F_{X_i}(x) \xrightarrow{i\uparrow\infty} F_X(x)$.

$\underline{Th^m}$: $M_{aX+b}(t) = e^{bt} M_X(at)$

pf.) trivial.

(e.g.) $X \sim B(n, p)$, n is very large and p is small ...

np is still small, but $np \approx \lambda$.

$$M_X(t) = (pe^t + 1 - p)^n \underset{(p \approx \lambda/n)}{\approx} \left(\lambda(e^t - 1)/n + 1\right)^n \approx e^{\lambda(e^t - 1)} : \text{Poisson}(\lambda)$$

(Poisson approx^n.)

c.f. (Cumulative generating function)

$\log M_X(t)$

c.f. (characteristic function)

$E e^{itX}$