On the ordering of credibility factors

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Outline

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Introduction

- Consider the following forecasting problem in the longitudinal setting:

$$\mathbb{E}\left[N_{T+1} \,|\, N_1, \cdots, N_T\right].$$

- Goal: This paper is about the linear approximation of the forecasting

$$\mathbb{E}\left[N_{T+1} \,|\, N_1, \cdots, N_T\right] \approx \alpha_0 + \alpha_1 N_1 + \cdots + \alpha_T N_t$$

  and its interpretation.
- Underlying model: Dynamic random effect model.
- Under certain condition, we show that credibility factors are ordered:

$$\alpha_1 < \cdots < \alpha_T.$$

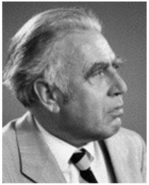
Introduction: Notation

For a policyholder,

- N_t : Frequency, the number of claims in the t-th policy year.
- $\overrightarrow{N}_{1:t}$: the vector of claim history upto time t.
- λ: priori rate. ($\lambda = \exp(X\beta)$)
- R: random effect, or risk characteristics not included in λ.

Homepage > ETH Zurich > Organisation > Who is who > Retired Professors > Details

Bühlmann, Hans, Prof. Dr.



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Additional information

Curriculum Vitae

Hans Bühlmann, born 1930, was Full Professor for Mathematics at ETH Zurich, starting 1966. From 1973 to 1977 he was Head of the Research Committee at the ETH Zurich. From 1981 to 1985 he was Chairman of the Department of Mathematics, from 1986 to 1987 he was Dean of the Faculty of Mathematics and Physics. From 1987 to 1990 he was President of the ETH Zurich an Vice-president of the board of ETH. He retired in 1997.

Introduction: Frequency Random Effect Model

- Model

$$N_t \middle| R, \lambda \overset{\text{i.i.d.}}{\sim} \text{Pois}\left(\lambda R\right)$$

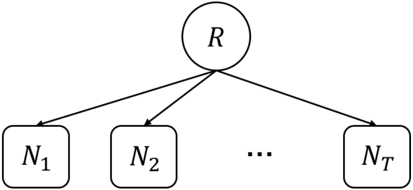
- Random Effect

$$R \sim \text{Gamma}(1, \psi)$$

- Prediction at time $T+1$

$$\mathbb{E}\left[N_{T+1} \middle| \overrightarrow{N}_{1:T},\ \lambda\right] = \lambda\,\mathbb{E}\left[R \middle| \overrightarrow{N}_{1:T},\ \lambda\right] = \lambda\,\frac{1/\psi + N_\Sigma}{1/\psi + \lambda T}$$

: i.e. prediction of random effect.



Bühlmann credibility: historical frequency

- This is the classical example of the Bühlmann credibility.
- Bühlmann premium based on $\mathcal{F}_t$:

$$\mathrm{Prem}_1 := \alpha_0\lambda + \alpha_1 N_1 + \cdots + \alpha_t N_t,$$

where

$$(\alpha_0, \cdots, \alpha_t) := \operatorname*{arg\,min}_{\alpha_0,\cdots,\alpha_t} \mathbb{E}\left[(N_{t+1} - (\alpha_0\lambda + \alpha_1 N_1 + \cdots + \alpha_t N_t))^2\right].$$

- It is easy to show that

$$\alpha_0 + \cdots + \alpha_t = 1.$$

- Prem_1 can be written as

$$\mathrm{Prem}_1 = Z_t\overline{N_t} + (1 - Z_t)\lambda$$

where the credibility Z_t can be analytically derived in our case.

Introduction: Frequency Random Effect Model

- The random effect model assumes static hidden effect.
- Past claims have the same weight in the prediction, regardless of their seniority (Li et al., 2020).
- $N_1, \cdots, N_T$ have same contribution to predict.
- e.g)

$$\mathbb{E}\left[N_4 \middle| N_1 = 5, N_2 = 0, N_3 = 0\right]$$

$$= \mathbb{E}\left[N_4 \middle| N_1 = 0, N_2 = 0, N_3 = 5\right]$$

Introduction: State Space Model

- Model

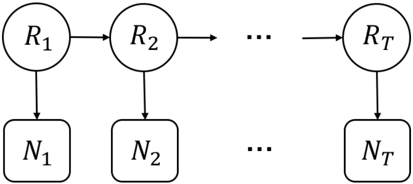
$$N_t \big| R_t^*, X \overset{\text{i.i.d.}}{\sim} \text{Pois}\left(\lambda R_t^*\right), \quad R_t^* = \exp(R_t)$$

- State space

$$R_t | R_{t-1} \sim \text{Normal}(\phi R_{t-1}, \psi)$$
$$R_0 \sim \text{N}(0, \psi/(1-\phi^2))$$

- Prediction at time $T+1$

$$\mathbb{E}\left[N_{T+1} \big| \overrightarrow{N}_{1:T}, X\right] = \lambda\, \mathbb{E}\left[R_{T+1}^* \big| \overrightarrow{N}_{1:T}, X\right]$$



Introduction: State Space Model

- Prediction at time $T+1$

$$\mathbb{E}\left[N_{T+1}\middle|\overrightarrow{N}_{1:T}, X\right]$$
$$= \lambda\,\mathbb{E}\left[R^*_{T+1}\middle|\overrightarrow{N}_{1:T}, X\right]$$
$$= \lambda \int\int R^*_{T+1} f(R_{T+1}|R_T, \overrightarrow{N}_{1:T}, X) f(R^*_T|\overrightarrow{N}_{1:T}, X) dR_T dR_{T+1}$$

- Filtering methods
  - Kalmann filter - for Gaussian linear model (dynamic linear model, DLM)
  - Particle filter, MCMC - for non-Gaussian, non-linear model.
- Gaussian SSM can obtain analytical solution but in general need simulation method.

Bühlmann credibility: historical frequency

- Bühlmann premium based on $\mathcal{F}_t$:

$$\mathrm{Prem}_2 := \alpha_0\lambda + \alpha_1 N_1 + \cdots + \alpha_t N_t,$$

where

$$(\alpha_0, \cdots, \alpha_t) := \underset{\alpha_0,\cdots,\alpha_t}{\arg\min}\, \mathbb{E}\left[\left(N_{t+1} - (\alpha_0\lambda + \alpha_1 N_1 + \cdots + \alpha_t N_t)\right)^2\right].$$

and

$$(\alpha_1, \cdots, \alpha_t)' = \mathbf{\Sigma}_t^{-1} \begin{pmatrix} \mathrm{cov}\left[N_1, N_{t+1}\right] \\ \mathrm{cov}\left[N_2, N_{t+1}\right] \\ \vdots \\ \mathrm{cov}\left[N_t, N_{t+1}\right] \end{pmatrix} \tag{1}$$

and

$$\widehat{\alpha}_0 = \lambda(1 - \sum_{z=1}^{t} \widehat{\alpha}_z) \tag{2}$$

where $\mathbf{\Sigma}_t$ is a covariance matrix of $(N_1, \cdots, N_t)$.

Question

- Under SSM, we clearly have the following **positive ordering of observations**:

$$\mathrm{cov}\left[N_{t+1}, N_1\right] < \mathrm{cov}\left[N_{t+1}, N_2\right] < \cdots < \mathrm{cov}\left[N_{t+1}, N_t\right].$$

- Natural question: Does the ordering of observations implies the **positive ordering of the credibility factor**:

$$\alpha_1 < \cdots < \alpha_t?$$

- Related literature: Does the positive covariance of observations implies the positive credibility factors (Pinquet, 2020)?
  - There is example where

$$\mathrm{cov}\left[N_{t+1}, N_z\right] > 0, \quad \text{for} \quad \text{all} \quad z = 1, \cdots, t$$

    but

$$\alpha_z < 0, \quad \text{for some} \quad z.$$

  - Issue of negative credibility factor.

Motivative example: INAR(1) model

INAR(1) model (McKenzie, 1985)

- INAR(1) is iteratively defined by

$$N_t = p \circ N_{t-1} + \epsilon_t$$

where $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{Pois}(\lambda)$.

- If $N_1 \sim \mathrm{Pois}\left(\lambda/(1-p)\right)$, then $N_t \sim \mathrm{Pois}\left(\lambda/(1-p)\right)$.
- Here, $p \circ N$ is a random variable distributed by $\mathrm{BN}(N, p)$.
- Ordering of covariance:

$$\operatorname{cov}\left[N_{t_1}, N_{t_2}\right] = \frac{p^{|t_1 - t_2|}}{1-p}\lambda.$$

Motivative example: heterogeneous INAR(1) model

Heterogeneous INAR(1) model (Gourieroux and Jasiak, 2004).

- In homogeneous INAR(1), we additionally use the random effect R.
- Conditional on R, we have the following iteration

$$N_t = p \circ N_{t-1} + \epsilon_t$$

where $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{Pois}(\lambda R)$.

- If $N_1 \sim \mathrm{Pois}\left(\lambda R/(1-p)\right)$, then $N_t \sim \mathrm{Pois}\left(\lambda R/(1-p)\right)$.
- Assume $R \sim \mathrm{Gamma}(1, \psi)$
- Ordering of covariance:

$$\mathrm{cov}\left[N_{t_1}, N_{t_2}\right] = \frac{\lambda}{1-p}\left(p^{|t_1 - t_2|} + \frac{\lambda}{1-p}\psi\right).$$

- Complicated but straightforward calculation shows

$$\alpha_1 > \alpha_j, \quad j = 2, \cdots, t-1.$$

Motivative example: conclusion

- It is obvious that the ordering of covariance does not guarantee the ordering of the credibility factors.
- Question: what is the models or conditions which guarantee the ordering of the credibility factors?
- We provide the model which guarantees the ordering of the credibility factors.

- A process $(R_t : t = 0, 1, \cdots)$ is said to have AR(1)-type autocorrelation if
  - Time invariant variance

$$\mathbb{V}ar\,(R_t) = \sigma^2, \quad t = 0, 1, \cdots .$$

  - Autocovariance function

$$\operatorname{cov}\left[N_{t_1}, N_{t_2}\right] = \sigma^2 \rho^{|t_1 - t_2|}.$$

- AR(1)-type autocorrelation can be characterized (Grunwald et al., 2000) by

$$\mathbb{E}\left[R_t | R_{t-1}\right] = \phi_1 R_{t-1} + \phi_2.$$

Example of AR(1)-type autocorrelation

- Gaussian AR(1) process.
- Beta-Gamma autoregressive process (Lewis et al., 1989):

$$R_t = B_t R_{t-1} + G_t$$

  with
  - $R_0 \sim \text{Gamma}\left(\frac{\gamma_1}{\gamma_2}, \frac{1}{\gamma_1}\right)$.
  - $B_t \sim \text{Beta}(r_1 \rho, r_1(1-\rho))$.
  - $G_t \sim \text{Gamma}\left(\frac{\gamma_1(1-\rho)}{\gamma_2}, \frac{1}{\gamma_1(1-\rho)}\right)$.
- The autoregressive gamma process (Lu, 2018)
- Gamma autoregressive process (Gaver and Lewis, 1980)

Poisson-AR(1) state-space model

Model

Poison-AR(1) state-space model with a state process $(R_t : t = 0, 1, \cdots)$ and observable time series $(N_t : t = 1, 2, \cdots)$

- *Conditional on R_t,*
$$N_t | R_t \sim \mathrm{Pois}(\lambda R_t).$$
- *A state process $(R_t : t = 0, 1, \cdots)$ has AR(1)-type autocorrelation.*

Main result

Theorem

Under Poisson-AR(1) state-space model, the credibility factors are ordered in a timely manner:

$$\alpha_1 < \cdots < \alpha_T.$$

Model

General AR(1) state-space model with a state process $(R_t : t = 0, 1, \cdots)$ *and observable time series* $(N_t : t = 1, 2, \cdots)$

- *Conditional on* R_t*,*

$$N_t | R_t \sim \mathrm{ED}(\lambda R_t, \psi).$$

- *A state process* $(R_t : t = 0, 1, \cdots)$ *has AR(1)-type autocorrelation.*

Theorem

Under general AR(1) state-space model, the credibility factors are ordered in a timely manner:

$$\alpha_1 < \cdots < \alpha_T.$$

Data Analysis: Data set

- We use a data set for collision coverage for old and new vehicle from the Wisconsin Local Government Property Insurance Fund (LGPIF) (Frees et al., 2016)
- The data is longitudinal from 2006 to 2010.

Categorical variables	Description	Proportions
Entity type	Type of local government entity	
	Miscellaneous	5.03%
	City	9.66%
	County	11.47%
	School	36.42%
	Town	16.90%
	Village	20.52%
Coverage	Collision coverage amount for old and new vehicles.	
	Coverage $\in (0, \quad 0.14] = 1$	33.40%
	Coverage $\in (0.14, 0.74] = 2$	33.20%
	Coverage $\in (0.74, \quad \infty] = 3$	33.40%

- Assume Poisson-AR(1) state-space model.

Table: Root mean squared prediction error (RMSPE)

	Estimate	Std. err	p-value
(Intercept)	-4.0315	0.3204	0.0000
TypeCity	0.9437	0.1907	0.0000
TypeCounty	1.7300	0.1993	0.0000
TypeMisc	-2.7326	1.0149	0.0071
TypeSchool	-0.9172	0.2776	0.0010
TypeTown	-0.3960	0.2772	0.1531
CoverageIM	0.0664	0.0074	0.0000
lnDeductIM	0.1353	0.0458	0.0031
NoClaimCreditIM	-0.3690	0.1313	0.0050

$\widehat{\rho} = 0.8831$ and $\widehat{\gamma} = 0.8360$.

Result

Table: Validation measures for the frequency models

	No RE (GLM)	Static RE (GLMM)	Dynamic RE (Dynamic GLMM)	Exact (Known RE)
RMSE	0.6439	0.5002	0.4263	0.4661
MAE	0.1220	0.1121	0.1046	0.1060

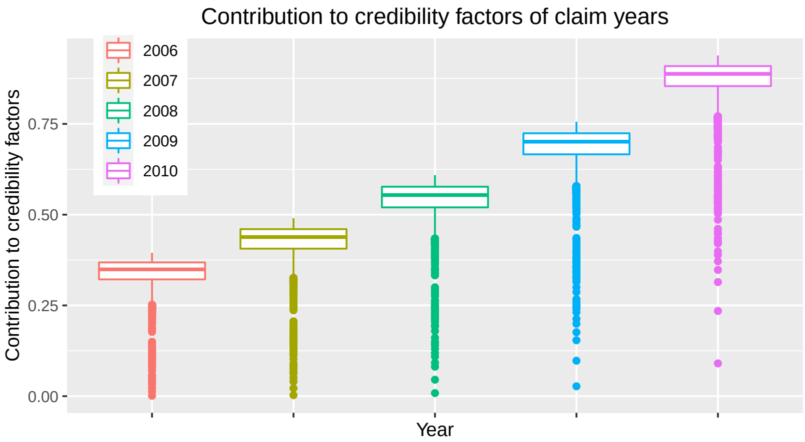


Figure: Contribution to credibility factors of past claim frequencies

Conclusion

- We consider the issue of seniority of claims.
- The ordering of covariance matrix does not guarantees the ordering of credibility factors.
- The state-space model having the state space with AR(1) correlation structure guarantees the ordering of credibility factors.
- The wider class of state space model other than the state space with AR(1) correlation structure?
- Immediate application in machine learning: credibility where credibility factors are constructed with the monotone increasing RNN.

Thank You.

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