10월 14일 과제

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1. 다음의 시계열 7, 6, 5, 8, 9, 4, 5, 5, 4, 6, 7, 8, 5, 6, 5 이 주어졌을 때, SACF $\hat{\rho}_h (h=1,2,3)$ 을 직접 구하시오.

$1 \ 0 \ -1 \ 2 \ 3 \ -2 \ -1 \ -1 \ -2 \ 0 \ 1 \ 2 \ -1 \ 0 \ -1$

solve) $\bar{Z} = \frac{1}{n}\sum_{t=1}^{n} Z_t = \frac{1}{15}(7+6+5+8+9+4+5+5+4+6+7+8+5+6+5) = 6.$

$$\hat{\rho}_h = \frac{\hat{\gamma}_h}{\hat{\gamma}_0} = \frac{\sum_{t=1}^{n-h}(Z_t-\bar{Z})(Z_{t+h}-\bar{Z})/n}{\sum_{t=1}^{n}(Z_t-\bar{Z})^2/n}$$

$* \sum_{t=1}^{n}(Z_t-\bar{Z})^2 = (7-6)^2+(6-6)^2+\cdots+(5-6)^2 = 32 \Rightarrow \hat{\gamma}_0 = \frac{32}{15}$

$\hat{\gamma}_1 = \frac{1}{15}\{(7-6)(6-6)+(6-6)(5-6)+\cdots+(6-6)(5-6)\} =$

$\frac{1}{15}(0+0-\cancel{2}+\cancel{6}-\cancel{6}+\cancel{2}+1+2+0+0+\cancel{2}-\cancel{2}+0+0) = \frac{3}{15}$

$\hat{\gamma}_2 = \frac{1}{15}\{(7-6)(5-6)+(6-6)(8-6)+\cdots+(5-6)(5-6)\} =$

$\frac{1}{15}(-1+0-3-\cancel{4}-3+\cancel{2}+\cancel{2}+0-2+0-\cancel{1}+0+\cancel{1}) = \frac{-9}{15}$

-7

$\hat{\gamma}_3 = \frac{1}{15}\{(7-6)(8-6)+(6-6)(9-6)+\cdots+(8-6)(5-6)\}$

$= \frac{1}{15}(2+0+\cancel{2}-\cancel{2}-3+\cancel{4}+0-1-\cancel{4}+0+0-2) = \frac{-4}{15}$

$-1 \quad -2$

$\therefore \hat{\rho}_1 = \frac{3}{32}, \hat{\rho}_2 = -\frac{9}{32}, \hat{\rho}_3 = -\frac{4}{32}$

2. 다음의 모형들에 의해 설명되는 확률과정 $\{Z_t\}$는 정상성을 갖는가? (단 $\varepsilon_t \sim WN(0,1^2)$)

1) $Z_t = \varepsilon_t - \varepsilon_{t-1} - \varepsilon_{t-2}$

① $E(Z_t)=0$, ② $Var(Z_t)=\gamma_0=3$

③ $Cov(Z_t, Z_{t+h}) = \gamma_h = \begin{cases} 0, & h=1 \\ -1, & h=2 \\ 0, & h\ge 3 \end{cases}$

$\Rightarrow$ 평균과 분산 일정, 자기공분산은 시차에만 의존 $\therefore$ 정상성 만족

2) $Z_t = \varepsilon_t \varepsilon_{t-1} + \varepsilon_{t-2}$

① $E(Z_t)=0$, $Var(Z_t) = Var(\varepsilon_t\varepsilon_{t-1}+\varepsilon_{t-2}) = E[(\varepsilon_t\varepsilon_{t-1}+\varepsilon_{t-2})^2]$

$= E[\varepsilon_t^2\varepsilon_{t-1}^2+\varepsilon_{t-2}^2+2\varepsilon_t\varepsilon_{t-1}\varepsilon_{t-2}] = 1\times1+1 = 2$

② $Cov(Z_t, Z_{t+h}) = \gamma_h = \begin{cases} Cov(\varepsilon_t\varepsilon_{t-1}+\varepsilon_{t-2}, \varepsilon_{t+1}\varepsilon_t+\varepsilon_{t-1}), & h=1 \\ Cov(\varepsilon_t\varepsilon_{t-1}+\varepsilon_{t-2}, \varepsilon_{t+2}\varepsilon_{t+1}+\varepsilon_t), & h=2 \\ 0, & h\ge 3 \end{cases}$

$\Rightarrow$ 평균과 분산 일정, 자기공분산은 시간과 시차에 모두 의존

$\therefore$ 정상성 만족 X

3) $Z_t = A\cdot \sin\left(\frac{2}{3}\pi t+U\right)$, $A \sim \cdot(0,1^2)$, U : constant

① $E(Z_t) = E(A)\cdot \sin\left(\frac{2}{3}\pi t + U\right) = 0$

② $Var(Z_t) = \gamma_0 = Var(A)\cdot\left\{\sin\left(\frac{2}{3}\pi t+U\right)\right\}^2 = \left\{\sin\left(\frac{2}{3}\pi t+U\right)\right\}^2 = \frac{1}{2}\left[1-\cos\left(\frac{4}{3}\pi t + 2U\right)\right]$

⇒ 분산이 시간에 따라 변함 ∴ 정상성 만족 X

4) $Z_t = A\sin(\pi t + U)$, $A \sim \cdot(0,1^2)$, $U \sim Uniform(-\pi, \pi)$, $A \perp U$

① $E(Z_t) = E(A)\cdot E[\sin(\pi t + U)] = 0$

② $Var(Z_t) = \gamma_0 = E(Z_t^2) - \{\underbrace{E(Z_t)}_{0}\}^2 = E\left[A^2\cdot\{\sin(\pi t+U)\}^2\right]$

$= \underbrace{E(A^2)}_{Var(A)+\{E(A)\}^2=1}\cdot E\left[\{\sin(\pi t+U)\}^2\right] = 1\cdot\int_{-\pi}^{\pi}\frac{1}{2\pi}\cdot\{\sin(\pi t+u)\}^2du$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\{1-\cos(2\pi t+2u)\}du = \frac{1}{4\pi}\left[u - \frac{1}{2}\sin(2\pi t+2u)\right]_{-\pi}^{\pi}$

$= \frac{1}{4\pi}\left[\left\{\pi - \frac{1}{2}\underbrace{\sin(2\pi t + 2\pi)}_{0}\right\} - \left\{-\pi - \frac{1}{2}\underbrace{\sin(2\pi t - 2\pi)}_{0}\right\}\right] = \frac{1}{2}$

③ $\gamma_h = Cov(Z_t, Z_{t+h}) = Cov\left(A\sin(\pi t+U), A\sin(\pi(t+h)+U)\right)$

↳ 시간과 시차에 모두 영향을 받음 ∴ 정상성 만족 X

5) $\begin{cases} P(Z_t=1) = P(Z_t=-1) = \frac{1}{2} & , t=\text{짝수} \\ Z_t = Z_{t-1} & , t=\text{홀수} \end{cases}$

① $E(Z_t)$ ⇒ If t=짝수 → $E(Z_t) = 1\times\frac{1}{2} - 1\times\frac{1}{2} = 0$
If t=홀수 → $E(Z_t) = E(\underbrace{Z_{t-1}}_{\text{짝수}}) = 0$
∴ $E(Z_t)=0$

② $Var(Z_t) = \gamma_0$ ⇒ If t=짝수 → $Var(Z_t) = 1^2\times\frac{1}{2} + (-1)^2\times\frac{1}{2} - 0 = 1$
If t=홀수 → $Var(Z_t) = Var(\underbrace{Z_{t-1}}_{\text{짝수}}) = 1$
∴ $Var(Z_t)=0$

③ $Cov(Z_t, Z_{t+h})$ ⇒ γ_1을 계산해보면 다음과 같다.

$\gamma_1 = Cov(Z_t, Z_{t+1})$ ⇒ If t=짝수 → $Z_{t+1} = Z_t$ → $Cov(Z_t, Z_t) = 0$
If t=홀수 → $Z_t = Z_{t-1}$ → $Cov(Z_{t-1}, Z_{t+1}) = \gamma_2$

↳ 자기공분산은 시간과 시차에 모두 의존함 ∴ 정상성 만족 X

3. $\varepsilon_t \sim WN(0,\sigma^2)$ 일때 다음 물음에 답하여라.

모형 1) $Z_t - 0.8Z_{t-1} = \varepsilon_t$

모형 2) $Z_t - 0.5Z_{t-1} = \varepsilon_t + 0.3\varepsilon_{t-1}$

1) 각 모형을 $\Phi(B)(Z_t-\mu)=\Theta(B)\varepsilon_t$ 의 형태로 표현하고, $\Phi(B)$, $\Theta(B)$, μ 명시

모형 1) $(1-0.8B)Z_t = \varepsilon_t \rightarrow \Phi(B)=1-0.8B,\ \Theta(B)=1,\ \mu=0$

모형 2) $(1-0.5B)Z_t = (1+0.3B)\varepsilon_t \rightarrow \Phi(B)=1-0.5B,\ \Theta(B)=1+0.3B,\ \mu=0$

2) 각 모형은 AR(p), MA(q), ARMA(p,q) 중 어느 것인가?

모형 1) AR(1), 모형 2) ARMA(1,1)

3) 각 모형에 대해 γ_h, $h=0,1,2,3$ 을 계산하여라. 그리고 ACF ρ_h, $h=0,1,2,3$ 을 계산하여라.

모형 1) $Z_t = 0.8Z_{t-1}+\varepsilon_t \rightarrow |0.8|<1$ 이므로 정상성 만족.

$E(Z_t)=0$, $Var(Z_t)=Var(0.8Z_{t-1}+\varepsilon_t)=(0.64)\cdot Var(Z_t)+\sigma^2 \quad (\because Z_{t-1}\perp\varepsilon_t)$

$$\therefore \gamma_0 = \frac{\sigma^2}{1-0.64} = \frac{\sigma^2}{0.36}$$

$\gamma_1 = Cov(Z_t, Z_{t+1}) = Cov(Z_t, 0.8Z_t+\varepsilon_{t+1}) = (0.8)\gamma_0 = (0.8)\cdot\frac{\sigma^2}{0.36} \quad (\because Z_t\perp\varepsilon_{t+1})$

$\gamma_2 = Cov(Z_t, Z_{t+2}) = Cov(Z_t, 0.8Z_{t+1}+\varepsilon_{t+2}) = (0.8)\cdot\gamma_1 = (0.8)^2\cdot\frac{\sigma^2}{0.36} \quad (\because Z_t\perp\varepsilon_{t+2})$

$\gamma_3 = Cov(Z_t, Z_{t+3}) = Cov(Z_t, 0.8Z_{t+2}+\varepsilon_{t+3}) = (0.8)\cdot\gamma_2 = (0.8)^3\cdot\frac{\sigma^2}{0.36} \quad (\because Z_t\perp\varepsilon_{t+3})$

$\Rightarrow$ ACF $\rho_h = \frac{\gamma_h}{\gamma_0} \Rightarrow \rho_0=1,\ \rho_1=0.8,\ \rho_2=(0.8)^2,\ \rho_3=(0.8)^3$

모형 2) $\Phi(B)=1-0.5B=0$ 과 $\Theta(B)=1+0.3B=0$ 의 근이 모두 절대값이 1보다 크기 때문에 정상성, 가역성 만족

$E(Z_t)=0$, $Var(Z_t)=Cov(Z_t,Z_t)=Cov(Z_t, 0.5Z_{t-1}+\varepsilon_t+0.3\varepsilon_{t-1})$

$=(0.5)\cdot\gamma_1 + Cov(Z_t,\varepsilon_t)+(0.3)\cdot Cov(Z_t,\varepsilon_{t-1})$

$=(0.5)\gamma_1 + Cov(0.5Z_{t-1}+\varepsilon_t+0.3\varepsilon_{t-1}, \varepsilon_t)+(0.3)\cdot Cov(0.5Z_{t-1}+\varepsilon_t+0.3\varepsilon_{t-1}, \varepsilon_{t-1})$

$=(0.5)\gamma_1+\sigma^2+(0.3)\cdot(0.5+0.3)\sigma^2 = (0.5)\cdot\gamma_1+(1.24)\cdot\sigma^2=\gamma_0 \quad \cdots ①$

$\gamma_1 = Cov(Z_t,Z_{t+1}) = Cov(Z_t, 0.5Z_t+\varepsilon_{t+1}+0.3\varepsilon_t) = (0.3)\sigma^2+(0.5)\gamma_0 \quad \cdots ②$

$\gamma_2 = Cov(Z_t,Z_{t+2}) = Cov(Z_t, 0.5Z_{t+1}+\varepsilon_{t+2}+0.3\varepsilon_{t+1}) = (0.5)\cdot\gamma_1 \quad \cdots ③$

$\gamma_3 = Cov(Z_t,Z_{t+3}) = Cov(Z_t, 0.5Z_{t+2}+\varepsilon_{t+3}+0.3\varepsilon_{t+2}) = (0.5)\cdot\gamma_2 \quad \cdots ④$

$\Rightarrow$ ①~④ 에 의하여 $\gamma_0=(1.853)\sigma^2,\ \gamma_1=(1.227)\sigma^2,\ \gamma_2=(0.613)\sigma^2,\ \gamma_3=(0.307)\sigma^2$

$\hookrightarrow \rho_0=1,\ \rho_1=0.662,\ \rho_2=0.331,\ \rho_3=0.165$

MA(2) process

$$\underbrace{Z_t-\mu}_{\dot{Z}_t} = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}\ ,\ \varepsilon_t \overset{iid}{\sim} WN(0,\sigma^2)$$

$$\Rightarrow \dot{Z}_t = (1-\theta_1 B-\theta_2 B^2)\varepsilon_t = \Theta(B)\varepsilon_t$$

* invertibility $\Rightarrow \theta_1+\theta_2<1\ ,\ \theta_2-\theta_1<1\ ,\ |\theta_2|<1$

① $E(Z_t)=\mu$ ② $Var(Z_t)=\gamma_0=(1+\theta_1^2+\theta_2^2)\sigma^2$

$$*\ Cov(\dot{Z}_t,\dot{Z}_{t+1}) = Cov(\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}\ ,\ \varepsilon_{t+1}-\theta_1\varepsilon_t-\theta_2\varepsilon_{t-1})$$
$$= -\theta_1\sigma^2+\theta_1\theta_2\sigma^2 = -\theta_1(1-\theta_2)\sigma^2=\gamma_1$$

$$Cov(\dot{Z}_t,\dot{Z}_{t+2}) = Cov(\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}\ ,\ \varepsilon_{t+2}-\theta_1\varepsilon_{t+1}-\theta_2\varepsilon_t)$$
$$= -\theta_2\sigma^2=\gamma_2$$

$$Cov(\dot{Z}_t,\dot{Z}_{t+k})=\gamma_k=0\ ,\ k\geq 3$$

$$\Rightarrow \text{ACF}\quad \rho_k=\frac{\gamma_k}{\gamma_0}=\begin{cases}\dfrac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2}\ , & k=1\\[2ex] \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2}\ , & k=2\\[2ex] 0\ , & k\geq 3\end{cases}$$

MA(q) process

$$\underbrace{Z_t - \mu}_{\dot{Z}_t} = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}$$

$$\Rightarrow \dot{Z}_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q) \cdot \varepsilon_t = \Theta(B) \cdot \varepsilon_t.$$

invertibility : $\Theta(B) = 0$ 의 응개 근들의 절대값이 모두 1보다 커야 함.

① $E(Z_t) = \mu$ ② $Var(Z_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2$

$$* \ Cov(\dot{Z}_t, \dot{Z}_{t+k}) = Cov(\varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}, \ \varepsilon_{t+k} - \theta_1 \varepsilon_{t+k-1} - \theta_2 \varepsilon_{t+k-2} - \cdots - \theta_q \varepsilon_{t+k-q})$$

$$= \gamma_k$$

If $k=1 \Rightarrow \gamma_1 = Cov(\varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}, \ \varepsilon_{t+1} - \theta_1 \varepsilon_t - \theta_2 \varepsilon_{t-1} - \cdots - \theta_q \varepsilon_{t+1-q})$

$$= (-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \cdots + \theta_{q-1}\theta_q)\sigma^2$$

If $k=2 \Rightarrow \gamma_2 = Cov(\varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}, \ \varepsilon_{t+2} - \theta_1 \varepsilon_{t+1} - \theta_2 \varepsilon_t - \cdots - \theta_q \varepsilon_{t+2-q})$

$$= (-\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \cdots + \theta_{q-2}\theta_q)\sigma^2$$

$$\vdots \quad \vdots$$

$$\gamma_k = \begin{cases} (-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q)\sigma^2, & k = 1, 2, \cdots, q \\ 0, & k \geq q+1 \end{cases}$$

$$\Rightarrow \text{ACF} \quad \rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2}, & k = 1, 2, \cdots, q \\ 0, & k \geq q+1 \end{cases}$$