Nonparametric specification tests for cross-sectional and panel stochastic frontier models

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KSS Meeting, 28 May 2021

Stochastic frontier models

In its cross-sectional setting, the stochastic frontier model is given by

$$Y = m(X) + V - U\,,$$

where

Y is the (logarithm of) output,

X is a p-dimensional covariate,

$m(\cdot)$ is an unknown production (or cost) frontier function,

U is the (unobservable) non-negative inefficiency term and V is the random error component.

Assume U and V are independent and V is symmetrically distributed, conditionally on X.

Distributional assumptions

- Half-normal (Aigner, Lovell, and Schmidt, 1977):
  $U|X = x \sim |N(0, \sigma_U^2(x))|$ and $V|X = x \sim N(0, \sigma_V^2(x))$.
- Exponential (Meeusen and van den Broeck,1977):
  $U|X = x \sim$ Exponential$(\beta(x))$ and $V|X = x \sim N(0, \sigma_V^2(x))$.
- Gamma (Greene,1990):
  $U|X = x \sim$ Gamma$(\alpha(x), \beta(x))$ and $V|X = x \sim N(0, \sigma_V^2(x))$.
- $\ldots$

Testing the distributional assumption

Methods based on *parametric frontier* and *homoscedasticity*:

- Lagrangean Multiplier tests for testing the half-normal specification against a four-parameter Pearson family alternative (Lee, 1983).
- Pearson chi-squared and Kolmogorov-Smirnov tests (Wang, Amsler and Schmidt, 2011).
- Centered residuals-based method of moments test (Chen and Wang, 2012).
- ...

Testing parametric frontiers

The only two existing tests for parametric frontiers introduced by Guo, Lee and Wong (2018) both impose some distributional assumption.

They may not work if the distributional assumption is violated.

Nonparametric specification testing

In the existing literature, to test distributional specifications we have no other choices but to assume a parametric frontier, without being able to check its validity.

All the existing specification tests assume the inefficiency and random error are both homoscedastic.

A way to overcome these challenges is not to make any parametric assumption on the frontier and to allow for heteroscedasticity.

Nonparametric stochastic frontier model

Recall that the stochastic frontier model is given by

$$Y = m(X) + V - U.$$

Define $\mu_U(X) = \mathrm{E}(U|X)$ and $\varepsilon = V - U + \mu_U(X)$. Then the composite error ε has $\mathrm{E}(\varepsilon|X) = 0$ and we can rewrite it as

$$Y = m(X) - \mu_U(X) + \varepsilon = m_1(X) + \sigma(X)\,\eta\,, \tag{1}$$

where

$m_1(X) = m(X) - \mu_U(X)$,
$\sigma^2(X) = \mathrm{Var}(\varepsilon|X) = \mathrm{Var}(U|X) + \mathrm{Var}(V|X)$,
$\eta = \varepsilon/\sigma(X)$ which has $\mathrm{E}(\eta|X) = 0$ and $\mathrm{Var}(\eta|X) = 1$.

Let $F_0\big(\cdot;\sigma_U^2(x)\big)$ and $G_0\big(\cdot;\sigma_V^2(x)\big)$ respectively denote the specified parametric distribution functions of U and V given $X = x$, where $\sigma_U(x)$ is a scale parameter and $\sigma_V^2(x) = \mathrm{Var}(V|X = x)$.

The null hypothesis is that the distributional assumption is correct:

$$\begin{aligned} \mathrm{H}_0: \quad & \text{For any given } x, \text{ there exists some positive constants } \sigma_U(x), \\ & \sigma_V(x) \text{ s.t. } \mathrm{P}(U \le u, V \le v|X = x) = F_0(u;\sigma_U^2(x))G_0(v;\sigma_V^2(x)) \\ & \text{for all } u, v \in \mathbb{R}, \end{aligned} \tag{2}$$

and the general fixed alternative hypothesis is that H_0 is incorrect:

$$\begin{aligned} \mathrm{H}_1: \quad & \text{For any given } x, \\ & \sup_{u\in\mathbb{R}} \big|\mathrm{P}(U \le u, V \le v|X = x) - F_0(u;\sigma_U^2)G_0(v;\sigma_V^2)\big| > 0 \\ & \text{for all } \sigma_U > 0, \sigma_V > 0. \end{aligned} \tag{3}$$

Estimation

Suppose (X_i, Y_i), $i = 1, \ldots, n$, are (possibly dependent) observations from model (1). Let U_i, V_i, ε_i, and η_i be the corresponding realizations of U, V, ε and η in the model, $i = 1, \ldots, n$.

For simplicity, we consider the case where $p = 1$ i.e. X is univariate.

Let K be a symmetric probability density function, h_0 be a bandwidth and $K_a(u) = K(u/a)/a$ for any positive constant a.

For a given point x of estimation, the local linear estimator of $m_1(x) = \mathrm{E}(Y|X = x) = m(x) - \mu_U(x)$ is defined as

$$\widehat{m}_1(x) = \frac{S_2(x)T_0(x) - S_1(x)T_1(x)}{S_2(x)S_0(x) - S_1(x)S_1(x)},$$

where $S_j(x) = \sum_{i=1}^n (X_i - x)^j K_{h_0}(X_i - x)$, $j = 0, 1, 2$, and $T_j(x) = \sum_{i=1}^n (X_i - x)^j K_{h_0}(X_i - x) Y_i$, $j = 0, 1$.

With the estimates $\widehat{m}_1(X_i)$, $i = 1, \ldots, n$, we get the following residuals:

$$\widehat{\varepsilon}_i = Y_i - \widehat{m}_1(X_i), \ i = 1, \ldots, n. \tag{4}$$

Then, $\mu_k(x) \equiv \mathsf{E}(\varepsilon^k | X = x)$ can be estimated by applying local linear regression to $\{(X_i, \widehat{\varepsilon}_i^k), i = 1, \ldots, n\}$ which yields

$$\widehat{\mu}_k(x) = \frac{S_{2k}(x)T_{0k}(x) - S_{1k}(x)T_{1k}(x)}{S_{2k}(x)S_{0k}(x) - S_{1k}(x)S_{1k}(x)}\,, \tag{5}$$

where $S_{jk}(x) = \sum_{i=1}^n (X_i - x)^j K_{h_k}(X_i - x)$, $j = 0, 1, 2$, and $T_{jk}(x) = \sum_{i=1}^n (X_i - x)^j K_{h_k}(X_i - x)\widehat{\varepsilon}_i^k$, $j = 0, 1$.

Therefore, we have the following nonparametric estimator of $\sigma^2(x) = \mathrm{Var}(\varepsilon | X = x) = \mu_2(x)$:

$$\widehat{\sigma}^2(x) = \widehat{\mu}_2(x)\,.$$

Recall that $\varepsilon = V - U + \mu_U(X)$.

If the third conditional moments of U and V given X both exist, then we have the following relationships under the null hypothesis:

$$\begin{aligned} \mu_U(X) &= \mathrm{E}(U|X) \equiv \kappa_1 \sigma_U(X), \\ \mu_2(X) &= \mathrm{E}(\varepsilon^2|X) = \mathrm{Var}(U|X) + \mathrm{Var}(V|X) \equiv \kappa_2 \sigma_U^2(X) + \sigma_V^2(X), \\ \mu_3(X) &= \mathrm{E}(\varepsilon^3|X) = -\mathrm{E}\left[(U - \mu_U(X))^3|X\right] \equiv -\kappa_3 \sigma_U^3(X), \end{aligned} \tag{6}$$

where κ_1, κ_2, κ_3 are constants dependent on the distributional assumption. Replacing $\mu_k(x) = \mathrm{E}(\varepsilon^k|X = x)$ in (6) with its estimator $\widehat{\mu}_k(x)$ given by (5), $k = 2, 3$, and recalling that $m(x) = m_1(x) + \mu_U(x)$, we obtain the following estimators under the null hypothesis:

$$\begin{aligned} \widetilde{\sigma}_U^2(x) &= \left[-\frac{\widehat{\mu}_3(x)}{\kappa_3}\right]_+^{2/3} \equiv g_1(\widehat{\mu}_3(x)), \;\; \widetilde{\mu}_U(x) = \kappa_1 \widetilde{\sigma}_U(x), \\ \widetilde{\sigma}_V^2(x) &= \left[\widehat{\mu}_2(x) - \kappa_2 \widetilde{\sigma}_U^2(x)\right]_+, \;\; \widetilde{m}(x) = \widehat{m}_1(x) + \widetilde{\mu}_U(x), \end{aligned} \tag{7}$$

where $a_+ = \max\{a, 0\}$ for any real number a. Kumbhakar et al. (2007) introduced this method under the half-normal specification.

Under the null hypothesis, if the fourth conditional moments of U and V both exist, then besides those given in (6) we have

$$\begin{aligned}
& \mathrm{E}(\varepsilon^4|X) \\
= \ & \mathrm{E}\left[(U-\mu_U(X))^4|X\right] + \mathrm{E}(V^4|X) + 6\mathrm{E}(V^2|X)\mathrm{E}\left[(U-\mu_U(X))^2|X\right] \\
= \ & \kappa_4\sigma_U^4(X) + \kappa_V\sigma_V^4(X) + 6\sigma_V^2(X)\kappa_2\sigma_U^2(X) \\
= \ & \kappa_4\sigma_U^4(X) - \frac{9\kappa_2^2}{\kappa_V}\sigma_U^4(X) + \kappa_V\left[\sigma^2(X)\right]^2,
\end{aligned} \tag{8}$$

where $\kappa_V = \mathrm{E}(V^4|X)/\sigma_V^4(X)$, $\kappa_4 = \mathrm{E}\left[(U-\mu_U(X))^4|X\right]/\sigma_U^4(X)$.

Solving the above equality for $\sigma^2(X)$ and noting that $\sigma_U^2(x) = \left[-\frac{\mu_3(x)}{\kappa_3}\right]^{2/3}$ we have

$$\sigma^2(x) = \left\{\frac{1}{\kappa_V}\left[\mu_4(x) - \left(\kappa_4 - \frac{9\kappa_2^2}{\kappa_V}\right)\sigma_U^4(x)\right]\right\}_+^{1/2} \equiv g_2(\mu_3(x), \mu_4(x)). \tag{9}$$

Therefore, under the null hypothesis we obtain another estimator of $\sigma^2(x) = \mathrm{E}(\varepsilon^2|X=x) = \mu_2(x)$:

$$\widetilde{\sigma}^2(x) = g_2(\widehat{\mu}_3(x), \widehat{\mu}_4(x)). \tag{10}$$

A test statistic is

$$T_S = \frac{1}{n}\sum_{i=1}^{n}\left[\widetilde{\sigma}^2(X_i) - \widehat{\sigma}^2(X_i)\right] = \frac{1}{n}\sum_{i=1}^{n}\left[g_2(\widehat{\mu}_3(X_i), \widehat{\mu}_4(X_i)) - \widehat{\mu}_2(X_i)\right],$$

and we would reject H_0 if $|T_S|$ is large.

Alternatively, we have the following standardized residuals constructed without the distributional assumption:

$$\widehat{\eta}_i = \frac{\widehat{\varepsilon}_i}{\widehat{\sigma}(X_i)},\ i = 1, \ldots, n\,.$$

Further, dividing the residuals $\widehat{\varepsilon}_i$ given in (4) by $\widetilde{\sigma}^2(X_i)$ obtained under the null hypothesis, we have another set of standardized residuals:

$$\widetilde{\eta}_i = \frac{\widehat{\varepsilon}_i}{\widetilde{\sigma}(X_i)},\ i = 1, \ldots, n\,.$$

The empirical distributions of $\{\widehat{\eta}_i^2\}$ and $\{\widetilde{\eta}_i^2\}$ are respectively:

$$\begin{aligned}\widehat{F}_{\eta^2}(u) &= \frac{1}{n\bar{w}}\sum_{i=1}^{n} w(X_i)\,\mathrm{I}(\widehat{\eta}_i^2 \le u) \quad \text{and} \\ \widetilde{F}_{\eta^2}(u) &= \frac{1}{n\bar{w}}\sum_{i=1}^{n} w(X_i)\,\mathrm{I}(\widetilde{\eta}_i^2 \le u),\ \ u \in \mathbb{R}\,,\end{aligned}$$

where $\mathrm{I}(\cdot)$ is the indicator function, $w(x)$ is a positive weight function and $\bar{w} = \sum_{i=1}^{n} w(X_i)/n$.

The the Kolmogorov-Smirnov type and Cramér-von Mises type test statistics are

$$\begin{aligned}T_{\eta,KS} &= \sqrt{n}\sup_{u\in\mathbb{R}}\left|\widetilde{F}_{\eta^2}(u) - \widehat{F}_{\eta^2}(u)\right|, \\ T_{\eta,CM} &= n\int_{-\infty}^{\infty}\left[\widetilde{F}_{\eta^2}(u) - \widehat{F}_{\eta^2}(u)\right]^2 d\widehat{F}_{\eta^2}(u)\,.\end{aligned}$$

Bootstrap tests

Given the sample $\mathcal{X} = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$, let

$$Y_i^* = \widetilde{m}(X_i) + V_i^* - U_i^* = \widehat{m}_1(X_i) + \widetilde{\mu}_U(X_i) + V_i^* - U_i^*,\ i = 1, \ldots, n,$$

where $U_i^* \sim F_0\big(\cdot; \widetilde{\sigma}_U^2(X_i)\big)$ and $V_i^* \sim G_0(\cdot; \widetilde{\sigma}_V^2(X_i))$ are independent.

Let $\widehat{m}_1^*(x)$, $\widehat{\varepsilon}_i^* = Y_i^* - \widehat{m}_1^*(X_i)$, $\widetilde{\sigma}_U^{*2}(X_i)$, $\widehat{\sigma}^{*2}(X_i)$, $\widetilde{\mu}_U^*(x)$ and $\widetilde{\sigma}^{*2}(X_i)$, be the version of $\widehat{m}_1(x)$, $\widehat{\varepsilon}_i$, $\widetilde{\sigma}_U^2(X_i)$, $\widehat{\sigma}^2(X_i)$, $\widetilde{\mu}_U(x)$ and $\widetilde{\sigma}^2(X_i)$ respectively based on the bootstrap sample $\mathcal{X}^* \equiv \{(X_1, Y_1^*), \ldots, (X_n, Y_n^*)\}$.

The two sets of bootstrap standardized residuals are
$\widehat{\eta}_i^* = \frac{\widehat{\varepsilon}_i^*}{\widehat{\sigma}^*(X_i)}$ and $\widetilde{\eta}_i^* = \frac{\widehat{\varepsilon}_i^*}{\widetilde{\sigma}^*(X_i)}$ $, i = 1, \ldots, n\,.$

Let $\widehat{F}_{\eta^2}^*(u)$ and $\widetilde{F}_{\eta^2}^*(u)$ denote respectively the empirical distributions of the bootstrap squared standardized residuals $\{(\widehat{\eta}_i^*)^2\}$ and $\{(\widehat{\eta}_i^*)^2\}$.

The bootstrap version of the test statistics T_S, $T_{\eta,KS}$ is

$$T_S^* = \frac{1}{n}\sum_{i=1}^{n}\left[\widetilde{\sigma}^{*2}(X_i) - \widehat{\sigma}^{*2}(X_i)\right] .$$

We repeat the above parametric bootstrap procedure B times to get B bootstrap realizations $T_S^{*,1},\ldots,T_S^{*,B}$.

Then, the bootstrap test using T_S rejects H_0 at significance level α if $|T_S| > |T_S^*|\big(\lfloor B(1-\alpha)\rfloor\big)$, where $|T_S^*|(1) < \cdots < |T_S^*|(B)$ are the order statistics of $|T_S^{*,1}|,\ldots,|T_S^{*,B}|$ and $\lfloor x\rfloor$ is the largest integer no greater than x.

The bootstrap Cramér-von Mises type test and the bootstrap Kolmogorov-Smirnov type test can be constructed similarly.

Recall that under the null hypothesis, as given in (9),

$$\mu_2(x) = \sigma^2(x) = \left\{ \frac{1}{\kappa_V} \left[\mu_4(x) - \left(\kappa_4 - \frac{9\kappa_2^2}{\kappa_V} \right) \sigma_U^4(x) \right] \right\}_+^{1/2} \equiv g_2(\mu_3(x), \mu_4(x)),$$

and so we have $\widetilde{\sigma}^2(x) = g_2(\widehat{\mu}_3(x), \widehat{\mu}_4(x))$.

Denote $g_{2j}(\mu_3(x), \mu_4(x)) \equiv \frac{\partial g_2(\mu_3(x), \mu_4(x))}{\partial \mu_j(x)}, j = 3, 4$, and define

$$b_H(x) = g_2(\mu_3(x), \mu_4(x)) - \mu_2(x).$$

Then, $b_H(x) \equiv 0$ under H_0 and $b_H(x) \neq 0$ at least for some x under H_1.

Consider the following local alternative hypothesis:

$\mathrm{H}_{1b} : U|X$ is such that $0 < \mathrm{E}\left|\sqrt{n} b_H(X)\right| < \infty$ and $\mathrm{E}[n b_H^2(X)] < \infty$.(11)

Theorem

Assume certain conditions. Then under the null hypothesis or the local alternative specification H_{1b}, *we have the following results conditionally on* $\mathcal{X} = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$:

(i) $\sqrt{n}T_S^*$ *converges weakly to a Gaussian process with the same asymptotic mean and covariance structures as that of* $\sqrt{n}T_S$;

(ii) $\sqrt{n}\big[\widetilde{F}^*_{\eta^2}(s) - \widehat{F}^*_{\eta^2}(s)\big]$ *converges weakly to a Gaussian process with the same mean and covariance structures as that of* $\sqrt{n}\big[\widetilde{F}_{\eta^2}(s) - \widehat{F}_{\eta^2}(s)\big]$.

The Gaussian limiting processes of $\sqrt{n}T_S$ and $\sqrt{n}\big[\widetilde{F}_{\eta^2}(s) - \widehat{F}_{\eta^2}(s)\big]$ both have the zero mean function under H_0, and they have constant mean functions under H_{1b}.

From the above theorem and the Continuous Mapping Theorem, we have the following results for the three bootstrap tests:

(i) they are asymptotically consistent level-α tests,

(ii) their powers are asymptotically the same as the respective level-α tests using $|T_S|$, T_{KS} and T_{CM}.

(iii) they have nontrivial powers against the local alternative H_{1b}, and their powers against the general fixed alternative H_1 tends to 1 as $n \to \infty$.

Asymptotic distribution-free Tests

Define $W(Y_i, X_i) = \left[\mu_2(X_i) - \varepsilon_i^2\right] - g_{23}(\mu_3(X_i), \mu_4(X_i)) \left[\mu_3(X_i) - \varepsilon_i^3\right] - g_{24}(\mu_3(X_i), \mu_4(X_i)) \left[\mu_4(X_i) - \varepsilon_i^4\right]$.

Assume $0 < \sigma_W^2 \equiv \mathrm{E}[W(Y,X)^2] < \infty$ and there exists a consistent estimator for σ_W^2, say $\widehat{\sigma}_W^2$.

A possible choice of $\widehat{\sigma}_W^2$ is the sample variance of $\{\widehat{W}_i'\}_{i=1}^n$, where $\widehat{W}_i' = \widehat{\varepsilon}_i^2 - g_{23}(\widehat{\mu}_3(X_i), \widehat{\mu}_4(X_i))\widehat{\varepsilon}_i^3 - g_{24}(\widehat{\mu}_3(X_i), \widehat{\mu}_4(X_i))\widehat{\varepsilon}_i^4$. Let

$$D_S = \sqrt{n}\frac{T_S}{\widehat{\sigma}_W}.$$

Then, based on the theorem stated below, we have the following asymptotic distribution-free test:

$$\text{reject } \mathrm{H}_0 \text{ at significance level } \alpha \text{ if } |D_S| = \sqrt{n}\left|\frac{T_S}{\widehat{\sigma}_W}\right| > z_{1-\alpha/2}, \tag{12}$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of the standard normal distribution.

Let $\mathcal{Z}$ denote a standard normal random variable.

Theorem

Under suitable conditions, as $n \to \infty$, the following asymptotic expansion of the test statistic T_S holds:

$$T_S = \frac{1}{n}\sum_{i=1}^{n}\left[W(Y_i, X_i) + b_H(X_i)\right] + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{13}$$

and we have

$$D_S = \sqrt{n}\frac{T_S}{\widehat{\sigma}_W} \xrightarrow{d} \mathcal{Z} + \delta(b_H) \text{ as } n \to \infty,$$

where $\delta(b_H) = \mathrm{E}\left[\sqrt{n} b_H\left(X\right)\right]/\sigma_W$.

Define $\widehat{C}(s) = \int_{-\infty}^{s} \sqrt{n}\big[\widetilde{F}_{\eta^2}(u) - \widehat{F}_{\eta^2}(u)\big]\, du$. The asymptotic expansion of $\widehat{C}(s)$ given in (17) involves the following unknown quantity:

$$M = \int_{-\infty}^{\infty} \big\{\mathrm{E}\big[\mathrm{I}(\eta^2 \le s)\eta^2\big]\big\}^2 dF_{\eta^2}(s), \tag{14}$$

which can be estimated by combining a method of moments and a numerical integration, e.g. an estimator of M is

$$\widehat{M} = \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{n}\sum_{j=1}^{n}\mathrm{I}(\widetilde{\eta}_j^2 \le \widetilde{\eta}_i^2)\widetilde{\eta}_j^2\Big]^2.$$

Then, we define the Kolmogorov-Smirnov type and the Cramér-von Mises type test statistics based on $\widehat{C}(s)$ as

$$D_{KS} = \sup_{-\infty<s<\infty} \frac{1}{\widehat{\sigma}_{W_1}}\big|\widehat{C}(s)\big| \text{ and } D_{CM} = \frac{1}{\widehat{M}\widehat{\sigma}_{W_1}^2}\int_{-\infty}^{\infty}\big[\widehat{C}(s)\big]^2 d\widetilde{F}_{\eta^2}(s),$$

where $\widehat{\sigma}_{W_1}^2$ is a consistent estimator of $\sigma_{W_1}^2$, for example, the sample variance of $\{\widehat{W}_{1i}'\}_{i=1}^n$ when $w(x) \equiv 1$, where
$\widehat{W}_{1i}' = \frac{\widehat{\varepsilon}_i^2}{\widetilde{\sigma}^2(X_i)} - \frac{g_{23}(\widehat{\mu}_3(X_i),\widehat{\mu}_4(X_i))\widehat{\varepsilon}_i^3}{\widetilde{\sigma}^2(X_i)} - \frac{g_{24}(\widehat{\mu}_3(X_i),\widehat{\mu}_4(X_i))\widehat{\varepsilon}_i^4}{\widetilde{\sigma}^2(X_i)}$.

Following from the asymptotic distributions of D_{KS} and D_{CM} given in the theorem stated below, we have two asymptotic distribution-free tests as given next.

The Cramér-von Mises type asymptotic test amounts to

$$\text{reject the null hypothesis at significance level } \alpha \text{ if } D_{CM} > \chi^2_{1,1-\alpha}, \quad (15)$$

where $\chi^2_{1,1-\alpha}$ denotes the $(1-\alpha)$-quantile of the chi-square distribution with one degree of freedom.

The Kolmogorov-Smirnov type asymptotic test is to

$$\text{reject the null hypothesis at significance level } \alpha \text{ if } D_{KS} > z_{1-\alpha/2}. \quad (16)$$

Theorem

Let $W_1(Y_i, X_i) = \frac{W(Y_i,X_i)}{g_2(\mu_3(X_i),\mu_4(X_i))}$ *and* $\delta_1(b_H) = \frac{\mathrm{E}[\sqrt{n}w(X)b_H(X)]}{(\mathrm{E}[w^2(X)W(Y,X)^2])^{1/2}}$. *Then, assuming certain conditions and independence of* η *and* X *,it holds uniformly in* $-\infty < s < \infty$ *that*

$$\widehat{C}(s) = \frac{\mathrm{E}\left[\mathrm{I}(\eta^2 \le s)\eta^2\right]}{\sqrt{n}\mathrm{E}[w(X)]} \sum_{i=1}^{n} w(X_i)\left[W_1(Y_i, X_i) + \frac{b_H(X_i)}{g_2(\mu_3(X_i), \mu_4(X_i))}\right] + o_p(1). \tag{17}$$

Further, if $0 < \sigma^2_{W_1} \equiv \frac{\mathrm{E}[w^2(X_i)W_1^2(Y_i,X_i)]}{[\mathrm{E}w(X_i)]^2} < \infty$ *and there exists a consistent estimator for* $\sigma^2_{W_1}$*, say* $\widehat{\sigma}^2_{W_1}$*, then we have*

$$D_{KS} \xrightarrow{d} |\mathcal{Z} + \delta_1(b_H)| \quad \text{and} \quad D_{CM} \xrightarrow{d} [\mathcal{Z} + \delta_1(b_H)]^2 \ \text{ as } n \to \infty.$$

The powers of our tests based on D_{KS} and D_{CM} against the general fixed alternative H_1 all tend to 1 as n increases i.e.

$$\mathsf{P}(|D_S| > z_{1-\alpha/2}|\mathsf{H}_1) \to 1,$$

$$\mathsf{P}(D_{KS} > z_{1-\alpha/2}|\mathsf{H}_1) \to 1,$$

$$\mathsf{P}(D_{CM} > \chi^2_{1,1-\alpha}|\mathsf{H}_1) \to 1,$$

as $n \to \infty$. And, they have nontrivial powers against the local alternative H_{1b}:

$$\mathsf{P}(|D_S| > z_{1-\alpha/2}|\mathsf{H}_{1b}) \to \mathsf{P}(|\mathcal{Z} + \delta(b_H)| > z_{1-\alpha/2}),$$

$$\mathsf{P}(D_{KS} > z_{1-\alpha/2}|\mathsf{H}_{1b}) \to \mathsf{P}(|\mathcal{Z} + \delta_1(b_H)| > z_{1-\alpha/2}),$$

$$\mathsf{P}(D_{CM} > \chi^2_{1,1-\alpha}|\mathsf{H}_{1b}) \to \mathsf{P}([\mathcal{Z} + \delta_1(b_H)]^2 > \chi^2_{1,1-\alpha})$$

as $n \to \infty$.

Proposition

Independence of η and X holds if the two scale parameters $\sigma_U(x)$ and $\sigma_V(x)$ satisfy $\sigma_V(x) = \lambda\sigma_U(x)$ for all $x \in R_X$ for some positive constant λ.

This gives a condition to ensure independence of η and X required by the asymptotic tests based on D_{KS} and D_{CM}.

We emphasize that it is unnecessary to estimate λ in the construction of these asymptotic tests.

Example: Testing the half-normal specification

Under the half-normal specification $U|X = x \sim |N(0, \sigma_U^2(x))|$, $V|X = x \sim N(0, \sigma_V^2(x))$, and we have

$$
\begin{aligned}
\mu_U(X) &= \sqrt{\frac{2}{\pi}}\sigma_U(X), \;\; \mathrm{E}(\varepsilon^2|X) = \frac{\pi-2}{\pi}\sigma_U^2(X) + \sigma_V^2(X), \\
\mathrm{E}(\varepsilon^3|X) &= \sqrt{\frac{2}{\pi}}\Big(1 - \frac{4}{\pi}\Big)\sigma_U^3(X), \\
\mathrm{E}(\varepsilon^4|X) &= \frac{8(\pi-3)}{\pi^2}\sigma_U^4(X) + 3\big[\sigma^2(X)\big]^2, \\
\widetilde{\sigma}_U^2(x) &= \Big[\sqrt{\frac{\pi}{2}}\big(\frac{\pi}{\pi-4}\big)\widehat{\mu}_3(x)\Big]_+^{\frac{2}{3}} = g_1(\widehat{\mu}_3(x))\,, \;\; \widetilde{\mu}_U(x) = \sqrt{\frac{2}{\pi}}\widetilde{\sigma}_U(x)\,, \\
\widetilde{\sigma}_V^2(x) &= \Big[\widehat{\mu}_2(x) - \frac{\pi-2}{\pi}\widetilde{\sigma}_U^2(x)\Big]_+\,, \;\; \widetilde{m}(x) = \widehat{m}_1(x) + \widetilde{\mu}_U(x), \\
\widetilde{\sigma}^2(x) &= \Big[\frac{1}{3}\widehat{\mu}_4(x) - \frac{8(\pi-3)}{3\pi^2}\widetilde{\sigma}_U^4(x)\Big]_+^{\frac{1}{2}} = g_2(\widehat{\mu}_3(x), \widehat{\mu}_4(x)).
\end{aligned}
$$

The explicit form of $W(Y_i, X_i)$ appearing in the asymptotic expansion of T_S given in (13) is $W(Y_i, X_i) =$
$\left[\mu_2(X_i) - \varepsilon_i^2\right] - \frac{16(\pi-3)[g_1(\mu_3(x))]^{1/2}[\mu_3(X_i)-\varepsilon_i^3]}{9\sqrt{2\pi}(4-\pi)g_2(\mu_3(X_i),\mu_4(X_i))} - \frac{\mu_4(X_i)-\varepsilon_i^4}{6g_2(\mu_3(X_i),\mu_4(X_i))}$.

Since $g_1(\mu_3(x)) = \sigma_U^2(x)$ and $g_2(\mu_3(x), \mu_4(x)) = \sigma^2(x)$ under H_0, the estimator $\widehat{\sigma}_W^2$ involved in the asymptotic test based on D_S can be taken as the sample variance of $\widehat{W}_i' = \widehat{\varepsilon}_i^2 - \frac{16(\pi-3)}{9\sqrt{2\pi}(4-\pi)} \frac{\widetilde{\sigma}_U(X_i)}{\widetilde{\sigma}^2(X_i)} \widehat{\varepsilon}_i^3 - \frac{1}{\widetilde{\sigma}^2(X_i)} \widehat{\varepsilon}_i^4$.

Letting $w(x) \equiv 1$, the estimator $\widehat{\sigma}_{W_1}^2$ involved in the asymptotic tests based on D_{KS} and D_{CM} can be taken as the sample variance of $\widehat{W}_{i1}' = \widehat{W}_i'/\widetilde{\sigma}^2(X_i)$.

Example: Testing the exponential specification

The exponential specification assumes $U|X = x \sim$ Exponential$(\beta(x))$ and $V|X = x \sim$ N$(0, \sigma_V^2(x))$. Let $\sigma_U^2(x) = $ Var$(U|X = x)$ in this case. Then, the equations given in (6) and (8) become

$$\mu_U(X) = \beta(X),\ \ \mathrm{E}(\varepsilon^2|X) = \sigma_U^2(X) + \sigma_V^2(X) = \beta^2(X) + \sigma_V^2(X),$$
$$\mathrm{E}(\varepsilon^3|X) = -2\beta^3(X),\ \ \mathrm{E}(\varepsilon^4|X) = 6\beta^4(X) + 3\left[\sigma^2(X)\right]^2,$$

and the estimators given in (7) and (10) are

$$\begin{aligned}
\widetilde{\sigma}_U^2(x) &= \widetilde{\beta}^2(x) = \Big[-\frac{\widehat{\mu}_3(x)}{2}\Big]_+^{\frac{2}{3}} = g_1(\widehat{\mu}_3(x))\,,\ \widetilde{\mu}_U(x) = \widetilde{\beta}(x)\,,\\
\widetilde{\sigma}_V^2(x) &= \left[\widehat{\mu}_2(x) - \widetilde{\beta}^2(x)\right]_+,\ \widetilde{m}(x) = \widehat{m}_1(x) + \widetilde{\mu}_U(x),\\
\widetilde{\sigma}^2(x) &= \left[\frac{1}{3}\widehat{\mu}_4(x) - 2\widetilde{\beta}^4(x)\right]_+^{\frac{1}{2}} = \left\{\frac{1}{3}\widehat{\mu}_4(x) - 2\Big[-\frac{\widehat{\mu}_3(x)}{2}\Big]_+^{\frac{4}{3}}\right\}_+^{\frac{1}{2}}\\
&= g_2(\widehat{\mu}_3(x), \widehat{\mu}_4(x))\,.
\end{aligned}$$

The explicit form of $W(Y_i, X_i)$ appearing in the asymptotic expansion (13) of T_S is

$$W(Y_i, X_i) = \left[\mu_2(X_i) - \varepsilon_i^2\right] - \frac{2[g_1(\mu_3(x))]^{1/2}\left[\mu_3(X_i) - \varepsilon_i^3\right]}{3g_2(\mu_3(X_i), \mu_4(X_i))} - \frac{\mu_4(X_i) - \varepsilon_i^4}{6g_2(\mu_3(X_i), \mu_4(X_i))}.$$

Since $[g_1(\mu_3(x))]^{1/2} = \beta(x)$ and $g_2(\mu_3(X_i), \mu_4(X_i)) = \sigma^2(x)$ under H_0, the estimator $\widehat{\sigma}_W^2$ involved in the test based on D_S can be taken as the sample variance of $\widehat{W}_i' = \widehat{\varepsilon}_i^2 - \frac{2\widetilde{\beta}(X_i)}{3\widetilde{\sigma}^2(X_i)}\widehat{\varepsilon}_i^3 - \frac{1}{6\widetilde{\sigma}^2(X_i)}\widehat{\varepsilon}_i^4$.

Letting $w(x) \equiv 1$, the estimator $\widehat{\sigma}_{W_1}^2$ involved in the tests based on D_{KS} and D_{CM} can be taken as the sample variance of $\widehat{W}_{i1}' = \widehat{W}_i' / \widetilde{\sigma}^2(X_i)$.

Example: Testing the lognormal specification

Under the lognormal specification $U|X = x \sim$ Lognormal$(\ln \sigma_U(x), \ln \psi(x))$ and $V|X = x \sim N(0, \sigma_V^2(x))$, where $\psi(x) \equiv \psi$ is a known constant. Then, (6) and (8) are as follows:

$$\begin{aligned} \mu_U(X) &= \psi^{1/2}\sigma_U(X), \;\; \mathrm{E}(\varepsilon^2|X) = \psi(\psi-1)\sigma_U^2(X) + \sigma_V^2(X), \\ \mathrm{E}(\varepsilon^3|X) &= -\psi^{3/2}(\psi-1)^2(\psi+2)\sigma_U^3(X), \\ \mathrm{E}(\varepsilon^4|X) &= (\psi^4 + 2\psi^3 + 3\psi^2 - 6)\psi^2(\psi-1)^2\sigma_U^4(X) + 3[\sigma^2(X)]^2, \end{aligned} \tag{18}$$

and the estimators given in (7) and (10) are

$$\begin{aligned} \widetilde{\sigma}_U^2(x) &= \frac{1}{\psi}\Big[-\frac{\widehat{\mu}_3(x)}{(\psi-1)^2(\psi+2)}\Big]_+^{\frac{2}{3}} = g_1(\widehat{\mu}_3(x)), \;\; \widetilde{\mu}_U(x) = \sqrt{\psi}\,\widetilde{\sigma}_U(x), \\ \widetilde{\sigma}_V^2(x) &= \big\{\widehat{\mu}_2(x) - \psi(\psi-1)\widetilde{\sigma}_U^2(x)\big\}_+, \; \widetilde{m}(x) = \widehat{m}_1(x) + \widetilde{\mu}_U(x), \\ \widetilde{\sigma}^2(x) &= \Big\{\frac{1}{3}\widehat{\mu}_4(x) - \frac{1}{3}(\psi^4 + 2\psi^3 + 3\psi^2 - 6)\Big[-\frac{\widehat{\mu}_3(x)}{(\psi-1)^{1/2}(\psi+2)}\Big]_+^{\frac{4}{3}}\Big\}_+^{\frac{1}{2}} \\ &= g_2(\widehat{\mu}_3(x), \widehat{\mu}_4(x)). \end{aligned} \tag{19}$$

The explicit form of $W(Y_i, X_i)$ appearing in (13) is
$W(Y_i, X_i) = [\mu_2(X_i) - \varepsilon_i^2] - \frac{2(\psi^4+2\psi^3+3\psi^2-6)\psi^{1/2}[g_1(\widehat{\mu}_3(x))]^{1/2}[\mu_3(X_i)-\varepsilon_i^3]}{9(\psi+2)g_2(\mu_3(X_i),\mu_4(X_i))} - \frac{\mu_4(X_i)-\varepsilon_i^4}{6g_2(\mu_3(X_i),\mu_4(X_i))}$.

Under H_0, $[g_1(\mu_3(x))]^{1/2} = \sigma_U(x)$ and $g_2(\mu_3(x), \mu_4(x)) = \sigma^2(x)$. So, the estimator $\widehat{\sigma}_W^2$ involved in the test based on D_S can be taken as the sample variance of $\widehat{W}_i' = \widehat{\varepsilon}_i^2 - \frac{2(\psi^4+2\psi^3+3\psi^2-6)\psi^{1/2}\widetilde{\sigma}_U(X_i)}{9(\psi+2)\widetilde{\sigma}^2(X_i)}\widehat{\varepsilon}_i^3 - \frac{1}{6\widetilde{\sigma}^2(X_i)}\widehat{\varepsilon}_i^4$.

Letting $w(x) \equiv 1$, the estimator $\widehat{\sigma}_{W_1}^2$ involved in the tests based on D_{KS} and D_{CM} can be taken as the sample variance of $\widehat{W}_{i1}' = \widehat{W}_i'/\widetilde{\sigma}^2(X_i)$.

In practice, we may not know the true value of ψ and we can estimate it by solving for ψ the equation

$$\sum_{i=1}^{n} \mu_4(X_i) = (\psi^4 + 2\psi^3 + 3\psi^2 - 6)\psi^2(\psi - 1)^2 \sum_{i=1}^{n} \widetilde{\sigma}_U^4(X_i) + 3 \sum_{i=1}^{n} [\widehat{\sigma}^2(X_i)]^2.$$

Then, by relating $\mu_5(x)$ with ψ, $\sigma^2(x)$ and $\sigma_U(x)$ similar to (18) we can construct an estimator of $\sigma^2(x)$ under H_0, which differs from that in (19) but is still denoted by $\widetilde{\sigma}^2(x)$ and used in our tests.

The asymptotic expansions given in (13) and (17) change now and thus $\widehat{\sigma}_W^2$ and $\widehat{\sigma}_{W_1}^2$ need to be modified accordingly.

Varying-Coefficient Stochastic Frontier Models for Panel Data

Suppose we have a panel data set
$\{(X_{it}^\top, Z_{it}^\top, Y_{it}),\ i = 1, \ldots, n\,,\ t = 1, 2, \ldots, T\}$.

Here, for the ith firm and at time t, Y_{it}, X_{it} and Z_{it} are respectively observations on

Y: (logarithm of) the output,

$X \in \mathbb{R}^p$: a vector of random regressors representing, for example, the traditional input variables, and

$Z \in \mathbb{R}^q$: a vector of exogenous environmental variables such as computer capital, $R\&D$, production subsidy or even time index.

The panel varying-coefficient stochastic frontier model is

$$Y_{it} = \alpha(Z_{it}) + X_{it}^\top \beta(Z_{it}) + V_{it} - U_{it},\ i = 1, 2, \ldots, n\,,\ t = 1, 2, \ldots, T\,, \tag{20}$$

where U_{it} is the time-varying (non-negative) inefficiency term and V_{it} is the random error term.

We assume that U_{it} and V_{it} are independent conditionally on X_{it} and Z_{it}, and both U_{it} and V_{it} are independent of X_{it} conditional on Z_{it}.

Define $\mu_U(Z_{it}) = \mathrm{E}(U_{it}|Z_{it})$ and $\varepsilon_{it} = V_{it} - U_{it} + \mu_U(Z_{it})$.

Then we have $\mathrm{E}(\varepsilon_{it}|X_{it}, Z_{it}) = 0$,
$\mathrm{Var}(\varepsilon_{it}|X_{it}, Z_{it}) = \mathrm{Var}\,(V_{it}|Z_{it}) + \mathrm{Var}\,(U_{it}|Z_{it}) \equiv \sigma^2(Z_{it})$.

And we can rewrite model (20) as

$$Y_{it} = \theta(Z_{it}) + X_{it}^{\top}\beta(Z_{it}) + \sigma(Z_{it})\,\eta_{it},\ i = 1, 2, \ldots, n\,,\ t = 1, 2, \ldots, T,$$

where $\theta(Z_{it}) = \alpha(Z_{it}) - \mu_U(Z_{it})$ and $\eta_{it} = \varepsilon_{it}/\sigma(Z_{it})$ which has $\mathrm{E}(\eta_{it}|X_{it}, Z_{it}) = 0$ and $\mathrm{Var}(\eta_{it}|X_{it}, Z_{it}) = 1$.

For simplicity, we consider the case where $q = 1$.

We can estimate $\theta(z)$ and $\beta(z)$ by $\widehat{\theta}(z) = \widehat{a}_1$ and $\widehat{\beta}(z) = \widehat{\mathbf{b}}_1$ with $\widehat{a}_1$, $\widehat{a}_2$, $\widehat{\mathbf{b}}_1$ and $\widehat{\mathbf{b}}_2$ minimizing the following locally linear least squares:

$$\sum_{i=1}^{n}\sum_{t=1}^{T}\left[Y_{it} - a_1 - a_2(Z_{it} - z) - X_{it}^{\top}\mathbf{b}_1 - X_{it}^{\top}\mathbf{b}_2(Z_{it} - z)\right]^2 K_{h_0}(Z_{it} - z).$$

Consequently, we obtain the following set of residuals:

$$\widehat{\varepsilon}_{it} = Y_{it} - \widehat{\theta}(z) - X_{it}^{\top}\widehat{\beta}(z),\ i = 1, 2, \ldots, n\,,\ t = 1, 2, \ldots, T\,,$$

and we can estimate $\mu_k(z) \equiv \mathsf{E}(\varepsilon^k | Z = z)$ by

$$\widehat{\mu}_k(z) = \frac{S_{2k}(z)T_{0k}(z) - S_{1k}(z)T_{1k}(z)}{S_{2k}(z)S_{0k}(z) - S_{1k}(z)S_{1k}(z)}\,,$$

where $S_{jk}(z) = \sum_{i=1}^{n}\sum_{t=1}^{T}(Z_{it} - z)^j K_{h_k}(Z_{it} - z)$, $j = 0, 1, 2$, and $T_{jk}(z) = \sum_{i=1}^{n}\sum_{t=1}^{T}(Z_{it} - z)^j K_{h_k}(Z_{it} - z)\widehat{\varepsilon}_{it}^{k}$, $j = 0, 1$.

For $k = 2$, we have a nonparametric estimator of the conditional variance $\mu_2(z) = \sigma^2(z) = \mathrm{Var}(\varepsilon|X = x, Z = z) = \mathrm{Var}(\varepsilon|Z = z)$:

$$\widehat{\sigma}^2(z) = \widehat{\mu}_2(z).$$

Then we have the following standardized residuals obtained by nonparametric estimation:

$$\widehat{\eta}_{it} = \frac{\widehat{\varepsilon}_{it}}{\widehat{\sigma}(Z_{it})},\ i = 1, 2, \ldots, n\,,\ t = 1, 2, \ldots, T.$$

Testing distributional assumptions

The null hypothesis H_0 specifies $U|Z \sim F_0\big(\cdot;\sigma_U^2(Z)\big)$ and $V|Z \sim G_0(\cdot;\sigma_V^2(Z))$.

We replace X by Z in the conditional moment equations (6) and (8). Then, in the definitions (7) and (10) of $\widetilde{\sigma}_U(x)$ and $\widetilde{\sigma}^2(x)$, replacing x and $\widehat{\mu}_k(x)$ by z and $\widehat{\mu}_k(z)$ respectively, we get an estimator of $\mu_2(z) = \sigma^2(z) = \mathrm{Var}(\varepsilon|Z = z)$ under the null hypothesis:

$$\widetilde{\sigma}^2(z) \quad = \quad g_2(\widehat{\mu}_3(z), \widehat{\mu}_4(z)),$$

where $g_2(\cdot,\cdot)$ is as defined in (9).

The corresponding standardized residuals are

$$\widetilde{\eta}_{it} = \frac{\widehat{\varepsilon}_{it}}{\widetilde{\sigma}(Z_{it})},\ \ i = 1, 2, \ldots, n\,,\ t = 1, 2, \ldots, T.$$

Then we have the following test statistic T_S to test the specification $U|Z \sim F_0\big(\cdot;\sigma_U^2(Z)\big)$ and $V|Z \sim G_0(\cdot;\sigma_V^2(Z))$:

$$\begin{aligned} T_S &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\widetilde{\sigma}^2(Z_{it}) - \widehat{\sigma}^2(Z_{it})\right] \\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[g_2(\widehat{\mu}_3(Z_{it}),\widehat{\mu}_4(Z_{it})) - \widehat{\mu}_2(Z_{it})\right]. \end{aligned}$$

Similarly, the Kolmogorov-Smirnov type and the Cramér-von Mises type test statistics are given by

$$\begin{aligned} T_{\eta,KS} &= \sqrt{nT}\sup_{u\in\mathbb{R}}\left|\widetilde{F}_{\eta^2}(u) - \widehat{F}_{\eta^2}(u)\right|, \\ T_{\eta,CM} &= nT\int\left[\widetilde{F}_{\eta^2}(u) - \widehat{F}_{\eta^2}(u)\right]^2 d\widehat{F}_{\eta^2}(u), \end{aligned}$$

where $\widehat{F}_{\eta^2}(u)$ and $\widetilde{F}_{\eta^2}(u)$ are respectively the empirical distribution functions based on $\{\widehat{\eta}_{it}^2\}$ and $\{\widetilde{\eta}_{it}^2\}$.

We can construct bootstrap tests using the test statistics T_S, $T_{\eta,KS}$ and $T_{\eta,CM}$ given here.

In the current context, the null hypothesis H_0 and the alternative hypothesis H_1 are the same as in (2) and (3) except that X is replaced by Z and

$$b_H(z) = g_2(\mu_3(z), \mu_4(z)) - \mu_2(z).$$

Then, $b_H(z) \equiv 0$ under H_0 and $b_H(z) \neq 0$ at least for some z under H_1.

Consider also the following local alternative hypothesis:

$$\mathsf{H}_{1b} : 0 < \mathrm{E}\left|\sqrt{nT} b_H(Z)\right| < \infty \text{ and } \mathrm{E}[nT b_H^2(Z)] < \infty.$$

Theorem

Suppose certain conditions hold. Then under the null hypothesis or the local alternative specification H_{1b}*, we have the following results conditionally on* $\mathcal{X} = \{(Y_{it}, X_{it}, Z_{it}), i = 1, \ldots, n, t = 1, \ldots, T\}$*:*
(i) $\sqrt{nT} T_S^*$ *converges weakly to a Gaussian process with the same asymptotic mean and covariance structure as that of* $\sqrt{nT} T_S$*;*
(ii) $\sqrt{nT}\left[\widetilde{F}^*_{\eta^2}(s) - \widehat{F}^*_{\eta^2}(s)\right]$ *converges weakly to a Gaussian process with the same mean and covariance structure as that of* $\sqrt{nT}\left[\widetilde{F}_{\eta^2}(s) - \widehat{F}_{\eta^2}(s)\right]$*.*

Theorem

Suppose certain conditions hold. Then, under the null hypothesis or the local alternative hypothesis, we have the following asymptotic expansion:

$$T_S = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[W(Y_{it}, X_{it}, Z_{it}) + b_H(Z_{it})\right] + o_p\left(\frac{1}{\sqrt{nT}}\right),$$

where
$W(Y_{it}, X_{it}, Z_{it}) = \left[\mu_2(Z_{it}) - \varepsilon_{it}^2\right] - g_{23}(\mu_3(Z_{it}), \mu_4(Z_{it}))\left[\mu_3(Z_{it}) - \varepsilon_{it}^3\right] - g_{24}(\mu_3(Z_{it}), \mu_4(Z_{it}))\left[\mu_4(Z_{it}) - \varepsilon_{it}^4\right]$. *Further, if* $0 < \sigma_W^2 \equiv \mathrm{E}[W^2(Y_{it}, X_{it}, Z_{it})] < \infty$ *and there exists a weakly consistent estimator for* σ_W^2, *say* $\widehat{\sigma}_W^2$, *then we have*

$$D_S = \sqrt{nT}\frac{T_S}{\widehat{\sigma}_W} \xrightarrow{d} \mathcal{Z} + \delta(b_H)\,, \text{ where } \delta(b_H) = \mathrm{E}\left[\sqrt{nT}b_H(Z)\right]/\sigma_W.$$

A natural estimator of σ_W^2 is the sample variance of $\{\widehat{W}'_{it}\}$, where $\widehat{W}'_{it} = \widehat{\varepsilon}_{it}^2 - g_{23}(\widehat{\mu}_3(Z_{it}), \widehat{\mu}_4(Z_{it}))\widehat{\varepsilon}_{it}^3 - g_{24}(\widehat{\mu}_3(Z_{it}), \widehat{\mu}_4(Z_{it}))\widehat{\varepsilon}_{it}^4$.

Therefore, we have the asymptotic distribution-free test based on the above constructions: reject the null hypothesis at significance level α if $D_S = \sqrt{nT}\left|\frac{T_S}{\widehat{\sigma}_W}\right| > z_{1-\alpha/2}$.

Further, It follows from the following theorem that we have two asymptotic distribution-free tests analogous to those given in (15) and (16).

Theorem

Suppose certain conditions hold and η *and* Z *are independent. Let* $W_1(Y_{it}, X_{it}, Z_{it}) = \frac{W(Y_{it}, X_{it}, Z_{it})}{g_2(\mu_3(Z_{it}), \mu_4(Z_{it}))}$. *Then, uniformly in* $-\infty < s < \infty$,

$$\widehat{C}(s) \equiv \int_{-\infty}^{s} \sqrt{nT}\left[\widetilde{F}_{\eta^2}(u) - \widehat{F}_{\eta^2}(u)\right] du$$

$$= \frac{\mathrm{E}\left[\mathrm{I}(\eta^2 \le s)\eta^2\right]}{\sqrt{nT}\mathrm{E}[w(Z)]} \sum_{i=1}^{n} \sum_{t=1}^{T} w(Z_{it}) \left[W_1(Y_{it}, X_{it}, Z_{it}) + \frac{b_H(Z_{it})}{g_2(\mu_3(Z_{it}), \mu_4(Z_{it}))}\right] + o_p(1).$$

Further, if $0 < \sigma^2_{W_1} \equiv \frac{\mathrm{E}[w^2(Z) W_1^2(Y_{it}, X_{it}, Z_{it})]}{\{\mathrm{E}[w(Z)]\}^2} < \infty$ *and* $\widehat{\sigma}^2_{W_1}$ *is a consistent estimator for* $\sigma^2_{W_1}$, *then we have*

$$D_{KS} \equiv \sup_{s \in \mathbb{R}} \frac{1}{\widehat{\sigma}_{W_1}} \left|\widehat{C}(s)\right| \xrightarrow{d} \left|\mathcal{Z} + \delta_1(b_H)\right|,$$

$$D_{CM} \equiv \frac{1}{\widehat{M}\widehat{\sigma}^2_{W_1}} \int_{-\infty}^{\infty} \left[\widehat{C}(s)\right]^2 d\widetilde{F}_{\eta^2}(s) \xrightarrow{d} \left[\mathcal{Z} + \delta_1(b_H)\right]^2,$$

where $\widehat{M}$ *is a consistent estimator for the version of* M *in (14) in the current context and* $\delta_1(b_H) = \frac{\mathrm{E}[\sqrt{n} w(Z) b_H(Z)]}{(\mathrm{E}[w(Z)^2 W(Y, X, Z)^2])^{1/2}}$.

Simulation study

For our cross-sectional data simulation, our model setting is as follows:

$$Y = 5 + 5X + \sin\{4\pi X\} - U + V,$$

where $X \sim U(0,1)$ and we considered several different cases.

We considered our bootstrap tests using T_S, T_{KS} and T_{CM}, our asymptotic tests using D_S, D_{KS} and D_{CM}, and the Kolmogorov-Smirnov type bootstrap test by Wang, et al. (2011), denoted by KS_W.

The kernel function was taken to be the Gaussian kernel. The nominal level was set as 0.05 in all cases. We ran 1000 simulations under each of the different configurations. For the bootstrap tests, we took the number of bootstrap repetitions B as 500.

Case 1: (H_0 is the half-normal specification, η not independent of X)

H_0 holds with $U|X = x \sim \left|N\left(0, (1+x)^2\right)\right|$ and
$V|X = x \sim N(0, (0.5+2x)^2)$; or

H_1 holds with $U|X = x \sim$ Lognormal $(0.25 + 0.6x, 0.64)$ and
$V|X = x \sim N(0, 0.64(0.5+2x)^2)$.

Table: Empirical sizes and powers under Case 1.

	test	$n = 100$	$n = 200$	$n = 400$	$n = 800$	$n = 1600$
size	T_S	0.055	0.055	0.047	0.048	0.047
	D_S	0.057	0.046	0.054	0.052	0.048
	T_{KS}	0.040	0.060	0.047	0.046	0.048
	T_{CM}	0.047	0.057	0.047	0.047	0.047
	KS_W	0.073	0.216	0.463	0.85	1.0
power	T_S	0.617	0.838	0.981	0.998	1.0
	D_S	0.512	0.784	0.950	0.994	0.998
	T_{KS}	0.345	0.680	0.961	0.998	1.0
	T_{CM}	0.433	0.751	0.973	0.998	1.0
	KS_W	0.580	0.871	0.993	1.0	1.0

Case 2: (H_0 is the half-normal specification, η independent of X)

H_0 holds, $U|X = x \sim \left|N\left(0, (1+2x)^2\right)\right|$,
$V|X = x \sim N(0, 0.36(1+2x)^2)$; or

H_1 holds, $U|X = x \sim$ Exponential $(1+2x)$,
$V|X = x \sim N(0, 0.36(1+2x)^2)$.

Table: Empirical sizes and powers under Case 2.

	test	$n = 200$	$n = 400$	$n = 800$	$n = 1600$	$n = 3200$
size	T_{KS}	0.052	0.042	0.048	0.049	0.051
	T_{CM}	0.058	0.046	0.051	0.050	0.052
	T_S	0.052	0.047	0.052	0.051	0.048
power	T_{KS}	0.430	0.714	0.872	0.969	0.998
	T_{CM}	0.460	0.746	0.878	0.978	0.999
	T_S	0.648	0.810	0.962	0.998	1.0
size	D_{KS}	0.112	0.102	0.090	0.064	0.053
	D_{CM}	0.094	0.072	0.068	0.054	0.052
	D_S	0.060	0.056	0.052	0.051	0.048
power	D_{KS}	0.472	0.612	0.792	0.916	0.974
	D_{CM}	0.526	0.642	0.846	0.938	0.981
	D_S	0.530	0.696	0.894	0.992	0.998

Case 3: (H_0 is the exponential specification, η independent of X)

H_0 holds, $U|X = x \sim$ Exponential $(0.6 + x)$,

$V|X = x \sim N(0, (0.6 + x)^2)$; or

H_1 holds, $U|X = x \sim$ Lognormal $(\ln(0.4 + x), 1)$,

$V|X = x \sim N(0, 0.16(e - 1)e(0.4 + x)^2)$.

Table: Empirical sizes and powers under Case 3.

	test	$n = 200$	$n = 400$	$n = 800$	$n = 1600$	$n = 3200$
size	T_{KS}	0.054	0.050	0.047	0.051	0.052
	T_{CM}	0.042	0.054	0.048	0.047	0.051
	T_S	0.048	0.052	0.050	0.051	0.049
power	T_{KS}	0.524	0.744	0.914	0.958	1.0
	T_{CM}	0.558	0.768	0.934	0.972	1.0
	T_S	0.623	0.788	0.940	0.989	1.0
size	D_{KS}	0.058	0.054	0.052	0.053	0.052
	D_{CM}	0.054	0.052	0.044	0.046	0.048
	D_S	0.048	0.052	0.050	0.050	0.049
power	D_{KS}	0.414	0.536	0.784	0.876	0.926
	D_{CM}	0.438	0.612	0.852	0.932	0.962
	D_S	0.412	0.617	0.886	0.978	1.0

Case 4: (H_0 is the lognormal specification, η independent of X)
H_0 holds, $U|X = x \sim$ Lognormal $(\ln(0.6 + 3x), 0.25)$,
$V|X = x \sim N(0, (0.3 + 1.5x)^2)$; or
H_1 holds, $U|X = x \sim$ Gamma $(2, 2 + x)$, $V|X = x \sim N(0, (0.8 + 0.4x)^2)$.

Table: Empirical sizes and powers in Case 4.

	test	$n = 200$	$n = 400$	$n = 800$	$n = 1600$	$n = 3200$
size	T_{KS}	0.056	0.047	0.050	0.046	0.052
	T_{CM}	0.050	0.043	0.047	0.048	0.051
	T_S	0.044	0.047	0.053	0.051	0.049
power	T_{KS}	0.820	0.884	0.912	0.983	0.998
	T_{CM}	0.830	0.876	0.917	0.982	0.998
	T_S	0.970	0.983	0.986	0.993	1.0
size	D_{KS}	0.082	0.068	0.056	0.046	0.047
	D_{CM}	0.074	0.062	0.060	0.050	0.052
	D_S	0.056	0.042	0.048	0.046	0.048
power	D_{KS}	0.372	0.522	0.854	0.940	0.980
	D_{CM}	0.366	0.584	0.854	0.940	0.974
	D_S	0.330	0.512	0.812	0.922	0.944

In the following, consider a parametric frontier model:

$$Y = 5 + 5X - U + V.$$

Case 01: H_0 is the half-normal specification and with homoscedasticity

H_0 holds, $U|X = x \sim |N(0,1)|$, $V|X = x \sim N(0, 0.49)$; or

H_1 holds, $U|X = x \sim$ Lognormal $(1, 0.55^2)$, $V|X = x \sim N(0, 0.49)$.

Table: Empirical sizes and powers under Case 01.

	test	$n = 200$	$n = 400$	$n = 800$	$n = 1600$	$n = 3200$
size	T_{KS}	0.048	0.052	0.045	0.052	0.051
	T_{CM}	0.042	0.043	0.050	0.046	0.047
	T_S	0.046	0.053	0.046	0.052	0.051
	KS_W	0.046	0.050	0.054	0.046	0.048
power	T_{KS}	0.668	0.912	0.992	0.969	1.0
	T_{CM}	0.740	0.940	0.996	0.978	1.0
	T_S	0.926	0.988	1.0	0.998	1.0
	KS_W	0.704	0.936	0.994	1.0	1.0
size	D_{KS}	0.080	0.056	0.048	0.052	0.051
	D_{CM}	0.078	0.050	0.048	0.050	0.050
	D_S	0.080	0.056	0.048	0.052	0.051
power	D_{KS}	0.432	0.530	0.824	0.928	0.976
	D_{CM}	0.458	0.550	0.850	0.958	0.991
	D_S	0.432	0.530	0.824	0.928	0.976

Case 02: H_0 is the half-normal specification and with heteroscedasticity)

H_0 holds, $U|X=x \sim \left|N\left(0,(1+x)^2\right)\right|$, $V|X=x \sim N(0,(0.5+2x)^2)$; or

H_1 holds, $U|X=x \sim$ Lognormal $(0.25+0.6x, 0.64)$,

$V|X=x \sim N(0, 0.64(0.5+2x)^2)$.

Table: Empirical sizes and powers under Case 02.

	test	$n=100$	$n=200$	$n=400$	$n=800$	$n=1600$
size	T_S	0.044	0.045	0.047	0.047	0.050
	D_S	0.048	0.054	0.048	0.050	0.049
	T_{KS}	0.050	0.044	0.046	0.052	0.047
	T_{CM}	0.053	0.046	0.054	0.046	0.048
	KS_W	0.108	0.294	0.698	0.956	1.0
power	T_S	0.573	0.863	0.990	1.0	1.0
	D_S	0.482	0.776	0.952	0.998	1.0
	T_{KS}	0.316	0.753	0.973	1.0	1.0
	T_{CM}	0.336	0.802	0.983	1.0	1.0
	KS_W	0.634	0.916	0.996	1.0	1.0

Application to US large commercial banks data

A sample of panel data on 66 banks from 2001 to 2015 was obtained from the reports of income and condition published by the Federal Reserve Bank at Chicago (Feng et al., 2017).

Input variables: the quantites of labor X_1, purchased funds and deposits X_2, and physical capital X_3 including premises and other fixed assets.

Output variables: consumer loans Y_1 (including all non-loan financial assets i.e. all financial assets minus the sum of all loans, securities and equity), non-consumer loans Y_2 (including industrial, commercial and real estate loans) and securities Y_3.

The translog frontier function (after normalisation) is given by

$$
\begin{aligned}
-\ln Y_3 = {} & \alpha_0 + \tau t + \frac{1}{2}\delta t^2 + \sum_{m=1}^{2} \alpha_m \ln \frac{Y_m}{Y_3} + \sum_{j=1}^{3} \gamma_j \ln X_j \\
& + \sum_{m=1}^{2} \phi_m t \ln \frac{Y_m}{Y_3} + \sum_{j=1}^{3} \varphi_j t \ln X_j + \frac{1}{2} \sum_{m=1}^{2} \sum_{k=1}^{2} \beta_{mk} \ln \frac{Y_m}{Y_3} \ln \frac{Y_k}{Y_3} \\
& + \frac{1}{2} \sum_{j=1}^{3} \sum_{l=1}^{3} \rho_{jl} \ln X_j \ln X_l + \sum_{m=1}^{2} \sum_{j=1}^{3} \psi_{mj} \ln \frac{Y_m}{Y_3} \ln X_j,
\end{aligned}
\tag{21}
$$

where $t \in [0, 1]$ is time, and $\beta_{mk} = \beta_{km}$ and $\rho_{jl} = \rho_{lj}$ according to symmetry.

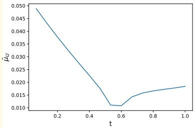
We suggest the following varying-coefficient frontier model:

$$
\begin{aligned}
-\ln Y_3 = {} & \alpha_0(t) + \sum_{m=1}^{2} \alpha_m(t) \ln \frac{Y_m}{Y_3} + \sum_{j=1}^{3} \gamma_j(t) \ln X_j \\
& + \frac{1}{2} \sum_{m=1}^{2} \sum_{k=1}^{2} \beta_{mk}(t) \ln \frac{Y_m}{Y_3} \ln \frac{Y_k}{Y_3} + \frac{1}{2} \sum_{j=1}^{3} \sum_{l=1}^{3} \rho_{jl}(t) \ln X_j \ln X_l \\
& + \sum_{m=1}^{2} \sum_{j=1}^{3} \psi_{mj}(t) \ln \frac{Y_m}{Y_3} \ln X_j, \qquad (22)
\end{aligned}
$$

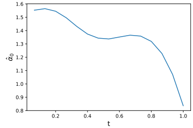
where $\beta_{mk}(t) = \beta_{km}(t)$ and $\rho_{jl}(t) = \rho_{lj}(t)$ by symmetry. The stochastic frontier model is then written as

$$
q_{it} = X_{it}^{\top} \boldsymbol{\theta}(t) - U_{it} + V_{it},\ i = 1, \ldots, n, t = 1, \ldots, T,
$$

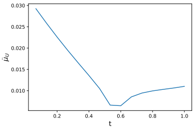
where, for the ith firm and at time t, q_{it} is the response on the LHS of equation (22), X_{it} is a vector comprising the constant 1 and the translog functions appearing on the RHS of equation (22), $\boldsymbol{\theta}(t)$ is a vector of the intercept function and the coefficient functions, U_{it} is the inefficiency term and V_{it} is the random error term.



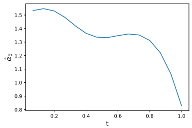
(a) $\widehat{\mu}_U(t)$ half-normal specification



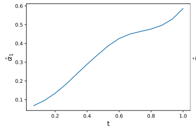
(b) $\widehat{\alpha}_0(t)$ half-normal specification



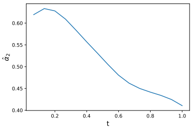
(c) $\widehat{\mu}_U(t)$ exponential specification



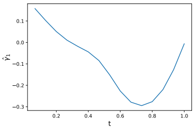
(d) $\widehat{\alpha}_0(t)$ exponential specification



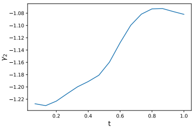
(e) $\widehat{\alpha}_1(t)$



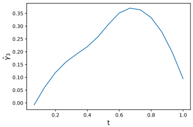
(f) $\widehat{\alpha}_2(t)$



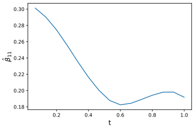
(g) $\widehat{\gamma}_1(t)$



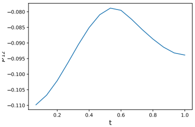
(h) $\widehat{\gamma}_2(t)$



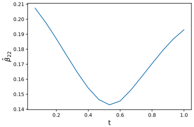
(i) $\widehat{\gamma}_3(t)$



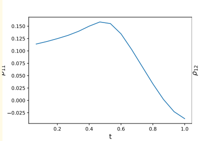
(j) $\widehat{\beta}_{11}(t)$



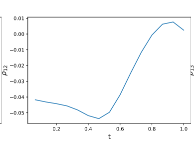
(k) $\widehat{\beta}_{12}(t)$



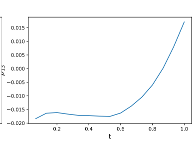
(l) $\widehat{\beta}_{22}(t)$



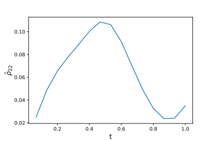
(m) $\widehat{\rho}_{11}(t)$



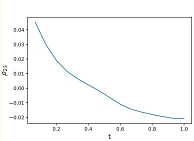
(n) $\widehat{\rho}_{12}(t)$



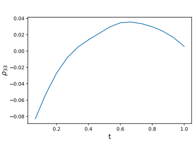
(o) $\widehat{\rho}_{13}(t)$



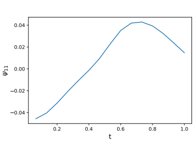
(p) $\widehat{\rho}_{22}(t)$



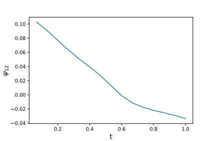
(q) $\widehat{\rho}_{23}(t)$



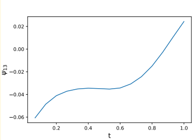
(r) $\widehat{\rho}_{33}(t)$



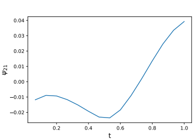
(s) $\widehat{\psi}_{11}(t)$



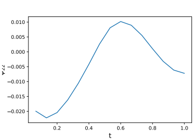
(t) $\widehat{\psi}_{12}(t)$



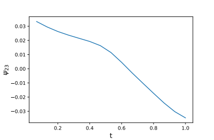
(u) $\widehat{\psi}_{13}(t)$



(v) $\widehat{\psi}_{21}(t)$



(w) $\widehat{\psi}_{22}(t)$



(x) $\widehat{\psi}_{23}(t)$

The translog frontier model (21) may not be suitable for this data set.

In addition, the estimate $1.3358 - 0.1330t + 0.1585t^2$, $t \in [0, 1]$, for the main effect of t in the frontier model (21) under the exponential specification is concave upward which may cause confusion as the output is $-\ln Y_3$.

The intercept function estimate $\widehat{\alpha}_0(t)$ under the exponential specification is roughly non-increasing which seems to be more reasonable.

Table: Banking data: test results. In the parentheses are the bootstrap critical values.

Specification	Test	Test statistic	p-value
half-normal	D_S	2.5440	0.0110
	D_{KS}	2.5075	0.0121
	D_{CM}	9.0392	0.0026
	KS_W	0.0586 (0.0162)	0.0000
	Adjusted KS_W	0.2111 (0.1747)	0.0000
exponential	D_S	0.6903	0.4900
	D_{KS}	0.6720	0.5016
	D_{CM}	0.7004	0.4026
	KS_W	0.0515 (0.0192)	0.0000
	Adjusted KS_W	0.1465 (0.1444)	0.0000
lognormal	D_S	3.1642	0.0016
	D_{KS}	2.2092	0.0272
	D_{CM}	6.0374	0.0140
	KS_W	0.3768 (0.3364)	0.0000
	Adjusted KS_W	0.3949 (0.3616)	0.0000

Note KS_W denotes the bootstrap KS type test by Wang et al. (2011) and Adjusted KS_W denotes its modification to accommodate heterscedasticity.

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