Estimation of the Weibull distribution based on Generalized Adaptive Progressive Hybrid censored Competing Risks Data

Author and Position

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Text

1. Introduction

- In this paper, presume that there are two competing risks factor for the failure of units, and the latent failure time follow a Weibull distribution with a common shape parameter $\boldsymbol{\alpha > 0}$ and different scale parameters $\boldsymbol{\beta_1 > 0, \beta_2 > 0, \beta_1 \neq \beta_2}$.
- To be more specific, suppose the random variable $\boldsymbol{X_{ji}, j = 1, 2}$ of the $\boldsymbol{j}$th cause of failure follows the Weibull distribution, of which the probability density function and the cumulative distribution function can be expressed as

$$f_j(x; \alpha, \beta_j) = \alpha\beta_j x^{\alpha-1} e^{-\beta_j x^\alpha}$$

$$F_j(x; \alpha, \beta_j) = 1 - e^{-\beta_j x^\alpha}, x > 0, j = 1,2$$

and the associate survival and hazard rate functions are

$$S_j(x; \alpha, \beta_j) = 1 - F_j(x; \alpha, \beta_j) = e^{-\beta_j x^\alpha}$$

$$h_j(x; \alpha, \beta_j) = \frac{f_j(x; \alpha, \beta_j)}{1 - F_j(x; \alpha, \beta_j)} = \alpha\beta_j x^{\alpha-1}.$$

- Because of its versatility in fitting time-to-failure distribution of a rather extensive variety of complex mechanisms, the Weibull distribution is one of the most popular distributions used in reliability and lifetime study.

2. Generalized Adaptive Progressive Hybrid Censoring Scheme

- Adaptive progressive censoring scheme also has disadvantages that the time of the experiment can be very long. Hence, Lee(2020) recently suggested Generalized adaptive progressive hybrid censoring scheme. Generalized adaptive progressive hybrid censoring scheme is the addition of time T2 that allows the experiment to proceed as much as possible in Adaptive progressive censoring scheme. If the m-th failure time is observed before T1, generalized progressive censoring scheme is used, if the m-th failure time is observed between T1 and T2, then adaptive progressive censoring scheme is used, if the m-th failure time is observed after T2, Adaptive progressive censoring scheme is used. And Generalized adaptive progressive hybrid censoring scheme can be thought of three cases.

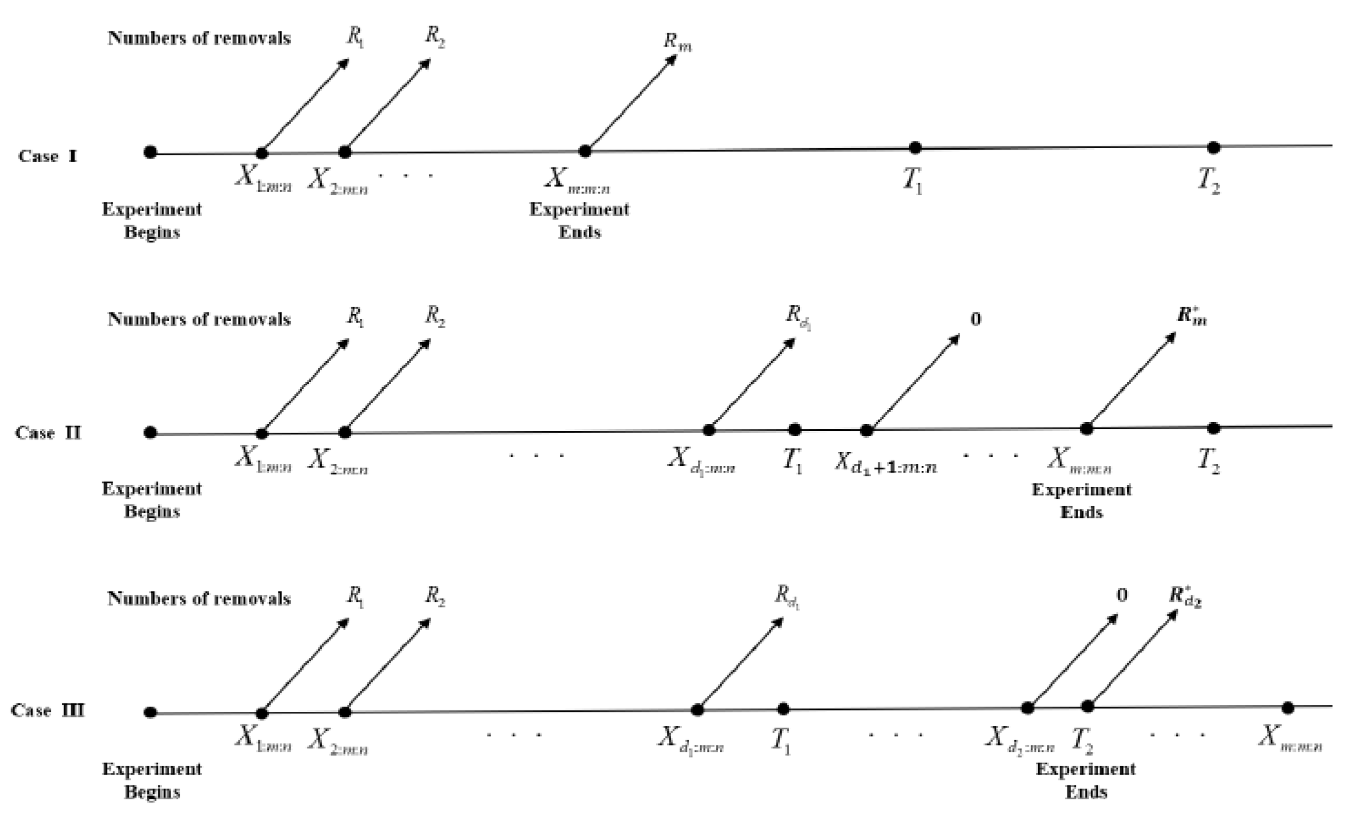

Figure 1.1 Schematic illustration of generalized adaptive progressive hybrid censoring scheme

3. Model description and Likelihood function

- Suppose $\boldsymbol{n}$ identical units are put in lifetime experiment, of which the associated lifetimes are described by identical and independent distributed random variables $\boldsymbol{X_1, X_2, \dots, X_n}$. Assuming there are two competing risks, the one has

$$X_i = \min\{X_{1i}, X_{2i}\}, i = 1, 2, \dots, n$$

where $\boldsymbol{X_{ji}, j = 1, 2}$, latent failure time of the $\boldsymbol{i}$th unit due to the $\boldsymbol{j}$th cause of failure

- From censoring type, following data from the generalized adaptive progressive hybrid censoring experiment can be observed

$$\text{Case I}: (X_{1:m:n}, \delta_1), (X_{2:m:n}, \delta_2), \dots, (X_{m:m:n}, \delta_m), \text{if } T < X_{k:m:n} < X_{m:m:n}$$

$$\text{Case II}: (X_{1:m:n}, \delta_1), \dots, (X_{k:m:n}, \delta_k), \dots, (X_{m:m:n}, \delta_m), \text{if } X_{k:m:n} < T < X_{m:m:n}$$

$$\text{Case III}: (X_{1:m:n}, \delta_1), \dots, (X_{k:m:n}, \delta_k), \dots, (X_{d:m:n}, \delta_d), \text{if } X_{k:m:n} < X_{m:m:n} < T$$

where $\boldsymbol{\delta_i}$ is the indicator for the cause of failure satisfying

$$\delta_i = \begin{cases} 1, \text{unit fails due to cause one} \\ 2, \text{unit fails due to cause two} \end{cases}$$

- And based generalized progressive hybrid censoring experiment, the likelihood function can be written as

$$L(\beta_1, \beta_2, \alpha) = \begin{cases} c_1 \prod_{i=1}^{m} \alpha e^{-\beta(1+r_j)x_{i:m:n}^\alpha} x_{i:m:n}^{\alpha-1} \beta_{\delta_i} & \text{(case I)} \\ c_2 \prod_{i=1}^{m} \alpha e^{-\beta(1+r_j)x_{i:m:n}^\alpha} x_{i:m:n}^{\alpha-1} \beta_{\delta_i} & \text{(case II)} \\ c_3 \prod_{i=1}^{d} \alpha e^{-\beta(1+r_j)x_{i:m:n}^\alpha - \beta r_{d+1}^* T^\alpha} x_{i:m:n}^{\alpha-1} \beta_{\delta_i} & \text{(case III)} \end{cases}$$

where

$$c_1 = \prod_{i=1}^{m} \sum_{j=i}^{m} (1 + r_j),\ c_2 = \prod_{i=1}^{m} \sum_{j=i}^{m} (1 + r_j),\ c_3 = \prod_{i=1}^{d} \sum_{j=i}^{m} (1 + c_j) \text{ and } \beta = \beta_1 + \beta_2.$$

- When competing risks data is available, one can also analyze the observations by using a mixture model. It may be assumed that the latent failure time follows a mixture model of two Weibull distribution as

$$F(x) = pF_1(x; \alpha, \beta_1) + (1 - p)F_2(x; \alpha, \beta_2), x > 0.$$

4. Maximum Likelihood function

- In order to find MLEs of unknown parameters in a concise way, the likelihood function can be rewritten in a compact expression as

$$L(\beta_1, \beta_2, \alpha; data) = c^* \alpha^{d^*} e^{-\beta w(\alpha)} \prod_{i=1}^{d^*} x_{i:m:n}^{\alpha-1} \prod_{i=1}^{d^*} \beta_{\delta_i}$$

where $\boldsymbol{c^* = \prod_{i=1}^{d^*} \sum_{k=j}^{m} (r_k + 1)}$, $\boldsymbol{d^* = m}$ in Case I, Case II, $\boldsymbol{d^* = d}$ in Case III and

$$w(\alpha) = \begin{cases} \sum_{i=1}^{d^*} (1 + r_i) x_{i:m:n}^\alpha & \text{(case I)} \\ \sum_{i=1}^{d^*} (1 + r_i) x_{i:m:n}^\alpha & \text{(case II)} \\ \sum_{i=1}^{d^*} (1 + r_i) x_{i:m:n}^\alpha + r_{d+1}^* T^\alpha & \text{(case III)} \end{cases}$$

- From rewritten likelihood function, the associated log likelihood function is given by

$$l(\beta_1, \beta_2, \alpha) = \ln c^* + d^* \ln\alpha + (\alpha - 1) \sum_{i=1}^{d^*} \ln x_{i:m:n} \sum_{i=1}^{d^*} \ln\beta_{\delta_i} - \beta w(\alpha)$$

suppose that $\boldsymbol{e_j = \sum_{i=1}^{d^*} 1_{(\delta_i = j)} \geq 1, j = 1, 2}$ the MLE of $\boldsymbol{\beta_j}$ given $\boldsymbol{\alpha : \widetilde{\beta}_j = e_j / w(\alpha)}$

- Under the similar approach, the MLE of $\boldsymbol{\beta_1 + \beta_2}$ for given $\boldsymbol{\alpha}$ can be expressed as

$$\widetilde{\beta_1 + \beta_2} = \frac{e_1 + e_2}{w(\alpha)}$$

- Using the above formular log-likelihood function can be rewritten as

$$l(\alpha) = \ln c^* + d^* \ln\alpha + (\alpha - 1) \sum_{i=1}^{d^*} \ln x_{i:m:n} + \sum_{j=1}^{2} e_j \ln\left[\frac{e_j}{w(\alpha)}\right] - d^*$$

- Following result establishes the MLE for parameter $\boldsymbol{\alpha}$

$$\frac{1}{\alpha} + \frac{1}{d^*} \sum_{i=1}^{d^*} \ln x_{i:m:n} - \frac{w'(\alpha)}{w(\alpha)} = 0$$

where

$$w'(\alpha) = \begin{cases} \sum_{i=1}^{m} (1 + r_i) x_{i:m:n}^\alpha \ln x_{i:m:n} & \text{(case I)} \\ \sum_{i=1}^{m} (1 + r_i) x_{i:m:n}^\alpha \ln x_{i:m:n} & \text{(case II)} \\ \sum_{i=1}^{d} (1 + r_i) x_{i:m:n}^\alpha \ln x_{i:m:n} + r_{d+1}^* T^\alpha \ln T & \text{(case III)} \end{cases}$$

- Denote that the MLE for parameter $\boldsymbol{\alpha}$, estimate the MLE of parameter $\boldsymbol{\beta_1, \beta_2}$

$$\hat{\beta}_1 = \frac{e_1}{w(\hat{\alpha})} \text{ and } \hat{\beta}_2 = \frac{e_2}{w(\hat{\alpha})}$$

5. Bayesian Inference

- In this section, independent priors for $\boldsymbol{\beta_1, \beta_2}$ and $\boldsymbol{\alpha}$ are considered as follows

$$\pi_1(\beta_1) \propto \beta_1^{a_1-1} e^{-b_1\beta_1}, a_1 > 0, b_1 > 0, \beta_1 > 0$$

$$\pi_2(\beta_2) \propto \beta_2^{a_2-1} e^{-b_2\beta_2}, a_2 > 0, b_2 > 0, \beta_2 > 0$$

$$\pi_3(\alpha) \propto \alpha^{a_3-1} e^{-b_3\alpha}, a_3 > 0, b_2 > 0, \alpha > 0$$

- The joint prior density function of $\boldsymbol{\beta_1, \beta_2}$ and $\boldsymbol{\alpha}$, namely $\boldsymbol{\pi(\beta_1, \beta_2, \alpha)}$ is given by

$$\pi(\beta_1, \beta_2, \alpha) = \pi_1(\beta_1)\, \pi_2(\beta_2)\, \pi_3(\alpha)$$

- And the associated joint PDF of $\boldsymbol{\beta_1, \beta_2}$ and $\boldsymbol{\alpha}$, namely $\boldsymbol{\pi(\beta_1, \beta_2, \alpha | data)}$ can be expressed as

$$\pi(\beta_1, \beta_2, \alpha | \text{data}) = \frac{\pi(\beta_1, \beta_2, \alpha) L(\beta_1, \beta_2, \alpha; \text{data})}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(\beta_1, \beta_2, \alpha)\, L(\beta_1, \beta_2, \alpha; \text{data})\, d\beta_1 d\beta_2 d\alpha}$$

- The Joint posterior PDF of $\boldsymbol{\beta_1, \beta_2}$ and $\boldsymbol{\alpha}$ can be rewritten as

$$\pi(\beta_1, \beta_2, \alpha | \text{data}) \propto \beta_1^{a_1+e_1-1} e^{-\beta_1[b_1+w(\alpha)]} \cdot \beta_2^{a_2+e_2-1} e^{-\beta_2[b_2+w(\alpha)]} \cdot \alpha^{a_3+d^*-1} e^{-\alpha[b_3 - \sum_{i=1}^{d^*} \ln x_{i:m:n}]}$$

- Furthermore, the posterior marginal PDF of $\boldsymbol{\alpha}$, namely $\boldsymbol{\pi(\alpha | data)}$ can be obtained as

$$\pi(\alpha | \text{data}) \propto \frac{\alpha^{a_3+d^*-1}}{[b_1 + w(\alpha)]^{a_1+e_1} [b_2 + w(\alpha)]^{a_2+e_2}} \cdot e^{-\alpha[b_3 - \sum_{i=1}^{d^*} \ln x_{i:m:n}]}, \alpha > 0$$

- Step2 : For given $\boldsymbol{\alpha}$, generate $\boldsymbol{\beta_1, \beta_2}$ from $\boldsymbol{\pi(\beta_1, \beta_2, \alpha | data)}$.
- Step1 : Generate $\boldsymbol{\alpha}$ from $\boldsymbol{\pi(\alpha | data)}$ by using the method proposed by Devroye.
- Step3 : Repeat above step1 and step 2 N times, and N samples of $\boldsymbol{(\beta_1, \beta_2, \alpha)}$ are generated as$\boldsymbol{(\beta_1^k, \beta_2^k, \alpha^k)}$, $\boldsymbol{k = 1, 2, \dots, N}$.
- Step4 : The Bayes estimate of $\boldsymbol{\eta(\beta_1, \beta_2, \alpha)}$ with respect to square error loss can be constructed as $\boldsymbol{\hat{\eta}(\beta_1, \beta_2, \alpha) = \sum_{k=1}^{N} \hat{\eta}_k / N}$ where $\boldsymbol{\hat{\eta}_k = \hat{\eta}(\beta_1^k, \beta_2^k, \alpha^k)}$.
- Step5 : To construct the HPD credible interval of $\boldsymbol{\eta(\beta_1, \beta_2, \alpha)}$, fist arrange all estimates $\boldsymbol{\hat{\eta}_k}$ in an ascending order, as $\boldsymbol{\hat{\eta}^{[1]}, \hat{\eta}^{[2]}, \dots, \hat{\eta}^{[N]}}$, then for arbitrary $\boldsymbol{0 < \gamma < 1}$, the $\boldsymbol{100(1 - \gamma)}$ credible interval of $\boldsymbol{\eta(\beta_1, \beta_2, \alpha)}$ can be obtained as $\boldsymbol{(\hat{\eta}^{[k]}, \hat{\eta}^{[k+N-[\gamma N+1]]})}$, $\boldsymbol{k = 1, 2, \dots, [N\gamma]}$. Therefore, the $\boldsymbol{100(1 - \gamma)\%}$ HPD credible interval can be constructed as $\boldsymbol{k^*}$th one satisfying $\boldsymbol{\hat{\eta}^{[k^*+N-[\gamma N+1]]} - \hat{\eta}^{[k^*]} = \min_{1 \leq k \leq \gamma N} (\hat{\eta}^{[k+N-[\gamma N+1]]} - \hat{\eta}^{[k]})}$.

6. Simulation Table