Min-max options and EIAs

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Overview

Introduction

- We derive min-max multi-step reflection principle and min-max probabilities.
- Prices of double barrier options are approximated by applying min-max probabilities.
- A new type of EIAs called rebounce EIA is introduced in this paper.
- Literature review
  - Kunitomo & Ikeda(1992) provided formulas for double barrier options.
  - Gulliaume(2010) provided formulas for step double barrier options.
  - Lee et al.(2020+) derived multi-step reflection principle for one direction.

Lee et al.(2020+)

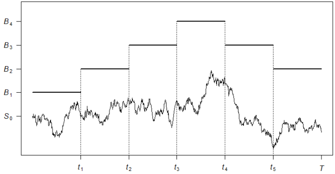

Figure: Max multi-step probabilities

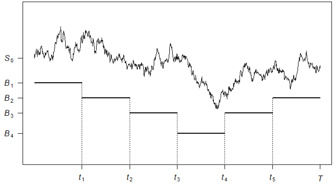

Figure: Min multi-step probabilities

In this paper,

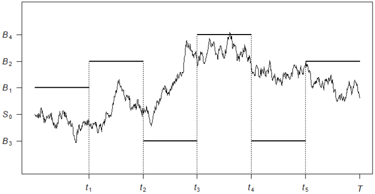

Figure: Min-max multi-step probabilities

Notations

- Underlying asset price at time t:

$$S(t) = S(0)e^{X(t)}, t \geq 0,$$

where $\{X(t)\}$ is a Brownian motion with drift μ, diffusion coefficient σ and $X(0) = 0$.

- The maximum and minimum of the Brownian motion from s to t:
$M(s,t) = max\{X(\tau) : s \leq \tau \leq t\}$ and $M(t) = M(0,t)$
$m(s,t) = min\{X(\tau) : s \leq \tau \leq t\}$ and $m(t) = m(0,t)$.

- The maximum and minimum of the geometric Brownian motion from s to t:
$M^S(s,t) = max\{S(\tau) : s \leq \tau \leq t\}$ and $M^S(t) = M^S(0,t)$
$m^S(s,t) = min\{S(\tau) : s \leq \tau \leq t\}$ and $m^S(t) = m^S(0,t)$.

Reflection principle and factorization formula

- Reflection principle

$$Pr(X(t) \leq x, M(0,t) \leq m) = \Phi(\frac{x \wedge m - \mu t}{\sigma\sqrt{t}}) - e^{\frac{2\mu}{\sigma^2}m}\Phi\left(\frac{x \wedge m - 2m - \mu t}{\sigma\sqrt{t}}\right).$$

for $m \geq 0$. Here $\Phi(\cdot)$ denotes the standard normal distribution function.

- Factorization formula

$$E[e^{kX(T)}I(B); h] = E[e^{kX(T)}; h]Pr(B; k+h)$$

for an event B whose condition is determined by $\{X(t), 0 \leq t \leq T\}$.

Min-max multi-step reflection principle

Theorem

For $1 \le i_1 < i_2 < \cdots < i_g \le n$, let $J = \{i_1, i_2, \cdots, i_g\} = J_{max} \cup J_{min}$ which are disjoint subsets of J where every element in J_{max} implies maximum subinterval or where every element in J_{min} implies minimum subinterval. For convenience, any ordered element in J alternatively belongs to J_{min} or J_{max}. Then

$$Pr(\cap_{i=1}^{n}\{x_i^* \le X(t_i) \le x_i\}, \cap_{\{i \in J_{max}\}}\{M(t_{i-1}, t_i) > m_i\}, \cap_{\{i \in J_{min}\}}\{m(t_{i-1}, t_i) < m_i\})$$
$$= e^{\frac{2\mu}{\sigma^2} m[n]} Pr(\cap_{i=1}^{n}\{x_i^* \le s_i X(t_i) + 2m[i] \le x_i\}),$$

where J is an arbitrary subset of $\{1, 2, \cdots, n\}$,

$$m_{[i]} := (m_i - m_{[i-1]}) I(i \in J) + m_{[i-1]} I(i \notin J), \tag{1}$$

and

$$s_i = \begin{cases} 1, & \text{if the number of elements that are greater than } i \text{ in } J \text{ is even,} \\ -1, & \text{otherwise,} \end{cases} \tag{2}$$

for $i = 1, \cdots, n$.

Min-max multi-step barrier probabilities

Proposition

For $1 \leq i_1 < i_2 < \cdots < i_g \leq n$, let $J = \{i_1, i_2, \cdots, i_g\} = J_{max} \cup J_{min}$ which are disjoint subsets of J where every element in J_{max} implies maximum subinterval or where every element in J_{min} implies minimum subinterval. For convenience, any ordered element in J alternatively belongs to J_{min} or J_{max}. Then, by using (1) and (2),

$$\begin{aligned}
& Pr(\cap_{i=1}^{n} \{x_i^* \leq X(t_i) \leq x_i\}, \cap_{i \in J_{max}} \{M(t_{i-1}, t_i) \leq m_i\}, \cap_{i \in J_{min}} \{m(t_{i-1}, t_i) > m_i\}) \\
& \quad = Pr(\cap_{i=1}^{n} \{x_i^* \leq X(t_i) \leq x_i\}) \\
& \qquad - Pr(\cap_{i=1}^{n} \{x_i^* \leq X(t_i) \leq x_i\}, \cup_{i \in J_{max}} \{M(t_{i-1}, t_i) > m_i\}, \cup_{i \in J_{min}} \{m(t_{i-1}, t_i) \leq m_i\}) \\
& \quad = \sum_{J \subset \{1,2,\cdots,n\}} (-1)^{|J|} Pr(\cap_{i=1}^{n} \{x_i^* \leq X(t_i) \leq x_i\}, \\
& \qquad \cap_{j \in J_{max}} \{M(t_{j-1}, t_j) > m_j\}, \cap_{j \in J_{min}} \{m(t_{j-1}, t_j) \leq m_j\}) \\
& \quad = \sum_{J \subset \{1,2,\cdots,n\}} (-1)^{|J|} e^{\frac{2\mu}{\sigma^2} m_{[n]}^J} Pr(\cap_{i=1}^{n} \{x_i^* \leq s_i^J X(t_i) + 2m_{[i]}^J \leq x_i\}).
\end{aligned}$$

Note that J is any subset of $\{1, 2, \cdots, n\}$ and $|J|$ is the cardinality of the set J.

Double barrier approximation

Corollary

For $1 \leq i_1 < i_2 < \cdots < i_g \leq n$, let $J = \{i_1, i_2, \cdots, i_g\} = J_{max} \cup J_{min}$ which are disjoint subsets of J where every element in J_{max} implies maximum subinterval or where every element in J_{min} implies minimum subinterval. For convenience, any ordered element in J alternatively belongs to J_{min} or J_{max}. n subintervals are divided into N segments where $n \leq N$. Then, by using (1) and (2),

$$Pr\Big(\bigcap_{i\in\{1,\cdots,n\}} \{x_i^* < X(t_i) \leq x_i, M(t_{i-1}, t_i) < m_i, m(t_{i-1}, t_i) > m_i^*\}\Big)$$

$$= \lim_{N\to\infty} \sum_{J_{\text{min}}, J_{\text{max}} \subset \{1,\cdots,N\}} (-1)^{|J|} e^{\frac{2\mu}{\sigma^2} m^{J}_{[N]}{}'} Pr\Big(\bigcap_{i=1}^{N} \{x_i^{*}{}' < s_i' X(t_i) + 2m^{J}_{[i]}{}' \leq x_i'\}\Big).$$

Note that J is any subset of $\{1, 2, \cdots, N\}$ and $|J|$ is the cardinality of the set J.

Double Barrier Out Call (DBOC)

- Option prices decrease as the difference between L and U decreases.
- The option prices converge faster with wider barriers.

L	U	K&I(1992)	$N=2$	$N=3$	$N=4$	$N=5$
40	160	6.8144	6.8144	6.8144	6.8144	6.8144
50	150	6.6129	6.6129	6.6129	6.6129	6.6129
60	140	6.0058	6.0058	6.0058	6.0058	6.0058
70	130	4.5654	4.5654	4.5654	4.5654	4.5654
80	120	2.2082	2.2082	2.2082	2.2082	2.2082
85	115	1.0221	1.0217	1.0221	1.0221	1.0221
90	110	0.1787	0.1710	0.1775	0.1785	0.1786
93	107	0.0098	0.0026	0.0078	0.0092	0.0096
95	105	0.0001	0.0146	-0.0001	0	0

Table: Double barrier Knock out Call option prices at various input levels with $S(0)=100$, $K=100$, $T=1/2$, $r=0.05$, and $\sigma=0.2$.

Double Barrier In Call (DBIC)

- Option prices increase as the difference between L and U decreases.
- The option prices converge faster with wider barriers.

L	U	K&I(1992)	$N=2$	$N=3$	$N=4$	$N=5$
40	160	0.0743	0.0743	0.0743	0.0743	0.0743
50	150	0.2758	0.2758	0.2758	0.2758	0.2758
60	140	0.8829	0.8829	0.8829	0.8829	0.8829
70	130	2.3233	2.3233	2.3233	2.3233	2.3233
80	120	4.6805	4.6805	4.6805	4.6805	4.6805
85	115	5.8666	5.8670	5.8666	5.8666	5.8666
90	110	6.7100	6.7177	6.7112	6.7103	6.7101
93	107	6.8789	6.8862	6.8809	6.8796	6.8791
95	105	6.8886	6.8742	6.8888	6.8887	6.8887

Table: Double barrier Knock in Call option prices at various input levels with $S(0)=100, K=100, T=1/2, r=0.05$, and $\sigma=0.2$.

Step double barrier options (Figures)

- Lower prices for out options are expected for Type2 as it has narrower barriers than Type 1 in the second period.

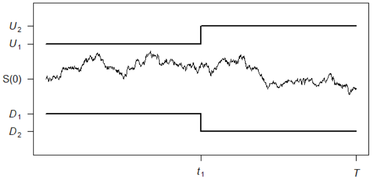

Figure: Type1 step double barrier

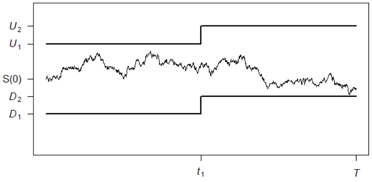

Figure: Type2 step double barrier

Type1 step double barrier out call options

- Option prices decrease with larger values of σ as the probability of being knocked out increases.

r	σ	U_1	L_1	U_2	L_2	$K = 100$	$K = 110$	$K = 120$
0.02	0.15	120	80	125	75	3.6823	0.7091	0.0231
		120	80	130	70	4.1136	0.9516	0.0896
		125	75	130	70	4.2611	1.0133	0.0992
		125	75	135	65	4.4996	1.1662	0.1676
	0.3	120	80	125	75	1.5205	0.3259	0.0115
		120	80	130	70	2.1354	0.6171	0.0733
		125	75	130	70	2.5946	0.7537	0.0901
		125	75	135	65	3.3284	1.1665	0.2381
0.03	0.15	120	80	125	75	3.8383	0.7547	0.0251
		120	80	130	70	4.3082	1.0195	0.0980
		125	75	130	70	4.4660	1.0861	0.1086
		125	75	135	65	4.7304	1.2558	0.1848
	0.3	120	80	125	75	1.5344	0.3304	0.0118
		120	80	130	70	2.1596	0.6272	0.0749
		125	75	130	70	2.6241	0.7660	0.0920
		125	75	135	65	3.3732	1.1880	0.2436

Table: Type 1 Step Double barrier at various input levels with S(0)=100

Type2 step double barrier out call options

- Prices with Type2 has smaller values than Type 1 as the probability of being knocked out increases with narrower barriers.
- Decreases as strike price increases since call option has payoff $(S(T) - K)_+$.

r	σ	U_1	L_1	U_2	L_2	$K = 100$	$K = 110$	$K = 120$
0.02	0.15	120	80	125	85	3.6801	0.7090	0.0231
		120	80	130	90	4.0274	0.9467	0.0894
		125	75	130	80	4.2610	1.0133	0.0992
		125	75	135	85	4.4964	1.1662	0.1676
	0.3	120	80	125	85	1.4026	0.3075	0.011
		120	80	130	90	1.5481	0.4798	0.0591
		125	75	130	80	2.5521	0.7467	0.0895
		125	75	135	85	3.0481	1.1011	0.2282
0.03	0.15	120	80	125	85	3.8361	0.7547	0.0251
		120	80	130	90	4.2201	1.0143	0.0979
		125	75	130	80	4.4660	1.0861	0.1086
		125	75	135	85	4.7272	1.2557	0.1848
	0.3	120	80	125	85	1.4156	0.3118	0.0112
		120	80	130	90	1.5666	0.4877	0.0604
		125	75	130	80	2.5813	0.7589	0.0914
		125	75	135	85	3.0901	1.1215	0.2335

Rebounce EIAs

- Payoff

$$\begin{cases} \alpha(1+(c_j^u - c_i^d))I(A_{i,j}) \text{ for } i \neq j \text{ and } i < j, & \text{at time } T, \\ (1+gT)I(A_{i,j}^c), & \text{at time } T, \end{cases}$$

for $i = 1, ..., n-1$ and $j = 2, ..., n$. α is a participation rate and g is a guaranteed rate. c^u and c^d are upper and lower coupon rates, respectively.

- Activating condition
$A_{i,j}$ is the triggering event for rebouncing with i^{th} lower barrier and j^{th} upper barrier where $i < j$.

$$\begin{aligned} A_{i,j} =& \{\bigcap_{k=1}^{i-1}\{m^S(t_{k-1}, t_k) > B_k^d\}, m^S(t_{i-1}, t_i) < B_i^d, \\ & \bigcap_{k=i+1}^{j-1} \{M^S(t_{k-1}, t_k) < B_k^u\}, M^S(t_{j-1}, t_j) > B_j^u\}. \end{aligned}$$

Activating conditions for rebounced payoff

- Description of $A_{1,3}$

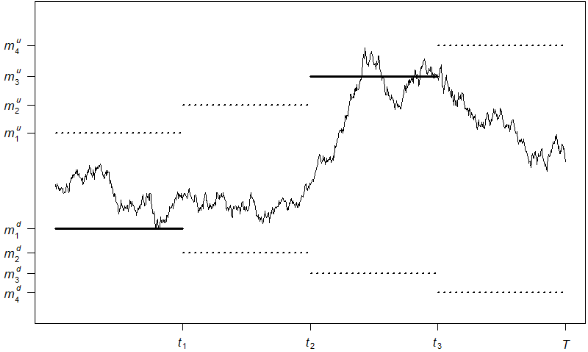

Figure: Activating conditions for the first lower touch and the first upper touch at the third upper barrier.

Activating conditions for guaranteed payoff

- Guaranteed payoff is paid as barriers are not being touched.

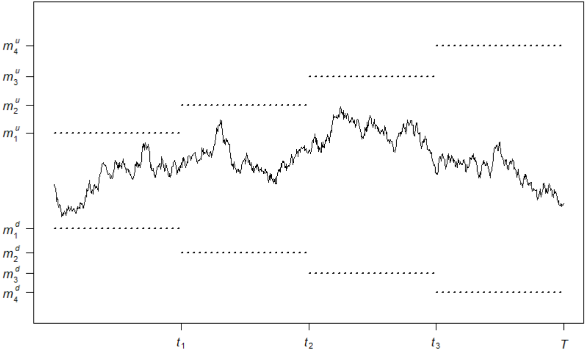

Figure: Activating conditions for guaranteed rate.

Break-even rates

- Break-even participation rates

$$\alpha_{BE} = \frac{1 - e^{-rT}(1+gT)Pr(A^c_{i,j})}{\sum_{i<j,i\neq j} e^{-rT}(1+(c^u_j - c^d_i))Pr(A_{i,j})}$$

- Break-even guaranteed rate

$$g_{BE} = (\frac{(1 - \sum_{i<j,i\neq j} e^{-rT}\alpha(1+(c^u_j - c^d_i))Pr(A_{i,j}))e^{rT}}{Pr(A^c_{i,j})} - 1)/T$$

- Lower break-even guaranteed rates are given with higher coupon rates.

α	c_{u_1} (c_{d_1})	c_{u_2} (c_{d_2})	c_{u_3} (c_{d_3})	c_{u_4} (c_{d_4})	i	j	$1+(c_{u_j}-c_{d_j})$	$1+gT$	$Pr(A_{i,j})$	$1-\sum Pr(A_{i,j})$
1	1.05 (0.95)	1.1 (0.9)	1.15 (0.85)	1.2 (0.8)	1	2	1.15	**1.2342**	0.2182	0.5847
					1	3	1.2		0.0556	
					1	4	1.25		0.0306	
					2	3	1.25		0.0094	
					2	4	1.3		0.0981	
					3	4	1.35		0.0035	
	1.1 (0.9)	1.15 (0.85)	1.2 (0.8)	1.25 (0.75)	1	2	1.25	**1.1622**	0.2004	0.5904
					1	3	1.3		0.0593	
					1	4	1.35		0.0333	
					2	3	1.35		0.0154	
					2	4	1.4		0.0960	
					3	4	1.45		0.0051	

Table: Break-even guaranteed rate with $r=0.02, \sigma=0.2$, and $T=10$

Break-even participation rates

- Lower break-even α is given with higher coupon rates.

$1+gT$	c_{u_1} (c_{d_1})	c_{u_2} (c_{d_2})	c_{u_3} (c_{d_3})	c_{u_4} (c_{d_4})	i	j	$1+(c_{u_j}-c_{d_j})$	α	$Pr(A_{i,j})$	$1-\sum Pr(A_{i,j})$
1.2	1.05 (0.95)	1.1 (0.9)	1.15 (0.85)	1.2 (0.8)	1	2	1.15	**0.9602**	0.2182	0.5847
					1	3	1.2		0.0556	
					1	4	1.25		0.0306	
					2	3	1.25		0.0094	
					2	4	1.3		0.0981	
					3	4	1.35		0.0035	
	1.1 (0.9)	1.15 (0.85)	1.2 (0.8)	1.25 (0.75)	1	2	1.25	**0.9583**	0.2004	0.5904
					1	3	1.3		0.0593	
					1	4	1.35		0.0333	
					2	3	1.35		0.0154	
					2	4	1.4		0.0960	
					3	4	1.45		0.0051	

Table: Break-even participation rate with $r=0.02, \sigma=0.2$, and $T=10$

Conclusion

- Min-max multi-step probabilities are useful to calculate prices for options and EIAs which deal with upper and lower barriers.
- We can approximate prices for double barrier options by using min-max probabilities.
- Rebounce EIA is newly introduced to which uses lower and upper barriers.

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Thank you