MA(2) process

$$\underbrace{Z_t - \mu}_{\dot{Z}_t} = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}, \quad \varepsilon_t \overset{iid}{\sim} WN(0, \sigma^2)$$

$$\Rightarrow \dot{Z}_t = (1 - \theta_1 B - \theta_2 B^2)\varepsilon_t = \Theta(B)\varepsilon_t$$

* invertibility $\Rightarrow \theta_1 + \theta_2 < 1, \ \theta_2 - \theta_1 < 1, \ |\theta_2| < 1$

① $E(Z_t) = \mu$ ② $Var(Z_t) = \gamma_0 = (1 + \theta_1^2 + \theta_2^2)\sigma^2$

$$* \, Cov(\dot{Z}_t, \dot{Z}_{t+1}) = Cov(\varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}, \varepsilon_{t+1} - \theta_1 \varepsilon_t - \theta_2 \varepsilon_{t-1})$$

$$= -\theta_1 \sigma^2 + \theta_1 \theta_2 \sigma^2 = -\theta_1 (1 - \theta_2)\sigma^2 = \gamma_1$$

$$Cov(\dot{Z}_t, \dot{Z}_{t+2}) = Cov(\varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}, \varepsilon_{t+2} - \theta_1 \varepsilon_{t+1} - \theta_2 \varepsilon_t)$$

$$= -\theta_2 \sigma^2 = \gamma_2$$

$$Cov(\dot{Z}_t, \dot{Z}_{t+k}) = \gamma_k = 0, \quad k \geq 3$$

$$\Rightarrow \text{ACF} \quad \rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \dfrac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2}, & k=1 \\ \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2}, & k=2 \\ 0, & k \geq 3 \end{cases}$$

MA(q) process

$$\underbrace{Z_t-\mu}_{\dot{Z}_t}=\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\cdots-\theta_q\varepsilon_{t-q}$$

$$\Rightarrow \dot{Z}_t=(1-\theta_1B-\theta_2B^2-\cdots-\theta_qB^q)\cdot\varepsilon_t=\Theta(B)\cdot\varepsilon_t.$$

invertibility : $\Theta(B)=0$ 의 q개 근들의 절대값이 모두 1보다 커야 함.

① $E(Z_t)=\mu$ ② $Var(Z_t)=(1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2)\sigma^2$

$$* \; Cov(\dot{Z}_t,\dot{Z}_{t+k})=Cov(\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\cdots-\theta_q\varepsilon_{t-q},\ \varepsilon_{t+k}-\theta_1\varepsilon_{t+k-1}-\theta_2\varepsilon_{t+k-2}-\cdots-\theta_q\varepsilon_{t+k-q})$$
$$=\gamma_k$$

If $k=1 \Rightarrow \gamma_k=Cov(\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\cdots-\theta_q\varepsilon_{t-q},\ \varepsilon_{t+1}-\theta_1\varepsilon_t-\theta_2\varepsilon_{t-1}-\cdots-\theta_q\varepsilon_{t+1-q})$

$$=(-\theta_1+\theta_1\theta_2+\theta_2\theta_3+\cdots+\theta_{q-1}\theta_q)\sigma^2$$

If $k=2 \Rightarrow \gamma_2=Cov(\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\cdots-\theta_q\varepsilon_{t-q},\ \varepsilon_{t+2}-\theta_1\varepsilon_{t+1}-\theta_2\varepsilon_t-\cdots-\theta_q\varepsilon_{t+2-q})$

$$=(-\theta_2+\theta_1\theta_3+\theta_2\theta_4+\cdots+\theta_{q-2}\theta_q)\sigma^2$$

$$\vdots \quad \vdots$$

$$\gamma_k=\begin{cases}(-\theta_k+\theta_1\theta_{k+1}+\theta_2\theta_{k+2}+\cdots+\theta_{q-k}\theta_q)\sigma^2, & k=1,2,\cdots,q\\ 0, & k\geq q+1\end{cases}$$

$$\Rightarrow \text{ACF} \quad \rho_k=\frac{\gamma_k}{\gamma_0}=\begin{cases}\dfrac{-\theta_k+\theta_1\theta_{k+1}+\cdots+\theta_{q-k}\theta_q}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2}, & k=1,2,\cdots,q\\ 0, & k\geq q+1\end{cases}$$