AR(1) process

$$Z_t - \mu = \phi(Z_{t-1} - \mu) + \varepsilon_t, \quad |\phi| < 1, \quad \varepsilon_t \overset{iid}{\sim} WN(0, \sigma^2)$$

$$\hookrightarrow Z_t - \mu = \phi\{\phi(Z_{t-2} - \mu) + \varepsilon_{t-1}\} + \varepsilon_t$$

$$= \phi^2(Z_{t-2} - \mu) + \phi\varepsilon_{t-1} + \varepsilon_t$$

$$= \phi^2\{\phi(Z_{t-3} - \mu) + \varepsilon_{t-2}\} + \phi\varepsilon_{t-1} + \varepsilon_t$$

$$= \phi^3(Z_{t-3} - \mu) + \phi^2\varepsilon_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t$$

$$= \cdots = \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \phi^3\varepsilon_{t-3} + \cdots = \sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j}$$

$$\Rightarrow Z_t = \mu + \sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j}$$

1) $E(Z_t) = E\left[\mu + \sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j}\right] = \mu$

2) $Var(Z_t) = Var\left[\mu + \sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j}\right] = \sum_{j=0}^{\infty}\phi^{2j}\cdot\underbrace{Var(\varepsilon_{t-j})}_{\sigma^2} = \sigma^2\cdot\frac{1}{1-\phi^2}$

3) $Cov(Z_t, Z_{t+h}) = Cov\left[\mu + \sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j}, \ \mu + \sum_{i=0}^{\infty}\phi^i\varepsilon_{t+h-i}\right]$

$$= E\left[\sum_{i=0}^{\infty}\phi^{h+2i}\cdot\varepsilon_{t-i}^2\right] = \sigma^2\cdot\frac{\phi^h}{1-\phi^2}$$

linear process

$$Z_t = \mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j},$$

$$\varepsilon_t \overset{iid}{\sim} WN(0, \sigma^2), \ \psi_0 = 1$$

$$E(Z_t) = E\left[\mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}\right] = \mu$$

$$Var(Z_t) = Var\left[\mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}\right] = \sum_{j=0}^{\infty} \psi_j^2 \cdot Var(\varepsilon_{t-j}) = \sigma^2 \cdot \sum_{j=0}^{\infty} \psi_j^2$$

$$Cov(Z_t, Z_{t+h}) = Cov[\mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots,$$

$$\mu + \varepsilon_{t+h} + \psi_1 \varepsilon_{t+h-1} + \psi_2 \varepsilon_{t+h-2} + \cdots]$$

$$= \{Cov(\varepsilon_t, \varepsilon_{t+h}) + Cov(\varepsilon_t, \psi_1 \varepsilon_{t+h-1}) + \cdots\}$$

$$+ \{Cov(\psi_1 \varepsilon_{t-1}, \varepsilon_{t+h}) + Cov(\psi_1 \varepsilon_{t-1}, \psi_1 \varepsilon_{t+h-1}) + \cdots\}$$

$$+ \cdots$$

$$= E\left[\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \psi_i \psi_j \varepsilon_{t-j} \varepsilon_{t+h-i}\right]$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \psi_i \psi_j E(\varepsilon_{t-j} \cdot \varepsilon_{t+h-i}) = \sigma^2 \cdot \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

j	i
0	h
1	$h+1$
2	$h+2$
$\vdots$	$\vdots$