Valuation of Piecewise Linear Double Barrier Options

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Contents

Introduction

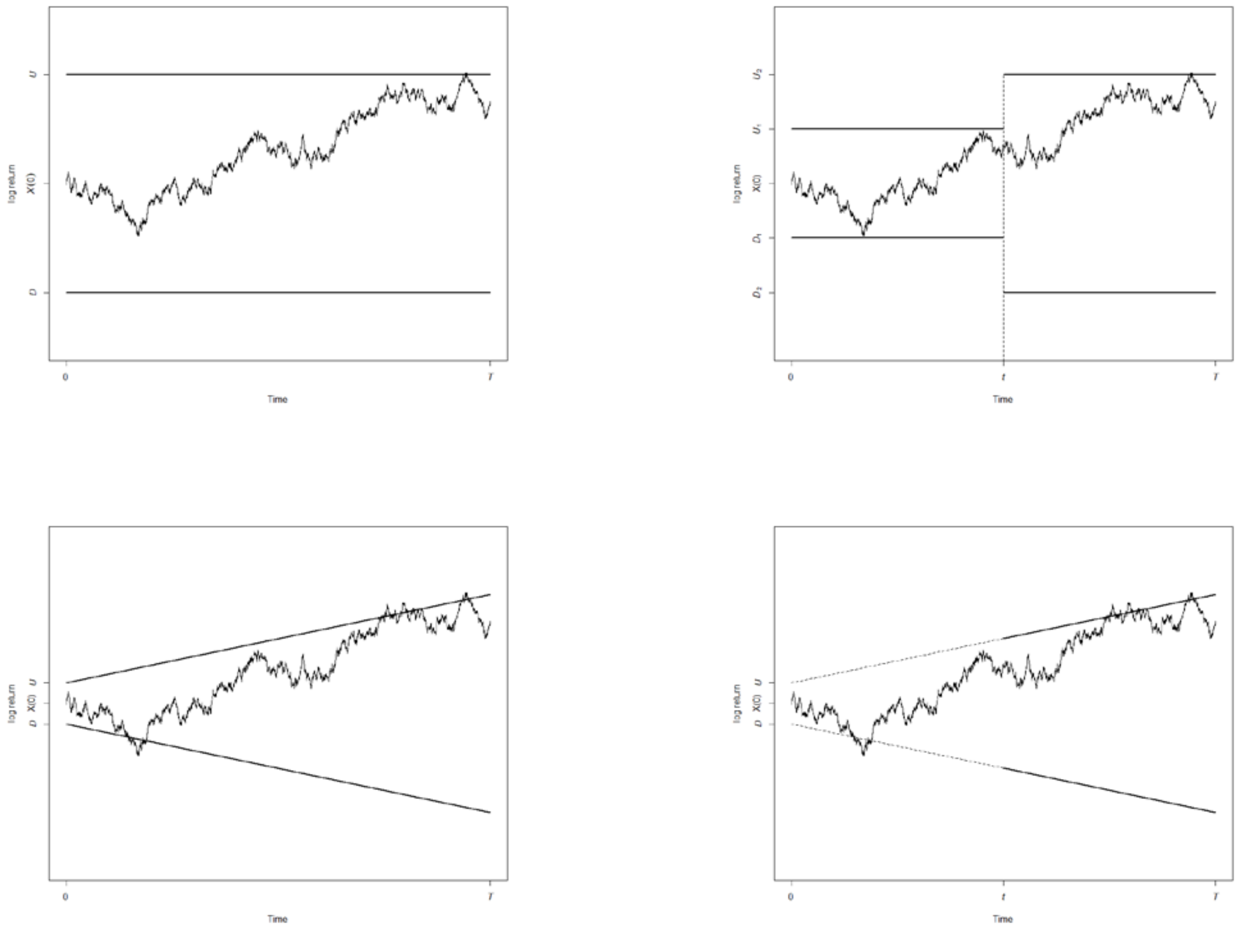
- The double barrier options are actively traded in the over-the-counter market by virtue of their tailoring capacity for risk management and investment strategy with low cost.
    - For example, double knock-out options provide a vehicle to materialize investment projects.
- It is possible to approximate the prices of complex double barrier options using numerical methods, but it may be costly to employ them.
- Hence, it is worth investing in a new type of non-trivial double barrier and developing a corresponding valuation formula from practical and theoretical perspectives.
- In this paper, we establish a closed-form pricing formula for piecewise linear double barrier options and their variants.

Literature review

- Kunitomo and Ikeda (1992) provide the closed-form solutions to the curved double barrier options.
- Buchen and Konstandatos (2009) derives prices of the exponential double barrier options including partial barrier.
- Guillaume (2010) provide non-crossing probability of a piecewise linear double barrier over two disjoint intervals.
- Lee et al. (2021) derived the closed-form solutions to the piecewise linear barrier options over three disjoint intervals.

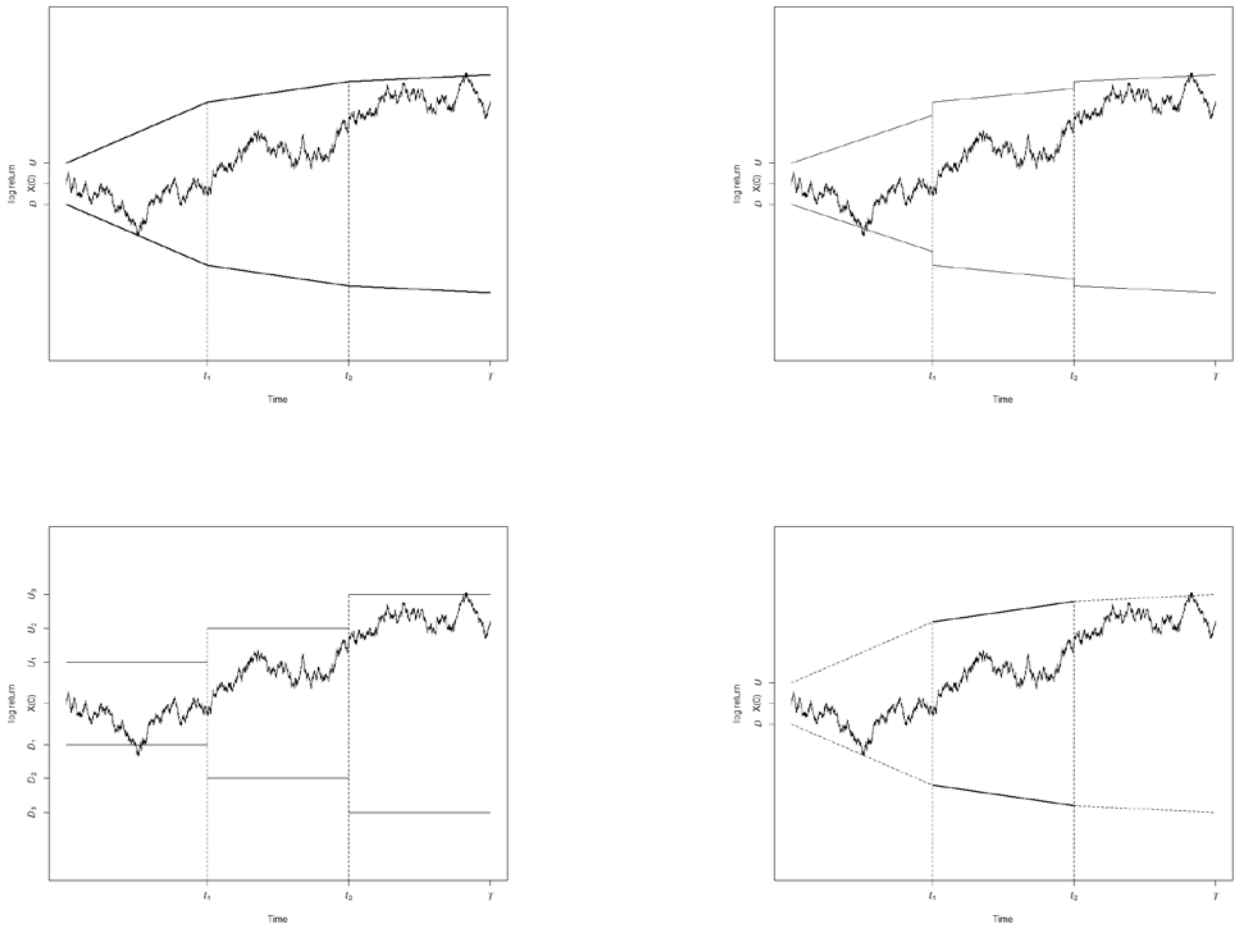
Literature review

- Illustrations of four classical double barriers



Literature review

- Illustrations of four double barriers discussed in this paper



Preliminaries

- One-dimensional Brownian motion
  - $S(t) = S(0)\, e^{X(t)}$, $t \geq 0$
  - $dX(t) = \mu\, dt + \sigma\, dZ(t)$, $dZ(t) \sim N(0, dt)$.

- Essher transform and factorization formula (see Gerber and Shiu (1994, 1996))
  - The moment generating function of $X(t)$ under Esshcer measure of parameter h is

    $\mathrm{E}[e^{zX(t)}; h] = \exp(z\mu t + z^2\sigma^2 t/2)$.
  - The **risk-neutral measure** is the Essher measure of parameter $h = h^*$ with respect to which the process $\{e^{-rt}S(t)\}$ is a martingale. Thus,

    $\mathrm{E}[e^{-rt}S(t); h^*] = S(0)$.

Preliminaries

- The process $\{X(t)\}$ under Esscher measure of parameter h is a one-dimensional Brownian motion with

  drift $\mu + h\sigma^2$ and diffusion coefficient $\sigma > 0$.

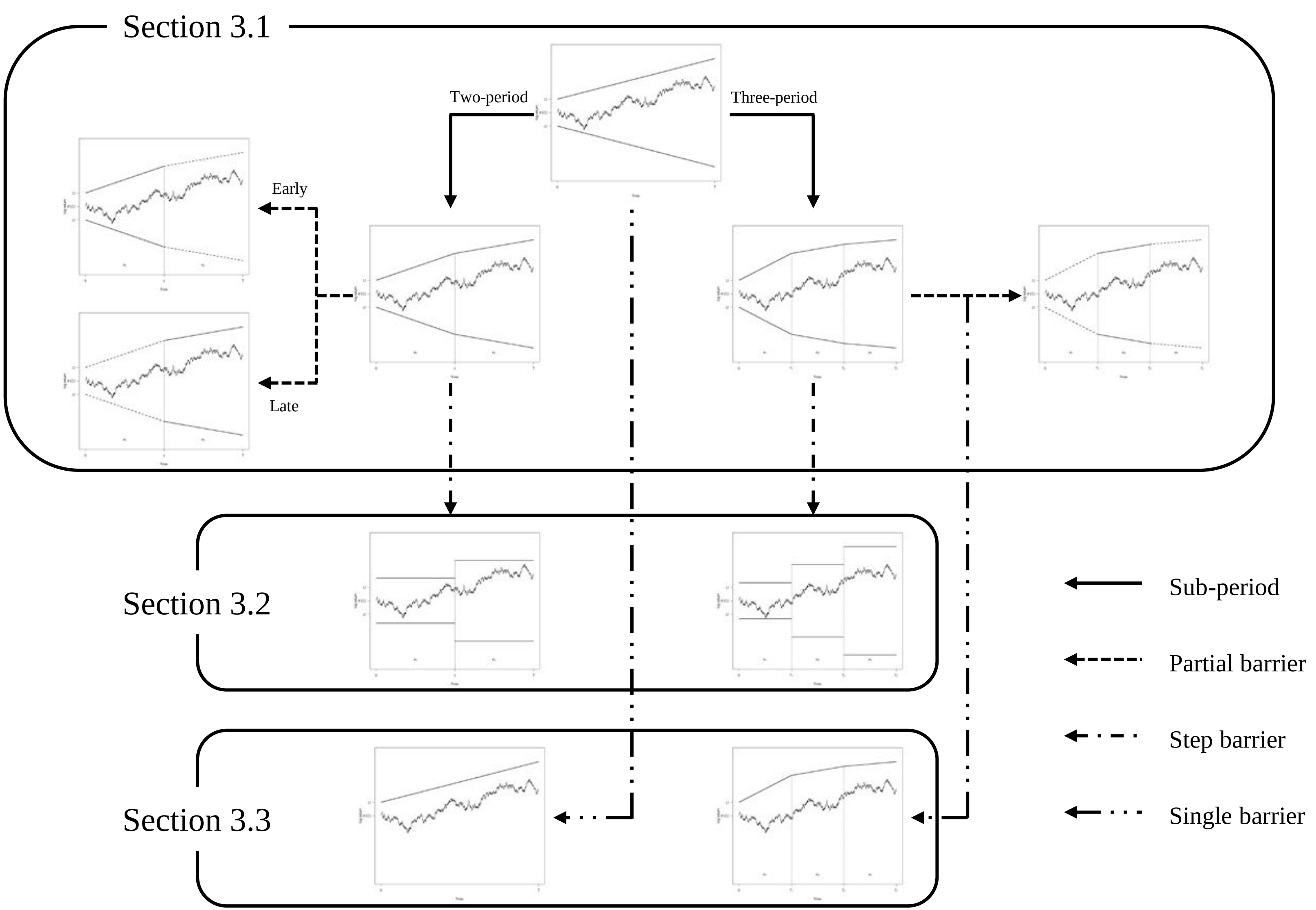
- Special case of the factorization formula (see Gerber and Shiu (1994, 1996))
  - For a random variable Y that is a real-valued function of $\{X(t)\}_{t\geq 0}$,

    $\mathrm{E}[e^{cX(t)}Y; h] = \mathrm{E}[e^{cX(t)}; h]\ \mathrm{E}[Y; h+c]$.
  - In particular, for and event B whose condition is determined by $\{X(t)\}_{t\geq 0}$,

    $\mathrm{E}[e^{cX(t)}I(B); h] = \mathrm{E}[e^{cX(t)}; h]\ \mathrm{E}[I(B); h + c] = \mathrm{E}[e^{cX(t)}; h]\ \Pr(B; h + c)$,

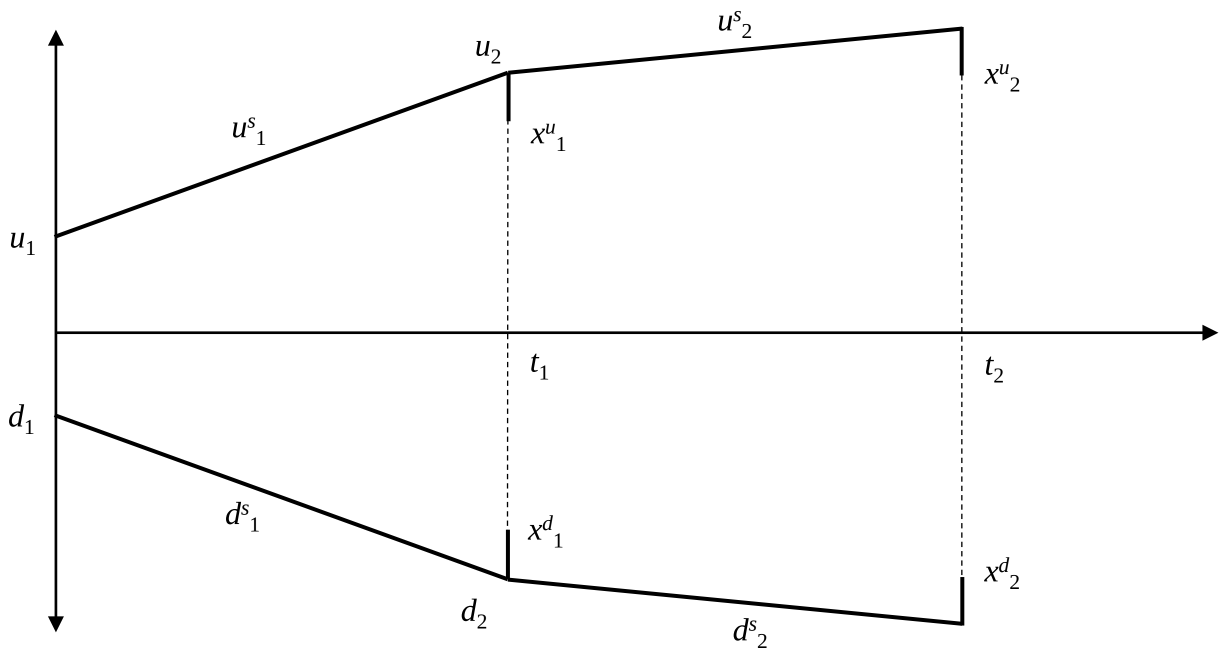
    where $I(\cdot)$ denotes the indicator function and $\Pr(B; h)$ is the probability of the even B under the parameter h.

Piecewise linear double barrier (Preview)



Piecewise linear double barrier

- Barrier expression
  - Up barrier $$u(t)=\sum_{i=1}^{n}u_i(t)I(t_{i-1}\le t<t_i)=\sum_{i=1}^{n}(u_i+u_i^s(t-t_{i-1}))I(t_{i-1}\le t<t_i)$$
  - Down barrier $$d(t)=\sum_{i=1}^{n}d_i(t)I(t_{i-1}\le t<t_i)=\sum_{i=1}^{n}(d_i+d_i^s(t-t_{i-1}))I(t_{i-1}\le t<t_i)$$



Piecewise linear double barrier

- Barrier expression
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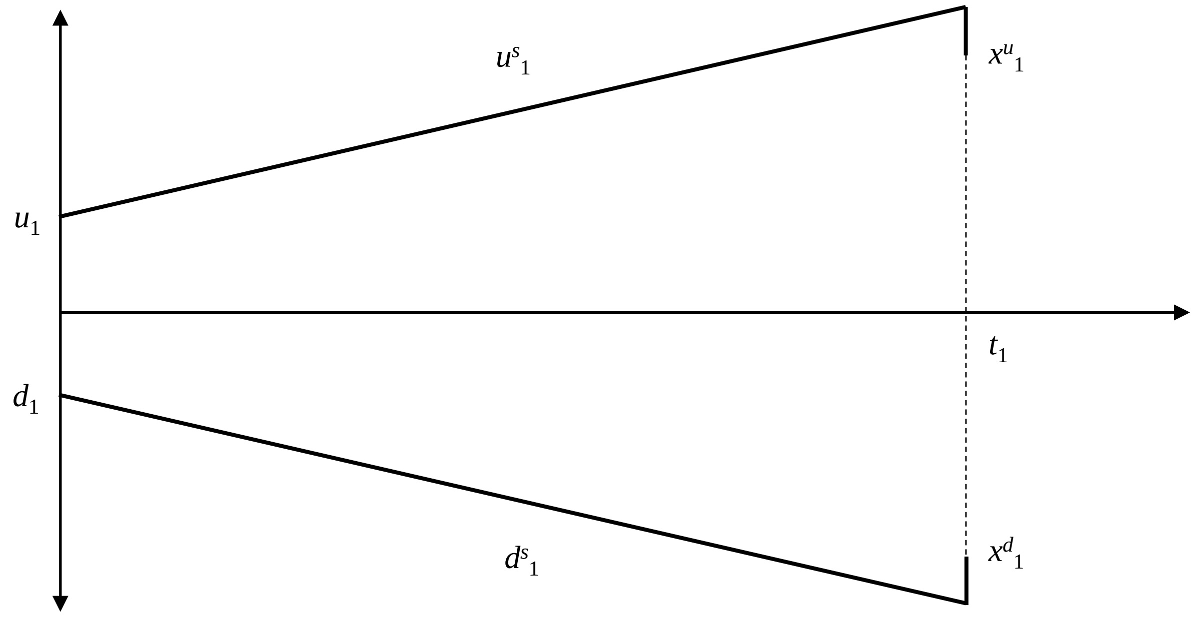
- Definition
  - $m_i^k=[u_i-(d_i-u_i)\frac{k-1}{2}]I(k:\mathrm{odd})+[(d_i-u_i)\frac{k}{2}]I(k:\mathrm{even})$
  - $v_i^k=[u_i^s-(d_i^s-u_i^s)\frac{k-1}{2}]I(k:\mathrm{odd})+[(d_i^s-u_i^s)\frac{k}{2}]I(k:\mathrm{even})$
  - $w_i^k=[(u_i d_i^s-d_i u_i^s)\frac{k}{2}]I(k:\mathrm{even}).$

Piecewise linear double barrier (Ordinary double barrier)

- Single period
  - Linear double barrier

$$\Pr(x_1^d < X(t_1) < x_1^u, \{d_1(t) < X(t) < u_1(t), 0 < t < t_1\})$$

$$= \sum_{k\in\mathbb{Z}} (-1)^k e^{\frac{2}{\sigma^2}[(\mu_1 - v_1^k)m_1^k + w_1^k]} \Pr(x_1^d < X(t_1) + 2m_1^k < x_1^u \mid \mu_1).$$



Piecewise linear double barrier (Ordinary double barrier)

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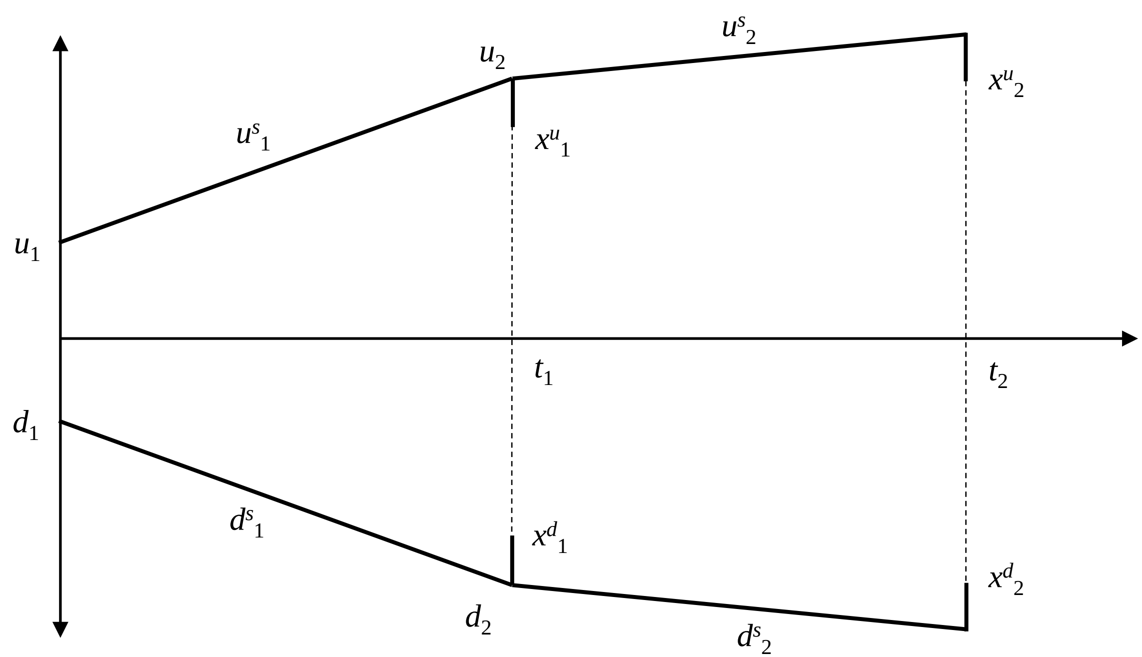
$$= \sum_{k \in \mathbb{Z}} (-1)^k e^{\frac{2}{\sigma^2}[(\mu_1 - v_1^k) m_1^k + w_1^k]} \Pr(x_1^d < X(t_1) + 2m_1^k < x_1^u \mid \mu_1).$$

  - In (A.10) of Kunitomo and Ikeda (1992), above formula can be obtained by replacing x_0, γ_1, γ_2, δ_1, δ_2, μ^*, and T by 0, u_1, d_1, $u^s{}_1$, $d^s{}_1$, μ_1, and t_1, respectively.

Piecewise linear double barrier (Ordinary double barrier)

- Two-period
  - Two-period piecewise linear double barrier

$$\Pr\left(\bigcap_{i=1}^{2}\{x_i^d < X(t_i) < x_i^u\}, \bigcap_{i=1}^{2}\{d_i(t) < X(t) < u_i(t), t_{i-1} < t < t_i\} \mid \mu_1, \mu_2\right)$$
$$= \sum_{\kappa=(k_1,k_2)\in\mathbb{Z}^2} (-1)^{k_1+k_2} e^{\sum_{i=1}^{2} 2\frac{(s_i^\kappa \mu_{[i]}^\kappa - v_i^{k_i}) m_i^{k_i} + w_i^{k_i}}{\sigma^2}} \mathrm{E}\left[e^{-\frac{2R_1^\kappa}{\sigma^2} X(t_1)}\right] \Pr\left(\bigcap_{i=1}^{2}\{x_i^d < s_i^\kappa X(t_i) + 2m_{[i]}^\kappa < x_i^u \mid \mu_{[1:2]}^\kappa\}\right).$$



Piecewise linear double barrier (Ordinary double barrier)

- Two-period
  - Two-period piecewise linear double barrier

$$\Pr\left(\bigcap_{i=1}^{2}\{x_i^d < X(t_i) < x_i^u\}, \bigcap_{i=1}^{2}\{d_i(t) < X(t) < u_i(t), t_{i-1} < t < t_i\} \mid \mu_1, \mu_2\right)$$

$$= \sum_{\kappa=(k_1,k_2)\in\mathbb{Z}^2} (-1)^{k_1+k_2} e^{\sum_{i=1}^{2} 2\frac{(s_i^\kappa \mu_{[i]}^\kappa - v_i^{k_i}) m_i^{k_i} + w_i^{k_i}}{\sigma^2}} \mathrm{E}\left[e^{-\frac{2R_1^\kappa}{\sigma^2}X(t_1)}\right] \Pr\left(\bigcap_{i=1}^{2}\{x_i^d < s_i^\kappa X(t_i) + 2m_{[i]}^\kappa < x_i^u \mid \mu_{[1:2]}^\kappa\}\right).$$

$$m_{[1]}^\kappa = m_1^{k_1},\ m_{[2]}^\kappa = (m_2^{k_2} - m_1^{k_1}) I(k_2 : \mathrm{odd}) + (m_2^{k_2} + m_1^{k_1}) I(k_2 : \mathrm{even})$$

$$s_2^\kappa = 1,\ s_1^\kappa = -I(k_2 : \mathrm{odd}) + I(k_2 : \mathrm{even})$$

$$R_1^\kappa = (\mu_2 - v_2^{k_2}) I(k_2 : \mathrm{odd})$$

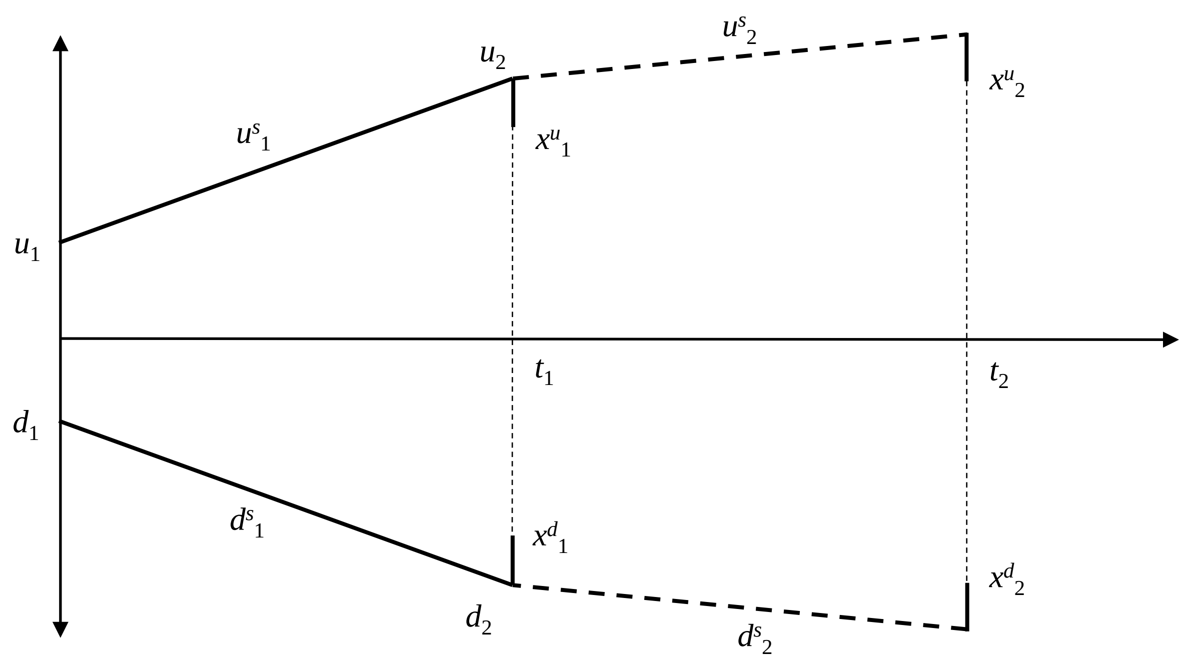
$$\mu_{[1]}^\kappa = s_1^\kappa(\mu_1 - 2R_1^\kappa),\ \mu_{[2]}^\kappa = \mu_2$$

Piecewise linear double barrier (Ordinary double barrier)

- Two-period
  - Early monitoring partial double barrier ($k_2 = 0$)

$$\Pr\left(\bigcap_{i=1}^{2}\{x_i^d < X(t_i) < x_i^u\},\{d_1(t) < X(t) < u_1(t), t_0 < t < t_1\} \mid \mu_1, \mu_2\right)$$

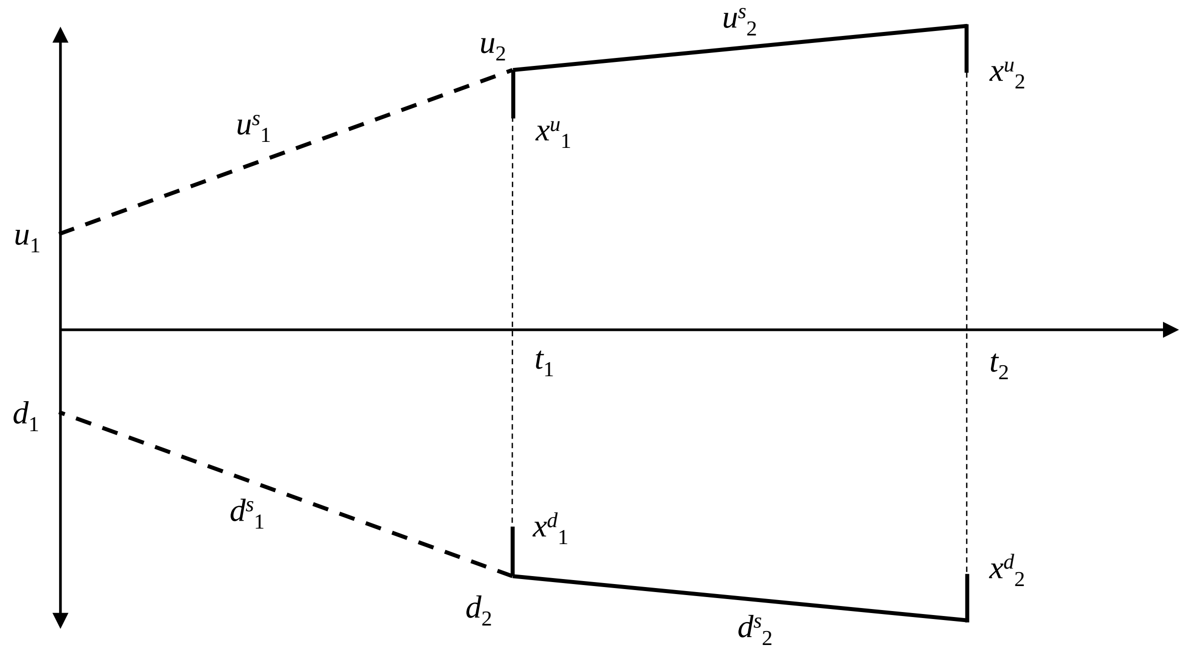
$$= \sum_{k_1\in\mathbb{Z}} (-1)^{k_1} e^{2\frac{(\mu_1 - v_1^{k_1})m_1^{k_1} + w_1^{k_1}}{\sigma^2}} \mathrm{E}\left[e^{-\frac{2R_1^{\kappa}}{\sigma^2}X(t_1)}\right] \Pr\left(\bigcap_{i=1}^{2}\{x_i^d < X(t_i) + 2m_1^{k_1} < x_i^u \mid \mu_1, \mu_2\}\right).$$



- Two-period
  - Late monitoring partial double barrier ($k_1 = 0$)

$$\Pr\left(\bigcap_{i=1}^{2}\{x_i^d < X(t_i) < x_i^u\},\{d_2(t) < X(t) < u_2(t), t_1 < t < t_2\} \mid \mu_1, \mu_2\right)$$

$$= \sum_{k_2\in\mathbb{Z}} (-1)^{k_2} e^{2\frac{(\mu_2 - v_2^{k_2}) m_2^{k_2} + w_2^{k\,2}}{\sigma^2}} \mathrm{E}\left[e^{-\frac{2R_1^{\kappa}}{\sigma^2} X(t_1)} \right] \Pr\left(\bigcap_{i=1}^{2} \{x_i^d < s_i^{\kappa} X(t_i) + 2m_{[i]}^{\kappa} < x_i^u \mid \mu_{[1:2]}^{\kappa}\} \right).$$



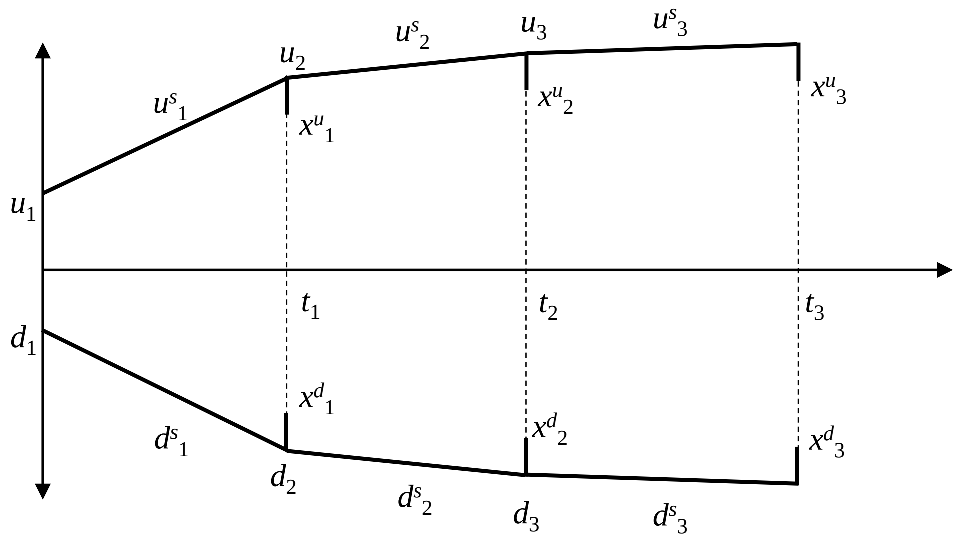
Piecewise linear double barrier (Ordinary double barrier)

- Three-period
  - Three-period piecewise linear double barrier

$$\Pr\left(\bigcap_{i=1}^{3}\{x_i^d < X(t_i) < x_i^u\}, \bigcap_{i=1}^{3}\{d_i(t) < X(t) < u_i(t), t_{i-1} < t < t_i\} \mid \mu_1, \mu_2, \mu_3\right)$$

$$= \sum_{\kappa=(k_1,k_2,k_3)\in\mathbb{Z}^3} (-1)^{k_1+k_2+k_{23}} e^{\sum_{i=1}^{3} 2\frac{(s_i^\kappa \mu_{[i]}^\kappa - v_i^{k_i}) m_i^{k_i} + w_i^{k_i}}{\sigma^2}} \mathrm{E}\left[e^{-\frac{2R_1^\kappa}{\sigma^2} X(t_1)}\right] \mathrm{E}\left[e^{-\frac{2R_2^\kappa}{\sigma^2}[X(t_2)-X(t_1)]}\right] \times$$

$$\Pr\left(\bigcap_{i=1}^{3}\{x_i^d < s_i^\kappa X(t_i) + 2m_{[i]}^\kappa < x_i^u \mid \mu_{[1:3]}^\kappa\}\right)$$



Piecewise linear double barrier (Ordinary double barrier)

- Three-period
  - Three-period piecewise linear double barrier

$$\Pr\left(\bigcap_{i=1}^{3}\{x_i^d < X(t_i) < x_i^u\}, \bigcap_{i=1}^{3}\{d_i(t) < X(t) < u_i(t), t_{i-1} < t < t_i\} \mid \mu_1, \mu_2, \mu_3\right)$$

$$= \sum_{\kappa=(k_1,k_2,k_3)\in\mathbb{Z}^3} (-1)^{k_1+k_2+k_{23}} e^{\sum_{i=1}^{3} 2\frac{(s_i^\kappa \mu_{[i]}^\kappa - v_i^{k_i})m_i^{k_i} + w_i^{k_i}}{\sigma^2}} \mathrm{E}\left[e^{-\frac{2R_1^\kappa}{\sigma^2}X(t_1)}\right] \mathrm{E}\left[e^{-\frac{2R_2^\kappa}{\sigma^2}[X(t_2)-X(t_1)]}\right] \times$$

$$\Pr\left(\bigcap_{i=1}^{3}\{x_i^d < s_i^\kappa X(t_i) + 2m_{[i]}^\kappa < x_i^u \mid \mu_{[1:3]}^\kappa\}\right)$$

$$m_{[1]}^\kappa = m_1^{k_1},\ m_{[i]}^\kappa = (m_i^{k_2} - m_{[i-1]}^{k_1})I(k_i : \mathrm{odd}) + (m_2^{k_2} + m_{[i-1]}^{k_1})I(k_i : \mathrm{even}),\ \ i = 2,3$$

$$s_3^\kappa = 1,\ s_1^\kappa = [-I(k_{i+1} : \mathrm{odd}) + I(k_{i+1} : \mathrm{even})]s_{i+1}^\kappa,\ \ i = 2,1$$

$$R_3^\kappa = 0,\ R_i^\kappa = (\mu_{i+1} - v_{i+1}^{k_{i+1}} - R_{i+1}^\kappa)I(k_{i+1} : \mathrm{odd}) + R_{i+1}^\kappa I(k_{i+1} : \mathrm{even}),\ \ i = 2,1$$

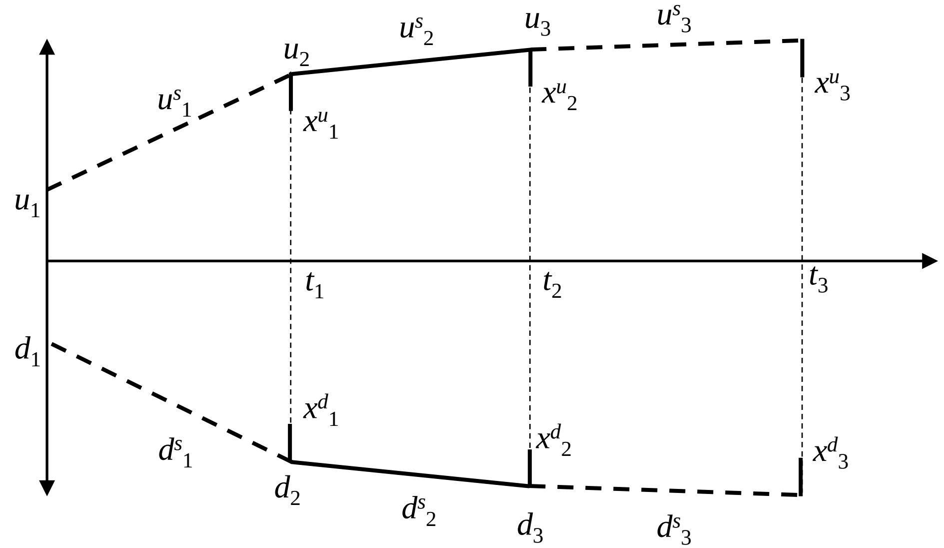
$$\mu_{[i]}^\kappa = s_i^\kappa(\mu_i - 2R_i^\kappa),\ \ i = 1,2,3$$

Piecewise linear double barrier (Ordinary double barrier)

- Three-period
  - Three-period partial double barrier ($k_1 = k_3 = 0$)

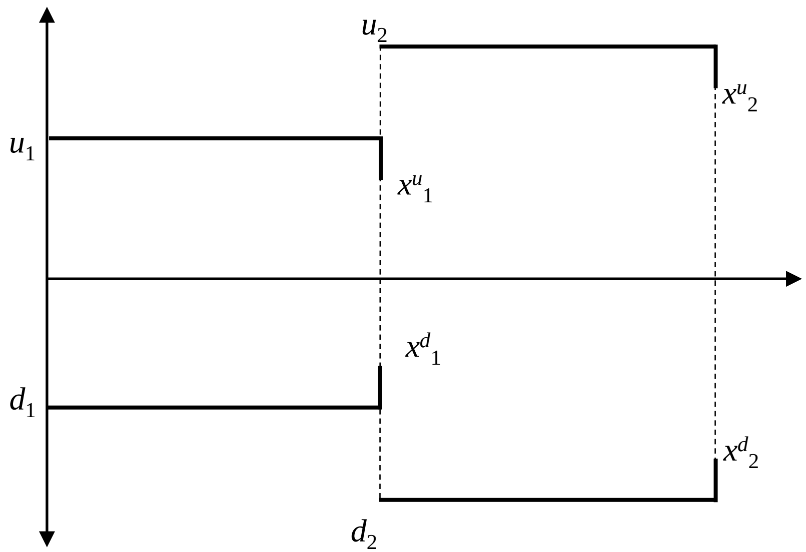
$$\Pr\left(\bigcap_{i=1}^{3}\{x_i^d < X(t_i) < x_i^u\},\{d_2(t) < X(t) < u_2(t), t_1 < t < t_2\} \mid \mu_1, \mu_2, \mu_3\right)$$

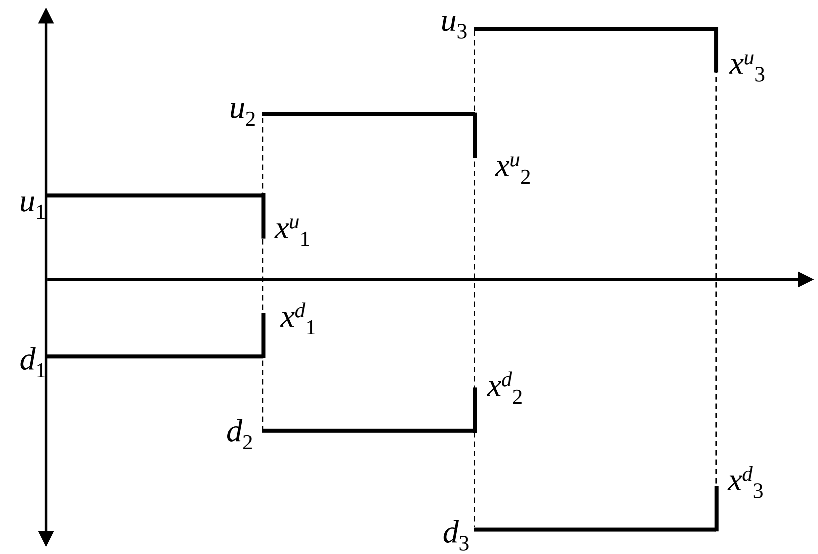
$$= \sum_{k_2 \in \mathbb{Z}} (-1)^{k_2} e^{2\frac{(\mu_2 - v_2^{k_2}) m_2^{k_2} + w_2^{k_2}}{\sigma^2}} \mathrm{E}\left[e^{-\frac{2R_1^{\kappa}}{\sigma^2} X(t_1)} \right] \Pr\left(\bigcap_{i=1}^{3} \{x_i^d < s_i^{\kappa} X(t_i) + 2m_{[i]}^{\kappa} < x_i^u \mid \mu_{[1:3]}^{\kappa}\} \right)$$



Piecewise linear double barrier (Step double barrier)

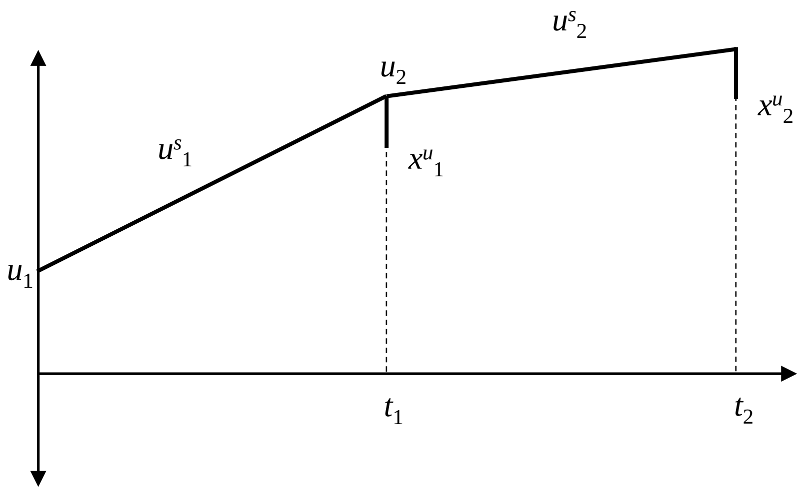
- Step double barrier
  - By setting each slope of the piecewise linear double barrier 0, (hence, $v^k_i = w^k_i = 0$) we can easily obtain the non-crossing probabilities for step double barrier.

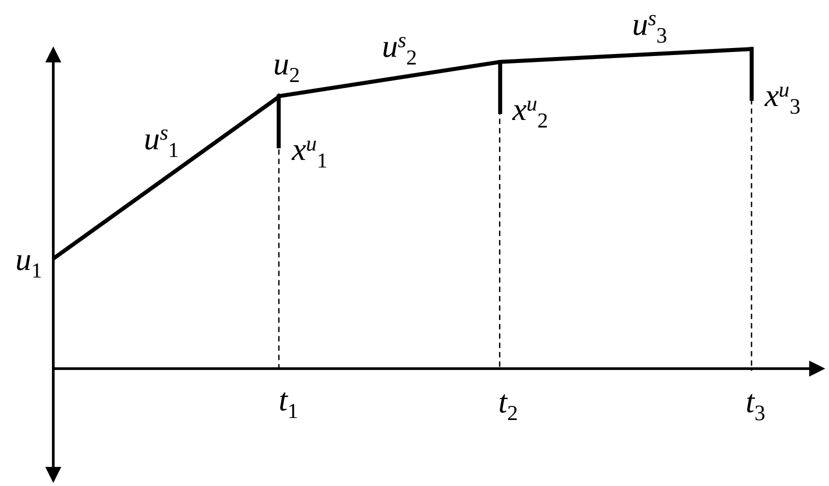




Piecewise linear double barrier (Single barrier)

- Piecewise linear up barrier
  - By reducing combination of integers $\mathbb{Z}$ to combination of $\{0, 1\}$ and letting $x^d{}_i$ = -∞, d_i = - ∞, we can easily obtain the non-crossing probabilities for piecewise linear up barrier.





Closed-form pricing formulas

- Black-Sholes framework
  - $\mu_i = r - \sigma^2 / 2$

- Activating event A (three-period)
  - $A = \bigcap_{i=1}^{3} \{x_i^d < X(t_i) < x_i^u\}, \bigcap_{i=1}^{3} \{d_i(t) < X(t) < u_i(t), t_{i-1} < t < t_i\}$

- Types of barrier option and their corresponding payoffs

Option type		Payoff
Knock-out	Put	$(K - S(T))_+ \, I(A)$
	Call	$(S(T) - K)_+ \, I(A)$
Knock-in	Put	$(K - S(T))_+ \, I(A^c)$
	Call	$(S(T) - K)_+ \, I(A^c)$

Note. K is the strike price, $S(T)$ is the price of underlying asset at time T, $(x)_+$ is the maximum of x and zero, and $I(A)$ is an indicator function of event A.

Closed-form pricing formulas

- Activating event for option pricing

  - $$A_p = \bigcap_{i=1}^{3} \{x_i^d < X(t_i) < x_i^{u*}\}, \bigcap_{i=1}^{3} \{d_i(t) < X(t) < u_i(t), t_{i-1} < t < t_i\}$$

  - $$A_c = \bigcap_{i=1}^{3} \{x_i^{d*} < X(t_i) < x_i^{u}\}, \bigcap_{i=1}^{3} \{d_i(t) < X(t) < u_i(t), t_{i-1} < t < t_i\}$$

  where $x^{u*}{}_i = x^u{}_i$ and $x^{d*}{}_i = x^d{}_i$ for $i = 1, 2$, $x^{u*}{}_3 = \min(x^u{}_3, k)$ and $x^{d*}{}_3 = \max(x^d{}_3, k)$, and $k = \ln(K / S(0))$.

- Types of barrier option and their corresponding pricing formulas

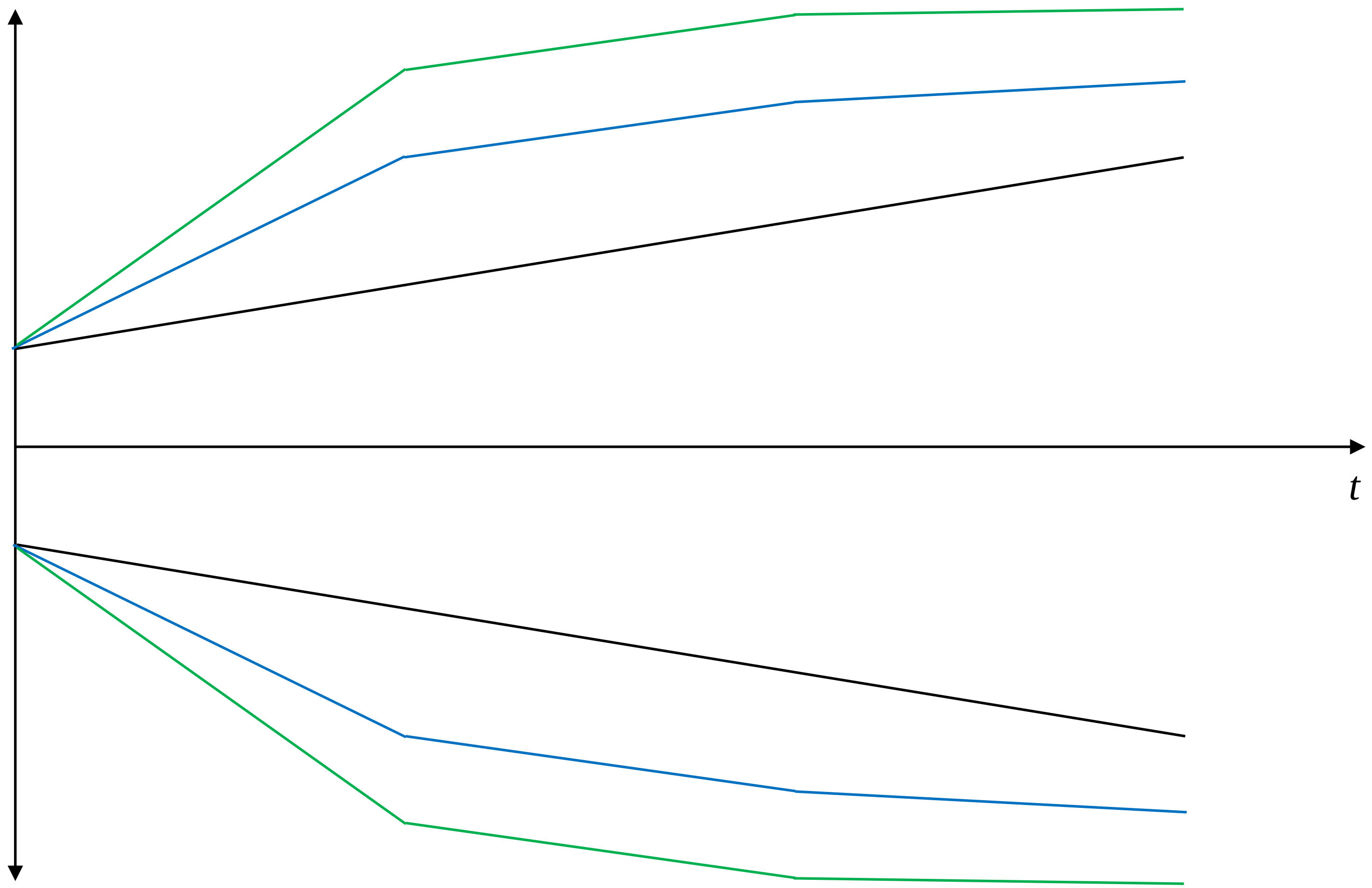
Option type		Pricing formula
Knock-out	Put	$e^{-rT} K \Pr(A_p) - S(0) \Pr(A_p; 1)$
	Call	$S(0) \Pr(A_c; 1) - e^{-rT} K \Pr(A_c)$
Knock-in	Put	$e^{-rT} K [\Phi(-d_2) - \Pr(A_p)] - S(0) [\Phi(-d_1) - \Pr(A_p; 1)]$
	Call	$S(0) [\Phi(d_1) - \Pr(A_c; 1)] - e^{-rT} K [\Phi(d_2) - \Pr(A_c)]$

Note. $d_1 = [-k + (r + \sigma^2 / 2)t_3] / (\sigma \sqrt{t_3})$, $d_2 = [-k + (r - \sigma^2 / 2)t_3] / (\sigma \sqrt{t_3})$.

$\Pr(; 1)$ means that the drift is shifted into $r + \sigma^2 / 2$.

Numerical Analysis

- Types of barrier (linear, concave, and more concave)



Numerical Analysis

- Numerical examples of option prices

r	σ	Option type	Knock - out		Knock - in	
			Put	Call	Put	Call
0.03	0.2	Linear	0.5213	0.6474	4.3609	5.7236
		Concave	3.3655	4.2634	1.5167	2.1075
		More concave	4.0275	5.1904	0.8547	1.1806
	0.3	Linear	0.0126	0.0141	7.6480	9.1353
		Concave	1.1158	1.2570	6.5448	7.8924
		More concave	2.0724	2.3748	5.5882	6.7746
0.04	0.2	Linear	0.4999	0.6673	4.1470	5.9597
		Concave	3.2173	4.4100	1.4296	2.2171
		More concave	3.8403	5.3860	0.8067	1.2411
	0.3	Linear	0.0123	0.0143	7.3980	9.3762
		Concave	1.0893	1.2769	6.3210	8.1136
		More concave	2.0177	2.4194	5.3926	6.9710

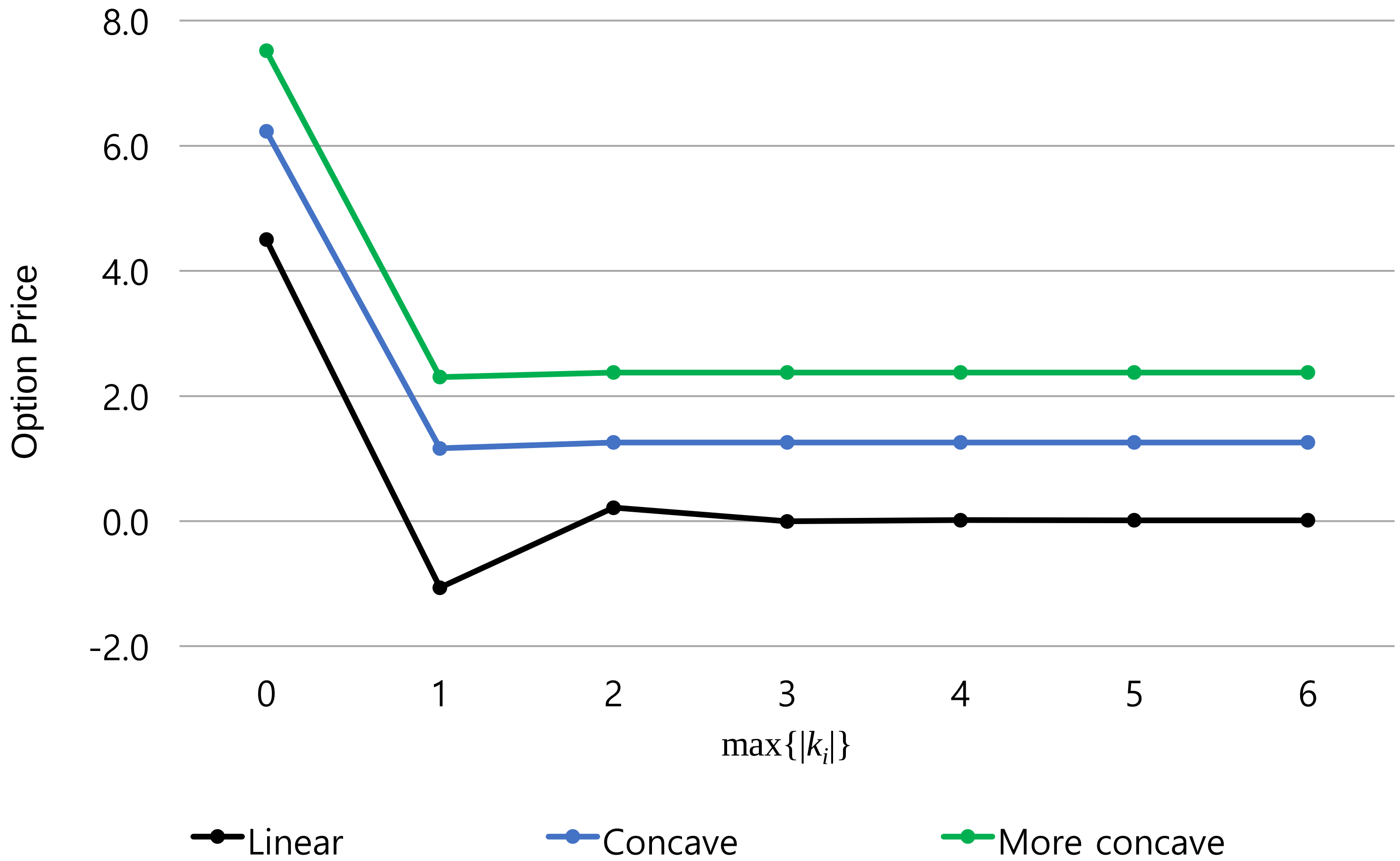
Numerical Analysis

- Convergence of knock-out call option

r	σ	Option type	$\max\{\lvert k_i\rvert\}$ 0	1	2	3	4	5	6
0.03	0.2	Linear	5.0895	0.2569	0.6562	0.6474	0.6474	0.6474	0.6474
		Concave	6.0351	4.2624	4.2634	4.2634	4.2634	4.2634	4.2634
		More concave	6.2865	5.1902	5.1904	5.1904	5.1904	5.1904	5.1904
	0.3	Linear	4.5009	-1.0634	0.2159	-0.0038	0.0148	0.0141	0.0141
		Concave	6.2338	1.1634	1.2574	1.2570	1.2570	1.2570	1.2570
		More concave	7.5190	2.3017	2.3749	2.3748	2.3748	2.3748	2.3748
0.04	0.2	Linear	5.2651	0.2646	0.6764	0.6673	0.6673	0.6673	0.6673
		Concave	6.2607	4.4089	4.4100	4.4100	4.4100	4.4100	4.4100
		More concave	6.5335	5.3857	5.3860	5.3860	5.3860	5.3860	5.3860
	0.3	Linear	4.5791	-1.0790	0.2191	-0.0038	0.0151	0.0143	0.0143
		Concave	6.3518	1.1818	1.2773	1.2769	1.2769	1.2769	1.2769
		More concave	7.6819	2.3450	2.4196	2.4194	2.4194	2.4194	2.4194

Numerical Analysis

- Convergence of knock-out call option ($r = 0.03$, $\sigma = 0.3$)



Conclusion

- Deriving distribution function including double barrier.
  - The function can also calculate the probability for special case of double barriers such as partial, step, and single barrier.
- Deriving closed-form pricing formula of piecewise linear double barrier options with the function.
- Numerical results show the relationship between parameter values and the price of the double barrier options.
  - It shows that the prices converge rapidly over a size of $\max\{|k_i|\}$.

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