CH9. 예측

최소 평균제곱오차 예측

- **평균제곱 오차 (mean square error : MSE)**
  - X의 함수 $g(X)$를 이용하여 Y를 예측
  - $MSE = E\left\{\left(Y - g(X)\right)^2\right\}$
- **최소 평균제곱 오차 (minimum MSE : MMSE) 예측**
  - $\hat{Y} = argmin_{g(X)} MSE = argmin_{g(X)} E\left\{\left(Y - g(X)\right)^2\right\}$
  - Fact : $\hat{Y} = E(Y|X)$

시계열에서의 예측

- **시계열에서의 예측**
  - 과거 관측값 $Z_n, Z_{n-1}, \ldots$을 이용하여 관측되지 않은 미래의 Z_{n+l} $(l > 0)$ 예측
  - 예측값
    - 표기 : $Z_n(l)$
    - 시점 n : 예측을 시작하는 원점 (origin)
    - 시차 l : 선행시차 (lead time)
  - 예측오차 (forecast error)
    - $e_n(l) = Z_{n+l} - Z_n(l)$
    - 예측오차가 작을수록 좋은 예측 모형임
    - MSE, MAPE, MAE 등의 측도를 최소로 하는 예측모형이 방법을 선택함
  - $MMSE$ 예측값 : $Z_n(l) = E(Z_{n+l} | Z_n, Z_{n-1}, \ldots)$

예측값과 예측구간

- **MMSE 예측과 예측오차**
  - 예측값 : $Z_n(l)$
  - 예측오차 : $e_n(l) = Z_{n+l} - Z_n(l)$

$$E\left(e_n(l)\right) = 0, Var\left(e_n(l)\right)$$

  - $100(1-\alpha)\%$ 예측구간 (정규분포 가정하에서)

$$Z_n(l) \pm z_{\alpha/2}\sqrt{Var\left(e_n(l)\right)}$$

추정된 예측값과 추정된 예측구간

- **추정된 예측값**
  - $\hat{Z}_n(l)$
  - $Z_n(l)$에 포함되어 있는 모수들을 추정값으로 대체

- **추정된 예측오차**
  - $\hat{e}_n(l) \ = Z_{n+l} - \hat{Z}_n(l)$
  - $100(1-\alpha)\%$ 예측구간 :

$$Z_n(l) \pm z_{\alpha/2}\sqrt{\widehat{Var}\big(e_n(l)\big)}$$

MMSE 예측 – AR(1)

- **AR(1) model**
  - $Z_t - \mu = \phi(Z_{t-1} - \mu) + \varepsilon_t$, $\varepsilon_t \sim WN(0, \sigma^2)$, $\varepsilon_t \perp \{Z_t, k < t\}$
  - 조건부 기대값
    - $E(Z_{n+1}|Z_n, Z_{n-1}, \dots) = E(\mu + \phi(Z_n - \mu) + \varepsilon_{n+1}|Z_n, Z_{n-1}, \dots)$

      $= \mu + \phi(Z_n - \mu)$
    - $E(Z_{n+2}|Z_n, Z_{n-1}, \dots) = E(\mu + \phi(Z_{n+1} - \mu) + \varepsilon_{n+2}|Z_n, Z_{n-1}, \dots)$

      $= E(\mu + \phi^2(Z_n - \mu) + \phi\varepsilon_{n+1} + \varepsilon_{n+2}|Z_n, Z_{n-1}, \dots)$

      $= \mu + \phi^2(Z_n - \mu)$
    - $E(Z_{n+l}|Z_n, Z_{n-1}, \dots) = \mu + \phi^l(Z_n - \mu)$
  - MMSE 예측값 : $Z_n(l) = \mu + \phi^l(Z_n - \mu)$

MMSE 예측 – AR(1)

- **AR(1) model**
  - MMSE 예측값 : $Z_n(l) = \mu + \phi^l(Z_n - \mu)$
  - 예측 오차
    - $e_n(l) = Z_{n+l} - Z_n(l) = Z_{n+l} - \{\mu + \phi^l(Z_n - \mu)\}$

      $= \varepsilon_{n+l} + \phi\varepsilon_{n+l-1} + \cdots + \phi^{l-1}\varepsilon_{n+1}$
    - $E\big(e_n(l)\big) = 0$
    - $Var\big(e_n(l)\big) = \big(1 + \phi^2 + \cdots + \phi^{2(l-1)}\big)\sigma^2 = \frac{\sigma^2(1-\phi^{2l})}{1-\phi^2} \leq \frac{\sigma^2}{1-\phi^2} = Var(Z_n)$

MMSE 예측 – AR(1)

- **AR(1) model**
  - 추정된 MMSE : $\hat{Z}_n(l) = \hat{\mu} + \hat{\phi}^l(Z_n - \hat{\mu})$
  - 추정된 예측 오차
    - $\hat{e}_n(l) \ = Z_{n+l} - \hat{Z}_n(l) = \varepsilon_{n+l} + \hat{\phi}\varepsilon_{n+l-1} + \cdots + \hat{\phi}^{l-1}\varepsilon_{n+1}$
    - $\widehat{Var}\big(e_n(l)\big) = \frac{\hat{\sigma}^2(1-\hat{\phi}^{2l})}{1-\hat{\phi}^2}$
    - $100(1-\alpha)\%$ 예측구간

$$\hat{Z}_n(l) \pm z_{\alpha/2}\hat{\sigma}\sqrt{\frac{1-\hat{\phi}^{2l}}{1-\hat{\phi}^2}}$$

MMSE 예측 – AR(1)

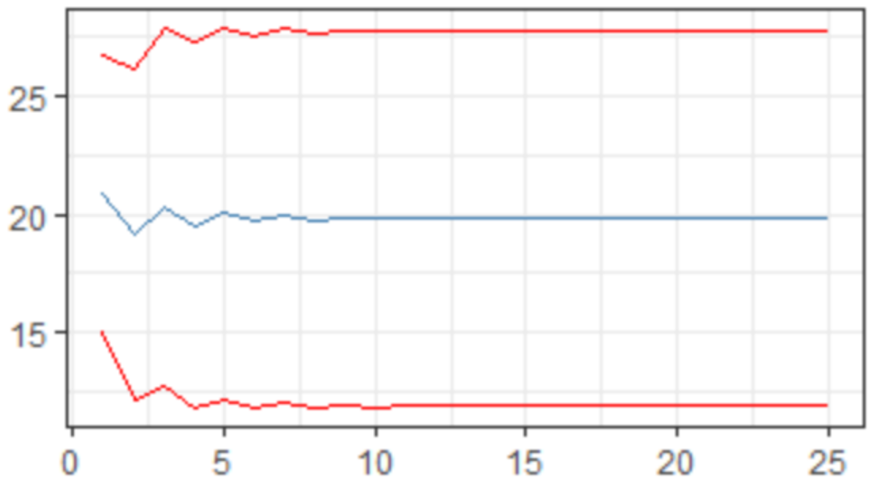
- **AR(1) model**
  - 예) $\hat{\phi} = -0.68, \hat{\mu} = 19.83, \hat{\sigma} = 2.99, n = 100$
  - 추정된 예측 오차
    - $\hat{Z}_n(l) = \hat{\mu} + \hat{\phi}^l(Z_n - \hat{\mu}), \widehat{Var}\left(e_n(l)\right) = \frac{\hat{\sigma}^2\left(1-\hat{\phi}^{2l}\right)}{1-\hat{\phi}^2}$
    - $\hat{Z}_{100}(1) =$
    - $\hat{Z}_{100}(2) =$
    - $\widehat{Var}\left(e_n(1)\right) =$
    - $\widehat{Var}\left(e_n(2)\right) =$
    - 98% 예측구간 : $\hat{Z}_n(l) \pm z_{\alpha/2}\hat{\sigma}\sqrt{\frac{1-\hat{\phi}^{2l}}{1-\hat{\phi}^2}}$

MMSE 예측 – AR(1)

- **AR(1) model**

  - 예) $\hat{\phi} = -0.68, \hat{\mu} = 19.83, \hat{\sigma} = 2.99, n = 100$

l	$\hat{Z}_n(l)$	upper_95	lower_95
1	20.824	26.642	15.005
2	19.153	26.179	12.128
3	20.284	27.798	12.770
4	19.519	27.246	11.792
5	20.036	27.859	12.214
6	19.686	27.552	11.820
7	19.923	27.809	12.037
8	19.763	27.658	11.868
9	19.871	27.771	11.972
10	19.798	27.699	11.897
11	19.848	27.750	11.945
12	19.814	27.717	11.911
13	19.837	27.739	11.934
14	19.821	27.724	11.918
15	19.832	27.735	11.929
16	19.825	27.728	11.922
17	19.829	27.732	11.927
18	19.826	27.729	11.923
19	19.828	27.731	11.926
20	19.827	27.730	11.924
21	19.828	27.731	11.925
22	19.827	27.730	11.924
23	19.828	27.731	11.925
24	19.827	27.730	11.925
25	19.828	27.731	11.925



MMSE 예측 – MA(1)

- **MA(1) model**
  - $Z_t - \mu = \varepsilon_t - \theta\varepsilon_{t-1},\ \varepsilon_t \sim WN(0, \sigma^2), |\theta| < 1$
  - 조건부 기대값
    - $E(Z_{n+1}|Z_n, Z_{n-1}, \ldots) = E(\mu + \varepsilon_{n+1} - \theta\varepsilon_n|Z_n, Z_{n-1}, \ldots)$

      $= \mu - \theta E(\varepsilon_n|Z_n, Z_{n-1}, \ldots) = \mu - \theta\varepsilon_n$
    - $E(Z_{n+2}|Z_n, Z_{n-1}, \ldots) = E(\mu + \varepsilon_{n+2} - \theta\varepsilon_{n+1}|Z_n, Z_{n-1} \ldots) = \mu$
    - $E(Z_{n+l}|Z_n, Z_{n-1}, \ldots) = \mu$
  - MMSE 예측값 : $Z_n(l) = \begin{cases} \mu - \theta\varepsilon_n, & l = 1 \\ \mu, & l > 1 \end{cases}$

MMSE 예측 – MA(1)

- **MA(1) model**

  - MMSE 예측값 : $Z_n(l) = \begin{cases} \mu - \theta\varepsilon_n, & l = 1 \\ \mu, & l > 1 \end{cases}$

  - 예측 오차

    - $e_n(l) = Z_{n+l} - Z_n(l) = Z_{n+l} - \mu - \theta\varepsilon_n = \varepsilon_{n+1}, l = 1$
    - $e_n(l) = Z_{n+l} - \mu = \varepsilon_{n+l} - \theta\varepsilon_{n+l-1}, l > 1$
    - $E\left(e_n(l)\right) = 0$
    - $Var\left(e_n(l)\right) = \begin{cases} \sigma^2, & l = 1 \\ (1+\theta^2)\sigma^2, & l > 1 \end{cases}$

MMSE 예측 – MA(1)

- **MA(1) model**

  - 추정된 MMSE : $\hat{Z}_n(l) = \begin{cases} \hat{\mu} - \hat{\theta}\hat{\varepsilon}_n, & l = 1 \\ \hat{\mu} &, l \geq 2 \end{cases}$

  - 예측 오차

    - $\widehat{Var}\big(e_n(l)\big) = \begin{cases} \hat{\sigma}^2 &, l = 1 \\ (1 + \hat{\theta}^2)\hat{\sigma}^2, & l \geq 2 \end{cases}$

    - $100(1-\alpha)\%$ 예측구간

$$\hat{Z}_n(l) \pm z_{\alpha/2}\sqrt{\widehat{Var}\big(e_n(l)\big)}$$

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