Linear Intrinsic Spherical Spline

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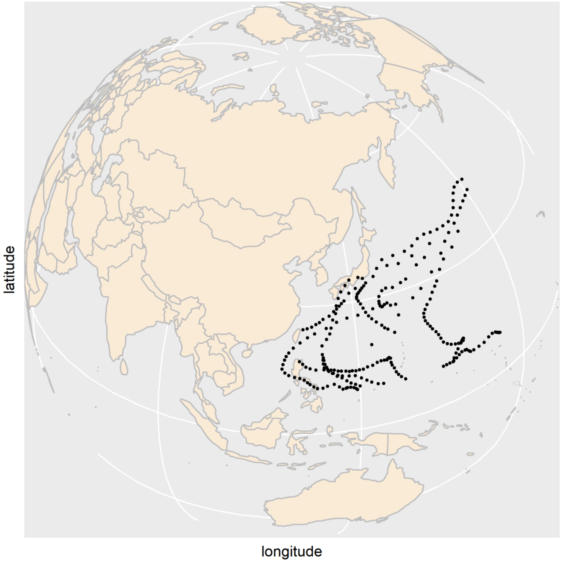
Agenda

Introduction

Tropical cyclones' path on a globe



Motivation

- Methods of mapping between the manifold and tangent space
- Fit a path curve by solving an intrinsic optimization problem
  - earth $\approx$ sphere
  - Riemannian geometry
- Application in animation, radiology, navigation

Linear Intrinsic Spherical Spline (LISS)

- Geodesic segment between v and w:

$$\alpha(t; v, w) \quad \text{for} \quad t \in [0, 1]$$

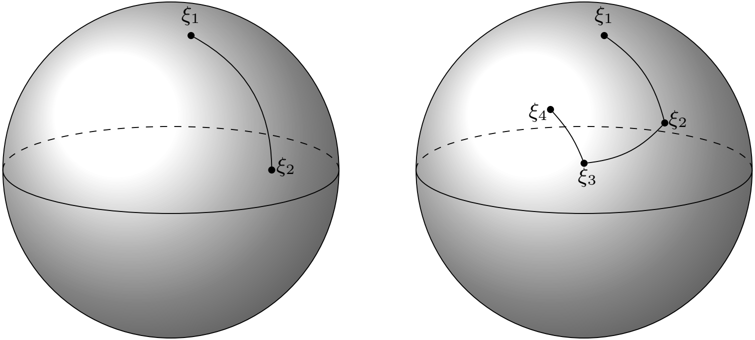
- Define $J - 1$ subintervals for knots points $\tau_1 < \cdots < \tau_J$ as

$$\mathcal{I}_j = [\tau_j, \tau_{j+1}) \quad \text{for} \quad j = 1, \ldots, J - 1$$

- Define LISS curve as below:

$$\gamma(t; \xi_1, \ldots, \xi_J) = \alpha\left(\frac{t - \tau_j}{\tau_{j+1} - \tau_j}; \xi_j, \xi_{j+1}\right) \quad \text{for} \quad t \in \mathcal{I}_j$$

LISS curve



(a) $J = 2$ (b) $J = 4$

Methodology

Penalty and loss function

- Data: $\{(t_i, y_i)\}_{i=1}^N$ for $t_i \in [0,1]$, $y_i \in \mathbb{S}$
- Loss function:

$$\ell(\xi) = \sum_{i=1}^{N} \mathsf{d}^2(y_i, \gamma(t_i; \xi)) \quad \text{for} \quad \xi = (\xi_1, \ldots, \xi_J),$$

  $\mathsf{d}(\cdot, \cdot)$: spherical distance

- Penalty function:

$$\mathsf{p}(\xi) = \sum_{j=2}^{J-1} |\alpha'(0; \xi_j, \xi_{j+1}) - \alpha'(1; \xi_{j-1}, \xi_j)|$$

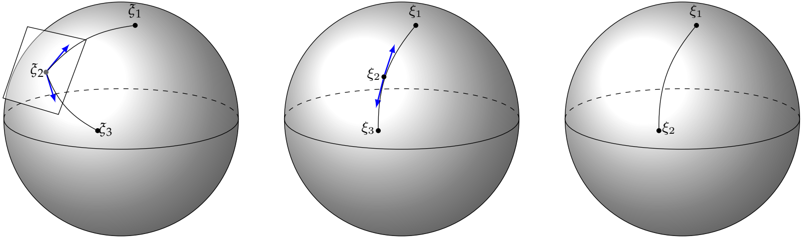
Penalized LISS path

- Fitted curve : $\gamma_J(\cdot; \hat{\xi}_\lambda)$ where

$$\hat{\xi}_\lambda = \operatorname*{argmin}_{\xi} \left\{ \ell(\xi) + \lambda \mathsf{p}(\xi) \right\},$$

  $\lambda > 0$: tuning parameter of penalty function

Elimination process



Implementation

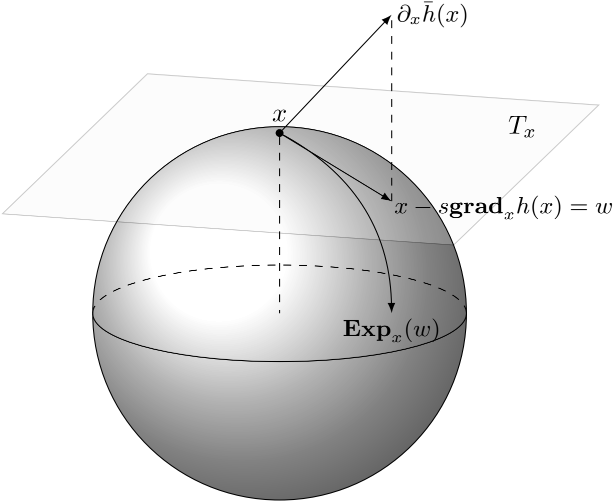
Implementation

- Objective function: penalized squared distance function
- Block coordinate-wise gradient descent algorithm
- Model selection by

$$\mathsf{BIC}_k = N \log \left(\sum_{i=1}^{N} \mathsf{d}(y_i, \gamma(t_i; \hat{\xi}_{\lambda_k}))^2 \right) + 3J_k \log N,$$

J_k : the number of control points for λ_k

Optimize control points

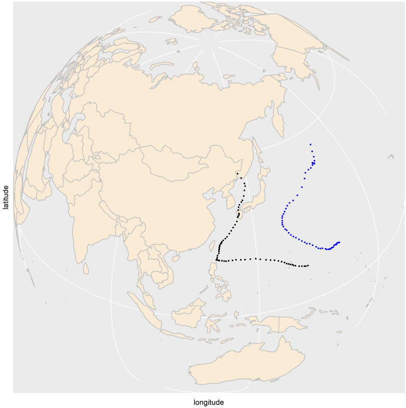


s = step size

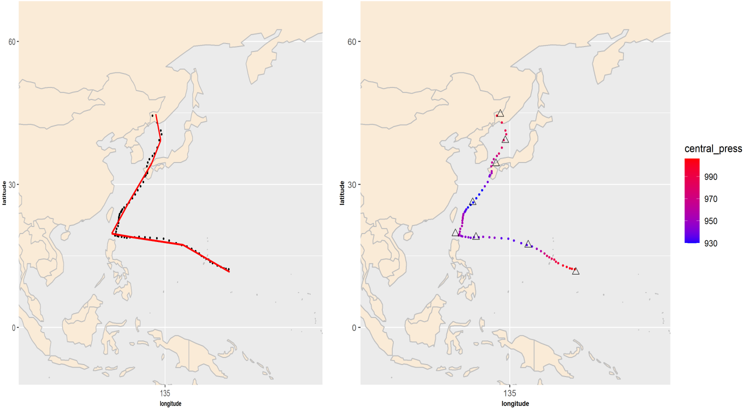
Real data analysis

Tropical Cyclone data in 2015

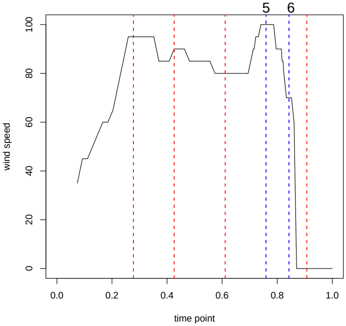
- Cyclone Goni
  - t_i : 08-13 $\sim$ 08-26
  - $N = 69$
- Cyclone Atsani
  - t_i : 08-14 $\sim$ 08-29
  - N $= 61$
- Factors: central pressure, wind speed, the longest and the shortest radius

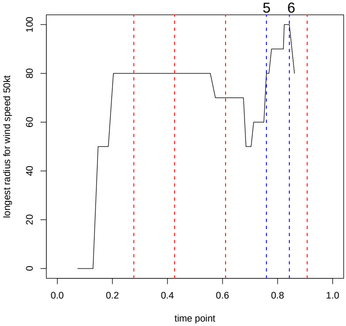


Goni's fitted path curve

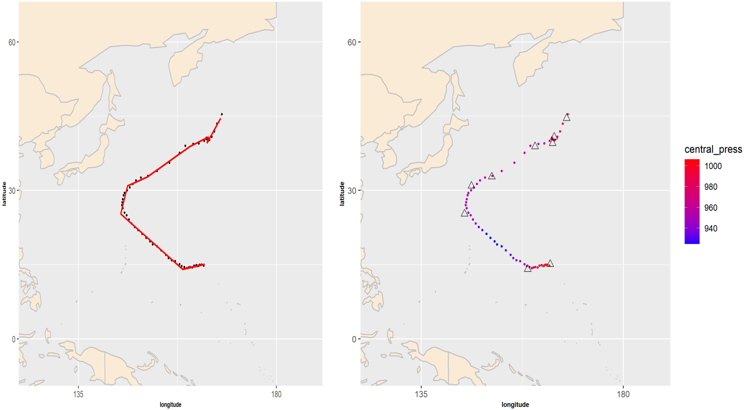


Goni's wind speed and the longest radius

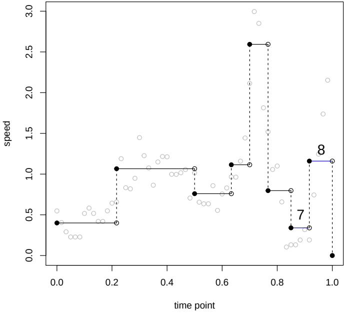


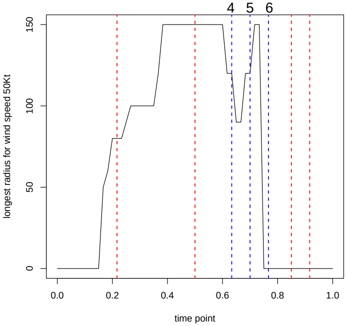


Atsani's fitted path curve

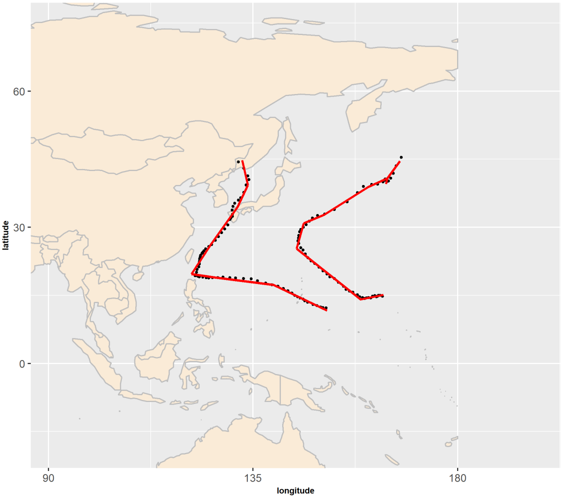


Atsani's speed and the longest radius





Fitted curves of cyclones



Conclusion

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- Investigate intrinsic smoothing method on the sphere based on data-adaptive way
- Fit cyclone's path curve by linear spherical spline and interpret with other factors
- Detect data trends using regularized spherical spline curve

Appendix

Spherical distance

- Define spherical distance between two points u, v as

$$\mathsf{d}(u, v) = \arccos(u^\top v)$$

Observe

$$\mathsf{d}(u, v) = \arccos(|u||v|\cos(\theta)) = \arccos(\cos(\theta)) = \theta$$

Geodesic curve

- Denote the 2-sphere as $\mathbb{S}$. A geodesic segment $\alpha : \mathcal{I} = [0, 1] \to \mathbb{S}$ between given two points u and v on $\mathbb{S}$ is defined as

$$\alpha(t; u, v) = \frac{\sin((1-t)\theta)}{\sin(\theta)} u + \frac{\sin(t\theta)}{\sin(\theta)} v, \quad t \in \mathcal{I}$$

where $\theta = \mathsf{d}(u, v)$