BAYESIAN STATISTICS

Chapter 10

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10. Simultaneous Estimation of Mean and Variance of Normal Population

It is not very common that mean and variance is known *apriori*. Here, we consider the Bayes inference the normal population without the prior information on them.

10.1. Prior distribution for mean and variance without the prior information

Suppose $X \sim \mathrm{N}(\mu, \sigma^2) = \mathrm{N}(\mu, 1/p_{rec})$ and μ, σ^2, p_{rec} are unknown. Prior distribution is the joint distribution of (μ, σ^2) or (μ, p_{rec}), which is specified as the multiplication of the conditional distribution of $\mu | p_{rec}$ and the marginal distribution of p_{rec}.

- **conditional prior for $\mu | p_{rec}$:** $\mu | p_{rec} \sim \mathrm{N}\left(\nu, \frac{1}{\pi_{rec} p_{rec}}\right)$

$$g(\mu | p_{rec}) \propto (\pi_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi_{rec} p_{rec} (\mu - \nu)^2}{2}\right\}$$

- **prior for p_{rec}:** $p_{rec} \sim \mathrm{Gamma}(\alpha, \beta)$

$$g(p_{rec}) = \frac{1}{\Gamma(\alpha)\beta^\alpha} p_{rec}^{\alpha - 1} e^{-p_{rec}/\beta}$$

- **joint prior for (μ, p_{rec}):**

$$\begin{aligned} g(\mu, p_{rec}) &\propto g(\mu | p_{rec}) g(p_{rec}) \\ &\propto (\pi_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi_{rec} p_{rec} (\mu - \nu)^2}{2}\right\} \times p_{rec}^{\alpha - 1} e^{-p_{rec}/\beta} \end{aligned}$$

10.2. Joint posterior for (μ, p_{rec})

The likelihood for a single observation $X = x$ is given as

$$\ell(x|\mu, p_{rec}) \propto p_{rec}^{1/2} \exp\left\{-\frac{p_{rec}(x-\mu)^2}{2}\right\}.$$

Thus, the posterior density function becomes

$$\begin{aligned}
h(\mu, p_{rec}|x) &\propto \ell(x|\mu, p_{rec}) g(\mu, p_{rec}) \\
&\propto p_{rec}^{1/2} \exp\left\{-\frac{p_{rec}(x-\mu)^2}{2}\right\} \\
&\quad \times (\pi_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi_{rec} p_{rec}(\mu-\nu)^2}{2}\right\} \times p_{rec}^{\alpha-1} e^{-p_{rec}/\beta} \\
&\propto (\pi'_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi'_{rec} p_{rec}(\mu-\nu')^2}{2}\right\} \times p_{rec}^{\alpha'-1} e^{-p_{rec}/\beta'},
\end{aligned}$$

where $\nu' = (\pi_{rec}\nu + x)/(\pi_{rec} + 1)$, $\pi'_{rec} = \pi_{rec} + 1$, $\alpha' = \alpha + 1/2$, and $\beta' = \left\{\frac{1}{\beta} + \frac{\pi_{rec}(x-\nu)^2}{2(\pi_{rec}+1)}\right\}^{-1}$.

10.3. Marginal posterior distributions

The marginal posterior distribution for p_{rec} is obtained by integrating out the joint posterior with respect to μ:

$$\begin{aligned} h(p_{rec}|x) &\propto \int_{-\infty}^{\infty} (\pi'_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi'_{rec} p_{rec}(\mu-\nu')^2}{2}\right\} \times p_{rec}^{\alpha'-1} e^{-p_{rec}/\beta'} d\mu \\ &\propto p_{rec}^{\alpha'-1} e^{-p_{rec}/\beta'} \sim \text{Gamma}(\alpha', \beta') \end{aligned}$$

It is little bit complicated to obtain the marginal posterior for μ since integration of the joint posterior with respect to p_{rec} is hard. Instead,

$$h(\mu|x) = \int h(\mu, p_{rec}|x) dp_{rec} = \int h(\mu|p_{rec}, x) h(p_{rec}|x) dp_{rec}.$$

From the joint posterior distribution, we get

$$h(\mu|p_{rec}, x) = \frac{h(\mu, p_{rec}|x)}{h(p_{rec}|x)} \propto (\pi'_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi'_{rec} p_{rec}(\mu-\nu')^2}{2}\right\}$$

which is $\mu|p_{rec}, x \sim \text{N}\left(\nu', \frac{1}{\pi'_{rec} p_{rec}}\right)$.

The marginal posterior for p_{rec} becomes

$$
\begin{aligned}
h(\mu|x) &= \int_0^\infty h(\mu|p_{rec}, x) h(p_{rec}|x) dp_{rec} \\
&= \frac{1}{\sqrt{2\alpha'} B(\alpha', 1/2)} \cdot \frac{\sqrt{\alpha' \beta' \pi'_{rec}}}{\{1+\beta' \pi'_{rec}(\mu-\nu')^2/2\}^{(2\alpha'+1)/2}}
\end{aligned}
$$

with the beta function $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha, \beta)}$. Thus,

$$(\alpha' \beta' \pi'_{rec})^{1/2}(\mu - \nu')|p_{rec}, x \sim t(2\alpha').$$

$t(r)$ is t-distribution with a degree of freedom r. Its probability density function is

$$f(t|r) = \frac{1}{\sqrt{r} B\left(\frac{r}{2}, \frac{1}{2}\right)} \left(1 + \frac{t^2}{r}\right)^{-(r+1)/2}$$

10.4. n samples case

Suppose the independent samples $X_1, \cdots, X_n$ are obtained from $\mathrm{N}(\mu, \sigma^2 = 1/p_{rec})$.

- prior:

$$g(\mu, p_{rec}) \propto (\pi_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi_{rec} p_{rec}(\mu - nu)^2}{2}\right\} \cdot p_{rec}^{\alpha-1} e^{-p_{rec}/\beta}$$

- likelihood:

$$\ell(\mathbf{x}|\mu, p_{rec}) \propto \prod_{i=1}^{n} p_{rec}^{1/2} \exp\left\{-\frac{p_{rec}(x_i - \mu)^2}{2}\right\} \propto p_{rec}^{n/2} \exp\left\{-\frac{p_{rec} \sum_{i=1}^{n}(x_i - \mu)^2}{2}\right\}$$

- posterior:

$$\begin{aligned} h(\mu, p_{rec}|\mathbf{x}) &= \ell(\mathbf{x}|\mu, p_{rec}) g(\mu, p_{rec}) \\ &\propto p_{rec}^{n/2} \exp\left\{-\frac{p_{rec} \sum_{i=1}^{n}(x_i - \mu)^2}{2}\right\} \\ &\quad \times (\pi_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi_{rec} p_{rec}(\mu - \nu)^2}{2}\right\} \cdot p_{rec}^{\alpha-1} e^{-p_{rec}/\beta} \\ &\propto (\pi_{rec}^{*} p_{rec})^{1/2} \exp\left\{-\frac{\pi_{rec}^{*} p_{rec}(\mu - \nu^{*})^2}{2}\right\} \cdot p_{rec}^{\alpha^{*}-1} e^{-p_{rec}/\beta^{*}} \end{aligned}$$

where $\nu^{*} = \frac{\pi_{rec}\nu + n\bar{x}}{\pi_{rec} + n}$, $\pi_{rec}^{*} = \pi_{rec} + n$, $\alpha^{*} = \alpha + n/2$, and

$\beta^{*} = \left\{\frac{1}{\beta} + \frac{nS^2}{2} + \frac{\pi_{rec} n(\bar{x} - \nu)^2}{2(\pi_{rec} + n)}\right\}^{-1}$ with $S^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/n$.

- marginal posterior for p_{rec}:

$$
\begin{aligned}
h(p_{rec}|\mathbf{x}) &\propto \int_{-\infty}^{\infty} (\pi^*_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi^*_{rec} p_{rec} (\mu - \nu^*)^2}{2}\right\} \cdot p_{rec}^{\alpha^*-1} e^{-p_{rec}/\beta^*} d\mu \\
&\propto p_{rec}^{\alpha^*-1} e^{-p_{rec}/\beta^*} \int_{-\infty}^{\infty} (\pi^*_{rec} p_{rec})^{1/2} \exp\left\{-\frac{\pi^*_{rec} p_{rec} (\mu - \nu^*)^2}{2}\right\} d\mu \\
&\propto p_{rec}^{\alpha^*-1} e^{-p_{rec}/\beta^*} \sim \mathrm{Gamma}(\alpha^*, \beta^*).
\end{aligned}
$$

- marginal posterior for μ: In the similar manner as in the single observation,

$$(\alpha^* \beta^* \pi^*_{rec})^{1/2} (\mu - \nu^*) \sim t(2\alpha^*). \quad \text{(show it!!)}$$

10.5. Bayes estimator

Bayes estimators for μ and p_{rec} are given as the marginal posterior mean.

$$\hat{p}_{rec,B} = \alpha^* \beta^* = \frac{\alpha + n/2}{\frac{1}{\beta} + \frac{nS^2}{2} + \frac{\pi_{rec} n(\bar{x} - \nu)^2}{2(\pi_{rec} + n)}}, \quad \hat{\mu}_B = \nu^* = \frac{\pi_{rec}\nu + n\bar{x}}{\pi_{rec} + n}.$$

Now suppose there is no prior information. To ignore the prior distribution by making it flat or vague, we consider $\alpha \to 0$, $\beta \to \infty$, $\pi_{rec} p_{rec} \to 0$ (or $\pi_{rec} \to 0$). This procedure makes the prior as

$$g(\mu, p_{rec}) \propto (\pi_{rec} p_{rec})^{1/2} \exp\left\{ -\frac{\pi_{rec} p_{rec} (\mu - \nu)^2}{2} \right\} \cdot p_{rec}^{\alpha - 1} e^{-p_{rec}/\beta} \longrightarrow \text{constant}$$

In this limiting case, the parameters for the posterior distribution become

$$\begin{aligned}
\nu^* &= \frac{\pi_{rec}\nu + n\bar{x}}{\pi_{rec} + n} \longrightarrow \bar{x} \\
\pi^*_{rec} &= \pi_{rec} + n \longrightarrow n \\
\alpha^* &= \alpha + n/2 \longrightarrow n/2 \\
\beta^* &= \left\{ \frac{1}{\beta} + \frac{nS^2}{2} + \frac{\pi_{rec} n(\bar{x} - \nu)^2}{2(\pi_{rec} + n)} \right\}^{-1} \longrightarrow \left\{ \frac{nS^2}{2} \right\}^{-1}
\end{aligned}$$

In the same limiting case, the marginal posteriors are following:

- marginal posterior for μ:

$$\frac{\mu - \bar{x}}{S/\sqrt{n}} \sim t(n)$$

- marginal posterior for p_{rec}:

$$p_{rec}|\mathbf{X} \sim \text{Gamma}\left(\tfrac{n}{2}, \tfrac{2}{nS^2}\right) \qquad \text{or} \qquad nS^2 p_{rec}|\mathbf{X} \sim \chi^2(n).$$