2. Life Insurance

Introduction

- Insurance systems aim to reduce the adverse financial impact of some random events.
- *Utility models* to represent preferences,
  Stochastic models to represent uncertain financial impact,
  Economic principles to guide pricing
  >> Agreements are reached after these analyses.
- In this chapter we shall develop models for life insurance designed to reduce the financial impact of the random event of untimely deaths.
- Our model will be built in terms of the insured's future-lifetime T.

Insurances payable at the moment of death

- The life insurance benefit will depend only on the length of the interval from the issue of the insurance to the death of the insured.
- t : the length of the interval from issue to death
- b_t : the benefit function
- v_t : the discount function
  - the interest discount factor from the time of payment back to the time of policy issue
  - the underlying force of interest is assumed deterministic

$$\text{(e.g.)} \quad v_t = v^t = \left(\frac{1}{1+r}\right)^t = e^{-\delta t}$$

(the force of interest)

- $z_t = b_t v_t$: the present value function
- In practice, we deal with $Z = z_T = b_T v_T$, a random variable. Hence we need to develop the probability model for Z .

(1) Level benefit insurance

- An n-year term life insurance:

$$b_t = \begin{cases} 1, & t \leq n \\ 0, & t > n \end{cases} \qquad\qquad v_t = v^t, \quad t \geq 0$$

$$Z = \begin{cases} v^T & T \leq n \\ 0, & T > n \end{cases}$$

- The expectation of PV of the payments $E(Z)$ is called the *net single premium*.
- The net single premium for the n-year term insurance with a unit payable at the moment of death of (x) is denoted by $\bar{A}^{1}_{x:\overline{n}|}$.

$$b_t = \begin{cases} 1, & t \le n \\ 0, & t > n \end{cases}$$

$$v_t = v^t, \quad t \ge 0$$

$$Z = \begin{cases} v^T & T \le n \\ 0, & T > n \end{cases}$$

$$\bar{A}^{1}_{x:\overline{n}|} = E(Z) = E(z_T) = \int_0^n v^t \, {}_tp_x \mu_{x+t} dt$$

$$\text{(c.f.)} \quad E(Z^j) = \int_0^n \left(v^t\right)^j {}_tp_x \mu_{x+t} dt = \int_0^n e^{-j\delta t} {}_tp_x \mu_{x+t} dt = {}^j\bar{A}^{1}_{x:\overline{n}|}$$

$$\mathrm{Var}(Z) = {}^2\bar{A}^{1}_{x:\overline{n}|} - \left(\bar{A}^{1}_{x:\overline{n}|}\right)^2$$

- <u>A whole life insurance</u>:

$$b_t = 1, \quad t \geq 0$$

$$v_t = v^t, \quad v \geq 0$$

$$Z = v^T$$

The net single premium is

$$\bar{A}_x = E(Z) = \int_0^\infty v^t \, {}_tp_x \mu_{x+t} dt.$$

Example

Assume that each of 100 independent lives

- is age x,
- is subject to a constant force of mortality $\mu = 0.04$,
- is insured for a death benefit amount of 10 units, payable at the moment of death.

The benefit payments are to be withdrawn from an investment fund earning $\delta = 0.06$.

Calculate the minimum amount at $t = 0$ so that the probability is approximately 0.95 that sufficient funds will be on hand to withdraw the benefit payment at the death of each individual.

(2) Endowment insurance

- An n-year pure endowment:

$$b_t = \begin{cases} 0, & t \le n \\ 1, & t > n \end{cases} \qquad v_t = v^n, \quad t \ge 0$$

$$Z = \begin{cases} 0, & T \le n \\ v^n, & T > n \end{cases}$$

$$A_{x:\overline{n}|}^{\;\;1} = E(Z) = v^n \Pr(T > n) = v^n {}_n p_x$$

c.f. $\mathrm{Var}(Z) = E(Z^2) - E(Z)^2 = {}^2A_{x:\overline{n}|}^{\;\;1} - \left(A_{x:\overline{n}|}^{\;\;1}\right)^2 = v^{2n} {}_n p_x \, {}_n q_x$

- An n-year endowment insurance:

$$b_t = 1, \quad t \geq 0 \qquad\qquad v_t = \begin{cases} v^t, & t \leq n \\ v^n, & t > n \end{cases}$$

$$Z = \begin{cases} v^T, & T \leq n \\ v^n, & T > n \end{cases}$$

$$\bar{A}_{x:\bar{n}|} = E(Z) = \bar{A}^{1}_{x:\bar{n}|} + A_{x:\overset{1}{\bar{n}|}}$$

Note: $Z = v^T I(T \leq n) + v^n I(T > n)$

n-year term life insurance + n-year pure endowment

HW

$$\mathrm{Var}(Z) = ?$$

(3) Deferred insurance

- An m-year deferred insurance:

$$b_t = \begin{cases} 0, & t \le m \\ 1, & t > m \end{cases} \qquad v_t = v^t, \quad t > 0$$

$$Z = \begin{cases} 0, & T \le m \\ v^T, & T > m \end{cases}$$

$$_{m|}\bar{A}_x = E(Z) = \int_m^\infty v^t \, {}_tp_x \mu_{x+t} dt$$

c.f. $\mathrm{Var}(Z) = {}^2_{m|}\bar{A}_x - \left({}_{m|}\bar{A}_x\right)^2$

HW

$$Z = Z_{\text{whole}} - Z_{\text{term}}$$

&

$${}_{m|}\bar{A}_x = \bar{A}_x - \bar{A}^{1}_{x:\overline{m}|}$$

(4) Varying benefit insurance

- An increasing whole life insurance:

benefit = 1 at the moment of death during the first year,
benefit = 2 at the moment of death in the second year,...

$$b_t = [t+1], \quad t \geq 0$$

$$v_t = v^t, \quad t \geq 0$$

$$Z = [T+1]v^T, \quad T \geq 0$$

$$(I\bar{A})_x = E(Z) = \int_0^\infty [t+1]v^t \, {}_tp_x \mu_{x+t} dt$$

- <u>An increasing *n*-year term life insurance</u>

$$Z = \begin{cases} [T+1]v^T, & 0 \le T < n \\ 0, & T > n \end{cases}$$

$$(I\bar{A})^{1}_{x:\bar{n}|} = E(Z)$$

- <u>An *m*thly increasing whole life insurance</u>

  benefit = 1/m at the moment of death during the first m-th of a year of the term of the insurance,
  benefit = 2/m at the moment of death during the second m-th of a year, ...

$$b_t = \frac{[mt+1]}{m}, \quad t \geq 0 \qquad\qquad v_t = v^t, \quad t \geq 0$$

$$Z = \frac{v^T [mT+1]}{m}$$

$$(I^{(m)} \bar{A})_x = E(Z)$$

The limiting case when *m* goes to infinity...

$$b_t = t, \quad t \geq 0 \qquad\qquad v_t = v^t, \quad t \geq 0$$

$$Z = T v^T, \quad T \geq 0$$

$$(\bar{I} \bar{A})_x = E(Z)$$

$$(\bar{I}\bar{A})_x = \int_0^\infty t v^t {}_t p_x \mu_{x+t} dt$$

$$= \int_0^\infty \left(\int_0^t ds \right) v^t {}_t p_x \mu_{x+t} dt$$

$$= \int_0^\infty \int_s^\infty v^t {}_t p_x \mu_{x+t} dt ds$$

$$= \int_0^\infty {}_{s|}\bar{A}_x ds$$

- A decreasing *n*-year term life insurance:

benefit = n at the moment of death during the first year,

benefit = n-1 at the moment of death in the second year,...

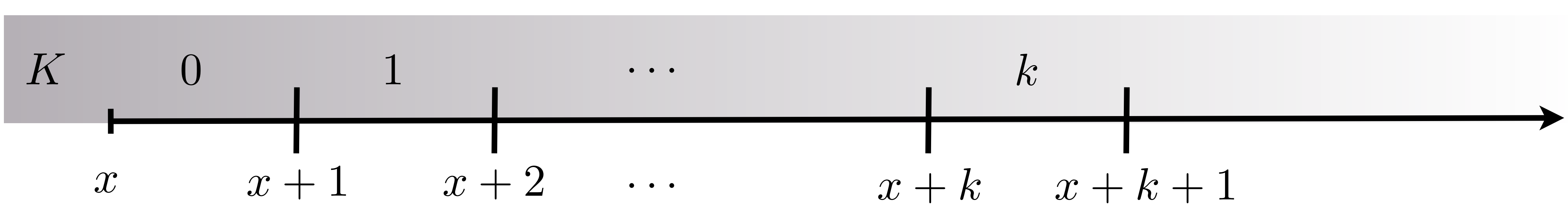
$$b_t = \begin{cases} n - [t], & t \leq n \\ 0, & t > n \end{cases} \qquad v_t = v^t, \quad t \geq 0$$

$$Z = \begin{cases} v^T(n - [T]), & T \leq n \\ 0, & T > n \end{cases}$$

$$(D\bar{A})^1_{x:\bar{n}|} = E(Z) = \int_0^n v^t(n - [t])_t p_x \mu_{x+t} dt$$

Insurances payable at the end of the year of death

- The size and time of payment of the benefits depend only on the number of complete years lived by the insured from policy issue up to the time of death.
- b_{k+1} : the benefit function
- v_{k+1} : the discount function
- At the time of policy issue, the insurance year of death is $K+1$.



(1) An *n*-year term insurance

$$b_{k+1} = \begin{cases} 1, & k = 0, 1, \cdots, n-1 \\ 0, & k = n, n+1, \cdots \end{cases}$$

$$v_{k+1} = v^{k+1}$$

$$Z = \begin{cases} v^{K+1}, & K = 0, 1, \cdots, n-1 \\ 0, & K = n, n+1, \cdots \end{cases}$$

$$A^{1}_{x:\bar{n}|} = \sum_{k=0}^{n-1} v^{k+1}\,{}_{k}p_{x}\,q_{x+k}$$

$$\mathrm{Var}(Z) = {}^{2}A^{1}_{x:\bar{n}|} - \left(A^{1}_{x:\bar{n}|}\right)^{2}$$

(2) A whole life insurance

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_k p_x q_{x+k}$$

(3) An *n*-year endowment insurance

$$b_{k+1} = 1, \quad k = 0, 1, \cdots$$

$$v_{k+1} = \begin{cases} v^{k+1}, & k = 0, 1, \cdots, n-1 \\ v^n, & k = n, n+1, \cdots, \end{cases}$$

$$Z = \begin{cases} v^{K+1}, & K = 0, 1, \cdots, n-1 \\ v^n, & K = n, n+1, \cdots, \end{cases}$$

$$A_{x:\bar{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x q_{x+k} + v^n {}_np_x$$

(4) An increasing whole life insurance

paying $k+1$ units at the end of insurance year $k+1$ provided the insured dies after k complete years

$$(IA)_x = \sum_{k=0}^{\infty}(k+1)v^{k+1}{}_kp_x q_{x+k}$$

(5) A decreasing *n*-year term insurance

$$b_{k+1} = \begin{cases} n-k, & k = 0, 1, \cdots, n-1 \\ 0, & k = n, n+1, \cdots \end{cases}$$

$$v_{k+1} = v^{k+1}, \quad k = 0, 1, 2, \cdots$$

$$Z = \begin{cases} (n-K)v^{K+1}, & K = 0, 1, \cdots, n-1 \\ 0, & K = n, n+1, \cdots \end{cases}$$

$$(DA)^{1}_{x:\bar{n}|} = \sum_{k=0}^{n-1} (n-k) v^{k+1} {}_k p_x q_{x+k}$$

Recall:

	Uniform	Constant force	Balducci
${}_t q_x$	tq_x	$1 - e^{-\mu t}$	$\dfrac{tq_x}{1-(1-t)q_x}$
${}_t p_x$	$1 - tq_x$	$e^{-\mu t}$	$\dfrac{p_x}{1-(1-t)q_x}$
${}_y q_{x+t}$	$\dfrac{yq_x}{1-tq_x}$	$1 - e^{-\mu y}$	$\dfrac{yq_x}{1-(1-y-t)q_x}$
μ_{x+t}	$\dfrac{q_x}{1-tq_x}$	μ	$\dfrac{q_x}{1-(1-t)q_x}$
${}_t p_x \mu_{x+t}$	q_x	$e^{-\mu t}\mu$	$\dfrac{p_x q_x}{\{1-(1-t)q_x\}^2}$

x : integer, $0 \le t \le 1$, $0 \le y \le 1$, $0 \le y + t \le 1$, $\mu = -\log p_x$.

Note: (Exercise 3.40) $T = K + S$

K and S are independent if and only if $\dfrac{{}_s q_{x+k}}{q_{x+k}}$ does not depend on k for $0 \le s \le 1$.

Proof.
$$\Pr(K = k,\, S \le s) = \Pr(k < T \le k + s)$$
$$= {}_{k+s}q_x - {}_k q_x$$
$$= {}_k q_x + {}_k p_x \times {}_s q_{x+k} - {}_k q_x$$
$$= {}_k p_x q_{x+k} \times \frac{{}_s q_{x+k}}{q_{x+k}}$$

(e.g.) Under the *uniform* & the *constant force of mortality* assumptions, K and S are independent.

Relationships between whole life insurances

Under the assumption of *a uniform distribution of deaths*,

$$\bar{A}_x = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt = \sum_{k=0}^\infty \int_k^{k+1} v^t {}_tp_x \mu_{x+t} dt$$

$$= \sum_{k=0}^\infty \int_0^1 v^{k+s} {}_{k+s}p_x \mu_{x+k+s} ds$$

$$= \sum_{k=0}^\infty v^{k+1} {}_kp_x \int_0^1 v^{s-1} {}_sp_{x+k} \, \mu_{x+k+s} ds$$

$$= \sum_{k=0}^\infty v^{k+1} {}_kp_x q_{x+k} \frac{i}{\delta} = \frac{i}{\delta} A_x$$

$${}_{k+s}p_x = {}_kp_x \times {}_sp_{x+k}$$

$${}_sp_{x+k}\mu_{x+k+s} = q_{x+k}$$

c.f. $\bar{A}_x = E(v^T) = E(v^{K+S}) = E(v^{K+1}v^{S-1}) = E(v^{K+1})E(v^{S-1}) = A_x \times \frac{i}{\delta}$

Under the assumption of *the constant force of mortality*,

HW: Exercise 4.16

Relationships between increasing *n*-year term insurances

Under the assumption of *a uniform distribution of deaths*,

$$(I\bar{A})^{1}_{x:\overline{n}|} = \frac{i}{\delta}(IA)^{1}_{x:\overline{n}|}$$

Proof: For the increasing *n*-year term insurance, the PV is

$$Z = \begin{cases} [T+1]v^T, & 0 \le T < n \\ 0, & T > n \end{cases}$$

$$= (K+1)v^T I(0 \le T < n)$$

$$= (K+1)v^{K+1}v^{S-1} I(K = 0, 1, \cdots, n-1)$$

$$\therefore \quad (I\bar{A})^{1}_{x:\overline{n}|} = E(Z) = (IA)^{1}_{x:\overline{n}|} \times \frac{i}{\delta}$$

Recursion equations

$$A_x = vq_x + vp_x A_{x+1}$$

Proof:
$$A_x = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x\ q_{x+k}$$
$$= vq_x + \sum_{k=1}^{\infty} v^{k+1}\,{}_kp_x\ q_{x+k}$$
$$= vq_x + vp_x \sum_{k=1}^{\infty} v^{k}\,{}_{k-1}p_{x+1}\ q_{x+k}$$
$$\scriptstyle (k' = k-1) \displaystyle = vq_x + vp_x \sum_{k'=0}^{\infty} v^{k'+1}\,{}_{k'}p_{x+1}\ q_{x+1+k'}$$
$$= vq_x + vp_x A_{x+1}$$

A_x *will provide either a unit in the event of death within the year or* A_{x+1} *in case of survival.*

$$l_x(1+i)A_x = l_x A_{x+1} + d_x(1 - A_{x+1})$$

Proof:

$$l_x(1+i)A_x = l_x q_x + l_x(1-q_x)A_{x+1} = l_x A_{x+1} + d_x(1 - A_{x+1})$$

With one year's interest, A_x *will provide* A_{x+1} *for all and an additional* $1 - A_{x+1}$ *for those expected to die within the year.*

c.f. $A_{x+1} - A_x = iA_x - q_x(1 - A_{x+1})$

$$A_x = \sum_{y=x}^{\infty} v^{y-x+1} q_y (1 - A_{y+1})$$

Proof: $A_{x+1} - vA_x = -q_x(1 - A_{x+1})$

$$\left.\begin{aligned} v^x A_{x+1} - v^{x-1} A_x &= -v^x q_x (1 - A_{x+1}) \\ v^{x+1} A_{x+2} - v^x A_{x+1} &= -v^{x+1} q_{x+1} (1 - A_{x+2}) \\ v^{x+2} A_{x+3} - v^{x+1} A_{x+2} &= -v^{x+2} q_{x+2} (1 - A_{x+3}) \\ &\ddots \end{aligned}\right\} \Sigma$$

A_x is the PV of the annual costs of insurance over the lifetime of the insured.

Recursion equations
for insurance payable at the moment of death

$$\frac{d}{dx}\bar{A}_x = \delta \bar{A}_x - \mu_x(1 - \bar{A}_x)$$

Proof: (Exercise 4.17)

c.f. $A_{x+1} - A_x = iA_x - q_x(1 - A_{x+1})$