3. Life Annuities

Payments contingent on ***death***

Payments contingent on ***survival***

annuity

noun (pl. **annuities**)

- a fixed sum of money paid to someone each year, typically for the rest of their life: *he left her an annuity of $1,000 in his will.*
- a specified income payable at stated intervals for a fixed or contingent period, often for the recipient's life, as in consideration of a premium paid

ORIGIN late Middle English: from French ***annuité***, from medieval Latin ***annuitas***, from Latin ***annuus 'yearly,'*** from ***annus 'year.'***

A *life annuity* is a series of payments made continuously or at equal intervals (such as months, quarters, years) while a given life survives.

- Important in life insurance operations:
  Life insurances are usually purchased by a life annuity of premiums rather than by a single premium.

- Central in pension systems:
  A retirement plan can be regarded as a system for purchasing deferred life annuities (payable during retirement).

Single payment contingent on survival

- Consider a unit payment due at the end of *n* years provided that (x) survives the *n* years.
- Such a benefit was called an *n*-year pure endowment of 1 in respect to (x).
- The *actuarial present value* of 1 due at the end of *n* years provided that (x) survives is

$$ {}_nE_x = A_{x:\overline{n}|}^{\;\;1} = v^n \, {}_np_x $$

c.f. $\quad l_x \; {}_nE_x \; (1+i)^n = l_{x+n}$

Example

Find the actuarial present value of 10,000 due at the end of 40 years if he/she aged 25 survives. Use the life table and the interest rate of 6%.

Solution:

$$10,000 \times {}_{40}E_{25} = 10,000 \times v^{40}\,{}_{40}p_{25}$$

$$= 10,000 \times (1+.06)^{-40} \times \frac{77107}{97110}$$

$$= 771.96$$

- The *actuarial accumulated value* at the end of *n* years of 1 contributed at age x:

$$S = \frac{1}{{}_nE_x} = \underbrace{(1+i)^n}_{\text{the interest accumulation factor}} \underbrace{\frac{l_x}{l_{x+n}}}_{\text{the survivorship accumulation factor}}$$

$$\frac{\partial}{\partial x}{}_nE_x = {}_nE_x(\mu_x - \mu_{x+n})$$

$$\frac{\partial}{\partial n}{}_nE_x = -{}_nE_x(\mu_{x+n} + \delta)$$

For $n > t$

$$ {}_{n}E_{x} = {}_{t}E_{x} \ {}_{n-t}E_{x+t} $$

$$ \frac{{}_{t}E_{x}}{{}_{n}E_{x}} = \frac{1}{{}_{n-t}E_{x+t}} $$

Continuous life annuities

- The actuarial PVs of continuous life annuities

An aggregate payment technique

1. Record the interest only PV of all payments to be made by the annuity if death occurs at time t;
2. Multiply the PV by the prob. of death at time t;
3. Add (integrate) over all times of death t.

A current payment technique

1. Record the amount of payment due at time t;
2. Determine the actuarial PV of the payment due at time t;
3. Add (integrate) these actuarial PVs for all payment times t.

The actuarial PV of a whole life annuity of 1 per annum payable continuously while (x) survives:

An aggregate payment technique gives...

$$\bar{a}_{\bar{t}|} = \int_0^t v^s ds$$ (the PV of the annuity payments made up to time t)

$$Y = \bar{a}_{\bar{T}|}$$ (the PV of the annuity payments made up to until death)

$$\bar{a}_x = E(Y) = \int_0^\infty \bar{a}_{\bar{t}|} \; {}_tp_x \mu_{x+t} dt$$

$$= \int_0^\infty v^t \; {}_tp_x dt$$

A current payment technique...

$$\bar{A}_x = \int_0^\infty v^t \ {}_tp_x \mu_{x+t} dt \ \ = 1 - \delta \bar{a}_x$$

or

$$1 = \delta \bar{a}_x + \bar{A}_x$$

c.f. $1 = \delta \bar{a}_{\overline{t}|} + v^t$

A unit invested now will produce

annual interest of δ payable continuously while (x) survives

plus

the repayment of the unit upon the death of (x).

The mortality risk in a continuous life annuity

$$\begin{aligned}
\mathrm{Var}\left(\bar{a}_{\bar{T}|}\right) &= \mathrm{Var}\left(\frac{1-v^T}{\delta}\right) \\
&= \frac{1}{\delta^2}\mathrm{Var}\left(v^T\right) \\
&= \frac{1}{\delta^2}\left({}^2\bar{A}_x - \bar{A}_x^2\right)
\end{aligned}$$

$$1 = \delta \bar{a}_{\bar{T}|} + v^T$$

$$1 = E\left(\delta \bar{a}_{\bar{T}|} + v^T\right)$$

$$0 = \mathrm{Var}\left(\delta \bar{a}_{\bar{T}|} + v^T\right)$$

No mortality risk for the combination of
a continuous life annuity of δ *per year*
and *a life insurance of 1 payable on death*

Example Assume a constant force of mortality, $\mu = 0.04$, and a constant force of interest, $\delta = 0.06$. Evaluate

(a) $\bar{a}_x$

(b) the standard deviation of $\bar{a}_{\bar{T}|}$

(c) the probability that $\bar{a}_{\bar{T}|}$ will exceed $\bar{a}_x$.

The actuarial PV of an n-year temporary life annuity of 1 per annum payable continuously while (x) survives during the next n years:

The current payment technique yields...

$$\bar{a}_{x:\bar{n}|} = \int_0^n v^t {}_tp_x dt$$

c.f.

the *n*-year term life insurance $\bar{A}^1_{x:\bar{n}|} = 1 - v^n {}_np_x - \delta \bar{a}_{x:\bar{n}|}$

$$\therefore \quad 1 = \delta \bar{a}_{x:\bar{n}|} + \bar{A}_{x:\bar{n}|}$$

the *n*-year endowment insurance

The aggregate payment technique yields...

PV: $Y = \bar{a}_{\overline{T}|} I(0 \leq T < n) + \bar{a}_{\overline{n}|} I(T \geq n)$

$$
\begin{aligned}
\bar{a}_{x:\overline{n}|} = E(Y) &= \int_0^n \bar{a}_{\overline{t}|} \; {}_tp_x \; \mu_{x+t} dt + \bar{a}_{\overline{n}|} \cdot {}_np_x \\
&= \int_0^n v^t \; {}_tp_x dt
\end{aligned}
$$

Since $\bar{a}_{\overline{t}|} = \dfrac{1 - v^t}{\delta}$, we have

$$Y = \frac{1 - Z}{\delta}$$

where Z is the PV for an *n*-year endowment insurance.

$$\therefore \quad \bar{a}_{x:\overline{n}|} = E(Y) = \frac{1}{\delta}\left(1 - \bar{A}_{x:\overline{n}|}\right)$$

$$\begin{aligned}\mathrm{Var}(Y) &= \frac{1}{\delta^2}\left({}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2\right) \\ &= \frac{2}{\delta}\left(\bar{a}_{x:\overline{n}|} - {}^2\bar{a}_{x:\overline{n}|}\right) - \left(\bar{a}_{x:\overline{n}|}\right)^2\end{aligned}$$

The actuarial PV of a deferred life annuity of 1 per annum payable continuously while (x) survives beyond age x+n:

The current payment technique yields...

$$ {}_{n|}\bar{a}_x = \int_n^\infty v^t \, {}_tp_x dt $$

c.f. $ {}_{n|}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\bar{n}|} = \dfrac{\bar{A}_{x:\bar{n}|} - \bar{A}_x}{\delta} $

By the aggregate payment technique,

PV: $Y = v^n \bar{a}_{\overline{T-n}|} I(T \geq n) = \left(\bar{a}_{\bar{T}|} - \bar{a}_{\bar{n}|}\right) I(T \geq n)$

$$_{n|}\bar{a}_x = \; E(Y) = v^n \int_n^\infty \bar{a}_{\overline{t-n}|} \; {}_tp_x \, \mu_{x+t} dt$$

$$= v^n \int_0^\infty \bar{a}_{\overline{s}|} \; {}_{s+n}p_x \, \mu_{x+n+s} ds$$

$$= v^n \, {}_np_x \int_0^\infty \bar{a}_{\overline{s}|} \; {}_sp_{x+n} \, \mu_{x+n+s} ds$$

$$= {}_nE_x \bar{a}_{x+n}$$

$$\mathrm{Var}(Y) = \int_n^\infty v^{2n}\, \bar{a}^2_{\overline{t-n|}}\, {}_tp_x\, \mu_{x+t} dt - \left({}_{n|}\bar{a}_x\right)^2$$

$$= \frac{2}{\delta}\, v^{2n}\, {}_np_x\, \left(\bar{a}_{x+n} - {}^2\bar{a}_{x+n}\right) - \left({}_{n|}\bar{a}_x\right)^2$$

The actuarial PV of a deferred temporary life annuity of 1 per annum payable continuously while (x) survives between ages x+m and x+m+n:

$$ {}_{m|n}\bar{a}_x = \int_m^{m+n} v^t \, {}_tp_x dt $$

c.f.

$$ {}_{m|n}\bar{a}_x = \bar{a}_{x:\overline{m+n}|} - \bar{a}_{x:\overline{m}|} $$

$$ = \frac{\bar{A}_{x:\bar{m}|} - \bar{A}_{x:\overline{m+n}|}}{\delta} $$

$$ = {}_mE_x \, \bar{a}_{x+m:\bar{n}|} $$

Derivatives of actuarial PV's:

$$\frac{d}{dx}\bar{a}_x = (\mu_x + \delta)\bar{a}_x - 1$$

$$\frac{\partial}{\partial x}\bar{a}_{x:\bar{n}|} = (\mu_x + \delta)\bar{a}_{x:\bar{n}|} - (1 - {}_nE_x)$$

$$\frac{\partial}{\partial n}{}_{n|}\bar{a}_x = -v^n\,{}_np_x$$

Discrete life annuities

The actuarial PV of a whole life annuity due of 1 payable at the beginning of each year while (x) survives:

The current payment technique yields...

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_k p_x$$

In terms of survivorship function, $\ddot{a}_x = \dfrac{1}{l_x} \sum_{k=0}^{\infty} v^k l_{x+k}.$

The aggregate payment technique gives...

$$Y = \ddot{a}_{\overline{K+1}|} = 1 + v + v^2 + \cdots + v^K$$

(the PV random variable of the annuity payment)

$$E(Y) = E(\ddot{a}_{\overline{K+1}|}) = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \; P(K = k)$$

$$= 1 + \sum_{k=0}^{\infty} v^{k+1} {}_{k+1}p_x = \sum_{k=0}^{\infty} v^k {}_k p_x = \ddot{a}_x$$

$$\sum_{k=0}^{\infty} z(k)g(k) = z(0) + \sum_{k=0}^{\infty} (1 - G(k))\, \Delta z(k)$$

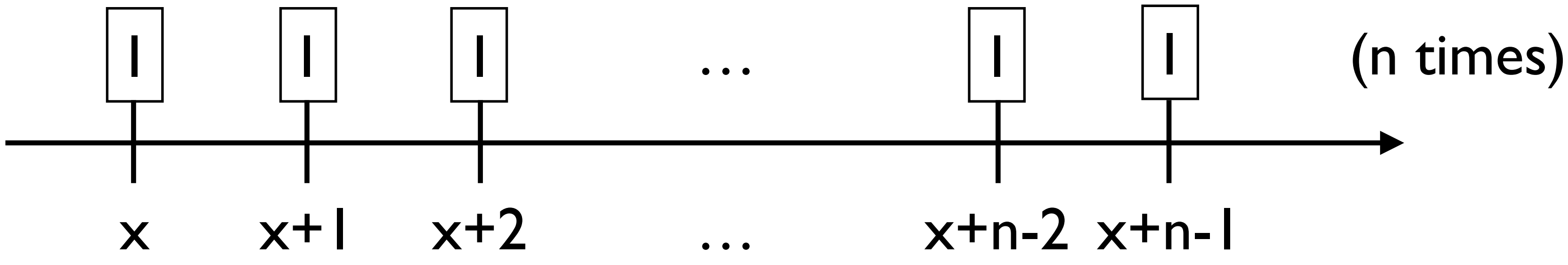
$$\ddot{a}_x = E\left(\frac{1-v^{K+1}}{1-v}\right) = \frac{1}{d}(1-A_x) \qquad \left(d = 1-v = \frac{1}{\ddot{a}_{\overline{\infty}|}}\right)$$

$$1 = d\ddot{a}_x + A_x$$

c.f. $\quad 1 = \delta\bar{a}_x + \bar{A}_x$

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{1-v^{K+1}}{d}\right) = \frac{1}{d^2}\mathrm{Var}(v^{K+1}) = \frac{1}{d^2}\left({}^2A_x - A_x^2\right)$$

The actuarial PV of an n-year temporary life annuity of 1 payable at the beginning of each year while (x) survives:



The current payment technique yields...

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_k p_x$$

The aggregate payment technique gives...

PV r.v.: $$Y = \ddot{a}_{\overline{K+1}|} I(K < n) + \ddot{a}_{\overline{n}|} I(K \geq n)$$
$$= (1 - Z)/d$$

where $$Z = v^{K+1} I(K < n) + v^n I(K \geq n).$$

c.f. PV r.v for n-year endowment insurance

$$E(Z) = \sum_{k=0}^{n-1} v^{k+1} P(K = k) + v^n P(K \geq n)$$
$$= \sum_{k=0}^{n-1} v^{k+1} \{P(K \geq k) - P(K \geq k+1)\} + v^n P(K \geq n)$$
$$= \sum_{k=0}^{n-1} v^{k+1} {}_k p_x - \sum_{k=0}^{n-1} v^{k+1} {}_{k+1} p_x + v^n {}_n p_x = \sum_{k=0}^{n-1} v^{k+1} {}_k p_x - \sum_{k=1}^{n-1} v^k {}_k p_x$$

$$1 - E(Z) = \sum_{k=0}^{n-1} v^k {}_k p_x - \sum_{k=0}^{n-1} v^{k+1} {}_k p_x = d \; \ddot{a}_{x:\overline{n}|}$$

$$E(Z) = A_{x:\overline{n}|}$$

$$d\ \ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|} = 1$$

$$\mathrm{Var}(Y) = \frac{1}{d^2}\mathrm{Var}(Z) = \frac{1}{d^2}\left({}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}^2\right)$$

The actuarial PV of a deferred life annuity of 1 payable at the beginning of each year while (x) survives from age x+n onward:

$$ {}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \, {}_kp_x $$

$$ = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} $$

$$ = \frac{A_{x:\overline{n}|} - A_x}{d} $$

$$ = {}_nE_x \ddot{a}_{x+n} $$

The actuarial PV of a whole life annuity of 1 payable at the end of each year while (x) survives:

$$a_x = \ddot{a}_x - 1 = \sum_{k=1}^{\infty} v^k \, {}_k p_x$$

The actuarial PV of an n-year temporary life annuity of 1 payable at the end of each year while (x) survives:

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_k p_x = \ddot{a}_{x:\overline{n}|} - 1 + {}_n E_x$$

The actuarial PV of a deferred life annuity of 1 payable at the end of each year while (x) survives after age x+n:

$${}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k \, {}_k p_x = a_x - a_{x:\overline{n}|} = {}_n E_x a_{x+n}$$