Actuarial Mathematics

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STAT, HUFS

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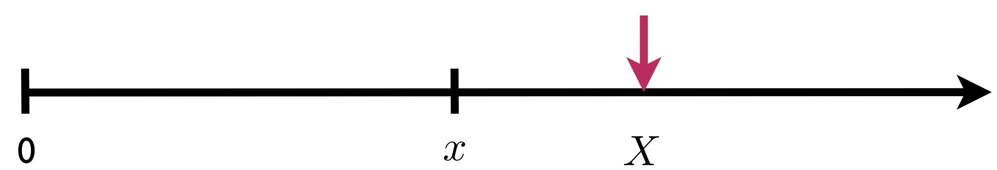
1. Survival Distributions and Life Tables

Introduction

- Age-at-death X
  Time-until-death $T(x) = X - x | X > x$



- Life table: a distribution of the age-at-death

Probability for *Age-at-Death*

- A newborn's age-at-death X
- Distribution function

$$F(x) = Pr(X \leq x)$$

- Survival function

$$s(x) = 1 - F(x) = Pr(X > x)$$

Probability for *Time-until-Death*

- (x): a life-aged-x
- Future lifetime of (x): $T(x) = X - x | X > x$

$$\Pr(T(x) \leq t) = \Pr(X \leq t + x | X > x)$$

$$= \frac{F(t+x) - F(x)}{1 - F(x)} = \frac{s(x) - s(t+x)}{s(x)}$$

$$ {}_tq_x = Pr(T(x) \le t) $$

$$ {}_tp_x = 1 - {}_tq_x = Pr(T(x) > t) $$

$$ {}_1q_x \equiv q_x, \qquad {}_1p_x \equiv p_x $$

Prob. that (x) will die within 1 year

Prob. that (x) will attain age x+1

$$ {}_tp_x = \frac{{}_{t+x}p_0}{{}_xp_0} = \frac{s(x+t)}{s(x)} $$

&

$$ {}_tq_x = 1 - \frac{s(x+t)}{s(x)} $$

$$ {}_{t+n}p_x = {}_np_x \, {}_tp_{x+n} $$

$$ {}_np_x = p_x \, p_{x+1} \cdots p_{x+n-1} \qquad (n = 1, 2, \dots) $$

$$ {}_{t|u}q_x = Pr(t < T(x) \leq t + u) = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x $$

Prob. that (x) will survive t years and die within the following u years

$$ {}_{t|}q_x \equiv {}_{t|1}q_x $$

$$ {}_{t|u}q_x = {}_tp_x \cdot {}_uq_{x+t} $$

Curtate-Future-Lifetime

the number of completed future years lived by (x)

$$\Pr(K(x) = k) = \Pr(k < T(x) \leq k+1)$$

$$= {}_kp_x - {}_{k+1}p_x = {}_kp_x \cdot q_{x+k} = {}_{k|}q_x$$

$$(k = 0, 1, 2, \ldots)$$

Force of Mortality

$$\Pr\left(x < X \leq x + \Delta x | X > x\right) = \frac{F(x + \Delta x) - F(x)}{1 - F(x)} \approx \frac{f(x)\Delta x}{1 - F(x)}$$

$$\mu_x = \frac{f(x)}{1 - F(x)}$$

(the force of mortality)

c.f. the hazard rate function

$$\mu_x = \frac{f(x)}{1-F(x)} = -\frac{s'(x)}{s(x)}$$

$$-\mu_x dx = d\log s(x)$$

$$-\int_x^{x+n} \mu_y dy = \log\frac{s(x+n)}{s(x)} = \log {}_np_x$$

$${}_np_x = \exp\left(-\int_x^{x+n} \mu_y dy\right) = \exp\left(-\int_0^{n} \mu_{x+y} dy\right)$$

$$ {}_xp_0 = s(x) = \exp\left(-\int_0^x \mu_y dy\right) $$

$$ F(x) = 1 - s(x) = 1 - \exp\left(-\int_0^x \mu_y dy\right) $$

$$ f(x) = F'(x) = \exp\left(-\int_0^x \mu_y dy\right)\mu_x = {}_xp_0\mu_x $$

Recall: ${}_tq_x = Pr(T(x) \leq t)$ **and**

${}_tp_x = 1 - {}_tq_x = Pr(T(x) > t)$.

$$\frac{d}{dt}{}_tq_x = \frac{d}{dt}\left(1 - \frac{s(x+t)}{s(x)}\right) = \frac{s(x+t)}{s(x)}\left(-\frac{s'(x+t)}{s(x+t)}\right) = {}_tp_x\mu_{x+t}$$

$$\therefore \quad \int_0^\infty {}_tp_x\,\mu_{x+t}dt = 1$$

$$\frac{d}{dt}(1 - {}_tp_x) = -\frac{d}{dt}{}_tp_x = {}_tp_x\mu_{x+t}$$

$$\lim_{n\to\infty} {}_np_x = 0$$

$$\lim_{n\to\infty} (-\log {}_np_x) = \infty$$

$$\therefore \quad \lim_{n\to\infty} \int_x^{x+n} \mu_y dy = \infty$$

Life Tables

- Consider a group of l_0 newborns
- Let $\mathcal{L}(x)$ denote the cohort's number of survivors to age x. Then,

$$\mathcal{L}(x) = \sum_{j=1}^{l_0} I_j \sim B(l_0, s(x))$$

where I_j is an indicator for the survival of life j to age x.

$$l_x = E\left[\mathcal{L}(x)\right] = l_0 s(x)$$

- Let ${}_n\mathcal{D}_x$ denote the number of deaths between ages x and $x+n$ among the initial l_0 lives.

$$ {}_nd_x = E\left[{}_n\mathcal{D}_x\right] = l_0\left\{s(x) - s(x+n)\right\} = l_x - l_{x+n} $$

$$ d_x = {}_1d_x, \quad \mathcal{D}_x = {}_1\mathcal{D}_x $$

$$\log l_x = \log l_0 + \log s(x)$$

$$-\frac{l'_x}{l_x}dx = -\frac{s'(x)}{s(x)}dx = \mu_x dx$$

$$-l'_x dx = \underbrace{l_x \mu_x}_{\text{(the expected density of deaths at the age } x\text{)}} dx = l_0 \, {}_x p_0 \, \mu_x dx$$

$$l_x = l_0 \exp\left(-\int_0^x \mu_y dy\right)$$

$$l_{x+n} = l_x \exp\left(-\int_x^{x+n} \mu_y dy\right)$$

$$l_x - l_{x+n} = \int_x^{x+n} l_y \mu_y dy$$

- Often assumed that there is an age ω such that $s(x) > 0$ for $x < \omega$ and $s(x) = 0$ for $x \geq \omega$.

  ω : the limiting age

- For convenience, call the concept of l_0 newborns, each with survival function $s(x)$, a random survivorship group.

$$l_x \mu_x = -l_0 s(x) \times \frac{s'(x)}{s(x)} = l_0 f(x) \quad \text{(curve of deaths)}$$

Local minimum of $l_x \mu_x$ at age 10,
and (local) maximum around age 80.

$$\frac{d}{dx} l_x \mu_x = -\frac{d^2}{dx^2} l_x$$

Deterministic Survivorship Group

A non-probabilistic interpretation of the life table

- The group initially consists of l_0 lives aged 0.
- The members are subject to effective annual rates of mortality (decrement) specified by q_x .
- The group is closed.

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$$
\begin{aligned}
l_1 &= l_0(1-q_0) = l_0 p_0 \\
l_2 &= l_1(1-q_1) = l_1 p_1 = l_0 p_0 p_1 \\
&\quad\vdots \qquad\qquad\qquad \vdots \\
l_x &= l_{x-1}(1-q_{x-1}) = l_{x-1}p_{x-1} = l_0\left(p_0 p_1 \cdots p_{x-1}\right) = l_0\, {}_xp_0
\end{aligned}
$$

Other Life Table Functions

- The complete-expectation-of-life

$$\mathring{e}_x = E[T(x)] = \int_0^\infty t \underbrace{{}_tp_x \mu_{x+t}}_{= \frac{d}{dt}\, {}_tq_x} dt$$

c.f. $$\mathring{e}_x = \int_0^\infty {}_tp_x dt$$

$$\mathrm{Var}[T(x)] = 2\int_0^\infty t\, {}_tp_x dt - \mathring{e}_x^2$$

- The median future lifetime of (x), $m(x)$

$$\Pr[T(x) > m(x)] = \frac{1}{2}$$

or

$$\frac{s(x + m(x))}{s(x)} = \frac{1}{2}$$

c.f. the mode of ${}_t p_x \mu_{x+t}$

- The curtate-expectation-of-life

Recall: $\Pr(K(x) = k) = \Pr(k < T(x) \le k+1)$

$$= {}_kp_x - {}_{k+1}p_x = {}_kp_x \cdot q_{x+k} = {}_{k|}q_x$$

$$(k = 0, 1, 2, \dots)$$

$$e_x = E[K(x)] = \sum_{k=0}^{\infty} k \; {}_kp_x \; q_{x+k} = \sum_{k=0}^{\infty} {}_{k+1}p_x$$

- The total expected number of years lived between ages x and $x+1$

$$L_x = \int_0^1 t \; l_{x+t} \;\; \mu_{x+t} \, dt + l_{x+1} = \int_0^1 l_{x+t} \, dt$$

the years lived of those who die between ages x and $x+1$

the years lived of those who survive to age $x+1$

- The central-death-rate at age x

$$m_x = \frac{\int_0^1 l_{x+t}\mu_{x+t}\,dt}{\int_0^1 l_{x+t}\,dt} = \frac{l_x - l_{x+1}}{L_x}$$

- The total number of years lived beyond age x

$$T_x = \int_0^\infty t\, l_{x+t}\mu_{x+t}\, dt = \int_0^\infty l_{x+t}\, dt$$

c.f. $\dfrac{T_x}{l_x} = \mathring{e}_x$

- The average number of years lived between age x and $x+1$ by those who die between those ages

$$a(x) = \frac{\int_0^1 t\ l_{x+t}\mu_{x+t}\,dt}{\int_0^1 l_{x+t}\mu_{x+t}\,dt}$$

Assumptions for Fractional Ages

- The life table specifies the probability distribution of K completely.
- To specify the distribution of T, we need some assumptions on the distribution between integers. For x : integer and $0 \le t \le 1$
  - Uniform distribution of deaths

$$s(x+t) = (1-t)s(x) + ts(x+1)$$

  - Constant force of mortality

$$s(x+t) = s(x)e^{-\mu t} \text{ where } \mu = -\log p_x$$

  - Balducci assumption

$$1/s(x+t) = (1-t)/s(x) + t/s(x+1)$$

	Uniform	Constant force	Balducci
${}_tq_x$	tq_x	$1-e^{-\mu t}$	$\dfrac{tq_x}{1-(1-t)q_x}$
${}_tp_x$	$1-tq_x$	$e^{-\mu t}$	$\dfrac{p_x}{1-(1-t)q_x}$
${}_yq_{x+t}$	$\dfrac{yq_x}{1-tq_x}$	$1-e^{-\mu y}$	$\dfrac{yq_x}{1-(1-y-t)q_x}$
μ_{x+t}	$\dfrac{q_x}{1-tq_x}$	μ	$\dfrac{q_x}{1-(1-t)q_x}$
${}_tp_x\mu_{x+t}$	q_x	$e^{-\mu t}\mu$	$\dfrac{p_xq_x}{\{1-(1-t)q_x\}^2}$

x : integer, $0 \le t \le 1$, $0 \le y \le 1$, $0 \le y+t \le 1$, $\mu = -\log p_x$.

Select and Ultimate Tables

- A special survival function that incorporates the *particular information* available at age x would be preferred.

- *Select life table*: the conditional probabilities of death for lives on which the special information became available at age x.

$$q_{[x]+i} \quad (i = 0, 1, 2, \cdots)$$

- *Select period*: the smallest integer n such that $\left| q_{[x]+n} - q_{[x-j]+n+j} \right|$ is small enough for all x and for all $j > 0$.

(The impact of selection may diminish following selection)

- Aggregate table: the functions given only for attained ages

- Select and ultimate life table:

$$l_{[x]},\ l_{[x]+1},\ l_{[x]+2},\cdots,l_{[x]+r-1},\ l_{[x]+r}=l_{x+r}$$

(e.g.)

$[x]$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
30	33829	33814	33795	32
31	33807	33791	33771	33
32	33784	33767	33746	34
33	33760	33742	33719	35
34	33734	33715	33690	36

- $l_{[x]+r-k-1}\times p_{[x]+r-k-1}=l_{[x]+r-k}$

$$(k=0,1,\cdots,r-1)$$