BAYESIAN STATISTICS

Chapter 9

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9. Population Variance Estimation

9.1. Introduction

In this new chapter, we continue to consider normal distribution. Contrast to the last chapter where μ is the parameter of interest and σ^2 is known, this chapter considers the inference on σ^2 with the known μ.

It is not common in practice that μ is known and σ^2 is not known. With this simplification, however, the Bayesian inference on σ^2 only is easy to derive and this will be a warm-start for the case for the unknown μ and unknown σ^2.

9.2. Bayes estimation for σ^2

Instead of using σ^2, we consider $p_{rec} = \frac{1}{\sigma^2}$. Suppose, thus, that

$$X \sim \mathrm{N}(\mu = m, \sigma^2) = \mathrm{N}\left(m, \frac{1}{p_{rec}}\right).$$

Then, the likelihood is going to be

$$\ell(x|m, p_{rec}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\} = \frac{p_{rec}^{1/2}}{\sqrt{2\pi}} \exp\left\{-\frac{p_{rec}(x-m)^2}{2}\right\}.$$

The conjugate prior for p_{rec} is gamma distribution, Gamma(α, β) for positive α and β. Thus, the posterior is obtained as following:

- prior: $g(p_{rec}) = \frac{1}{\Gamma(\alpha)\beta^\alpha} p_{rec}^{\alpha-1} e^{-p_{rec}/\beta}$
- likelihood:
  $\ell(x_1, \ldots, x_n | p_{rec}) = \prod_{i=1}^n f(x_i | p_{rec}) = (2\pi)^{-n/2} p_{rec}^{n/2} \exp\left\{ -p_{rec} \frac{\sum_{i=1}^n (x_i - m)^2}{2} \right\}$
- posterior:

$$
\begin{aligned}
h(p_{rec} | x_1, \ldots, x_n) &\propto p_{rec}^{\alpha-1} e^{-p_{rec}/\beta} \times p_{rec}^{n/2} \exp\left\{ -p_{rec} \frac{\sum_{i=1}^n (x_i - m)^2}{2} \right\} \\
&\propto p_{rec}^{n/2+\alpha-1} \exp\left\{ -p_{rec} \left(\frac{\sum_{i=1}^n (x_i - m)^2}{2} + \frac{1}{\beta} \right) \right\} \\
&\sim \text{Gamma}\left(\frac{n}{2} + \alpha, \left\{ \frac{\sum_{i=1}^n (x_i - m)^2}{2} + \frac{1}{\beta} \right\}^{-1} \right)
\end{aligned}
$$

Thus, the Bayes estimate is

$$\hat{p}_{rec,B} = \frac{n/2+\alpha}{ns^2/2+1/\beta}$$

with $s^2 = \sum_{i=1}^n (x_i - m)^2/n$. This becomes $1/s^2$ when n gets large.

The posterior variance is given as

$$\frac{n/2+\alpha}{(ns^2/2+1/\beta)^2}$$

and, so, it is going to be close to zero when n gets large.

9.3. No prior information on σ^2

Suppose we consider the prior distribution for $p_{rec} = 1/\sigma^2$ as gamma distribution Gamma(α, β) as we did previously.

Note that no information on p_{rec} implies that there is no preference on any value of $p_{rec} > 0$. This can be interpreted as that the prior must "spread out" and be "flat".

β is the scale parameter. The larger β, the more the gamma distribution spreads out. So the natural choice is $\beta \to \infty$.

α is the shape parameter. The gamma distribution becomes symmetric when α gets larger. If $\alpha = 0$ (this makes the prior improper, anyway), then the gamma distribution becomes flat. Thus, the natural choice for α is 0.

Thus the vague, and improper, prior for p_{rec} is

$$g(p_{rec}) = \frac{1}{p_{rec}}, \quad p_{rec} > 0$$

and the associated posterior is

$$\text{Gamma}\left(\frac{n}{2} + \alpha, \left(\frac{ns^2}{2} + \frac{1}{\beta}\right)^{-1}\right) \xrightarrow{\alpha \to 0, \beta \to \infty} \text{Gamma}\left(\frac{n}{2}, \left(\frac{ns^2}{2}\right)^{-1}\right).$$

Note that

$$\frac{ns^2}{\sigma^2}\Big|(X_1, X_2, \ldots, X_n) \sim \chi^2(n). \quad \text{(prove it!)}$$

9.5. Credible interval for σ^2 with the vague prior

From the fact that $\frac{ns^2}{\sigma^2}\Big|(X_1, X_2, \ldots, X_n) \sim \chi^2(n)$,

$$\begin{aligned} & \Pr\left(\chi^2_{n,\alpha/2} \le \frac{ns^2}{\sigma^2} \le \chi^2_{n,1-\alpha/2}\Big| X_1, \ldots, X_n\right) \\ = \; & \Pr\left(\frac{ns^2}{\chi^2_{n,1-\alpha/2}} \le \sigma^2 \le \frac{ns^2}{\chi^2_{n,\alpha/2}}\Big| X_1, \ldots, X_n\right) \\ = \; & 1-\alpha. \end{aligned}$$

Thus, 95% EPD credible interval for σ^2 is

$$\left(\frac{ns^2}{\chi^2_{n,1-\alpha/2}}, \frac{ns^2}{\chi^2_{n,\alpha/2}}\right).$$

In the same manner, 95% EPD credible interval for p_{rec} is

$$\left(\frac{\chi^2_{n,\alpha/2}}{ns^2}, \frac{\chi^2_{n,1-\alpha/2}}{ns^2}\right).$$

Example (9-1)

Suppose we obtained 12 samples from normal distribution with mean $m = 6$ and the unknown σ^2 as follow:

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7.12  5.62  4.31  8.22  6.39  5.91  6.55  5.25  7.02  4.99  6.02  7.00
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If there is no prior information on σ^2, find the posterior for σ^2 and the corresponding 95% EPD credible interval.

(solution) The posterior distribution with the vague prior is

$$p_{rec}|X_1, \ldots, X_n \sim \text{Gamma}(n/2, ns^2/2).$$

From the data, we get $s^2 = 1.1058$ and $n = 12$. Thus, the posterior of p_{rec} is Gamma$(6, 6.6348)$. The 95% EPD credible interval for p_{rec} is

$$\left(\frac{\chi^2_{n,\alpha/2}}{ns^2}, \frac{\chi^2_{n,1-\alpha/2}}{ns^2} \right) = (0.3316, 1.7353)$$

and the 95% EPD credible interval for σ^2 is

$$\left(\frac{ns^2}{\chi^2_{n,1-\alpha/2}}, \frac{ns^2}{\chi^2_{n,\alpha/2}} \right) = (0.5762, 3.0157).$$