<최소분산불편추정량 관련 문제>

1. $X_1, X_2, \ldots, X_n$ 을 확률밀도함수 $f(x;\theta) = \theta x^{\theta-1}\ (\theta > 0,\ 0 < x < 1)$ 으로부터 추출한 크기 n 인 확률표본이라 할 때, θ 의 UMVUE를 구하여라.

sol) $f(x_1, \cdots, x_n|\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1} = \exp\left[n log\,\theta + (\theta-1)\sum_i \log x_i\right]$

$\therefore \sum \log X_i$ *is a C.S.S of* θ

Let $Y = -\log X$, *then* $X = e^{-Y}$ *and* $|J| = e^{-y}$.

$\therefore\ f_Y(y) = \theta e^{-\theta y}$

$\therefore Y \sim \mathrm{Exp}(\theta)$

Let $Z = \sum Y_i$

$\therefore Z \sim GAMMA\ (n, \theta)$.

$\therefore\ E(Z) = E(\sum Y_i) = E(-\sum \log X_i) = \sum E(-\log X_i) = n\theta$

$\therefore\ -\frac{1}{n}\sum \log X_i$ *is an U.E of* θ

$\therefore\ -\frac{1}{n}\sum \log X_i$ *is the UMVUE.*

5. $X_1, X_2, \ldots, X_n$ 을 확률질량함수 $f(x;\theta) = \dfrac{(\ln\theta)^x}{\theta x!}$ $(\theta > 1,\ x = 0, 1, 2, 3, \cdots)$ 를 가지는 분포로부터 추출한 크기 n인 확률표본이다. 이때 $\ln\theta$ 의 UMVUE를 구하여라.

sol) $f(x_1, \cdots, x_n|\theta) = \dfrac{(\ln\theta)^{\sum x_i}}{\theta \prod x_i!} = \exp\left[\sum x_i \log(\ln\theta) - n\log\theta - \sum \log x_i!\right]$

$\therefore \sum X_i \ is \ a \ C.S.S \ of \ \ln\theta$

$Let \ Y = \sum X_i$

$\therefore \ E(Y) = E(\sum X_i) = \sum E(X) = n\ln\theta$

$\therefore \ \dfrac{1}{n}\sum X_i \ is \ an \ U.E \ of \ \ln\theta$

$\therefore \ \dfrac{1}{n}\sum X_i \ is \ the \ UMVUE.$

※ $E(X) = \displaystyle\sum_{x=0}^{\infty} x \frac{(\ln\theta)^x}{\theta x!} = \sum_{x=1}^{\infty} x \frac{(\ln\theta)^x}{\theta x!}$

$= \dfrac{\ln\theta}{\theta} \displaystyle\sum_{x=1}^{\infty} \frac{(\ln\theta)^{x-1}}{(x-1)!}, \quad t = x-1$

$= \dfrac{\ln\theta}{\theta} \displaystyle\sum_{t=0}^{\infty} \frac{(\ln\theta)^t}{t!}, \quad Tayor\,Expansion$에 의하여

$= \dfrac{\ln\theta}{\theta} e^{\ln\theta} = \ln\theta$