Nonparametric Statistics

Ch.4 One-way ANOVA (k-sample problem)

Motivation

- Sometimes, one's interest is centered on comparison for the locations of three or more populations. This is called the k-sample location problem.

- A nonparametric method designed for k-sample location problem is an alternative to the one-way ANOVA.

- Our interest is to see if there is no differences among the location parameters of multiple distributions.

Review : One-way ANOVA

Let $X_{i1}, \dots, X_{in_i}$ be a random sample from $N(\mu_i, \sigma^2)$ $(i = 1, \cdots, k)$. A test for the hypothesis

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k \qquad vs \qquad H_1: not\ \ H_0$$

or equivalently,

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_k = 0 \qquad vs \qquad H_1: not\ \ H_0$$

where $\mu_i = \mu + \tau_i$ can be performed with the test statistic

$$F = \frac{SS_{tr}/(k-1)}{SSE\ /(N-k)},$$

where $SS_{tr} = \sum_{i=1}^{k} n_i(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2$, $SSE = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(X_{ij} - \bar{X}_{i\cdot}\right)^2$,

$N = \sum_{i=1}^{k} n_i$, $\bar{X}_{i\cdot} = n_i^{-1}\sum_{j=1}^{n_i} X_{ij}$, $\bar{X}_{\cdot\cdot} = N^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}$.

- F has the $F\ (k-1, N-k)$ distribution under H_0.
- Rejection region for the test at the significance level α is $\{f: f > f_\alpha(k-1, N-k)\}$ where $f_\alpha(k-1, N-k)$satisfies $P(F > f_\alpha(k-1, N-k))= \alpha$ with $F \sim F\ (k-1, N-k)$.
- p-value is $P(F > f_0)$ where f_0 is the observed value of the test statistic.
- τ_i's are called the treatment effects.

Review : One-way ANOVA

- This test is based on the assumption that all random samples are independent.

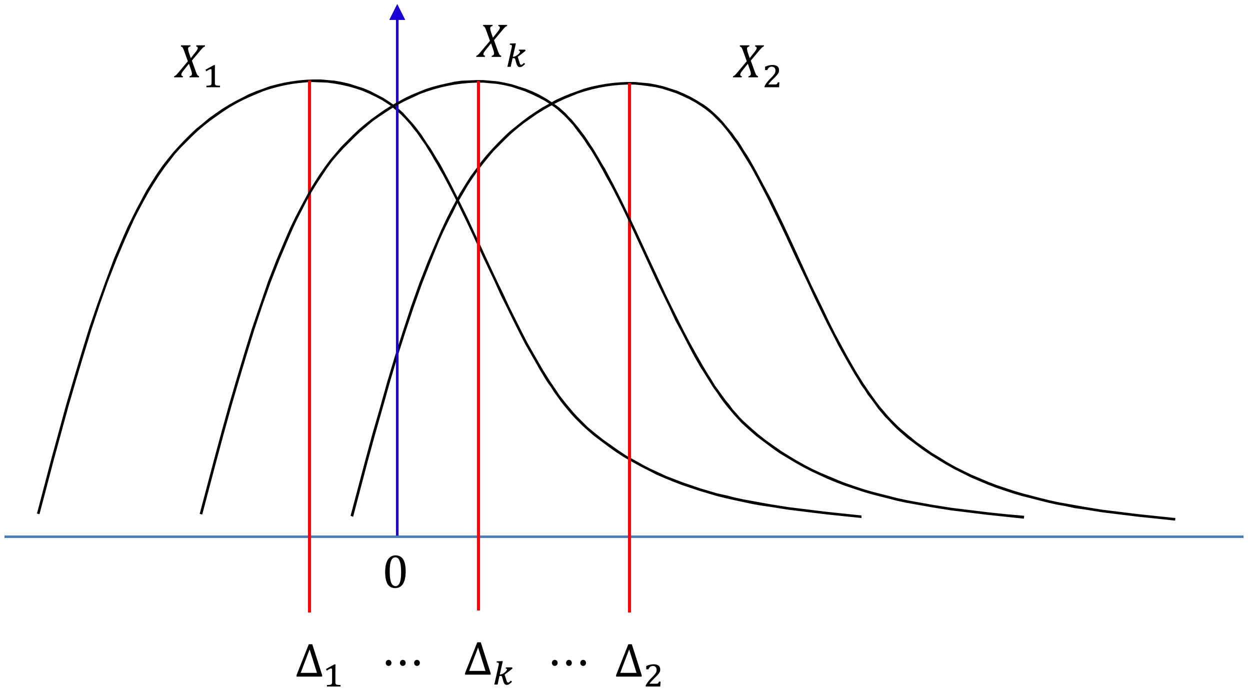
- The basic idea of this test is to see how big the group variation (SS_{tr}) is in the total variation ($SST = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(X_{ij} - \bar{X}_{..}\right)^2$).

- If $k = 2$, this reduces to the independent two-sample t-test with equal variances we have learned in the previous chapter.

Kruskal-Wallis test

Let $X_{i1}, \dots, X_{in_i}$ be a random sample from $F(\cdot - \Delta_i)$ $(i = 1, \dots, k)$.

- We assume that F is continuous.



Kruskal-Wallis test

A test for the hypothesis

$$H_0 : \Delta_1 = \cdots \Delta_k \qquad vs \qquad H_1 : not \;\; H_0$$

can be performed with the test statistic

$$KW_N = \frac{12}{N(N+1)} \sum_{i=1}^{k} n_i \left(\bar{R}_{i\cdot} - \frac{N+1}{2} \right)^2$$

$$= \frac{12}{N(N+1)} \sum_{i=1}^{k} n_i \bar{R}_{i\cdot}^2 - 3(N+1),$$

$$= \frac{12}{N(N+1)} \sum_{i=1}^{k} \frac{R_{i\cdot}^2}{n_i} - 3(N+1),$$

where $N = \sum_{i=1}^{k} n_i$, $\bar{R}_{i\cdot} = n_i^{-1} \sum_{j=1}^{n_i} R_{ij} \; = n_i^{-1} R_{i\cdot}$ and R_{ij} is the rank of X_{ij} in the pooled sample.

Kruskal-Wallis test

- Large values of W_n support the alternative hypothesis.
- Note that we assume that all distributions are the same in shape.
- In the case of $k = 2$, this test is equivalent to the Wilcoxon rank sum test and the Mann-Whitney test.
- In fact, the Kruskal-Wallis test borrows a similar idea with the one-way ANOVA. It is nothing but replacing X_{ij} with R_{ij} . Note that if we treat the rank R_{ij} 's as observations,

$$SS_{tr} = \sum_{i=1}^{k} n_i(\bar{R}_{i\cdot} - \bar{R}_{\cdot\cdot})^2 = \sum_{i=1}^{k} n_i\left(\bar{R}_{i\cdot} - \frac{N+1}{2}\right)^2,$$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(R_{ij} - \bar{R}_{\cdot\cdot})^2 = \frac{N(N+1)(N-1)}{12},$$

Then, the test statistic can be expressed as

$$KW_N = \frac{(N-1)SS_{tr}}{SST}.$$

Kruskal-Wallis test

- If $\min\{n_i : 1 \leq i \leq k\}$ is sufficiently large,
$$KW_N \approx \chi^2(k-1)$$
under H_0.

- Rejection region
$$\{\chi_0^2 :\ \chi_0^2 > \chi_\alpha^2(k-1)\}$$

- p-value
$$P(\chi^2 > kw_N)$$
where kw_N is the observed value of the test statistic and $\chi^2 \sim \chi^2(k-1)$.

Kruskal-Wallis test : Example

Ex1] The tensile strength of 12 wires were measured. They were produced by three different machines. Is there a significant difference among the machines?

	A	B	C	
	56	48	52	
	60	61	50	
	57	49	44	
	64	53	46	
mean	59.25	52.75	48.00	53.33

(1) One-way ANOVA

From the table,

$$SST = 438.67, SS_{tr} = 255.17, SSE = 183.50$$

The observed test statistic is

$$f_0 = \frac{255.17/(3-1)}{183.50/(12-3)} = 6.29$$

The p-value is $P(F > 6.29) = 0.0195$ where $F \sim F(2,9)$.

Kruskal-Wallis test : Example

(2) Kruskal-Wallis test

– Add ranks to the table

	A	B	C	
	56 (8)	48 (3)	52 (6)	
	60 (10)	61 (11)	50 (5)	
	57 (9)	49 (4)	44 (1)	
	64 (12)	53 (7)	46 (2)	
Rank sum	(39)	(25)	(14)	78

– The observed test statistic:

$$KW_N = \frac{12}{12(12+1)}\left(\frac{39^2}{4} + \frac{25^2}{4} + \frac{14^2}{4}\right) - 3(12+1) = 6.038$$

– The p-value by large sample approximation is $P(\chi^2 > 6.038) = 0.0489$ where $\chi^2 \sim \chi^2(2)$.

Kruskal-Wallis test : Example

Ex2] Calculate observed Kruskal-Wallis test statistics in the following data sets.

(1) Effect levels by 3 sleeping pills.

A	B	C
2	1	5
4	3	7
	6	

(2) Fuel consumption on 4 buses

A	B	C	D
114	111	107	105
108	109	106	98
110	112	116	102
113	114	111	101
113	115	102	96