Chap 4. Expected Value (기댓값)

4.1 Expected value of a random variable

Def 4.1 이산형 확률변수 X의 기댓값 :

$$E(X) = \mu = \sum_i x_i p(x_i)$$

연속형 확률변수 X의 기댓값 :

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

[예 4.1]

Expected values of several probability distributions :

- $X \sim B(n,p)$

$$\begin{aligned} E(X) &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\ &= \sum_{l=0}^{n-1} \frac{n!}{l!(n-l-1)!} p^{l+1} (1-p)^{n-l-1} \quad (let \ \ l = k-1) \\ &= np \sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-l-1)!} p^l (1-p)^{n-l-1} \\ &= np \end{aligned}$$

- $X \sim Poisson(\lambda)$

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} \\ &= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} e^{-\lambda} \\ &= \lambda \end{aligned}$$

- $X \sim Geometric(p)$

$$E(X) = \sum_{k=1}^{\infty} k(1-p)^{k-1} = p\sum_{k=1}^{\infty} k(1-p)^{k-1} \quad (4.4)$$

$$(1-p)E(X) = p\sum_{k=1}^{\infty} k(1-p)^{k} \quad (4.5)$$

(4.4)식에서 (4.5)식을 빼면,

$$pE(X) = p(\sum_{k=1}^{\infty} k(1-p)^{k-1} - \sum_{k=1}^{\infty} (k-1)(1-p)^{k-1}) = p(\sum_{k=1}^{\infty} (1-p)^{k-1}) = 1$$

$$\therefore E(X) = \frac{1}{p}$$

참고] Another way to get $E(X)$ (미분이용)

- $X \sim Gamma(\alpha, \frac{1}{\lambda})$

$$\begin{aligned} E(X) &= \int_0^{\infty} x \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \int_0^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha} e^{-\lambda x} dx \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{1}{\lambda} \int_0^{\infty} \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{\alpha} e^{-\lambda x} dx \\ &= \frac{\alpha}{\lambda} \end{aligned}$$

- $X \sim Exp(\frac{1}{\lambda})$

$$E(X) = \frac{1}{\lambda} \quad (\because Exp(\frac{1}{\lambda}) \equiv Gamma(1, \frac{1}{\lambda}))$$

- $X \sim \chi^2(r)$

$$E(X) = r \quad (\because \chi^2(r) \equiv Gamma(\frac{r}{2}, 2))$$

- $X \sim N(\mu, \sigma^2)$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \mu + \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \mu + \int_{-\infty}^{\infty} t \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} dx \quad (let \ t = x - \mu) \\ &= \mu \end{aligned}$$

참고]

- $X \sim NB(p, r) \Rightarrow E(X) = \frac{r(1-p)}{p}$
- $X \sim HyperGeometric(n, r, m) \Rightarrow E(X) = \frac{r}{n} m$
- $X \sim U(a, b) \Rightarrow E(X) = \frac{a+b}{2}$
- $X \sim Beta(\alpha, \beta) \Rightarrow E(X) = \frac{\alpha}{\alpha + \beta}$
- $X \sim C(\theta, 1) \Rightarrow$ It does not exist!
- $X \sim DE(\mu, \sigma) \Rightarrow E(X) = \mu$

4.2 Expected value of a function of random variable

확률변수 X의 함수인 $Y=g(X)$의 기댓값 구하는 방법

$$E[g(X)]=E(Y)=\begin{cases}\sum_i y_i p_Y(y_i) & \text{(이산형 확률변수의 경우)}\\ \int_{-\infty}^{\infty} y f_Y(y)dy & \text{(연속형 확률변수의 경우)}\end{cases}$$

예 : $X\sim Exp(\frac{1}{\lambda})$일 때 $Y=2X$의 기댓값?

$f_X(x)=\lambda e^{-\lambda x}$, $x\geq 0$이므로 $Y=2X$의 확률밀도함수는

$$f_Y(y)=\frac{1}{2}f_X(\frac{y}{2})=\frac{\lambda}{2}e^{-\frac{\lambda}{2}y},\ y\geq 0$$

$$\therefore E(Y)=\int_0^{\infty} y f_Y(y)dy=\int_0^{\infty} y\frac{\lambda}{2}e^{-\frac{\lambda}{2}y}dy\ =\frac{2}{\lambda}$$

Thm 4.1 이산형 확률변수 X의 확률질량함수 : $p_X(x)$일 때

$$E[g(X)]=\sum_i g(x_i)p_X(x_i)$$

연속형 확률변수 X의 확률밀도함수 : $f_X(x)$일 때,

$$E[g(X)]=\int_{-\infty}^{\infty} g(x)f_X(x)dx$$

(proof)

[예 4.2]

Thm 4.2 이산형 확률변수 $(X_1,...,X_n)$의 결합확률질량함수를 $p(x_1,...,x_n)$이라 할 때 $g(X_1,...,X_n)$의 기댓값은

$$E[g(X_1,...,X_n)] = \sum_{x_1} \cdots \sum_{x_n} g(x_1,...,x_n)p(x_1,...,x_n)$$

또, 연속형 확률변수 $(X_1,...,X_n)$의 결합확률밀도함수를 $f(x_1,...,x_n)$이라 할 때 $g(X_1,...,X_n)$의 기댓값은

$$E[g(X_1,...,X_n)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1,...,x_n)f(x_1,...,x_n)dx_1 \cdots dx_n$$

Thm 4.3 When X and Y are independent,

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

(proof) Consider both X and Y are continuous case. Since X and Y are independent, the joint p.d.f is $f(x,y) = f_X(x)f_Y(y)$. Therefore,

$$\begin{aligned} E[g(X)h(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy \\ &= \left(\int_{-\infty}^{\infty} g(x)f_X(x)dx\right)\left(\int_{-\infty}^{\infty} h(y)f_Y(y)dy\right) \\ &= E[g(X)]E[h(Y)] \end{aligned}$$

If $g(x) = x$ and $h(y) = y$, for independent X and Y,

$$E(XY) = E(X)E(Y)$$

[예 4.3] $X \sim Exp(\frac{1}{\lambda_1}), Y \sim Exp(\frac{1}{\lambda_2}), X \perp Y \quad \Rightarrow \; E(XY)?$

Thm 4.4 For constants $a, b_1, ..., b_n$,

$$E\left(a + \sum_{i=1}^{n} b_i X_i\right) = a + \sum_{i=1}^{n} b_i E(X_i)$$

(proof)

4.3 Variance and Standard Deviation

Def 4.2 The variance of a random variable X is defined by

$$Var(X) = E(X - \mu)^2$$

Brief caculation

$$\begin{aligned} Var(X) &= E(x-\mu)^2 = E(X^2 - 2\mu X + \mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - \mu^2 \\ &= E(X^2) - [E(X)]^2 \end{aligned}$$

[예 4.4]

Some examples of calculating variances :

- $X \sim Bernoulli(p)$

$$Var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$$

- $X \sim B(n,p)$

$$Var(X) = E[X(X-1)] + E(X) - E(X)^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

Note that

$$\begin{aligned} E[X(X-1)] &= \sum_{k=0}^{n} k(k-1)\binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} \\ &= \sum_{l=0}^{n-2} \frac{n!}{l!(n-l-2)!} p^{l+2} (1-p)^{n-l-2} \quad (let\ \ l = k-2) \\ &= n(n-1)p^2 \sum_{l=0}^{n-2} \frac{(n-2)!}{l!(n-l-2)!} p^l (1-p)^{n-l-2} \\ &= n(n-1)p^2 \end{aligned}$$

- $X \sim Poisson(\lambda)$

$$Var(X) = \lambda$$

- $X \sim Geometric(p)$

$$Var(X) = \frac{1-p}{p^2}$$

- $X \sim Gamma(\alpha, \frac{1}{\lambda})$

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha+1} e^{-\lambda x} dx \\ &= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{1}{\lambda^2} \int_0^\infty \frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{(\alpha+2)-1} e^{-\lambda x} dx \\ &= \frac{(\alpha+1)\alpha}{\lambda^2} \end{aligned}$$

$$\therefore Var(X) = \frac{\alpha(\alpha+1)}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}$$

- $X \sim Exp(\frac{1}{\lambda})$

$$Var(X) = \frac{1}{\lambda^2} \quad (\because Exp(\frac{1}{\lambda}) \equiv Gamma(1, \frac{1}{\lambda}))$$

- $X \sim \chi^2(r)$

$$Var(X) = 2r \quad (\because \chi^2(r) \equiv Gamma(\frac{r}{2}, 2))$$

- $X \sim N(\mu, \sigma^2)$

$$\begin{aligned} Var(X) &= \int_{-\infty}^\infty (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \sigma^2 \int_{-\infty}^\infty t^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \quad (let\ \ t = \frac{x-\mu}{\sigma}) \\ &= 2\sigma^2 \int_0^\infty t^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \quad (symmetric\ \ about\ \ zero) \\ &= \sigma^2 \int_0^\infty u \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} \frac{1}{\sqrt{u}} du \quad (let\ \ u = t^2) \\ &= \sigma^2 \int_0^\infty \frac{1}{(1/2)\sqrt{\pi}} (\frac{1}{2})^{\frac{3}{2}} u^{\frac{3}{2}-1} e^{-\frac{u}{2}} du \\ &= \sigma^2 \end{aligned}$$

Note that $\Gamma(\frac{3}{2}) = (1/2)\Gamma(1/2) = (1/2)\sqrt{\pi}$.

참고]

- $X \sim NB(p,r) \Rightarrow Var(X) = \frac{r(1-p)}{p^2}$
- $X \sim HyperGeometric(n,r,m) \Rightarrow Var(X) = m\frac{r}{n}\frac{n-r}{n}\frac{n-m}{n-1}$
- $X \sim U(a,b) \Rightarrow Var(X) = \frac{(b-a)^2}{12}$
- $X \sim Beta(\alpha,\beta) \Rightarrow Var(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$
- $X \sim C(\theta,1) \Rightarrow$ It does not exist!
- $X \sim DE(\mu,\sigma) \Rightarrow Var(X) = 2\sigma^2$

Thm 4.5 For any constant a and b,

(i) $Var(a+X) = Var(X)$

(ii) $Var(bX) = b^2 Var(X)$

(iii) $Var(a+bX) = b^2 Var(X)$

(proof) Since $E(a+X) = a + E(X)$,

$$Var(a+X) = E[a+X-(a+E(X))]^2 = E[X-E(X)]^2 = Var(X).$$

And $E(bX) = bE(X)$ implies

$$Var(bX) = E[bX - bE(X)]^2 = b^2 E[X-E(X)]^2 = b^2 Var(X).$$

Therefore, $Var(a+bX) = Var(bX) = b^2 Var(X)$.

Thm 4.6 [체비셰프 부등식, Chebyshev inequality] Let $E(X) = \mu$ and $Var(X) = \sigma^2$. For any real number $t > 0$.

$$P(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}.$$

(proof) Consider only when X is continuous. Then,

$$\begin{aligned} P(|X-\mu| \geq t) &= P(X \leq \mu - t \text{ or } X \geq \mu + t) \\ &= \int_{-\infty}^{\mu-t} f(x)dx + \int_{\mu+t}^{\infty} f(x)dx \end{aligned}$$

When $x \leq \mu - t$ or $x \geq \mu + t$, $\left(\dfrac{x-\mu}{t}\right)^2 \geq 1$ holds and this implies

$$\begin{aligned} &\int_{-\infty}^{\mu-t} f(x)dx + \int_{\mu+t}^{\infty} f(x)dx \\ &\quad \leq \int_{-\infty}^{\mu-t} \left(\frac{x-\mu}{t}\right)^2 f(x)dx + \int_{\mu+t}^{\infty} \left(\frac{x-\mu}{t}\right)^2 f(x)dx \\ &\quad \leq \int_{-\infty}^{\infty} \left(\frac{x-\mu}{t}\right)^2 f(x)dx = \frac{\sigma^2}{t^2} \end{aligned}$$

In the Chebyshev inequality, if we let $t = k\sigma$, then

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

4.4 Delta method (델타방법)

Taylor expansion of function $Y = g(X)$ at $x = \mu_X$:

$$g(X) = g(\mu_X) + g'(\mu_X)(X-\mu_X) + \frac{g''(\mu_X)}{2}(X-\mu_X)^2 + \cdots$$

Approximation of expected value $Y = g(X)$ (Use the first three components)

$$\begin{aligned} E(g(X)) &\approx g(\mu_X) + g'(\mu_X)E(X-\mu_X) + \frac{g''(\mu_X)}{2}E(X-\mu_X)^2 \\ &= g(\mu_X) + \frac{g''(\mu_X)}{2}\sigma_X^2 \end{aligned}$$

Approximation of variance of $Y = g(X)$ (Use the first two components)

$$\begin{aligned} Var(g(X)) &\approx Var\big(g(\mu_X) + g'(\mu_X)(X-\mu_X)\big) \\ &= (g'(\mu_X))^2\sigma_X^2 \end{aligned}$$

[예 4.5] $X \sim U(0,1)$. Calculate $E(Y)$ & $Var(Y)$ when $Y=\sqrt{X}$

(a) Exact calculation

(b) Approximation by delta method

4.5 Covariance (공분산)

Def 4.3 Covariance of a random variable X and Y :

$$Cov(X,Y)=E[(X-\mu_X)(Y-\mu_Y)]$$

Simple formula

$$\begin{aligned} Cov(X,Y) &= E(x-\mu)^2 = E[X-\mu_X)(Y-\mu_Y)] = E(XY-\mu_X Y-\mu_Y X+\mu_X\mu_{Y)} \\ &= E(XY)-\mu_X\mu_{Y-}\mu_Y\mu_X+\mu_X\mu_Y = E(XY)-\mu_X\mu_Y \\ &= E(XY)-E(X)E(Y) \end{aligned}$$

[예 4.6]

[예 4.7] $f(x,y) = x + y,\ 0 \le x \le 1,\ 0 \le y \le 1,\ Cov(X,Y)?$

Thm 4.7 For a constant a and random variables X, X_1, X_2, Y,

(i) $Cov(X+a, Y) = Cov(X, Y)$

(ii) $Cov(aX, Y) = aCov(X, Y)$

(iii) $Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$

Generalization of calculating covariance :

$$Cov(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j) = \sum_{i=1}^{m} Cov(a_i X_i, \sum_{j=1}^{n} b_j Y_j)$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} Cov(a_i X_i, b_j Y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j Cov(X_i, Y_j)$$

$$Var(X+Y) = Cov(X+Y, X+Y)$$
$$Cov(X,X) + 2Cov(X,Y) + Cov(Y,Y) = Var(X) + 2Cov(X,Y) + Var(Y)$$

$$Var(\sum_{i=1}^{n} X_i) = Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(X_i, X_j)$$

For independent random variables $X_1, \ldots, X_n$

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$$

Def 4.4 Correlation coefficient of r.v. X and Y :

$$\rho = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

[예 4.8] $f(x,y)=x+y,\ 0 \le x \le 1,\ 0 \le y \le 1,\ Corr(X,Y)$?

Thm 4.8 (Properties of correlation coefficient)

(i) For constants $a,b,c,d,$

$$Corr(aX+b, cY+d) = Corr(X,Y)$$

(ii) $-1 \le \rho \le 1$

(iii) $\rho = 1$

$\Leftrightarrow$ For real number a and positive real number b, $P(Y=a+bX)=1$

(iii) $\rho = -1$

$\Leftrightarrow$ For real number a and negative real number b, $P(Y=a+bX)=1$

Here, $P(Y=a+bX)=1$ means that (X,Y) can take the value only on the line $y=a+bx$

(proof)

Correlation coefficients of four distributions : 그림 4.1

4.6 Expected value and variance of conditional distribution

Def 4.5 Given $Y=y$, conditional expectation of discrete r.v. X :

$$E(X|Y=y)=\sum_i x_i p_{X|Y}(x_i|y)$$

Given $Y=y$, conditional expectation of continuous r.v. X :

$$E(X|Y=y)=\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

Given $Y=y$, conditional expectation of $g(X)$:

$$E(g(X)|Y=y)=\begin{cases}\sum_i g(x_i)p_{X|Y}(x_i|y) & (Discrete)\\ \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)dx & (Continuous)\end{cases}$$

Given $Y=y$, conditional variance of X ($Var(X|Y=y)$) :

$$Var(X|Y=y)=E(X^2|Y=y)-(E(X|Y=y))^2$$

[예 4.9]

[예 4.10] $(X,Y)\sim N_2(\cdot,\cdot)$. $E(Y|X=x)$?

[예 4.11] $f(x,y)=\lambda^2 e^{-\lambda y},\ 0 \le x \le y$

(a) $E(X|Y=y),\ \ Var(X|Y=y)$?

(b) $E(Y|X=x),\ \ Var(Y|X=x)$?

Thm 4.9 For random variables X and Y

$$E[E(X|Y)] = E(X)$$

(proof) For continuous random variable

$$E[E(X|Y)] = \int_{-\infty}^{\infty} E(X|Y=y) f_Y(y)dy$$

$$E(X|Y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

implies

$$E[E(X|Y)] = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx\right) f_Y(y)dy$$

Here, $f_{X|Y}(x|y) f_Y(y) = f(x,y)$,

$$E[E(X|Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)dxdy = E(X)$$

[예 4.12]

[예 4.13] $f(x,y)=\lambda^2 e^{-\lambda y},\ 0\le x\le y$

(a) $E[E(X|Y)]$? $E[E(X|Y)]=E(X)$?

(a) $E[E(Y|X)]$? $E[E(Y|X)]=E(Y)$?

Thm 4.10 For random variables X and Y,

$$Var(X)=Var[E(X|Y)]+E[Var(X|Y)]$$

(proof)

$$\begin{aligned}Var[E(X|Y)] &= E[E(X|Y)]^2-[E(E(X|Y))]^2\\ &= E[E(X|Y)]^2-[E(X)]^2\end{aligned}$$

$$\begin{aligned}E[Var(X|Y)] &= E[E(X^2|Y)-(E(X|Y))^2]\\ &= E(X^2)-E[E(X|Y)]^2\end{aligned}$$

Therefore,

$$Var[E(X|Y)]+E[Var(X|Y)]=E(X^2)-[E(X)]^2=Var(X)$$

[예 4.14] (revisit 예 4.9)

(a) $Var(E(Y|X))$?

(b) $E(Var(Y|X))$?

(c) $(a)=(b)$?

[예 4.15]

4.7 Moment Generating Function : m.g.f (적률생성함수)

Def 4.6 Moment Generating Function $M(t)$: For a r.v. X, the expected value of e^{tX}, that is,

$$M(t)=E(e^{tX})$$

- Properties of m.g.f

(i) m.g.f specifies the probability distribution

(ii) $M(0)=1$ (when $t=0$, $e^{tX=1}$.)

(iii) $M'(0)=E(X)$

$$\because M'(t)=\frac{d}{dt}\int_{-\infty}^{\infty}e^{tx}f(x)dx=\int_{-\infty}^{\infty}\frac{d}{dt}e^{tx}f(x)dx$$
$$=\int_{-\infty}^{\infty}xe^{tx}f(x)dx$$

(iv) $M^{(k)}(0)=E(X^k)$ ($E(X^k)$: k-th moment of a r.v. X)

$$\because M^{(k)}(t)=\frac{d^k}{dt^k}\int_{-\infty}^{\infty}e^{tx}f(x)dx=\int_{-\infty}^{\infty}\frac{d^k}{dt^k}e^{tx}f(x)dx$$
$$=\int_{-\infty}^{\infty}x^ke^{tx}f(x)dx$$

- Finding m.g.f of probability distributions

(i) $X\sim Bernoulli(p)$:

$$M(t)=E(e^{tX})=(1-p)+pe^t$$

(ii) $X\sim B(n,p)$:

$$\begin{aligned}M(t)&=E(e^{tX})=\sum_{k=0}^{n}e^{tk}\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}\\&=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(pe^t)^k(1-p)^{n-k}\\&=(1-p+pe^t)^n\end{aligned}$$

$$\begin{aligned}M'(t)&=n(1-p+pe^t)^{n-1}pe^t\\M''(t)&=n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2+n(1-p+pe^t)^{n-1}pe^t.\end{aligned}$$

Therefore,

$$\begin{aligned}E(X)&=M'(0)=np\\E(X^2)&=M''(0)=n(n-1)p^2+np\end{aligned}$$

(iii) $X\sim Poisson(\lambda)$

$$M(t)=\sum_{k=0}^{\infty}e^{tk}\frac{\lambda^k}{k!}e^{-\lambda}=e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!}=e^{-\lambda}e^{\lambda e^t}=e^{\lambda(e^t-1)}$$

(Note that $\sum_{k=0}^{\infty}\frac{a^k}{k!}=e^a$ for a real number a)

(iv) $X \sim Gamma(\alpha, \frac{1}{\lambda})$

$$
\begin{aligned}
M(t) &= \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx \\
&= \left(\frac{1}{1-(t/\lambda)}\right)^\alpha \int_0^\infty \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx
\end{aligned}
$$

$t < \lambda$일 때 적분값이 1이 됨.

$$\therefore M(t) = \left(\frac{1}{1-(t/\lambda)}\right)^\alpha, \ t < \lambda$$

(v) $X \sim N(0,1)$

$$
\begin{aligned}
M(t) &= \int_{-\infty}^\infty e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2} dx \\
&= e^{-\frac{1}{2}t^2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx
\end{aligned}
$$

$$\therefore M(t) = e^{\frac{1}{2}t^2}$$

Thm 4.11 (i) When the m.g.f of X is $M_X(t)$, the m.g.f of $Y = a + bX$ is $M_Y(t) = e^{at} M_X(bt)$

(ii) Suppose that two r.v.s X and Y are independent and each m.g.f is $M_X(t)$ and $M_Y(t)$, respectively. The m.g.f of $Z = X + Y$ is $M_Z(t) = M_X(t) M_Y(t)$.

(proof)

(i) From the definition, m.g.f of $Y = a + bX$:

$$M_Y(t) = E(e^{tY}) = E(e^{at+btX}) = e^{at} E(e^{btX}) = e^{at} M_X(bt)$$

(ii) m.g.f of $Z = X + Y$:

$$
\begin{aligned}
M_Z(t) &= E(e^{tZ}) = E(e^{tX+tY}) \\
&= E(e^{tX}) E(e^{tY}) = M_X(t) M_Y(t)
\end{aligned}
$$

[예 4.16] Find m.g.f of $N(\mu,\sigma^2)$.

[예 4.17] Find m.g.f of $Z=X+Y$ when $X\perp Y$.

(a) $X\sim Poisson(\lambda_1)$, $Y\sim Poisson(\lambda_2)$

(b) $X\sim Gamma(\alpha_1,\frac{1}{\lambda})$, $Y\sim Gamma(\alpha_2,\frac{1}{\lambda})$

(c) $X\sim N(\mu_1,\sigma_1^2)$, $Y\sim N(\mu_2,\sigma_2^2)$