- Fact 1 : If y is normally distributed, then $ay+b$ is also normally distributed.

- Fact 1 : If $y_1, y_2, \cdots, y_n$ are independently normally distributed, then

  $\sum_i^n a_i y_i$ is also normally distributed.

- Fact 2 :

$$\mathrm{Cov}(\sum_i^n a_i y_i, \sum_j^n b_j z_j) = \sum_i^n \sum_j^n a_i b_j \mathrm{Cov}(y_i, z_j)$$

- Fact 3 :

$$\mathrm{Cov}(y_i, y_i) = \mathrm{V}(y_i) = \sigma^2$$

- Fact 4 : If $y_1, y_2, \cdots, y_n$ are independent,

$$\mathrm{Cov}(\sum_i^n a_i y_i, \sum_j^n b_j y_j) = \sum_i^n \sum_j^n a_i b_j \mathrm{Cov}(y_i, y_j)$$

$$= \sum_i^n a_i b_i \mathrm{V}(y_i) = \sum_i^n a_i b_i\, \sigma^2$$

If $y_i = \beta_0 + \beta_1 x_i + e_i, \quad e_i \sim \text{i.i.d.}\, N(0, \sigma^2)$

- Prove : $y_i \sim \text{ind. } N(\beta_0 + \beta_1 x_i,\, \sigma^2)$

- Prove : If $b_1 = \dfrac{\sum_i^n (x_i - \bar{x}) y_i}{\sum_i^n (x_i - \bar{x})^2}, \qquad b_0 = \bar{y} - b_1 \bar{x},$

  $b_1,\, b_0$ are normally distributed.

- Prove : $E(b_1) = \beta_1$

$$\because \qquad E(b_1) = \frac{\sum_j^n (x_j - \bar{x})}{\sum_i^n (x_i - \bar{x})^2} E(y_j) = \frac{\sum_j^n (x_j - \bar{x})}{\sum_i^n (x_i - \bar{x})^2} E(\beta_0 + \beta_1 x_j)$$

$$= \frac{\sum_j^n (x_j - \bar{x})}{\sum_i^n (x_i - \bar{x})^2} \beta_0 + \frac{\sum_j^n (x_j - \bar{x}) x_j}{\sum_i^n (x_i - \bar{x})^2} \beta_1 = \frac{\sum_i^n (x_i - \bar{x})(x_i - \bar{x})}{\sum_i^n (x_i - \bar{x})^2} \beta_1$$

$$= \beta_1$$

- Prove : $V(b_1) = \frac{\sigma^2}{\sum_i^n (x_i - \bar{x})^2}$

$$\because \qquad V(b_1) = V\left(\frac{\sum_j^n (x_j - \bar{x}) y_i}{(n-1)S_x^2}\right), \quad \text{where } S_x^2 = \frac{1}{n-1}\sum_i^n (x_i - \bar{x})^2$$

$$V(b_1) = \frac{1}{(n-1)^2 S_x^4} V\left(\sum_j^n (x_j - \bar{x}) y_i\right) = \frac{1}{(n-1)^2 S_x^4} \sum_j^n (x_j - \bar{x})^2 V(y_i)$$

$$= \frac{\sigma^2}{(n-1)S_x^2}, \qquad \text{where } S_x^2 = \frac{1}{n-1}\sum_i^n (x_i - \bar{x})^2$$

$$= \frac{\sigma^2}{\sum_i^n (x_i - \bar{x})^2}$$

Therefore,

$$b_1 \sim N\left(\beta_1, \frac{\sigma^2}{\sum_i^n (x_i - \bar{x})^2}\right)$$

- Prove：　$E(b_0) = \beta_0$

$$\because \quad E(b_0) = E(\bar{y} - b_1\bar{x}) = E(\bar{y}) - \bar{x}E(b_1)$$

$$= \frac{1}{n}\sum_i^n E(y_i) - \bar{x}\beta_1 = \frac{1}{n}\sum_i^n E(\beta_0 + \beta_1 x_i + e_i) - \bar{x}\beta_1$$

$$= \beta_0 + \beta_1\bar{x} - \bar{x}\beta_1$$

$$= \beta_0$$

- Prove：　$\mathrm{Cov}(\bar{y}, b_1) = 0$

$$\because \quad \mathrm{Cov}\left(\frac{1}{n}\sum_i^n y_i, \frac{1}{(n-1)S_x^2}\sum_j^n (x_j - \bar{x})y_j\right) = \frac{1}{n(n-1)S_x^2}\sum_i^n\sum_j^n \mathrm{Cov}(y_i, (x_j - \bar{x})y_j)$$

$$= \frac{1}{n(n-1)S_x^2}\sum_i^n\sum_j^n (x_j - \bar{x})\mathrm{Cov}(y_i, y_j) = \frac{1}{n(n-1)S_x^2}\sum_i^n (x_i - \bar{x})\mathrm{Cov}(y_i, y_i)$$

$$= \frac{\sum_i^n (x_i - \bar{x})\sigma^2}{n(n-1)S_x^2} = 0$$

- Prove : $\mathrm{V}(b_0) = \left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_i^n (x_1 - \bar{x})^2}\right)\sigma^2$

$$\because \quad \mathrm{V}(b_0) = \mathrm{V}\left(\bar{y} - b_1\bar{x}\right) = \mathrm{V}(\bar{y}) + \bar{x}^2\mathrm{V}(b_1) - 2\,\bar{x}\,\mathrm{Cov}(\bar{y}, b_1)$$

$$= \mathrm{V}(\bar{y}) + \bar{y}^2\,\mathrm{V}(b_1) = \frac{\sigma^2}{n} + \frac{\bar{x}^2\,\sigma^2}{\sum_i^n (x_i - \bar{x})^2}$$

$$= \left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_i^n (x_i - \bar{x})^2}\right)\sigma^2$$

Therefore,

$$b_0 \sim N\left(\beta_0,\, \left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_i^n (x_i - \bar{x})^2}\right)\sigma^2\right)$$

In Summary :

$$y_i = \beta_0 + \beta_1 x_i + e_i, \qquad e_i \sim \text{i.i.d.}\, N(0, \sigma^2)$$

$$b_1 = \frac{\sum_i^n (x_i - \bar{x}) y_i}{\sum_i^n (x_i - \bar{x})^2}, \qquad b_0 = \bar{y} - b_1 \bar{x},$$

$$b_1 \sim N\left(\beta_1, \frac{\sigma^2}{\sum_i^n (x_i - \bar{x})^2}\right)$$

$$b_0 \sim N\left(\beta_0, \left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_i^n (x_i - \bar{x})^2}\right) \sigma^2\right)$$

In general, the Confidence Interval for θ (the parameter)

$$\hat{\theta} \pm t_{\alpha/2,\nu} \times \text{ S.E.}\,(\hat{\theta})$$

ν is the degree of freedom

Therefore,

The Confidence Interval for β_1 is

$$b_1 \pm t_{\alpha/2,\nu} \sqrt{\frac{MSE}{\sum_i^n (x_i - \bar{x})^2}}$$

The Confidence Interval for β_1 is

$$b_1 \pm t_{\alpha/2,\nu} \sqrt{\left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_i^n (x_i - \bar{x})^2}\right) MSE}$$