Chap 3. Joint Distribution

Joint distribution (결합분포) or Joint probability distribution (결합확률분포) : Probability of two or more random variables defined in the same sample space

Example of joint distribution : p.86 Tossing coins

Joint cumulative distribution function of two r.v. X and Y :

$$F(x,y) = P(X \le x, Y \le y)$$

Joint cumulative distribution function of n r.v. $(X_1,...,X_n)$:

$$F(x_1,...,x_n) = P(X_1 \le x_{1,} \ ...,X_n \le x_n)$$

Marginal distribution (주변분포) : Partial (single) distribution of r.v. among several random variables. (결합분포를 갖는 여러 확률변수 중에 일부 확률변수의 확률분포)

Some Notations

$F(x,y)$, $f(x,y)$: Joint cumulative distribution function and joint probability distribution function of (X, Y)

$F_X(x)$, $f_X(x)$: Marginal cumulative distribution function and marginal probability distribution function of X (X의 주변누적분포함수와 주변확률밀도함수)

$F_Y(y)$, $f_Y(y)$: Marginal cumulative distribution function and marginal probability distribution function of Y (Y의 주변누적분포함수와 주변확률밀도함

수)

Marginal cumulative distribution function of X :

$$\begin{aligned} F_X(x) &= P(X \le x) = P(X \le x, Y < \infty) \\ &= \lim_{y \to \infty} F(x,y) \\ &= F(x,\infty) \end{aligned}$$

[예 3.1]

3.1 Discrete joint distribution (이산형 결합분포)

(X,Y) : Discrete joint random variable

$(x_{i,}y_j)$, $i=1,2,\ldots$ $j=1,2,\ldots$: (X,Y)가 취할 수 있는 값

$p(x_i,y_j)$ or $f(x_i,y_j)$: joint probability mass function (결합확률질량함수)

$$p(x_i,y_j) = P(X=x_{i,}\, Y=y_j)$$

X의 주변확률질량함수 $p_X(x_i)$ 구하기

$$\begin{aligned} p_X(x_i) &= P(X=x_i) = P(\cup_j \{X=x_i, Y=y_j\}) \\ &= \sum_j P(X=x_i, Y=y_j) \\ &= \sum_j p(x_i,y_j) \end{aligned}$$

$(X_1,\ldots,X_n)$의 결합확률질량함수가 $p(x_1,\ldots,x_n)$일 때, X_1의 marginal p.m.f :

$$p_{X_1}(x_1) = \sum_{x_2, x_3, \ldots, x_n} p(x_1,\ldots,x_n)$$

3.2 Continuous joint distribution (연속형 결합분포)

Joint probability density function (결합확률밀도함수) : $f(x,y)$

$$f(x,y) = \frac{d^2}{dxdy}F(x,y), \quad F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)dtds$$

Properties of joint p.d.f (결합확률밀도함수의 성질)

(i) $f(x,y) \geq 0$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)dtds = 1$

(iii) $P((X,Y)\in A) = \int\int_A f(x,y)dxdy$

[예 3.2]

Marginal p.d.f (주변확률밀도함수) :

$$\begin{aligned} F_X(x) &= P(X \le x, Y < \infty) \\ &= \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(s,y)\,dy\,ds \end{aligned}$$

Therefore,

$$\begin{aligned} f_X(x) &= \frac{d}{dx}F_X(x) = \frac{d}{dx}\left(\int_{-\infty}^{x}\int_{-\infty}^{\infty} f(s,y)\,dy\,ds\right) \\ &= \int_{-\infty}^{\infty} f(x,y)\,dy \end{aligned}$$

[예 3.3]

3.3 Bivariate Normal Distribution (이변량정규분포)

Joint p.d.f of bivariate normal distribution :

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right)\right]$$

Notation : When the joint r.v. (X, Y) follows bivariate normal distribution,

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}\right)$$

The marginal p.d.f of X when (X, Y) follows $N_2(\cdot, \cdot)$:

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \\
&\quad \cdot \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right)\right]dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \\
&\quad \cdot \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{y-\mu_Y}{\sigma_Y} - \rho\frac{x-\mu_X}{\sigma_X}\right)^2 + (1-\rho^2)\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right)\right]dy \\
&= \frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right) \\
&\quad \cdot \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_Y^2(1-\rho^2)}} \exp\left(-\frac{1}{2\sigma_Y^2(1-\rho^2)}\left(y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)^2 dy.\right)
\end{aligned}
$$

$$
f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right)
$$

That is, $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$.

3.4 Independence of Random Variable

- Independence of two events

  Independence of two events A and $B \Leftrightarrow P(A \cap B) = P(A)P(B)$

- Independence of two random variables

  Independence of two random variables X and Y

  $\Leftrightarrow$ For all subset A and B of real number

  $$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

  $\Leftrightarrow$ For all x and y, $F(x,y) = F_X(x)F_Y(y)$

  $\Leftrightarrow$ For all x_i and y_j, $p(x_i, y_j) = p_X(x_i)p_Y(y_j)$ (Discrete)

$\Leftrightarrow$ For all x and y, $f(x,y) = f_X(x) f_Y(y)$ (Continuous)

- Independence of n random variables

Independence of n random variables $X_1, \ldots, X_n$

$$\Leftrightarrow\ F(x_1, \ldots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

[예 3.4]

[예 3.5]

3.5 Conditional Distribution

Conditional distribution (조건부분포) : Distribution of a random variable under the condition that the other random variable is given (어떤 확률변수가 주어졌을 때, 다른 확률변수의 확률분포)

Ex] $Y = y$일 때, X의 조건부분포가 $N(y, \sigma^2)$ 이면, $X|Y = y \sim N(y, \sigma^2)$로 표기

3.5.1 Conditional distribution of discrete random variable

- Sample space of (X, Y) : $\{(x_i, y_j),\ i=1,2,\ldots,\ j=1,2,\ldots\}$

$$p_{X|Y}(x_i|y_j) = \frac{P(X=x_i, Y=y_j) = P(X=x_i, Y=y_j)}{P(Y=y_j)} = \frac{p(x_i, y_j)}{p_Y(y_j)}$$

: Conditional probability mass function. In the same way,

$$p_{Y|X}(y_j|x_i) = \frac{p(x_i, y_j)}{p_X(x_i)}$$

- Properties of conditional p.m.f $p_{X|Y}(x_i|y_j)$

(i) $p_{X|Y}(x_i|y_j) \ge 0$

(ii) $\sum_i p_{X|Y}(x_i|y_j) = 1$

(iii) $P(a \le X \le b | Y=y_j) = \sum_{x_i : a \le x_i \le b} P_{X|Y}(x_i|y_j)$

(iv) $p(x_i, y_j) = p_{X|Y}(x_i, y_j) p_Y(y_j)$

[예 3.6]

[예 3.7]

3.5.2 Conditional distribution of continuous random variable

- Conditional p.d.f of X given $Y=y$:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

- Conditional p.d.f of Y given $X=x$:

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

- Properties of conditional p.d.f $f_{X|Y}(x|y)$

(i) $f_{X|Y}(x|y) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1$

(iii) $P(a < X < b | Y=y) = \int_a^b f_{X|Y}(x|y)dx$

(iv) $f(x,y) = f_{X|Y}(x|y)f_Y(y)$

[예 3.8]

[예 3.9]

3.6 Transformation of joint random variables

What is the p.d.f of $Z=g(X,Y)$ when the joint p.d.f of X and Y is given?

3.6.1 합의 분포

- Discrete case

X와 Y가 이산형 확률변수일 때 $Z=X+Y$의 확률분포는?

$$P(Z=z)=P(X+Y=z)=\sum_x P(X=x,Y=z-x)$$

즉,

$$p_Z(z)=\sum_x p(x,z-x)$$

If X and Y are independent

$$p_Z(z)=\sum_x p_X(x)p_Y(z-x)$$

[예 3.10]

- Continuous case

X와 Y가 연속형 확률변수일 때 $Z=X+Y$의 확률밀도함수?

$$\begin{aligned}
F_Z(z) &= P(Z\le z)=P(X+Y\le z)\\
&= P(-\infty<X<\infty, Y\le z-X)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\,dydx\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{z} f(x,v-x)\,dvdx\\
&= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(x,v-x)\,dxdv
\end{aligned}$$

$$\therefore f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx$$

If X and Y are independent

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

[예 3.11]

[예 3.12]

3.6.2 곱의 분포

- When $(X,Y) \sim f(x,y)$, what is the p.d.f of $Z = XY$?

$$
\begin{aligned}
F_Z(z) &= P(XY \le z) = P(Y \le z/X, X > 0) + P(Y \ge z/X, X < 0) \\
&= \int_0^\infty \int_{-\infty}^{\frac{z}{x}} f(x,y)dydx + \int_{-\infty}^0 \int_{\frac{z}{x}}^\infty f(x,y)dydx \\
&= \int_0^\infty \int_{-\infty}^{z} f(x,\frac{v}{x})\frac{1}{x}dvdx + \int_{-\infty}^0 \int_{-\infty}^{z} f(x,\frac{v}{x})\left(-\frac{1}{x}\right)dvdx \quad (let\ v = xy) \\
&= \int_{-\infty}^{z} \int_{-\infty}^{\infty} \left|\frac{1}{x}\right| f(x,\frac{v}{x})dxdv
\end{aligned}
$$

$$
\therefore f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{x}\right| f(x,\frac{z}{x})dx
$$

- When $(X,Y) \sim f(x,y)$, what is the p.d.f of $Z = \frac{Y}{X}$?

[예 3.13]

3.6.3 Transformation of bivariate random variables

$(X,Y) \sim f_{X,Y}(x,y)$: continuous r.v.

$$U = g_1(X,Y),\quad V = g_2(X,Y)$$

What is the joint p.d.f of U,V?

Thm 3.1 Let $U = g_1(X,Y)$, $V = g_2(X,Y)$ and there exist h_1 and h_2 such that $X = h_1(U,V)$, $Y = h_2(U,V)$. When h_1 and h_2 are differentiable, the joint p.d.f of U,V is

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))|J(u,v)|,$$

$$\text{where } J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial u}h_1(u,v) & \frac{\partial}{\partial v}h_1(u,v) \\ \frac{\partial}{\partial u}h_2(u,v) & \frac{\partial}{\partial v}h_2(u,v) \end{vmatrix}$$

참고. $f_X(x)$: p.d.f of continuous r.v. X, let $Y = g(X)$ and $g^{-1}(\cdot)$ exist and differentiable. Then,

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

<u>Example of bivariate transformation</u>

Example 1.

Let $X \sim N(0,1)$, $Y \sim N(0,1)$, X & Y are independent.

$$\Rightarrow f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2}) \frac{1}{\sqrt{2\pi}} exp(-\frac{y^2}{2}),\quad -\infty < x,y < \infty$$

Define $U = X + Y$, $V = X - Y$.

What is the joint p.d.f of (U,V)?

$\Rightarrow$ Express X and Y in terms of U,V

$$\Rightarrow X = \frac{U+V}{2},\quad Y = \frac{U-V}{2}.$$

$$|J| = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{1}{2}$$

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{\sqrt{2\pi}} exp\left(-\left(\frac{u+v}{2}\right)^2/2\right) \frac{1}{\sqrt{2\pi}} exp\left(-\left(\frac{u-v}{2}\right)^2/2\right) \cdot \frac{1}{2} \\ &= \frac{1}{\sqrt{2\pi}\sqrt{2}} exp(-u^2/(2 \cdot 2)) \frac{1}{\sqrt{2\pi}\sqrt{2}} exp(-v^2/(2 \cdot 2)) \end{aligned}$$

Hence, U and V are independent, $U \sim N(0,2)$, $V \sim N(0,2)$.

Example 2.

$X \sim \text{Gamma}(\alpha_1, \beta)$, $Y \sim \text{Gamma}(\alpha_2, \beta)$ and independent.

$$f_{X,Y}(x,y) = \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} x^{\alpha_1 - 1} e^{-\frac{x}{\beta}} \frac{1}{\Gamma(\alpha_2)\beta^{\alpha_2}} y^{\alpha_2 - 1} e^{-\frac{y}{\beta}}$$

Let $U = X + Y$. What is the p.d.f of U?

$\Rightarrow$ Invent pseudo variable

Let $V = Y$.

Then, $U = X + Y$, $V = Y$. That is, $X = U - V$, $Y = V$.

$$\{(x,y) : x > 0, y > 0\} \Rightarrow \{(u,v) : 0 < v < y < \infty\}$$

$$|J| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Hence,

$$f_{U,V}(u,v) = \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} (u-v)^{\alpha_1 - 1} e^{-\frac{u-v}{\beta}} \frac{1}{\Gamma(\alpha_2)\beta^{\alpha_2}} v^{\alpha_2 - 1} e^{-\frac{v}{\beta}}, \ 0 < v < u < \infty.$$

The marginal p.d.f of U can be obtained by integration as follows

$$\begin{aligned} f_U(u) &= \int_0^u \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} (u-v)^{\alpha_1 - 1} e^{-\frac{u-v}{\beta}} \frac{1}{\Gamma(\alpha_2)\beta^{\alpha_2}} v^{\alpha_2 - 1} e^{-\frac{v}{\beta}} dv \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1 + \alpha_2}} e^{-\frac{u}{\beta}} \int_0^u u^{\alpha_1 + \alpha_2 - 2} \left(\frac{u-v}{u}\right)^{\alpha_1 - 1} \left(\frac{v}{u}\right)^{\alpha_2 - 1} dv \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1 + \alpha_2}} e^{-\frac{u}{\beta}} u^{\alpha_1 + \alpha_2 - 2} u \int_0^1 (1-z)^{\alpha_1 - 1} (z)^{\alpha_2 - 1} dz \quad (let\ z = \frac{v}{u}) \\ &= \frac{1}{\Gamma(\alpha_1 + \alpha_2)\beta^{\alpha_1 + \alpha_2}} u^{\alpha_1 + \alpha_2 - 1} e^{-\frac{u}{\beta}} \end{aligned}$$

So, $U \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$