$$f(\mathbf{u},\mathbf{e}) = \frac{1}{(2\pi)^{(n+g)/2}} \begin{vmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{vmatrix}^{-1/2} \exp\left\{ -\frac{1}{2} \begin{bmatrix} \mathbf{u} \\ \mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{Z}\mathbf{u} \end{bmatrix}' \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{Z}\mathbf{u} \end{bmatrix} \right\}$$

Maximization of $f(\mathbf{u},\mathbf{e})$ with respect to $\boldsymbol{\beta}$ and $\mathbf{u}$ requires minimization of

$$Q = \begin{bmatrix} \mathbf{u} \\ \mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{Z}\mathbf{u} \end{bmatrix}' \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{Z}\mathbf{u} \end{bmatrix}$$

$$= \mathbf{u}'\mathbf{G}^{-1}\mathbf{u} + (\mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{Z}\mathbf{u})' \mathbf{R}^{-1} (\mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{Z}\mathbf{u})$$

where we have taken advantage of the independence of **u** and **e**. This leads to the equations

$$\partial Q/\partial\boldsymbol{\beta} = 0 \Leftrightarrow \mathbf{X}'\mathbf{R}^{-1}\mathbf{X}\tilde{\boldsymbol{\beta}} + \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z}\tilde{\mathbf{u}} = \mathbf{X}'\mathbf{R}^{-1}\mathbf{y}$$

$$\partial Q/\partial\mathbf{u} = 0 \Leftrightarrow \left(\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1}\right)\tilde{\mathbf{u}} + \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X}\tilde{\boldsymbol{\beta}} = \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y}$$

H.W. #2

After some rearranging, these can be written as

$$\begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y} \end{bmatrix} \tag{A1.4}$$