Computations for the double pendulum equations

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1 Formulation of the problem for a general mechanical system

Consider the mechanical system given by

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = \tau$$

Doing the computations step by step, with $x_1 = q$ and $x_2 = \dot{q}$ we obtain:

$$\dot{x_1} = \dot{q} = x_2$$

and for the mechanical system:

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = \tau \iff M(x_1)\dot{x_2} + c(x_1, x_2) + g(x_1) = \tau$$

If we consider that the matrix M is invertible, then we multiply the last equation with $M^{-1}(x_1)$ (on the left) and get:

$$\dot{x_2} + M^{-1}(x_1)c(x_1, x_2) + M^{-1}(x_1)g(x_1) = M^{-1}(x_1)\tau$$

or:

$$\dot{x_2} = -M^{-1}(x_1)c(x_1, x_2) - M^{-1}(x_1)g(x_1) + M^{-1}(x_1)\tau$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -M^{-1}(x_1)[c(x_1, x_2) + g(x_1)] \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(x_1) \end{bmatrix} \tau$$

2 Computations for a double pendulum

For the double pendulum, we will use the equations from *Appendix A: Equations of motion of a double pendulum.* We define 4 states and 2 outputs, as following:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

For the first two states, it holds: $\dot{x_1} = \dot{\theta}_1 = x_3$ and $\dot{x_2} = \dot{\theta}_2 = x_4$ For the computation of the derivatives of the third and fourth state, we define then the following matrices;

$$\mathbf{M(x)} = \begin{bmatrix} \alpha + 2\beta cos(\theta_2) & \delta + \beta cos(\theta_2) \\ \delta + \beta cos(\theta_2) & \delta \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta cos(x_2) & \delta + \beta cos(x_2) \\ \delta + \beta cos(x_2) & \delta \end{bmatrix}$$

$$\mathbf{C(x)} = \begin{bmatrix} -\beta sin(\theta_2)\dot{\theta_2} + b_1 & -\beta sin(\theta_2)(\dot{\theta_1} + \dot{\theta_2}) \\ \beta sin(\theta_2)\dot{\theta_1} & b_2 \end{bmatrix} = \begin{bmatrix} -\beta sin(x_2)x_4 + b_1 & -\beta sin(x_2)(x_3 + x_4) \\ \beta sin(x_2)x_3 & b_2 \end{bmatrix}$$

$$\mathbf{H(x)} = \begin{bmatrix} m_1 g r_1 cos(\theta_1) + m_2 g(l_1 cos(\theta_1) + r_2 cos(\theta_1 + \theta_2)) \\ m_2 g r_2 cos(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} m_1 g r_1 cos(x_1) + m_2 g(l_1 cos(x_1) + r_2 cos(x_1 + x_2)) \\ m_2 g r_2 cos(x_1 + x_2) \end{bmatrix}$$

With these matrices, we rewrite the last equation from the appendix:

$$\mathbf{M(x)} \begin{bmatrix} \dot{x_3} \\ \dot{x_4} \end{bmatrix} + \mathbf{C(x)} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} + \mathbf{H(x)} = \mathbf{u} \iff \mathbf{M(x)} \begin{bmatrix} \dot{x_3} \\ \dot{x_4} \end{bmatrix} = -\mathbf{C(x)} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} - \mathbf{H(x)} + \mathbf{u}$$

If we assume that the matrix $M(x)$ is invertible, then we can multiple the equation from the left-hand side with the inverse of M and obtain:

$$\begin{bmatrix} \dot{x_3} \\ \dot{x_4} \end{bmatrix} = -M(x)^{-1}\mathbf{C(x)} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} - M(x)^{-1}\mathbf{H(x)} + M^{-1}(x)\mathbf{u} = -M(x)^{-1}\mathbf{C(x)} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} - M(x)^{-1}\mathbf{H(x)} + M^{-1}(x)\mathbf{u}$$

If we summarize then, the whole system looks like:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -M(x)^{-1}\mathbf{C(x)} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} - M(x)^{-1}\mathbf{H(x)} + M^{-1}(x)\mathbf{u} \end{bmatrix}$$

We want to write our system in the form $\dot{x} = f(x) + g(x)u$, so the matrices are then defined as following:

$$f = \begin{bmatrix} x_3 \\ x_4 \\ -M(x)^{-1}\mathbf{C(x)} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} - M(x)^{-1}\mathbf{H(x)} \end{bmatrix}$$

and

$$g = \begin{bmatrix} 0 \\ 0 \\ M(x)^{-1} \end{bmatrix}$$