Need to prove the following for $m < n$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m = 0 \tag{1}$$

Begin from

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \tag{2}$$

Differentiate both sides with respect to p,

$$p\frac{d}{dp}(p+q)^n = pn(p+q)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} kp^k q^{n-k} \tag{3}$$

Now we let $p = -1$ and $q = -1$ to give,

$$\sum_{k=0}^{n} \binom{n}{k} k(-1)^k = 0 \tag{4}$$

which corresponds to $m = 1$ and only holds true when $n > 1$. For the general case m, from Eq. 2 we apply a nested derivative such that

$$\left(p\frac{d}{dp}\right)^m (p+q)^n = p\frac{d}{dp}\left(p\frac{d}{dp}\left(...\left(p\frac{d}{dp}(p+q)^n\right)\right)\right) = \sum_{k=0}^{n} \binom{n}{k} k^m (-1)^k \tag{5}$$

for m=2, the LHS becomes

$$p\frac{d^2}{dp^2}(p+q)^n = pn(p+q)^{n-1} + p^2 n(n-1)(p+q)^{n-2} \tag{6}$$

Generally, all terms contain some form of the differential where

$$\frac{d^m}{dp^m}(p+q)^n = n(n-1)...(n-m+1)(p+q)^{n-m} \tag{7}$$

Since $p = -1$ and $q = 1$, for $m < n$,

$$\sum_{k=0}^{n} \binom{n}{k} k^m (-1)^k = 0 \tag{8}$$