Intra Automne 2016 IFT-1575 pratique

Tarik Hireche : 202 301 89

17. January 2026

Rapport écrit par Tarik Hireche

1 Question 3. (15 points)

Solve the following problem with the simplex algorithm:

$$\text{Min } z = -x_1 - x_2 \tag{1.1}$$

s.a

$$x_1 + 0x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_4 = \frac{1}{2} \tag{1.2}$$

$$0x_1 + x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = \frac{1}{2} \tag{1.3}$$

$$x_1, x_2, x_3, x_4 \geq 0 \tag{1.4}$$

We can see that Z, the function to minimize is in function of two variables:

- x_1 and x_2.

Therefore, locally, theses variables are independant to Z but they are dependant in their own equations. Meaning:

- In this case equation, $x_1 + 0x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_4 = \frac{1}{2}$, our dependant variable $\rightarrow x_1$:
  - is implictely in function of x_2,x_3 and x_4. Rewriting it in function of the independant variables, gives us:
    - $x_1 = \frac{1}{2} + \frac{3}{2}x_3 - \frac{1}{2}x_4 = \frac{1}{2}(3x_3 - x_4 + 1)$
- As for our second dependant variable, x_2, we can also rewrite it in function of the others independant variables:
  - $x_2 = \frac{1}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_4 = \frac{1}{2}(x_4 - x_3 + 1)$

Alright, we have rewritten our equation in a more meaningful way. We can now clearly see our two sets of variables. The first one, let's call A, the set of dependant variables and the second one B, the set of independant variables are:

$A = \{x_1, x_2\}$, $B = \{x_3, x_4\}$

Now, the next step is to **clean the objective function**, it is **MANDATORY**. Before starting the algorithm, we must express Z only in terms of independant variables from set $B = \{x_3, x_4\}$ to avoid wrong reads of the actual costs of resources. So we need to substitute x_1 and x_2 into Z:

Let's rewrite Z in function of theses variables, as they are the real resources used.

so $Z = -\left[\frac{1}{2}(3x_3 - x_4 + 1)\right] - \left[\frac{1}{2}(x_4 - x_3 + 1)\right]$

Let's do some cleaning:

$$Z = -\left[\frac{3}{2}x_3 - \frac{1}{2}x_4 + \frac{1}{2}\right] - \left[-\frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}\right] \tag{1.5}$$

$$Z = -\frac{3}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2} + \frac{1}{2}x_3 - \frac{1}{2}x_4 - \frac{1}{2} \tag{1.6}$$

$$Z = x_3\left(\frac{1}{2} - \frac{3}{2}\right) + x_4\left(\frac{1}{2} - \frac{1}{2}\right) - \frac{1}{2} - \frac{1}{2} \tag{1.7}$$

$$Z = x_3\left(-\frac{2}{2}\right) + x_4\left(\frac{1}{2} - \frac{1}{2}\right) - 1 = -x_3 - 1 \tag{1.8}$$

So the objective function is: $Z = -x_3 - 1$

So in the end, our actions are x_3 and x_4, and they use resources x_1, x_2.

1.1 Starting the loop, while(!optimal)

1. **function A**: `Select_Input_variabe()`;
   - In this case, it is a minimization problem, therefore we need to minimize Z, and to minimize Z, **we look at the coefficients in the CLEANED Z equation** $Z = -1 - x_3 + 0x_4$. We need to select the variable with **the most negative coefficient** as it is the one that will have the most impact in lowering the total costs of resources.

- Let's look at the coefficients of the independant variables:
  - for $x_3 \to -x_3$, the coefficient is -1
  - for $x_4 \to 0x_4$, the coefficent is zero, no impact at all on the cost.
- The decision is kinda forced here. The variable x_3 is the only one that lowers Z.
- **Therefore the input variable is**:

$$x_{\text{in}} = x_3 \tag{1.9}$$

2. **function B**: `Select_output_variable(`x_{in}`)`; The **ratio test**

   Let's remember that our set A constitutes the set of resources! Therefore x_1 and x_2 are physical, limited resources. We need to find the bound at which they will be exhausted to make sure to not go over it.

   - We need to scan **all** of the dependant variables equations (so for the equations fo all elements of set A) and compute the ratios for each of them : $\frac{\text{Constant}}{\text{coefficient}}$.
     - $x_1 = \frac{1}{2} + \frac{3}{2}x_3 - \frac{1}{2}x_4 = \frac{1}{2}(3x_3 - x_4 + 1)$
     - $x_2 = \frac{1}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_4 = \frac{1}{2}(x_4 - x_3 + 1)$

   We need to find which physical resources (x_1 or x_2) will break (hit zero) first as we keep making the same action x_3, so as we increase x_3. We scan the dependant equations while assuming other independant variables, in this case x_4 stay at zero.

- **Analyzing first resource x_1** while $x_4 = 0$:
  - $x_1 = \frac{1}{2} + \frac{3}{2}x_3$
  - The coefficient of x_3 is $+\frac{3}{2}$, **positive**, therefore **it will never cancel with $\frac{1}{2}$, the constant**.
  - That means that the resource x_1 is not a bottleneck in our problem.

$$\text{Ratio} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \tag{1.10}$$

- **Analyzing second resource x_2** while $x_4 = 0$:
  - $x_2 = \frac{1}{2} - \frac{1}{2}x_3$
  - As we decide to make an action (x_3 increases), our resource x_2 that x_3 uses, decreases. **So there is a limit to x_3 on x_2.**
  - Ratio $= \frac{\text{Constant}}{\text{Coefficient}}$:

    Ratio $= \frac{\frac{1}{2}}{|}(-\frac{1}{2})| = 1$

    Decision: The minimum ratio is 1, coming from x_2, we have found our output variable! :]
  - **Output variable:** x_2

3. **function C:** `Pivot_Operation`($x_{\text{in}} = x_3, x_{\text{out}} = x_2$

   On this step, if I remember correctly, I have to rewrite the whole system by substituting the input variable that we've found (x_3) with its corresponding equation in the system

   - $x_1 = \frac{1}{2} + \frac{3}{2}x_3 - \frac{1}{2}x_4 = \frac{1}{2}(3x_3 - x_4 + 1)$
   - $x_2 = \frac{1}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_4 = \frac{1}{2}(x_4 - x_3 + 1)$

   Why are we doing this? And what do we mean by input variable and output variable?

A common confusion that I also was a culprit of, is to think that the terms *Input Variable* and *Output variable* are the input and output of a function $f(x)$, like our objective function. No!, we are talking about the **BASE**.

But what's the base? The base is simply the list of the variables that are **allowed to be active** (different than zero).

- In our system, we have a total of 4 variables (x_1, x_2, x_3, x_4)
- But this base can only contain **2 places**, why 2 and not 10501951 or 3? or 5?
  - Because the number of places is **equal to the number of constraints**.
  - In our case **we have TWO constraints** that limit our system, so we have **2 places in our base**, if we had 100 constraints, we'd have 100 places. Z could have 1 BILLION variables, it doesn't matter. **What matter is the number of constraints**.

In linear algebra, the logic is GEOMETRIC! Meaning that **it is the constraints that define a literal polygon**. The **Objective function** just gives us the direction.

Alright, let's rewrite the system by isolating x_3 in the x_2 equation, **why not the x_1 equation?** Because we rewrite the **input** variable in function of the **output** variable. **Why again?** Because we saw that the x_1 resource can never be exhausted by x_3 not matter how much it uses it.

- Alright, from the x_2 equation we have:

- $x_2 = \frac{1}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_4$, let's write x_3 in function of x_2.

$$\frac{1}{2}x_3 = \frac{1}{2} - x_2 + \frac{1}{2}x_4 \tag{1.11}$$

Multiplying by 2 gives us:

$$x_3 = \frac{2}{2} - 2x_2 + \frac{2}{2}x_4 \tag{1.12}$$

$$x_3 = 1 - 2x_2 + x_4 \tag{1.13}$$

We didn't finish substituting it everywhere, we missed x_1.

- $x_1 = \frac{1}{2} + \frac{3}{2}x_3 - \frac{1}{2}x_4$

substitute the new x_3 equation in it.

- $x_1 = \frac{1}{2} + \frac{3}{2}[1 - 2x_2 + x_4] - \frac{1}{2}x_4$
- $x_1 = \frac{1}{2} + \frac{3}{2} - 3x_2 + \frac{3}{2}x_4 - \frac{1}{2}x_4$
- $x_1 = \frac{4}{2} - 3x_2 + x_4\left(\frac{3}{2} - \frac{1}{2}\right)$
- $x_1 = 2 - 3x_2 + x_4$

Now in $Z = -1 - x_3$, replace x_3 with what we have just found.

$$Z = -1 - [1 - 2x_2 + x_4] \tag{1.14}$$

$$Z = -1 - 1 + 2x_2 - x_4 \tag{1.15}$$

$$Z = -2 + 2x_2 - x_4 \tag{1.16}$$

Now if we remember, x_4 is asleep, we set it at zero when we chose our input variable. And x_2 is also at zero because we have used up this resource, we use 1 x_3 and we found that the ratio was 1 to exhaust x_2.

Therefore :

$$Z = -2 \tag{1.17}$$

But, **are we done?**. Nope! What is the stopping rule if we remember correctly? Well sicnce its a minimization problem, we stop when our objective function has no negative coefficient that we can use to pull it down. In our case, Z's current value is at 2 but it can still be lowered (potentially) by x_4 as its coefficient is -1 (negative).

1.2 Second iteration

First of all things, rewrite the system in **its current state**.

- **Objective function**: $Z = -2 + 2x_2 - x_4$

What are our resources now? The resources come from the pivot step. We isolated x_3 with the equation of x_2 then we injected it everywhere.

- We have found that $x_3 = 1 - 2x_2 + x_4$ and that
- $x_1 = 2 - 3x_2 + x_4$

1. **Choose the lowest most negative coefficient as our input variable**

   Z is now equal to

$$Z = -2 + 2x_2 - x_4 \tag{1.18}$$

Our only choice is x_4, so $x_{\text{in}} = x_4$

2. **Ratio test**: finding the output variable x_{out}

   We have two equations:
   - $x_3 = 1 - 2x_2 + x_4$
   - $x_1 = 2 - 3x_2 + x_4$

   Since we chose our input variable to be x_4, x_2 will be set at 0, leaving us with:

   - $x_3 = 1 + x_4$
   - $x_1 = 2 + x_4$

   We can stop here! As we can see, **there are no negative coefficient**. Meaning that there are **no limited resources!!!**The entry variable x_4 is the action, it has positives coefficients in all of the equations of the base variables x_1, x_3, the resources. x_4 doesn't consume any resource, in fact it increases them. Nothing stops how far x_4 can be actionned. Since x_4 can improve the objective Z, we can improve Z indefinitely $(-\inf)$.