FURTHER TOPICS in LINEAR REGRESSION ANALYSIS: Functional Forms, Quadratic Models, Interaction Terms, Prediction

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Further Issues in MLR: Outline

- Data scaling
- Standardized regression
- Additional topics in functional form: quadratic models, models with interaction terms
- Goodness of fit: Adjusted R^2
- Prediction

Effects of Data Scaling on OLS Statistics

- Changing the units of measurements changes the OLS intercept and slope estimates.
- Why may we be interested in changing the units of measurements: cosmetic purposes, such as reducing the number of zeros on coefficient estimates, easier interpretation.
- Rescaling data does not change the testing outcomes.
- Rescaling data does not change the significance of coefficient estimates. t statistics do not change.
- R^2 remains the same.
- SSR and SER would change if we rescale the data.
- F test statistic remains the same.

Example

- Consider the following model:

$$bwght = \beta_0 + \beta_1 cigs + \beta_2 faminc + u$$

where $bwght$ is measured in ounces, $cigs$ is the number of cigarettes smoked per day, $faminc$ is measured in 1000 US Dollars.
- Let us change the the unit of measurement for the dependent variable from ounces to grams. Because $1\ ounce = 28.3495231\ grams$ we define a new variable

$$bwghtgrams = bwght \times\ 28.3495231$$

and estimate

$$bwghtgrams = \beta_0 + \beta_1 cigs + \beta_2 faminc + u$$

- Coefficient estimates and standard errors will accordingly change. See the R script in the next slide.

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R Example

```
> model1 <- lm(bwght ~ cigs + faminc, data=bwght)
--------------------------------------------------
                       Est.    S.E.    t val.      p
----------------- -------- ------ -------- ------
(Intercept)          116.97    1.05   111.51   0.00
cigs                  -0.46    0.09    -5.06   0.00
faminc                 0.09    0.03     3.18   0.00
--------------------------------------------------
```

$$bwght = 116.97 - 0.46(cigs) + 0.09(faminc) + residual$$

```
> bwght$bwghtgrams <- bwght$bwght*28.3495231
> model2 <- lm(bwghtgrams ~ cigs + faminc, data=bwght)
---------------------------------------------------
                        Est.     S.E.    t val.      p
----------------- --------- ------- -------- ------
(Intercept)          3316.16    29.74   111.51   0.00
cigs                  -13.14     2.60    -5.06   0.00
faminc                  2.63     0.83     3.18   0.00
---------------------------------------------------
```

$$bwghtgrams = 3316.16 - 13.14(cigs) + 2.63(faminc) + residual$$

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R Example: changing the units of measurement

```
# change faminc to dollars instead of 1000$
> bwght$famincdollars <- bwght$faminc*1000
> model3 <- lm(bwghtgrams ~ cigs + famincdollars, data=bwght)
-------------------------------------------------------------
                          Est.       S.E.      t val.         p
------------------ ------------ ---------- ----------- ---------
(Intercept)          3316.16081   29.73820   111.51182   0.00000
cigs                  -13.13738    2.59616    -5.06031   0.00000
famincdollars           0.00263    0.00083     3.17819   0.00151
-------------------------------------------------------------
```

$$bwghtgrams = 3316.16081 - 13.13738(cigs) + 0.00263(famincdollars) + residual$$

```
# change cigs to packs
> bwght$packs <- bwght$cigs/20
> model4 <- lm(bwghtgrams ~ packs + famincdollars, data=bwght)
-------------------------------------------------------------
                          Est.       S.E.      t val.         p
------------------ ------------ ---------- ----------- ---------
(Intercept)          3316.16081   29.73820   111.51182   0.00000
packs                -262.74766   51.92319    -5.06031   0.00000
famincdollars           0.00263    0.00083     3.17819   0.00151
-------------------------------------------------------------
```

$$bwghtgrams = 3316.16081 - 262.74766(packs) + 0.00263(famincdollars) + residual$$

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Standardized Regression

- If x_j changes 1 standard deviation, instead of 1 unit, how would y change?
- Answer: Standardize all variables in the regression model (y and all xs) and estimate the model using standardized variables.
- Standardization: subtract the arithmetic mean and divide by sample standard deviation:

$$z_y = \frac{y - \bar{y}}{\hat{\sigma}_y},$$

$$z_1 = \frac{x_1 - \bar{x}_1}{\hat{\sigma}_1}, z_2 = \frac{x_2 - \bar{x}_2}{\hat{\sigma}_2}, \ldots, z_k = \frac{x_k - \bar{x}_k}{\hat{\sigma}_k},$$

- $\hat{\sigma}_j$ is the sample standard deviation of x_j.

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Standardized Regression

We wand to standardize the following model:

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik} + \hat{u}_i$$

Subtracting the sample averages from the model we obtain:

$$y_i - \bar{y} = \hat{\beta}_1(x_{i1} - \bar{x}_1) + \hat{\beta}_2(x_{i2} - \bar{x}_2) + \ldots + \hat{\beta}_k(x_{ik} - \bar{x}_k) + \hat{u}_i$$

since the sample average of $\hat{u}$ is zero. Notice that there is no intercept term in the model. Dividing by the sample standard deviations and using simple algebra gives:

$$\frac{y_i - \bar{y}}{\hat{\sigma}_y} = \frac{\hat{\sigma}_1}{\hat{\sigma}_y}\hat{\beta}_1\frac{(x_{i1} - \bar{x}_1)}{\hat{\sigma}_1} + \frac{\hat{\sigma}_2}{\hat{\sigma}_y}\hat{\beta}_2\frac{(x_{i2} - \bar{x}_2)}{\hat{\sigma}_2} + \ldots + \frac{\hat{\sigma}_k}{\hat{\sigma}_y}\hat{\beta}_k\frac{(x_{ik} - \bar{x}_k)}{\hat{\sigma}_k} + \frac{\hat{u}_i}{\hat{\sigma}_y}$$

Standardized Regression

- Rewrite the model as follows:
$$z_y = \hat{b}_1 z_1 + \hat{b}_2 z_2 + \ldots + \hat{b}_k z_k + error,$$
where
$$z_y = \frac{y - \bar{y}}{\hat{\sigma}_y}, \quad z_j = \frac{x_j - \bar{x}_j}{\hat{\sigma}_j}, \ j = 1, 2, \ldots, k$$
- Slope coefficients: known as standardized coefficients or beta coefficients
$$\hat{b}_j = \frac{\hat{\sigma}_j}{\hat{\sigma}_y} \hat{\beta}_j, \ j = 1, 2, \ldots, k$$
- Interpretation: In response to a one standard deviation in x_j, y is predicted to change by $\hat{b}_j$ standard deviations.
- Original units of measurements are irrelevant. They are now measured in terms of standard deviation and they can be compared.

Standardized Regression: Example

Air pollution and house prices: hprice2.gdt

Dependent variable: median house prices in the region (price)
Explanatory Variables:
nox: measure of air pollution,
dist: distance to city business centers,
crime: crime rate in the community,
rooms: average number of rooms in the community,
stratio: average student-teacher ratio in the community

In levels:
$$price = \beta_0 + \beta_1 nox + \beta_2 crime + \beta_3 rooms + \beta_4 dist + \beta_5 stratio + u$$
Standardized model:
$$zprice = b_1 znox + b_2 zcrime + b_3 zrooms + b_4 zdist + b_5 zstratio + zu$$

Standardized Regression: Example

Standardized model results

$$\begin{aligned}\widehat{zprice} = &-0.340\ znox - 0.143\ zcrime + 0.514\ zrooms \\ &- 0.235\ zdist - 0.270\ zstratio\end{aligned}$$

- One standard deviation increase in air pollution decreases price by 0.34 standard deviation.
- One standard deviation increase in crime reduces price by 0.143 standard deviation.
- The same relevant movement of pollution in the population has a larger effect on housing prices than crime does.
- Size of the house (measured by the number of rooms) has the largest standardized effect.

Example cont.: Unstandardized regression results

$$\begin{aligned}\widehat{\text{price}} = &\underset{(5054.6)}{20871.1} - \underset{(354.09)}{2706.43}\,\text{nox} - \underset{(32.929)}{153.601}\,\text{crime} + \underset{(393.60)}{6735.50}\,\text{rooms} \\ &- \underset{(188.11)}{1026.81}\,\text{dist} - \underset{(127.43)}{1149.20}\,\text{stratio}\end{aligned}$$
$$n = 506 \quad \bar{R}^2 = 0.6320 \quad F(5, 500) = 174.47 \quad \hat{\sigma} = 5586.2$$
(standard errors in parentheses)

More on Functional Form: Logarithmic specifications

- In our previous lectures, we learned how to allow for nonlinear relationships between variables using logarithmic transformation.
- Example: house price model

$$\log(price) = \beta_0 + \beta_1 \log(nox) + \beta_3 rooms + u$$

- β_1: elasticity of prices with respect to air pollution
- $100\beta_2$: approximate percentage change in price in response to a one unit increase in rooms (semi-elasticity)

Example

House prices: hprice2.gdt

$$\widehat{\log(\text{price})} = \underset{(0.188)}{9.234} - \underset{(0.066)}{0.718} \, \text{log(nox)} + \underset{(0.019)}{0.306} \, \text{rooms}$$

$$n = 506 \quad \bar{R}^2 = 0.512 \quad F(2, 503) = 265.69 \quad \hat{\sigma} = 0.28596$$

(standard errors in parentheses)

- Holdings rooms fixed, 1% increase in nox price falls by 0.718%. The elasticity of price with respect to air pollution is 0.718.
- Holding nox fixed, when rooms increases by one, price increases by 30.6% (100×0.306).
- The approximation $\%\Delta y \approx 100 \times \Delta \log(y)$ becomes inaccurate as the change in $\log(y)$ becomes larger and larger.

Approximation Error in Logarithmic Changes

- We can use the following formula for big changes in logarithmic dependent variable:

$$\widehat{\%\Delta y} = 100 \times [\exp(\hat{\beta}_2) - 1]$$

- In the previous example we obtained $\hat{\beta}_2 = 0.306$:

$$\widehat{\%\Delta y} = 100 \times [\exp(0.306) - 1] = \%35.8$$

- Now the semi-elasticity is larger.
- $\exp(\hat{\beta}_2)$ is a biased but consistent estimator (why?)

Advantages of Logarithmic Transformation

- There are many advantages of using logarithms of strictly positive variables $(y > 0)$.
- Interpretation of coefficients is easier: independent of the units of measurements of xs (elasticity or semi-elasticity).
- When $y > 0$, $\log(y)$ often satisfies CLM assumptions more closely than y in levels. Strictly positive variables (prices, income, etc.) often have heteroscedastic or skewed distributions. Taking logs can mitigate these problems.
- Log transformation reduces or eliminates skewness and reduces variance.
- Taking logs narrows the range of the variable leading to less sensitive estimates to outliers (extreme observations).

Some Rules of Thumb for Taking Logs

- Strictly positive variables such as wage, income, population, production, sales etc. are generally included in the model using log transformation.
- Proportions or rates such as unemployment rate, interest rate, etc. usually appear in their original form. But sometimes they may be included in log form if strictly positive.
- If rates or proportions are included in levels: **a percentage point increase** or change.
- If logarithms of rates are taken (e.g. log(unemployment rate)): 1% (**a percentage increase**) or change.
- This distinction is important: if unemployment rate increases from 8% to 9% the increase is 1 percentage point.But in log form there is 100×(log(9)-log(8))=100×0.1177=11.77% increase in unemployment.

Log Transformation

- If the variable takes nonnegative values (≥ 0), i.e. it is 0 for some observations, we cannot use log transformation because $\log(0)$ is not defined.
- In this case we can use $\log(1+y)$ transformation instead of $\log(y)$.
- If the data contain relatively few 0 values we can use this approach. The interpretation is the same (except for the changes beginning at 0)
- We cannot compare the R^2s from two models in which we have $\log(y)$ as the dependent variable in one of the models and y in the other.

Functional Form: Quadratic Models

- Quadratic functions are generally used to capture decreasing or increasing marginal effects.
- In quadratic models slope coefficient is not constant. It depends on the value of x:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 x^2$$

- The slope between x and y can be approximated as follows:

$$\Delta\hat{y} \approx (\hat{\beta}_1 + 2\hat{\beta}_2 x)\Delta x$$

- Or,

$$\frac{\Delta\hat{y}}{\Delta x} \approx (\hat{\beta}_1 + 2\hat{\beta}_2 x)$$

- If $x = 0$ then $\hat{\beta}_1$ is the slope estimated for the change from $x = 0$ to $x = 1$. For values larger than $x = 1$ we need to consider the second term.

Quadratic Models: Example

$$\widehat{wage} = 3.73 + 0.298\ exper - 0.0061\ exper^2$$

- If $\beta_1 > 0, \beta_2 < 0$ then the relationship is $\frown$-shaped.
- If $\beta_1 < 0, \beta_2 > 0$ then the relationship is $\smile$-shaped.
- The regression above implies that exper has a diminishing marginal effect on wage.
- Slope estimate is

$$\frac{\Delta\widehat{wage}}{\Delta exper} \approx 0.298 - (2 \times 0.0061) exper$$

- The first year of experience is worth approximately \$0.298. The second year of experience is worth less:

$$\frac{\Delta\widehat{wage}}{\Delta exper} = 0.298 - 0.0122(1) = 0.286$$

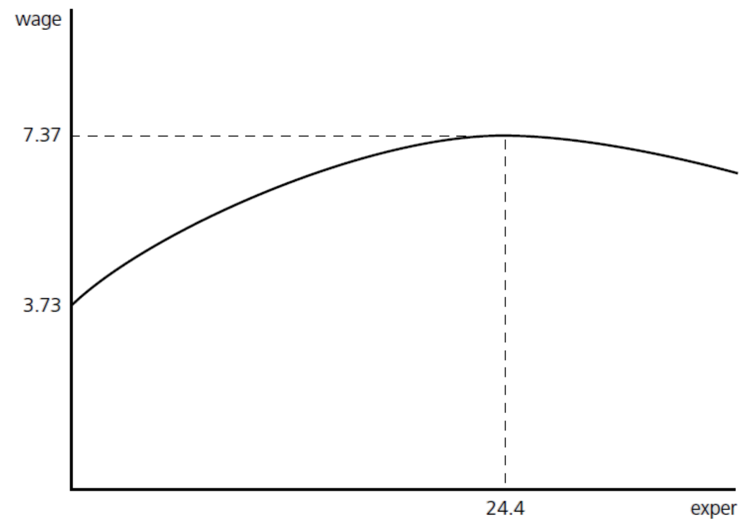
Quadratic Models: Example

$$\widehat{wage} = 3.73 + 0.298\ exper - 0.0061\ exper^2$$

- If exper changes from 10 to 11, wage is predicted to change by:
$$\frac{\Delta\widehat{wage}}{\Delta exper} = 0.298 - 0.0122(10) = 0.176$$
- Turning point:
$$\frac{\Delta\hat{y}}{\Delta x} \approx (\hat{\beta}_1 + 2\hat{\beta}_2 x) = 0 \Rightarrow x^* = \left|\frac{\hat{\beta}_1}{2\hat{\beta}_2}\right|$$
- Estimated turning point for the wage-exper relationship:
$$exper^* = 0.298/0.0122 = 24.4$$

Quadratic Models: Wage-Experience

$$\widehat{wage} = 3.73 + 0.298\ exper - 0.0061\ exper^2$$

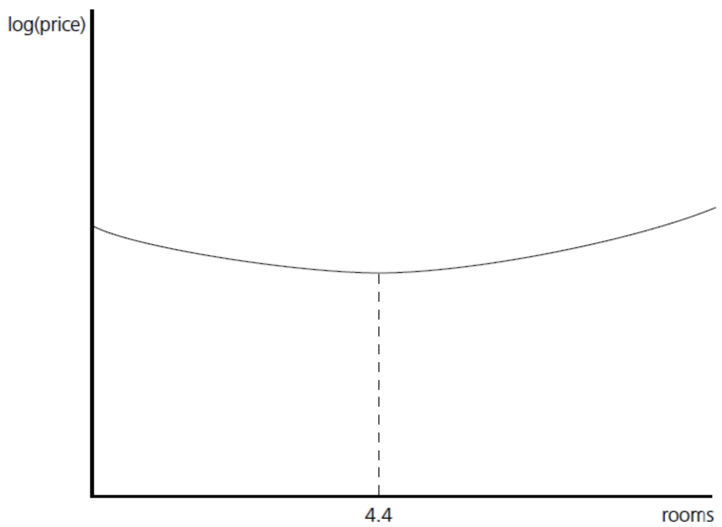


Quadratic Models: Example

$$\widehat{\log(\text{price})} = \underset{(0.566)}{13.386} - \underset{(0.115)}{0.902}\,\text{log(nox)} - \underset{(0.043)}{0.0868}\,\text{log(dist)} - \underset{(0.0059)}{0.0476}\,\text{stratio}$$
$$- \underset{(0.1655)}{0.5451}\,\text{rooms} + \underset{(0.0128)}{0.0623}\,\text{rooms}^2$$
$$n = 506 \quad \bar{R}^2 = 0.5988 \quad F(5,500) = 151.77 \quad \hat{\sigma} = 0.25921$$

- House value and rooms: First decreasing then increasing.
- As the number of rooms changes from 3 to 4, price is predicted to change by:
$$\frac{\Delta\widehat{\log(price)}}{\Delta rooms} = -0.5451 + 0.1246(3) = -0.1713 \approx -17.13\%$$
- At Rooms=3, an additional room leads to approximately 17.13% decrease in price.
- Turning point:
$$rooms^* = 0.5451/0.1246 = 4.37 \approx 4.4$$

Quadratic Models: House Price



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Quadratic Models: House Price

- The impact of an additional room on price:

$$\Delta \widehat{\log(price)} = [-0.545 + 2(0.062)rooms]\Delta rooms$$

$$\%\Delta \widehat{price} = 100 \times [-0.545 + 2(0.062)rooms]\Delta rooms$$
$$= (-54.5 + 12.4rooms)\Delta rooms$$

- For example as the number of rooms changes from 5 to 6, price increases by $-54.5 + 12.4 \times 5 = 7.5\%$. Notice that here, $\Delta rooms = 1$.
- Going from 6 to 7: $-54.5 + 12.4 \times 6 = 19.9\%$.
- Going from 5 to 7: $(-54.5 + 12.4 \times 5)2 = 15\%$. Notice that in this case $\Delta rooms = 2$.

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Models with Interaction Terms

- In some cases, the partial impact of one variable may depend on the magnitude of another explanatory variable.
- To capture this we add interaction terms into the regression model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 \underbrace{x_1 \times x_2}_{interaction} + \beta_4 x_3 + u$$

- Interaction variables: x_1 and x_2. The partial impact of x_1 on y depends on x_2:
$$\frac{\Delta y}{\Delta x_1} = \beta_1 + \beta_3 x_2$$
- To compute this interaction effect we need to plug in a value for x_2. In practice, we generally use mean or median of x_2.
- Similarly, the partial impact of x_2 depends on x_1:
$$\frac{\Delta y}{\Delta x_2} = \beta_2 + \beta_3 x_1$$

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Models with Interaction Effects

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 \underbrace{x_1 \times x_2}_{interaction} + \beta_4 x_3 + u$$

$$\frac{\Delta y}{\Delta x_1} = \beta_1 + \beta_3 x_2$$

- Let the sample mean of x_2 be $\bar{x}_2$. Using this value:
$$\frac{\Delta y}{\Delta x_1} = \beta_1 + \beta_3 \bar{x}_2$$
- This gives us the interaction effect at $x_2 = \bar{x}_2$. Is this effect statistically significant?
- To test this we rewrite the model using $x_1 \times (x_2 - \bar{x}_2)$ instead of $x_1 \times x_2$:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 \underbrace{x_1 \times (x_2 - \bar{x}_2)}_{interaction} + \beta_4 x_3 + u$$

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Models with Interaction Terms

- The model now is

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 \underbrace{x_1 \times (x_2 - \bar{x}_2)}_{interaction} + \beta_4 x_3 + u$$

- Simple significance t-test

$$H_0 : \beta_1 = 0$$

- Other effects can be tested similarly.

Interaction Effects: Example, attend.gdt

Variable Definitions:
stndfnl: Standardized final score; **atndrte**: attendance rate (%); **priGPA**: cumulative GPA in the previous semester (out of 4); **ACT**: achievement test score,

$$\widehat{stndfnl} = \underset{(1.36)}{2.05} - \underset{(.0102)}{.0067}\ atndrte - \underset{(0.48)}{1.63}\ priGPA - \underset{(.098)}{.128}\ ACT$$
$$+ \underset{(.101)}{.296}\ priGPA^2 + \underset{(.0022)}{.0045}\ ACT^2 + \underset{(.0043)}{.0056}\ priGPA \cdot atndrte$$
$$n = 680, R^2 = .229, \bar{R}^2 = .222.$$

The coefficient estimate on $atndrte$ (-0.0067) measures the impact when $priGPA = 0$. Since there is no 0 in $priGPA$ its sign is unimportant. This coefficient alone does not measure the impact of attendance rate because there is interaction term with priGPA.

Interaction Effects: Example, attend.gdt

$$\widehat{stndfnl} = \underset{(1.36)}{2.05} - \underset{(.0102)}{.0067}\ atndrte - \underset{(0.48)}{1.63}\ priGPA - \underset{(.098)}{.128}\ ACT$$
$$+ \underset{(.101)}{.296}\ priGPA^2 + \underset{(.0022)}{.0045}\ ACT^2 + \underset{(.0043)}{.0056}\ priGPA \cdot atndrte$$
$$n = 680, R^2 = .229, \bar{R}^2 = .222.$$

- We need to take into account the interaction term (β_6). Note that β_1 and β_6 cannot pass individual t-statistics but they are jointly significant (The null hypothesis $H_0 : \beta_1 = \beta_2 = 0$ can be rejected using F test with p-value=0.014).
- The sample mean of $priGPA$ is 2.59. Using this:
$$\Delta\widehat{stndfnl} = -0.0067 + (0.0056)(\mathbf{2.59}) = 0.0078$$
- Interpretation: At the mean GPA, $priGPA$ 2.59, a 10 percentage point increase in atndrte increases $\widehat{stndfnl}$ by 0.078 standard deviations from the mean final score.

Interaction Effects: Example, attend.gdt

- The partial effect of the attendance rate at the mean GPA is estimated as 0.0078. Is this effect statistically different from zero?
- To test this we will re-estimate the model using $(priGPA - 2.59) \times atndrte$ instead of $priGPA \times atndrte$.
- In this regression, the coefficient estimate on $atndrte$ (ie., $\hat{\beta}_1$) will measure the predicted partial effect when $priGPA = 2.59$, its sample mean.
- This can easily be tested using the standard t-test.

Interaction Effects: Example, attend.gdt

$$\widehat{\text{stndfnl}} = \underset{(1.36)}{2.05} + \underset{(0.0026)}{\mathbf{0.0078}}\,\textbf{atndrte} - \underset{(0.481)}{1.6285}\,\text{priGPA} + \underset{(0.101)}{0.2959}\,\text{priGPA}^2$$
$$- \underset{(0.098)}{0.1280}\,\text{ACT} + \underset{(0.0022)}{0.0045}\,\text{ACT}^2 + \underset{(0.004)}{0.0056}\,\textbf{(priGPA}-\textbf{2.59)}\cdot\ \textbf{atndrte}$$
$$n = 680 \quad \bar{R}^2 = 0.2218 \quad F(6, 673) = 33.250 \quad \hat{\sigma} = 0.87287$$

- Test:
- $t = 0.0078/0.0026 = 3$, Therefore we reject $H_0 : \beta_1 = 0$, the effect is significant ($p - value = 0.003$).

Goodness-of-Fit: R-Squared

- The coefficient of determination, R^2, is simply an estimate of "how much variation in y is explained by $x_1, x_2, \ldots, x_k$ in the population".
- A low R^2 value does not automatically imply that the MLR assumptions fail.
- As the number of explanatory variables (k) increases, R^2 always increases (it never decreases). Thus, R^2 has a limited role in choosing between alternative models.
- The relative change in the R-squared when variables added to an equation may be very helpful (e.g. F-statistic for exclusion restrictions depends on the difference in R^2s).

Adjusted R-Squared: $\bar{R}^2$

- Recall the definition of R^2:

$$R^2 = 1 - \frac{SSR}{SST}$$

- Dividing the numerator and denominator by n:

$$R^2 = 1 - \frac{SSR/n}{SST/n} = 1 - \frac{\sigma_u^2}{\sigma_y^2}$$

- Since SST/n and SSR/n are biased estimators of respective population variances we will instead use:

$$\frac{SST}{n-1}, \quad \frac{SSR}{n-k-1}$$

Adjusted R-Squared: $\bar{R}^2$

- Adjusted R-squared is defined as

$$\bar{R}^2 = 1 - \frac{SSR/(n-k-1)}{SST/(n-1)} = 1 - (1 - R^2)\frac{n-1}{n-k-1}$$

- Adjusted R^2, or R-bar squared may increase or decrease when a new variable is added to the regression. Recall that, in contrast, R^2 never decreases.
- The reason is that when a new variable is added, while SSR decreases, the degrees of freedom $(n-k-1)$ also decreases.
- Basically, it imposes a penalty for adding additional variables to a model. $SSR/(n-k-1)$ can go up or down.
- When a new x variable is added, R-bar square increases if, and only if, the t statistic on the new variable is greater than one in absolute value.
- Extension: when a group of x variables is added, $\bar{R}^2$ increases if, and only if, the F statistic for joint significance of the new variables is greater than 1.

Example

Model 1: OLS, using observations 1–506
Dependent variable: lprice

	Coefficient	Std. Error	t-ratio	p-value
const	8.95348	0.181147	49.4266	0.0000
lnox	−0.304841	0.0821638	−3.7102	0.0002
proptax	−0.00760708	0.000977765	−7.7801	0.0000
rooms	0.288707	0.0181186	15.9343	0.0000

Mean dependent var	9.941057	S.D. dependent var	0.409255
Sum squared resid	36.70511	S.E. of regression	0.270403
R^2	0.566042	Adjusted R^2	0.563449
$F(3, 502)$	218.2650	P-value(F)	1.35e–90

Example

Model 2: OLS, using observations 1–506
Dependent variable: lprice

	Coefficient	Std. Error	t-ratio	p-value
const	8.85532	0.172131	51.4452	0.0000
lnox	−0.275421	0.0779513	−3.5332	0.0004
proptax	−0.00422185	0.00102745	−4.1090	0.0000
rooms	0.281587	0.0171939	16.3771	0.0000
crime	−0.0124893	0.00163861	−7.6219	0.0000

Mean dependent var	9.941057	S.D. dependent var	0.409255
Sum squared resid	32.89123	S.E. of regression	0.256225
R^2	0.611133	Adjusted R^2	**0.608028**
$F(4, 501)$	196.8397	P-value(F)	2.7e–101

Example

Model 3: OLS, using observations 1–506
Dependent variable: lprice

	Coefficient	Std. Error	t-ratio	p-value
const	9.76749	0.222071	43.9837	0.0000
lnox	−0.355701	0.0763150	−4.6610	0.0000
proptax	−0.00185202	0.00106268	−1.7428	0.0820
rooms	0.251409	0.0172902	14.5405	0.0000
crime	−0.0122323	0.00158140	−7.7351	0.0000
stratio	−0.0370699	0.00599178	−6.1868	0.0000

Mean dependent var	9.941057	S.D. dependent var	0.409255
Sum squared resid	30.55237	S.E. of regression	0.247194
R^2	0.638785	Adjusted R^2	**0.635173**
$F(5, 500)$	176.8435	P-value(F)	4.2e–108

Exclusion test for **crime** and **stratio**: $F = 50.35$, $pval < 0.0001$

Adjusted R^2

- When comparing two models using $\bar{R}^2$s the dependent variables must be the same.
- Adjusted R^2 can especially be useful when comparing non-nested models (if the dependent variables are the same)
- For example consider the following non-nested models:

$$y = \beta_0 + \beta_1 \log(x), \quad \bar{R}_A^2$$

$$y = \beta_0 + \beta_1 x + \beta_2 x^2, \quad \bar{R}_B^2$$

- F statistic can only be used to test nested models.
- We can choose the model with larger $\bar{R}^2$.

Prediction

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \ldots + \hat{\beta}_k x_k$$

- When we plug in particular x values into the model above we obtain a prediction for y which is an estimate of the expected value of y given the particular values for the explanatory variables, $\mathsf{E}(y|x)$.
- Let particular values be $x_1 = c_1$, $x_2 = c_2, \ldots, x_k = c_k$. Also let the prediction value for y be θ_0:

$$\theta_0 = \beta_0 + \beta_1 c_1 + \beta_2 c_2 + \ldots + \beta_k c_k$$
$$= \mathsf{E}[y|x_1 = c_1, x_2 = c_2, \ldots, x_k = c_k]$$

- The OLS estimator of θ_0 is:

$$\hat{\theta}_0 = \hat{\beta}_0 + \hat{\beta}_1 c_1 + \hat{\beta}_2 c_2 + \ldots + \hat{\beta}_k c_k$$

Predicting E$(y|x)$

- 95% confidence interval for θ_0:

$$\hat{\theta}_0 \pm \ 2 \ se(\hat{\theta}_0)$$

- To compute this we need the standard error of $\hat{\theta}_0$.
- This standard error can easily be calculated using an auxiliary regression. By definiton

$$\beta_0 = \theta_0 - \beta_1 c_1 - \beta_2 c_2 - \ldots - \beta_k c_k$$

- Substituting into the model and rearranging we get

$$y = \theta_0 + \beta_1(x_1 - c_1) + \beta_2(x_2 - c_2) + \ldots + \beta_k(x_k - c_k) + u$$

- The standard error on the intercept estimate will give us the standard error of the prediction.

Prediction: Example, gpa2.gdt

$$\begin{aligned} \widehat{colgpa} = \underset{(0.075)}{1.493} + \underset{(.00007)}{.00149}\ sat - \underset{(.00056)}{.01386}\ hsperc \\ - \underset{(.01650)}{.06088}\ hsize + \underset{(.00227)}{.00546}\ hsize^2 \end{aligned}$$

$$n = 4{,}137,\ R^2 = .278,\ \bar{R}^2 = .277,\ \hat{\sigma} = .560,$$

- Prediction points: $sat = 1200, hsperc = 30, hsize = 5$ (hsrank:rank in class; hsize:size of class; hsper:100*(hsrank/hssize)
- Plugging into the estimated regression we get $colGPA = 2.70$.
- To compute the standard error of this prediction we define: $sat0 = sat - 1200$, $hsperc0 = hsperc - 30$, $hsize0 = hsize - 5$, $hsizesq0 = hsize - 25$. Then, we regress colGPA on these variables.

Prediction: Example, gpa2.gdt

sat0 = *sat* − 1,200, *hsperc0* = *hsperc* − 30, *hsize0* = *hsize* − 5, and *hsizesq0* = *hsize*² − 25.

$$\begin{aligned} \widehat{colgpa} = \underset{(0.020)}{2.700} + \underset{(.00007)}{.00149}\ sat0 - \underset{(.00056)}{.01386}\ hsperc0 \\ - \underset{(.01650)}{.06088}\ hsize0 + \underset{(.00227)}{.00546}\ hsizesq0 \end{aligned}$$

$$n = 4{,}137,\ R^2 = .278,\ \bar{R}^2 = .277,\ \hat{\sigma} = .560.$$

- 95% Confidence Interval: $2.70 \pm 1.96(0.020) = [2.66, 2.74]$.
- The variance of $\hat{\theta}_0$ reaches its smallest value at the arithmetic means of x variables ($c_j = \bar{x}_j$).
- Thus, as the values of c_j get farther away from the $\bar{x}_j$, Var$(\hat{y})$ gets larger and larger.
- The standard error and the confidence interval computed above are for the average value of y for the subpopulation with a given set of covariates.
- This is not the same as the confidence interval for the **individual** predictions of y.

Confidence Interval (CI) for Individual Predictions

- In forming a CI for an unknown outcome on y, we must account for another source of variation: the variance in the unobserved error u in addition to the variance in $\hat{y}$.
- Let y^o represent a new cross-sectional unit (individual, firm, region, country, etc.) not in our original sample:

$$y^o = \beta_0 + \beta_1 x_1^o + \beta_2 x_2^o + \ldots + \beta_k x_k^o + u^o$$

- The OLS prediction of y^o at the values x_j^o:

$$\hat{y}^o = \hat{\beta}_0 + \hat{\beta}_1 x_1^o + \hat{\beta}_2 x_2^o + \ldots + \hat{\beta}_k x_k^o.$$

- The prediction error is

$$\hat{e}^o = y^o - \hat{y}^o = \beta_0 + \beta_1 x_1^o + \beta_2 x_2^o + \ldots + \beta_k x_k^o + u^o - \hat{y}^o$$

- Taking expectations we obtain

$$\mathsf{E}(\hat{e}^o) = 0$$

Confidence Interval (CI) for Individual Predictions

- The variance of the prediction error

$$\text{Var}(\hat{e}^o) = \text{Var}(\hat{y}^o) + \text{Var}(u^o) = \text{Var}(\hat{y}^o) + \sigma^2$$

- $\text{Var}(\hat{y}^o)$ is inversely related to the sample size n. It gets smaller as n increases.
- σ^2 is the variance of the unobserved error term. It does not decrease as n increases.
- Thus, σ^2 is the dominant term in the variance of the prediction error.
- The standard error of the prediction error

$$se(\hat{e}^o) = \sqrt{\text{Var}(\hat{y}^o) + \hat{\sigma}^2}$$

- 95% CI is

$$[\hat{y}^o \pm t_{0.025} \cdot se(\hat{e}^o)]$$

Confidence Interval (CI) for Individual Predictions: Example

- The confidence intervals for the individual predictions will be much larger than the CI for the conditional average of y. The reason is that $\hat{\sigma}^2$ is much larger than $\text{Var}(\hat{y}^o)$.
- For example, suppose that we want to construct a 95% CI for the colGPA of a high school student with $sat = 1200, hsperc = 30, hsize = 5$.
- Plugging these values in the regression model we obtain $colGPA = 2.70$ ($\hat{y}^o$) as before.
- From our earlier calculations we know $se(\hat{y}^o) = 0.02$ and $\hat{\sigma} = 0.56$. Thus, $se(\hat{e}^o) = \sqrt{0.02^2 + 0.56^2} = 0.56$ and the 95% CI is

$$2.70 \pm 1.96 \cdot (0.56) = [1.6,\ 3.8]$$

- This is a very wide confidence interval. It is so large that it is almost impossible to accurately pin down an individual's future college grade point average.