Conjugate priors and statistical inference

Show that for the Poisson distribution, the Gamma distribution (for ϑ)is a conjugate prior.

In the terms of poisson distribution x! is constant with respect to ϑ .

$$p(x|\vartheta) = \frac{\vartheta^x * e^{-\vartheta}}{x!} \ \alpha \ \vartheta^x * e^{-\vartheta}$$

In the terms of gamma distribution β^α and $\Gamma(\alpha)$ is constant with respect to ϑ .

$$p(\vartheta|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} * \vartheta^{\alpha-1}e^{-\beta\vartheta} \ \alpha \ \vartheta^{\alpha-1}e^{-\beta\vartheta}$$

Posterior is

$p(\vartheta|x) = p(x|\vartheta)p(\vartheta|\alpha,\beta) = \frac{1}{x!}\vartheta^x e^{-\vartheta}\frac{\beta^\alpha}{\Gamma(\alpha)}\vartheta^{\alpha-1}e^{-\beta\vartheta}$ is proportional to

$$\vartheta^x e^{-\vartheta}\vartheta^{\alpha-1}e^{-\beta\vartheta} \ = \ \vartheta^{x+\alpha-1}e^{-\vartheta(\beta+1)}$$

Which is equal to $gamma(\vartheta|(x+\alpha),(\beta+1))$. So, prior is conjugate

What parameters of α and β correspond to a flat prior

We have to choose α and β close to zero as possible to neglect the effects of prior to posterior. In this case I have choose the values as followings:
$\alpha = 0.0001$
$\beta = 0.0001$

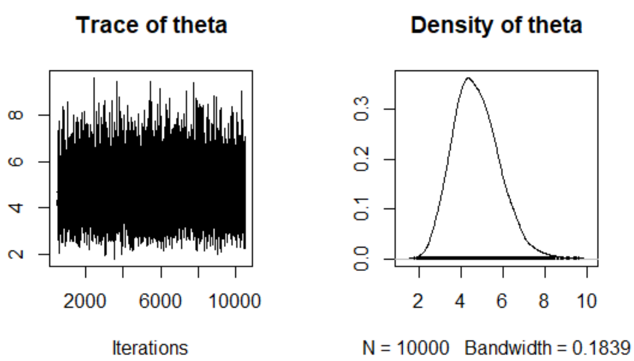

Figure 0.1 – Posterior distribution with the values of $\alpha = 0.0001$ and $\beta = 0.0001$

Which value of α and β does one have to choose, in order to reach the asymptotic limit as quick as possible

We have to choose high α and β values to reach the asymptotic limit. $\alpha = 1000$ and $\beta = 1000$

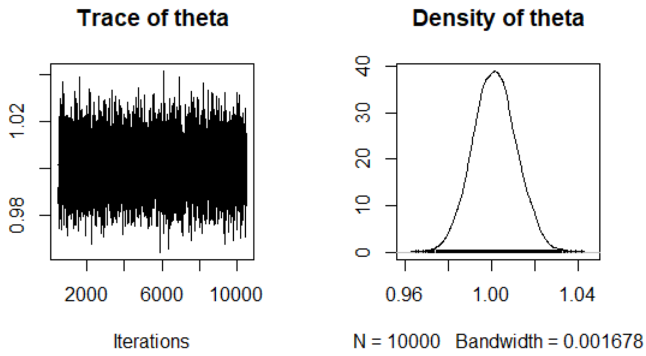

Figure 0.2 – Posterior distribution with the values of $\alpha = 0.0001$ and $\beta = 0.0001$

Bayesian parameter estimation using a loss function

Plot the loss function

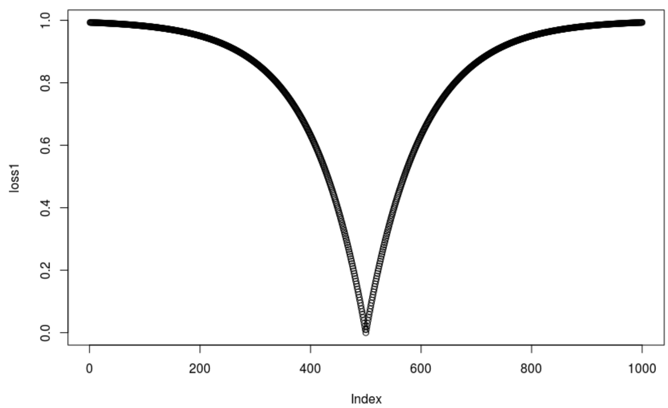

Figure 0.3 – Loss function of ϕ=1

Calculate the optimal estimated parameter for that loss function for $\phi = 1 and \phi = 5$

Optimal parameter is 0.46 for $\phi = 1$ and 0.65 for $\phi = 5$

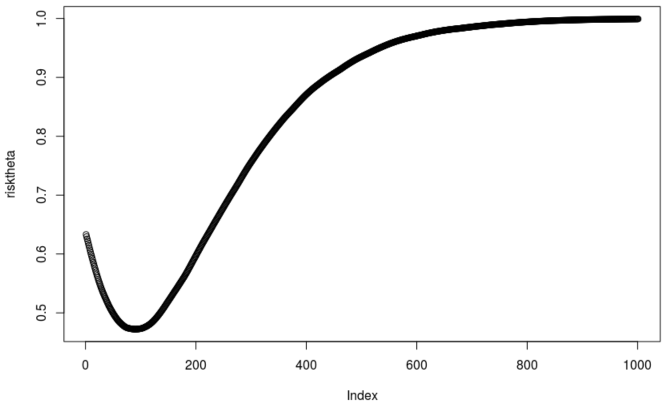

Figure 0.4 – Loss function of ϕ=1

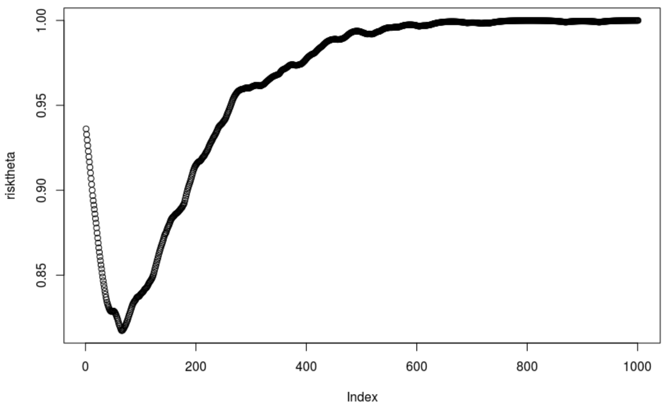

Figure 0.5 – Loss function of ϕ=1

If ϕ value is too low, I would expect a high estimated parameter, also risk graph would be smooth and lack of detail. If ϕ value is too high graph would be rough and hard to analyze, and estimated parameter would be low.