陈宏泰

(1) 1° $\vec{r}_x = (1, 0, f_x)$ $\vec{r}_y = (0, 1, f_y)$

$E = 1 + f_x^2$ $F = f_x f_y$ $G = 1 + f_y^2$

$\vec{n} = \vec{e}_3 = \dfrac{\vec{r}_x \wedge \vec{r}_y}{|\vec{r}_x \wedge \vec{r}_y|} = \dfrac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$ 记 $W = \sqrt{1 + f_x^2 + f_y^2}$

$\vec{r}_{xx} = (0, 0, f_{xx})$ $\vec{r}_{xy} = (0, 0, f_{xy})$ $\vec{r}_{yy} = (0, 0, f_{yy})$

$L = \dfrac{f_{xx}}{W}$ $M = \dfrac{f_{xy}}{W}$ $N = \dfrac{f_{yy}}{W}$

$$H = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2} = \frac{1}{2W^3}\left(f_{xx}(1 + f_y^2) - 2f_{xy}f_x f_y + f_{yy}(1 + f_x^2)\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(\frac{f_x}{W}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{W}\right)\right)$$

于是 Σ 为极小曲面 $\Leftrightarrow H = 0$

$$\Leftrightarrow \frac{\partial}{\partial x}\left(\frac{f_x}{W}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{W}\right) = 0$$

$$\Leftrightarrow f_{xx}(1 + f_y^2) - 2f_{xy}f_x f_y + f_{yy}(1 + f_x^2) = 0$$

2° I) $f_x = \dfrac{-\frac{y}{x^2}}{1 + \frac{y^2}{x^2}} = \dfrac{-y}{x^2 + y^2}$ $f_y = \dfrac{\frac{1}{x}}{1 + \frac{y^2}{x^2}} = \dfrac{x}{x^2 + y^2}$

$f_{xx} = \dfrac{2xy}{(x^2 + y^2)^2}$ $f_{xy} = \dfrac{-x^2 + y^2}{(x^2 + y^2)^2}$ $f_{yy} = \dfrac{-2xy}{(x^2 + y^2)^2}$

$$\frac{1}{(x^2 + y^2)^4}\left(2xy(2x^2 + y^2) - 2(-x^2 + y^2)(-xy) + (-2xy)(x^2 + 2y^2)\right) = 0$$

✓

II) $f_x = \dfrac{x}{\sqrt{x^2 + y^2 - 1} \cdot \sqrt{x^2 + y^2}}$ $f_y = \dfrac{y}{\sqrt{x^2 + y^2 - 1} \cdot \sqrt{x^2 + y^2}}$

$f_{xx} = \dfrac{y^4 - x^4 - y^2}{\left((x^2 + y^2 - 1)(x^2 + y^2)\right)^{\frac{3}{2}}}$ $f_{xy} = \dfrac{-xy(2x^2 + 2y^2 - 1)}{\left((x^2 + y^2 - 1)(x^2 + y^2)\right)^{\frac{3}{2}}}$ $f_{yy} = \dfrac{x^4 - y^4 - x^2}{\left((x^2 + y^2 - 1)(x^2 + y^2)\right)^{\frac{3}{2}}}$

$$\frac{1}{\left((x^2 + y^2 - 1)(x^2 + y^2)\right)^{\frac{3}{2}}}\Big((y^4 - x^4 - y^2)\left((x^2 + y^2 - 1)(x^2 + y^2) + y^2\right) - 2(-xy)(2x^2 + 2y - 1)xy$$
$$+ (x^4 - y^4 - x^2)\left((x^2 + y^2 - 1)(x^2 + y^2) + x^2\right)\Big) = 0$$

✓

(Ⅲ) $f_x = -\tan x \qquad f_y = \tan y$

$$f_{xx} = -\frac{1}{\cos^2 x} \quad f_{xy} = 0 \quad f_{yy} = \frac{1}{\cos^2 y}$$

$$\left(-\frac{1}{\cos^2 x}(1+\tan^2 y) + \frac{1}{\cos^2 y}\cdot(1+\tan^2 x)\right) = 0$$

✓

(2) 1° 证: 设 $u = x + \varphi_x(x,y) \quad v = y + \varphi_y(x,y)$

$$F = (x - \varphi_x(x,y)) - i(y - \varphi_y(x,y))$$

$$\frac{\partial u}{\partial x} = 1 + \varphi_{xx} \qquad \frac{\partial u}{\partial y} = \varphi_{xy}$$

$$\frac{\partial v}{\partial x} = \varphi_{xy} \qquad \frac{\partial v}{\partial y} = 1 + \varphi_{yy}$$

$$\Longrightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} 1+\varphi_{xx} & \varphi_{xy} \\ \varphi_{xy} & 1+\varphi_{yy} \end{pmatrix}$$

$$\Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2+\varphi_{xx}+\varphi_{yy}} \begin{pmatrix} 1+\varphi_{yy} & -\varphi_{xy} \\ -\varphi_{xy} & 1+\varphi_{xx} \end{pmatrix}$$

$$\frac{\partial Re(F)}{\partial u} = \frac{\partial Re(F)}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial Re(F)}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{1}{2+\varphi_{xx}+\varphi_{yy}}(\varphi_{yy} - \varphi_{xx})$$

$$\frac{\partial Re(F)}{\partial v} = \frac{-2\varphi_{xy}}{2+\varphi_{xx}+\varphi_{yy}} \qquad \frac{\partial Im(F)}{\partial u} = \frac{2\varphi_{xy}}{2+\varphi_{xx}+\varphi_{yy}}$$

$$\frac{\partial Im(F)}{\partial v} = \frac{\varphi_{yy} - \varphi_{xx}}{2+\varphi_{xx}+\varphi_{yy}}$$

满足 Cauchy-Riemann 方程. 故 $F(u+vi)$ 为复解析函数.

Ⅱ) $F'(w) = \frac{\partial F}{\partial w} = \frac{1}{2}\left(\frac{\partial F}{\partial u} - i\frac{\partial F}{\partial v}\right) = \frac{1}{2+\varphi_{xx}+\varphi_{yy}}(\varphi_{yy} - \varphi_{xx} + 2i\varphi_{xy})$

在 $\varphi_{xx} > 0$ 时, $\varphi_{yy} > 0$, $|Re(F)_u| \leq \frac{1}{2+\varphi_{xx}+\varphi_{yy}}(|\varphi_{xx}| + |\varphi_{yy}|) \leq 1$

$$|Re(F)_v| = \frac{1}{2+\varphi_{xx}+\varphi_{yy}} 2\varphi_{xy} \leq \frac{\sqrt{\varphi_{xx}\varphi_{yy}-1}}{1+\sqrt{\varphi_{xx}\varphi_{yy}}} \leq 1$$

那么 $|F'|$ 有界, 根据 Liouville 定理, F' 为常数

从而知 $\varphi_{xx}, \varphi_{xy}, \varphi_{yy}$ 皆为常数

2° Σ 为极小曲面 $\Longleftrightarrow$ Monge-Ampere 方程

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = \det H_\varphi = \frac{(1+f_x^2)(1+f_y^2)}{W^2} - \frac{f_x^2 f_y^2}{W^2} = 1$$

再由1°知 $\varphi_{xx}, \varphi_{xy}, \varphi_{yy}$ 均为常数,

从而 f_x, f_y 均为常数. $\Rightarrow$ f为线性函数 Bernstein ✓.

(3) 1° 证: $dl = \langle d\vec{e}_3, \vec{a}\rangle = \langle w_{3i}\vec{e}_i, \vec{a}\rangle$

$$= \langle -h_{ij} w_j \vec{e}_i, \vec{a}\rangle = -h_{ij}\langle \vec{e}_i, \vec{a}\rangle w_j$$

$$\Rightarrow \quad l_i = -h_{ij}\langle \vec{e}_j, \vec{a}\rangle$$

再求一次共变导数

$$l_{ij} w_j = D l_i = d l_i + l_j w_{ji}$$

$$= d(-h_{ik}\langle \vec{e}_k, \vec{a}\rangle) + w_{ji}(-h_{jk}\langle \vec{e}_k, \vec{a}\rangle)$$

$$= -dh_{ik}\cdot\langle \vec{e}_k, \vec{a}\rangle - h_{ki}\cdot\sum_{j=1}^{3} w_{kj}\langle \vec{e}_j, \vec{a}\rangle - w_{ji}h_{jk}\langle \vec{e}_k, \vec{a}\rangle$$

$k \leftrightarrow j$

$$= -(dh_{ik} + h_{jk}w_{ji} + h_{ij}w_{jk})\langle \vec{e}_k, \vec{a}\rangle - h_{ki}w_{k3}\langle \vec{e}_3, \vec{a}\rangle$$

$$= -h_{ik,j}w_j\langle \vec{e}_k, \vec{a}\rangle - h_{ki}h_{kj}w_j\langle \vec{e}_3, \vec{a}\rangle$$

$$\Rightarrow \quad l_{ij} = -h_{ik,j}\langle \vec{e}_k, \vec{a}\rangle - h_{ik}h_{jk}\langle \vec{e}_3, \vec{a}\rangle$$

2° 由于 $\Delta_\Sigma l = \langle \Delta_\Sigma \vec{e}_3, \vec{a}\rangle$

那么 $\Delta_\Sigma l = l_{11} + l_{22}$

$$= -h_{1k,1}\langle \vec{e}_k, \vec{a}\rangle - h_{1k}h_{1k}\langle \vec{e}_3, \vec{a}\rangle$$

$$-h_{2k,2}\langle \vec{e}_k, \vec{a}\rangle - h_{2k}h_{2k}\langle \vec{e}_3, \vec{a}\rangle$$

$$= -\langle (h_{11,k} + h_{22,k})\vec{e}_k - |A|^2\vec{e}_3, \vec{a}\rangle$$

$$\Rightarrow \quad \Delta_\Sigma \vec{e}_3 = -2H_k\vec{e}_k - |A|^2\vec{e}_3$$

Codazzi: $h_{ik,j} = h_{ij,k}$

$h_{ik,j} = h_{ki,j}$

(由定义)

3° 取 $\vec{a} = (0,0,1)$, 那么 $l = \frac{1}{W}$

由2°, $\Delta_\Sigma l = -|A|^2\langle \vec{e}_3, \vec{a}\rangle = -|A|^2 l = -\frac{|A|^2}{W}$

而 $|A|^2 = h_{ik}^2 = (h_{11}+h_{22})^2 - 2(h_{11}h_{22} - h_{12}^2)$

$$= 4H^2 - 2K = -2K$$

$$\Rightarrow \quad K = \frac{1}{2} W \Delta_z \left(\frac{1}{W}\right)$$

4° 取 $\vec{a} = (0,0,1)$ $\quad l = \frac{1}{W}$

$$\nabla l = l_1 \vec{e_1} + l_2 \vec{e_2}$$

$$\begin{aligned}
|\nabla l|^2 &= l_1^2 + l_2^2 \\
&= (-h_{1j}\langle e_j, \vec{a}\rangle)^2 + (-h_{2j}\langle e_j, \vec{a}\rangle)^2 \\
&= h_{11}^2 \langle \vec{e_1}, \vec{a}\rangle^2 + h_{12}^2 \langle \vec{e_2}, \vec{a}\rangle^2 + 2h_{11}h_{12}\langle \vec{e_1}, \vec{a}\rangle\langle \vec{e_2}, \vec{a}\rangle \\
&\quad\ h_{21}^2 \langle \vec{e_1}, \vec{a}\rangle^2 + h_{22}^2 \langle \vec{e_2}, \vec{a}\rangle^2 + 2h_{21}h_{22}\langle \vec{e_1}, \vec{a}\rangle\langle \vec{e_2}, \vec{a}\rangle \\
&= K_1^2(\langle \vec{e_1}, \vec{a}\rangle^2 + \langle \vec{e_2}, \vec{a}\rangle^2) \\
&= K_1^2(1 - \langle \vec{e_3}, \vec{a}\rangle^2) \\
&= -K\left(1 - \frac{1}{W^2}\right)
\end{aligned}$$

$h_{11} + h_{22} = 0 = K_1 + K_2$

$h_{12} = h_{21}$

$-K_1^2 = K_1K_2 = K = h_{11}h_{22} - h_{12}^2$

$= -h_{11}^2 - h_{12}^2$

$\Rightarrow K_1^2 = K_2^2 = h_{11}^2 + h_{12}^2$

$-K \quad = h_{21}^2 + h_{22}^2$

$$\Longrightarrow \quad \left|\nabla\left(\frac{1}{W}\right)\right|^2 = -K\left(1 - \frac{1}{W^2}\right)$$

$$\begin{aligned}
K = \Delta_z \log\left(1 + \frac{1}{W}\right) &= \frac{W}{W+1}\Delta_z\left(\frac{1}{W}\right) - \left(\frac{W}{W+1}\right)^2 \left|\nabla\left(\frac{1}{W}\right)\right|^2 \\
&= \frac{W}{W+1}\Delta_z\left(\frac{1}{W}\right) - \left(\frac{W}{W+1}\right)^2 \cdot (-K)\left(1 - \frac{1}{W^2}\right) \\
&= \frac{W}{W+1}\Delta_z\left(\frac{1}{W}\right) + K \cdot \frac{W-1}{W+1}
\end{aligned}$$

$$\Longleftrightarrow \quad K = \frac{W}{2}\Delta_z\left(\frac{1}{W}\right) \quad \checkmark$$

(4) Weierstrass 表示

$$x_1 = \mathrm{Re}\,\frac{1}{2}\int f(1-g^2)\,dz \qquad x_2 = \mathrm{Re}\,\frac{i}{2}\int f(1+g^2)\,dz \qquad x_3 = \mathrm{Re}\int fg\,dz$$

1° $x_1 = \frac{u}{2} \qquad x_2 = \frac{v}{2} \qquad x_3 = 0$

2° $x_1 = \frac{1}{2}u - \frac{1}{6}u^3 + \frac{1}{2}uv^2 \qquad x_2 = \frac{1}{2}v - \frac{1}{6}v^3 + \frac{1}{2}u^2v \qquad x_3 = \frac{u^2 - v^2}{2}$

3° $x_1 = \frac{1}{2}u + \frac{1}{2}\cdot\frac{u}{u^2+v^2} \qquad x_2 = \frac{v}{2} + \frac{1}{2}\cdot\frac{v}{u^2+v^2} \qquad x_3 = \frac{1}{2}\ln(u^2+v^2)$

4° $\quad x_1 = -\frac{1}{2}v + \frac{1}{2}\cdot\frac{v}{u^2+v^2}$ $\quad x_2 = \frac{u}{2} - \frac{1}{2}\cdot\frac{u}{u^2+v^2}$ $\quad x_3 = -\arctan\frac{v}{u}$

5° $\quad x_1 = \frac{1}{2}Re(\arctan z)$ $\quad x_2 = \frac{1}{2}Re(\arctan iz)$ $\quad x_3 = \frac{1}{2i}Re(\arctan iz^2)$

$\quad = \frac{1}{2}Im(\arctan z)$ $\quad = \frac{1}{2}Re(\arctan z^2)$

(5)

$$g = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} \frac{1}{\lambda^2} & \\ & \frac{1}{\lambda^2} \end{pmatrix}$$

$$\Delta_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^\beta}\left(\sqrt{g}\, g^{\alpha\beta}\frac{\partial}{\partial u^\alpha}\right)$$

$$= \frac{1}{\lambda^2}\left(\frac{\partial}{\partial u^1}\left(\lambda^2 g^{11}\frac{\partial}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\sqrt{g}\, g^{22}\frac{\partial}{\partial u^2}\right)\right)$$

$$= \frac{1}{\lambda^2}\left(\partial_{ii} + \partial_{jj}\right)$$