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习题二

10. 设 $\tilde{r}$ 的弧长参数为 $\tilde{s}$

$$\tilde{s}=\int_a^b\langle\frac{d\tilde{r}}{ds},\frac{d\tilde{r}}{ds}\rangle^{\frac{1}{2}}ds$$

$$=\int_a^b\langle\frac{d\vec{r}}{ds}T,\frac{d\vec{r}}{ds}T\rangle^{\frac{1}{2}}ds$$

$$=\int_a^b\left(\frac{d\vec{r}}{ds}T\cdot T^t\left(\frac{d\vec{r}}{ds}\right)^t\right)^{\frac{1}{2}}ds \qquad (T\cdot T^t=I)$$

$$=\int_a^b\langle\frac{d\vec{r}}{ds},\frac{d\vec{r}}{ds}\rangle^{\frac{1}{2}}ds=s$$

故 s 也是 $\tilde{r}$ 的弧长参数.

关于曲线 $\tilde{r}$ 的Frenet标架我们有

$$\tilde{t}=\frac{d}{ds}\tilde{r}=\frac{d\vec{r}}{ds}T=\vec{t}\cdot T,$$

$$\frac{d\tilde{t}}{ds}=\frac{d\vec{t}}{ds}\cdot T$$

故 $\tilde{t}=\vec{t}\cdot T$, $\tilde{n}=\vec{n}\cdot T$

$$\tilde{k}=\frac{\frac{d\tilde{t}}{ds}}{\tilde{n}}=\frac{\frac{d\vec{t}}{ds}\cdot T}{\vec{n}\cdot T}=\frac{k\cdot\vec{n}\cdot T}{\vec{n}\cdot T}=k$$

而 $\tilde{b}=\tilde{t}\wedge\tilde{n}=(tT)\wedge(nT)\xlongequal{\text{习题一.5}}\det T(\vec{t}\wedge\vec{n})T=-(\vec{t}\wedge\vec{n})T$

所以 $\tilde{b}=-\vec{b}T$, 而 $\frac{d\tilde{n}}{ds}=\frac{d\vec{n}}{ds}\cdot T$

$$\tilde{\tau}=\langle\frac{d\tilde{n}}{ds},\tilde{b}\rangle=\langle\frac{d\vec{n}}{ds}\cdot T,-\vec{b}T\rangle=\langle\frac{d\vec{n}}{ds},-\vec{b}\rangle=-\tau$$

综上,弧长参数、曲率不变,挠率变为相反数.

11. (1) 证：设 $\tilde{C}$ 的弧长参数为 $\tilde{s}$

则 $$\tilde{s}(s_0)=\int_0^{s_0}\left\langle \frac{d\tilde{r}}{ds}, \frac{d\tilde{r}}{ds}\right\rangle^{\frac{1}{2}} ds$$
$$=\int_0^{s_0}\langle \vec{b}(s), \vec{b}(s)\rangle^{\frac{1}{2}} ds$$
$$=\int_0^{s_0} ds = s_0$$

于是 $\tilde{s}=s$，s 为 $\tilde{C}$ 的弧长参数

设曲线 $\tilde{C}$ 的 Frenet 标架为

$$\vec{\tilde{t}}=\frac{d\tilde{r}}{ds}=\vec{b} \quad \dot{\vec{\tilde{t}}}=\frac{d\vec{\tilde{t}}}{ds}=\frac{d\vec{b}}{ds}=-\tau\vec{n}$$

$$\tilde{\kappa}=|\dot{\vec{\tilde{t}}}|=|-\tau|=\tau$$

$$\vec{\tilde{n}}=\frac{\dot{\tilde{t}}}{\tilde{\kappa}}=-\vec{n}$$

$$\vec{\tilde{b}}=\vec{\tilde{t}}\wedge\vec{\tilde{n}}=\vec{b}\wedge(-\vec{n})=\vec{t}$$

$$\dot{\vec{\tilde{n}}}=\frac{d\vec{\tilde{n}}}{ds}=-\frac{d\vec{n}}{ds}$$

$$\tilde{\tau}=\langle\dot{\vec{\tilde{n}}}, \vec{\tilde{b}}\rangle=\left\langle-\frac{d\vec{n}}{ds}, \vec{t}\right\rangle$$
$$=\langle \kappa\vec{t}-\tau\vec{b}, \vec{t}\rangle=\kappa$$

#

(2) 由(1)立得 Frenet 标架 $\{\tilde{r}(s); \vec{b}(s), -\vec{n}(s), \vec{t}(s)\}$

12. 证：运用习题二.5　　　　tip：s 并非 $\tilde{\vec{r}}$ 的弧长参数

$$\tilde{\vec{r}}'(s) = \dot{\vec{t}}(s) = k\vec{n}$$

$$\tilde{\vec{r}}''(s) = \frac{d(k\vec{n})}{ds} = \dot{k}\vec{n} + k\dot{\vec{n}}$$
$$= -k^2\vec{t} + \dot{k}\vec{n} + k\tau\vec{b}$$

$$\tilde{\vec{r}}'''(s) = \frac{d(-k^2\vec{t} + \dot{k}\vec{n} + k\tau\vec{b})}{ds}$$
$$= -2k\dot{k}\vec{t} - k^2\dot{\vec{t}} + \ddot{k}\vec{n} + \dot{k}\dot{\vec{n}} + \dot{k}\tau\vec{b} + k\dot{\tau}\vec{b} + k\tau\dot{\vec{b}}$$
$$= -3k\dot{k}\vec{t} + (-k^3 + \ddot{k} - k\tau^2)\vec{n} + (2\dot{k}\tau + k\dot{\tau})\vec{b}$$

从而 $\tilde{\vec{r}}'(s)\wedge\tilde{\vec{r}}''(s) = k^2\tau\vec{t} + k^3\vec{b}$

$$|\tilde{\vec{r}}'(s)\wedge\tilde{\vec{r}}''(s)| = k^2\sqrt{\tau^2+k^2}$$

$$k(s) = \frac{|\tilde{\vec{r}}'\wedge\tilde{\vec{r}}''|}{|\tilde{\vec{r}}'|^3} = \frac{k^2\sqrt{\tau^2+k^2}}{k^3} = \sqrt{1+\left(\frac{\tau}{k}\right)^2}$$

而 $(\tilde{\vec{r}}'(s), \tilde{\vec{r}}''(s), \tilde{\vec{r}}'''(s))$

$$= -3k^3\dot{k}\tau + k^3(2\dot{k}\tau + k\dot{\tau}) = k^3(k\dot{\tau} - \dot{k}\tau)$$

故有 $$\tau(s) = \frac{(\tilde{\vec{r}}', \tilde{\vec{r}}'', \tilde{\vec{r}}''')}{|\tilde{\vec{r}}'\wedge\tilde{\vec{r}}''|^2} = \frac{k^3(k\dot{\tau} - \dot{k}\tau)}{k^4(\tau^2+k^2)}$$

$$= \frac{k\dot{\tau} - \dot{k}\tau}{k^2}\cdot\frac{1}{k\left(1+\left(\frac{\tau}{k}\right)^2\right)} = \frac{\frac{d}{ds}\left(\frac{\tau}{k}\right)}{k\left(1+\left(\frac{\tau}{k}\right)^2\right)}$$

16. 证：设 C 的弧长参数表达式为 $\vec{r}(s)$

且 $\vec{r}(0)=P_0$

于是 $\lim\limits_{P\to P_0}\dfrac{2d(P,l)}{d^2(P_0,P)}=\lim\limits_{s\to 0}\dfrac{2\langle\vec{r}(s)-\vec{r}(0),\vec{n}(0)\rangle}{|\vec{r}(s)-\vec{r}(0)|^2}$

$\xlongequal{L'H\hat{o}pital}\lim\limits_{s\to 0}\dfrac{2\langle\vec{t}(s),\vec{n}(0)\rangle}{2\langle\vec{r}(s)-\vec{r}(0),\vec{t}(s)\rangle}$

$=\lim\limits_{s\to 0}\dfrac{\kappa(s)\langle\vec{n}(s),\vec{n}(0)\rangle}{1+\langle\vec{r}(s)-\vec{r}(0),\vec{n}(s)\rangle}=\kappa(s)$. #

19. 设 $\vec{v}(s)=a_1(s)\vec{t}(s)+a_2(s)\vec{n}(s)+a_3(s)\vec{b}(s)$

$\kappa\vec{n}=\dot{\vec{t}}=\vec{v}\wedge\vec{t}=a_3(s)\vec{n}-a_2(s)\vec{b}$

$-\kappa\vec{t}+\tau\vec{b}=\dot{\vec{n}}=\vec{v}\wedge\vec{n}=-a_3(s)\vec{t}+a_1(s)\vec{b}$

$-\tau\vec{n}=\dot{\vec{b}}=\vec{v}\wedge\vec{b}=a_2(s)\vec{t}-a_1(s)\vec{b}$

于是有 $\begin{cases}a_1(s)=\tau(s)\\ a_2(s)\equiv 0\\ a_3(s)=\kappa(s)\end{cases}$

故 $\vec{v}(s)=\tau(s)\vec{t}(s)+\kappa(s)\vec{b}(s)$

20. 由曲线论基本定理，只需两条曲线的曲率与挠率相等即可

$\vec{r}'(t)=(1+\sqrt{3}\cos t,\ -2\sin t,\ \sqrt{3}-\cos t)$ $|\vec{r}'(t)|=2\sqrt{2}$

$\vec{r}''(t)=(-\sqrt{3}\sin t,\ -2\cos t,\ \sin t)$

$\vec{r}'''(t)=(-\sqrt{3}\cos t,\ 2\sin t,\ \cos t)$

$\vec{r}'(t)\wedge\vec{r}''(t)=(2\sqrt{3}\cos t-2,\ -4\sin t,\ -2\cos t-2\sqrt{3})$

$|\vec{r}'(t)\wedge\vec{r}''(t)|=4\sqrt{2}$

$(\vec{r}'(t), \vec{r}''(t), \vec{r}'''(t)) = -8$

$$\kappa(t) = \frac{|\vec{r}' \wedge \vec{r}''|}{|\vec{r}'|^3} = \frac{4\sqrt{2}}{(2\sqrt{2})^3} = \frac{1}{4}$$

$$\tau(t) = \frac{(\vec{r}', \vec{r}'', \vec{r}''')}{|\vec{r}' \wedge \vec{r}''|^2} = \frac{-8}{(4\sqrt{2})^2} = -\frac{1}{4}$$

而由例 3.2 知

，$\tilde{\vec{r}}(t)$ 的曲率为 $\tilde{\kappa}(t) = \frac{1}{4}$，挠率 $\tilde{\tau}(t) = -\frac{1}{4}$

$t' = \frac{t}{2}$

$(\tilde{\vec{r}}(t') = (2\cos t', 2\sin t', -2t')$

由于 $\kappa(s) \equiv \tilde{\kappa}(s)$，$\tau(s) \equiv \tilde{\tau}(s)$，由曲线论基本定理知

$\vec{r}(t)$ 与 $\tilde{\vec{r}}(t)$ 是合同的.

22. (1) 证：$\vec{t}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$$\vec{r}'(t) = |\vec{r}'(t)| \vec{t}(t)$$

$$\vec{\alpha}'(t) = \vec{r}'(t) + \frac{\dot{\vec{n}}(t)\kappa(t) - \dot{\kappa}(t)\vec{n}(t)}{\kappa^2(t)}$$

$$= |\vec{r}'(t)| \vec{t}(t) - \frac{\dot{\kappa}(t)\vec{n}(t)}{\kappa^2(t)} + \frac{\dot{\vec{n}}(t)}{\kappa(t)}$$

$\langle \vec{t}(t), \vec{t}(t) \rangle = 1$ $\qquad$ $\langle \vec{t}(t), \vec{n}(t) \rangle = 0$

而由 $\dot{\vec{n}}(s) = -\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)$

(PS: 平面曲线中 $\tau(t) \equiv 0$)

代入 $s = s(t)$，得

$$\frac{\dot{\vec{n}}(t)}{|\vec{r}'(t)|} = -\kappa(t)\vec{t}(t) + \tau(t)\vec{b}(t)$$

$$\Rightarrow \langle \vec{t}(t), \dot{\vec{n}}(t) \rangle = -\kappa(t)|\vec{r}'(t)|$$

$\Rightarrow \langle \vec{\alpha}'(t), \vec{r}'(t)\rangle$

$= \langle |\vec{r}'(t)|\vec{t}(t) - \frac{k'(t)}{k^2(t)}\vec{n}(t) + \frac{1}{k(t)}\dot{\vec{n}}(t), |\vec{r}'(t)|\vec{t}(t)\rangle$

$= |\vec{r}'(t)|^2 + \frac{1}{k(t)}\cdot(-k(t)\cdot|\vec{r}'(t)|^2) = 0$

$\Rightarrow$ $\vec{\alpha}'(t)$ 与 $\vec{t}(t)$ 垂直. #

(2) 证:

$$\alpha(t,t') - r(t) = k_1\vec{n}(t)$$

$$\alpha(t,t') - r(t') = k_2\vec{n}(t')$$

固定 t, k_1, k_2 是 t' 的函数 (不妨设其都光滑)

$k_1 = k_1(t')$ $k_2 = k_2(t')$ 函数 k 由 t 决定 且易知 $k_1(t) = k_2(t)$

$$\Rightarrow r(t) + k_1\vec{n}(t) = r(t') + k_2\vec{n}(t')$$

$$\Rightarrow r(t') - r(t) = k_1\vec{n}(t) - k_2\vec{n}(t')$$

$$\Rightarrow \frac{r(t') - r(t)}{t'-t} = \frac{k_1(\vec{n}(t) - \vec{n}(t')) + (k_1 - k_2)\vec{n}(t')}{t'-t}$$

令 $t' \to t$, 有 $\vec{r}'(t) = -k_1(t)\dot{\vec{n}}(t) - k'(t)\vec{n}(t)$

其中 $k'(t) = \lim\limits_{t'\to t}\frac{k_1(t') - k_2(t')}{t'-t} = \lim\limits_{t'\to t}\frac{k_1(t') - k_1(t) + k_2(t) - k_2(t')}{t'-t} = k_1'(t) - k_2'(t)$

又 $\langle \vec{r}'(t), \vec{t}(t)\rangle = |\vec{r}'(t)|$

$\langle -k_1(t)\dot{\vec{n}}(t) - k'(t)\vec{n}(t), \vec{t}(t)\rangle$

$= -k_1(t)\langle \dot{\vec{n}}(t), \vec{t}(t)\rangle = -k_1(t)(-k(t)|\vec{r}'(t)|)$

可知 $k_1(t)\cdot k(t) \equiv 1 \Rightarrow k_1(t) = \frac{1}{k(t)}$

$\Rightarrow \alpha(t,t') - r(t) = k_1(t)\vec{n}(t)$

令 $t' \to t$, 有 $\alpha(t,t') \to r(t) + \frac{\vec{n}(t)}{k(t)} = \alpha(t)$

习题三

2. (1) 注意到 $\vec{r}=(x,y,z)$ 满足

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2(u+v)^2}{a^2}-\frac{b^2(u-v)^2}{b^2}=4uv=z$$

故由1. 知，为双曲抛物面

(2) 再次注意到 $\vec{r}=(x,y,z)$ 满足

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2u^2\cosh^2 v}{a^2}-\frac{b^2u^2\sinh^2 v}{b^2}=u^2(\cosh^2 v-\sinh^2 v)$$
$$=u^2=z$$

故也为双曲抛物面

7. 椭球面的参数化为

$$\vec{r}(u,v)=(a\sin u\cos v,\ b\sin u\sin v,\ c\cos u)$$
$$u\in[0,\pi]\quad v\in[0,2\pi)$$

$$\begin{aligned}\vec{r}_u&=(a\cos u\cos v,\ b\cos u\sin v,\ -c\sin u)\\ \vec{r}_v&=(-a\sin u\sin v,\ b\sin u\cos v,\ 0)\end{aligned}$$

$$E=\langle\vec{r}_u,\vec{r}_u\rangle=a^2\cos^2u\cos^2v+b^2\cos^2u\sin^2v+c^2\sin^2u$$

$$F=\langle\vec{r}_u,\vec{r}_v\rangle=(b^2-a^2)\sin u\cos u\sin v\cos v$$

$$G=\langle\vec{r}_v,\vec{r}_v\rangle=a^2\sin^2u\sin^2v+b^2\sin^2u\cos^2v$$

第一基本形式为

$$I(u,v)=(a^2\cos^2u\cos^2v+b^2\cos^2u\sin^2v+c^2\sin^2u)du^2$$
$$2(b^2-a^2)\sin u\cos u\sin v\cos v\,dudv$$
$$a^2\sin^2u\sin^2v+b^2\sin^2u\cos^2v\,dv^2$$

8. (1) $\vec{r}_u = (\cos v, \sin v, 0)$

$\vec{r}_v = (-u\sin v, u\cos v, b)$

$E = \langle \vec{r}_u, \vec{r}_u \rangle = 1$

$F = \langle \vec{r}_u, \vec{r}_v \rangle = 0$

$G = \langle \vec{r}_v, \vec{r}_v \rangle = u^2 + b^2$

第一基本形式为

$I(u,v) = du^2 + (u^2+b^2)dv^2$

(2) $\vec{r}_u = (a, b, 2u)$

$\vec{r}_v = (a, -b, 2v)$

$E = \langle \vec{r}_u, \vec{r}_u \rangle = a^2+b^2+4u^2$

$F = \langle \vec{r}_u, \vec{r}_v \rangle = a^2-b^2+4uv$

$G = \langle \vec{r}_v, \vec{r}_v \rangle = a^2+b^2+4v^2$

第一基本形式为

$I(u,v) = (a^2+b^2+4u^2)du^2 + (a^2-b^2+4uv)dudv$
$+ (a^2+b^2+4v^2)dv^2$