陈宏泰

6. 解: $\vec{e}_1 = \dfrac{\vec{r}_1}{\sqrt{E}}$ $\quad \vec{e}_2 = \dfrac{\vec{r}_2 - \langle \vec{r}_2, \vec{e}_1 \rangle \vec{e}_1}{|\vec{r}_2 - \langle \vec{r}_2, \vec{e}_1 \rangle \vec{e}_1|} = \dfrac{E\vec{r}_2 - F\vec{r}_1}{\sqrt{E(EG-F^2)}}$

$$d\vec{r} = \vec{r}_\alpha du^\alpha = \left(\sqrt{E}\,du^1 + \frac{F}{\sqrt{E}}du^2\right)\vec{e}_1 + \sqrt{\frac{EG-F^2}{E}}\,du^2\,\vec{e}_2$$

又 $d\vec{r} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2$

有 $\begin{cases} \omega_1 = \sqrt{E}\,du^1 + \dfrac{F}{\sqrt{E}}du^2 \\ \omega_2 = \sqrt{\dfrac{EG-F^2}{E}}\,du^2 \end{cases}$

又由结构方程 $\begin{cases} d\omega_1 = \omega_{12}\wedge\omega_2 \\ d\omega_2 = \omega_{21}\wedge\omega_1 \\ \omega_{12}+\omega_{21} = 0 \end{cases} \Longrightarrow \omega_{12} = \dfrac{d\omega_1}{\omega_1\wedge\omega_2}\omega_1 + \dfrac{d\omega_2}{\omega_1\wedge\omega_2}\omega_2$

$$d\omega_1 = -(\sqrt{E})_2\,du^1\wedge du^2 + \left(\frac{F}{\sqrt{E}}\right)_1 du^1\wedge du^2 = \left(-\frac{E_2}{2\sqrt{E}} + \frac{2F_1E - FE_1}{2E\sqrt{E}}\right)du^1\wedge du^2$$

$$\omega_1\wedge\omega_2 = \sqrt{EG-F^2}\,du^1\wedge du^2$$

$$d\omega_2 = \left(\sqrt{\frac{EG-F^2}{E}}\right)_1 du^1\wedge du^2 = \sqrt{\frac{E}{EG-F^2}}\cdot\frac{E^2G_1 - 2FF_1E + E_1F^2}{2E^2}$$

$$\frac{\dfrac{(E_1G+EG_1-2FF_1)E - E_1(EG-F^2)}{E^2}}{2\sqrt{\dfrac{EG-F^2}{E}}} = \frac{E^2G_1 - 2FF_1E + E_1F^2}{2E^2\sqrt{\dfrac{EG-F^2}{E}}}$$

$$\omega_{12} = \frac{d\omega_1}{\omega_1\wedge\omega_2}\omega_1 + \frac{d\omega_2}{\omega_1\wedge\omega_2}\omega_2$$

$$= \left(-\frac{E_2}{2\sqrt{E}} + \frac{2F_1E-FE_1}{2E\sqrt{E}}\right)\cdot\frac{1}{\sqrt{EG-F^2}}\left(\sqrt{E}\,du^1 + \frac{F}{\sqrt{E}}du^2\right)$$

$$+ \frac{\sqrt{E}}{EG-F^2}\cdot\frac{E^2G_1 - 2FF_1E + E_1F^2}{2E^2}\cdot\sqrt{\frac{EG-F^2}{E}}\,du^2$$

$$= \frac{1}{2\sqrt{EG-F^2}}\left(-E_2 + 2F_1 - \frac{FE_1}{E}\right)du^1$$

$$+ \left(\frac{F}{2\sqrt{EG-F^2}}\left(-\frac{E_2}{E} + \frac{2F_1E - FE_1}{E^2}\right) + \frac{1}{2\sqrt{EG-F^2}}\cdot\frac{E^2G_1 - 2FF_1E + E_1F^2}{E^2}\right)du^2$$

$$\frac{1}{2\sqrt{EG-F^2}}\left(-\frac{F}{E}E_2 + G_1\right)$$

$$= \frac{\sqrt{EG-F^2}}{E}\Gamma^2_{11}\,du^1 + \frac{\sqrt{EG-F^2}}{E}\Gamma^2_{12}\,du^2$$

$$\frac{-FEE_2 + \cancel{2FF_1E} - \cancel{F^2E_1} + E^2G_1 - \cancel{2FF_1E} + \cancel{E_1F^2}}{E^2}$$

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9. 取沿曲线 C 的与 $\vec{v}$ 垂直的单位方向场 $\vec{w}$，

则 $\dfrac{D\vec{v}}{dt}=0 \iff \begin{cases} \langle \dfrac{D\vec{v}}{dt}, \vec{v} \rangle = 0 & (1) \\ \langle \dfrac{D\vec{v}}{dt}, \vec{w} \rangle = 0 & (2) \end{cases}$

而 $0 = d|\vec{v}|^2 = \langle \dfrac{d\vec{v}}{dt}, \vec{v} \rangle = \langle \dfrac{D\vec{v}}{dt}, \vec{v} \rangle$，故 (1) 式恒成立.

$$\left(\begin{aligned} & D\vec{v} = \langle d\vec{v}, \vec{v} \rangle \vec{v} + \langle d\vec{v}, \vec{w} \rangle \vec{w} \\ & d\vec{v} = \langle d\vec{v}, \vec{v} \rangle \vec{v} + \langle d\vec{v}, \vec{w} \rangle \vec{w} + \langle d\vec{v}, \vec{e}_3 \rangle \vec{e}_3 \\ & d\vec{v} - D\vec{v} = \langle d\vec{v}, \vec{e}_3 \rangle \vec{e}_3 \qquad \frac{d\vec{v}}{dt} - \frac{D\vec{v}}{dt} = \langle \frac{d\vec{v}}{dt}, \vec{e}_3 \rangle \vec{e}_3 \\ & \Rightarrow \langle \frac{d\vec{v}}{dt}, \vec{v} \rangle = \langle \frac{D\vec{v}}{dt}, \vec{v} \rangle \end{aligned} \right)$$

故 $\vec{v}$ 沿曲线平行当且仅当 (2) 成立.

$$而\ \langle \frac{D\vec{v}}{dt}, \vec{w} \rangle = \langle \frac{D(\frac{1}{\sqrt{E}}\vec{r}_1)}{dt}, \vec{w} \rangle = \frac{d}{dt}\left(\frac{1}{\sqrt{E}}\right)\langle \vec{r}_1, \vec{w} \rangle + \frac{1}{\sqrt{E}} \langle \frac{d\vec{r}_1}{dt}, \vec{w} \rangle$$

$$= \frac{1}{\sqrt{E}} \langle \Gamma_{1\alpha}^{\beta} \frac{du^\alpha}{dt} \cdot \vec{r}_\beta, \vec{w} \rangle$$

$$= \frac{1}{\sqrt{E}} \Gamma_{1\alpha}^{2} \frac{du^\alpha}{dt} \langle \vec{r}_2, \vec{w} \rangle$$

由曲面参数的选取，可知 $\langle \vec{r}_2, \vec{w} \rangle \neq 0$.

从而 (2) 式成立 $\iff \Gamma_{1\alpha}^{2} \dfrac{du^\alpha}{dt} = 0$ #

也可用 6　有 $\langle \dfrac{D\vec{v}}{dt}, \vec{w} \rangle = \langle \dfrac{D\vec{e}_1}{dt}, \vec{e}_2 \rangle = \langle d\vec{e}_1, \vec{e}_2 \rangle \vec{e}_2$，$\langle d\vec{e}_1, \vec{e}_2 \rangle = \omega_{12}$.

10. (1) 证：$I = a^2 du du + a^2 \cos^2 u\, dv dv$　(u,v) 为正交参数

$$k_g = \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}} \frac{\partial \ln E}{\partial v} \cos\theta + \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial u} \sin\theta$$

$$= \frac{d\theta}{ds} - \frac{\sin u}{a\cos u} \sin\theta$$

$$\vec{r}_u = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)$$
$$\vec{r}_v = (-a\cos u\sin v, a\cos u\cos v, 0)$$
$$\frac{d\vec{r}}{ds} = \vec{r}_u \frac{du}{ds} + \vec{r}_v \frac{dv}{ds}$$

$$而\ \sin\theta = \frac{1}{|\vec{r}_v|} \langle \frac{d\vec{r}}{ds}, \vec{r}_v \rangle = a\cos u \frac{dv}{ds}$$

$$\therefore k_g = \frac{d\theta}{ds} - \sin u \frac{dv}{ds}$$

(2) 纬线：$C:(u_0, \frac{1}{a\cos u_0}s)$ $\theta \equiv \frac{\pi}{2}$

代入(1)中结论，即有 $k_g = -\frac{1}{a}\tan u_0$.

12. (1) $k_g = \langle \frac{D\vec{e}_1}{ds}, \vec{e}_2 \rangle = \langle \frac{d\vec{e}_1}{ds}, \vec{e}_2 \rangle = \lambda_1$

$k_n = \langle \frac{d\vec{e}_1}{ds}, \vec{e}_3 \rangle = \lambda_2$

(2) 证：

$$(e_3, \dot{e}_3, e_1) = (e_1, e_3, \dot{e}_3)$$
$$= \langle e_1 \wedge e_3, \dot{e}_3 \rangle$$
$$= \langle -e_2, \dot{e}_3 \rangle$$
$$= \lambda_3 = \tau_g.$$

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13. 证：设 $\vec{w} = e_1$ $\vec{n} \wedge \vec{w} = e_2$

$\vec{v} = \overline{e_1}$ $\vec{n} \wedge \vec{v} = \overline{e_2}$

$$\begin{cases} \overline{e_1} = \cos\theta\, e_1 + \sin\theta\, e_2 \\ \overline{e_2} = -\sin\theta\, e_1 + \cos\theta\, e_2 \end{cases}$$

$$\langle \frac{D\overline{e_1}}{dt}, \overline{e_2} \rangle = \langle \frac{D(\cos\theta\, e_1 + \sin\theta\, e_2)}{dt}, -\sin\theta\, e_1 + \cos\theta\, e_2 \rangle$$

$$= \langle (-\sin\theta\, e_1 + \cos\theta\, e_2)\frac{d\theta}{dt}, -\sin\theta\, e_1 + \cos\theta\, e_2 \rangle$$

$$+ \langle \frac{\cos\theta\, De_1 + \sin\theta\, De_2}{dt}, -\sin\theta\, e_1 + \cos\theta\, e_2 \rangle$$

$$= \frac{d\theta}{dt} + \cos^2\theta \langle \frac{De_1}{dt}, e_2 \rangle - \sin^2\theta \langle \frac{De_2}{dt}, e_1 \rangle$$

$\langle e_1, e_2 \rangle = 0$
$\downarrow$
$\langle De_1, e_2 \rangle + \langle e_1, De_2 \rangle = 0$

$$= \frac{d\theta}{dt} + \langle \frac{De_1}{dt}, e_2 \rangle$$

(用$-\theta$代替θ即可)

从而 $\langle \frac{D\vec{w}}{dt}, \vec{n} \wedge \vec{w} \rangle - \langle \frac{D\vec{v}}{dt}, \vec{n} \wedge \vec{v} \rangle = \frac{d\theta}{dt}$

14. 证: $\vec{r}_u = (f'(u)\cos v, f'(u)\sin v, g'(u))$

$\vec{r}_v = (-f(u)\sin v, f(u)\cos v, 0)$

$E = (f')^2 + (g')^2 \qquad F = 0 \qquad G = f^2$

利用 Liouville 公式 $\qquad \vec{r}_u \dfrac{du}{ds}$

$$k_g = \frac{d\theta}{ds} + \frac{1}{2\sqrt{(f')^2+(g')^2}} \cdot \frac{1}{f^2} \cdot 2ff'\sin\theta$$

$$= \frac{d\theta}{ds} + \frac{f'}{f\sqrt{(f')^2+(g')^2}}\sin\theta = 0 \Rightarrow f\sqrt{(f')^2+(g')^2}\frac{d\theta}{ds} + f'\sin\theta = 0$$

$$\cos\theta = \frac{1}{|\vec{r}_u|} \cdot \left\langle \frac{d\vec{r}}{ds}, \vec{r}_u \right\rangle = \sqrt{(f')^2+(g')^2}\frac{du}{ds}$$

于是 $\qquad f\cos\theta \cdot \dfrac{d\theta}{ds} + f'\sin\theta\dfrac{du}{ds} = 0$

$$\Rightarrow (f(u)\sin\theta)' = 0$$

$$f(u)\sin\theta = \text{常数}$$

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