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1.(2) 取参数化

$$\vec{r}(t) = (a\cos t,\ b\sin t) \qquad (0 \le t \le 2\pi)$$

$$s(t) = \int_0^t |\vec{r}'(t)|\,dt = \int_0^t \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$$

$$= a\int_0^t \sqrt{1-e^2\sin^2 t}\,dt \qquad e=\sqrt{1-\frac{b^2}{a^2}} \text{ 为离心率,}$$

弧长为 $s(2\pi) = a\int_0^{2\pi}\sqrt{1-e^2\sin^2 t}\,dt$

$$= 4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2 t}\,dt =: 4aE(e)$$

$E(e)$ 为熟知的第二类完全椭圆积分

$$\frac{ds}{dt} = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$\vec{t} = \frac{d\vec{r}}{dt}\cdot\frac{dt}{ds} = \frac{(-a\sin t,\ b\cos t)}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

$$\Rightarrow \vec{n} = \frac{(-b\cos t,\ -a\sin t)}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

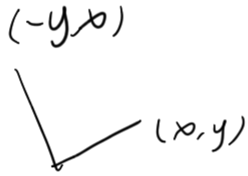
$$\dot{\vec{t}} = \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{dt}\cdot\frac{dt}{ds}$$

$$= \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}\cdot\frac{ab(-b\cos t,\ a\sin t)}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}$$

$$K = \frac{\dot{\vec{t}}}{\vec{n}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}$$

(3) $\vec{r}(t) = (a\cosh t, b\sinh t)$

$$s(t) = \int_0^t |\vec{r}\,'(t)|\,dt = \int_0^t \sqrt{a^2\sinh^2 t + b^2\cosh^2 t}\,dt$$



$$\frac{ds}{dt} = \sqrt{a^2\sinh^2 t + b^2\cosh^2 t}$$

$$\vec{t} = \frac{d\vec{r}}{s} = \frac{d\vec{r}}{dt}\cdot\frac{dt}{ds} = \frac{(a\sinh t, b\cosh t)}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}$$

$$\vec{n} = \frac{(-b\cosh t, a\sinh t)}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}$$

$$\dot{\vec{t}} = \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{dt}\cdot\frac{dt}{ds} = \frac{1}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}\cdot\frac{ab(b\cosh t, -a\sinh t)}{(a^2\sinh^2 t + b^2\cosh^2 t)^{\frac{3}{2}}}$$

$$= \frac{ab(b\cosh t, -a\sinh t)}{(a^2\sinh^2 t + b^2\cosh^2 t)^2}$$

$$\kappa = \frac{\dot{\vec{t}}}{\vec{n}} = -\frac{ab}{(a^2\sinh^2 t + b^2\cosh^2 t)^{\frac{3}{2}}}$$

2. 证: $s(t) = \int_0^t \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2} =: \sqrt{(x')^2 + (y')^2}$$

$$\vec{t} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\cdot\frac{dt}{ds} = \frac{(x'(t), y'(t))}{\sqrt{(x')^2 + (y')^2}}$$

$$\vec{n} = \frac{(-y'(t), x'(t))}{\sqrt{(x')^2 + (y')^2}}$$

$$\dot{\vec{t}} = \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{dt}\cdot\frac{dt}{ds} = \frac{1}{\sqrt{(x')^2 + (y')^2}}\cdot\frac{(x'(t)y''(t) - x''(t)y'(t))(-y'(t), x'(t))}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}}$$

$$\kappa(t) = \frac{\dot{\vec{t}}}{\vec{n}} = \frac{x'(t)y''(t) - x''(t)y'(t)}{[(x')^2 + (y')^2]^{\frac{3}{2}}}$$

3. 证: 作变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\Rightarrow \vec{r}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$$

由 Z.
$$K(\theta)=\frac{x'(\theta)y''(\theta)-x''(\theta)y'(\theta)}{[(x')^2+(y')^2]^{\frac{3}{2}}}$$

$$x'=f'(\theta)\cos\theta-f(\theta)\sin\theta \qquad y'=f'(\theta)\sin\theta+f(\theta)\cos\theta$$

$$x''=f''(\theta)\cos\theta-2f'(\theta)\sin\theta-f(\theta)\cos\theta$$

$$y''=f''(\theta)\sin\theta+2f'(\theta)\cos\theta-f(\theta)\sin\theta$$

$$=\frac{(f'(\theta)\cos\theta-f(\theta)\sin\theta)(f''(\theta)\sin\theta+2f'(\theta)\cos\theta)-f(\theta)\sin\theta) - (f'(\theta)\sin\theta+f(\theta)\cos\theta)(f''(\theta)\cos\theta-2f'(\theta)\sin\theta-f(\theta)\cos\theta)}{[(f'(\theta)\cos\theta-f(\theta)\sin\theta)^2+(f'(\theta)\sin\theta+f(\theta)\cos\theta)^2]^{\frac{3}{2}}}$$

$$=\frac{f^2(\theta)+2(f'(\theta))^2-f(\theta)f''(\theta)}{[f^2(\theta)+(f'(\theta))^2]^{\frac{3}{2}}}=\frac{f^2(\theta)+2\left(\frac{df}{d\theta}\right)^2-f(\theta)\frac{d^2f}{d\theta^2}}{[f^2(\theta)+\left(\frac{df}{d\theta}\right)^2]^{\frac{3}{2}}}$$

5. 证:
$$s(t)=\int_0^t|\vec{r}'(u)|du$$

$$d(\vec{r}'\cdot\vec{r}')=0$$
$$\vec{r}\cdot\vec{r}''+\vec{r}'\cdot\vec{r}'=0$$
$$\frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}}{\sqrt{\ }}$$

$$\frac{ds}{dt}=|\vec{r}'(t)|$$

$$\vec{t}(t)=\frac{d\vec{r}}{ds}=\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}=\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$K(t)=\left|\frac{d\vec{t}}{dt}\cdot\frac{dt}{ds}\right|=\frac{1}{|\vec{r}'(t)|}\cdot\left|\frac{d\frac{\vec{r}'(t)}{|\vec{r}'(t)|}}{dt}\right|$$

$$\frac{d\frac{\vec{r}'(t)}{|\vec{r}'(t)|}}{dt}=\frac{(\dot{y}(\dot{x}\ddot{y}-\ddot{x}\dot{y})-\dot{z}(\ddot{x}\dot{z}-\dot{x}\ddot{z}),\ \dot{z}(\ddot{y}\dot{z}-\dot{y}\ddot{z})-\dot{x}(\dot{y}\ddot{x}-\ddot{y}\dot{x}),\ \dot{x}(\ddot{z}\dot{x}-\dot{z}\ddot{x})-\dot{y}(\dot{z}\ddot{y}-\ddot{z}\dot{y}))}{|\vec{r}'(t)|^3}$$

取模长, 即有 $\left|\frac{d\frac{\vec{r}'(t)}{|\vec{r}'(t)|}}{dt}\right|=\frac{|\vec{r}'\wedge\vec{r}''|}{|\vec{r}'|^2}$

过于难算了, 感觉可以不用算.

$$\Rightarrow K(t)=\frac{|\vec{r}'\wedge\vec{r}''|}{|\vec{r}'|^3}$$

$$\dot{\vec{t}}(t)=\frac{d\left(\vec{r}'(t)\cdot\frac{dt}{ds}\right)}{ds}=\vec{r}''(t)\left(\frac{dt}{ds}\right)^2+\vec{r}'(t)\cdot\frac{d^2t}{ds^2}$$

$$\vec{n}(t)=\frac{1}{K(t)}\dot{\vec{t}}(t)$$

$$\vec{b}(t)=\vec{t}(t)\wedge\vec{n}(t)=\left(\vec{r}'(t)\cdot\frac{dt}{ds}\right)\wedge\left(\frac{1}{K(t)}\cdot\left(\vec{r}''(t)\left(\frac{dt}{ds}\right)^2+\vec{r}'(t)\cdot\frac{d^2t}{ds^2}\right)\right)$$
$$=\frac{1}{K(t)}\cdot\left(\frac{dt}{ds}\right)^3\vec{r}'(t)\wedge\vec{r}''(t)$$

又 $|\vec{b}|=1$

$$\Rightarrow \quad \kappa(t)=\left(\frac{dt}{ds}\right)^3\left|\vec{r}'(t)\wedge\vec{r}''(t)\right|=\frac{|\vec{r}'\wedge\vec{r}''|}{|\vec{r}'|^3}$$

聪明多了

$$\dot{\vec{t}}(t)=\kappa(t)\vec{n}(t)$$

$$\ddot{\vec{t}}(t)=\dot{\kappa}(t)\vec{n}(t)+\kappa(t)\dot{\vec{n}}(t)=\dot{\kappa}(t)\vec{n}(t)+\kappa(t)\left(-\kappa(t)\vec{t}(t)+\tau(t)\vec{b}(t)\right)$$

$$=-\kappa^2(t)\vec{t}(t)+\dot{\kappa}(t)\vec{n}(t)+\kappa(t)\tau(t)\vec{b}(t)$$

$$\Rightarrow \quad \kappa(t)\tau(t)=\langle\ddot{\vec{t}}(t),\vec{b}(t)\rangle$$

$$\ddot{\vec{t}}(t)=\frac{d(\dot{\vec{t}}(t))}{ds}=\vec{r}'''(t)\cdot\left(\frac{dt}{ds}\right)^3+3\cdot\vec{r}''(t)\cdot\frac{dt}{ds}\cdot\left(\frac{d^2t}{ds^2}\right)+\vec{r}'(t)\cdot\frac{d^3t}{ds^3}$$

$$\langle\ddot{\vec{t}}(t),\vec{b}(t)\rangle$$

$$=\left\langle\vec{r}'''(t)\cdot\left(\frac{dt}{ds}\right)^3+3\cdot\vec{r}''(t)\cdot\frac{dt}{ds}\cdot\left(\frac{d^2t}{ds^2}\right)+\vec{r}'(t)\cdot\frac{d^3t}{ds^3},\ \frac{1}{\kappa(t)}\cdot\left(\frac{dt}{ds}\right)^3\vec{r}'(t)\wedge\vec{r}''(t)\right\rangle$$

$$=\frac{1}{\kappa(t)}\cdot\left(\frac{dt}{ds}\right)^6\left(\vec{r}',\vec{r}''(t),\vec{r}'''(t)\right)$$

$$\Rightarrow \quad \tau(t)=\frac{1}{\kappa^2(t)}\left(\frac{dt}{ds}\right)^6\left(\vec{r}'(t),\vec{r}''(t),\vec{r}'''(t)\right)$$

$$=\frac{(\vec{r}',\vec{r}'',\vec{r}''')}{|\vec{r}'\wedge\vec{r}''|^2}$$

b. $|\vec{r}'(s)|=\sqrt{\frac{1+s}{4}+\frac{1-s}{4}+\frac{1}{2}}=1$ (验证 s 为弧长参数)

$$\vec{r}'(s)=\left(\frac{\sqrt{1+s}}{2},-\frac{\sqrt{1-s}}{2},\frac{1}{\sqrt{2}}\right)$$

$$\vec{r}''(s)=\left(\frac{1}{4\sqrt{1+s}},\frac{1}{4\sqrt{1-s}},0\right)$$

$$\vec{r}'''(s)=\left(-\frac{1}{8(1+s)^{\frac{3}{2}}},\frac{1}{8(1-s)^{\frac{3}{2}}},0\right)$$

$$\kappa(t)=\frac{|r'\wedge r''|}{|r'|^3}=\left|\left(-\frac{1}{4\sqrt{2(1-s)}},\frac{1}{4\sqrt{2(1+s)}},\frac{1}{4\sqrt{(1-s)(1+s)}}\right)\right|=\frac{1}{2\sqrt{2(1-s^2)}}$$

$$\tau(t)=\frac{(r',r'',r''')}{|r'\wedge r''|^2}=\frac{\frac{1}{16\sqrt{2}(1-s^2)^{\frac{3}{2}}}}{\frac{1}{8(1-s^2)}}=\frac{1}{2\sqrt{2}(1-s^2)}$$

而 $\vec{t}(s)=\dfrac{\vec{r}'(s)}{|\vec{r}'(s)|}=\vec{r}'(s)=\left(\dfrac{\sqrt{1+s}}{2},-\dfrac{\sqrt{1-s}}{2},\dfrac{1}{\sqrt{2}}\right)$

$$\vec{n}(s)=\frac{\vec{r}''(s)}{|\vec{r}''(s)|}=\left(\sqrt{\frac{1-s}{2}},\sqrt{\frac{1+s}{2}},0\right)$$

$$\vec{b}(s)=\frac{\vec{r}'\wedge\vec{r}''}{|\vec{r}'\wedge\vec{r}''|}=\left(-\frac{\sqrt{1+s}}{2},\frac{\sqrt{1-s}}{2},\frac{1}{\sqrt{2}}\right)$$

Frenet标架为 $\{\vec{r}(s);\ \vec{t}(s),\vec{n}(s),\vec{b}(s)\}$

7. (1) 要证 $\vec{r}(t)$ 是一条正则曲线，只需证 $\vec{r}(t)$ 在 $t=0$ 处是 C^∞ 的.

记 $\vec{r}(t)=(x(t),y(t),z(t))$

当 $t\neq 0$ 时，$x(t),y(t),z(t)$ 显然都是 C^∞ 的.

而容易证明：$\left(e^{-\frac{1}{t^2}}\right)^{(k)}=\dfrac{P_k(t)}{t^{n_k}}\cdot e^{-\frac{1}{t^2}}$，$n_k$ 为正整数

y(t) 显然 C^∞

而对于 $z(t)$，

$z'_+(t)=\lim\limits_{t\to0^+}\dfrac{z(t)-z(0)}{t}=\lim\limits_{t\to0^+}\dfrac{e^{-\frac{1}{t^2}}}{t}=\lim\limits_{t\to0^+}\dfrac{\frac{2}{t}e^{-\frac{1}{t^2}}}{1}$

证明：归纳法 $k=1$ 时，$\left(e^{-\frac{1}{t^2}}\right)'=\dfrac{2}{t^3}e^{-\frac{1}{t^2}}$ $\quad P_1(t)=2\quad n_1=3$

设 $k=n$ 时成立，

则 $\left(e^{-\frac{1}{t^2}}\right)^{(k+1)}=\left(\dfrac{P_k(t)}{t^{n_k}}e^{-\frac{1}{t^2}}\right)'$

$2P_k(t)+t^3P_k'(t)$
$-n_kt^2P_k(t)$

$$=\frac{\left(\frac{2P_k(t)}{t^3}+P_k'(t)\right)e^{-\frac{1}{t^2}}\cdot t^{n_k}-n_kt^{n_k-1}P_k(t)\cdot e^{-\frac{1}{t^2}}}{(t^{n_k})^2}$$

$$=\frac{(2-n_kt^2)P_k(t)+t^3P_k'(t)}{t^{n_k+3}}e^{-\frac{1}{t^2}}$$

$P_{k+1}(t)=(2-n_kt^2)P_k(t)+t^3P_k'(t)\qquad n_{k+1}=n_k+3=\cdots=3(k+1)$

#.

① $t<0$ 时 故 $\vec{r}'(t)=\left(\dfrac{2}{t}e^{-\frac{1}{t^2}},1,0\right)\neq0$.

$\vec{r}^{(k)}(t)=\left(\dfrac{P_k(t)}{t^{3k}}\cdot e^{-\frac{1}{t^2}},0,0\right)\neq0\quad k>2$

② $t>0$ 时，$\vec{r}'(t)=\left(0,1,\dfrac{2}{t}e^{-\frac{1}{t^2}}\right)\neq0$

$\vec{r}^{(k)}(t)=\left(0,0,\dfrac{P_k(t)}{t^{3k}}e^{-\frac{1}{t^2}}\right)\neq0$

故当 $t\neq0$ 时，$r(t)$ 是光滑的，且 $\vec{r}'(t)\neq0$. 故是正则的.

当 $t=0$ 时，$y(t)$ 是 C^∞ 的，只需证明 $x(t)$ 与 $z(t)$ 光滑.

$$z^{(k)}(t)=\frac{P_k(t)}{t^{k+1}}e^{-\frac{1}{t^2}},\quad t>0$$

$$z^{(k)}(t)=0,\quad t<0$$

显然 $t=0$ 处各阶左导数 $z_-^{(k)}(t)=0$

下面计算各阶右导数

$$z_+'(0)=\lim_{t\to 0^+}\frac{z(t)-0}{t}=\lim_{t\to 0^+}\frac{e^{-\frac{1}{t^2}}}{t}=\lim_{u\to+\infty}\frac{u}{e^{u^2}}=\lim\frac{1}{2ue^{u^2}}=0=z_-'(0)$$

递推地，$$z_+^{(k)}(t)=\lim_{t\to 0^+}\frac{z^{(k-1)}(t)-z^{(k-1)}(0)}{t}=\lim_{t\to 0^+}\frac{\frac{P_{k-1}(t)}{t^{3(k-1)}}e^{-\frac{1}{t^2}}}{t}=\lim_{u\to+\infty}\frac{P_{k-1}'(u)}{e^{u^2}}=0=z_-^{(k)}(0)$$

（易知 $\deg P_k=2(k-1)$，P_{k-1}' 也是一个多项式）

故 $z(t)$ 在 $t=0$ 处也光滑

类似地，$x(t)$ 在 $t=0$ 处也光滑

$t=0$ 时，有 $\vec r'(0)=(0,1,0)$

$\vec r''(0)=(0,0,0)$

$$\Rightarrow k(0)=\frac{|\vec r'(0)\wedge\vec r''(0)|}{|\vec r'(0)|^3}=0$$ #

(2)

$$\vec t(t)=\frac{\vec r'(t)}{|\vec r'(t)|}=\begin{cases}\frac{1}{(1+\frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac12}}\left(\frac{2}{t^3}e^{-\frac{1}{t^2}},1,0\right), & t<0\\ \frac{1}{(1+\frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac12}}\left(0,1,\frac{2}{t^3}e^{-\frac{1}{t^2}}\right), & t>0\end{cases}$$

判断 $\vec t(t)$ 方向变化，判断 $\vec n(t)$，有

$$\vec n(t)=\begin{cases}\frac{1}{(1+\frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac12}}\left(\mathrm{sgn}(-6t^2+4),\frac{2\,\mathrm{sgn}(6t^2-4)}{t^3}e^{-\frac{1}{t^2}},0\right), & t<0\\ \frac{1}{(1+\frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac12}}\left(0,\frac{2\,\mathrm{sgn}(6t^2-4)}{t^3}e^{-\frac{1}{t^2}},\mathrm{sgn}(-6t^2+4)\right), & t>0\end{cases}$$

$$\vec b(t)=\vec t(t)\wedge\vec n(t)=\begin{cases}(0,0,\mathrm{sgn}(6t^2-4)), & t<0\\ (\mathrm{sgn}(-6t^2+4),0,0), & t>0\end{cases}$$

Frenet 标架即为 $(\vec r(t);\vec t(t),\vec n(t),\vec b(t))$

当 $t\to 0^-$ 时，Frenet 标架的极限为 $\{\vec r(0)=0;\vec t(0^-),\vec n(0^-),\vec b(0^-)\}$

$\vec t(0^-)=(0,1,0)$ $\quad\vec n(0^-)=(1,0,0)$ $\quad\vec b(0^-)=(0,0,-1)$

当 $t \to 0^+$ 时，Frenet 标架的极限为 $\{\vec{r}(0^+)=0;\vec{t}(0^+),\vec{n}(0^+),\vec{b}(0^+)\}$

$\vec{t}(0^+)=(0,1,0) \quad \vec{n}(0^+)=(0,0,1) \quad \vec{b}(0^+)=(1,0,0)$

9. (1) 证：设 C 曲线方程为 $\vec{r}(s)$，s 为弧长参数

定点为 $\vec{p}_0$

有 $\vec{r}(s)-\vec{p}_0=\lambda(s)\vec{t}(s)$

其中 $\lambda(s)=\langle\vec{r}(s)-\vec{p}_0,\vec{t}(s)\rangle$

两边对 s 求导，有

$$\vec{t}(s)=\lambda'(s)\vec{t}(s)+\lambda(s)\vec{t}'(s)$$

$$=\lambda'(s)\vec{t}(s)+\lambda(s)\kappa(s)\vec{n}(s)$$

由于 $\vec{t}$ 与 $\vec{n}$ 线性无关

故 $\lambda'(s)=1 \qquad \lambda(s)\kappa(s)=0$

$\Rightarrow \lambda(s)$ 不恒为零（至多一处） $\kappa(s)=0$

又由 $\kappa(s)$ 连续性，$\Rightarrow \kappa(s)\equiv 0$

$\Rightarrow C$ 为直线

(2) 证：设 C 曲线方程为 $\vec{r}(s)$

$$\vec{r}(s)-\vec{p}_0=\lambda(s)\vec{n}(s)$$

$$\lambda(s)=\langle\vec{r}(s)-\vec{p}_0,\vec{n}(s)\rangle$$

对两边求导

$$\vec{t}(s)=\lambda'(s)\vec{n}(s)+\lambda(s)\vec{n}'(s)$$

$-\kappa(s)\vec{t}(s)+\tau(s)\vec{b}(s)$

$$=-\lambda(s)\kappa(s)\vec{t}(s)+\lambda'(s)\vec{n}(s)+\lambda(s)\tau(s)\vec{b}(s)$$

$\Rightarrow -\lambda(s)\kappa(s)=1$

$\lambda'(s)=0 \qquad \lambda(s)\tau(s)=0$

$\lambda(s)\equiv c$（常数） 若 $\lambda(s)=c=0$，则 C 为 p_0 点，不是曲线

则 $\lambda(s) \neq 0$， $\Rightarrow$ $\tau(s) \equiv 0$

$\Rightarrow$ C 在某个平面中

$K(s) = -\frac{1}{\lambda(s)} = -\frac{1}{c} \neq 0$ $K(s)$ 为常数 $-\frac{1}{c}$ $(\neq 0)$

$\Rightarrow$ C 为以 p_0 为圆心，半径为 $|c|$ 的圆