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习题四

7. 证：设主曲率为 k_1, k_2

若 $k_1 = k_2$，则曲面为全脐点曲面

若 $k_1 \neq k_2$，则可设 $k_1 > H > k_2$

取 (u,v) 是曲面的正交曲率线网，由 6，有 $\begin{cases} L_v = HE_v \\ N_u = HG_u \end{cases}$

故 $L = HE + f(u) \quad N = HG + g(v)$

而 $k_1 = \frac{L}{E} = H + \frac{f(u)}{E}$，$k_2 = \frac{N}{G} = H + \frac{g(v)}{G}$ $\quad f(u) > 0 \quad g(v) < 0$.

而由 $k_1 + k_2 = 2H$，得 $\frac{f(u)}{E} + \frac{g(v)}{G} = 0$

不妨设 $\frac{1}{\lambda(u,v)} := \frac{f(u)}{E} = -\frac{g(v)}{G}$

有 $E = \lambda f(u) \quad G = -\lambda g(v)$

于是 $\mathrm{I} = \lambda f(u)\,du\,du - \lambda g(v)\,dv\,dv$

$\mathrm{II} = f(u)(1+\lambda H)\,du\,du + g(v)(1-\lambda H)\,dv\,dv$

于是作参数变换：

$$\begin{cases} \tilde{u} = \int \sqrt{f(u)}\,du \\ \tilde{v} = \int \sqrt{-g(v)}\,dv \end{cases}$$

便有 $\mathrm{I} = \lambda(\tilde{u}, \tilde{v})(d\tilde{u}\,d\tilde{u} + d\tilde{v}\,d\tilde{v})$

$\mathrm{II} = (1+\lambda H)\,d\tilde{u}\,d\tilde{u} - (1-\lambda H)\,d\tilde{v}\,d\tilde{v}$ #

8. 证：取正交曲率线网 (u,v)，

$\mathrm{I} = E\,du\,du + G\,dv\,dv \quad \mathrm{II} = L\,du\,du + N\,dv\,dv \quad E, G, L, N$ 都为常数

故 $k_1 = \frac{L}{E}$，$k_2 = \frac{N}{G}$ 都为常数.

由 Lecture 12 例 2，

知曲面只能是平面、球面、圆柱面

而球面的第一、二基本形式不为常数，

平面和圆柱面符合条件. 故只能是平面或圆柱面.

9. (1) $E=1 \quad G=1 \quad L=1 \quad N=1$

这显然不符合 Gauss 方程: $-\frac{1}{\sqrt{EG}}\left\{\left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v+\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u\right\}=\frac{LN-M^2}{EG-F^2}$

故不存在

(2) $E=1 \quad G=\cos^2 u \quad L=\cos^2 u \quad N=1$

代入 Gauss方程

$$-\frac{1}{\sqrt{\cos^2 u}}\left((\sqrt{\cos^2 u})_{uu}\right)=1=\frac{\cos^2 u}{\cos^2 u} \quad \text{满足}$$

代入 Codazzi 方程 $\quad L_v=HE_v \quad N_u=HG_u$

$$H=\frac{1}{2}\frac{LG+NE}{EG}=\frac{\cos^4 u+1}{2\cos^2 u}$$

而 $\quad N_u=0\neq\frac{\cos^4 u+1}{2\cos^2 u}\cdot\sin 2u$ 故也不存在.

10. 可知 z 局部可写成 $z=z(x,y) \quad (F_z\neq 0)$

$$F(x,y,z)=F(x,y,z(x,y))=0$$

对 x 求偏导 $\quad F_x+F_z z_x=0 \quad\Rightarrow\quad z_x=-\frac{F_x}{F_z}$

对 y 求偏导 $\quad F_y+F_z z_y=0 \qquad\quad z_y=-\frac{F_y}{F_z}$

求二阶导. 有

$$F_{xx}+F_{xz}\cdot z_x+(F_{zz}z_x+F_{zx})z_x+F_z z_{xx}=0$$

$$F_{xy}+F_{xz}\cdot z_y+(F_{zz}z_y+F_{zy})z_x+F_z z_{xy}=0$$

$$(\,F_{yx}+F_{yz}\cdot z_x+(F_{zz}z_x+F_{zx})z_y+F_z z_{yx}=0\,)$$

$$F_{yy}+F_{yz}\cdot z_y+(F_{zz}z_y+F_{zy})z_y+F_z z_{yy}=0$$

$$\Rightarrow\quad z_{xx}=\frac{-F_{xx}F_z^2+2F_{xz}F_xF_z-F_{zz}F_x^2}{F_z^3}=\frac{A}{F_z^3}$$

$$Z_{xy} = \frac{-F_{xy}\cdot F_z^2 + F_{xz}\cdot F_y\cdot F_z + F_{yz}\cdot F_x\cdot F_z - F_{zz}\cdot F_x\cdot F_y}{F_z^3} = \frac{B}{F_z^3}$$

$$Z_{yy} = \frac{-F_{yy}F_z^2 + 2F_{yz}F_y\cdot F_z - F_{zz}F_y^2}{F_z^3} = \frac{C}{F_z^3}$$

$$\vec{r}_x = (1, 0, Z_x) \quad \vec{r}_y = (0, 1, Z_y)$$

$$\vec{r}_{xx} = (0, 0, Z_{xx}) \quad \vec{r}_{xy} = (0, 0, Z_{xy}) \quad \vec{r}_{yy} = (0, 0, Z_{yy})$$

$$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{(F_x, F_y, F_z)}{\sqrt{F_x^2 + F_y^2 + F_y^2}} \quad \left(\vec{n} = \frac{\vec{r}_x \wedge \vec{r}_y}{|\vec{r}_x \wedge \vec{r}_y|} = \frac{(-Z_x, -Z_y, 1)}{\sqrt{1 + Z_x^2 + Z_y^2}}\right.$$

$$\left.= \frac{(F_x, F_y, F_z)}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \text{ 是一致的}\right)$$

$$E = 1 + Z_x^2 \quad F = Z_x Z_y \quad G = 1 + Z_y^2$$

$$L = \frac{1}{|\nabla F|}\cdot F_z \cdot Z_{xx} = \frac{A}{F_z^2 |\nabla F|} \quad M = \frac{B}{F_z^2|\nabla F|} \quad N = \frac{C}{F_z^2|\nabla F|}$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{AC - B^2}{F_z^2 |\nabla F|^4}$$

其中 $AC - B^2 = (-F_{xx}F_z^2 + 2F_{xz}F_xF_z - F_{zz}F_x^2)(-F_{yy}F_z^2 + 2F_{yz}F_y\cdot F_z - F_{zz}F_y^2)$

$$- (-F_{xy}\cdot F_z^2 + F_{xz}\cdot F_y\cdot F_z + F_{yz}\cdot F_x\cdot F_z - F_{zz}\cdot F_x\cdot F_y)^2$$

$$= (F_{xx}F_{yy}F_z^4 + F_{xx}F_{zz}F_z^2F_y^2 + F_{yy}F_{zz}\cdot F_x^2F_z^2 + F_{zz}^2F_x^2F_y^2 + 4F_{xz}F_{yz}F_xF_yF_z^2$$

$$-2F_{xx}F_{yz}F_yF_z^3 - 2F_{zz}F_{yz}F_x^2F_yF_z - 2F_{yy}F_{xz}F_xF_z^3 - 2F_{zz}F_{xz}F_xF_y^2F_z)$$

$$-(F_{xy}^2F_z^4 + (F_{xz}\cdot F_y\cdot F_z)^2 + (F_{yz}\cdot F_x\cdot F_z)^2 + (F_{zz}\cdot F_x\cdot F_y)^2 + 2F_{xy}F_{zz}F_xF_yF_z^2$$

$$+2F_{xz}F_{yz}F_xF_yF_z^2 - 2F_{xy}F_{xz}F_yF_z^3 - 2F_{xy}F_{yz}F_xF_z^3 - 2F_{xz}F_{zz}F_xF_y^2F_z$$

$$-2F_{yz}F_{zz}F_x^2F_yF_z)$$

$$
\begin{aligned}
&\big(F_{xx}F_{yy}F_z^4 + F_{xx}F_{zz}F_z^2F_y^2 + F_{yy}F_{zz}\cdot F_xF_z^3 + \cancel{F_{zz}^2F_x^2}F_y^2 + \underline{\cancel{4}^{\,2}F_{xz}F_{yz}F_xF_yF_z^2} \\
&-2F_{xx}F_{yz}F_yF_z^3 - 2\cancel{F_{zz}}F_{yz}F_x^2F_yF_z - 2F_{yy}F_{xz}F_xF_z^3 - 2\cancel{F_{zz}F_{xz}}F_xF_y^2F_z\big) \\
&-\big(F_{xy}^2F_z^4 + (F_{xz}\cdot F_y\cdot F_z)^2 + (F_{yz}\cdot F_x\cdot F_z)^2 + (\cancel{F_{zz}\cdot F_x}\cdot F_y)^2 + 2F_{xy}F_{zz}F_xF_yF_z^2 \\
&\quad \underline{+2F_{xz}\cancel{F_{yz}F_x}F_yF_z^2} - 2F_{xy}^{F_{xz}}\cdot F_yF_z^3 - 2F_{xy}F_{yz}F_xF_z^3 - 2\cancel{F_{xz}F_{zz}}F_xF_y^2F_z \\
&\quad -2F_{yz}\cancel{F_{zz}}F_x^2F_yF_z\big)
\end{aligned}
$$

$$
\begin{aligned}
= F_z^2\big(&F_{xx}F_{yy}F_z^2 + F_{xx}F_{zz}F_y^2 + F_{yy}F_{zz}\cdot F_x^2 \\
&-F_{xy}^2F_z^2 - F_{xz}^2F_y^2 - F_{yz}^2F_x^2 \\
&+2F_{xz}F_{yz}F_xF_y + 2F_{xy}F_{xz}F_yF_z + 2F_{xy}F_{yz}F_xF_z \\
&-2F_{xx}F_{yz}F_yF_z - 2F_{yy}F_{xz}F_xF_z - 2F_{xy}F_{zz}F_xF_y\big)
\end{aligned}
$$

$$
= -F_z^2 \det\begin{bmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{yx} & F_{yy} & F_{yz} & F_y \\ F_{zx} & F_{zy} & F_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{bmatrix} = -F_z^2 \det\begin{bmatrix} \nabla^2F & (\nabla F)^T \\ \nabla F & 0 \end{bmatrix}
$$

于是 $K = -\dfrac{1}{|\nabla F|}\cdot\det\begin{bmatrix} \nabla^2F & (\nabla F)^T \\ \nabla F & 0 \end{bmatrix}$ #

12. (1) 代入 Gauss-Codazzi 方程， $H=\frac{1}{2}\frac{LG+NE}{EG}=\lambda$

有 $-\frac{1}{\sqrt{EG}}\left\{\left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v+\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u\right\}=\frac{LN}{EG}=\lambda^2$

$L_v=HE_v \Rightarrow \lambda_v E+\lambda E_v=\lambda E_v \quad \lambda_v E=0$ $\xRightarrow{E,G>0} \lambda_v=\lambda_u=0$

$N_u=HG_u \Rightarrow \lambda_u G+\lambda G_u=\lambda G_u \quad \lambda_u G=0$

于是 λ 为常数. 并且 $-\frac{1}{\sqrt{EG}}\left\{\left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v+\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u\right\}=\lambda^2$ 时,

φ,ψ 可以作为曲面的第一、第二基本形式.

(2) $E=G$ 时，有 $-\frac{1}{E}\left((\ln\sqrt{E})_{vv}+(\ln\sqrt{E})_{uu}\right)=\lambda^2$

$$\Rightarrow \quad \Delta \ln E+2E\lambda^2=0$$

由于 $\mathrm{II}=\lambda\mathrm{I}$，故曲面为全脐点曲面，只能是平面与球面

若是平面. $\lambda=0$， $\Rightarrow E=G=1$

若是球面. $\lambda\neq0$，球半径为 $\frac{1}{|\lambda|}$ 由第三章，例2.5

$$E=G=\frac{4}{(1+\lambda^2(u^2+v^2))^2}$$

13. 解: $\vec{r}_u=(\cos v,\sin v,f'(u))$ $\vec{r}_v=(-u\sin v,u\cos v,0)$

取正交标架 $\vec{e}_1=\frac{1}{1+(f')^2}(\cos v,\sin v,f'(u))=\frac{1}{1+(f')^2}\vec{r}_u$

$\vec{e}_2=(-\sin v,\cos v,0)=\frac{1}{u}\vec{r}_v$

$\omega_1=\sqrt{1+(f'(u))^2}\,du \qquad \omega_2=u\,dv \qquad E=1+(f')^2 \quad G=u^2 \quad F=0$

$L=\frac{f''}{\sqrt{1+(f')^2}} \qquad M=0 \qquad N=\frac{uf'}{\sqrt{1+(f')^2}}$

注意到 (u,v) 是正交参数 $(F=0)$ 由例6.3

于是 $\omega_{12}=-\frac{(\sqrt{E})_v}{\sqrt{G}}du+\frac{(\sqrt{G})_u}{\sqrt{E}}dv=\frac{du}{\sqrt{1+(f'(u))^2}}$

$\omega_{13}=\frac{L}{\sqrt{E}}du+\frac{M}{\sqrt{E}}dv=\frac{f''(u)}{1+(f'(u))^2}du$

$$\omega_{23} = \frac{M}{\sqrt{G}}du + \frac{N}{\sqrt{G}}dv = \frac{f'}{\sqrt{1+(f'(u))^2}}dv$$

14. 证：设 $\{\vec{e}_1, \vec{e}_2\}$ 与 $\{\tilde{e}_1, \tilde{e}_2\}$ 是两组正交标架，

有 $\begin{cases} \tilde{e}_1 = \cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2 \\ \tilde{e}_2 = -\sin\theta\, \vec{e}_1 + \cos\theta\, \vec{e}_2 \end{cases}$

$$\tilde{\omega}_1 = \langle d\vec{r}, \tilde{e}_1 \rangle = \langle d\vec{r}, \cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2 \rangle = \cos\theta\, \omega_1 + \sin\theta\, \omega_2$$

$$\tilde{\omega}_2 = -\sin\theta\, \omega_1 + \cos\theta\, \omega_2 \quad \Rightarrow \quad \tilde{\omega}_1 \wedge \tilde{\omega}_2 = \omega_1 \wedge \omega_2$$

$$d\tilde{e}_1 = (\cos\theta\, d\vec{e}_1 + \sin\theta\, d\vec{e}_2) + (-\sin\theta\, \vec{e}_1 + \cos\theta\, \vec{e}_2)\, d\theta$$

$$\tilde{\omega}_{12} = \langle d\tilde{e}_1, \tilde{e}_2 \rangle = \omega_{12} + d\theta$$

于是 $d\tilde{\omega}_{12} = d\omega_{12}$

书中也有 $d\omega_{12} = K\omega_1 \wedge \omega_2$
但未注意到.

$$\Rightarrow \frac{d\tilde{\omega}_{12}}{\tilde{\omega}_1 \wedge \tilde{\omega}_1} = \frac{d\omega_{12}}{\omega_1 \wedge \omega_2}$$

故 $\dfrac{d\omega_{12}}{\omega_1 \wedge \omega_2}$ 与正交标架 $\vec{e}_1, \vec{e}_2$ 的选取无关.

15. (1) $\vec{r}_u = (-a\sin u\cos v,\ -a\sin u\sin v,\ a\cos u)$

$\vec{r}_v = (-a\cos u\sin v,\ a\cos u\cos v,\ 0)$

由 $\langle \vec{r}_u, \vec{r}_v \rangle = 0$，可取

$$\vec{e}_1 = \frac{1}{a}\vec{r}_u = (-\sin u\cos v, -\sin u\sin v, \cos u) \quad \Rightarrow \quad \omega_1 = a\,du$$

$$\vec{e}_2 = \frac{1}{a\cos u}\vec{r}_v = (-\sin v, \cos v, 0) \quad \Rightarrow \quad \omega_2 = a\cos u\,dv$$

$$\vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2 = (-\cos u\cos v, -\cos u\sin v, -\sin u)$$

一组正交活动标架为 $\{\vec{r}, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$

(2) $\omega_1 = a\,du \qquad \omega_2 = a\cos u\,dv$

$$d\vec{e}_1 = (-\cos u\cos v, -\cos u\sin v, -\sin u)\,du + (\sin u\sin v, -\sin u\cos v, 0)\,dv$$

$$d\vec{e}_2 = (-\cos v, -\sin v, 0)\,dv$$

$$\omega_{12} = \langle d\vec{e}_1, \vec{e}_2 \rangle = -\sin u\,dv$$

$$\omega_{13} = \langle d\vec{e}_1 , \vec{e}_3 \rangle = du$$

$$\omega_{23} = \langle d\vec{e}_2 , \vec{e}_3 \rangle = \cos u \, dv$$

(3) $\mathrm{II} = \omega_1 \omega_{13} + \omega_2 \omega_{23} = a\,du\,du + a\cos^2 u\,dv\,dv$ #