2. 证: (1)　$|\vec{a}| \equiv C$ (常数

$\Leftrightarrow \vec{a}^2 \equiv C^2$

$\Leftrightarrow \dfrac{d(\vec{a}^2)}{dt} \equiv 0 \quad \Leftrightarrow \dfrac{d(\vec{a}\cdot\vec{a})}{dt} \equiv 0$

$\Leftrightarrow 2\dfrac{d\vec{a}}{dt}\cdot\vec{a} \equiv 0 \quad \Leftrightarrow \langle \vec{a}(t), \vec{a}'(t)\rangle \equiv 0$

(2) "$\Rightarrow$" $\vec{a}(t)$的方向不变 $\Leftrightarrow \vec{a}(t) = \varphi(t)\vec{u}$ 　$\vec{u}$为单位向量 $\exists \varphi(t)$

$\vec{a}'(t) = \varphi'(t)\vec{u}$

$\vec{a}(t)\wedge\vec{a}'(t) = \varphi(t)\varphi'(t)\,\vec{u}\wedge\vec{u} = \vec{0}$

"$\Leftarrow$" 若 $\vec{a}(t)\wedge\vec{a}'(t) = 0$

设 $\vec{a}(t) = \varphi(t)\vec{u}(t)$ 　　$\vec{u}(t)$为单位向量 $\varphi(t)\neq0$

$\vec{a}' = \varphi'\vec{u} + \varphi\vec{u}'$

$\vec{a}\wedge\vec{a}' = \varphi\vec{u}\wedge(\varphi'\vec{u}+\varphi\vec{u}')$

$\qquad = \varphi\varphi'(\vec{u}\wedge\vec{u}) + \varphi^2(\vec{u}\wedge\vec{u}')$

$\qquad = \varphi^2(\vec{u}\wedge\vec{u}') = \vec{0}$

由 $\varphi\neq0$ 　$\vec{u}\wedge\vec{u}' = 0$

同时 由假设 $\vec{u}$ 为单位向量, 由(1) $\langle\vec{u},\vec{u}'\rangle=0$ $\Big\}$

$\qquad\qquad\qquad\qquad\qquad \Rightarrow \vec{u}' = 0$

$\qquad\qquad\qquad\qquad\qquad \vec{u}$不变

即 $\vec{a}(t)$ 的方向不变

4. 证: (1)　完全显然 　　　　　$\sigma(i)\neq\sigma(j)$

有 $|\vec{e}_{\sigma(i)}| = 1$ 　　$\langle\vec{e}_{\sigma(i)}, \vec{e}_{\sigma(j)}\rangle = 0$

(2) $\{O; e_1, e_2, e_3\}$ 与 $\{O; e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}\}$ 定向相同

$\iff |e_1, e_2, e_3| = |e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}|$

又 $|e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}| = (-1)^{N(\sigma)} |e_1, e_2, e_3|$

$N(\sigma)$ 代表置换 σ 的逆序数

$\iff$ $N(\sigma)$ 为偶数 $\iff$ σ 为偶置换.

5. 猜测 $(T\vec{v}) \wedge (T\vec{w}) = \pm T(\vec{v} \wedge \vec{w})$

设: T 为 f 对应的正交矩阵.

$f: X \to XT + P$

设 $\vec{v} = (v^1, v^2, v^3)$ $\vec{w} = (w^1, w^2, w^3)$

记 $T = \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}$ $\vec{a}, \vec{b}, \vec{c}$ 两两正交 且都为单位向量

$(T\vec{v}) \wedge (T\vec{w}) = (v^1\vec{a} + v^2\vec{b} + v^3\vec{c}) \wedge (w^1\vec{a} + w^2\vec{b} + w^3\vec{c})$

$= (v^1 w^2 - v^2 w^1) \vec{a} \wedge \vec{b}$
$+ (v^2 w^3 - v^3 w^2) \vec{b} \wedge \vec{c}$
$+ (v^3 w^1 - v^1 w^3) \vec{c} \wedge \vec{a}$

$T(\vec{v} \wedge \vec{w}) = (v^2 w^3 - v^3 w^2)\vec{a} + (v^3 w^1 - v^1 w^3)\vec{b} + (v^1 w^2 - v^2 w^1)\vec{c}$

① $\vec{a} \wedge \vec{b} = \vec{c}$ $\vec{b} \wedge \vec{c} = \vec{a}$ $\vec{c} \wedge \vec{a} = \vec{b}$ $|T| = 1$

$\Rightarrow (T\vec{v}) \wedge (T\vec{w}) = T(\vec{v} \wedge \vec{w})$

② $\vec{a} \wedge \vec{b} = -\vec{c}$ $\vec{b} \wedge \vec{c} = -\vec{a}$ $\vec{c} \wedge \vec{a} = -\vec{b}$ $|T| = -1$

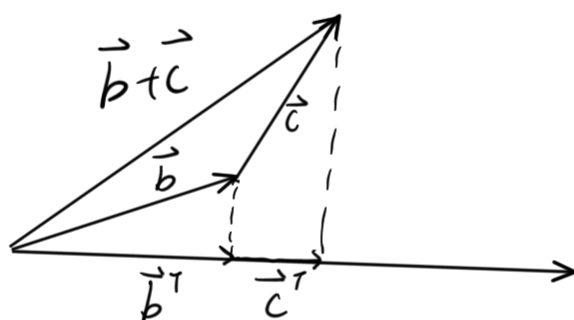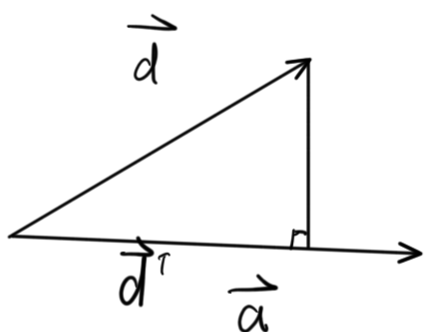$\Rightarrow (T\vec{v}) \wedge (T\vec{w}) = -T(\vec{v} \wedge \vec{w})$

综上， $(T\vec{v}) \wedge (T\vec{w}) = |T| T(\vec{v} \wedge \vec{w})$

当 T 不改变定向时，$(T\vec{v}) \wedge (T\vec{w}) = T(\vec{v} \wedge \vec{w})$

当 T 改变定向时，$(T\vec{v}) \wedge (T\vec{w}) = -T(\vec{v} \wedge \vec{w})$

补充题

1. $\mathbb{R}^3$ 中的内积符合分配律： $\langle \vec{a}, \vec{b} + \vec{c} \rangle = \langle \vec{a}, \vec{b} \rangle + \langle \vec{a}, \vec{c} \rangle$

证：

记 $\vec{d}^T$ 为 $\vec{d}$ 在 $\vec{a}$ 方向上的投影

那么有 $\langle \vec{a}, \vec{d}^T \rangle = \langle \vec{a}, \vec{d} \rangle$
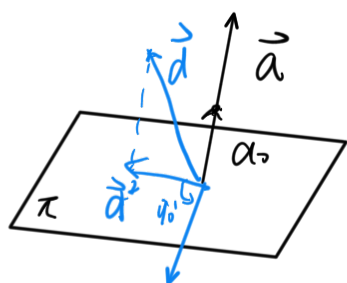
$(\vec{b} + \vec{c})^T = \vec{b}^T + \vec{c}^T$

故 $\langle \vec{a}, \vec{b} + \vec{c} \rangle = \langle \vec{a}, (\vec{b} + \vec{c})^T \rangle = \langle \vec{a}, \vec{b}^T + \vec{c}^T \rangle$

$\qquad = |\vec{a}| \cdot |\vec{b}^T + \vec{c}^T|$

$\qquad = |\vec{a}||\vec{b}^T| + |\vec{a}||\vec{c}^T| = \langle \vec{a}, \vec{b}^T \rangle + \langle \vec{a}, \vec{c}^T \rangle$

$\qquad = \langle \vec{a}, \vec{b} \rangle + \langle \vec{a}, \vec{c} \rangle$

2. $\mathbb{R}^3$ 中的外积满足分配律：

$\qquad \vec{a} \wedge (\vec{b} + \vec{c}) = \vec{a} \wedge \vec{b} + \vec{a} \wedge \vec{c}$

证：若 $\vec{a}, \vec{b}, \vec{c}$ 中有两者共线，或有一者为零向量，容易验

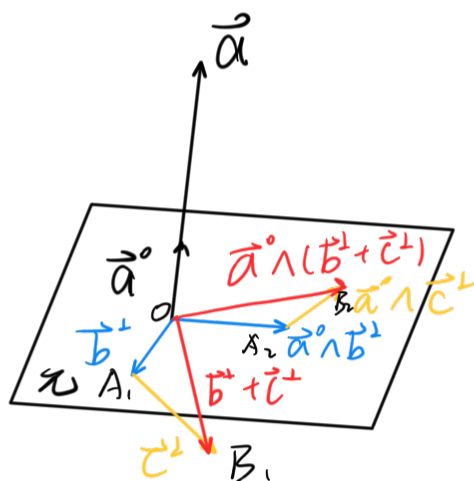下设不是上述情况，记 $\vec{a}^\circ = \dfrac{\vec{a}}{|\vec{a}|}$，$\vec{d}$ 沿 $\vec{a}$ 的投影部分为 $\vec{d}^\perp$

如图，由右手法则，$\vec{a}_\circ \wedge \vec{d}$ 与 $\vec{a}_\circ \wedge \vec{d}^\top$ 方向相同

又 $|\vec{a}^\circ \wedge \vec{d}| = |\vec{a}^\circ| \cdot |\vec{d}| \sin\theta = |\vec{a}^\circ| |\vec{d}^\top| = |\vec{a}^\circ \wedge \vec{d}^\perp|$，

因此 $\vec{a}^\circ \wedge \vec{d} = \vec{a}^\circ \wedge \vec{d}^\perp$．

由 $\vec{a}^\circ \perp \vec{d}^\perp$，且 $|\vec{a}^\circ \wedge \vec{d}^\perp| = |\vec{a}^\circ| \cdot |\vec{d}^\perp| \sin\dfrac{\pi}{2} = |\vec{d}^\perp|$．

可知：$\vec{a}^\circ \wedge \vec{d}^\perp$ 为向量 $\vec{d}^\perp$ 在与 $\vec{a}$ 垂直的平面 π 中，沿满足右手法则的时针方向旋转 $\dfrac{\pi}{2}$ 所得

如图，设 $\overrightarrow{OA_1} = \vec{b}^\perp$，$\overrightarrow{A_1B_1} = \vec{c}^\perp$，则

$$\overrightarrow{OB_1} = \overrightarrow{OA_1} + \overrightarrow{A_1B_1} = \vec{b}^\perp + \vec{c}^\perp.$$

由前面讨论，有

$$\overrightarrow{OA_2} = \vec{a}^\circ \wedge \vec{b}^\perp, \quad \overrightarrow{A_2B_2} = \vec{a}^\circ \wedge \vec{c}^\perp, \quad \overrightarrow{OB_2} = \vec{a}^\circ \wedge (\vec{b}^\perp + \vec{c}^\perp)$$

因此. $\overrightarrow{OB_2} = \overrightarrow{OA_2} + \overrightarrow{A_2B_2}$

$\Rightarrow \vec{a}^\circ \wedge (\vec{b}^\perp + \vec{c}^\perp) = \vec{a}^\circ \perp \vec{b}^\perp + \vec{a}^\circ + \vec{c}^\perp$.

从而, $\vec{a} \wedge (\vec{b} + \vec{c}) = \vec{a} \wedge (\vec{b} + \vec{c})^\perp = \vec{a} \wedge (\vec{b}^\perp + \vec{c}^\perp)$

$$= |\vec{a}| \, \vec{a}^\circ \wedge (\vec{b}^\perp + \vec{c}^\perp)$$

$$= |\vec{a}| \vec{a}^\circ \wedge \vec{b}^\perp + |\vec{a}| \vec{a}^\circ \wedge \vec{c}^\perp$$

$$= \vec{a} \wedge \vec{b}^\perp + \vec{a} \wedge \vec{c}^\perp$$

$$= \vec{a} \wedge \vec{b} + \vec{a} \wedge \vec{c}$$