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(1) 1° 对环面参数化

$$\vec{r}(u,v)=\left((a+r\cos v)\cos u,(a+r\cos v)\sin u,\ r\sin v\right)$$

可分为

$(0<u<2\pi,\ 0<v<2\pi)$ 与 $(-\pi<u<\pi,\ -\pi<v<\pi)$ 两个曲面片.

其有共同的单位法向量场

$$\vec{n}=(-\cos u\cos v,\ -\cos u\sin v,\ -\sin u)$$

其显然连续,故有相同的定向

2° 参考 习题三. 21.(2)

有 $\mathrm{I}=a^2dudu+(a+r\cos u)^2dvdv$

$\mathrm{II}=a^2dudu+(a+r\cos u)\cos u\,dvdv$

$$K=\frac{\cos u}{a+r\cos u}$$

$\omega_1=adu\quad \omega_2=(a+r\cos u)dv$

$dA=\omega_1\wedge\omega_2=a(a+r\cos u)dudv$

$$\Longrightarrow \int_{T^2}KdA=\int_0^{2\pi}\int_0^{2\pi}\frac{\cos u}{a+r\cos u}\quad a(a+r\cos u)dudv=0$$

(2) 熟知 直纹面的 Gauss 曲率

$$K=\frac{-M^2}{EG-F^2}\leq 0$$

但 $\mathbb{R}^3$ 中紧致曲面上必有一点 P_0, 其 Gauss 曲率 $K(P_0)>0$ 矛盾!

故 $\mathbb{R}^3$ 中不存在紧致直纹面

(3) 1° $\vec{r}_\theta=-\lambda\sin\theta\vec{n}+\lambda\cos\theta\vec{b}$

$\vec{r}_s=\vec{t}+\lambda\cos\theta(-\kappa\vec{t}+\tau\vec{b})+\lambda\sin\theta\cdot(-\tau)\vec{n})$

$=(1-\lambda\kappa\cos\theta)\vec{t}-\lambda\tau\sin\theta\vec{n}+\lambda\tau\cos\theta\vec{b}$

$\vec{r}_\theta\wedge\vec{r}_s=\lambda\cos\theta(1-\lambda\kappa\cos\theta)\vec{n}+\lambda\sin\theta(1-\lambda\kappa\cos\theta)\vec{b}$

$$\vec{n}_{T^2} = \frac{\vec{r}_\theta \wedge \vec{r}_s}{|\vec{r}_\theta \wedge \vec{r}_s|} = \cos\theta\,\vec{n} + \sin\theta\,\vec{b}$$

这都可以猜到

$$E = \lambda^2 \quad F = \lambda^2\tau \quad G = (1-\lambda k\cos\theta)^2 + \lambda^2\tau^2$$

于是 $I = \lambda^2 d\theta^2 + 2\lambda^2\tau\, d\theta ds + \left((1-\lambda k\cos\theta)^2 + \lambda^2\tau^2\right) ds^2$

$$= (1-\lambda k\cos\theta)^2 ds^2 + \lambda^2(\tau ds + d\theta)^2$$

取 $w_1 = (1-\lambda k\cos\theta)ds \qquad w_2 = \lambda(\tau ds + d\theta)$

于是 $w_{12} = \frac{dw_1}{w_1\wedge w_2} w_1 + \frac{dw_2}{w_1 \wedge w_2} w_2 = -k\sin\theta\, ds$

$dw_1 = \lambda k\sin\theta\, d\theta\wedge ds$

$dw_2 = 0$

$\underline{-k\cos\theta\, d\theta\wedge ds}$

$w_1\wedge w_2 = \lambda(1-\lambda k\cos\theta)\, ds\wedge d\theta$

$$\Rightarrow K = -\frac{dw_{12}}{w_1\wedge w_2} = \frac{-k\cos\theta}{\lambda(1-\lambda k\cos\theta)}$$

2° $\vec{r}_{\theta\theta} = -\lambda\cos\theta\,\vec{n} - \lambda\sin\theta\,\vec{b}$

$\vec{r}_{\theta s} = -\lambda\sin\theta(-k\vec{t} + \tau\vec{b}) + \lambda\cos\theta(-\tau)\cdot\vec{n}$

$= \lambda k\sin\theta\,\vec{t} - \lambda\tau\cos\theta\,\vec{n} - \lambda\tau\sin\theta\,\vec{b}$

$\vec{r}_{ss} = \lambda\dot{k}\cos\theta\,\vec{t} + (1-\lambda k\cos\theta)(k\vec{n}) - \lambda\dot{\tau}\sin\theta\,\vec{n}$

$- \lambda\tau\sin\theta(-k\vec{t} + \tau\vec{b}) + \lambda\dot{\tau}\cos\theta\,\vec{b} + \lambda\tau\cos\theta(-\tau\vec{n})$

$= \lambda(\dot{k}\cos\theta + k\tau\sin\theta)\vec{t} +$

$[k(1-\lambda k\cos\theta) - \lambda\dot{\tau}\sin\theta - \lambda\tau^2\cos\theta]\vec{n}$

$(-\lambda\tau^2\sin\theta + \lambda\dot{\tau}\cos\theta)\vec{b}$

$\Rightarrow L = -\lambda \qquad M = -\lambda\tau$

$N = k\cos\theta(1-\lambda k\cos\theta) - \lambda\tau^2$

$\Rightarrow II = -\lambda d\theta^2 - \lambda\tau\, d\theta ds + [k\cos\theta(1-\lambda k\cos\theta) - \lambda\tau^2]ds^2$

$= k\cos\theta(1-\lambda k\cos\theta)ds^2 - \lambda(\tau ds + d\theta)^2$

结合 $I = (1-\lambda\kappa\cos\theta)^2 ds^2 + \lambda^2(\tau ds + d\theta)^2$

作参数变化 $\begin{cases} u = s \\ v = \tau s + \theta \end{cases}$

即得 曲线的正交参数化.

$\Rightarrow$ 主曲率为 $k_1 = -\frac{1}{\lambda}$ $\quad k_2 = \dfrac{\kappa\cos\theta}{1-\lambda\kappa\cos\theta}$

$$H = \frac{1}{2}(k_1+k_2) = \frac{1}{2}\cdot\frac{2\lambda\kappa\cos\theta - 1}{\lambda(1-\lambda\kappa\cos\theta)} \qquad dA = \lambda(1-\lambda\kappa\cos\theta)\,d\theta\,ds$$

$$\int_{T^2} H^2 dA = \frac{1}{4}\int_{T^2}\left(\frac{1-2\lambda\kappa\cos\theta}{\lambda(1-\lambda\kappa\cos\theta)}\right)^2\cdot\left(\lambda(1-\lambda\kappa\cos\theta)\,d\theta\,ds\right)$$

$$= \frac{1}{4}\int_{T^2}\frac{(1-\lambda\kappa\cos\theta)^2 - 2\lambda\kappa\cos\theta(1-\lambda\kappa\cos\theta) + (\lambda\kappa\cos\theta)^2}{\lambda(1-\lambda\kappa\cos\theta)}\,d\theta\,ds$$

$$= \frac{1}{4}\int_{T^2}\left(\frac{1-\lambda\kappa\cos\theta}{\lambda} - 2\kappa\cos\theta + \frac{\lambda\kappa^2\cos^2\theta}{1-\lambda\kappa\cos\theta}\right)d\theta\,ds$$

注意到 κ 与 θ 无关，仅与 s 有关，先对 $d\theta$ 积分

$$\Rightarrow \int_{T^2} H^2 dA = \frac{1}{4}\int_0^l\int_0^{2\pi}\left(\frac{1}{\lambda} + \frac{\lambda\kappa^2\cos^2\theta}{1-\lambda\kappa\cos\theta}\right)d\theta\,ds$$

$$= \frac{1}{4}\int_0^l\int_0^{2\pi}\left(-\kappa\cos\theta + \frac{1}{\lambda(1-\lambda\kappa\cos\theta)}\right)d\theta\,ds$$

$$= \frac{1}{4}\int_0^l 2\left(\frac{2\arctan\frac{\sqrt{1+\lambda\kappa}\tan\frac{\theta}{2}}{\sqrt{1-\lambda\kappa}}}{\lambda\sqrt{1-(\lambda\kappa)^2}}\right)\Bigg|_0^{\pi} ds$$

$2\kappa\lambda\sqrt{1-\lambda^2\kappa^2} \le \lambda^2 + 1 - \lambda^2\kappa^2 = 1$

$$= \frac{1}{4}\int_0^l \frac{2\pi}{\lambda\sqrt{1-\lambda^2\kappa^2}}\,ds$$

$$= \frac{1}{2}\int_0^l \frac{\pi}{\lambda\sqrt{1-\lambda^2\kappa^2}}\,ds$$

$$\geq \int_0^l \kappa\pi\,ds$$

Fenchel $\geq 2\pi^2$

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(4) 证：由Gauss-Bonnet公式推论，知

$$\int_\Sigma K dA = 4\pi(1-g)$$

又 Σ 是紧致曲面，其上必有一点 P_0，使得 $K(P_0)>0$

于是 $\int_\Sigma K dA > 0$，结合 $g \geq 0$ 且为整数，

故 $\int_\Sigma K dA = 4\pi$

(5) 证：首先由于 Σ 紧致，Σ 上存在椭圆点，Σ_+ 非空

考虑 Gauss 映射 $g: \Sigma_+ \to S^2$，若 g 为满射

则 $\int_{\Sigma_+} K dA = \int_{g(\Sigma_+)} d\sigma \geq Area(S^2) = 4\pi$.

于是只需证明 g 为满射.

$\forall \vec{e} \in S^2$，定义函数

$$f(P) = \langle \vec{r}(P), \vec{e} \rangle$$

其中 $\vec{r}$ 是 P 的位置向量.

由 Σ 紧致，f 可在某一点 P_1 处取得极大值，则

$$df(P_1) = \langle d\vec{r}(P_1), \vec{e} \rangle = 0$$

$$\Rightarrow \vec{n}(P_1) \parallel \vec{e}$$

又 $D^2 f(P_1) \leq 0$ （f 的二阶导数矩阵在 P_1 处半负定）

于是 $K(P_1) \geq 0 \Rightarrow P_1 \in \Sigma_+$

而曲面的 Gauss 映射取各点处的外法向量.

将以 $\vec{e}$ 为法向量的平面沿 $\vec{e}$ 平移，

由 $f(P) = \langle \vec{r}(P), \vec{e} \rangle$ 在 P_1 处达到最大，所以平移过程中首个接触到的点为 P_1.

因此，曲面在 $P_1 \in \Sigma_+$ 的单位外法向量为 $\vec{e}$，$g: \Sigma_+ \to S^2$ 为满射

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(6) 证明需结合 (3) 与 (5) 的部分证明与结论

如 (3) 题干所述构造管状曲面.

有 $K = -\frac{\kappa\cos\theta}{\lambda(1-\lambda\kappa\cos\theta)}$ $\omega_1\wedge\omega_2 = \lambda(1-\lambda\kappa\cos\theta)ds\wedge d\theta$

由 (5) $\int_{\Sigma_+} K dA = \int_{\Sigma_+} K\omega_1\wedge\omega_2 \geq 4\pi$

此时 $\Sigma_+ = \{(s,\theta) \mid \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$

$$4\pi \leq \int_{\Sigma_+} K\omega_1\wedge\omega_2 = -\int_{\Sigma_+} \kappa\cos\theta\, ds\wedge d\theta$$

$$= \int_C \kappa\, ds \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\cos\theta)\, d\theta = 2\int_C \kappa\, ds$$

$$\Longrightarrow \int_C \kappa\, ds \geq 2\pi$$

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