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(1) 证: 考虑 $(\vec{r},\vec{n},d\vec{r})$

有 $d(\vec{r},\vec{n},d\vec{r}) = (d\vec{r},\vec{n},d\vec{r}) + (\vec{r},d\vec{n},d\vec{r})$ (*)

$$(d\vec{r},\vec{n},d\vec{r}) = (\omega_1\vec{e}_1+\omega_2\vec{e}_2,\ \vec{e}_3,\ \omega_1\vec{e}_1+\omega_2\vec{e}_2)$$
$$= \omega_1\wedge\omega_2(\vec{e}_1,\vec{e}_3,\vec{e}_2) + \omega_2\wedge\omega_1(\vec{e}_2,\vec{e}_3,\vec{e}_1)$$
$$= -2\,\omega_1\wedge\omega_2$$

$$(\vec{r},d\vec{n},d\vec{r}) = (\vec{r},\ \omega_{31}\vec{e}_1+\omega_{32}\vec{e}_2,\ \omega_1\vec{e}_1+\omega_2\vec{e}_2)$$
$$= \omega_{31}\wedge\omega_2(\vec{r},\vec{e}_1,\vec{e}_2) + \omega_{32}\wedge\omega_1(\vec{r},\vec{e}_2,\vec{e}_1)$$
$$= (-k_1\omega_1\wedge\omega_2)\cdot(-\varphi) + k_2\omega_1\wedge\omega_2\varphi$$
$$= 2H\,\omega_1\wedge\omega_2$$

对(*)两边积分, 即得

$$\int_\Sigma dA = \int_\Sigma H\varphi\, dA$$

再考虑 $(\vec{r},\vec{n},d\vec{n})$

$d(\vec{r},\vec{n},d\vec{n}) = (d\vec{r},\vec{n},d\vec{n}) + (\vec{r},d\vec{n},d\vec{n})$ (**)

$$(d\vec{r},\vec{n},d\vec{n}) = (\omega_1\vec{e}_1+\omega_2\vec{e}_2,\ \vec{e}_3,\ \omega_{31}\vec{e}_1+\omega_{32}\vec{e}_2)$$
$$= \omega_1\wedge\omega_{32}(\vec{e}_1,\vec{e}_3,\vec{e}_2) + \omega_2\wedge\omega_{31}(\vec{e}_2,\vec{e}_3,\vec{e}_1)$$
$$= k_2\omega_1\wedge\omega_2 + k_1\omega_1\wedge\omega_2$$
$$= 2H\,\omega_1\wedge\omega_2$$

$$(\vec{r},d\vec{n},d\vec{n}) = (\vec{r},\ \omega_{31}\vec{e}_1+\omega_{32}\vec{e}_2,\ \omega_{31}\vec{e}_1+\omega_{32}\vec{e}_2)$$
$$= \omega_{31}\wedge\omega_{32}(\vec{r},\vec{e}_1,\vec{e}_2) + \omega_{32}\wedge\omega_{31}(\vec{r},\vec{e}_2,\vec{e}_1)$$
$$= k_1k_2\omega_1\wedge\omega_2(-\varphi) - k_1k_2\omega_1\wedge\omega_2\cdot\varphi$$
$$= -2K\varphi\,\omega_1\wedge\omega_2$$

对(**)两边积分, 即有

$$\int_\Sigma H\, dA = \int_\Sigma K\varphi\, dA$$

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(2) 证：考虑 $(\vec{r}, d\vec{r}, \vec{a})$，$(\vec{r}, d\vec{n}, \vec{a})$，$(\vec{n}, d\vec{n}, \vec{a})$

与 $(\vec{n}, d\vec{r}, a)$ 等价

其中 $\vec{a}$ 是任意向量

$$d(\vec{r}, d\vec{r}, \vec{a}) = (d\vec{r}, d\vec{r}, \vec{a})$$
$$= (w_1\vec{e}_1 + w_2\vec{e}_2, w_1\vec{e}_1 + w_2\vec{e}_2, \vec{a})$$
$$= w_1 \wedge w_2 (\vec{e}_1, \vec{e}_2, \vec{a}) + w_2 \wedge w_1 (\vec{e}_2, \vec{e}_1, \vec{a})$$
$$= 2 w_1 \wedge w_2 \langle \vec{n}, \vec{a} \rangle$$

积分即得 $\int_\Sigma \langle \vec{n}, \vec{a} \rangle dA = 0$

$$d(\vec{r}, d\vec{n}, \vec{a}) = (d\vec{r}, d\vec{n}, \vec{a})$$
$$= (w_1\vec{e}_1 + w_2\vec{e}_2, w_{31}\vec{e}_1 + w_{32}\vec{e}_2, \vec{a})$$
$$= w_1 \wedge w_{32} (\vec{e}_1, \vec{e}_2, \vec{a}) + w_2 \wedge w_{31} (\vec{e}_2, \vec{e}_1, \vec{a})$$
$$= -k_2 w_1 \wedge w_2 \langle \vec{n}, \vec{a} \rangle + k_1 w_1 \wedge w_2 \cdot (-\langle \vec{n}, \vec{a} \rangle)$$
$$= -2H \langle \vec{n}, \vec{a} \rangle$$

积分即得 $\int_\Sigma H \langle \vec{n}, \vec{a} \rangle dA = 0$

$$d(\vec{n}, d\vec{n}, \vec{a}) = (d\vec{n}, d\vec{n}, \vec{a})$$
$$= (w_{31}\vec{e}_1 + w_{32}\vec{e}_2, w_{31}\vec{e}_1 + w_{32}\vec{e}_2, \vec{a})$$
$$= w_{31} \wedge w_{32} (\vec{e}_1, \vec{e}_2, \vec{a}) + w_{32} \wedge w_{31} (\vec{e}_2, \vec{e}_1, \vec{a})$$
$$= k_1 k_2 w_1 \wedge w_2 \langle \vec{n}, \vec{a} \rangle - k_1 k_2 w_1 \wedge w_2 \cdot (-\langle \vec{n}, \vec{a} \rangle)$$
$$= 2K \langle \vec{n}, \vec{a} \rangle$$

积分即得 $\int_\Sigma K \langle \vec{n}, \vec{a} \rangle dA = 0$

再由 $\vec{a}$ 的任意性，即有

$$\int_\Sigma \vec{n}\, dA = 0, \quad \int_\Sigma H\vec{n}\, dA = 0, \quad \int_\Sigma K\vec{n}\, dA = 0$$

(3) Σ $K \equiv$ cosnt，又由紧致曲面上必有椭圆点 P_0，$K(P_0) > 0$

于是 $K \equiv \text{const} > 0 \xrightarrow{\text{Hadamard}} \Sigma$ 为凸曲面，

$H^2 - K = \frac{1}{4}(k_1 - k_2)^2 \geq 0$，选取 $\vec{n}$，s.t. $\varphi > 0$ $H > 0$

$$\Rightarrow \int_\Sigma dA = \int_\Sigma H\varphi\, dA \geq \int_\Sigma \sqrt{K}\varphi\, dA$$
$$= \frac{1}{\sqrt{K}} \int_\Sigma K\varphi\, dA = \frac{1}{\sqrt{K}} \int_\Sigma H\, dA \geq \int_\Sigma dA$$

以上不等式均取到等号. $\Rightarrow H^2 \equiv K \Rightarrow k_1 \equiv k_2 > 0 \Rightarrow \Sigma$ 是球面.

(4) 只能读懂PPT上的证明:

证明一: 构造 $n-1$ 次微分形式

$$\varphi_0 = (\vec{r}, \vec{n}, \underbrace{d\vec{r}, \cdots, d\vec{r}}_{n-1})$$

$$\varphi_1 = (\vec{r}, \vec{n}, d\vec{n}, \underbrace{d\vec{r}, \cdots, d\vec{r}}_{n-2})$$

$$\vdots$$

$$\varphi_{n-1} = (\vec{r}, \vec{n}, d\vec{n}, \cdots, d\vec{n})$$

$$\vec{r} = -\varphi \vec{e}_{n+1} + \langle \vec{r}, e_i \rangle \vec{e}_i$$

$$d\varphi_0 = (d\vec{r}, \vec{n}, d\vec{r}, \cdots, d\vec{r}) + (\vec{r}, d\vec{n}, d\vec{r}, \cdots, d\vec{r})$$

$$= (w_{i_1}\vec{e}_{i_1}, \vec{e}_{n+1}, w_{i_2}\vec{e}_{i_2}, \cdots, w_{i_n}\vec{e}_{i_n}) + (-\varphi \vec{e}_{n+1}, W_{n+1,i_1}\vec{e}_{i_1}, w_{i_2}\vec{e}_{i_2}, \cdots, w_{i_n}\vec{e}_{i_n})$$

$$= \delta^{123\cdots n+1}_{i_1(n+1)i_2\cdots i_n} w_{i_1}\wedge w_{i_2}\wedge\cdots\wedge w_{i_n} - \varphi\,\delta^{123\cdots n+1}_{(n+1)i_1\cdots i_n} W_{n+1,i_1}\wedge W_{i_2}\wedge\cdots W_{i_n}$$

$$= (-1)^{n-1}\delta^{123\cdots n+1}_{i_1i_2\cdots i_n} w_{i_1}\wedge w_{i_2}\wedge\cdots\wedge w_{i_n} - \varphi\cdot(-1)^n\delta^{123\cdots n}_{i_1\cdots i_n} -K_{i_1} W_{i_1}\wedge W_{i_2}\wedge\cdots\wedge W_{i_n}$$

$$= (-1)^{n-1} n!\, w_1\wedge\cdots\wedge w_n + \varphi(-1)^n (n-1)!(K_1+K_2+\cdots+K_n) W_1\wedge\cdots\wedge W_n$$

$$= (-1)^{n-1} n!\,(1-H_1\varphi) w_1\wedge\cdots\wedge w_n$$

$$= (-1)^{n-1} n!\,(1-H\varphi)\,dA$$

两边积分, 则有

$$\int_M dA = \int_M H_1 \varphi\, dA$$

$$d\varphi_1 = (d\vec{r}, \vec{e}_{n+1}, d\vec{e}_{n+1}, d\vec{r}, \cdots, d\vec{r}) + (\vec{r}, d\vec{e}_{n+1}, d\vec{e}_{n+1}, d\vec{r}, \cdots, d\vec{r})$$

$$= (W_{i_1}\vec{e}_{i_1}, \vec{e}_{n+1}, -K_{i_2}W_{i_2}\vec{e}_{i_2}, w_{i_3}\vec{e}_{i_3}, \cdots, w_{i_n}\vec{e}_{i_n})$$

$$+ (-\varphi\vec{e}_{n+1}, -K_{i_1}W_{i_1}\vec{e}_{i_1}, -K_{i_2}W_{i_2}\vec{e}_{i_2}, W_{i_3}\vec{e}_{i_3}, \cdots, W_{i_n}\vec{e}_{i_n})$$

$$= (-1)^n \delta^{123\cdots n}_{i_1\cdots i_n} K_{i_2} W_{i_1}\wedge\cdots\wedge W_{i_n} + \varphi(-1)^{n+1}\delta^{123\cdots n}_{i_1\cdots i_n} K_{i_1}K_{i_j} W_{i_1}\wedge\cdots\wedge W_{i_n}$$

$$= (-1)^n (n-1)!\,(K_1+\cdots+K_n) W_1\wedge\cdots\wedge W_n + \varphi(-1)^{n+1}(n-2)!\,\Big(\sum_{i\neq j} K_iK_j\Big) W_1\wedge\cdots\wedge W_n$$

$$= (-1)^n (H_1 - \varphi H_2)\, dA$$

$$\Longrightarrow \int_\Sigma H_1\, dA = \int_\Sigma H_2 \varphi\, dA$$

其他等式类似可证.

证明二: 证明一中的第一个 Minkowski 积分公式是本质的

设 $\vec{r}: M\to \mathbb{R}^{n+1}$ 是闭的定向超曲面，令

$$\vec{r}_t=\vec{r}-t\vec{n}=\vec{r}-t\vec{e}_{n+1}$$

其中 $\vec{r}_t$ 是 M_t 的位置向量，t 很小

则有 $d\vec{r}_t=d(\vec{r}-t\vec{e}_{n+1})=w_i\vec{e}_i+tk_iw_i\vec{e}_i=(1+tk_i)w_i\vec{e}_i$

从而 M_t 与 M 在对应点的切空间平行，

令 $\vec{e}_i(t)=\vec{e}_i$，$\vec{e}_{n+1}(t)=\vec{e}_{n+1}$，则

$$\widetilde{w}_i(t)=(1+tk_i)w_i$$

$$dA(t)=\widetilde{w}_1(t)\wedge\cdots\wedge\widetilde{w}_n(t)=(1+tk_1)\cdots(1+tk_n)dA$$

$$d\vec{e}_{n+1}(t)=\widetilde{w}_{n+1,i}(t)\vec{e}_i=-\widetilde{k}_i(t)\widetilde{w}_i(t)\vec{e}_t$$

结合 $d\vec{e}_{n+1}=-k_iw_i\vec{e}_i$，得

$$\widetilde{w}_{i,n+1}(t)=\widetilde{k}_i(t)\widetilde{w}_i(t)=k_iw_i\Longrightarrow\widetilde{k}_i(t)=\frac{k_i}{1+tk_i}$$

以上在之前习题有过完全类似题目

记 $W(t):=\prod_{i=1}^{n}(1+tk_i)=\sum_{r=0}^{n}\binom{n}{r}H_rt^r$，则 M_t 的平均曲率是

$$\widetilde{H}_1=\frac{1}{n}\sum\widetilde{k}_i=\frac{1}{n}\cdot\frac{W'(t)}{W(t)}=\frac{1}{nW(t)}\sum_{r=1}^{n}r\binom{n}{r}H_rt^{r-1}$$

key point

$\Rightarrow$ 在 M_t 上使用第一个 Minkowski 积分公式，有

$$0=\int_{M_t}(1+\widetilde{H}_1\langle\vec{r}_t,\vec{e}_{n+1}\rangle)dA(t)$$

$$=\int_M\left(1+\frac{W'(t)}{nW(t)}\langle\vec{r}-t\vec{e}_{n+1},\vec{e}_{n+1}\rangle\right)W(t)dA$$

$$=\int_M\left(W(t)-\frac{1}{n}W'(t)\varphi-\frac{t}{n}W'(t)\right)dA$$

$$=\frac{1}{n}\int_M\left(\sum_{r=0}^{n}n\binom{n}{r}H_rt^r-\sum_{r=0}^{n}r\binom{n}{r}H_rt^{r-1}\varphi-\sum_{r=1}^{n}r\binom{n}{r}H_rt^r\right)dA$$

$$=\frac{1}{n}\int_M\left(\sum_{r=0}^{n-1}(n-r)\binom{n}{r}H_rt^r-\sum_{r=0}^{n-1}(r+1)\binom{n}{r+1}H_{r+1}\varphi t^r\right)dA$$

$$=\frac{1}{n}\sum_{r=0}^{n-1}\int_M\left((n-r)\binom{n}{r}H_r-(r+1)\binom{n}{r+1}H_{r+1}\varphi\right)dA\cdot t^r$$

由 t 的任意性，t^r 系数皆为 0. 又 $(n-r)\binom{n}{r}=(r+1)\binom{n}{r+1}$

于是 $\int_M H_rdA=\int_M H_{r+1}\varphi\, dA$，$0\le r\le n-1$

(5) $\forall \vec{p_0} \in \Sigma$，考虑 $f_{p_0}(p) = \langle \vec{r}(p) - \vec{p_0}, \vec{n_0} \rangle$.

由于 Σ 为凸超曲面，于是 f 在 p_0 处取得极小值.

$\Rightarrow$ f_{p_0} 的Hessen矩阵在 p_0 处是半正定的.

在 p_0 处取主方向 $\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}$.

$$\nabla f_{p_0} = \langle \omega_i \vec{e_i}, \vec{n_0} \rangle = \langle \vec{e_i}, \vec{n_0} \rangle \omega_i \qquad f_i = \langle \vec{e_i}, \vec{n_0} \rangle$$

$$\begin{aligned} Df_i &= df_i + f_j \omega_{ji} \\ &= \langle \omega_{ij} \vec{e_j} + \omega_{i,n+1} \vec{e_{n+1}}, \vec{n_0} \rangle + f_j \omega_{ji} \\ &= \omega_{i,n+1} + f_j \omega_{ji} \end{aligned}$$

p_0 处，有 $Df_i = \omega_{i,n+1} = \sum_{j=1}^{n} h_{ij} \omega_j = \sum_{j=1}^{n} k_i \delta_{ij} \omega_j$

$$Df_i = \sum_{j=1}^{n} f_{ij} \omega_j$$

$$\Rightarrow f_{ij} = k_i \delta_{ij}$$

由 Hessen 矩阵半正定，$k_i = f_{ii} \geq 0$.

(6) ① 若 $H_i = \text{const}$，$1 \leq i \leq n-1$，由Minkowski积分公式得

$$\int_M dA = \int_M H_1 \varphi \, dA \qquad (*)$$

$$\int_M H_i \, dA = \int_M H_{i+1} \varphi \, dA \qquad (**)$$

$(*) \times H_i - (**)$，得

$$\int_M (H_1 H_i - H_{i+1} \varphi) \, dA = 0$$

M 为凸超曲面，可取 $\varphi > 0$，且由(5)知主曲率 k_i 均非负

$$H_1 H_i - H_{i+1} = \frac{1}{n \cdot \binom{n}{i}} \left(\sum k_j \right) \left(\sum k_{j_1} k_{j_2} \cdots k_{j_i} \right) - \frac{1}{\binom{n}{i+1}} \left(\sum k_{j_1} k_{j_2} \cdots k_{j_{i+1}} \right)$$

$$= \frac{i! \, (n-i-1)!}{n \cdot n!} \sum_{j_1 < \cdots < j_{i+1}} \sum_{s<t} (k_{j_s} - k_{j_t})^2 \prod_{\substack{l=1 \\ l \neq s,t}}^{i+1} k_{j_l} \geq 0$$

这取到等号.

由 M 紧致可知，存在椭圆点 P_0，在 P_0 邻域 U 内有主曲率 $k_i > 0$，

由取得条件知 $k_1=k_2=\cdots=k_n=(H_i)^{\frac{1}{i}}$，从而U是n维球面的一部分.

利用取得条件和主曲率的连续性，可知紧致超曲面M是n维球面.

② 若 $H_n = const$，有Minkowski积分公式

$$\int_M dA = \int_M H_1 \varphi dA$$

$$\int_M H_{n-1} dA = \int_M H_n \varphi dA$$

由M为凸超曲面，可取 $\varphi>0$，可知主曲率 $k_i \geq 0$；

由M紧致，可知存在椭圆点 P_0，则 $H_n \equiv H_n(P_0) > 0$，从而主曲率 $k_i>0$

又

$$H_1 = \frac{1}{n}\sum_i k_i \geq H_n^{\frac{1}{n}} = (k_1\cdots k_n)^{\frac{1}{n}}$$

$$\frac{H_{n-1}}{H_n} = \frac{1}{n}\sum_i \frac{1}{k_i} \geq H_n^{-\frac{1}{n}} \quad \Rightarrow \quad H_{n-1} \geq H_n^{\frac{n-1}{n}}$$

$$\Rightarrow \int_M dA = \int_M H_1 \varphi dA \geq \int_M H_n^{\frac{1}{n}} \varphi dA = H_n^{\frac{1-n}{n}} \int_M H_n \varphi dA$$

$$= H_n^{\frac{1-n}{n}} \int_M H_{n-1} dA$$

$$\geq H_n^{\frac{1-n}{n}} \int_M H_n^{\frac{n-1}{n}} dA = \int_M dA$$

以上不等式均取等，$k_1 = \cdots = k_n \Rightarrow$ M是n维球面.