14. $\vec{r}_u = (f'(u)\cos v, f'(u)\sin v, g'(u))$

$\vec{r}_v = (-f(u)\sin v, f(u)\cos v, 0)$

$E = (f')^2 + (g')^2 \qquad F = 0 \qquad G = f^2$

利用 Liouville 公式 $\qquad\qquad \vec{r}_u \frac{du}{ds}$

$k_g = \frac{d\theta}{ds} + \frac{1}{2\sqrt{(f')^2+(g')^2}} \cdot \frac{1}{f} \cdot 2ff'\sin\theta$

$\quad = \frac{d\theta}{ds} + \frac{f'}{f\sqrt{(f')^2+(g')^2}}\sin\theta = 0 \implies f\sqrt{(f')^2+(g')^2}\frac{d\theta}{ds} + f'\sin\theta = 0$

$\cos\theta = \frac{1}{|\vec{r}_u|}\cdot\langle\frac{d\vec{r}}{ds}, \vec{r}_u\rangle = \sqrt{(f')^2+(g')^2}\frac{du}{ds}$

于是 $\quad f\cos\theta\cdot\frac{d\theta}{ds} + f'\sin\theta\frac{du}{ds} = 0$

$\implies (f(u)\sin\theta)' = 0$

$\qquad f(u)\sin\theta = 常数 \quad \#$

15. 解: 由14题, 有

$$0 = k_g = \frac{d\theta}{ds} + \frac{\sin\theta}{u\sqrt{1+(f')^2}}$$

其中 θ 是测地线 与经线 (u 线) 的夹角.

$$\begin{cases} \dfrac{d\vec{r}}{ds} = \vec{r}_u\dfrac{du}{ds} + \vec{r}_v\dfrac{dv}{ds} \\[2mm] \dfrac{d\vec{r}}{ds} = \cos\theta\,\vec{e}_u + \sin\theta\,\vec{e}_v \\[2mm] \vec{e}_u = \dfrac{\vec{r}_u}{\sqrt{1+(f')^2}} \qquad \vec{e}_v = \dfrac{\vec{r}_v}{u} \end{cases} \implies \begin{cases} \dfrac{du}{ds} = \dfrac{\cos\theta}{\sqrt{1+(f')^2}} \\[2mm] \dfrac{dv}{ds} = \dfrac{\sin\theta}{u} \end{cases}$$

$u\sin\theta = C$ 为常值

从而 $\dfrac{dv}{du} = \dfrac{\sqrt{1+(f')^2}}{u}\tan\theta = \dfrac{\sqrt{1+(f')^2}}{u}\cdot\dfrac{C}{\sqrt{u^2-c^2}}$

即 $v - v_0 = \displaystyle\int_0^u \dfrac{\sqrt{1+(f')^2}}{u}\cdot\dfrac{C}{\sqrt{u^2-c^2}}\,du$.

16. 证: 由 Gauss 方程

$$K = -\frac{1}{\sqrt{G}}(\sqrt{G})_{uu}$$

$$K(0,v) = -\frac{1}{\sqrt{G(0,v)}}(\sqrt{G(0,v)})_{uu} = -\left(\frac{G_u(0,v)}{2\sqrt{G(0,v)}}\right)_u$$

$$= -\frac{2G_{uu}G - G_u^2}{4G^{\frac{3}{2}}}\bigg|_{u=0} = -\frac{G_{uu}}{2}$$

$G(u,v)$ 在 $u=0$ 处, 作 Taylor 展开, 有

$$G(u,v) = G(0,v) + G_u(0,v)\cdot u + \frac{1}{2!}G_{uu}(0,v)\cdot u^2 + o(u^2)$$

$$= 1 + u^2 K(0,v) + o(u^2)$$

18. 证: 在测地极坐标 (ρ, θ) 中，第一基本型式为

$$I = d\rho^2 + G\,d\theta^2$$

Gauss 方程为 $K = -\dfrac{G_{\rho\rho}}{\sqrt{G}}$

其中 G 满足 $\lim\limits_{\rho \to 0} \sqrt{G} = 0$, $\lim\limits_{\rho \to 0} (\sqrt{G})_\rho = 1$

于是 $\sqrt{G} = 0 + \rho - \dfrac{1}{2} K \sqrt{G}\, \rho^2 - \dfrac{1}{6}(K\sqrt{G})_\rho\, \rho^3 + o(\rho^3)$

$$= \rho - K(P)\rho^3 + o(\rho^3)$$

$$\dfrac{1}{r^3}\big(2\pi r - L(r)\big) = \dfrac{1}{r^3}\int_0^{2\pi}(r - \sqrt{G})\,d\theta$$

$$= \dfrac{1}{r^3}\int_0^{2\pi}\dfrac{1}{6}K r^3\,d\theta + o(1) \longrightarrow \dfrac{\pi}{3}K(P) \quad , \ r \to 0$$

从而 $K(P) = \lim\limits_{r \to 0}\dfrac{3}{\pi}\cdot\dfrac{2\pi r - L(r)}{r^3}$

$$\dfrac{1}{r^4}(\pi r^2 - A(r)) = \dfrac{1}{r^4}\int_0^r ds \int_0^{2\pi}(s - \sqrt{G})\,d\theta$$

$$= \dfrac{1}{r^4}\int_0^r\int_0^{2\pi}\dfrac{1}{6}K s^3\,d\theta\,ds + o(1) \longrightarrow \dfrac{\pi}{12}K(P) \quad, \ r \to 0$$

从而 $K(P) = \lim\limits_{r \to 0}\dfrac{12}{\pi}\cdot\dfrac{\pi r^2 - A(r)}{r^4}$.

20. $\forall P \in S$, 取 P 附近的正交参数 (u, v), 使得 u-线是一族测地线

由 Liouville 公式

$$0 = k_g = -\dfrac{1}{2\sqrt{G}}\dfrac{\partial \ln E}{\partial u}$$

设另一族测地线与 u-线夹角为 $\theta_0 \in (0, \pi)$
对过 P 点的测地线 C_P, 有

$$0 = k_g(C_P) = -\dfrac{1}{2\sqrt{G}}\dfrac{\partial \ln E}{\partial v}\cos\theta_0 + \dfrac{1}{2\sqrt{E}}\dfrac{\partial \ln G}{\partial u}\sin\theta_0$$

$$= \dfrac{1}{2\sqrt{E}}\cdot\dfrac{\partial \ln G}{\partial u}\sin\theta_0$$

$$\Rightarrow \dfrac{1}{2\sqrt{E}}\cdot\dfrac{\partial \ln G}{\partial u} = 0$$

可从两式得 $E_v = G_u = 0$

$$\Rightarrow \quad E(u,v) = E(u) \qquad G(u,v) = G(v)$$

作变换 $\quad \tilde{u} = \int \sqrt{E(u)}\, du \qquad \tilde{v} = \int \sqrt{G(v)}\, dv$

从而 $\quad 1 = E du^2 + G dv^2 = d\tilde{u}^2 + d\tilde{v}^2$

这是从 曲面到 平面的等距变换，从而 曲面为可展曲面.

也可由 Gauss 方程得： $K = 0$ ，故曲面为可展曲面.

21. 反证：假设存在，那么 两条测地线构成了一条分段光滑的简单闭曲线.

由 Gauss-Bonnet 公式

$$0 > \int_D K dA + \int_{\partial D} k_g ds = 2\pi - \alpha_1 - \alpha_2$$

而 $\alpha_1, \alpha_2 < \pi$. 从而 矛盾.

22. α_i 是内角，那么 $(\pi - \alpha_i)$ 为外角

由 Gauss-Bonnet 公式，有

$$\int_A K d\sigma + \int_{\partial A} k_g ds + \sum_{i=1}^{4} (\pi - \alpha_i) = 2\pi$$

$$\Rightarrow \int_A K d\sigma + \int_{\partial A} k_g ds = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2\pi$$

23. 先找一个特解 $\sigma = (\tilde{u}(u,v), \tilde{v}(u,v))$，记 $J_\sigma = \dfrac{\partial(\tilde{u}, \tilde{v})}{\partial(u,v)}$ #

有 $\quad \begin{pmatrix} \frac{a^2}{v^2} & \\ & \frac{a^2}{v^2} \end{pmatrix} = J_\sigma \begin{pmatrix} 1 & \\ & e^{\frac{2\tilde{u}}{a}} \end{pmatrix} J_\sigma^T$

$$\Rightarrow \begin{pmatrix} \frac{a^2}{v^2} & \\ & \frac{a^2}{v^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix} \begin{pmatrix} 1 & \\ & e^{\frac{2\tilde{u}}{a}} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} (\frac{\partial \tilde{u}}{\partial u})^2 + e^{\frac{2\tilde{u}}{a}}(\frac{\partial \tilde{v}}{\partial u})^2 & \frac{\partial \tilde{u}}{\partial u} \cdot \frac{\partial \tilde{u}}{\partial v} + e^{\frac{2\tilde{u}}{a}} \frac{\partial \tilde{v}}{\partial u} \cdot \frac{\partial \tilde{v}}{\partial v} \\ \frac{\partial \tilde{u}}{\partial u} \cdot \frac{\partial \tilde{u}}{\partial v} + e^{\frac{2\tilde{u}}{a}} \frac{\partial \tilde{v}}{\partial u} \cdot \frac{\partial \tilde{v}}{\partial v} & (\frac{\partial \tilde{u}}{\partial v})^2 + e^{\frac{2\tilde{u}}{a}}(\frac{\partial \tilde{v}}{\partial v})^2 \end{pmatrix}$$

可取 $\begin{cases} \tilde{u} = -a \ln v \\ \tilde{v} = au \end{cases}$ 记为 σ_0.

$D = \{(u,v) \mid v > 0\}$ 上的等距变换群为 $\{T_b(u,v) = (u+b, v) \mid b \in \mathbb{R}\}$.

$\therefore$ 等距变换全体为 $\{\theta_0 \cdot T_b\}$

24. 证：取 $\vec{e_1} = \dfrac{\vec{v}}{|\vec{v}|}$，$\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ 为其正交标架.

$\Rightarrow k_g = \langle \dfrac{d\vec{e_1}}{ds}, e_2 \rangle = \langle \dfrac{dt}{ds} \cdot \dfrac{D}{dt}(\dfrac{\vec{v}}{|\vec{v}|}), \vec{e_2} \rangle$

$\qquad\qquad = \dfrac{dt}{ds} \cdot \dfrac{1}{|\vec{v}|} \langle \dfrac{D\vec{v}}{dt}, \vec{e_2} \rangle$

提 $k_g = 0 \Longleftrightarrow \dfrac{D\vec{v}}{dt} = \lambda' \vec{e_1} = \dfrac{\lambda'}{|\vec{v}|} \cdot \vec{v} = (-\lambda)\vec{v}$

$\qquad\qquad \Longleftrightarrow \dfrac{D\vec{v}}{dt} + \lambda \vec{v} = 0 \qquad\qquad$ #

25. (1) $\quad E = G = v \qquad F = 0$

$\Gamma_{11}^1 = \dfrac{1}{2} \dfrac{\partial \ln E}{\partial u} = 0 \qquad \Gamma_{12}^1 = \Gamma_{21}^1 = \dfrac{1}{2} \dfrac{\partial \ln E}{\partial v} = \dfrac{1}{2v}$

$\Gamma_{11}^2 = -\dfrac{1}{2G} \dfrac{\partial E}{\partial v} = -\dfrac{1}{2v} \qquad \Gamma_{12}^2 = \Gamma_{21}^2 = \dfrac{1}{2} \cdot \dfrac{\partial \ln G}{\partial u} = 0$

$\Gamma_{22}^1 = -\dfrac{1}{2E} \dfrac{\partial G}{\partial u} = 0 \qquad \Gamma_{22}^2 = \dfrac{1}{2} \cdot \dfrac{\partial \ln G}{\partial v} = \dfrac{1}{2v}$

代入测地线方程，得

$\begin{cases} \ddot{u} + \dfrac{\dot{u}\dot{v}}{v} = 0 & (1) \\[2mm] \ddot{v} + \dfrac{(\dot{v})^2 - (\dot{u})^2}{2v} = 0 & (2) \end{cases}$

由(1)，$\quad \ddot{u}v + \dot{u}\dot{v} = (\dot{u}v)' = 0 \quad \Rightarrow \dot{u}v \equiv const$

由(2)，$\quad 2v\ddot{v} + 2(\dot{v})^2 = (\dot{v})^2 + (\dot{u})^2$

从而得到方程 $(\dot{u} \neq 0$ 时$)$：

$\dfrac{d^2 v}{du^2} = \dfrac{1}{c^2} v \dfrac{1}{2}(\dot{u}^2 + \dot{v}^2) = C_1$，

解得

$\qquad v = C_1 u^2 + C_2 u + C_3, \qquad C_1 > 0,$

$\dot{u} = 0$ 时，得到 v 有非平凡解，即 $v-$曲线也是测地线.

(2) 用几何法.

　　设二维球面 S^2 在北极点的球极投影为 $T: S^2 \setminus \{N\} \to \mathbb{R}^2$,

这个映射可以看成 $(S^2 \setminus \{N\}, g_{\mathbb{R}^3}|_{S^2})$ 到 (D, ds^2) 的等距同构.

从而 T 保测地线、保圆.

而 S^2 上测地线为大圆, 每个大圆均与赤道相交于某对对径点,

所以 (D, ds^2) 上测地线为所有与单位圆相交于单位圆某对对径点的圆.

由解析几何二次曲线结论可以得到, 这些曲线的方程为

　　　$\{ a(x^2 + y^2 - 1) + bx + cy = 0 : a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 \neq 0 \}$.