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习题四

16. 证：由于S没有脐点，可取S的两个主方向为$\vec{e}_1, \vec{e}_2$

于是 $\begin{cases} \omega_{13} = k_1\omega_1 \\ \omega_{23} = k_2\omega_2 \end{cases}$

取$\vec{e}_3 = \vec{n}(u,v)$，由于曲面$\tilde{S}$与曲面S切平面平行

故不妨设$\{\tilde{\vec{e}}_1, \tilde{\vec{e}}_2, \tilde{\vec{e}}_3\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \Rightarrow \tilde{\omega}_{ij} = \omega_{ij}$

$$\begin{aligned} d\tilde{r} &= d\vec{r} + \lambda d\vec{n} \\ &= \omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \lambda(-\omega_{13}\vec{e}_1 - \omega_{23}\vec{e}_2) \\ &= (\omega_1 - \lambda\omega_{13})\vec{e}_1 + (\omega_2 - \lambda\omega_{23})\vec{e}_2 \end{aligned}$$

$$\Rightarrow \tilde{\omega}_1 = \omega_1 - \lambda\omega_{13} = (1-\lambda k_1)\omega_1 \qquad \tilde{\omega}_2 = \omega_2 - \lambda\omega_{23} = (1-\lambda k_2)\omega_2$$

从而 $\tilde{\omega}_{13} = \omega_{13} = k_1\omega_1 = \dfrac{k_1}{1-\lambda k_1}\tilde{\omega}_1$

$\tilde{\omega}_{23} = \omega_{23} = k_2\omega_2 = \dfrac{k_2}{1-\lambda k_2}\tilde{\omega}_2$

从而 $\tilde{k}_1 = \dfrac{k_1}{1-\lambda k_1} \qquad \tilde{k}_2 = \dfrac{k_2}{1-\lambda k_2}$

$$\tilde{K} = \tilde{k}_1\cdot\tilde{k}_2 = \frac{k_1k_2}{(1-\lambda k_1)(1-\lambda k_2)} = \frac{K}{1-\lambda(k_1+k_2)+\lambda^2k_1k_2} = \frac{K}{1-2\lambda H+\lambda^2K}$$

$$\tilde{H} = \frac{1}{2}(\tilde{k}_1+\tilde{k}_2) = \frac{H-\lambda K}{1-2\lambda H+\lambda^2K}$$

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18. 解：$d\vec{r} = \vec{r}_u du + \vec{r}_v dv$

$= \vec{e}_1 du + \vec{e}_2 dv$

$\Rightarrow \omega_1 = du \qquad \omega_2 = dv$

$d\omega_1 = \omega_{12}\wedge\omega_2 = 0$

$d\omega_2 = \omega_{21}\wedge\omega_1 = 0$

$\omega_{12} = \dfrac{d\omega_1}{\omega_1\wedge\omega_2}\omega_1 + \dfrac{d\omega_2}{\omega_1\wedge\omega_2}\omega_2$

设 $\omega_{12} = a\omega_1 + b\omega_2$

$d\omega_1 = \omega_{12}\wedge\omega_2 = a\omega_1\wedge\omega_2 \qquad a = \dfrac{d\omega_1}{\omega_1\wedge\omega_2} = 0$

同理 $b=0$ 从而 $\omega_{12}=0 \Rightarrow K = \dfrac{d\omega_{12}}{\omega_1\wedge\omega_2} = 0$

19. 由例 6.3（再算一遍）

$$\omega_1=\sqrt{E}\,du \qquad \omega_2=\sqrt{G}\,dv$$

$$d\omega_1=(\sqrt{E})_v\,dv\wedge du \qquad d\omega_2=(\sqrt{G})_u\,du\wedge dv.$$

$$\omega_{12}=-\omega_{21}=\frac{d\omega_1}{\omega_1\wedge\omega_2}\omega_1-\frac{d\omega_2}{\omega_1\wedge\omega_2}\omega_2$$

$$=-\frac{(\sqrt{E})_v}{\sqrt{E}\cdot\sqrt{G}}\cdot\sqrt{E}\,du+\frac{(\sqrt{G})_u}{\sqrt{E}\cdot\sqrt{G}}\cdot\sqrt{G}\,dv$$

$$=-\frac{(\sqrt{E})_v}{\sqrt{G}}\cdot du+\frac{(\sqrt{G})_u}{\sqrt{E}}\cdot dv$$

$$\omega_{13}=\langle d\vec{e}_1,\vec{e}_3\rangle=\langle(\frac{\vec{r}_{uu}}{\sqrt{E}}+(\frac{1}{\sqrt{E}})_u\vec{r}_u)du+(\frac{\vec{r}_{uv}}{\sqrt{E}}+(\frac{1}{\sqrt{E}})_v\vec{r}_v)dv,\ \vec{n}\rangle$$

$$=\frac{L}{\sqrt{E}}du+\frac{M}{\sqrt{E}}dv$$

$$\omega_{23}=\langle d\vec{e}_2,\vec{e}_3\rangle=\langle(\frac{\vec{r}_{vu}}{\sqrt{G}}+(\frac{1}{\sqrt{G}})_u\vec{r}_u)du+(\frac{\vec{r}_{vv}}{\sqrt{G}}+(\frac{1}{\sqrt{G}})_v\vec{r}_v)dv,\vec{n}\rangle$$

$$=\frac{M}{\sqrt{G}}du+\frac{N}{\sqrt{G}}dv$$

提 $$d\omega_{13}=\left[-(\frac{L}{\sqrt{E}})_v+(\frac{M}{\sqrt{E}})_u\right]du\wedge dv$$

$$\omega_{12}\wedge\omega_{23}=-\left[\frac{(\sqrt{E})_v}{\sqrt{G}}\cdot\frac{N}{\sqrt{G}}+\frac{M}{\sqrt{G}}\cdot\frac{(\sqrt{G})_u}{\sqrt{E}}\right]du\wedge dv$$

$d\omega_{13}=\omega_{12}\wedge\omega_{23}$ 即

$$(\frac{L}{\sqrt{E}})_v-(\frac{M}{\sqrt{E}})_u-N\frac{(\sqrt{E})_v}{G}-M\frac{(\sqrt{G})_u}{\sqrt{EG}}=0$$

$$d\omega_{23}=\left[-(\frac{M}{\sqrt{G}})_v+(\frac{N}{\sqrt{G}})_u\right]du\wedge dv$$

$$\omega_{21}\wedge\omega_{13}=\left[\frac{(\sqrt{E})_v}{\sqrt{G}}\cdot\frac{M}{\sqrt{E}}+\frac{(\sqrt{G})_u}{\sqrt{E}}\cdot\frac{L}{\sqrt{E}}\right]du\wedge dv$$

从而 $d\omega_{23}=\omega_{21}\wedge\omega_{13}$，得

$$(\frac{M}{\sqrt{G}})_v-(\frac{N}{\sqrt{G}})_u+M\frac{(\sqrt{E})_v}{\sqrt{EG}}+L\frac{(\sqrt{G})_u}{E}=0 \qquad \text{即为(3.17)}.$$

即 $$(\frac{N}{\sqrt{G}})_u-(\frac{M}{\sqrt{G}})_v-L\frac{(\sqrt{G})_u}{E}-M\frac{(\sqrt{E})_v}{\sqrt{EG}}=0$$

20. 证: $\omega_{13} = k_1 \omega_1 \quad \omega_{23} = k_2 \omega_2$

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}$$

$$\Rightarrow d(k_1 \omega_1) = \omega_{12} \wedge (k_2 \omega_2) = k_2\, \omega_{12} \wedge \omega_2$$

$$\parallel$$

$$dk_1 \wedge \omega_1 + k_1 d\omega_1 = dk_1 \wedge \omega_1 + k_1 (\omega_{12} \wedge \omega_2)$$

$$\Rightarrow dk_1 \wedge \omega_1 = (k_2 - k_1)\, \omega_{12} \wedge \omega_2$$

$$d\omega_{23} = \omega_{21} \wedge \omega_{13}$$

$$\Rightarrow d(k_2 \omega_2) = \omega_{21} \wedge (k_1 \omega_1) = k_1\, \omega_{21} \wedge \omega_1$$

$$\parallel$$

$$dk_2 \wedge \omega_2 + k_2 d\omega_2 = dk_2 \wedge \omega_2 + k_2 (\omega_{21} \wedge \omega_1)$$

$$\Rightarrow dk_2 \wedge \omega_2 = (k_1 - k_2)\, \omega_{21} \wedge \omega_1$$

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习题五

1. (3) $\vec{e}_1=(c+u^2+v^2)\vec{r}_u \quad \vec{e}_2=(c+u^2+v^2)\vec{r}_v$

$$\omega_1=\frac{1}{c+u^2+v^2}du \quad \omega_2=\frac{1}{c+u^2+v^2}dv$$

$$\omega_{12}=\frac{\frac{2v}{(c+u^2+v^2)^2}}{\frac{1}{c+u^2+v^2}}du-\frac{\frac{2u}{(c+u^2+v^2)^2}}{\frac{1}{c+u^2+v^2}}dv$$

$$=\frac{2v\,du-2u\,dv}{c+u^2+v^2}$$

$$d\omega_{12}=-\frac{4c}{(c+u^2+v^2)^2}du\wedge dv$$

$$\omega_1\wedge\omega_2=\frac{1}{(c+u^2+v^2)^2}du\wedge dv$$

$$\Rightarrow K=-\frac{d\omega_{12}}{\omega_1\wedge\omega_2}=4c$$

注意到 这是等温参数.

$$K=-\frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\ln\lambda$$

$$=(c+u^2+v^2)^2\frac{4c}{(c+u^2+v^2)^2}=4c$$

2. 取 $\tilde{S}$ 上的局部正交参数 (u,v), 则有

$$\tilde{I}=E(du)^2+F(dv)^2$$

那么 $\quad I=\lambda\tilde{I}=\lambda E(du)^2+\lambda F(dv)^2$

于是 $$K=-\frac{1}{\sqrt{\lambda E\cdot\lambda G}}\left\{\left[\frac{(\sqrt{\lambda E})_v}{\sqrt{\lambda G}}\right]_v+\left[\frac{(\sqrt{\lambda G})_u}{\sqrt{\lambda E}}\right]_u\right\}$$

$$=-\frac{1}{\lambda\sqrt{EG}}\cdot\left\{\left[\frac{(\sqrt{E})_v}{\sqrt{G}}\right]_v+\left[\frac{(\sqrt{G})_u}{\sqrt{E}}\right]_u\right\}$$

$$=\frac{1}{\lambda}\tilde{K}$$

$$\Rightarrow \quad K=\frac{1}{\lambda}\tilde{K} \qquad \#$$

$$\begin{matrix} a & 0 & au & a \\ 0 & b & bv & 0 \end{matrix}$$

3. (1) 证: $\quad \vec{r}_u=(a,\ 0,\ au) \quad \vec{r}_v=(0,b,\ bv)$

$$\vec{n}=\frac{\vec{r}_u\wedge\vec{r}_v}{|\vec{r}_u\wedge\vec{r}_v|}=\frac{(-abu,-abv,ab)}{ab\sqrt{1+u^2+v^2}}=\frac{(-u,-v,1)}{\sqrt{1+u^2+v^2}}$$

$$EG-F^2=|\vec{r}_u\wedge\vec{r}_v|^2=a^2b^2(1+u^2+v^2)$$

$$\vec{r}_{uu}=(0,0,a) \quad \vec{r}_{uv}=0 \quad \vec{r}_{vv}=(0,0,b)$$

$$L=\frac{a}{\sqrt{1+u^2+v^2}} \quad M=0 \quad N=\frac{b}{\sqrt{1+u^2+v^2}} \quad \Rightarrow LN-M^2=\frac{ab}{1+u^2+v^2}$$

$$\Rightarrow K_S = \frac{LN-M^2}{EG-F^2} = \frac{1}{ab(1+u^2+v^2)^2}$$

完全相同的过程，有 $K_{\tilde{S}} = \frac{1}{\tilde{a}\tilde{b}(1+\tilde{u}^2+\tilde{v}^2)^2}$

故 $ab=\tilde{a}\tilde{b}$ 时，对在 $(u,v)=(\tilde{u},\tilde{v})$ 下，

$$K_S = K_{\tilde{S}}$$

(2) $T: S \longrightarrow \tilde{S}$ 是等距变换，$T(u,v)=(\xi,\eta)$

于是 由 $K_S(u,v)=K_{\tilde{S}}(\xi,\eta)$

$T(0,0)=(0,0)$，有

$$ab=\tilde{a}\tilde{b}$$

又由 $K_S(u,v)=K_{\tilde{S}}(\xi,\eta)$，得

$$u^2+v^2=\xi^2+\eta^2$$

令 $(u,v)=r(\cos\theta,\ \sin\theta)$，$(\xi,\eta)=r(\cos\varphi,\ \sin\varphi)$

即有 $T^*\frac{\partial}{\partial r}=\frac{\partial}{\partial r}$

$T^*\frac{\partial}{\partial\theta}=\varphi_\theta\frac{\partial}{\partial\varphi}$

$$\Rightarrow I\left(\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)=\tilde{I}\left(T^*\frac{\partial}{\partial r},\ T^*\frac{\partial}{\partial\theta}\right).$$

其中 $I/(dr\,d\theta) = -(a^2-b^2)\sin 2\theta + r^2\frac{a^2-b^2}{2}\sin 2\theta + r^2\left(\frac{a^2-b^2}{2}\right)^2\sin 4\theta.$

于是 有微分方程

$$-(a^2-b^2)\sin 2\theta + r^2\frac{a^2-b^2}{2}\sin 2\theta + r^2\left(\frac{a^2-b^2}{2}\right)^2\sin 4\theta.$$

$$= \varphi_\theta\left(-(\tilde{a}^2-\tilde{b}^2)\sin 2\varphi + r^2\frac{\tilde{a}^2-\tilde{b}^2}{2}\sin 2\varphi + r^2\left(\frac{\tilde{a}^2-\tilde{b}^2}{2}\right)^2\sin 4\varphi\right)$$

固定 r，有如下情况

(a) $\varphi=\theta$，$(\tilde{a},\tilde{b})=\pm(a,b)$

(b) $\varphi=\theta+\frac{\pi}{2}$，$(\tilde{a},\tilde{b})=\pm(b,a)$

5. $$d\vec{r}_\alpha = (\Gamma^\gamma_{\alpha\beta}\vec{r}_\gamma + b_{\alpha\beta}\vec{n})\,du^\beta$$

$$D\vec{r}_\alpha = \Gamma^\gamma_{\alpha\beta}\vec{r}_\gamma\,du^\beta$$

$$f^\alpha D\vec{r}_\alpha = f^\alpha \Gamma^\gamma_{\alpha\beta}\vec{r}_\gamma\,du^\beta = f^\beta \Gamma^\alpha_{\beta\gamma}\vec{r}_\alpha\,du^\gamma$$

$$\begin{aligned} D\vec{v} &= D(f^\alpha \vec{r}_\alpha) \\ &= df^\alpha \vec{r}_\alpha + f^\alpha D\vec{r}_\alpha \\ &= (df^\alpha + \Gamma^\alpha_{\beta\gamma} f^\beta\,du^\gamma)\vec{r}_\alpha \end{aligned}$$

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