AMS 316.01: Homework 4

Due on 10/28

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Problem 1

Find the ac.f of the first-order AR process defined by

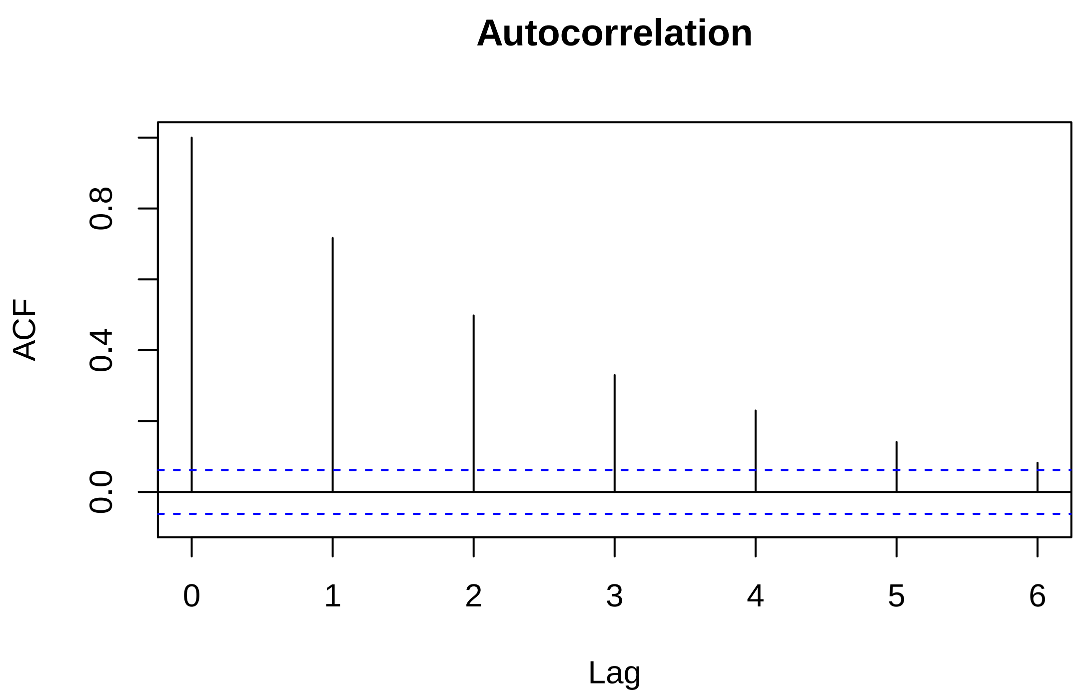
$$X_t = 0.7X_{t-1} + Z_t$$

Plot $\rho(k)$ for $k = -6, -5, ..., 0, +1, ..., +6$.

Solution

$$\rho(k) = \alpha^{|k|}$$
$$\rho(k) = 0.7^{|k|}, k = 0, \pm 1, \pm 2, ...$$

```r
set.seed(12345)
sim1 <- arima.sim(list(ar=c(0.7)),n=1000, innov=rnorm(1000))
acf(sim1, xlim = c(0,6), main='Autocorrelation')
```



Same plot for $k = 0, -1, -2, ..., -6$

Problem 2

If $X_t = \mu + Z_t + \beta Z_{t-1}$, where μ is a constant, show that the ac.f does not depend on μ.

Solution

The ac.f does not depend on μ. Proof in the equation to solve for $\rho(k)$

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \sum_{i=0}^{q-k} \beta_i \beta_{i+k} / \sum_{i=0}^{q} \beta_i^2 & k = 1, ..., q \\ 0 & k > q \end{cases}$$

μ is not involved in the expression to solve for $\rho(k)$

Problem 3

Consider the second-order AR process defined by

$$X_t = \lambda_1 X_{t-1} + \lambda_2 X_{t-2} + Z_t$$

If $\lambda_1 = 1/3, \lambda_2 = 2/9$, show that the ac.f of X_t is given by

$$\rho(k) = \frac{16}{21}\left(\frac{2}{3}\right)^{|k|} + \frac{5}{21}\left(-\frac{1}{3}\right)^{|k|}, k = 0, \pm 1, \pm 2, ...$$

Solution

For AR(2)

$$p(k) = A_1 \pi_1^{|k|} + A_2 \pi_2^{|k|}$$
$$y^2 - \alpha_1 y - \alpha_2 = 0$$
$$y^2 - \frac{1}{3} y - \frac{2}{9} = 0$$
$$\pi_1 = \frac{2}{3}, \pi_2 = -\frac{1}{3}$$

For A's,

$$A_1 = \frac{\alpha_1/(1-\alpha_2) - \pi_2}{\pi_1 - \pi_2}$$
$$A_1 = \frac{(1/3)/(7/9) + (1/3)}{(1/3)}$$
$$A_1 = \frac{16}{21}$$
$$A_2 = 1 - A_1$$
$$A_2 = \frac{5}{21}$$

$$\rho(k) = \frac{16}{21}\left(\frac{2}{3}\right)^{|k|} + \frac{5}{21}\left(-\frac{1}{3}\right)^{|k|}, k = 0, \pm 1, \pm 2, ...$$

Problem 4

Explain what is meant by a weakly (or second-order) stationary process, and define the ac.f $\rho(k)$ for such a process. Show that $\rho(k) = \rho(-k)$ and that $|\rho(k)| \leq 1$. Show that the ac.f of the stationary second-order AR process

$$X_t = \frac{1}{12}X_{t-1} + \frac{1}{12}X_{t-2} + Z_t$$

is given by

$$\rho(k) = \frac{45}{77}\left(\frac{1}{3}\right)^{|k|} + \frac{32}{77}\left(-\frac{1}{4}\right)^{|k|} \quad k = 0, \pm 1, \pm 2, ...$$

Solution

A weakly stationary process is when the time series has a constant mean and variance.The autocovariance does not vary and is finite.

Proof of $\rho(k) = \rho(-k)$

$$\begin{aligned}\gamma(-k) &= Cov(X_{t-k}, X_t)\\ &= E[X_{t-k}X_t] - \mu_{t-k}\mu_t\\ &= E[X_{t+k}X_t] - \mu_{t+k}\mu\\ &= \gamma(k)\end{aligned}$$

For ρ

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$
$$\rho(-k) = \frac{\gamma(-k)}{\gamma(0)} = \frac{\gamma(k)}{\gamma(0)} = \rho(k)$$

Therefore $\rho(k) = \rho(-k)$.
For $|\rho(k)| \leq 1$

$$-1 \leq \gamma(k) \leq 1$$
$$-1 \leq \rho(k) \leq 1$$

Therefore $|\rho(k)| \leq 1$
For AR(2)

$$p(k) = A_1\pi_1^{|k|} + A_2\pi_2^{|k|}$$
$$y^2 - \alpha_1 y - \alpha_2 = 0$$
$$y^2 - \frac{1}{12}y - \frac{1}{12} = 0$$
$$\pi_1 = \frac{1}{3}, \pi_2 = -\frac{1}{4}$$

Problem 4 continued on next page...

For A's,

$$A_1 = \frac{\alpha_1/(1-\alpha_2) - \pi_2}{\pi_1 - \pi_2}$$
$$A_1 = \frac{(1/12)/(11/12) + (1/4)}{(7/12)}$$
$$A_1 = \frac{45}{77}$$
$$A_2 = 1 - A_1$$
$$A_2 = \frac{32}{77}$$

$$\rho(k) = \frac{45}{77}\left(\frac{1}{3}\right)^{|k|} + \frac{32}{77}\left(-\frac{1}{4}\right)^{|k|} \; k = 0, \pm 1, \pm 2, ...$$

Problem 5

Show that AR(2) process

$$X_t = X_{t-1} + cX_{t-2} + Z_t$$

is stationary provided $-1 < c < 0$. Find the autocorrelation function when $c = -3/16$. Show that the AR(3) process

$$X_t = X_{t-1} + cX_{t-2} - cX_{t-3} + Z_t$$

is non-stationary for all values of c.

Solution

Stationary condition is solutions of x from below equation must be outside unit circle

$$\phi(x) = 1 - x - cx^2 > 1$$
$$root: -1 < c < 0$$

ac.f for when $c = -\frac{3}{16}$

$$p(k) = A_1 \pi_1^{|k|} + A_2 \pi_2^{|k|}$$
$$y^2 - \alpha_1 y - \alpha_2 = 0$$
$$y^2 - 1y - \frac{3}{16} = 0$$
$$\pi_1 = \frac{1}{4}, \pi_2 = \frac{3}{4}$$

Problem 5 continued on next page...

For A's,

$$A_1 = \frac{\alpha_1/(1-\alpha_2) - \pi_2}{\pi_1 - \pi_2}$$
$$A_1 = \frac{(1)/(19/16) - (3/4)}{(-1/2)}$$
$$A_1 = \frac{-7}{38}$$
$$A_2 = 1 - A_1$$
$$A_2 = \frac{45}{38}$$

$$\rho(k) = -\frac{7}{38}\left(\frac{1}{4}\right)^{|k|} + \frac{45}{38}\left(\frac{3}{4}\right)^{|k|} \; k = 0, \pm 1, \pm 2, ...$$