# AMS 316.01: Homework 3

Due on 10/07

*Haipeng Xing*

Harris Temuri

In the following equations, $Z_t$ is a discrete-time, purely random process such that $E(Z_t) = 0, Var(Z_t) = s_Z^2$, and successive values of $Z_t$ are independent so that $Cov(Z_t, Z_{t+k}) = 0, k \neq 0$.

# Problem 1

Show that the ac.f of the second-order MA process

$$X_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2}$$

is given by

$$p(k) = \begin{cases} 1, & k = 0 \\ 0.37, & k = \pm 1 \\ -0.13, & k = \pm 2 \\ 0, & otherwise \end{cases}$$

## Solution

$$p(k) = \begin{cases} 1, & k = 0 \\ \sum_{i=0}^{q-k} \beta_i \beta_{i+k} / \sum_{i=0}^{q} \beta_i^2, & k = 1, ..., q \\ 0, & k \geq q \end{cases}$$

for k=1,

$$\begin{aligned} &= \sum_{i=0}^{q-1} \beta_i \beta_{i+k} \\ &= (1 * 0.7) + (0.7 * (-.2)) \\ &= 0.56 \end{aligned}$$

$$\begin{aligned} &= \sum_{i=0}^{q} \beta_i^2 \\ &= 1^2 + 0.7^2 + (-0.2)^2 \\ &= 1.53 \end{aligned}$$

$$\begin{aligned} &= \sum_{i=0}^{q-1} \beta_i \beta_{i+k} / \sum_{i=0}^{q} \beta_i^2 \\ &= {}^{0.56}/_{1.53} \\ &= 0.366 \cong \mathbf{0.37} \end{aligned}$$

for k=2,

$$= \sum_{i=0}^{q-2} \beta_i \beta_{i+k}$$
$$= (1 * (-0.2))$$
$$= -0.2$$

$$= \sum_{i=0}^{q-2} \beta_i \beta_{i+k} / \sum_{i=0}^{q} \beta_i^2$$
$$= {-0.2}/{1.53}$$
$$= -0.1307 \cong \mathbf{-0.13}$$

$$p(k) = \begin{cases} 1, & k = 0 \\ 0.37, & k = \pm 1 \\ -0.13, & k = \pm 2 \\ 0, & otherwise \end{cases}$$

# Problem 2

Consider the MA($m$) process, with equal weights $\frac{1}{m+1}$ at all lags (so it is a real moving average), given by,

$$X_t = \sum_{k=0}^{m} \frac{Z_{t-k}}{m+1}$$

Show that the ac.f of this process is

$$p(k) = \begin{cases} \frac{m+1-k}{m+1}, & k = 0, ..., m \\ 0, & k \geq m \end{cases}$$

## Solution

$$p(k) = \begin{cases} 1, & k = 0 \\ \sum_{i=0}^{q-k} \beta_i \beta_{i+k} / \sum_{i=0}^{q} \beta_i^2, & k = 1, ..., m \\ 0, & k \geq q \end{cases}$$

for $k = 0, ..., m$,

$$= \sum_{i=0}^{m} \beta_i \beta_{i+k}$$
$$= \frac{m+1-k}{(m+1)^2}$$

$$= \sum_{i=0}^{m} \beta_i^2$$
$$= \frac{1}{m+1}$$

$$= \sum_{i=0}^{q-1} \beta_i \beta_{i+k} / \sum_{i=0}^{q} \beta_i^2$$
$$= \frac{m+1-k}{m+1}$$

$$p(k) = \begin{cases} 1, & k = 0 \\ \frac{m+1-k}{m+1}, & k = 1, ..., m \\ 0, & otherwise \end{cases}$$

# Problem 3

Consider the infinite-order MA process $X_t$, defined by

$$X_t = Z_t + C(Z_{t-1} + Z_{t-2} + ...)$$

where $C$ is a constant. Show that the process is non-stationary. Also show that the series of the first differences $Y_t$ defined by

$$Y_t = X_t - X_{t-1}$$

is a first-order MA process and is stationary. Find the ac.f of $Y_t$.

## Solution

$X_t$ is not stationary because its variance changes with time. $Y_t$ is stationary because it forms a purely random process, which is stationary.

$$X_t = Z_t + C(Z_{t-1} + Z_{t-2} + ...)Y_t = X_t - X_{t-1}$$

$$\begin{aligned} Y_t &= Z_t + C(Z_{t-1} + Z_{t-2} + ...) - (Z_{t-1} + C(Z_{t-2} + Z_{t-3} + ...)) \\ &= Z_t + CZ_{t-1} - Z_{t-1} \\ &= Z_t + (C-1)Z_{t-1} \end{aligned}$$

$$p(k) = \begin{cases} 1, & k = 0 \\ \sum_{i=0}^{q-k} \beta_i\beta_{i+k} / \sum_{i=0}^{q} \beta_i^2, & k = 1, ..., q \\ 0, & k \geq q \end{cases}$$

for $k = 1$,

$$\begin{aligned} &= \sum_{i=0}^{q-1} \beta_i\beta_{i+k} \\ &= C - 1 \end{aligned}$$

$$\begin{aligned} &= \sum_{i=0}^{q} \beta_i^2 \\ &= 1 + (C-1)^2 \end{aligned}$$

$$
\begin{aligned}
&= \textstyle\sum_{i=0}^{q-1} \beta_i \beta_{i+k} \Big/ \sum_{i=0}^{q} \beta_i^2 \\
&= \frac{C-1}{1+(C-1)^2}
\end{aligned}
$$

$$
p(k) = \begin{cases} 1, & k = 0 \\ \frac{C-1}{1+(C-1)^2}, & k = \pm 1 \\ 0, & otherwise \end{cases}
$$