Kümeler

$s(A) = n$ (A kümesinin eleman sayısı n olsun)

Alt küme sayısı $= 2^n$

Öz alt küme sayısı $= 2^n - 1$ (Kendi hariç)

m elemanlı alt küme sayısı $= C\binom{n}{m} = \binom{n}{m} = \dfrac{n!}{(n-m)! \cdot m!}$ Kombinezon

$P\binom{n}{m} = \dfrac{n!}{(n-m)!} = m! \cdot C\binom{n}{m}$ Permütasyon

---

$(x+y)^n \xrightarrow{m.\text{terimi}} \binom{n}{m} \cdot x^{n-m+1} \cdot y^{m-1}$ Binom açılımından yararlanılır.

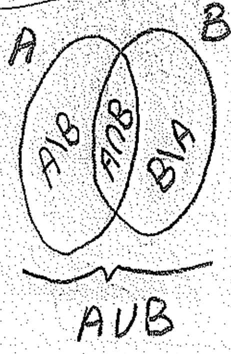
A ve B farklı iki küme olsun.

$\forall x \in A$ için $x \in B$ ise $A \subset B$. B kümesi A kümesini kapsar. Yani A kümesi B kümesinin bir alt kümesidir

$A \subset B \Longleftrightarrow [x \in A \Rightarrow x \in B]$

$A \subset B$ ve $A \supset B$ ise $A = B$ (Eşit kümeler)

$A \neq B$ fakat $s(A) = s(B)$ ise $A \equiv B$ (Denk kümeler)



$A \cup B = \{x : x \in A \text{ veya } x \in B\}$ A bileşim B

$A \cap B = \{x : x \in A \text{ ve } x \in B\}$ A kesişim B

$A \setminus B = \{x : x \in A \text{ ve } x \notin B\}$ A fark B

$B \setminus A = \{x : x \notin A \text{ ve } x \in B\}$ B fark A

$s(A \times B) = s(A) \cdot s(B)$

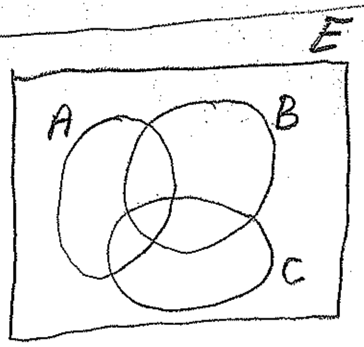
$s(A \cup B) = s(A) + s(B) - s(A \cap B)$

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

$A \times B = \{(x,y) : x \in A, y \in B\}$ Kartezyen çarpım

$A = B$ ise $A \times B = B \times A$ olur.



$E = A \cup A' = B \cup B' = C \cup C'$ Evrensel küme

$\phi = A \cap A' = B \cap B' = C \cap C'$ Boş küme

$A \times B \times C = (A \times B) \times C = A \times (B \times C)$

$= \{(x,y,z) : x \in A, y \in B, z \in C\}$ Sıralı üçlü

$s(A \cup B \cup C) = s(A) + s(B) + s(C) - s(A \cap B) - s(A \cap C) - s(B \cap C) + s(A \cap B \cap C)$

$N=\{0,1,2,3,4,\dots\}$ Doğal sayılar kümesi

$Z=\{\dots,-3,-2,-1,0,1,2,3,\dots\}$ Tamsayılar kümesi

$Z = Z^- \cup \{0\} \cup Z^+$

Z^+ → pozitif tamsayılar kümesi

Z^- → negatif tamsayılar kümesi

$R = Q \cup I$

I → irrasyonel sayılar kümesi

Q → rasyonel sayılar kümesi

R → reel sayılar kümesi

$Q=\{q : q=\frac{m}{n}\ ;\ m,n\in Z,\ n\neq 0\}$ iki tamsayının oranı şeklinde gösterilen sayılardır. $\frac{m}{n}$ en sade halde olmalı.

$n=1$ için tamsayı $(Z\subset Q)$

<u>İrrasyonel sayılar kümesi</u>

Ondalıklı kısmın uzunluğu sonsuz ve peryodik değil.

$$\left.\begin{aligned} e &\approx 2.718281828 \\ \pi &\approx 3.141592654 \\ \sqrt{2} &\approx 1.41421356 2 \\ \sqrt{3} &\approx 1.732050808 \end{aligned}\right\}$$

Hesap makinası yuvarlar. Gerçekte irrasyonel ama biz rasyonel kabul ederiz.

$a,b\in A$ için $a*b\in A$ ise $*$ işlemi A kümesinde kapalıdır.

Reel sayılar kümesi 4 işleme göre de kapalıdır.

Çift sayının karesi çift sayıdır

n çift sayı olsun

$n=2k,\ k\in Z$

$n^2=(2k)^2=4k^2=2(2k^2)$

$=2m,\ m\in Z$

n^2 çift sayı

Tek sayının karesi tek sayıdır.

n tek sayı olsun

$n=2k+1,\ k\in Z$

$n^2=(2k+1)^2=4k^2+4k+1$

$=2(2k^2+2k)+1$

$=2m+1,\ m\in Z$

n^2 tek sayı

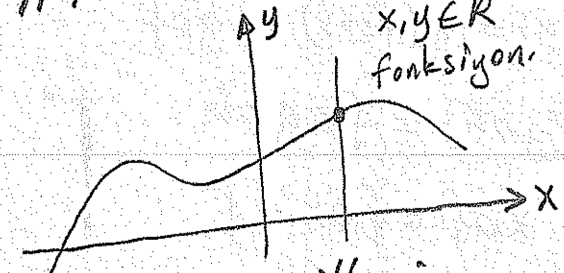
Fonksiyonlar

A (Tanım kümesi) ve B (Değer kümesi) boş olmayan iki küme olsun. A kümesindeki her elemanı B kümesinde bir elemana eşleyen f bağıntısına fonksiyon denir.

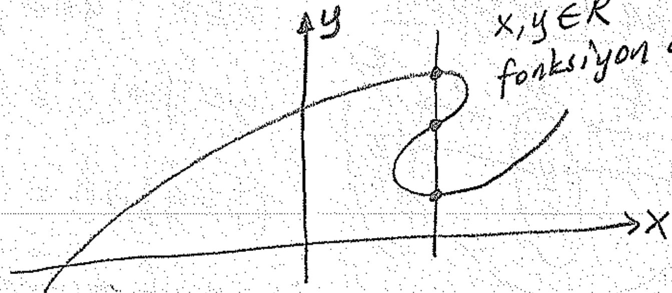
$f: A \rightarrow B$ veya $x \rightarrow y = f(x)$, $x \in A$, $y \in B$

A kümesindeki her elemanın B kümesinde bir karşılığı olmalı.
A kümesindeki her eleman B kümesinde birden fazla yere gidemez.

$x, y \in \mathbb{R}$
fonksiyon.

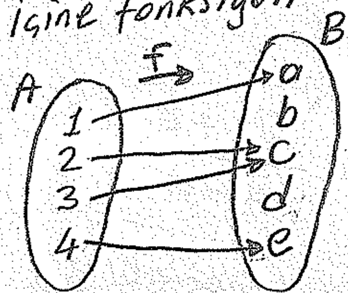


$x, y \in \mathbb{R}$
fonksiyon değil.



Fonksiyon Çeşitleri

① İçine fonksiyon



$A = \{1,2,3,4\}$

$B = \{a,b,c,d,e\}$

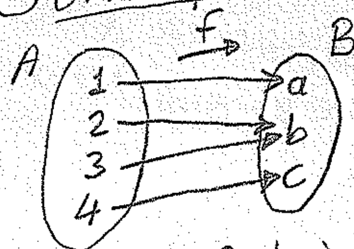
$f(A) = \{a,c,e\}$

$f: A \rightarrow B$ olsun

$f(A) \subset B$ ve $f(A) \neq B$ ise

f içine fonksiyondur.

② Örten fonksiyon



$A = \{1,2,3,4\}$

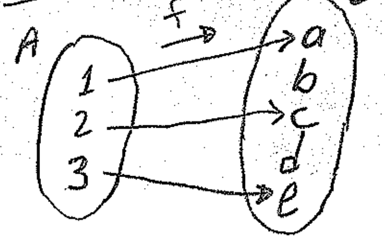
$B = f(A) = \{a,b,c\}$

$f: A \rightarrow B$ olsun

$f(A) = B$ ise

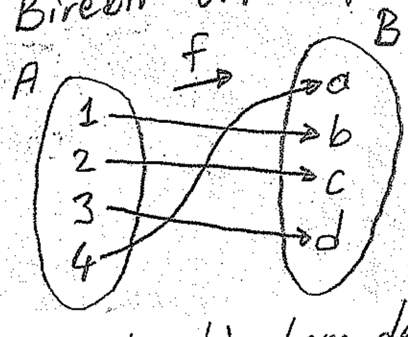
f örten fonksiyondur.

③ Birebir fonksiyon



A kümesinin her elemanı B kümesinde farklı bir eleman ile eşleşiyor.

④ Birebir örten fonksiyon



Hem birebir hem de örten ise f birebir örten fonksiyon

Eşit fonksiyonlar

$f, g : A \to B$ olsun.

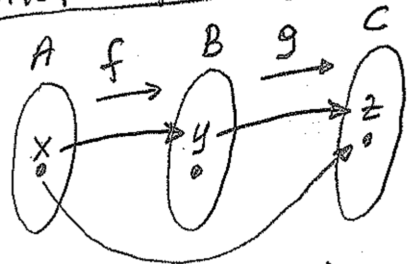
$\forall x \in A$ için $f(x) = g(x)$ ise f ve g eşit fonksiyonlardır.

Sabit fonksiyon : $f(x) = k$

Birim (Etkisiz) fonksiyon : $f(x) = x$

Bileşke fonksiyon

$x \in A$, $y \in B$, $z \in C$ olsun



$gof = g(f(x))$

$y = f(x)$, $z = g(y) = g(f(x)) = gof$

$(fog)oh = fo(goh) = fogoh$

$gof \neq fog \quad xof = fox = f$

Ters fonksiyon

$fof^{-1} = f^{-1}of = x$

$(f^{-1})^{-1} = f$

$(fog)^{-1} = g^{-1}of^{-1}$

$(fogoh)^{-1} = h^{-1}og^{-1}of^{-1}$

$fog = h$ ise

$f^{-1}ofog = f^{-1}oh$

$g = f^{-1}oh$

$fogog^{-1} = hog^{-1}$

$f = hog^{-1}$

$f(x) = \dfrac{ax+b}{cx+d}$ ise $f^{-1}(x) = \dfrac{-dx+b}{cx-a}$

→ x değişkenini çek. Sonra x yerine y, y yerine x koy.

Bir fonksiyonun tersi $y = x$ doğrusuna göre simetrisidir. Bu yüzden x yerine y, y yerine x konur ve y çekilir.

Parçalı fonksiyon

$$f(x) = \begin{cases} g(x) & x < a \\ h(x) & a \leq x < b \\ p(x) & x \geq b \end{cases}$$

parçalı fonksiyonda parça sayısı en az iki olmalı.

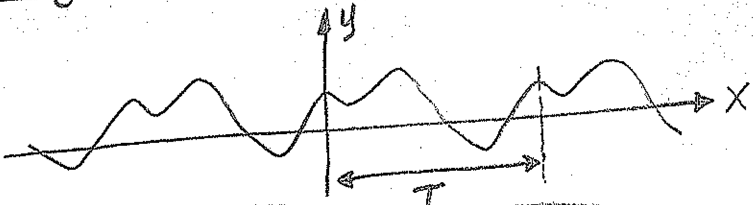
Mutlak fonksiyon

$$f(x) = |g(x)| = \begin{cases} g(x) & g(x) \geq 0 \\ -g(x) & g(x) < 0 \end{cases}$$

$$|x| = \sqrt{x^2} = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\left| \frac{f(x) \cdot g(x)}{h(x)} \right| = \frac{|f(x)| \cdot |g(x)|}{|h(x)|}, \quad h(x) \neq 0$$

Peryodik fonksiyon



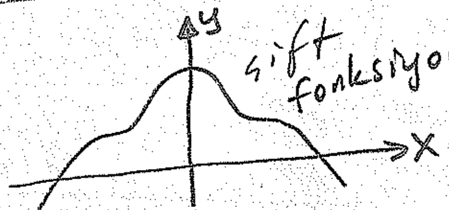
T : Peryod

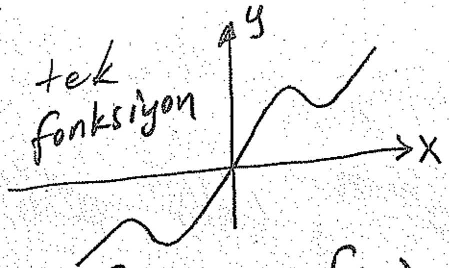
f : Frekans

$$f = \frac{1}{T}$$

$$f(x) = f(x+nT) \;,\; n \in \mathbb{Z}$$

Tek ve Çift fonksiyonlar





$f(-x) = f(x)$ çift fonk.

$f(-x) = -f(x)$ tek fonks.

$$f(x) = f_ç(x) + f_t(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{çift fonk}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{tek fonk.}}$$

$$y = f(x) = x^3 + 3x^2 - 2x + 5 = \underbrace{3x^2+5}_{\text{çift}} + \underbrace{x^3-2x}_{\text{tek}}$$

$y = \cos x$ çift fonksiyon.

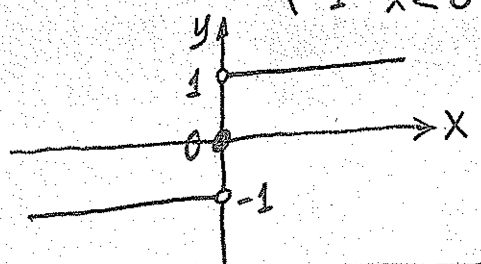
$\cos(-x) = \cos x$

$y = \sin x$ tek fonksiyon.

$\sin(-x) = -\sin x$

Signum fonksiyonu (işaret fonksiyonu)

$$y = \operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



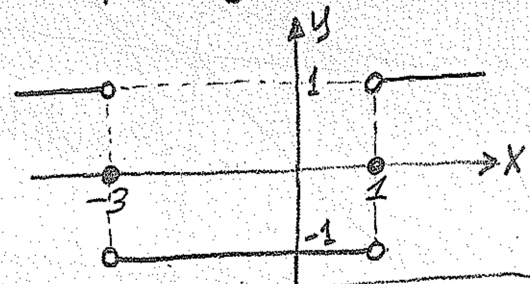
$y = \operatorname{sgn}(x^2+2x-3)$ fonk. grafiğini çiz.

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3 \;,\; x=1$

	-3		1	
$+$		$-$		$+$



Tam değer fonksiyonu

$$f(x) = [\![g(x)]\!] = \begin{cases} g(x) & , \; g(x) \in \mathbb{Z} \\ g(x)\text{'den küçük en büyük tamsayı} & , \; g(x) \notin \mathbb{Z} \end{cases}$$

$[\![x]\!] = a \;,\; a \in \mathbb{Z}$ ise $a \leq x < a+1$

$[\![3.2]\!] = [\![3.8]\!] = 3$

$[\![-3.2]\!] = [\![-3.8]\!] = -4$

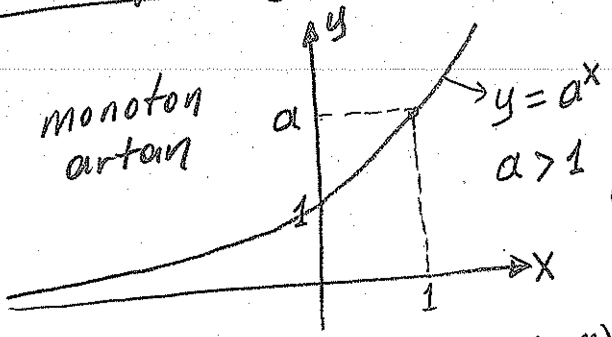
$[\![x+y]\!] \geq [\![x]\!] + [\![y]\!]$

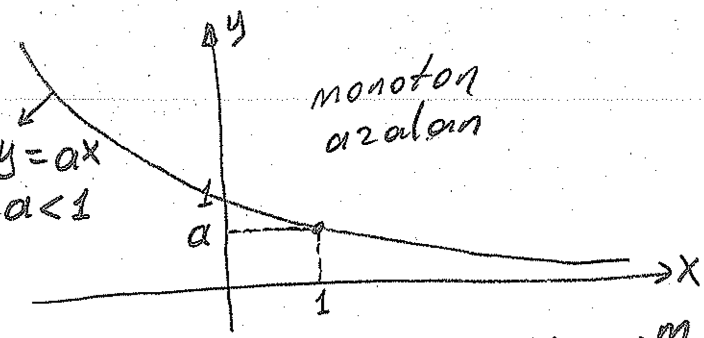
$[\![x]\!] \leq x < [\![x]\!] + 1$

$[\![x+a]\!] = [\![x]\!] + a \;,\; a \in \mathbb{Z}$

$[\![x + [\![y]\!]]\!] = [\![x]\!] + [\![y]\!]$

Üstel fonksiyonlar





$a^m a^n = a^{m+n}$, $(a^m)^n = a^{mn}$, $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

$\frac{a^m}{a^n} = a^m a^{-n} = a^{m-n}$, $(ab)^m = a^m b^m$, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Logaritmik fonksiyonlar

$x = a^y \Longleftrightarrow y = \lg_a x$

$x = e^y \Longleftrightarrow y = \ln x$

$\lg_a a = 1$ $\quad \lg_a 1 = 0$

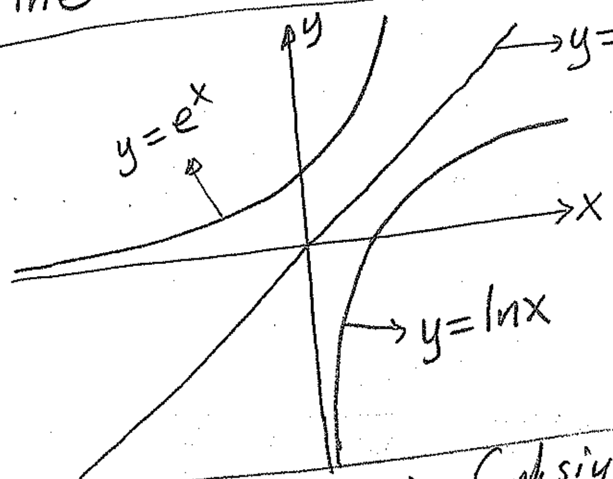
$\ln e = 1$ $\quad \ln 1 = 0$

$\lg\left(\frac{ab}{c}\right) = \lg a + \lg b - \lg c$

$\lg_a b = \frac{\lg b}{\lg a} = \frac{1}{\lg_b a}$ $\qquad e^{\ln x} = x$

$\lg x = \lg_{10} x$

$\lg_{a^m} b^n = \frac{n}{m} \lg_a b$ $\qquad a^{\lg_a x} = x$



$y = e^x \xleftrightarrow{\text{tersi}} y = \ln x$

Bu iki fonksiyon $y = x$ fonksiyonuna göre simetriktir.

$f \circ f^{-1} = f^{-1} \circ f = x$

$y = \ln(x + \sqrt{5+x^2})$ fonksiyonunun tersi.

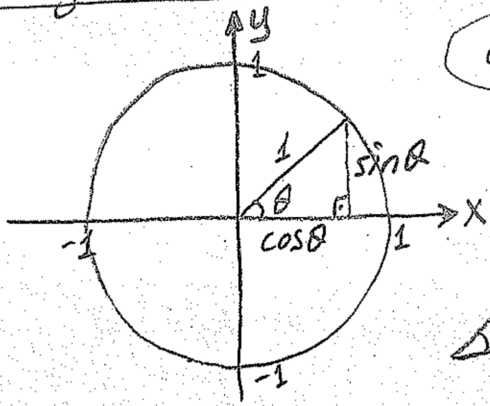
$y = \ln(x + \sqrt{5+x^2}) \longrightarrow e^y = x + \sqrt{5+x^2}$

$-y = -\ln(x + \sqrt{5+x^2}) \longrightarrow e^{-y} = \frac{1}{x + \sqrt{5+x^2}} = \frac{x - \sqrt{5+x^2}}{-5}$

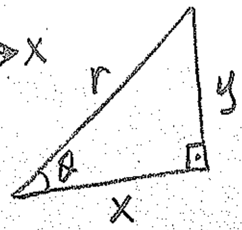
$e^y - 5e^{-y} = x + \cancel{\sqrt{5+x^2}} + x - \cancel{\sqrt{5+x^2}} = 2x$

$x = \frac{e^y - 5e^{-y}}{2} \longrightarrow f^{-1}(x) = \frac{e^x - 5e^{-x}}{2}$

Trigonometrik fonksiyonlar



$$\cos^2\theta + \sin^2\theta = 1 \rightarrow \cos^2\theta = 1 - \sin^2\theta$$
$$\rightarrow \sin^2\theta = 1 - \cos^2\theta$$



$$r = \sqrt{x^2+y^2}$$

$$\cos\theta = x/r \qquad \sin\theta = y/r$$

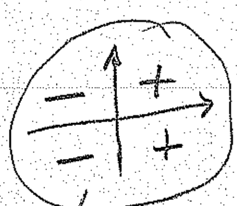
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{x}{y}$$

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

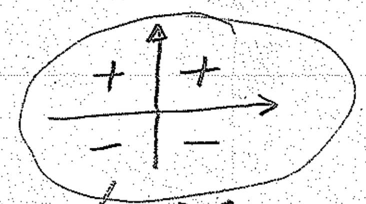
$\cos(-\theta) = \cos\theta$ çift fonk.

$\sin(-\theta) = -\sin\theta$ tek fonk.



$\rightarrow \cos\theta$

$-1 \le \cos\theta \le 1$



$\rightarrow \sin\theta$

$-1 \le \sin\theta \le 1$

$\theta_1 + \theta_2 = \pi/2$ ise $\sin\theta_1 = \cos\theta_2$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$
$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$
$$2\sin\alpha\sin\beta = -\cos(\alpha+\beta) + \cos(\alpha-\beta)$$

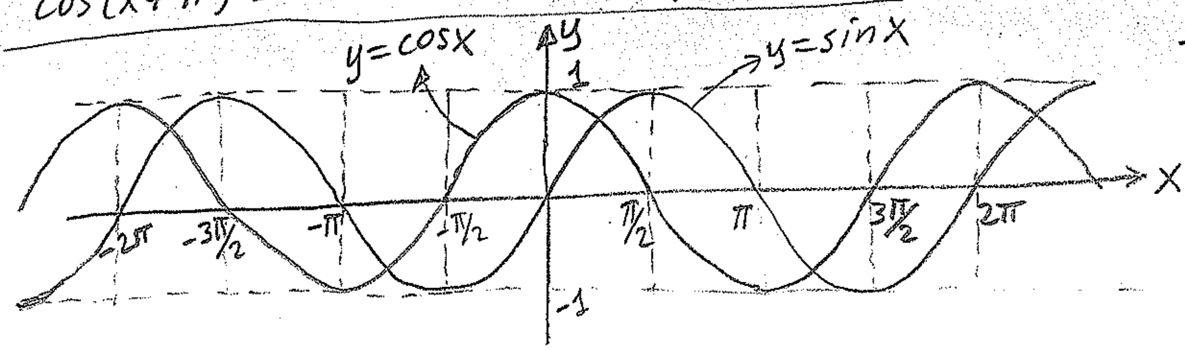
$$\cos 2X = \cos^2 X - \sin^2 X = 2\cos^2 X - 1 = 1 - 2\sin^2 X$$

$$\cos^2 X = \frac{1+\cos 2X}{2}$$
$$\sin^2 X = \frac{1-\cos 2X}{2}$$

$$\sin 2X = 2\cos X \sin X$$

$$\sin(x+\pi) = -\sin x \qquad \sin(x+\pi/2) = \cos x$$
$$\cos(x+\pi) = -\cos x \qquad \cos(x+\pi/2) = -\sin x$$

$$\tan(\alpha \mp \beta) = \frac{\tan\alpha \mp \tan\beta}{1 \pm \tan\alpha\tan\beta}$$



$T = 2\pi$

peryodik

Ters trigonometrik fonksiyonlar

$f(x) = \sin x \longrightarrow f^{-1}(x) = \arcsin x$

$f(x) = \cos x \longrightarrow f^{-1}(x) = \arccos x$

$f(x) = \tan x \longrightarrow f^{-1}(x) = \arctan x$

$f(x) = \cot x \longrightarrow f^{-1}(x) = \operatorname{arccot} x$

$f(x) = \sec x \longrightarrow f^{-1}(x) = \operatorname{arcsec} x$

$f(x) = \csc x \longrightarrow f^{-1}(x) = \operatorname{arccsc} x$

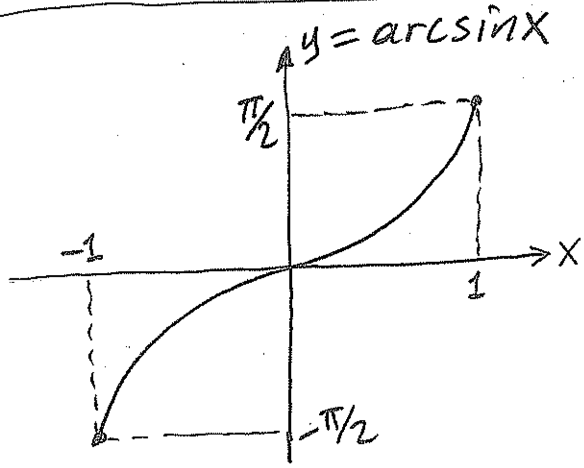
$f \circ f^{-1} = f^{-1} \circ f = x$

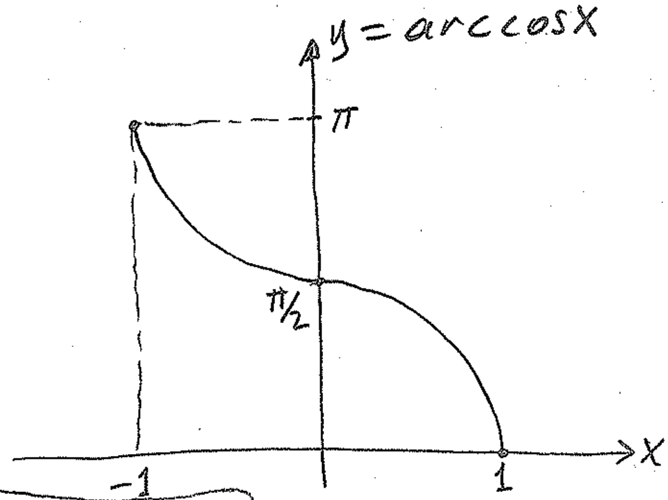
$\sin(\arcsin x) = x$

$\arcsin(\sin x) = x$

$\tan(\arctan x) = x$

$\arctan(\tan x) = x$



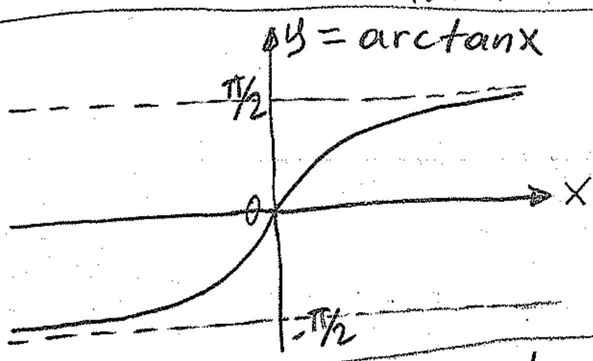


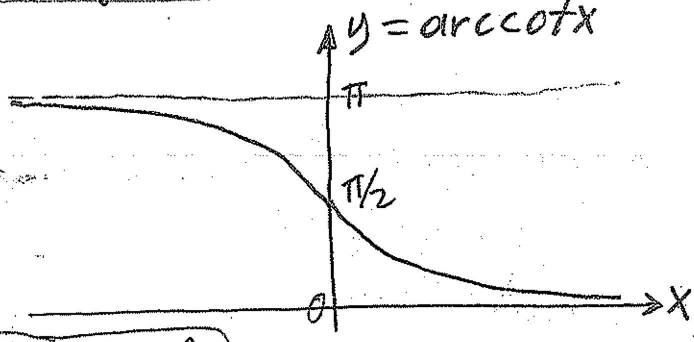
$$\boxed{\arcsin x + \arccos x = \pi/2 \,,\ x \in [-1,1]}$$

$y = \arcsin x \longrightarrow x \in [-1,1] \,,\ y \in [-\pi/2, \pi/2]$

$y = \arccos x \longrightarrow x \in [-1,1] \,,\ y \in [0, \pi]$

Tanım aralığı   Değer aralığı





$$\boxed{\arctan x + \operatorname{arccot} x = \pi/2 \,,\ x \in R}$$

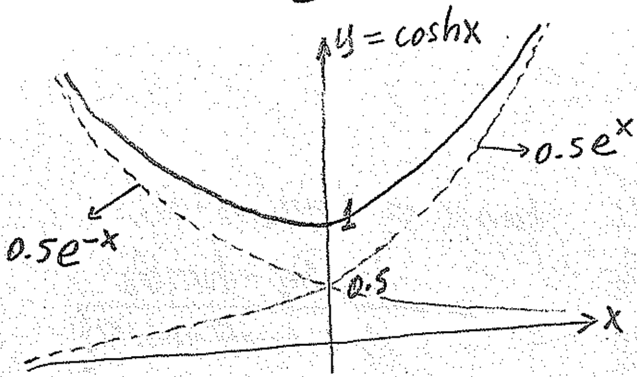
$y = \arctan x \longrightarrow x \in R \,,\ y \in (-\pi/2, \pi/2)$

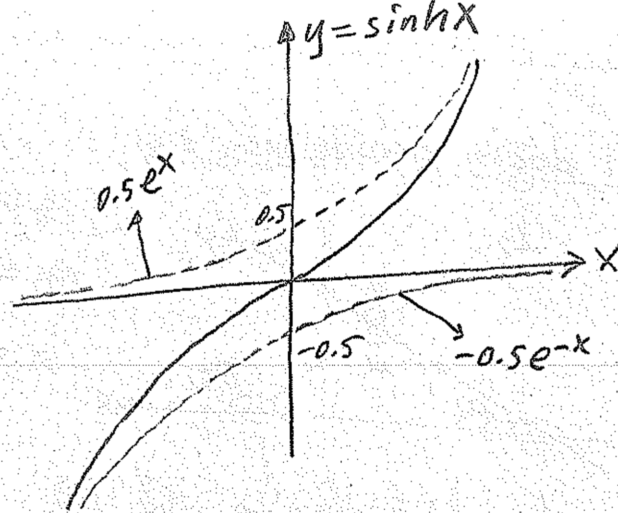
$y = \operatorname{arccot} x \longrightarrow x \in R \,,\ y \in (0, \pi)$

Tanım aralığı   değer aralığı

Hiperbolik fonksiyonlar

$$\left.\begin{aligned}\cosh x &= \frac{e^x+e^{-x}}{2}\\ \sinh x &= \frac{e^x-e^{-x}}{2}\end{aligned}\right\} \quad f(x)=e^x = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{çift fonk.}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{tek fonk.}}$$

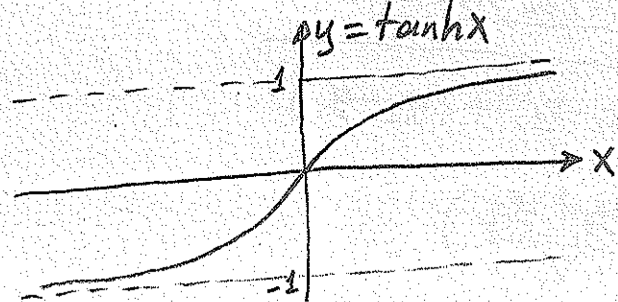


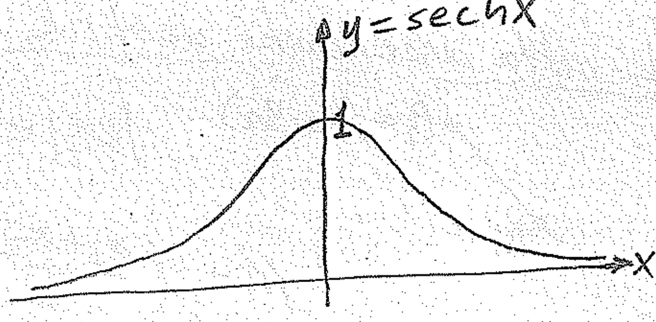


$$\cosh x + \sinh x = e^x$$
$$\cosh x - \sinh x = e^{-x}$$
$$\overline{\cosh^2 x - \sinh^2 x = 1}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1} \qquad \coth x = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x-e^{-x}}$$





$$\cosh(a+b) = \cosh a\cosh b \mp \sinh a \sinh b$$
$$\sinh(a+b) = \sinh a \cosh b \mp \cosh a \sinh b$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x \qquad \cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh 2x = 2\cosh x \sinh x \qquad \sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$
$$\operatorname{csch}^2 x = \coth^2 x - 1$$

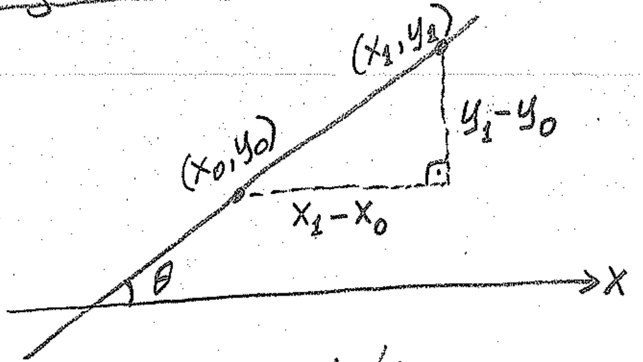
$$\operatorname{arcsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ 0<x\le 1$$

$$\operatorname{arcsinh} x = \ln\left(x+\sqrt{x^2+1}\right),\ x\in R$$

$$\operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ |x|<1$$

$$\operatorname{arccosh} x = \ln\left(x+\sqrt{x^2-1}\right),\ x\ge 1$$

Doğrunun analitik incelenmesi

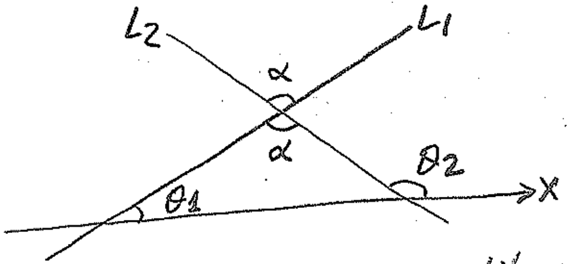


$$m = \tan\theta = \frac{y_1 - y_0}{x_1 - x_0}$$

$$\left.\begin{aligned} y &= y_0 + m(x - x_0) \\ y &= mx + b \end{aligned}\right\} \text{Doğru denklemleri}$$

$$Ax + By + C = 0 \quad \} \text{ Doğrunun genel denklemi}$$

iki doğru arasındaki açı



$$m_1 = \tan\theta_1 \qquad \tan\alpha = \tan(\theta_2 - \theta_1)$$

$$m_2 = \tan\theta_2 \qquad = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2}$$

$$\alpha = \theta_2 - \theta_1 \qquad = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Doğrular birbirlerine dik ise

$$\alpha = \pi/2 \longrightarrow \tan\pi/2 = \infty$$

$1 + m_1 m_2 = 0$ olmalı

$m_1 m_2 = -1$ veya $m_2 = -\dfrac{1}{m_1}$

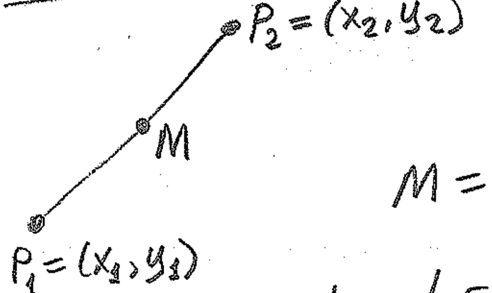
Doğrular birbirlerine paralel ise

$$\alpha = \theta_2 - \theta_1 = 0 \longrightarrow \theta_2 = \theta_1$$

$$\tan\theta_2 = \tan\theta_1$$

$$m_2 = m_1$$

iki nokta arasındaki uzaklık ve orta nokta

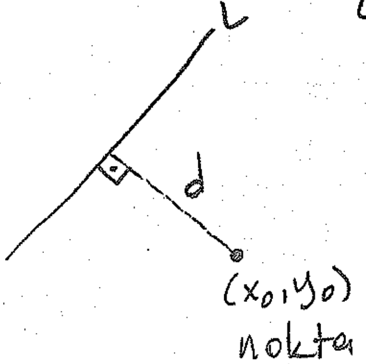


$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

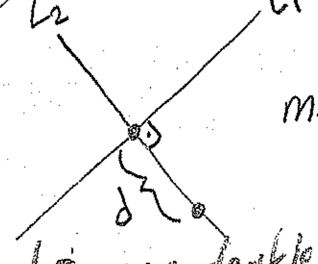
Bir noktanın bir doğruya en yakın uzaklığı

$ax + by + c = 0$ doğru denklemi



$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

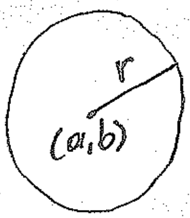
diğer yol



$$m_2 = -\frac{1}{m_1}$$

Diğer doğrunun denklemini bul.
Kesiştikleri noktayı bul.
iki nokta arasındaki uzaklık.

Çember Denklemi



$$(x-a)^2+(y-b)^2=r^2 \Rightarrow x^2+y^2+Dx+Ey+F=0$$

$$M=(a,b)=\left(-\frac{D}{2},-\frac{E}{2}\right)$$

$$r=\frac{1}{2}\sqrt{D^2+E^2-4F}$$

Küre Denklemi

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

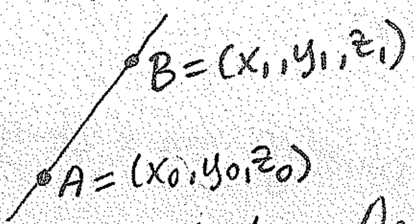
$$x^2+y^2+z^2+Dx+Ey+Fz+G=0$$

$$M=(a,b,c)=\left(-\frac{D}{2},-\frac{E}{2},-\frac{F}{2}\right) \qquad r=\frac{1}{2}\sqrt{D^2+E^2+F^2-4G}$$

Uzayda Doğru Denklemi

$$\left.\begin{aligned} x&=x_0+at \\ y&=y_0+bt \\ z&=z_0+ct \end{aligned}\right\} \quad \frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}=t$$

$$|AB|=\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}$$

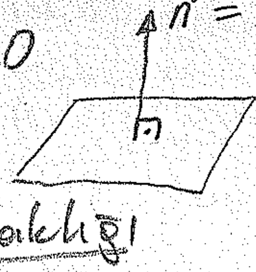


$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}$$

Uzayda Düzlem Denklemi

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

$$Ax+By+Cz+D=0$$



$\vec{n}=(A,B,C)$ düzlemin normali

Bir noktanın bir düzleme en yakın uzaklığı

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

Bir doğru ile bir düzlemin kesişme noktası

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{d}=t \quad \text{Doğru}$$

$$Ax+By+Cz+D=0 \quad \text{Düzlem}$$

$x=x_0+at$, $y=y_0+bt$, $z=z_0+ct$ çekilip düzlem denkleminde yerine konur. t bulunur. sonra diğer parametreler.

Düzlemde Simetriler

x eksenine göre simetri

(x,y) noktasının x eksenine göre simetriği $(x,-y)$ noktasıdır.

$y=f(x)$ ise simetriği $y=-f(x)$

y eksenine göre simetri

(x,y) noktasının y eksenine göre simetriği $(-x,y)$ noktasıdır.

$y=f(x)$ ise simetriği $y=f(-x)$

Orjine göre simetri

(x,y) noktasının orjine göre simetriği $(-x,-y)$ noktasıdır.

$y=f(x)$ ise simetriği $y=-f(-x)$

x=a doğrusuna göre simetri

(x,y) noktasının $x=a$ doğrusuna göre simetriği $(2a-x,y)$

$y=f(x)$ ise simetriği $y=f(2a-x)$

y=b doğrusuna göre simetri

(x,y) noktasının $y=b$ doğrusuna göre simetriği $(x,2b-y)$

$y=f(x)$ ise simetriği $y=2b-f(x)$

Bir noktaya göre simetri

(x,y) noktasının (a,b) noktasına göre simetriği $(2a-x,2b-y)$

$y=f(x)$ ise simetriği $y=2b-f(2a-x)$

y=x doğrusuna göre simetri

(x,y) noktasının $y=x$ doğrusuna göre simetriği (y,x)

$y=f(x)$ ise simetriği $x=f(y)$

Bir fonksiyonun $y=x$ doğrusuna göre simetriği fonksiyonun tersidir.

y=ax+b doğrusuna göre simetri

(x,y) noktasının $y=ax+b$ doğrusuna göre simetriği $\left(\frac{y-b}{a}, ax+b\right)$ noktasıdır.

$y=f(x)$ ise simetriği $ax+b=f\left(\frac{y-b}{a}\right)$

Ör $y=1-2x$, $y=3x-5$ doğruları arasındaki açı.

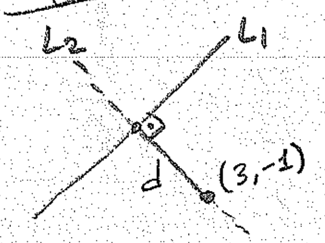
$m_2=-2$, $m_1=3$

$$\tan\alpha=\frac{m_2-m_1}{1+m_1m_2}=\frac{-2-3}{1-2\times3}=\frac{-5}{-5}=1\Rightarrow\alpha=\pi/4$$

Ör $P=(3,-1)$ noktasının $4x-3y+5=0$ doğrusuna en yakın uzaklığı.

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}=\frac{|4\times3+3\times1+5|}{\sqrt{(4)^2+(-3)^2}}=\frac{20}{5}=4$$

Diğer yol



$L_1: 4x-3y+5=0\rightarrow y=\frac{4x+5}{3}\longrightarrow m_1=4/3$

$m_2=-\frac{1}{m_1}=-3/4$ (3,-1) noktası için

$L_2: y=m_2x+b=-\frac{3x}{4}+b$ $\quad -1=-\frac{9}{4}+b\rightarrow b=\frac{5}{4}$

$y=\frac{5-3x}{4}$ bulunur.

iki doğrunun çakışma noktası

$$y=y\Longrightarrow\frac{4x+5}{3}=\frac{5-3x}{4}\Rightarrow16x+20=15-9x\Rightarrow x=-0{,}2,\ y=1.4$$

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(3+0.2)^2+(-1-1.4)^2}=\sqrt{(3.2)^2+(2.4)^2}=4$$

Ör $y=5+3x-x^2$ fonksiyonunun (2,5) noktasına göre simetriği.

$x\longrightarrow4-x$

$y\longrightarrow10-y$

$$10-y=5+3(4-x)-(4-x)^2$$
$$=5+12-3x-16+8x-x^2$$
$$=1+5x-x^2\Longrightarrow y=x^2-5x+9$$

Ör $P=(1,2)$ noktasından geçen ve denklemi $2x-3y+5=0$ doğrusuna dik olan doğrunun denklemi

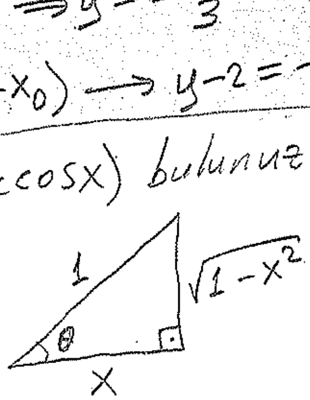
$2x-3y+5=0\Rightarrow y=\frac{2x+5}{3}\longrightarrow m_1=2/3\longrightarrow m_2=-\frac{1}{m_1}=-3/2$

$y-y_0=m_2(x-x_0)\longrightarrow y-2=-\frac{3}{2}(x-1)\longrightarrow y=\frac{7-3x}{2}$

Ör $y=\sin(\arccos x)$ bulunuz.

$\theta=\arccos x$

$x=\cos\theta$



$$y=\sin(\arccos x)$$
$$=\sin\theta$$
$$=\sqrt{1-x^2}$$

Ör: $f(x) = \arcsin\left(\frac{3x-4}{5}\right)$ ise ters fonksiyonu bul.

$$y = f(x) = \arcsin\left(\frac{3x-4}{5}\right) \longrightarrow \sin y = \frac{3x-4}{5} \longrightarrow x = \frac{4+5\sin y}{3}$$

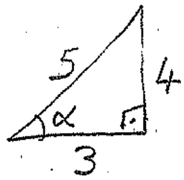
$$f^{-1}(x) = \frac{4+5\sin x}{3}$$

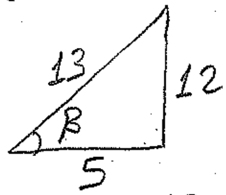
doğruya göre simetri ne?

Ör: $x = \cos\left(\arcsin\left(\frac{4}{5}\right) + \arccos\left(\frac{5}{13}\right)\right)$ hesapla

$\alpha = \arcsin\left(\frac{4}{5}\right) \longrightarrow \sin\alpha = 4/5$

$\beta = \arccos\left(\frac{5}{13}\right) \longrightarrow \cos\beta = \frac{5}{13}$





$\cos\alpha = \frac{3}{5}$ $\quad\sin\beta = \frac{12}{13}$

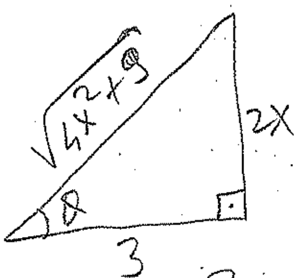
$$x = \cos\left(\arcsin\left(\tfrac{4}{5}\right) + \arccos\left(\tfrac{5}{13}\right)\right)$$

$$= \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{3}{5}\frac{5}{13} - \frac{4}{5}\frac{12}{13} = -\frac{33}{65}$$

Ör: $\arctan\left(\frac{2x}{3}\right) + \arcsin\left(\frac{\sqrt{4x^2+9}}{12}\right) = \pi/2$ ise $x = ?$

$\theta = \arctan\left(\frac{2x}{3}\right)$

$\tan\theta = \frac{2x}{3}$



$\cos\theta = \frac{3}{\sqrt{4x^2+9}}$

$\sqrt{4x^2+9} = 3\sec\theta$

$\theta + \arcsin\left(\frac{3\sec\theta}{12}\right) = \pi/2$

$\arcsin\left(\frac{\sec\theta}{4}\right) = \pi/2 - \theta$

$\frac{\sec\theta}{4} = \sin(\pi/2 - \theta)$

$\frac{1}{4\cos\theta} = \cos\theta \Longrightarrow \cos^2\theta = 1/4$

$\cos\theta = \mp 1/2 \Rightarrow \theta = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

$x = \frac{3}{2}\tan\theta \Rightarrow x = \left\{-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right\}$

Ör: $f(x) = x^2 - 6x + 14$, $x \geq 3$ için ters fonksiyon

$y = x^2 - 6x + 14$

$y = (x-3)^2 + 5$

$(x-3)^2 = y - 5$

$x - 3 = \mp\sqrt{y-5}$

$\to x = 3 \mp \sqrt{y-5}$

$x \geq 3$ için $x = 3 + \sqrt{y-5}$

x yerine y, y yerine x koy.

$y = 3 + \sqrt{x-5}$

$f^{-1}(x) = 3 + \sqrt{x-5}$

Ör $x^2+2|y-3|=1+5y$ ise $y=?$

$y-3=0$
$y=3$

$y<3$ için
$x^2-2(y-3)=1+5y$
$x^2-2y+6=1+5y$
$y=\dfrac{x^2+5}{7}<3$
$x^2<16 \Rightarrow |x|<4$

$y\geq 3$ için
$x^2+2(y-3)=1+5y$
$x^2+2y-6=1+5y$
$y=\dfrac{x^2-7}{3}\geq 3$
$x^2\geq 16 \Rightarrow |x|\geq 4$

$$y=\begin{cases}\dfrac{x^2+5}{7} & -4<x<4\\[2mm] \dfrac{x^2-7}{3} & x\leq -4,\ x\geq 4\end{cases}$$

Ör $f(x)=e^x+3x$ fonksiyonunu çift ve tek fonksiyonlarına ayır.

$f(x)=e^x+3x$
$f(-x)=e^{-x}-3x$

$$f_\text{ç}(x)=\frac{f(x)+f(-x)}{2}=\frac{e^x+3x+e^{-x}-3x}{2}=\frac{e^x+e^{-x}}{2}=\cosh(x)$$

$$f_t(x)=\frac{f(x)-f(-x)}{2}=\frac{e^x+3x-e^{-x}+3x}{2}=\frac{e^x-e^{-x}}{2}+3x=\sinh(x)+3x$$

Ör $3x+2y-7=0$ doğrusu üzerinde $P=(4,4)$ noktasına en yakın noktanın koordinatları

$3x+2y-7=0$
$y=\dfrac{7-3x}{2}$
$m_1=-3/2$

$m_2=-\dfrac{1}{m_1}=\dfrac{2}{3}$
$y-y_0=m_2(x-x_0)$
$y-4=\frac{2}{3}(x-4)$
$y=\dfrac{2x+4}{3}$

$y=y$
$\dfrac{7-3x}{2}=\dfrac{2x+4}{3}$
$21-9x=4x+8$
$x=1,\ y=2$

Ör $y=2x-1$ doğrusuna $(2,3)$ noktasında teğet ve yarıçapı $\sqrt{5}$ olan çemberlerin denklemleri.

$y=2x-1$
$m_1=2$
$m_2=-\dfrac{1}{m_1}=-\dfrac{1}{2}$

$y-y_0=m_2(x-x_0)$
$y-3=-\frac{1}{2}(x-2)$
$y=4-x/2$

$x=x_1$ ise $y=y_1=4-\dfrac{x_1}{2}$
$\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}=\sqrt{5}$
$(x_1-2)^2+(4-\frac{x_1}{2}-3)^2=5$
$(x_1^2-4x_1+4)+(1-x_1+\frac{x_1^2}{4})=5$
$x_1(x_1-4)=0 \rightarrow x_1=0,\ x_1=4$

$(x-x_1)^2+(y-y_1)^2=r^2=5$

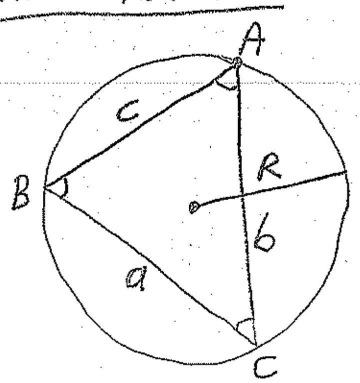
$x_1=0,\ y_1=4 \Rightarrow x^2+(y-4)^2=5$
$x_1=4,\ y_1=2 \Rightarrow (x-4)^2+(y-2)^2=5$

Ör $y=\sqrt{x-5}$ tanım aralığı ve değişme aralığı

$x-5\geq 0$
$x\geq 5$

$T.A=[5,\infty)$
$D.A=[0,\infty)$

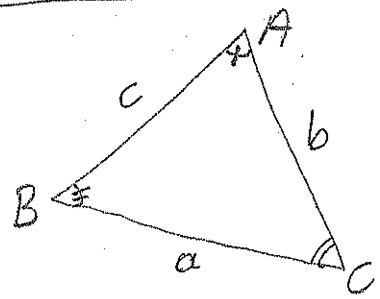
Sinüs Teoremi



ABC üçgeninin çevrel çemberinin yarıçapı R olsun.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Kosinüs Teoremi



$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Ör ABC üçgeninde $m(A) = \frac{\pi}{4}$, $m(B) = \frac{5\pi}{12}$ ve $c = |AB| = 5$ ise $a = |BC| = ?$

$$m(A) + m(B) + m(C) = \pi \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{\pi}{4} + \frac{5\pi}{12} + m(C) = \pi \qquad \frac{a}{\sin\frac{\pi}{4}} = \frac{5}{\sin\frac{\pi}{3}} \Rightarrow a = 5\,\frac{\sin\pi/4}{\sin\pi/3}$$

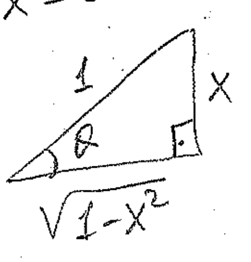
$$m(C) = \frac{\pi}{3} \qquad = 5\,\frac{\sqrt{2}/2}{\sqrt{3}/2} = \frac{5}{3}\sqrt{6}$$

Ör ABC üçgeninde $a = 4$, $b = 3$ ve $m(C) = \pi/3$ ise $c = ?$

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$= 16 + 9 - 2 \times 4 \times 3 \times \cos\pi/3$$
$$\to c^2 = 25 - 24\cos\pi/3$$
$$= 25 - 24 \times \tfrac{1}{2} = 13$$
$$c = \sqrt{13}$$

Ör $y = \sin(2\arcsin x)$ sadeleştir.

$$\theta = \arcsin x \qquad y = \sin(2\arcsin x) = \sin 2\theta$$
$$x = \sin\theta \qquad = 2\cos\theta\sin\theta$$



$$= 2 \cdot \frac{\sqrt{1-x^2}}{1} \cdot x$$
$$= 2x \cdot \sqrt{1-x^2}$$

Ör $y=\sqrt{9-4x^2}$ tanım ve değer kümeleri.

$9-4x^2 \geq 0$ $\quad$ $x\in[-\frac{3}{2},\frac{3}{2}]$ için $x^2 \leq 9/4$ dır.

$4x^2 \leq 9$ $\quad$ $0\leq x^2 \leq 9/4$ 4 ile çarp.

$x^2 \leq 9/4$ $\quad$ $0 \leq 4x^2 \leq 9$ -1 ile çarp

$|x| \leq 3/2$ $\quad$ $-9 \leq -4x^2 \leq 0$ 9 ekle.

$T.K=[-\frac{3}{2},\frac{3}{2}]$ $\quad$ $0\leq 9-4x^2 \leq 9$ kök al.

$0 \leq \sqrt{9-4x^2} \leq 3 \Longrightarrow D.K=[0,3]$

Ör $y=\frac{3}{x-2}$ tanım ve değer kümeleri

$x-2 \neq 0 \longrightarrow x\neq 2 \longrightarrow T.K.=R-\{2\}$

$x=\mp\infty$ için $y=0 \longrightarrow D.K.=R-\{0\}$

Ör $y=\sqrt{5-x}+\lg(x-3)$ tanım aralığı

$5-x\geq 0$ $\quad$ $x-3>0$ $\quad$ $\text{Ç.K.}=\text{Ç}_1\cap \text{Ç}_2$

$x\leq 5$ $\quad$ $x>3$ $\quad$ $=(3,5]$

$\text{Ç}_1=(-\infty,5]$ $\quad$ $\text{Ç}_2=(3,\infty)$

Ör $|x^2-17|>8$ eşitsizliğinin çözüm kümesi

$x^2-17>8$ $\quad$ $x^2-17<-8$ $\quad$ $\text{Ç.K}=\text{Ç}_1\cup \text{Ç}_2$

$x^2>25$ $\quad$ $x^2<9$ $\quad$ $=(-\infty,-5)\cup(-3,3)\cup(5,\infty)$

$|x|>5$ $\quad$ $|x|<3$

$\text{Ç}_1=(-\infty,-5)\cup(5,\infty)$ $\quad$ $\text{Ç}_2=(-3,3)$

Ör $|x^2-6x-1|\leq 6$ eşitsizliğinin çözüm kümesi

$-6 \leq x^2-6x-1 \leq 6$, 10 ekle

$4 \leq x^2-6x+9 \leq 16$

$4 \leq (x-3)^2 \leq 16$

$2 \leq |x-3| \leq 4$

$|x-3|\leq 4$ için

$-4 \leq x-3 \leq 4$

$-1 \leq x \leq 7$ $\quad$ $\text{Ç}_1=[-1,7]$

$|x-3|\geq 2$ için

$x-3\geq 2$ $\quad$ $x-3\leq -2$

$x\geq 5$ $\quad$ $x\leq 1$

$\text{Ç}_2=(-\infty,1]\cup[5,\infty)$

$\text{Ç.K}=\text{Ç}_1\cap \text{Ç}_2$

$=[-1,1]\cup[5,7]$

Ör $[\![3x-5]\!]=-1$ ç.k ve aralık genişliği

$-1 \le 3x-5 < 0$, 5 ekle

$4 \le 3x < 5$ $\quad$ ç.k $= [\frac{4}{3}, \frac{5}{3})$

$\frac{4}{3} \le x < \frac{5}{3}$ $\quad$ aralık gen. $= \frac{5}{3} - \frac{4}{3} = \frac{1}{3}$

Ör $[\![\frac{1}{3x-2}]\!]=2$ çözüm kümesi ve aralık genişliği

$2 \le \frac{1}{3x-2} < 3$

$\frac{1}{3} < 3x-2 \le \frac{1}{2}$

$2 < 18x-12 \le 3$

$14 < 18x \le 15$

$\frac{14}{18} < x \le \frac{15}{18}$

ç.k $= (\frac{7}{9}, \frac{15}{18}]$

ara. gen. $= \frac{15}{18} - \frac{7}{9} = \frac{1}{18}$

Ör $[\![x + 2[\![x]\!]]\!] \le 12$ çözüm kümesi

$[\![x]\!] + 2[\![x]\!] \le 12$

$[\![x]\!] \le 4$

$x < 5$

ç.k $= (-\infty, 5)$

Ör $y = \sqrt{x^2-9} + \sqrt{25-x^2}$ fonk. tanım kümesi

$x^2-9 \ge 0$

$x^2 \ge 9$

$|x| \ge 3$

$Ç_1 = (-\infty,-3] \cup [3,\infty)$

$25-x^2 \ge 0$

$x^2 \le 25$

$|x| \le 5$

$Ç_2 = [-5,5]$

ç.k $= Ç_1 \cap Ç_2$

$= [-5,-3] \cup [3,5]$

Ör $y = \sqrt{4-\sqrt{x^2-9}}$ çözüm kümesi

$4 - \sqrt{x^2-9} \ge 0$

$\sqrt{x^2-9} \le 4$

$0 \le \sqrt{x^2-9} \le 4$

$0 \le x^2-9 \le 16$

$9 \le x^2 \le 25$

$3 \le |x| \le 5$

$Ç_1 = (-\infty,-3] \cup [3,\infty)$

$Ç_2 = [-5,5]$

ÇK $= Ç_1 \cap Ç_2$

$= [-5,-3] \cup [3,5]$

Ör: $y=\sqrt{5-|3-x|}$ çözüm kümesi

$5-|3-x| \geq 0 \Rightarrow |x-3| \leq 5$

$|3-x| \leq 5$

$-5 \leq x-3 \leq 5$

$-2 \leq x \leq 8$

$\text{ç.k} = [-2, 8]$

Ör: $\left|\dfrac{x-2}{x+3}\right| > 0$ çözüm kümesi

$\dfrac{|x-2|}{|x+3|} > 0$

$|x-2| \neq 0 \rightarrow x \neq 2$

$|x+3| \neq 0 \rightarrow x \neq -3$

$\text{ç.k.} = \mathbb{R} - \{-3, 2\}$

Ör: $|x+2| + x|x-3| \geq 2x-1$ çözüm kümesi

$x+2=0$
$x=-2$

$$\begin{array}{c|c} \multicolumn{2}{c}{-2} \\ - & + \end{array}$$

$x-3=0$
$x=3$

$$\begin{array}{c|c} \multicolumn{2}{c}{3} \\ - & + \end{array}$$

$x < -2$ için

$-(x+2) - x(x-3) \geq 2x-1$

$-x^2-1 \geq 0$

$x^2+1 \leq 0$

$\text{ç}_1 = \emptyset$

$-2 \leq x < 3$ için

$(x+2) - x(x-3) \geq 2x-1$

$-x^2+2x+3 \geq 0$

$x^2-2x-3 \leq 0$

$(x+1)(x-3) \leq 0$

$$\begin{array}{c|c|c} & -1 & 3 \\ + & - & + \end{array}$$

$\text{ç}_2 = [-1, 3)$

$x \geq 3$ için

$(x+2) + x(x-3) \geq 2x-1$

$x^2-4x+3 \geq 0$

$(x-1)(x-3) \geq 0$

$$\begin{array}{c|c|c} & 1 & 3 \\ + & - & + \end{array}$$

$\text{ç}_2 = [3, \infty)$

$\text{ç} = \text{ç}_1 \cup \text{ç}_2 \cup \text{ç}_3$
$= [-1, \infty)$

Ör: $|5-3x| - |2x+3| \geq 7$ çözüm kümesi

$5-3x=0$
$x=5/3$

$$\begin{array}{c|c} \multicolumn{2}{c}{5/3} \\ + & - \end{array}$$

$2x+3=0$
$x=-3/2$

$$\begin{array}{c|c} \multicolumn{2}{c}{-3/2} \\ - & + \end{array}$$

$x < -3/2$ için

$(5-3x) + (2x+3) \geq 7$

$8-x \geq 7$

$x \leq 1 \quad \text{ç}_1 = (-\infty, -3/2)$

$-3/2 \leq x < 5/3$ için

$(5-3x) - (2x+3) \geq 7$

$2-5x \geq 7$

$x \leq -1$

$\text{ç}_2 = [-3/2, -1]$

$x \geq 5/3$ için

$-(5-3x) - (2x+3) \geq 7$

$-5+3x-2x-3 \geq 7$

$x-8 \geq 7$

$x \geq 15 \quad \text{ç}_2 = [15, \infty)$

$\text{ç.k.} = \text{ç}_1 \cup \text{ç}_2 \cup \text{ç}_3$
$= (-\infty, -3/2) \cup [-3/2, -1] \cup [15, \infty)$
$= (-\infty, -1] \cup [15, \infty)$

sadece eşitlik olsaydı
$\text{ç.k} = \{-1, 15\}$ olurdu.

Ör $\left|2|x|+5x\right|<21$ çözüm kümesi

$-21<2|x|+5x<21$

$x<0$ için

$-21<3x<21$

$-7<x<7$

$Ç_1=(-7,0)$

$x\geq 0$ için

$-21<7x<21$

$-3<x<3$

$Ç_2=[0,3)$

$Ç.K.=Ç_1\cup Ç_2$

$=(-7,0)\cup[0,3)$

$=(-7,3)$

Ör $\dfrac{1}{|x+3|}<1-x$ çözüm kümesi

$\dfrac{1}{|x+3|}+x-1<0$

$x+3=0$

$x=-3$

$\begin{array}{c|c} \multicolumn{2}{c}{-3} \\ - & + \end{array}$

$x<-3$ için

$\dfrac{-1}{x+3}+\dfrac{x-1}{\underset{(x+3)}{1}}<0$

$\dfrac{x^2+2x-4}{x+3}<0$ $\quad x+3 \to \ominus$

$x^2+2x-4>0$

$(x+1)^2-5>0$

$|x+1|>\sqrt{5}$

$\begin{array}{c|c|c} \multicolumn{1}{c}{} & -1-\sqrt{5} \quad -1+\sqrt{5} & \\ + & - & + \end{array}$

$Ç_1=(-\infty,-1-\sqrt{5})$

$Ç=Ç_1\cup Ç_2$

$=(-\infty,-1-\sqrt{5})$

$x\geq 3$ için

$\dfrac{1}{x+3}+\dfrac{x-1}{\underset{(x+3)}{1}}<0$

?

$\dfrac{x^2+2x-2}{x+3}<0$ $\quad x+3 \to \oplus$

$x^2+2x-2<0$

$(x+1)^2-3<0$

$|x+1|<\sqrt{3}$

$\begin{array}{c|c|c} \multicolumn{1}{c}{} & -1-\sqrt{3} \quad -1+\sqrt{3} & \\ + & - & + \end{array}$

$Ç_2=\emptyset$

Ör $\left|\dfrac{x-1}{x-2}\right|>3-x$ çözüm kümesi

$x-2=0$

$x=2$

$\begin{array}{c|c} \multicolumn{2}{c}{2} \\ - & + \end{array}$

$x-1=0$

$x=1$

$\begin{array}{c|c} \multicolumn{2}{c}{1} \\ - & + \end{array}$

	1		2	
−		−		+
−		+		+
+		−		+

$x\neq 2$ olmalı

$x-3+\left|\dfrac{x-1}{x-2}\right|>0$

$1<x<2$

$x-3-\dfrac{x-1}{x-2}>0$

$\dfrac{x^2-6x+7}{x-2}>0$ $\quad x-2 \to \ominus$

$x^2-6x+7<0$

$(x-3)^2-2<0$

$|x-3|\leq\sqrt{2}$

$\begin{array}{c|c|c} \multicolumn{1}{c}{} & 3-\sqrt{2} \quad 3+\sqrt{2} & \\ + & - & + \end{array}$

$Ç_1=(3-\sqrt{2},2)$

$Ç=Ç_1\cap Ç_2$

$=(3-\sqrt{2},2)\cup(2,\infty)$

$x\leq 1,\ x\geq 2$ için

$x-3+\dfrac{x-1}{x-2}>0$

$\dfrac{x^2-4x+5}{x-2}>0$

$\dfrac{(x-2)^2+1}{x-2}>0$ $\quad (x-2)^2+1 \to \oplus$

$x-2>0$

$x>2$

$Ç_2=(2,\infty)$

ör $\left|\frac{2-x}{3x+1}\right| \le 4$ çözüm kümesi

$\frac{|x-2|}{|3x+1|} \le 4 \longrightarrow |x-2|-4|3x+1| \le 0, \; x \ne -\frac{1}{3}$

$x-2=0$
$x=2$

$3x+1=0$
$x=-\frac{1}{3}$

$x<-\frac{1}{3}$ için

$-(x-2)+4(3x+1) \le 0$

$11x+6 \le 0$

$x \le -\frac{6}{11}$ $\quad Ç_1=(-\infty,-\frac{6}{11}]$

$-\frac{1}{3}<x<2$ için

$-(x-2)-4(3x+1) \le 0$

$-13x-2 \le 0$

$13x+2 \ge 0$

$x \ge -\frac{2}{13}$ $\quad Ç_2=[-\frac{2}{13},2)$

$x \ge 2$ için

$(x-2)-4(3x+1) \le 0$

$-11x-6 \le 0$

$11x+6 \ge 0$

$x \ge -\frac{6}{11}$ $\quad Ç_3=[2,\infty)$

$Ç=Ç_1 \cup Ç_2 \cup Ç_3=(-\infty,-\frac{6}{11}] \cup [-\frac{2}{13},\infty)$

ör $x-\left|\frac{x+4}{x-2}\right| \le 3$ çözüm kümesi

$x+4=0$
$x=-4$
$x-2=0$
$x=2$ $\quad x \ne 2$

$-4<x<2$ için

$x-3+\frac{x+4}{x-2} \le 0$

$\frac{x^2-4x+10}{x-2} \le 0$ $\quad x-2 \to \ominus$

$x^2-4x+10 \ge 0$

$(x-2)^2+6 \ge 0$ ✓

$Ç_1=(-4,2)$

$x \le -4, \; x>2$ için

$x-3-\frac{x+4}{x-2} \le 0$

$\frac{x^2-6x+2}{x-2} \le 0$

$\frac{(x-3)^2-7}{x-2} \le 0$

$Ç_2=(-\infty,-4] \cup (2,3+\sqrt{7}]$

$Ç=Ç_1 \cup Ç_2=(-\infty,2) \cup (2,3+\sqrt{7}]$

ör $y=\sqrt{3x+5}+\lg(2x+4)+\arccos(x+2)$ tanım aralığı

$3x+5 \ge 0$
$x \ge -\frac{5}{3}$

$2x+4>0$
$x>-2$

$-1 \le x+2 \le 1$
$-3 \le x \le -1$

$Ç.k=[-\frac{5}{3},\infty) \cap (-2,\infty) \cap [-3,-1]=[-\frac{5}{3},-1]$

Ör $y = \dfrac{\sqrt{\operatorname{sgn}(x^2-4)}}{[\![2x-5]\!]} + \ln(x^2+x-6)$ tanım aralığı

$\operatorname{sgn}(x^2-4) \geq 0$

$x^2-4 \geq 0$

$x^2 \geq 4$

$|x| \geq 2$

$G_1 = (-\infty, -2] \cup [2, \infty)$

$[\![2x-5]\!] \neq 0$ olmalı

$[\![2x-5]\!] = 0$ için

$0 \leq 2x-5 < 1$

$5 \leq 2x < 6$

$5/2 \leq x < 3$

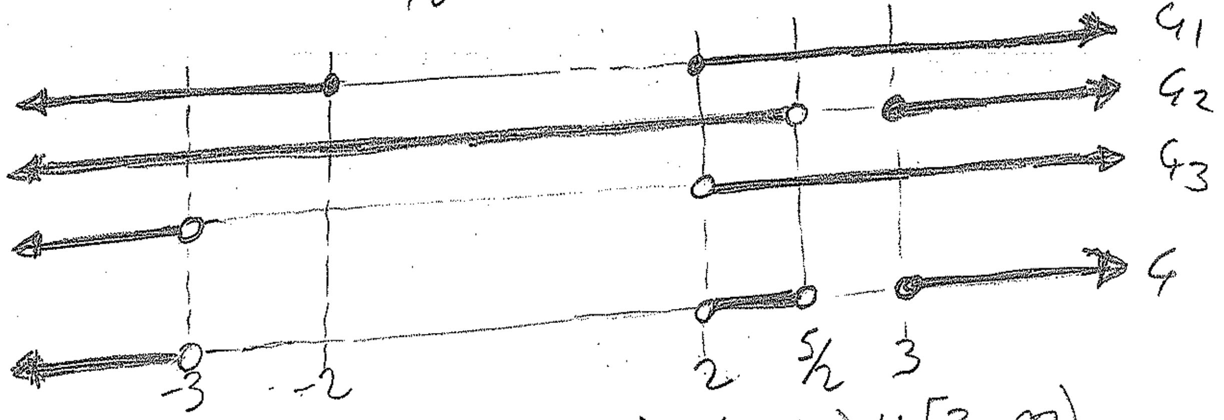
$G_2 = (-\infty, \frac{5}{2}) \cup [3, \infty)$

$x^2+x-6 > 0$

$(x+3)(x-2) > 0$

	-3		2	
$+$		$-$		$+$

$G_3 = (-\infty, -3) \cup (2, \infty)$



$G = G_1 \cap G_2 \cap G_3 = (-\infty, -3) \cup (2, 5/2) \cup [3, \infty)$

Ör $y = \sqrt{5 - [\![x+2]\!]} + \ln(x+3) + \arcsin\left(\dfrac{x+1}{3}\right)$ tanım aralığı

$5 - [\![x+2]\!] \geq 0$

$[\![x+2]\!] \leq 5$

$x+2 < 6$

$x < 4$

$G_1 = (-\infty, 4)$

$x+3 > 0$

$x > -3$

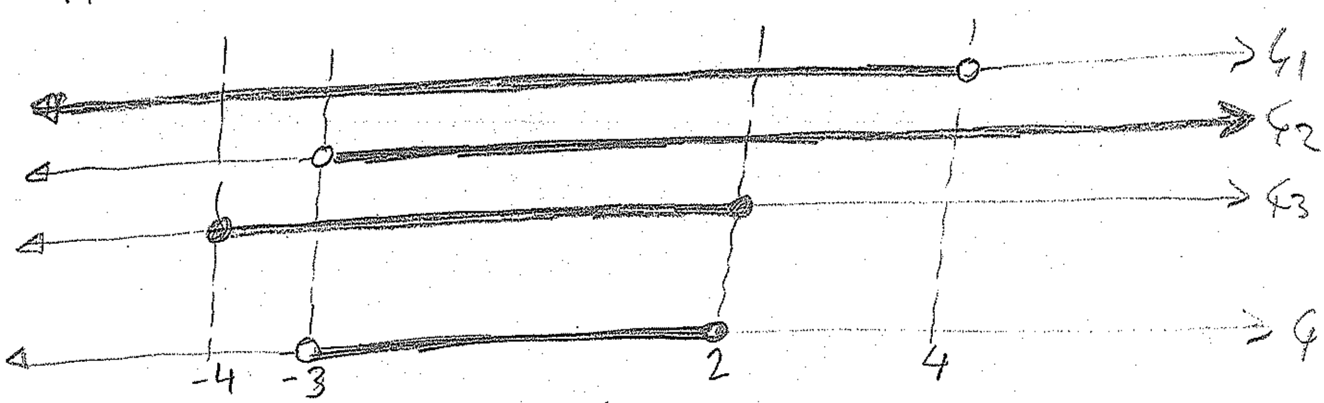
$G_2 = (-3, \infty)$

$-1 \leq \dfrac{x+1}{3} \leq 1$

$-3 \leq x+1 \leq 3$

$-4 \leq x \leq 2$

$G_3 = [-4, 2]$



$G = G_1 \cap G_2 \cap G_3 = (-3, 2]$

Limit

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ ise $\lim_{x\to a} f(x) = L$ limit var.

Süreklilik

$\lim_{x\to a} f(x) = f(a)$ ise $f(x)$ fonksiyonu $x=a$ için süreklidir.

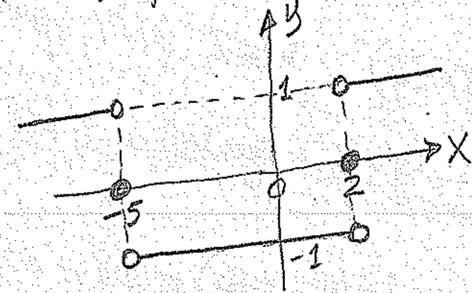
Ör $f(x) = \text{sgn}(x^2+3x-10)$ fonksiyonunu süreklilik bakımından incele.

$x^2+3x-10=0$

$(x+5)(x-2)=0$

$x=-5 \quad x=2$

-5		2	
+	-	+	



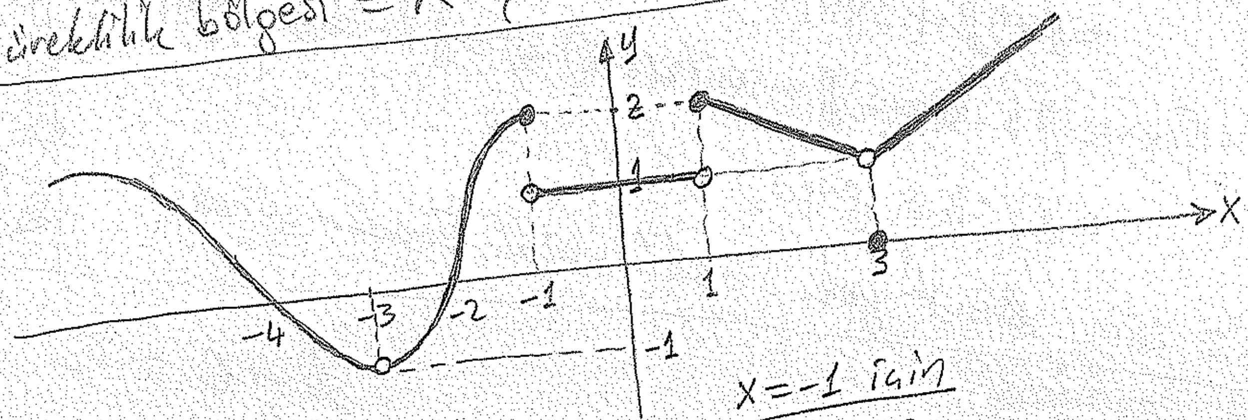
$\lim_{x\to -5^-} f(x) = 1 \neq \lim_{x\to 5^+} f(x) = -1$

$f(-5)=0$

$\lim_{x\to 2^-} f(x) = -1 \neq \lim_{x\to 2^+} f(x) = 1$

$f(2)=0$

süreklilik bölgesi $= R - \{-5, 2\}$



$x=-3$ için

$f(-3)$'ün değeri yok.

$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = -1$

limit var. Sürekli değil

$x=-1$ için

$f(-1)=2$

$\lim_{x\to -1^-} f(x) = 2$, $\lim_{x\to -1^+} f(x) = 1$

limit yok. Sürekli değil. Soldan Sürekli.

$x=1$ için

$f(1)=2$

$\lim_{x\to 1^-} f(x) = 1$, $\lim_{x\to 1^+} f(x) = 2$

Limit yok. Sürekli değil.

Sağdan sürekli.

$x=3$ için

$f(3)=0$

$\lim_{x\to 3^-} f(x) = 1$, $\lim_{x\to 3^+} f(x) = 1$

limit var ve $\lim_{x\to 3} f(x) = 1$

sürekli değil.

Diğer durumlarda limit var ve süreklidir.

Süreklilik bölgesi $= R - \{-3, -1, 1, 3\}$

Ör $f(x) = \begin{cases} [\![x]\!] & x<0 \\ sgn(x) & x \geq 0 \end{cases}$ $x=0$ için fonksiyonu incele.

$f(0) = sgn(0) = 0$ tanımlı.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} [\![x]\!] = [\![0^-]\!] = -1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} sgn(x) = sgn(0^+) = 1$

limit yok. aynı zamanda sürekli de değil

Ör $f(x) = [\![7-3x]\!] + sgn(2x-4)$ $x=2$ için fonk. incele.

$f(2) = [\![7-6]\!] + sgn(4-4) = [\![1]\!] + sgn(0) = 1+0 = 1$

$\lim_{x\to 2^-} f(x) = [\![7-3(2-\delta)]\!] + sgn(2(2-\delta)-4) = [\![1+3\delta]\!] + sgn(-2\delta)$
$= [\![1^+]\!] + sgn(0^-) = 1-1 = 0$

$\lim_{x\to 2^+} f(x) = [\![7-3(2+\delta)]\!] + sgn(2(2+\delta)-4) = [\![1-3\delta]\!] + sgn(2\delta)$
$= [\![1^-]\!] + sgn(0^+) = 0+1 = 1$

$x=2$ için limit yok. Sürekli değil. Sağdan sürekli.

$\lim_{x\to a} f(x) = L$ ve $x=L$ için $g(x)$ sürekli ise $\lim_{x\to a} g(f(x)) = g(L)$ olur.

Limit Özellikleri

$\lim_{x\to a} f(x) = L$ ise

$\lim_{x\to a} c.f(x) = c \lim_{x\to a} f(x) = cL$

$\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right| = |L|$

$\lim_{x\to a} C = C$

$\lim_{x\to a} f(x) = L$ ve $\lim_{x\to a} g(x) = M$ ise

$\lim_{x\to a} (f(x) \mp g(x)) = \lim_{x\to a} f(x) \mp \lim_{x\to a} g(x) = L \mp M$

$\lim_{x\to a} (f(x).g(x)) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x) = L.M$

$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$, $M \neq 0$

ε−δ kuralı ile limit

$\lim_{x\to a} f(x) = L$, $\delta>0$, $\varepsilon>0$ ise

$|x-a|<\delta$ için $|f(x)-L|<\varepsilon$ olmalı.

ε ve δ birbirleri cinsinden ifade edilebilmeli.

Ör $\lim_{x\to 3}(5x-6) = 9$ olduğunu ε−δ kuralı ile ispatla.

$|x-a|<\delta$ için $|f(x)-L|<\varepsilon$ olmalı.

$|x-3|<\delta$ için $|(5x-6)-9|<\varepsilon$

$|5x-15|<\varepsilon$

$5|x-3|<\varepsilon$

$|x-3|<\varepsilon/5$

Sonuç olarak $\delta = \varepsilon/5$ limit doğru.

Ör $\lim_{x\to 2}(x^2-4x+7) = 3$ olduğunu ε−δ kuralı ile ispatla

$|x-a|<\delta$ için $|f(x)-L|<\varepsilon$ olmalı.

$|x-2|<\delta$ için $|(x^2-4x+7)-3|<\varepsilon$

$|x^2-4x+4|<\varepsilon \Rightarrow |(x-2)^2|<\varepsilon$

$|x-2|<\sqrt{\varepsilon}$

Sonuç olarak $\delta = \sqrt{\varepsilon}$ limit doğru

Ör $\lim_{x\to 2}\dfrac{x^2+3x-5}{3x-1}$ bulunuz.

$$\lim_{x\to 2}\frac{x^2+3x-5}{3x-1} = \frac{4+6-5}{6-1} = \frac{5}{5} = 1$$

Ör $$\lim_{x\to \pi/2}\frac{1+\sin x}{1-\cos x} = \frac{1+\sin \pi/2}{1-\cos \pi/2} = \frac{1+1}{1-0} = 2$$

Ör $$\lim_{x\to 0} x^2.\sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x^2 . \lim_{x\to 0}\sin(1/x) = 0 \times \text{Sayı} = 0$$

Ör $$\lim_{x\to 0}\frac{5e^x-2e^{-x}}{3-x} = \frac{5e^0-2e^0}{3-0} = \frac{5-2}{3} = 1$$

Ör $\lim\limits_{x\to 2}\left(x^2-\operatorname{sgn}\left(\frac{x+3}{x-1}\right)+[\![x/3]\!]\right)=?$

x^2 için kritik nokta yok.

$\operatorname{sgn}\left(\frac{x+3}{x-1}\right)$ için $x=-3$ ve $x=1$ kritik.

$[\![x/3]\!]$ için $x=3k,\ k\in Z$ kritik.

sonuç olarak $x=2$ kritik nokta değil.

$\lim\limits_{x\to 2} f(x)=4-\operatorname{sgn}(5)+[\![2/3]\!]=4-1+0=3$

Ör $\lim\limits_{x\to 3}\frac{5}{3-x}=?$ $\quad 3-x=0\rightarrow x=3$ kritik nokta

$$\left.\begin{aligned}\lim_{x\to 3^-}\frac{5}{3-x}&=\frac{5}{3-(3-\delta)}=\frac{5}{\delta}=\infty\\ \lim_{x\to 3^+}\frac{5}{3-x}&=\frac{5}{3-(3+\delta)}=\frac{5}{-\delta}=-\infty\end{aligned}\right\}\text{ limit yok}$$

Ör $\lim\limits_{x\to 5}\frac{2}{(5-x)^2}=?$ $\quad 5-x=0\rightarrow x=5$ kritik.

$$\left.\begin{aligned}\lim_{x\to 5^-}\frac{2}{(5-x)^2}&=\frac{2}{(5-(5-\delta))^2}=\frac{2}{\delta^2}=\infty\\ \lim_{x\to 5^+}\frac{2}{(5-x)^2}&=\frac{2}{(5-(5+\delta))^2}=\frac{2}{\delta^2}=\infty\end{aligned}\right\}\text{ limit yok } \infty\notin R$$

Ör $f(x)=\begin{cases}x^2-1 & x<2\\ 3 & x=2\\ 2x-1 & x>2\end{cases}$ $\quad x=2$ için $f(x)$ fonksiyonunu incele.

$f(2)=3$

$$\left.\begin{aligned}\lim_{x\to 2^-}f(x)&=\lim_{x\to 2^-}(x^2-1)=4-1=3\\ \lim_{x\to 2^+}f(x)&=\lim_{x\to 2^+}(2x-1)=4-1=3\end{aligned}\right\}\ \lim_{x\to 2}f(x)=3$$

limit var ve sürekli.

Belirsiz Durumlarda Limit

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0.\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0 durumları.

Ör $\lim\limits_{x\to\infty} \dfrac{3x^2-5x+4}{2x^2+3} = \lim\limits_{x\to\infty} \dfrac{3-\frac{5}{x}+\frac{4}{x^2}}{2+\frac{3}{x^2}} = \dfrac{3-0+0}{2+0} = \dfrac{3}{2}$

Ör $\lim\limits_{x\to\infty} \dfrac{2x+5}{3x^2-4} = \lim\limits_{x\to\infty} \dfrac{2/x+5/x^2}{3-4/x^2} = \dfrac{0+0}{3-0} = \dfrac{0}{3} = 0$

Ör $\lim\limits_{x\to\infty} \dfrac{x^2-3}{2x+1} = \lim\limits_{x\to\infty} \dfrac{1-3/x^2}{\frac{2}{x}+1/x^2} = \dfrac{1-0}{0+0} = \dfrac{1}{0} = \infty$ limit yok.

Ör $\lim\limits_{x\to 0} \dfrac{2x}{3-\sqrt{x+9}} = ?$ $\frac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{2x}{3-\sqrt{x+9}} \cdot \dfrac{3+\sqrt{x+9}}{3+\sqrt{x+9}} = \lim\limits_{x\to 0} \dfrac{2x(3+\sqrt{x+9})}{9-(x+9)} = \lim\limits_{x\to 0} \dfrac{2x(3+\sqrt{x+9})}{-x}$

$= -2\lim\limits_{x\to 0}(3+\sqrt{x+9}) = -12$

Ör $\lim\limits_{x\to 0} \dfrac{\sqrt{9-x^2}-3}{2x} = ?$ $\frac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{\sqrt{9-x^2}-3}{2x} \cdot \dfrac{\sqrt{9-x^2}+3}{\sqrt{9-x^2}+3} = \lim\limits_{x\to 0} \dfrac{(9-x^2)-9}{2x(\sqrt{9-x^2}+3)}$

$= \lim\limits_{x\to 0} \dfrac{-x^2}{2x(\sqrt{9-x^2}+3)} = -\dfrac{1}{2}\lim\limits_{x\to 0} \dfrac{x}{\sqrt{9-x^2}+3} = -\dfrac{1}{2} \cdot \dfrac{0}{6} = 0$

Ör $\lim\limits_{x\to\infty} \dfrac{2x}{[\![x]\!]} = ?$

$[\![x]\!] \le x < [\![x]\!]+1$

$1 \le \dfrac{x}{[\![x]\!]} < 1 + \dfrac{1}{[\![x]\!]}$

$2 \le \dfrac{2x}{[\![x]\!]} < 2 + \dfrac{2}{[\![x]\!]}$

$\lim\limits_{x\to\infty}\left(2+\dfrac{2}{[\![x]\!]}\right) = 2$ sandiviç teoremine göre

$\lim\limits_{x\to\infty} \dfrac{2x}{[\![x]\!]} = 2$ olur.

Ör $$\lim_{x\to\pi} \frac{\sin^2 x}{1+\cos^3 x} = ? \quad \frac{0}{0}$$

$$\lim_{x\to\pi} \frac{1-\cos^2 x}{1+\cos^3 x} = \lim_{x\to\pi} \frac{\cancel{(1+\cos x)}(1-\cos x)}{\cancel{(1+\cos x)}(1-\cos x+\cos^2 x)}$$

$$= \lim_{x\to\pi} \frac{1-\cos x}{1-\cos x+\cos^2 x} = \frac{1+1}{1+1+1} = 2/3$$

Ör $$\lim_{x\to 3} \frac{x^2-9}{x^3-11x+6} = ? \quad \frac{0}{0}$$

$$\lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x^2+3x-2)} = \lim_{x\to 3} \frac{x+3}{x^2+3x-2} = \frac{3+3}{9+9-2} = \frac{3}{8}$$

Ör $$\lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-5x+4} = ? \quad \frac{0}{0}$$

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{(x-4)(x-1)} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(x-1)(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{1}{(x-1)(\sqrt{x}+2)} = \frac{1}{3\times 4} = \frac{1}{12}$$

$a^3-b^3 = (a-b)(a^2+ab+b^2)$
$a^3+b^3 = (a+b)(a^2-ab+b^2)$
$a^2-b^2 = (a-b)(a+b)$

Ör $$\lim_{x\to 8} \frac{x^2-5x-24}{\sqrt[3]{x}-2} = ? \quad \frac{0}{0}$$

$$\lim_{x\to 8} \frac{(x-8)(x+3)}{\sqrt[3]{x}-2} \cdot \frac{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}{\sqrt[3]{x^2}+2\sqrt[3]{x}+4} = \lim_{x\to 8} \frac{\cancel{(x-8)}(x+3)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{\cancel{x-8}}$$

$$= \lim_{x\to 8} (x+3)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4) = 11\times(4+4+4) = 132$$

Ör $$\lim_{x\to 4} \frac{x-\sqrt{3x+4}}{\sqrt{2x+1}-3} = ? \quad \frac{0}{0}$$

$$\lim_{x\to 4} \frac{x-\sqrt{3x+4}}{\sqrt{2x+1}-3} \cdot \frac{x+\sqrt{3x+4}}{x+\sqrt{3x+4}} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3} = \lim_{x\to 4} \frac{(x^2-3x-4)(\sqrt{2x+1}+3)}{(2x-8)(x+\sqrt{3x+4})}$$

$$= \lim_{x\to 4} \frac{\cancel{(x-4)}(x+1)(\sqrt{2x+1}+3)}{2\cancel{(x-4)}(x+\sqrt{3x+4})} = \lim_{x\to 4} \frac{x+1}{2} \cdot \frac{\sqrt{2x+1}+3}{x+\sqrt{3x+4}}$$

$$= \frac{5}{2} \cdot \frac{3+3}{4+4} = \frac{15}{8}$$

ör $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{e^{2x}-1} = ?$ $0/0$

$$\lim_{x\to 0} \frac{\cancel{(e^x-1)}(e^{2x}+e^x+1)}{\cancel{(e^x-1)}(e^x+1)} = \lim_{x\to 0} \frac{e^{2x}+e^x+1}{e^x+1} = \frac{1+1+1}{1+1} = 3/2$$

ör $\lim\limits_{x\to\infty} \left(\sqrt{x^2+9}-\sqrt{x^2-4}\right) = ?$ $\infty-\infty$

$$\lim_{x\to\infty} \left(\sqrt{x^2+9}-\sqrt{x^2-4}\right) \frac{\sqrt{x^2+9}+\sqrt{x^2-4}}{\sqrt{x^2+9}+\sqrt{x^2-4}} = \lim_{x\to\infty} \frac{(x^2+9)-(x^2-4)}{\sqrt{x^2+9}+\sqrt{x^2-4}}$$

$$= \lim_{x\to\infty} \frac{13}{\sqrt{x^2+9}+\sqrt{x^2-4}} = \frac{13}{\infty+\infty} = \frac{13}{\infty} = 0$$

ör $\lim\limits_{x\to-\infty} \left(\sqrt{x^2-3x+1}+x\right) = ?$ $\infty-\infty$

$$\lim_{x\to-\infty} \left(\sqrt{x^2-3x+1}+x\right) \frac{\sqrt{x^2-3x+1}-x}{\sqrt{x^2-3x+1}-x} = \lim_{x\to-\infty} \frac{(x^2-3x+1)-x^2}{\sqrt{x^2-3x+1}-x}$$

$$= \lim_{x\to-\infty} \frac{3x-1}{x-\sqrt{x^2-3x+1}} = \lim_{x\to-\infty} \frac{3x-1}{x-|x|\cdot\sqrt{1-\frac{3}{x}+1/x^2}}$$

$$= \lim_{x\to-\infty} \frac{3x-1}{x+x\sqrt{1-3/x+1/x^2}} = \lim_{x\to-\infty} \frac{3-1/x}{1+\sqrt{1-\frac{3}{x}+1/x^2}} = \frac{3-0}{1+1} = 3/2$$

ör $\lim\limits_{x\to-\infty} \dfrac{\sqrt{9x^2-5x+2}}{\sqrt[3]{8x^3+1}} = ?$ $\dfrac{\infty}{\infty}$

$$\lim_{x\to-\infty} \frac{|x|\sqrt{9-5/x+2/x^2}}{x\sqrt[3]{8+1/x^3}} = -\lim_{x\to-\infty} \frac{\sqrt{9-\frac{5}{x}+2/x^2}}{\sqrt[3]{8+1/x^3}} = -\frac{3}{2}$$

ör $\lim\limits_{x\to-\infty} \dfrac{2x+7}{\sqrt{x^2+3}-5x} = ?$ $\dfrac{\infty}{\infty}$

$$\lim_{x\to-\infty} \frac{2x+7}{|x|\sqrt{1+3/x^2}-5x} = \lim_{x\to-\infty} \frac{x(2+7/x)}{-x\sqrt{1+3/x^2}-5x} = \lim_{x\to-\infty} \frac{x(2+7/x)}{-x\left(\sqrt{1+3/x^2}+5\right)}$$

$$= -\lim_{x\to-\infty} \frac{2+7/x}{\sqrt{1+\frac{3}{x^2}}+5} = -\frac{2+0}{1+5} = -\frac{1}{3}$$

Kurallar

$x \to 0$ iken

$\sin(ax) \sim ax$

$\tan(ax) \sim ax$

$\cos(ax) \sim 1 - \frac{(ax)^2}{2}$

$\ln(1+ax) \sim ax$

$e^{ax} \sim 1 + ax$

$c^{ax} \sim 1 + ax \cdot \ln c, \quad c>0, c \neq 1$

$(1+ax)^n \sim 1 + n.ax, \quad n \neq 0$

$\log_b(1+ax) \sim \frac{ax}{\ln b}$

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

ispat

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}} = e^{\lim_{x\to 0}\frac{x}{x}} = e$$

$$\lim_{x\to\infty}\sqrt[x]{x} = 1$$

ispat

$$\lim_{x\to\infty}\sqrt[x]{x} = \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln x^{1/x}} = e^{\lim_{x\to\infty}\frac{\ln x}{x}} = e^0 = 1$$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x}{\sin x} = 1$$

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{x}{\tan x} = 1$$

$$\lim_{x\to 0}\frac{\sin(nx)}{mx} = \frac{n}{m}$$

$$\lim_{x\to 0}\frac{nx}{\tan(mx)} = \frac{n}{m}$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0}\frac{a^x-1}{x} = \ln a, \quad a>0, a\neq 1$$

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x} = n, \quad n\neq 0$$

$$\lim_{x\to 0}\frac{\sin(nx)}{\tan(mx)} = \frac{n}{m}$$

$$\lim_{x\to 0}\frac{e^{ax}-1}{\ln(1+ax)} = 1$$

ör $\lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2} = ?\ \dfrac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2} = \lim\limits_{x\to 0} \dfrac{1-(1-x^2/2)}{3x^2} = \lim\limits_{x\to 0} \dfrac{x^2/2}{3x^2} = \dfrac{1}{6}$

ör $\lim\limits_{x\to 0} \dfrac{2x \cdot \sin(5x)}{1-\cos(3x)} = ?\ \dfrac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{2x \cdot 5x}{1-\left(1-\frac{9x^2}{2}\right)} = \lim\limits_{x\to 0} \dfrac{10x^2}{9x^2/2} = \dfrac{20}{9}$

ör $\lim\limits_{x\to 0} \dfrac{e^{1-\cos x}-1}{x \cdot \sin x} = ?\ 0/0$

$\lim\limits_{x\to 0} \dfrac{e^{1-(1-x^2/2)}-1}{x \cdot x} = \lim\limits_{x\to 0} \dfrac{e^{x^2/2}-1}{x^2} = \lim\limits_{x\to 0} \dfrac{x^2/2}{x^2} = 1/2$

ör $\lim\limits_{x\to 0} \dfrac{\ln(1+5x)}{e^{3x}-1} \cdot \dfrac{1-\cos x+\sin x}{(x+4)\cdot \tan x} = ?\ \dfrac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{5x}{3x} \cdot \dfrac{1-(1-x^2/2)+x}{x(x+4)} = \dfrac{5}{3} \lim\limits_{x\to 0} \dfrac{x^2/2+x}{x(x+4)} = \dfrac{5}{3} \lim\limits_{x\to 0} \dfrac{\frac{x}{2}+1}{x+4} = \dfrac{5}{12}$

ör $\lim\limits_{x\to 0} \dfrac{\cos(3x)-\cos(2x)}{x \cdot \sin x} = ?\ 0/0$

$\lim\limits_{x\to 0} \dfrac{\left(1-\frac{9x^2}{2}\right)-\left(1-\frac{4x^2}{2}\right)}{x \cdot x} = \lim\limits_{x\to 0} \dfrac{-\frac{5x^2}{2}}{x^2} = -5/2$

ör $\lim\limits_{x\to 0} \dfrac{1-\cos(3x)+\ln(1+5x)}{e^{2x} \cdot \sin x} = ?\ \dfrac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{1-\left(1-\frac{9x^2}{2}\right)+5x}{(1+2x)\cdot x} = \lim\limits_{x\to 0} \dfrac{\frac{9x^2}{2}+5x}{x(2x+1)} = \lim\limits_{x\to 0} \dfrac{\frac{9x}{2}+5}{2x+1} = 5$

ör $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{3x^2+5x^3} = \dfrac{0}{0}$

$\lim\limits_{x\to 0} \dfrac{\sin^2 x}{3x^2+5x^3} = \lim\limits_{x\to 0} \dfrac{x^2}{3x^2+5x^3} = \lim\limits_{x\to 0} \dfrac{1}{3+5x} = \dfrac{1}{3}$

Ör $\lim_{x\to 2}(x-2)\tan\left(\frac{x\pi}{4}\right)=?\quad 0\times\infty \qquad u=x-2$

$$\lim_{x\to 2}(x-2)\tan\left(\frac{x\pi}{4}\right)=\lim_{u\to 0}u\tan\left(\frac{\pi}{4}(u+2)\right)$$

$$=\lim_{u\to 0}u.\tan\left(\frac{\pi u}{4}+\frac{\pi}{2}\right)=\lim_{u\to 0}u.\frac{\sin\left(\frac{\pi u}{4}+\pi/2\right)}{\cos\left(\frac{\pi u}{4}+\pi/2\right)}$$

$$=\lim_{u\to 0}u.\frac{\cos\left(\frac{\pi u}{4}\right)}{-\sin\left(\frac{\pi u}{4}\right)}=-\lim_{u\to 0}\frac{u}{\tan\left(\frac{\pi u}{4}\right)}=-\lim_{u\to 0}\frac{u}{\frac{\pi u}{4}}=-\frac{4}{\pi}$$

Ör $\lim_{x\to\infty}x.\sin\left(\frac{3}{2x}\right)=?\quad \infty.0 \qquad u=\frac{3}{2x}\to x=\frac{3}{2u}$

$$\lim_{x\to\infty}x.\sin\left(\frac{3}{2x}\right)=\lim_{u\to 0}\frac{3}{2u}\sin u=\lim_{u\to 0}\frac{3u}{2u}=3/2$$

Ör $\lim_{x\to\infty}x.\ln\left(1+\frac{2}{3x}\right)=?\quad \infty.0 \qquad u=\frac{2}{3x}\to x=\frac{2}{3u}$

$$\lim_{x\to\infty}x.\ln\left(1+\frac{2}{3x}\right)=\lim_{u\to 0}\frac{2}{3u}\ln(1+u)=\lim_{u\to 0}\frac{2u}{3u}=\frac{2}{3}$$

Ör $\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^{2x}=?\quad 1^\infty \qquad u=\frac{3}{x}\to x=\frac{3}{u}$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}=\lim_{u\to 0}(1+u)^{6/u}=\left(\lim_{u\to 0}(1+u)^{1/u}\right)^6=e^6$$

Ör $\lim_{x\to\pi/2}\frac{\sin 2x}{\pi-2x}=?\quad\frac{0}{0} \qquad u=\frac{\pi}{2}-x\to x=\pi/2-u$

$$\lim_{x\to\pi/2}\frac{\sin 2x}{\pi-2x}=\lim_{x\to\pi/2}\frac{2\cos x\sin x}{\pi-2x}=\lim_{x\to\pi/2}\frac{\cancel{2}\sin(\pi/2-x)\sin x}{\cancel{2}(\pi/2-x)}$$

$$=\lim_{u\to 0}\frac{\sin u}{u}\sin(\pi/2-u)=\lim_{u\to 0}\sin(\pi/2-u)=1$$

Ör $f(x)=\begin{cases}\frac{\sin x}{|x|} & x\neq 0\\ 1 & x=0\end{cases}$

$x=0$ için sürekli mi?

$f(0)=1$

$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\frac{\sin x}{|x|}=\lim_{x\to 0^-}\frac{\sin x}{-x}=-1$

$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\frac{\sin x}{|x|}=\lim_{x\to 0^+}\frac{\sin x}{x}=1$

limit yok
sürekli değil.
sağdan sürekli

$$\lim_{x\to 2}\frac{\ln(x^2-5x+7)+e^{x-2}-1}{x^2+2x-8}$$

Tümevarım

$$\sum_{k=1}^{n} a_k = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \dots + a_n \Rightarrow n \text{ terimli}$$

$$\sum_{k=m}^{n} a_k = \sum_{k=m-p}^{n-p} a_{k+p} \quad , \quad m < n \Rightarrow (n-m+1) \text{ terimli}$$

$$\sum_{k=1}^{n} (a_k \mp b_k) = \sum_{k=1}^{n} a_k \mp \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} c.a_k = c.\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{p} a_k + \sum_{k=p+1}^{n} a_k \quad 1 < p < n$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right)$$

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + \dots + r^n = S_n$$

$$S_n - rS_n = 1 - r^{n+1}$$

$$S_n = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad , \quad |r| < 1$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{k=1}^{n} (2k-1) = n^2$$

$$\sum_{k=1}^{n} 2k = n(n+1)$$

$$\sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \frac{n(n+1)(n+2)}{3}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$x = 0.3\overline{45}$ değerini kesirli hale getir.

$$\left.\begin{aligned} 1000x &= 345.\overline{45} \\ 10x &= 3.\overline{45} \end{aligned}\right\} \quad x = \frac{345.\overline{45} - 3.\overline{45}}{1000-10} = \frac{342}{990} = \frac{19}{55}$$

Tümevarım yöntemiyle ispat

Ör $1+2+3+\dots+n=\dfrac{n(n+1)}{2}$, $n\in Z^+$ eşitliğini tümevarım yöntemiyle ispatla.

a) $n=1$ için $1=\dfrac{1\times 2}{2}\Rightarrow 1=1$ ✓

b) $n=k$ için doğru olduğunu kabul et.

c) $n=k+1$ için

$$\underbrace{1+2+3+\dots+k}+(k+1)=\frac{(k+1)(k+2)}{2} \text{ olmalı.}$$

$$\frac{k(k+1)}{2}+(k+1)=(k+1)\left(\frac{k}{2}+1\right)=\frac{(k+1)(k+2)}{2} \checkmark$$

Ör $1^2+2^2+3^2+\dots+n^2=\dfrac{n(n+1)(2n+1)}{6}$ eşitliğini tümevarım yöntemiyle ispatla.

a) $n=1$ için $1^2=\dfrac{1\times 2\times 3}{6}\Rightarrow 1=1$ ✓

b) $n=k$ için doğru olduğunu kabul et.

c) $n=k+1$

$$\underbrace{1^2+2^2+3^2+\dots+k^2}+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6} \text{ olmalı}$$

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{k+1}{6}\Big(k(2k+1)+6(k+1)\Big)$$

$$=\frac{k+1}{6}\left(2k^2+7k+6\right)=\frac{(k+1)(k+2)(2k+3)}{6} \checkmark$$

Ör $1\times 2+2\times 3+3\times 4+\dots+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$ eşitliğini tümevarım yöntemiyle ispatla.

a) $n=1$ için $1\times 2=\dfrac{1\times 2\times 3}{3}\Rightarrow 2=2$ ✓

b) $n=k$ için doğru olduğunu kabul et

$$\underbrace{1\times 2+2\times 3+3\times 4+\dots+k(k+1)}+(k+1)(k+2)=\frac{(k+1)(k+2)(k+3)}{3} \text{ olmalı}$$

$$\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)=(k+1)(k+2)\left(\frac{k}{3}+1\right)$$

$$=(k+1)(k+2)\frac{k+3}{3} \checkmark$$

Diziler

Tanım kümesi N^+ olan fonksiyona dizi denir.
Fonksiyonun değer kümesi R ise bu diziye gerçel sayı dizisi denir.

$f: N^+ \longrightarrow R$ Gerçel sayı dizisi

f fonksiyonunun görüntü kümesi $= \{f(1), f(2), f(3), \ldots\}$

$a_n = f(n)$ yazılırsa a_n genel terim olur.

$(a_n) = (a_1, a_2, a_3, \ldots)$ Dizi gösterimi

Sonlu Dizi : Eleman sayısı sabit olan dizi

Sabit Dizi : $a_n = k$ yani $(a_n) = (k, k, k, \ldots)$

Eşit Diziler : $a_n = b_n$ ise (a_n) ve (b_n) dizileri eşit dizilerdir.

Dizilerde işlemler

$(a_n) \mp (b_n) = (a_n \mp b_n)$

$k(a_n) = (k a_n)$

$(a_n)(b_n) = (a_n b_n)$

$\dfrac{(a_n)}{(b_n)} = \left(\dfrac{a_n}{b_n}\right), \quad b_n \neq 0$

$a_{n+1} > a_n$ ise (a_n) monoton artan dizi

$a_{n+1} < a_n$ ise (a_n) monoton azalan dizi

$a_{n+1} = a_n$ ise (a_n) sabit dizi

Alt dizi

(k_n) pozitif tamsayıların artan bir dizisi olmak üzere (a_{k_n}) dizisine (a_n) dizisinin bir alt dizisi denir.

$\forall k_n \in N^+$ için $1 \leq k_1 \leq k_2 \leq k_3 \leq \ldots$

Her dizi kendinin bir alt dizisidir.

(a_{2n}), (a_{2n-1}), (a_{n^2+1}) dizileri birer alt dizilerdir.

(a_{4n-9}) dizisi alt dizi değil. $n=1$ ve $n=2$ için $k_n=-5$ $\quad k_n=-1$

Aritmetik Dizi

Ardışık iki terimi arasındaki fark sabit olan dizi.

$$a_2-a_1=a_3-a_2=a_4-a_3=\cdots=a_n-a_{n-1}=r \text{ : Ortak fark}$$

$$a_n=a_1+(n-1).r=a_p+(n-p).r\,,\quad n>p$$

$$r=\frac{a_n-a_1}{n-1}=\frac{a_n-a_p}{n-p}\,,\quad a_k=\frac{a_{k-p}+a_{k+p}}{2}\,,\quad k>p$$

$$S_n=\frac{a_1+a_n}{2}.n=\frac{n}{2}\left(2a_1+(n-1)r\right)$$ ilk n teriminin toplamı

Geometrik Dizi

Ardışık iki terimi arasındaki oran sabit olan dizi.

$$\frac{a_2}{a_1}=\frac{a_3}{a_2}=\frac{a_4}{a_3}=\cdots=\frac{a_n}{a_{n-1}}=r \text{ : Ortak çarpan}$$

$$a_n=a_1r^{n-1}=a_pr^{n-p}\,,\quad n>p$$

$$r=\sqrt[n-1]{\frac{a_n}{a_1}}=\sqrt[n-p]{\frac{a_n}{a_p}}\qquad a_k=\sqrt{a_{k-p}\cdot a_{k+p}}\,,\quad k>p$$

$$S_n=\sum_{k=1}^{n}a_k=a_1+a_2+a_3+\cdots+a_n$$

$$=a_1+a_1r+a_1r^2+\cdots+a_1r^{n-1}=a_1\sum_{k=0}^{n-1}r^k$$

$$=a_1\frac{1-r^n}{1-r}\,,\quad r\neq 1$$

Dizilerde Limit

$\lim\limits_{n\to\infty}a_n=L$ ve $\lim\limits_{n\to\infty}b_n=M$ olsun

$$\lim_{n\to\infty}k.a_n=k\lim_{n\to\infty}a_n=kL$$

$$\lim_{n\to\infty}(a_n\mp b_n)=\lim_{n\to\infty}a_n\mp\lim_{n\to\infty}b_n=L\mp M$$

$$\lim_{n\to\infty}(a_nb_n)=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}b_n=LM$$

$$\lim\left(\frac{a_n}{b_n}\right)=\frac{\lim a_n}{\lim b_n}=\frac{L}{M}\,,\quad M\neq 0$$

Ör $(a_{3n-2}) = \left(\frac{2n-1}{5-n}\right)$ ise $(a_n) = ?$

$$k = 3n-2 \longrightarrow n = \frac{k+2}{3}$$

$$a_k = \frac{2\,\frac{k+2}{3} - 1}{5 - \frac{k+2}{3}} = \frac{2(k+2)-3}{15-(k+2)} = \frac{2k+1}{13-k} \Rightarrow (a_n) = \left(\frac{2n+1}{13-n}\right)$$

Ör 5. terimi 16, ortak çarpanı $3/2$ olan geometrik dizinin 9. terimi nedir?

$$a_n = a_p\, r^{n-p}$$

$$a_9 = a_5\, r^4 \rightarrow a_9 = 16 \times \left(\frac{3}{2}\right)^4$$

$$= 16\,\frac{3^4}{2^4} = 16\,\frac{81}{16} = 81$$

Ör 13. terimi 35, ortak fark 3 olan aritmetik dizinin 4. terimi nedir?

$$a_n = a_p + (n-p).r$$

$$a_{13} = a_4 + 9r \rightarrow a_4 = a_{13} - 9r$$

$$= 35 - 9 \times 3$$

$$= 35 - 27 = 8$$

Ör $(a_n) = (3n-2)$ dizisinin ilk 35 teriminin toplamı.

$(a_n) = (3n-2) = (1, 4, 7, 10, 13, \ldots)$ aritmetik dizi

$$r = 3,\ a_1 = 1 \quad S_n = \frac{n}{2}\left(2a_1 + (n-1)r\right) = \frac{35}{2}(2 + 34 \times 3) = 1820$$

Ör Genel terimi $a_n = \frac{1}{(n+1)(n+2)}$ olan dizinin ilk 48 teriminin toplamı.

$$a_n = \frac{1}{(n+1)(n+2)} = \frac{a}{n+1} + \frac{b}{n+2} \quad \begin{matrix} a = 1 \\ b = -1 \end{matrix} \quad a_n = \frac{1}{n+1} - \frac{1}{n+2}$$

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \ldots + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2} - \frac{1}{n+2}$$

$$S_{48} = \underset{(25)}{\frac{1}{2}} - \underset{(1)}{\frac{1}{50}} = \frac{24}{50} = 0.48$$

Ör $\left(\frac{3}{n+5}\right)$ dizisinin kaç terimi $(0.25, 0.75)$ aralığı içindedir.

$$0.25 < \frac{3}{n+5} < 0.75$$

$$\frac{1}{4} < \frac{3}{n+5} < \frac{3}{4} \quad \text{ters çevir}$$

$$\frac{4}{3} < \frac{n+5}{3} < 4$$

→ 3 ile çarp

$$4 < n+5 < 12, \quad 5 \text{ çıkar}$$

$$-1 < n < 7$$

$$n = 1, 2, 3, 4, 5, 6$$

ilk 6 terimi

Ör $\left(\frac{2n-15}{7-3n}\right)$ dizisinin kaç terimi pozitiftir?

$$f(x) = \frac{2x-15}{7-3x}$$

$$2x - 15 = 0$$
$$x = 15/2$$
$$7 - 3x = 0$$
$$x = 7/3$$

		$\frac{7}{3}$		$\frac{15}{2}$	
$2x-15$	$-$		$-$		$+$
$7-3x$	$+$		$-$		$-$
$f(x)$	$-$		$+$		$-$

$$\frac{7}{3} < x < \frac{15}{2}$$

$$\frac{7}{3} > 2, \quad \frac{15}{2} < 8$$

$$n = 3, 4, 5, 6, 7$$

5 terim pozitif

Ör $\left(\frac{n^2-7n+1}{n+2}\right)$ dizisinin kaç terimi $\frac{1}{2}$'den küçüktür.

$$\frac{n^2-7n+1}{n+2} < \frac{1}{2} \Rightarrow \underset{(2)}{\frac{n^2-7n+1}{n+2}} - \underset{(n+2)}{\frac{1}{2}} < 0$$

$$\frac{2n^2-15n}{2(n+2)} < 0 \Rightarrow \frac{n(2n-15)}{2(n+2)} < 0$$

pozitif

$$2n - 15 < 0 \Rightarrow n < 15/2 \longrightarrow n = 1, 2, 3, 4, 5, 6, 7$$

ilk 7 terim.

Ör $(a_n) = \left(\frac{3n-2}{n+1}\right)$ dizisini monotonluk yönünden incele.

$$a_{n+1} - a_n = \frac{3(n+1)-2}{(n+1)+1} - \frac{3n-2}{n+1} = \underset{(n+1)}{\frac{3n+1}{n+2}} - \underset{(n+2)}{\frac{3n-2}{n+1}}$$

$$= \frac{(n+1)(3n+1)-(n+2)(3n-2)}{(n+1)(n+2)} = \frac{5}{(n+1)(n+2)} > 0$$

monoton artan

Ör $(a_n) = \left(\frac{n}{e^n}\right)$ dizisini monotonluk yönünden incele.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \frac{n+1}{n} \cdot \frac{e^n}{e^{n+1}} = \frac{1}{e}\left(1+\frac{1}{n}\right) < \frac{2}{e} < 1$$ monoton azalan

Ör $\left(\frac{4n+3}{10n+2}\right)$ dizisi $\left(\frac{2n+1}{5n-1}\right)$ dizisinin bir alt dizisi midir?

$a_n = \frac{2n+1}{5n-1}$ dersek $a_{k_n} = \frac{2k_n+1}{5k_n-1} = \frac{4n+3}{10n+2}$ olmalı

$$\cancel{20nk_n} + 4k_n + 10n + 2 = \cancel{20nk_n} + 15k_n - 4n - 3$$

$$11k_n = 14n+5 \to k_n = \frac{14n+5}{11}$$

$\forall n \in N^+$ için $\frac{14n+5}{11} \notin N^+$ Bu yüzden alt dizisi değil.

Ör $\left(\frac{3n-1}{n+2}\right)$ dizisi yakınsak mı?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n-1}{n+2} = \lim_{n\to\infty} \frac{3-1/n}{1+2/n} = 3 < \infty$$ yakınsak

Ör $\lim_{n\to\infty} \frac{2^{n+1} + 3\times 5^{n+1}}{3^n + 2\times 5^n} = ?$

$$\lim_{n\to\infty} \frac{2^{n+1} + 3\times 5^{n+1}}{3^n + 2\times 5^n} = \lim_{n\to\infty} \frac{5^{n+1}\left(\left(\frac{2}{5}\right)^{n+1} + 3\right)}{5^n\left(\left(\frac{3}{5}\right)^n + 2\right)}$$

$$= 5 \lim_{n\to\infty} \frac{3+(2/5)^{n+1}}{2+(3/5)^n} = 5\times\frac{3+0}{2+0} = \frac{15}{2}$$

Ör $(a_n) = \left(\sqrt{n^2+3n} - \sqrt{n^2+4}\right)$ dizisinin limitini bul.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sqrt{n^2+3n} - \sqrt{n^2+4}\right)\frac{\sqrt{n^2+3n}+\sqrt{n^2+4}}{\sqrt{n^2+3n}+\sqrt{n^2+4}}$$

$$= \lim_{n\to\infty} \frac{(n^2+3n)-(n^2+4)}{\sqrt{n^2+3n}+\sqrt{n^2+4}} = \lim_{n\to\infty} \frac{3n-4}{|n|\left(\sqrt{1+3/n}+\sqrt{1+4/n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{3-4/n}{\sqrt{1+3/n}+\sqrt{1+4/n^2}} = 3/2$$

Teorem
(a_n) pozitif terimli bir dizi olsun

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = r$ ise $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = r$ olur.

$(a_n) = \left(\frac{n}{e^n}\right)$ dizisinini monotonluk yönünden incele.

① $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \lim\limits_{n\to\infty} \frac{n+1}{n} \frac{e^n}{e^{n+1}}$

$= \frac{1}{e} \lim\limits_{n\to\infty} \frac{n+1}{n} = \frac{1}{e} \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right) = \frac{1}{e} < 1$ monoton azalan.

②
$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{e^n}} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{n}}{e} = \frac{1}{e} < 1$ monoton azalan

Ör $(a_n) = \left(\frac{3}{2} + \frac{5}{n}\right)^n$ dizisini monotonluk yönünden incele.

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \left(\frac{3}{2} + \frac{5}{n}\right) = \frac{3}{2} > 1$ monoton artan

Ör $(a_n) = \left(\frac{n^2}{3^n}\right)$ dizisini monotonluk yönünden incele.

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n^2}{3^n}} = \lim\limits_{n\to\infty} \frac{(\sqrt[n]{n})^2}{3} = \frac{1}{3} < 1$ monoton azalan

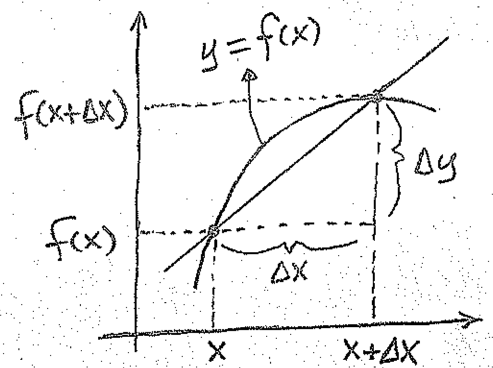
Ör $\left(\frac{5n-2}{3n+\sqrt{n^2+4}}\right)$ dizisinin limitini bul.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{5n-2}{3n+\sqrt{n^2+4}} = \lim\limits_{n\to\infty} \frac{5n-2}{3n+|n|\sqrt{1+4/n^2}}$

$= \lim\limits_{n\to\infty} \frac{5n-2}{3n+n\sqrt{1+4/n^2}} = \lim\limits_{n\to\infty} \frac{n(5-2/n)}{n(3+\sqrt{1+4/n^2})}$

$= \frac{5-0}{3+1} = \frac{5}{4}$

Türev ve Uygulamaları



$$\Delta y = \Delta f(x) = f(x+\Delta x) - f(x)$$

$\dfrac{\Delta y}{\Delta x}$: Değişim oranı

$$y' = \frac{dy}{dx} = f'(x) = \frac{df(x)}{dx} = D_x(f(x)) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Teorem
Bir fonksiyonun herhangi bir noktadaki türevi fonksiyona o noktada teğet olan doğrunun eğimidir.

$u = x + \Delta x$

$$y=f(x) \text{ ise } y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{u \to x} \frac{f(u)-f(x)}{u-x}$$

Teorem
Fonksiyonun herhangi bir noktada sağdan ve soldan türevleri birbirine eşit ise fonksiyon o noktada türevlidir.

Soldan türev

$$f'(a^-) = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a}$$

Sağdan türev

$$f'(a^+) = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$$

$f'(a^-) = f'(a^+)$ ise $f(x)$ fonksiyonu $x=a$'da türevlidir.

Teoremler

1. Fonksiyon türevli olduğu noktalarda süreklidir.
2. Fonksiyonun sürekli olduğu halde türevli olmayan noktaları kırılma noktalarıdır.
3. Fonksiyon süreksiz olduğu noktalarda türevsizdir.

Ör $f(x) = x^3 - x^2 + 3x + 5$ ise $f'(x) = ?$ Limit kullan

$$f'(x) = \lim_{u \to x} \frac{f(u)-f(x)}{u-x} = \lim_{u \to x} \frac{(u^3-u^2+3u+5)-(x^3-x^2+3x+5)}{u-x}$$

$$= \lim_{u \to x} \frac{(u^3-x^3)-(u^2-x^2)+3(u-x)}{u-x}$$

$$= \lim_{u \to x} \frac{(u-x)(u^2+ux+x^2)-(u-x)(u+x)+3(u-x)}{u-x}$$

$$= \lim_{u \to x} \frac{(u-x)(u^2+ux+x^2-u-x+3)}{u-x} = \lim_{u \to x}(u^2+ux+x^2-u-x+3)$$

$$= x^2+x^2+x^2-x-x+3 = 3x^2-2x+3$$

Kurallar

$(f(x) \mp g(x))' = f'(x) \mp g'(x)$ Toplama ve çıkarmada türev

$(f(x) \cdot g(x))' = f'(x)g(x) + f(x) \cdot g'(x)$ Çarpmada türev

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ Bölmede türev

$(uv)' = u'v + uv'$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$

Bileşke fonksiyonun türevi

$(f(g(x)))' = g'(x) \cdot f'(g(x))$

$(f(u))' = u' \cdot f'(u)$, $u = g(x)$

Çarpmanın türevinin ispatı

$p(x) = f(x) \cdot g(x)$ olsun

$$p'(x) = \lim_{u \to x} \frac{p(u) - p(x)}{u - x} = \lim_{u \to x} \frac{f(u)g(u) - f(x)g(x)}{u - x}$$

$$= \lim_{u \to x} \frac{f(u)g(u) - f(x)g(x) + f(x)g(u) - f(x)g(u)}{u - x}$$

$$= \lim_{u \to x} \frac{f(u)g(u) - f(x)g(u)}{u - x} + \lim_{u \to x} \frac{f(x)g(u) - f(x)g(x)}{u - x}$$

$$= \lim_{u \to x} g(u) \frac{f(u) - f(x)}{u - x} + \lim_{u \to x} f(x) \frac{g(u) - g(x)}{u - x}$$

$$= g(x) \lim_{u \to x} \frac{f(u) - f(x)}{u - x} + f(x) \lim_{u \to x} \frac{g(u) - g(x)}{u - x}$$

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$y = x^n$ ise $y' = n \cdot x^{n-1}$, $n \in R$, $n \neq 0$

$y = u^n$ ise $y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = n \cdot u^{n-1} \cdot u'$, $n \in R$, $n \neq 0$

Ör: $y = x^4 + \frac{1}{x} - \frac{3}{x^2} + \sqrt{x} - \sqrt[3]{x^2}$ ise $y' = ?$

$y = x^4 + x^{-1} - 3x^{-2} + x^{1/2} - x^{2/3}$

$y' = 4x^3 - \frac{1}{x^2} + \frac{6}{x^3} + \frac{1}{2\sqrt{x}} - \frac{2}{3 \cdot \sqrt[3]{x}}$

Çift fonksiyonun türevi tek fonksiyon, tek fonksiyonun türevi çift fonksiyondur.

$y = x^2 + 3$	$y = x^3 + 2x$
$y' = 2x$	$y' = 3x^2 + 2$

Ör $y=\sqrt{x^2+9}$ ise $y'=?$

$y=(x^2+9)^{1/2}$ yazılır. $y'=\frac{1}{2}(x^2+9)^{-1/2}\cdot 2x=\dfrac{x}{\sqrt{x^2+9}}$

Ör $f(x)=\dfrac{3x-2}{2x+1}$ ise $f'(2)=?$

$f'(x)=\dfrac{3\cdot(2x+1)-(3x-2)\cdot 2}{(2x+1)^2}=\dfrac{7}{(2x+1)^2}\Rightarrow f'(2)=\dfrac{7}{25}$

Parçalı fonksiyonun türevi

$$f(x)=\begin{cases} g(x) & x<a \\ h(x) & a\le x<b \\ p(x) & x\geq b \end{cases} \quad \text{ise} \quad f'(x)=\begin{cases} g'(x) & x<a \\ h'(x) & a\le x<b \\ p'(x) & x\geq b \end{cases}$$

a ve b noktalarında f(x) sürekli ise bakılır.

$g(a)=h(a)$ ise $x=a$'da sürekli.

$h(b)=p(b)$ ise $x=b$'de sürekli.

Mutlak değer fonksiyonunun türevi

$$f(x)=|g(x)|=\begin{cases} g(x) & g(x)\geq 0 \\ -g(x) & g(x)<0 \end{cases}$$

$$f'(x)=\begin{cases} g'(x) & g(x)>0 \\ -g'(x) & g(x)<0 \end{cases}$$

$x=a$'da $g(x)=0$ olsun.

$g'(a)=-g'(a)$ olması için

$g'(a)=0$ ise türevli.

Signum fonksiyonunun türevi

$$f(x)=\operatorname{sgn}(g(x))=\begin{cases} -1 & g(x)<0 \\ 0 & g(x)=0 \\ 1 & g(x)>0 \end{cases} \Rightarrow f'(x)=\begin{cases} 0 & g(x)\neq 0 \\ \text{türevsiz} & g(x)=0 \end{cases}$$

Tam değer fonksiyonunun türevi

$$f(x)=[\![g(x)]\!] \Rightarrow f'(x)=\begin{cases} 0 & g(x)\notin \mathbb{Z} \\ \text{türevsiz} & g(x)\in \mathbb{Z} \end{cases}$$

Ör $f(x)=\begin{cases} x^2+5x-7 & x<2 \\ 9x-11 & x\geq 2 \end{cases}$ ise $f'(x)=?$

$f'(x)=\begin{cases} 2x+5 & x<2 \\ 9 & x\geq 2 \end{cases}$

$f(2^-)=4+10-7=7$

$f(2^+)=18-11=7$

süreklidir.

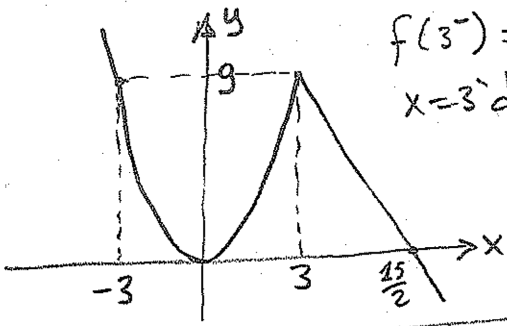
$f'(2^-)=4+5=9$

$f'(2^+)=9$

soldan ve sağdan türev birbirine eşit

Kırılma noktası yoktur.

Ör $f(x) = \begin{cases} x^2, & x < 3 \\ 15-2x, & x \geq 3 \end{cases}$ ise $f'(x)$



$f(3^-) = f(3^+) = 9$

$x=3$'de sürekli

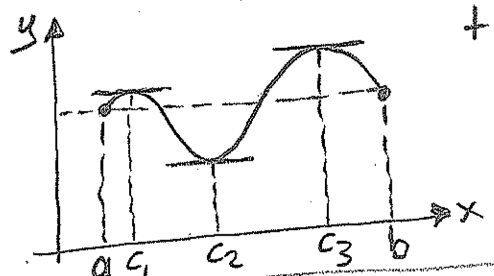
$f'(3^-)=6 \quad f'(3^+)=-2$

$x=3$'de türevsiz

$f'(x) = \begin{cases} \text{türevsiz}, & x=3 \\ 2x, & x<3 \\ -2, & x>3 \end{cases}$

$x=3$'de fonksiyon sürekli fakat türevsiz olduğundan kırılma noktası.

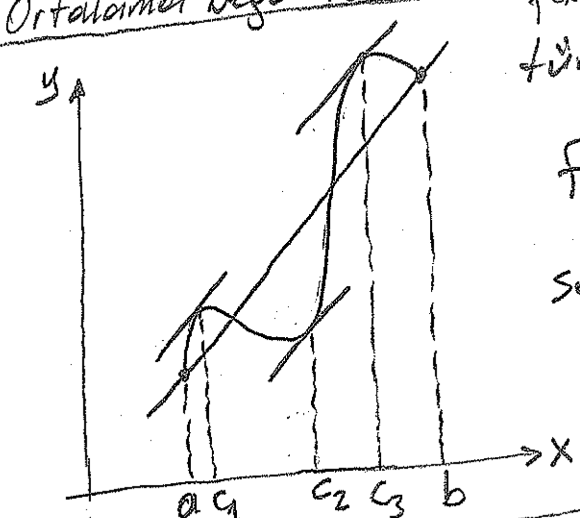
Rolle Teoremi



$f(x)$ fonksiyonu $x \in [a,b]$ için sürekli ve türevli olsun.

$f(a)=f(b)$ ise $f'(c)=0$ yapan en az bir tane $(c, f(c))$ noktası vardır.

Ortalama Değer Teoremi



$f(x)$ fonksiyonu $x \in [a,b]$ için sürekli ve türevli olsun

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$ eşitliğini sağlayan en az bir tane $(c, f(c))$ noktası vardır.

Ör $f(x) = x^2-3x+5$ fonksiyonunun $x \in [-1,3]$ aralığında ortalama değer teoremini sağlayan c noktasını bul.

$f(x) = x^2-3x+5 \qquad f'(c) = \dfrac{f(b)-f(a)}{b-a} \qquad 2c-3 = \dfrac{f(3)-f(-1)}{(3)-(-1)}$

$f'(x) = 2x-3 \qquad\qquad 2c-3 = \dfrac{5-9}{4} = -1$

$f(3) = 9-9+5 = 5$

$f(-1) = 1+3+5 = 9$

$c=1$ için $f(x)$ fonk. ortalama değer teoremini sağlıyor. $\quad c=1$

Ör $f(x) = x^3-3x^2+2x+3$ fonksiyonu $x \in [0,1]$ için Rolle teoremini sağlar mı?

$f(0)=f(1)=3 \qquad f'(x) = 3x^2-6x+2 = 0 \rightarrow x_1 = \dfrac{3-\sqrt{3}}{3}$ ✓, $\quad x_2 = \dfrac{3+\sqrt{3}}{3}$ ✗

$c = \dfrac{3-\sqrt{3}}{3}$

Trigonometrik Fonksiyonların Türevi

$y=\sin x \longrightarrow y=\cos x$

$y=\cos x \longrightarrow y'=-\sin x$

$y=\tan x \longrightarrow y'=\sec^2 x$

$y=\cot x \longrightarrow y'=-\csc^2 x$

$y=\sec x \longrightarrow y'=\sec x.\tan x$

$y=\csc x \longrightarrow y'=-\csc x.\cot x$

$y=\sin u \longrightarrow y'=u'.\cos u$

$y=\cos u \longrightarrow y'=-u'.\sin u$

$y=\tan u \longrightarrow y'=u'.\sec^2 u$

$y=\cot u \longrightarrow y'=-u'.\csc^2 u$

$y=\sec u \longrightarrow y'=u'.\sec u.\tan u$

$y=\csc u \longrightarrow y'=-u'.\csc u.\cot u$

Ör $f(x)=\cos x$ fonk. türevini limit kuralını kullanarak bul.

$$f'(x)=\lim_{u\to x}\frac{f(u)-f(x)}{u-x}=\lim_{u\to x}\frac{\cos u-\cos x}{u-x}=\lim_{u\to x}\frac{-2\sin\left(\frac{u+x}{2}\right)\sin\left(\frac{u-x}{2}\right)}{u-x}$$

$$=-\lim_{u\to x}\sin\left(\frac{u+x}{2}\right).\lim_{u\to x}\frac{\sin\left(\frac{u-x}{2}\right)}{\frac{u-x}{2}}=-\sin x.\lim_{v\to 0}\frac{\sin v}{v}=-\sin x$$

Ters trigonometrik fonksiyonların türevi

$y=\arcsin x \longrightarrow y'=\dfrac{1}{\sqrt{1-x^2}}$

$y=\arccos x \longrightarrow y'=\dfrac{-1}{\sqrt{1-x^2}}$

$y=\arctan x \longrightarrow y'=\dfrac{1}{1+x^2}$

$y=\operatorname{arccot} x \longrightarrow y'=\dfrac{-1}{1+x^2}$

$y=\operatorname{arcsec} x \longrightarrow y'=\dfrac{1}{x\sqrt{x^2-1}}$

$y=\operatorname{arccsc} x \longrightarrow y'=\dfrac{-1}{x\sqrt{x^2-1}}$

$y=\arcsin u \longrightarrow y'=\dfrac{u'}{\sqrt{1-u^2}}$

$y=\arccos u \longrightarrow y'=\dfrac{-u'}{\sqrt{1-u^2}}$

$y=\arctan u \longrightarrow y'=\dfrac{u'}{1+u^2}$

$y=\operatorname{arccot} u \longrightarrow y'=\dfrac{-u'}{1+u^2}$

$y=\operatorname{arcsec} u \longrightarrow y'=\dfrac{u'}{u.\sqrt{u^2-1}}$

$y=\operatorname{arccsc} u \longrightarrow y'=\dfrac{-u'}{u\sqrt{u^2-1}}$

$\arcsin(\sin x)=\sin(\arcsin x)=x$

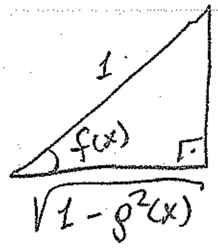
$\arctan(\tan x)=\tan(\arctan x)=x$

$\operatorname{arcsec}(\sec x)=\sec(\operatorname{arcsec} x)=x$

ör $f(x) = \arcsin(g(x))$ için $f'(x) = ?$

$f(x) = \arcsin(g(x))$

$g(x) = \sin(f(x))$



$g'(x) = f'(x) \cdot \cos(f(x))$

$$f'(x) = \frac{g'(x)}{\cos(f(x))} = \frac{g'(x)}{\sqrt{1 - g^2(x)}}$$

Üstel fonksiyonların türevi

$y = e^x \rightarrow y' = e^x$

$y = a^x \rightarrow y' = a^x . \ln a \quad a > 0, \ a \neq 1$

$y = e^u \rightarrow y' = u' . e^u$

$y = a^u \rightarrow y' = u' . a^u . \ln a \quad a > 0, \ a \neq 1$

Logaritmik fonksiyonların türevi

$y = \ln x \rightarrow y' = \frac{1}{x}$

$y = \log_a x \rightarrow y' = \frac{1}{x . \ln a} \quad a > 0, \ a \neq 1$

$y = \ln u \rightarrow y' = \frac{u'}{u}$

$y = \log_a u \rightarrow y' = \frac{u'}{u . \ln a} \quad a > 0, \ a \neq 1$

Logaritma Yardımıyla Türev

$y = u^v$ ise $\ln y = \ln u^v = v . \ln u \qquad u = u(x), \ v = v(x)$

$$\frac{y'}{y} = v' \ln u + v . \frac{u'}{u} \Rightarrow y' = u^v . \left(v' \ln u + v \frac{u'}{u}\right)$$

Hiperbolik fonksiyonlarda türev

$y = \sinh x \rightarrow y' = \cosh x$

$y = \cosh x \rightarrow y' = \sinh x$

$y = \tanh x \rightarrow y' = \mathrm{sech}^2 x$

$y = \coth x \rightarrow y' = -\mathrm{csch}^2 x$

$y = \mathrm{sech} x \rightarrow y' = -\mathrm{sech} x . \tanh x$

$y = \mathrm{csch} x \rightarrow y' = -\mathrm{csch} x . \coth x$

$y = \sinh u$ ise

$y' = u' . \cosh u$

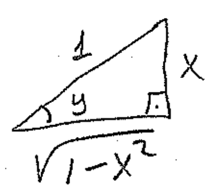
Ters fonksiyonların türevi

$y = f(x)$ için $x = g(y)$ olsun

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \Rightarrow y_x = \frac{1}{x_y}$$

$y = \arcsin x$ ise $x = \sin y$

$$y' = y_x = \frac{1}{x_y} = \frac{1}{\cos y}$$



$\cos y = \sqrt{1 - x^2}$

Ör $y = x . \arctan\left(\frac{2x}{3}\right) - \frac{3}{4}\ln(4x^2+9)$ ise $y'' = ?$

$$y' = \arctan\left(\frac{2x}{3}\right) + x\,\frac{2/3}{1+\left(\frac{2x}{3}\right)^2} - \frac{3}{4}\,\frac{8x}{4x^2+9}$$

$$= \arctan\left(\frac{2x}{3}\right) + \frac{6x}{4x^2+9} - \frac{6x}{4x^2+9} = \arctan\left(\frac{2x}{3}\right)$$

$$y'' = \frac{2/3}{1+\left(\frac{2x}{3}\right)^2} = \frac{6}{4x^2+9}$$

Ör $y = x^2 . \operatorname{arcsec}x - \sqrt{x^2-1}$ ise $y'' = ?$

$$y' = 2x . \operatorname{arcsec}x + x^2\,\frac{1}{x\sqrt{x^2-1}} - \frac{1}{2}(x^2-1)^{-1/2} . 2x$$

$$= 2x . \operatorname{arcsec}x + \frac{x}{\sqrt{x^2-1}} - \frac{x}{\sqrt{x^2-1}} = 2x . \operatorname{arcsec}x$$

$$y'' = 2 . \operatorname{arcsec}x + 2x\,\frac{1}{x\sqrt{x^2-1}} = 2\operatorname{arcsec}x + \frac{2}{\sqrt{x^2-1}}$$

Ör $y = x^2\ln x$ ise $y' = ?$

$$y' = 2x . \ln x + x^2 . \frac{1}{x} = 2x\ln x + x = x\,(1+2\ln x)$$

Ör $f(x) = \sqrt{x} . \cos x$ ise $f'(\pi/4) = ?$

$$f'(x) = \frac{1}{2\sqrt{x}}\cos x - \sqrt{x} . \sin x = \frac{\cos x - 2x . \sin x}{2\sqrt{x}}$$

$$f'(\pi/4) = \frac{\cos(\pi/4) - \pi/2 . \sin(\pi/4)}{2\sqrt{\pi/4}} = \frac{1-\pi/2}{\sqrt{2\pi}}$$

Ör $y = \cos^2 x . \sin^3 x$ ise $y' = ?$

$$y' = (-2\cos x\sin x) . \sin^3 x + \cos^2 x . (3\sin^2 x . \cos x)$$

$$= -2\cos x\sin^4 x + 3\cos^3 x\sin^2 x$$

$$= \cos x . \sin^2 x\,(3\cos^2 x - 2\sin^2 x) = \cos x . \sin^2 x . (5\cos^2 x - 2)$$

Ör. $y = \arctan\left(\frac{x-4}{x+4}\right)$ ise $y' = ?$

$$y' = \frac{\left(\frac{x-4}{x+4}\right)'}{1 + \left(\frac{x-4}{x+4}\right)^2} = \frac{\frac{(x+4)-(x-4)}{(x+4)^2}}{\frac{(x+4)^2+(x-4)^2}{(x+4)^2}} = \frac{4}{x^2+16}$$

Ör. $y = \ln\left(\frac{\sec^2 x}{1-\tan^2 x}\right)$ ise $y' = ?$

$$y = 2\ln(\sec x) - \ln(1-\tan^2 x)$$

$$y' = 2\frac{\sec x \tan x}{\sec x} - \frac{-2\tan x . \sec^2 x}{1-\tan^2 x}$$

$\cos^2 x$ ile çarp

$$= 2\tan x + \frac{2\tan x}{\cos^2 x - \sin^2 x} = 2\tan x\,(1+\sec 2x)$$

Ör. $y = x.\arcsin x + \sqrt{1-x^2}$ ise $y' = ?$

$$y' = \arcsin x - x\frac{1}{\sqrt{1-x^2}} + \frac{-2x}{2\sqrt{1-x^2}} = \arcsin x$$

Ör. $y = x.\arctan x - \ln\sqrt{1+x^2}$ ise $y' = ?$

$$y' = \arctan x + x\frac{1}{1+x^2} - \frac{\frac{1}{2}(1+x^2)^{-1/2}.2x}{\sqrt{1+x^2}}$$

$$= \arctan x + \frac{x}{1+x^2} - \frac{x}{1+x^2} = \arctan x$$

Ör. $y = e^{\sqrt{x}} - \sqrt{x}.\ln x$ ise $y' = ?$

$$y' = \frac{1}{2\sqrt{x}}e^{\sqrt{x}} - \left(\frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x}\right)$$

$$= \frac{e^{\sqrt{x}}}{2\sqrt{x}} - \frac{\ln x}{2\sqrt{x}} - \frac{1}{\sqrt{x}} = \frac{e^{\sqrt{x}} - \ln x - 2}{2\sqrt{x}}$$

Ör $y = x^{\cos x}$ ise $y' = ?$

$\ln y = \ln x^{\cos x} = \cos x \cdot \ln x \rightarrow \frac{y'}{y} = -\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}$

$y' = y\left(\frac{\cos x}{x} - \sin x \cdot \ln x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \cdot \ln x\right)$

Ör $y = (x^2+1)^3 / (x^3-2)^2$ ise $y' = ?$

$\ln y = 3\ln(x^2+1) - 2\ln(x^3-2)$

$\frac{y'}{y} = 3\frac{2x}{x^2+1} - 2\frac{3x^2}{x^3-2} = -\frac{6x(x+2)}{(x^2+1)(x^3-2)}$

$y' = -y\frac{6x(x+2)}{(x^2+1)(x^3-2)} = -\frac{6x(x+2)(x^2+1)^2}{(x^3-2)^3}$

Ör $y = \sqrt{x^7 e^{3x}}$ ise $y' = ?$

$y = x^{7/2} e^{3x/2} \longrightarrow \ln y = \frac{7}{2}\ln x + \frac{3x}{2}$

$\frac{y'}{y} = \frac{7}{2x} + \frac{3}{2} = \frac{3x+7}{2x} \longrightarrow y' = \frac{3x+7}{2}\sqrt{x^5 e^{3x}}$

Ör $f(x) = \frac{1}{x-2} + |x+3| + \operatorname{sgn}(x^2+x-2)$

a) Türevsiz bölge b) $f'(0) = ?$

a) $x-2=0$, $x=2$ $\quad x+3=0$, $x=-3$ $\quad x^2+x-2=0$, $(x+2)(x-1)=0$, $x=-2$ $x=1$

Türevsiz bölge $= \{-3, -2, 1, 2\}$

b) $x=0$ için $f(x) = \frac{1}{x-2} + x + 3 + \operatorname{sgn}(-2) = \frac{1}{x-2} + x + 2$

$f'(x) = \frac{-1}{(x-2)^2} + 1 \longrightarrow f'(0) = -\frac{1}{4} + 1 = 3/4$

Ör $f(x) = [\![\frac{7x+3}{5}]\!]\frac{\operatorname{sgn}\left(\frac{3x-1}{2}\right)}{x} + |x^2-5x+3|$ ise $f'(2) = ?$

$x=2$ için $f(x) = [\![\frac{17}{5}]\!]\frac{\operatorname{sgn}(5/2)}{x} - (x^2-5x+3) = \frac{3}{x} - x^2 + 5x - 3$

$f'(x) = -\frac{3}{x^2} - 2x + 5 \rightarrow f'(2) = -\frac{3}{4} - 4 + 5 = 1/4$

Ör $f(x) = |x^2-2x-3| + \operatorname{sgn}(x^2-4) + \frac{2}{x+3}$ türevsiz bölge

$x^2-2x-3=0$
$(x+1)(x-3)=0$
$x=-1 \quad x=3$

$x^2-4=0$
$(x-2)(x+2)=0$
$x=2, x=-2$
$\{-2,2\}$ türevsiz

$x+3=0$
$x=-3$
$\{-3\}$ türevsiz

$$\begin{array}{c|c|c} & -1 & 3 \\ + & - & + \end{array}$$

$g(x)=x^2-2x-3$
$g'(x)=2x-2$
$g'(-1)=-4 \neq 0$
$g'(3)=4 \neq 0$
$\{-1,3\}$ türevsiz

Türevsiz bölge $= \{-1,3\} \cup \{-2,2\} \cup \{-3\}$
$= \{-3,-2,-1,2,3\}$

Ör $f(x) = \frac{1}{x^2-2x-3} + |x^2-5x+6| + \operatorname{sgn}(x-5)$ Türevsiz bölge

$x^2-2x-3=0$
$(x+1)(x-3)=0$
$x=-1 \quad x=3$
$\{-1,3\}$ türevsiz

$x^2-5x+6=0$
$(x-2)(x-3)=0$
$x=2 \quad x=3$
$\{2,3\}$ türevsiz

$g(x)=x^2-5x+6$
$g'(x)=2x-5$
$g'(2)=-1 \neq 0$
$g'(3)=1 \neq 0$

$x-5=0$
$x=5$
$\{5\}$ türevsiz

Türevsiz Bölge $= \{-1,3\} \cup \{2,3\} \cup \{5\}$
$= \{-1,2,3,5\}$

Ör Diferansiyel kullanarak $\sqrt[3]{7.95}$ değerini yaklaşık olarak hesapla.

$f(x) = x^{1/3} \longrightarrow f'(x) = \frac{1}{3}x^{-2/3}$ $\quad x=8$, $\Delta x = -0.05$

$f(x+\Delta x) = f(x) + f'(x)\cdot\Delta x$

$f(7.95) = f(8) - 0.05 f'(8) = 2 - 0.05 \times \frac{1}{3} 8^{-2/3} = 2 - \frac{1}{240} \approx 1.996$

Ör $\sqrt{10} \approx ?$

$f(x)=\sqrt{x} \quad f'(x) = \frac{1}{2\sqrt{x}}$

$f(x+\Delta x) = f(x) + f'(x)\cdot\Delta x$ $\quad x=9$, $\Delta x = 1$

$f(10) = f(9) + f'(9)$
$= \sqrt{9} + \frac{1}{2\sqrt{9}} = 3 + \frac{1}{6} \approx 3.167$

Parametrik fonksiyonların türevleri

$z = z(y)$, $y = y(x)$, $x = x(t)$ olsun

$$\frac{dz}{dt} = \frac{dz}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\} \; y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{d(y')/dt}{dx/dt}$$

$f = f(x,y,z)$; $x = x(t)$, $y = y(t)$, $z = z(t)$ olsun.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = f_x \cdot dx + f_y \cdot dy + f_z \cdot dz$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$f = f(x,y,z)$; $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

Ör $z = 2y^2 + 3y + 5$, $y = 3x - 4$, $x = t^2 - 3$ ise $\left.\frac{dz}{dt}\right|_{t=2} = ?$

$t = 2$ için $x = 1$, $y = -1$, $z = 4$ olur.

$$\frac{dz}{dt} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dt} = (4y+3)(3)(2t) \Rightarrow \left.\frac{dz}{dt}\right|_{t=2} = -12$$

Ör $y = 2x - \frac{3}{x}$, $x = 2t + 3$, $t = r^2 - 2r - 5$ ise $\left.\frac{dy}{dr}\right|_{r=3} = ?$

$r = 3$ için $t = -2$, $x = -1$, $y = 1$ olur

$$\frac{dy}{dr} = \frac{dy}{dx}\frac{dx}{dt}\frac{dt}{dr} = \left(2 + \frac{3}{x^2}\right)(2)(2r-2)$$

$$\left.\frac{dy}{dr}\right|_{r=3} = (2+3)(2)(4) = 40$$

$\underline{\text{ör}}$ $x=\sqrt{t}$, $y=2t-3/\sqrt{t}$ ise $\dfrac{dy}{dx}=?$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2+\frac{3}{2}t^{-3/2}}{\frac{1}{2}t^{-1/2}}=4\sqrt{t}+\frac{3}{t}=4x+\frac{3}{x^2}$$

<u>diğer yol</u>

$x=\sqrt{t}$ ise $t=x^2$

$$y=2t-3/\sqrt{t}=2x^2-\frac{3}{x}\Rightarrow\frac{dy}{dx}=4x+\frac{3}{x^2}$$

$\underline{\text{ör}}$ $x=t^2+2t+3$, $y=3t^2+5t-2$ ise $x=6$ için $y'=?$

$$y'=\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{6t+5}{2t+2}$$

$x=6$

$t^2+2t+3=6$

$t^2+2t-3=0$

$(t-1)(t+3)=0$

$t=1,\ t=-3$

$$y'\Big|_{t=1}=\frac{11}{4}\qquad y'\Big|_{t=-3}=\frac{13}{4}$$

$$y'(6)=\left\{\frac{11}{4},\ 13/4\right\}$$

$\underline{\text{ör}}$ $x=5-2\sin t$, $y=3+\cos 2t$ ise $y'=?$

$$y'=\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-2\sin 2t}{-2\cos t}=\frac{\sin 2t}{\cos t}=\frac{2\cos t\sin t}{\cos t}$$

$$=2\sin t=5-x$$

$\underline{\text{ör}}$ $x=t-\sin t$, $y=1-\cos t$ ise $y''=?$

$$y'=\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sin t}{1-\cos t}$$

$$y''=\frac{d^2y}{dx^2}=\frac{d}{dx}(y')=\frac{d(y')/dt}{dx/dt}=\frac{\dfrac{\cos t(1-\cos t)-\sin t.\sin t}{(1-\cos t)^2}}{1-\cos t}$$

$$=\frac{\cos t-\cos^2 t-\sin^2 t}{(1-\cos t)^3}=\frac{\cos t-1}{(1-\cos t)^3}=-\frac{1}{(1-\cos t)^2}$$

$$=-\frac{1}{y^2}$$

Ör $y = \ln z$, $z = 3x + y^2$ ise $\frac{dz}{dx} = ?$

$$\frac{dz}{dx} = \frac{d}{dx}(3x + y^2) = 3 + 2y\frac{dy}{dx} = 3 + 2y\frac{dy}{dz}\frac{dz}{dx}$$

$$= 3 + 2y\cdot\frac{1}{z}\frac{dz}{dx} = 3 + \frac{2y}{z}\frac{dz}{dx}$$

$$\frac{dz}{dx}\left(1 - \frac{2y}{z}\right) = 3 \Rightarrow \frac{dz}{dx} = \frac{3}{1 - 2y/z} = \frac{3z}{z - 2y}$$

Ör $f(x,y,z) = \ln(x^2+y^2+z^2)$, $x = \cos t$, $y = \sin t$, $z = t$ ise $\left.\frac{df}{dt}\right|_{t=2} = ?$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{2x}{x^2+y^2+z^2}(-\sin t) + \frac{2y}{x^2+y^2+z^2}(\cos t) + \frac{2z}{x^2+y^2+z^2}\cdot(1)$$

$$= \frac{-2x\sin t + 2y\cos t + 2z}{x^2+y^2+z^2} = \frac{\cancel{-2\cos t\sin t} + \cancel{2\sin t\cos t} + 2t}{\cos^2 t + \sin^2 t + t^2}$$

$$= \frac{2t}{t^2+1} \qquad \left.\frac{df}{dt}\right|_{t=2} = \frac{4}{5}$$

Ör $f(x,y) = x^2 + \frac{y}{x}$, $x = u + 2v + 3$, $y = 2u - v + 2$ ise $\frac{\partial f}{\partial u}$ ve $\frac{\partial f}{\partial v}$ deperlerini hesapla.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \left(2x - y/x^2\right)(1) + \left(\frac{1}{x}\right)(2)$$

$$= 2x + \frac{2}{x} - y/x^2$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = \left(2x - \frac{y}{x^2}\right)(2) + \left(\frac{1}{x}\right)(-1)$$

$$= 4x - \frac{1}{x} - \frac{2y}{x^2}$$

Kapalı Fonksiyonlarda Türev

$f(x,y) = C$ ise $y = y(x)$ şeklinde yazılabilir.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = f_x\, dx + f_y\, dy = 0$$

$$y' = \frac{dx}{dx} = -\frac{f_x}{f_y} \quad \text{olur.}$$

$f(x,y,z) = C$ ise $z = z(x,y)$ şeklinde yazılabilir.

$$\frac{\partial f}{\partial x} = f_x \frac{\partial x}{\partial x} + f_y \frac{\partial y}{\partial x} + f_z \frac{\partial z}{\partial x} = f_x + f_z \frac{\partial z}{\partial x} = 0$$

$$z_x = \frac{\partial z}{\partial x} = -\frac{f_x(x,y,z)}{f_z(x,y,z)}$$

$$\frac{\partial f}{\partial y} = f_x \frac{\partial x}{\partial y} + f_y \frac{\partial y}{\partial y} + f_z \frac{\partial z}{\partial y} = f_y + f_z \frac{\partial z}{\partial y} = 0$$

$$z_y = \frac{\partial z}{\partial y} = -\frac{f_y(x,y,z)}{f_z(x,y,z)}$$

Ör $x^2 + 3xy + y^2 = 5$ ise $y' = ?$

$$f(x,y) = x^2 + 3xy + y^2 = 5$$

$$df = f_x\, dx + f_y\, dy = (2x+3y)dx + (3x+2y)dy = 0$$

$$y' = \frac{dy}{dx} = -\frac{2x+3y}{3x+2y}$$

Ör $5x^3y + 7xy^2 - 10y = 18$ ise $y'\big|_{(1,2)} = ?$

$$f(x,y) = 5x^3y + 7xy^2 - 10y = 18$$

$$y' = -\frac{f_x}{f_y} = -\frac{15x^2y + 7y^2}{5x^3 + 14xy - 10} \Rightarrow y'\big|_{(1,2)} = -\frac{30+28}{5+28-10} = -\frac{58}{23}$$

Ör $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ise $y' = ?$

$$f(x,y) = \frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$df = f_x\, dx + f_y\, dy = 0$$

$$df = \frac{2x}{4} dx + \frac{2y}{9} dy = 0$$

$$y' = \frac{dy}{dx} = -\frac{9x}{4y}$$

Ör $e^{-xy}+5z-e^{z}=3$ ise z_x ve z_y bulunuz.

$f(x,y,z)=e^{-xy}+5z-e^{z}=3$

$f_x=-ye^{-xy}$, $f_y=-xe^{-xy}$, $f_z=5-e^{z}$

$z_x=\frac{\partial z}{\partial x}=-\frac{f_x}{f_z}=\frac{ye^{-xy}}{5-e^{z}}$, $z_y=\frac{\partial z}{\partial y}=-\frac{f_y}{f_z}=\frac{xe^{-xy}}{5-e^{z}}$

Ör $x^2+z^2+3xz-y^2z=4$ ise z_x ve z_y bulunuz.

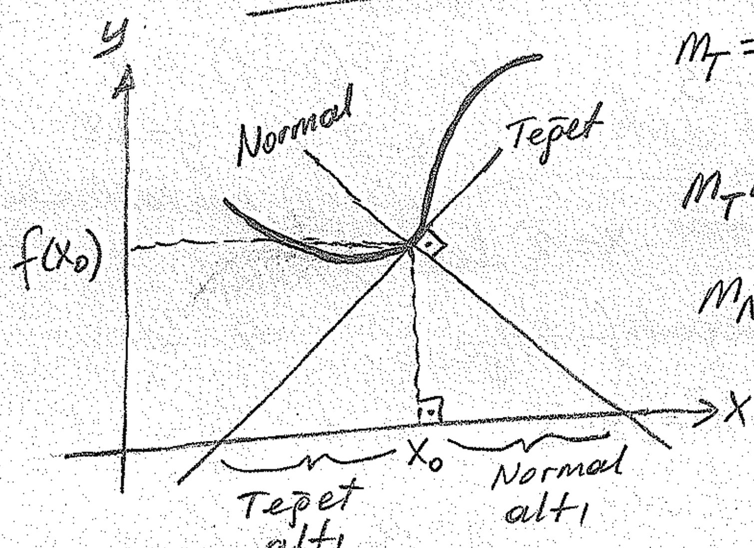
$f(x,y,z)=x^2+z^2+3xz-y^2z=4$

$f_x=2x+3z$ $\qquad z_x=\frac{\partial z}{\partial x}=-\frac{f_x}{f_z}=\frac{2x+3z}{y^2-3x-2z}$

$f_y=-2yz$

$f_z=2z+3x-y^2$ $\qquad z_y=\frac{\partial z}{\partial y}=-\frac{f_y}{f_z}=\frac{2yz}{2z+3x-y^2}$

Türevin Geometrik Anlamı



$m_T=f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

$m_T m_N=-1$

$m_N=-\frac{1}{m_T}=-\frac{1}{f'(x_0)}$

$y=y_0+m(x-x_0)$ Doğru denklemi

Ör $x^2+2xy+y^2+2x-4y=17$ eğrisine $(2,3)$ noktasında teğet olan doğrunun denklemi.

$y'=-\frac{f_x}{f_y}=-\frac{2x+2y+2}{2x+2y-4}$

$y'\big|_{(2,3)}=-\frac{4+6+2}{4+6-4}$

$=-2=m_T$

$y=y_0+m_T(x-x_0)$

$=3-2(x-2)$

$=7-2x$

teğet denklemi

Ör $x^3-x^2y-2xy^2-y^3=3$ eğrisine $(1,-2)$ noktasında teğet olan doğru, $y=x^2+3x+2$ eğrisinin hangi noktasındaki teğetine diktir.

$$f(x,y)=x^3-x^2y-2xy^2-y^3=3$$

$$y'=-\frac{f_x}{f_y}=-\frac{3x^2-2xy-2y^2}{-x^2-4xy-3y^2}=\frac{3x^2-2xy-2y^2}{x^2+4xy+3y^2}$$

$$y'\Big|_{(1,-2)}=\frac{3+4-8}{1-8+12}=-\frac{1}{5}=m_{T1} \qquad m_{T2}=-\frac{1}{m_{T1}}=5$$

$$y=x^2+3x+2 \rightarrow y'=2x+3=m_{T2}=5 \rightarrow \begin{array}{l} x=1 \\ y=6 \end{array}$$

$(1,6)$ noktasındaki

Ör $y=5x^2-x^3$ eğrisinin $(1,4)$ noktasındaki teğeti eğriyi başka hangi noktada keser.

$$f(x)=5x^2-x^3$$

$$f'(x)=10x-3x^2$$

$$f'(1)=7=m_T$$

$$y=y_0+m_T(x-x_0)$$

$$=4+7(x-1)$$

$$=7x-3$$

teğet denk.

$$y=y$$

$$5x^2-x^3=7x-3$$

$$x^3-5x^2+7x-3=0$$

$$(x-1)^2(x-3)=0$$

$x=3$ için $y=18$

$(3,18)$ noktasında

Ör $y=|x^2-5x+3|$ fonksiyonunun $x=2$'deki normalinin denklemi.

$$y=f(x)=|g(x)|$$

$$g(x)=x^2-5x+3$$

$$g(2)=4-10+3$$

$$=-3$$

$x=2$ negatif bölgede

$$f(x)=-x^2+5x-3$$

$$f'(x)=-2x+5$$

$$f'(2)=1=m_T$$

$$m_N=-\frac{1}{m_T}=-1$$

$$y=y_0+m_N(x-x_0)$$

$$=3-(x-2)$$

$$=5-x$$

Ör $y=\ln x$ eğrisinin orjinden geçen teğetinin eğimi.

$$f(x)=\ln x$$

$$f'(x)=\frac{1}{x}$$

$x_0=a$ için $y_0=\ln a$

$$f'(a)=1/a=m_T$$

$$y=y_0+m_T(x-x_0)$$

$$=\ln a+\frac{x-a}{a}$$

$x=0,\ y_0=0$ için

$$0=\ln a-1$$

$$\ln a=1$$

$$a=e$$

$$m_T=1/e$$

Ör $y=x^3+3x^2-5x+8$ eğrisine teğet ve $x+4y=7$ doğrusuna dik olan doğruların denklemleri:

$x+4y=7$

$y=\frac{7-x}{4}$

$m_1=-1/4$

$m_2=-1/m_1=4$

$y=x^3+3x^2-5x+8$

$y'=3x^2+6x-5=m_2=4$

$3x^2+6x-9=0$

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3,\ x=1$

$y=23\quad y=7$

$y=y_0+m_2(x-x_0)$

$(-3,23)$ noktasındaki

$y=23+4(x+3)$

$=4x+35$

$(1,7)$ noktasındaki

$y=7+4(x-1)$

$=4x+3$

Ör $y=x^3+5x^2+5x+2$ eğrisinin eğimi 2 olan teğetlerinin değme noktaları ve bu noktalardaki teğet denklemleri.

$y=x^3+5x^2+5x+2$

$y'=3x^2+10x+5=m_T=2$

$3x^2+10x+3=0$

$(3x+1)(x+3)$

$x=-1/3\quad x=-3$

$y=23/27\quad y=5$

$y=y_0+m_T(x-x_0)$

$$\left.\begin{aligned} y&=\frac{23}{27}+2\left(x+\frac{1}{3}\right)\\ &=2x+\frac{41}{27}\\ y&=5+2(x+3)=2x+11\end{aligned}\right\}\ \text{Teğet denklemleri}$$

Ör $y=\frac{4x}{x^2+4}$ eğrisinin hangi noktalarındaki teğetleri $y=\sqrt{4x-x^2}$ eğrisinin $x=2$ noktasındaki teğetine paraleldir.

$y=\sqrt{4x-x^2}$

$y'=\frac{2-x}{\sqrt{4x-x^2}}$

$y'(2)=m_T=\frac{2-2}{\sqrt{8-4}}=0$

$y=\frac{4x}{x^2+4}$

$y'=\frac{(4)(x^2+4)-(4x)(2x)}{(x^2+4)^2}=\frac{4(4-x^2)}{(x^2+4)^2}=m_T=0$

$\to$ pozitif

$4-x^2=0$ olmalı

$x^2=4\to x=\mp 2$

$x=-2$ için $y=-1$

$x=2$ için $y=1$

$(-2,-1)$ ve $(2,1)$ noktaları

L'Hospital Kuralı

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0} \text{ veya } \frac{\infty}{\infty} \text{ ise } \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

ör $$\lim_{x\to 0}\frac{e^{3x}-\cos 2x}{\ln(1+5x)}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{(e^{3x}-\cos 2x)'}{(\ln(1+5x))'}=\lim_{x\to 0}\frac{3e^{3x}+2\sin 2x}{\frac{5}{1+5x}}=\frac{3+0}{5/1}=3/5$$

ör $$\lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}=\frac{0}{0}$$

$$\lim_{x\to 7}\frac{(2-\sqrt{x-3})'}{(x^2-49)'}=\lim_{x\to 7}\frac{-\frac{1}{2\sqrt{x-3}}}{2x}=\frac{-\frac{1}{4}}{14}=-\frac{1}{56}$$

ör $$\lim_{x\to 0}(0.7e^{x}+0.3e^{-x})^{3/2x}=1^{\infty}$$

$$\lim_{x\to 0}e^{\ln(0.7e^{x}+0.3e^{-x})^{2/3x}}=e^{\lim_{x\to 0}\frac{3\ln(0.7e^{x}+0.3e^{-x})}{2x}}$$

$$=e^{\lim_{x\to 0}\frac{3\frac{0.7e^{x}-0.3e^{-x}}{0.7e^{x}+0.3e^{-x}}}{2}}=e^{\frac{3\times 0.4}{2}}=e^{0.6}$$

ör $$\lim_{x\to 0^+}x\ln x=0\cdot\infty$$

$$\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0$$

ör $$\lim_{x\to \pi/6}\frac{\sin(x-\pi/6)}{\frac{\sqrt{3}}{2}-\cos x}=0/0$$

$$\lim_{x\to \pi/6}\frac{(\sin(x-\pi/6))'}{(\frac{\sqrt{3}}{2}-\cos x)'}=\lim_{x\to \pi/6}\frac{\cos(x-\pi/6)}{+\sin x}=\frac{\cos 0}{\sin \pi/6}=\frac{1}{1/2}=2$$

ör $$\lim_{x\to 1}\frac{\arctan(x^2+2x-3)}{x^2-\cos(x-1)}=\frac{0}{0}$$

$$\lim_{x\to 1}\frac{(\arctan(x^2+2x-3))'}{(x^2-\cos(x-1))'}=\lim_{x\to 1}\frac{\frac{2x+2}{1+(x^2+2x-3)^2}}{2x+\sin(x-1)}=\frac{4/1}{2+0}=2$$

Ör $\lim\limits_{x\to 0} \dfrac{e^{5x}-\cos 3x}{e^{7x}-\cos 2x} = 0/0$

$$\lim_{x\to 0} \frac{(e^{5x}-\cos 3x)'}{(e^{7x}-\cos 2x)'} = \lim_{x\to 0} \frac{5e^{5x}+3\sin 3x}{7e^{7x}+2\sin 2x} = \frac{5+0}{7+0} = 5/7$$

Ör $\lim\limits_{x\to 0} \dfrac{x\cos x-\sin x}{x^2\sin x} = 0/0$

$$\lim_{x\to 0} \frac{(x\cos x-\sin x)'}{(x^2\sin x)'} = \lim_{x\to 0} \frac{\cancel{\cos x}-x\sin x-\cancel{\cos x}}{2x\sin x+x^2\cos x} = \lim_{x\to 0} \frac{-\sin x}{2\sin x+x\cos x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{(-\sin x)'}{(2\sin x+x\cos x)'} = \lim_{x\to 0} \frac{-\cos x}{2\cos x+\cos x-x\sin x} = \frac{-1}{2+1-0} = -1/3$$

Ör $\lim\limits_{x\to 1} \left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right) = \infty-\infty$

$$\lim_{x\to 1} \frac{\ln x-x+1}{(x-1)\ln x} = \lim_{x\to 1} \frac{(\ln x-x+1)'}{((x-1)\ln x)'} = \lim_{x\to 1} \frac{\frac{1}{x}-1}{\ln x+\frac{x-1}{x}}$$

$$= \lim_{x\to 1} \frac{1-x}{x\ln x+x-1} = \lim_{x\to 1} \frac{(1-x)'}{(x\ln x+x-1)'} = \lim_{x\to 1} \frac{-1}{\ln x+1+1} = -1/2$$

Ör $\lim\limits_{x\to 0} \left(\dfrac{1}{x}-\dfrac{3}{e^{3x}-1}\right) = \infty-\infty$

$$\lim_{x\to 0} \frac{e^{3x}-1-3x}{x(e^{3x}-1)} = \lim_{x\to 0} \frac{(e^{3x}-1-3x)'}{(x(e^{3x}-1))'} = \lim_{x\to 0} \frac{3e^{3x}-3}{e^{3x}-1+3xe^{3x}}$$

$$= \lim_{x\to 0} \frac{(3e^{3x}-3)'}{(e^{3x}-1+3xe^{3x})'} = \lim_{x\to 0} \frac{9e^{3x}}{3e^{3x}+3e^{3x}+9xe^{3x}} = \lim_{x\to 0} \frac{3}{2+3x} = 3/2$$

Ör $\lim\limits_{x\to\infty} \dfrac{x^2+\ln x}{e^x} = \dfrac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{(x^2+\ln x)'}{(e^x)'} = \lim_{x\to\infty} \frac{2x+1/x}{e^x} = \lim_{x\to\infty} \frac{(2x+1/x)'}{(e^x)'}$$

$$= \lim_{x\to\infty} \frac{2-1/x^2}{e^x} = \frac{2-0}{\infty} = 0$$

Ör $\lim_{x\to 0^+} x^{\sin x} = 0^0$

$$\lim_{x\to 0^+} e^{\ln x^{\sin x}} = \lim_{x\to 0^+} e^{\sin x . \ln x} = e^{\lim_{x\to 0^+} \frac{\ln x}{1/\sin x}}$$

$$= e^{\lim_{x\to 0^+} \frac{(\ln x)'}{(1/\sin x)'}} = e^{\lim_{x\to 0^+} \frac{1/x}{-\frac{\cos x}{\sin^2 x}}} = e^{\lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x}}$$

$$= e^{\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)\left(-\frac{\sin x}{\cos x}\right)} = e^{1\times 0} = 1$$

Ör $\lim_{x\to \pi/2} (\sin x)^{\tan x} = 1^\infty$

$$\lim_{x\to \pi/2} e^{\ln (\sin x)^{\tan x}} = \lim_{x\to \pi/2} e^{\tan x . \ln(\sin x)} = e^{\lim_{x\to \pi/2} \frac{\ln(\sin x)}{\cot x}}$$

$$= e^{\lim_{x\to \pi/2} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin^2 x}}} = e^{\lim_{x\to \pi/2} (-\cos x \sin x)} = e^0 = 1$$

Ör $\lim_{x\to \infty} \left(\frac{x+3}{x-3}\right)^x = 1^\infty$

$$\lim_{x\to \infty} e^{\ln\left(\frac{x+3}{x-3}\right)^x} = \lim_{x\to \infty} e^{x \ln\left(\frac{x+3}{x-3}\right)} = e^{\lim_{x\to \infty} \frac{\ln\left(\frac{x+3}{x-3}\right)}{1/x}} = e^{0/0}$$

$$= e^{\lim_{x\to \infty} \frac{\left(\ln\left(\frac{x+3}{x-3}\right)\right)'}{(1/x)'}} = e^{\lim_{x\to \infty} \frac{\frac{(x-3)-(x+3)}{(x-3)^2} \Big/ \frac{x+3}{x-3}}{-1/x^2}} = e^{\lim_{x\to \infty} \frac{6x^2}{x^2-9}} = e^6$$

Ör $\lim_{x\to 0} (3x + e^{5x})^{2/7x} = 1^\infty$

$$\lim_{x\to 0} e^{\ln (3x+e^{5x})^{2/7x}} = e^{\lim_{x\to 0} \frac{2\ln(3x+e^{5x})}{7x}} = e^{\lim_{x\to 0} \frac{(\ln(3x+e^{5x}))'}{(7x/2)'}}$$

$$= e^{\lim_{x\to 0} \frac{\frac{3+5e^{5x}}{3x+e^{5x}}}{7/2}} = e^{\frac{8/1}{7/2}} = e^{16/7}$$

Fonksiyonlarda Grafik Çizimi

Ör $y = 2x + 3$ doğrusunun grafiğini çiz.

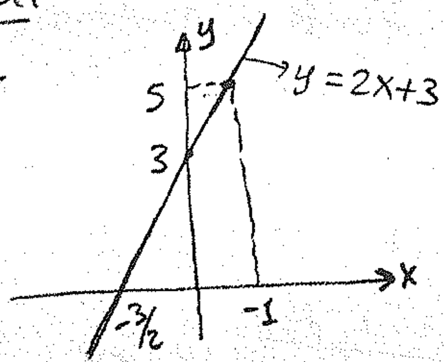
$y' = 2 = m > 0$ sürekli artan

$x = 0 \longrightarrow y = 3$

$y = 0 \longrightarrow 2x + 3 = 0$

$x = -3/2$

$x = 1 \longrightarrow y = 5$



Ör $y = |2x+3|$ fonk. grafiğini çiz

$2x + 3 = 0$

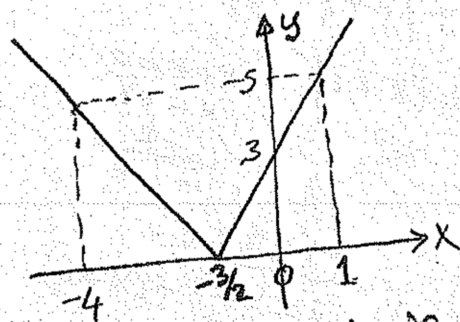
$x = -3/2$

$$y = \begin{cases} 2x+3 & x \geq -3/2 \\ -2x-3 & x < -3/2 \end{cases}$$

$$\begin{array}{c|c} \multicolumn{2}{c}{-3/2} \\ - & + \end{array}$$

$x = 0 \longrightarrow y = 3$

$x = 1, x = -4 \longrightarrow y = 5$



Ör $y = \big||x-1|-1\big|$ fonk. kesikli hale getirip grafiğini çiz.

$x - 1 = 0$

$x = 1$

$$\begin{array}{c|c} \multicolumn{2}{c}{1} \\ - & + \end{array}$$

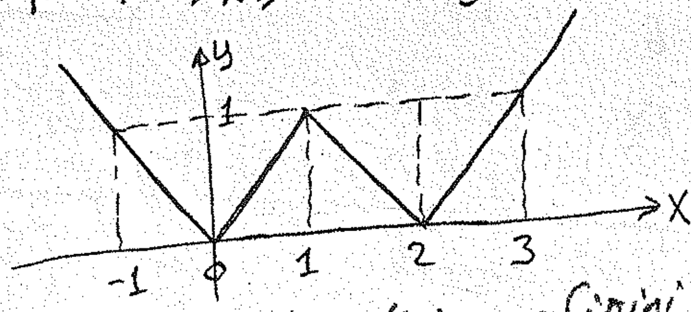
$x < 1$ için

$y = |-(x-1)-1| = |x|$ → $x < 0$ için $y = -x$; $0 \leq x < 1$ için $y = x$

$x \geq 1$ için

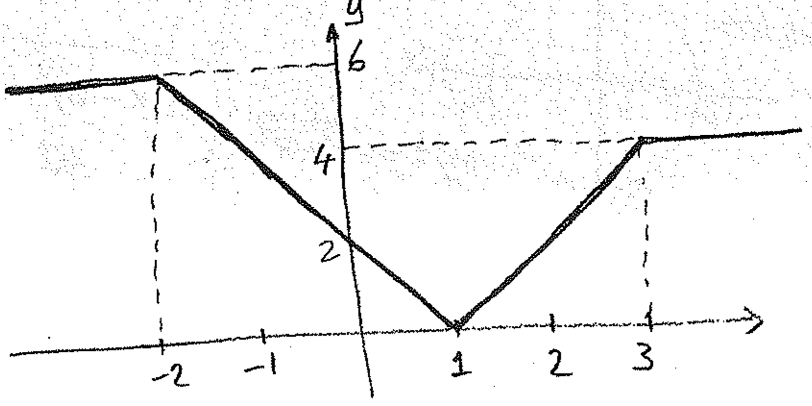
$y = |(x-1)-1| = |x-2|$ → $1 \leq x < 2$ için $y = 2 - x$; $x \geq 2$ için $y = x - 2$

$$f(x) = \begin{cases} -x & x < 0 \\ x & 0 \leq x < 1 \\ 2-x & 1 \leq x < 2 \\ x-2 & x \geq 2 \end{cases}$$



Ör $y = \big|1 - |x+2| + |x-3|\big|$ fonk. kesikli hale getirip grafiğini çiz.

$$y = \begin{cases} 6 & x < -2 \\ 2-2x & -2 \leq x \leq 1 \\ 2x-2 & 1 < x < 3 \\ 4 & x \geq 3 \end{cases}$$



Ör: $y = |x+3| + |2-|x-1||$ fonk. kesikli hale getirip grafiğini çiz.

$x+3=0$
$x=-3$
$\begin{array}{c|c} & -3 \\ \hline - & + \end{array}$

$x-1=0$
$x=1$
$\begin{array}{c|c} & 1 \\ \hline - & + \end{array}$

$x<-3$ için

$y = -(x+3) + |2+(x-1)| = -x-3 + |x+1|$ ($x<-3$ için negatif)

$= -x-3-x-1 = -2x-4$

$-3 \le x < 1$ için

$y = (x+3) + |2+(x-1)|$

$= x+3+|x+1|$

$x+1=0$
$x=-1$
$\begin{array}{c|c} & -1 \\ \hline - & + \end{array}$

$-3 \le x < -1$

$y = x+3-(x+1)$

$= x+3-x-1$

$= 2$

$-1 \le x < 1$

$y = x+3+(x+1)$

$= 2x+4$

$x \geq 1$ için

$y = (x+3) + |2-(x-1)|$

$= x+3+|x-3|$

$x-3=0$
$x=3$
$\begin{array}{c|c} & 3 \\ \hline - & + \end{array}$

$1 \le x < 3$

$y = x+3-(x-3)$

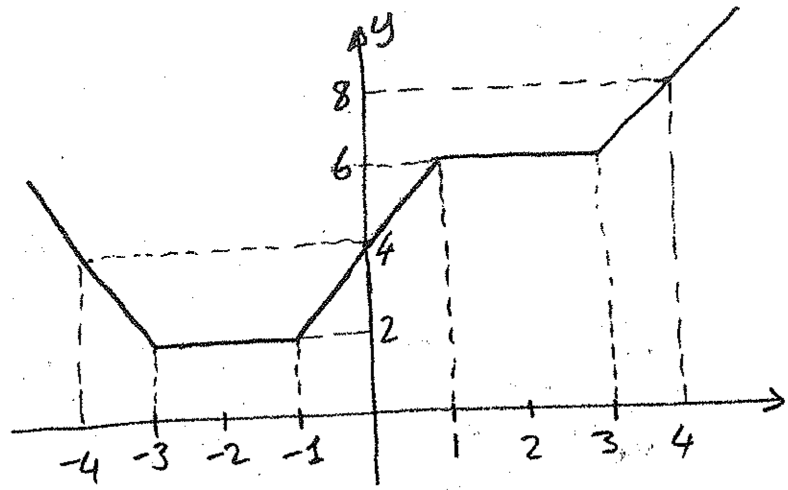
$= 6$

$x \geq 3$

$y = x+3+(x-3)$

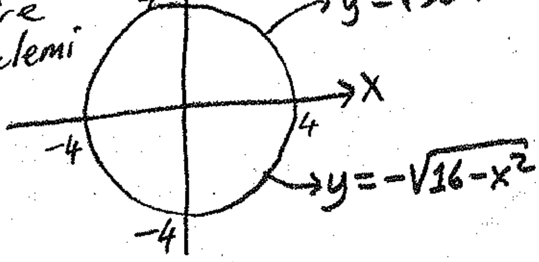
$= 2x$

$$y = \begin{cases} -2x-4 & x<-3 \\ 2 & -3 \le x < -1 \\ 2x+4 & -1 \le x < 1 \\ 6 & 1 \le x < 3 \\ 2x & x \geq 3 \end{cases}$$



$x^2+y^2=16$ grafik

daire denklemi

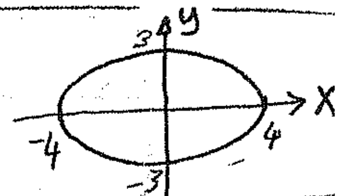


$x^2+y^2=16 \longrightarrow y^2=16-x^2$

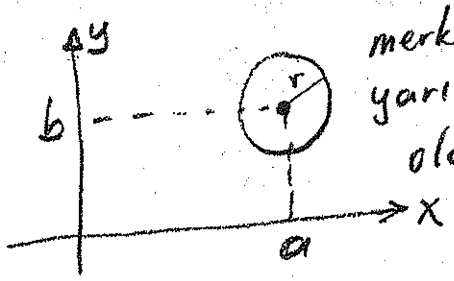
$y = \mp\sqrt{16-x^2}$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Elips denklemi



$(x-a)^2+(y-b)^2=r^2$



merkezi (a,b) yarıçapı r olan daire denklemi

Fonksiyonlarda artan azalan bölge

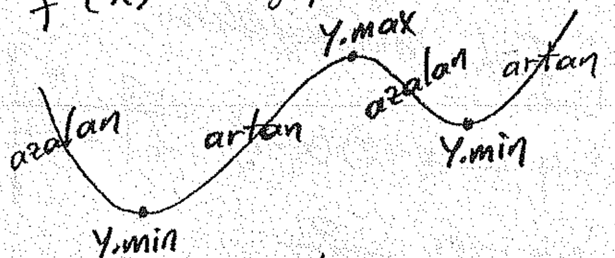
$\forall x \in [a,b]$ için $f'(x) > 0$ ise $f(x)$ artan

$f'(x) < 0$ ise $f(x)$ azalan

$f'(x) = 0$ ise $f(x)$ sabit

Yerel minimum, Yerel maksimum

$f'(x) = 0$ yapan noktalar



Yerel minimum: önce azalıyor, sonra artıyor.

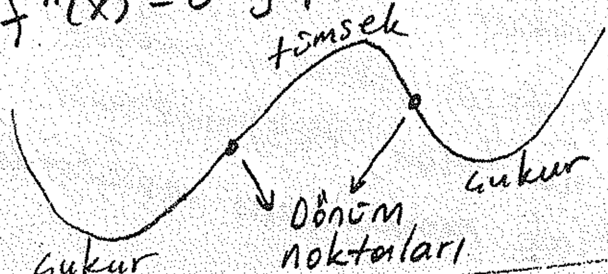
Yerel maksimum: önce artıyor, sonra azalıyor.

$f''(x) > 0$ ise çukur

$f''(x) < 0$ ise tümsek

Dönüm Noktaları

$f''(x) = 0$ yapan noktalar



Çukur iken tümsek, tümsek iken çukur oluyorsa dönüm noktası vardır.

Ör. $y = x^2 + 2x - 3$ fonk. grafiğini çiz.

$y' = 2x + 2 = 0$

$x = -1$

$y'' = 2 > 0$ çukur

	-1	
y'	$-$	$+$
y	$\searrow$	$\nearrow$

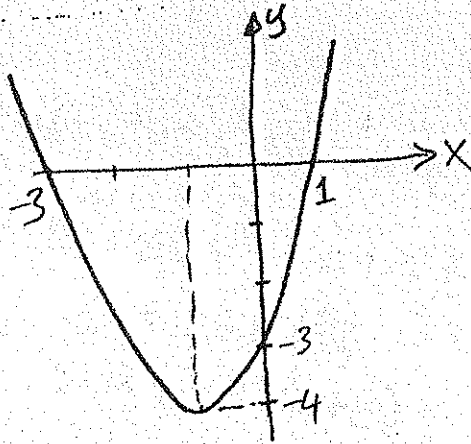
$x = -1$ için $y = -4$

$x = 0 \longrightarrow y = -3$

$y = 0 \longrightarrow x^2 + 2x - 3 = 0$

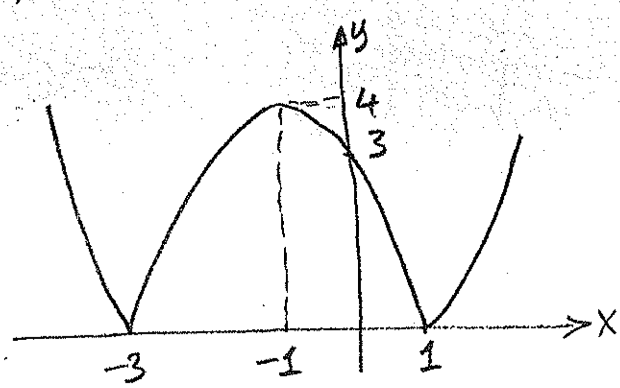
$(x+3)(x-1) = 0$

$x = -3,\ x = 1$



$y = |x^2 + 2x - 3|$ grafiği istenseydi

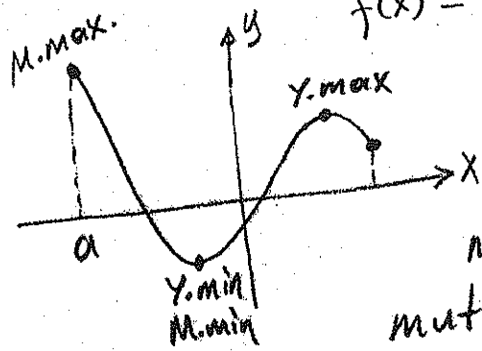
$$y = \begin{cases} x^2 + 2x - 3 & x \le -3,\ x \ge 1 \\ -x^2 - 2x + 3 & -3 < x < 1 \end{cases}$$



Mutlak Minimum, Mutlak Maksimum (Uç noktalar)

$\forall x \in [a,b]$ için $f(x) \geq f(c)$ ise $(c, f(c))$ mutlak min. nokta

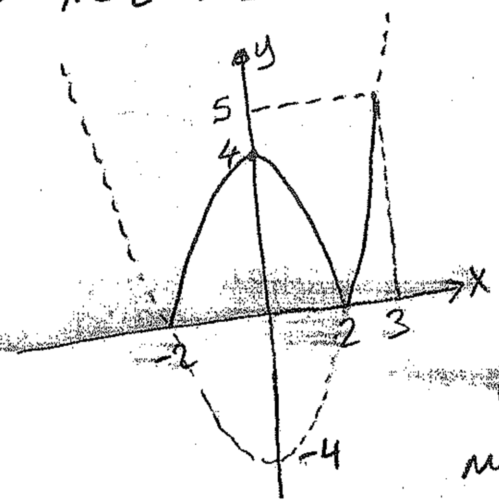
$f(x) \leq f(c)$ ise $(c, f(c))$ mutlak max. nokta



$[a,b]$ aralığında $f(x)$ fonksiyonunun en küçük olduğu nokta mutlak minimum, en büyük olduğu nokta mutlak maksimumdur.

$f(x) = |x^2-4|$ fonk. inceleyelim.

Ör: $x \in [-2,3]$ için $f(x) = |x^2-4|$



$g(x) = x^2-4 = (x+2)(x-2)$ $\quad$ -2, 2: $+ \mid - \mid +$

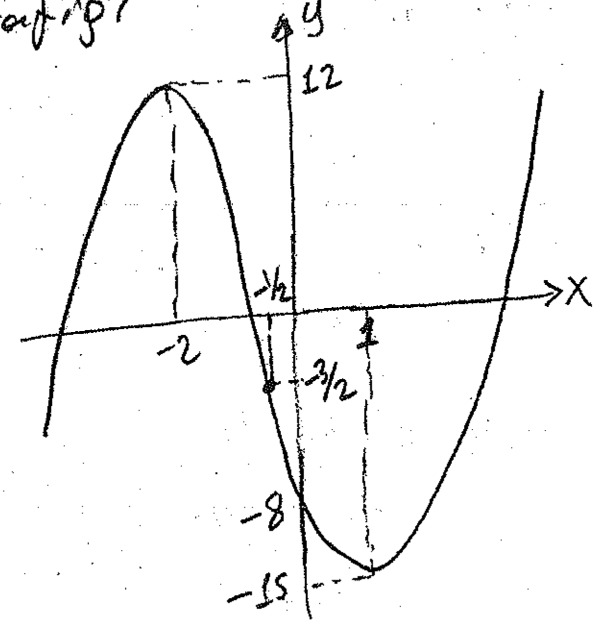
$g'(x) = 2x = 0$ $\quad$ 0: $- \mid +$ $\quad \searrow \ \nearrow$

$g''(x) = 2 > 0$

çukur

$(-2,0)$ ve $(2,0)$ ~~noktaları mutlak~~ noktası minimum noktalar, $(3,5)$ mutlak maksimum noktadır.

Ör: $f(x) = 2x^3+3x^2-12x-8$ fonk. grafiği



$f'(x) = 6x^2+6x-12 = 0$

$x^2+x-2 = 0$

$(x+2)(x-1) = 0$

-2, 1: $+ \mid - \mid +$ $\quad \nearrow \ \searrow \ \nearrow$

$f(0) = -8$

$f(-2) = 12$

$f(1) = -15$

$f(-\frac{1}{2}) = -\frac{3}{2}$

$f''(x) = 12x+6 = 0$ $\quad$ $-\frac{1}{2}$: $- \mid +$ $\quad \cap \ \cup$

$x = -\frac{1}{2}$

Ör $y=|x^2-1|$ fonk. grafiğini çiz.

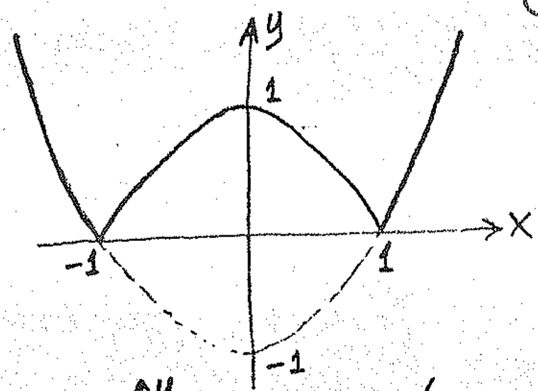
$x^2-1=0$ $\quad g(x)=x^2-1$

$(x+1)(x-1)=0$ $\quad g'(x)=2x=0$

$x=-1$, $x=1$

-1		$+1$	
$+$	$-$		$+$

	0	
$-$		$+$
$\searrow$		$\nearrow$



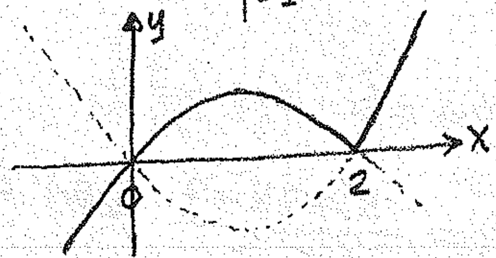
Ör $y=x|x-2|$ fonk. grafiğini çiz.

$x-2=0$ $\quad x<2 \to y=-x(x-2)$

$x=2$ $\quad x\geq 2 \to y=x(x-2)$

	2	
$-$		$+$

$$y=\begin{cases}-x(x-2) & x<2\\ x(x-2) & x\geq 2\end{cases}$$



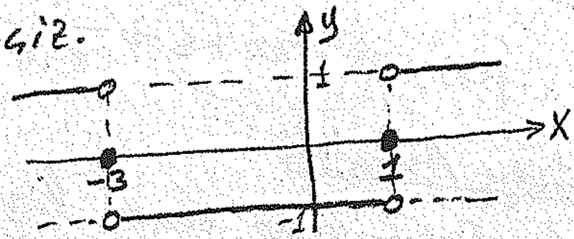
Ör $y=\mathrm{sgn}(x^2+2x-3)$ fonk. grafiğini çiz.

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3$, $x=1$

	-3		1	
$+$		$-$		$+$



Yatay Asimtot

$\lim\limits_{x\to\infty} f(x)=a$ veya $\lim\limits_{x\to-\infty} f(x)=a$ ise $y=a$ doğrusu yatay asimtottur.

Eğik veya Eğri Asimtotlar

$\lim\limits_{x\to\infty}(f(x)-g(x))=0$ veya $\lim\limits_{x\to-\infty}(f(x)-g(x))=0$ ise

$y=g(x)$ doğru ise eğik asimtot, eğri ise eğri asimtottur.

$$f(x)=\frac{a_nx^n+\dots+a_1x+a_0}{b_mx^m+\dots+b_1x+b_0}$$ fonksiyonu için

$n=m$ ise $y=\frac{a_n}{b_m}$ doğrusu yatay asimtottur.

$n<m$ ise $y=0$ doğrusu yatay asimtottur.

$n>m$ ise yatay asimtot yok.

eğik veya eğri asimtot vardır.

$n=m+1$ ise eğik, $n>m+1$ ise eğri asimtot

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Ör $f(x)=\dfrac{2x^2+3x+5}{x^2-2x+4}$ yatay asimtot nedir?

$$\lim_{x\to\mp\infty}\frac{2x^2+3x+5}{x^2-2x+4}=\lim_{x\to\mp\infty}\frac{x^2\left(2+\cancel{\frac{3}{x}}+\cancel{\frac{5}{x^2}}\right)}{x^2\left(1-\cancel{\frac{2}{x}}+\cancel{\frac{4}{x^2}}\right)}=2$$

$y=2$ doğrusu yatay asimtot.

Ör $f(x)=\dfrac{x^2+2x+8}{x-3}$ eğik asimtot nedir?

$$f(x)=x+5+\frac{23}{x-3}$$

$y=x+5$ doğrusu eğik asimtottur.

Ör $f(x)=\dfrac{x^3+2x-7}{x-2}$ eğri asimtot nedir?

$$f(x)=x^2+2x+6+\frac{5}{x-2}$$

$y=x^2+2x+6$ eğrisi eğri asimtottur.

<u>Düşey Asimtot</u>

$\lim\limits_{x\to a^+}f(x)=\mp\infty$ veya $\lim\limits_{x\to a^-}f(x)=\mp\infty$ ise $x=a$ doğrusu düşey asimtottur.

~~Ör $f(x)=\dfrac{5x-3}{1-2x}+\dfrac{2}{x}$ Asimtotları bul.~~

$$\lim_{x\to\mp\infty}\left(\frac{5x-3}{1-2x}+\frac{2}{x}\right)=-\frac{5}{2}\text{ , } y=-\frac{5}{2}\text{ doğrusu } \text{~~yatay asimtotlar~~.}$$

$$f(x)=\underset{(x)}{\frac{5x-3}{1-2x}}+\underset{(1-2x)}{\frac{2}{x}}=\frac{5x^2-7x+2}{x(1-2x)}=\frac{(5x-2)(x-1)}{x(1-2x)}$$

$\left.\begin{array}{l}x=0\\ x=\frac{1}{2}\end{array}\right\}$ düşey asimtotla

Ör $f(x)=x\ln\left(1+\dfrac{2}{x}\right)$ asimtotları bulunuz.

$$\lim_{x\to\mp\infty}x.\ln\left(1+\frac{2}{x}\right)=\lim_{x\to\mp\infty}\frac{\ln\left(1+2/x\right)}{1/x}=\lim_{u\to0}\frac{\ln(1+2u)}{u}$$

$u=1/x$, $x\to\mp\infty$ iken $u\to0$

$$=\lim_{u\to0}\frac{(\ln(1+2u))'}{(u)'}=\lim_{u\to0}\frac{\frac{2}{1+2u}}{1}=2$$

$y=2$ doğrusu yatay asimtottur.

$1+\dfrac{2}{x}=0$ için $\ln\left(1+\dfrac{2}{x}\right)=\ln 0=-\infty$

$\hookrightarrow x=-2$ düşey asimtottur.

Ör $y=x^2(x-2)^2$ grafik

Yatay ve düşey asimtot yok.

$y=x^2(x-2)^2 \geq 0$

$x=0 \rightarrow y=4$

$y=0 \rightarrow x=0, x=2$

$y'=4x(x-1)(x-2)=0$

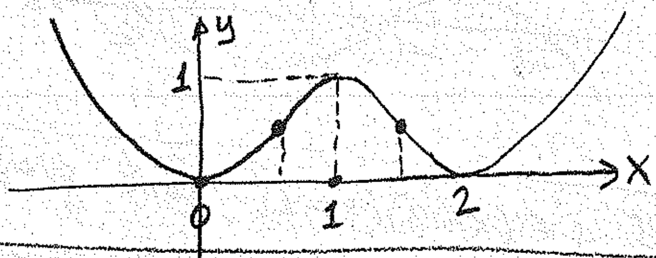
0	1	2	
−	+	−	+
↘	↗	↘	↗

$y(0)=4,\ y(1)=1,\ y(2)=0$

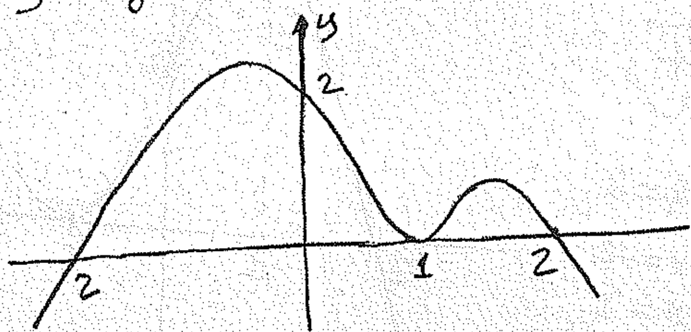
$y'=4x(x-1)(x-2)$
$=4x(x^2-3x+2)$

$y''=4(x^2-3x+2)+4x(2x-3)$
$=12x^2-24x+8=0$

$x^2-2x+\frac{2}{3}=0 \rightarrow x_{1,2}=1\mp\frac{\sqrt{3}}{3}$



Ör $y=\frac{1}{8}(x+2)(x-1)^2(2-x)^3$



$x=0 \rightarrow y=2$

$y=0 \rightarrow x=-2, x=1, x=2$

$(2-x)^3=\underbrace{(2-x)^2}_{+}(2-x)$

6. dereceden olmasına rağmen 4. dereceden görünür

$y'=0 \rightarrow$ min, max noktaları bulunur.

$y''=0 \rightarrow$ dönüm noktaları bulunur.

Yatay ve düşey asimtot yoktur.

Ör $f(x)=(1-x)^2(x^2+4)$ grafik

Yatay ve düşey asimtot yok.

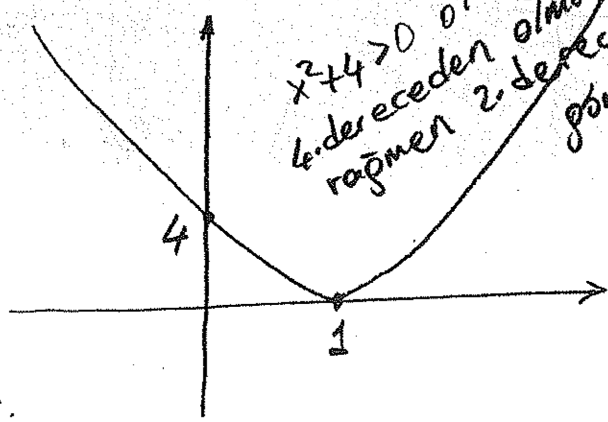
$f'(x)=4(x-1)\underbrace{(x^2-\frac{x}{2}+2)}_{+}=0$

1	
−	+
↘	↗

$x=0 \rightarrow y=4$

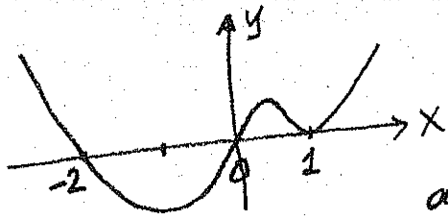
$y=0 \rightarrow x=1$

$x^2+4>0$ olduğundan 4. dereceden olmasına rağmen 2. dereceden görünüyo

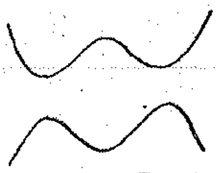


$f''(x)=12x^2-12x+6$
$=12(x^2-x+\frac{1}{2})$
$=12((x-\frac{1}{2})^2+\frac{1}{4})>0$ çukur.

Ör $y = x(x+2)(x-1)^2$ fonk. grafiği



4. dereceden polinom
4 tane azalan artan bölge var

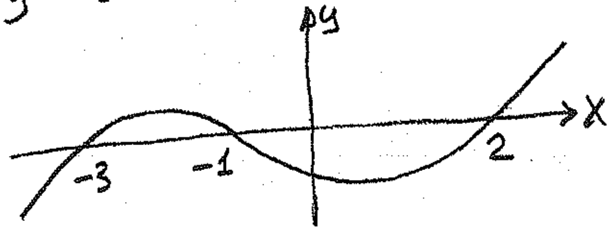


$a>0$

$a<0$

$y = ax^4 + bx^3 + cx^2 + dx + e$

Ör $y = (x+3)(x+1)(x-2)$ fonk. grafiği



3. dereceden polinom
3 tane artan azalan bölge var

$a>0$

$a<0$

$y = ax^3 + bx^2 + cx + d$

Ör $y = (x+2)(x-1)^2$ fonk. grafiğini çiz.

$y' = 3x^2 - 3 = 0$

$x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$x = -1,\ x = 1$

-1		$+1$	
$+$	$-$		$+$
↗	↘		↗

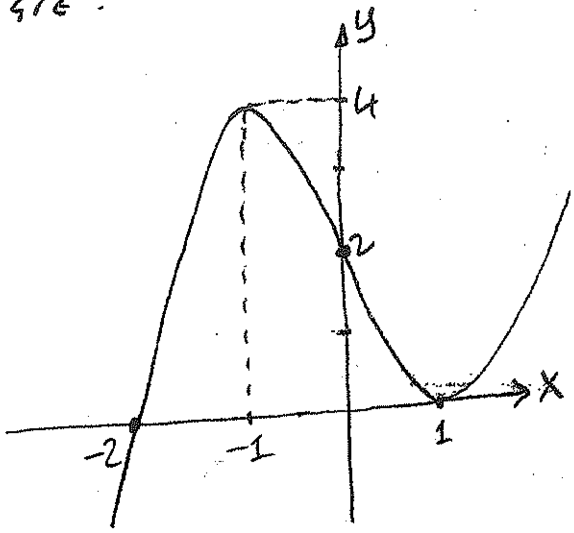
$y(-1) = 4,\ y(1) = 0$

$y'' = 6x = 0$

0	
$-$	$+$
∩	∪

$x = 0 \rightarrow y = 2$

$y = 0 \rightarrow x = -2$
$x = 1$



Ör $y = (x+1)^2(2-x)$ fonk. grafiğini çiz.

$y' = (x+1)(3-3x) = 0$

$x = -1,\ x = 1$

-1		1	
$+$	$-$		$+$
↗	↘		↗

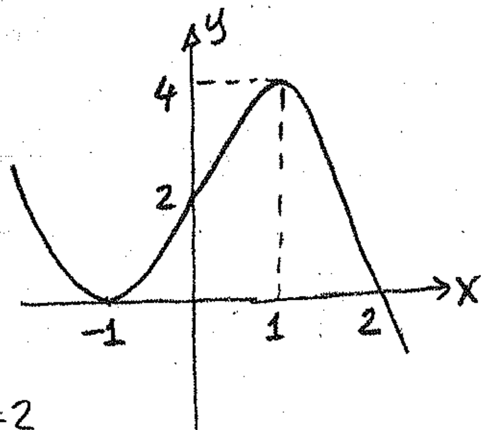
$x = 0 \rightarrow y = 2$

$y = 0 \rightarrow x = -1$
$x = 2$

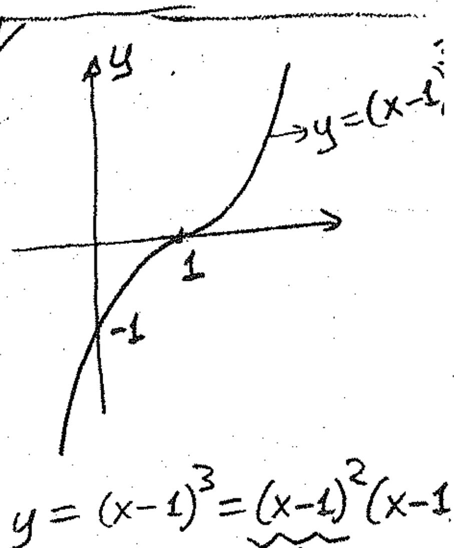
$y'' = -6x = 0$

0	
$+$	$-$
∪	∩

$y(0) = 2$



Ör



$y = (x-1)^3 = \underbrace{(x-1)^2}_{+}(x-1$

$y' = 3(x-1)^2 > 0$ artan

$y'' = 6(x-1)$ $\frac{1}{-\,|\,+}$

Ör. $y = \dfrac{x-1}{x+1}$ fonk. grafiğini çiz.

$x+1=0 \rightarrow x=-1$ düşey asimtot.

$\lim\limits_{x\to\mp\infty} \dfrac{x-1}{x+1} = 1 \rightarrow y=1$ yatay asimtot.

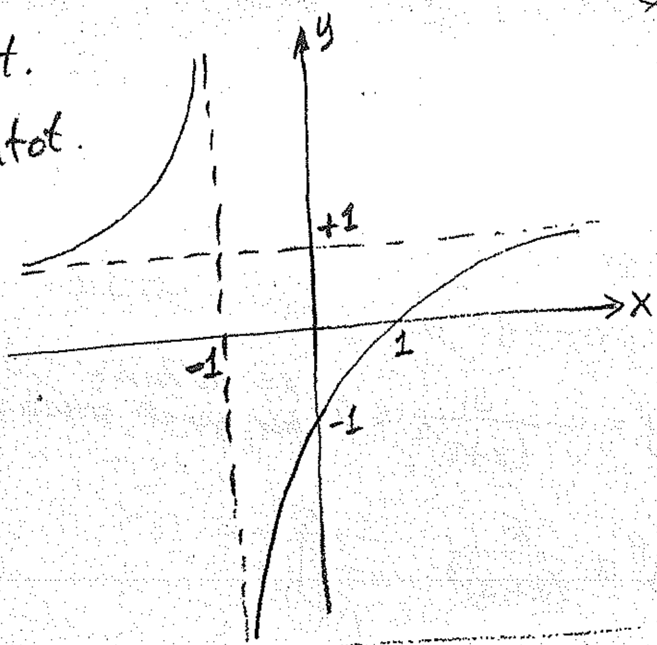
$x=0 \rightarrow y=-1$

$y=0 \rightarrow x=1$

$y' = \dfrac{2}{(x+1)^2} > 0$ $x \neq -1$ hariç her yerde artan

$y'' = -\dfrac{4}{(x+1)^3}$

	-1	
$+$ ∪		$-$ ∩



Ör. $y = \dfrac{3-2x}{x-1}$ fonk. grafiğini çiz.

$x-1=0 \rightarrow x=1$ düşey asimtot

$\lim\limits_{x\to\mp\infty} \dfrac{3-2x}{x-1} = -2 \rightarrow y=-2$ yatay asimtot

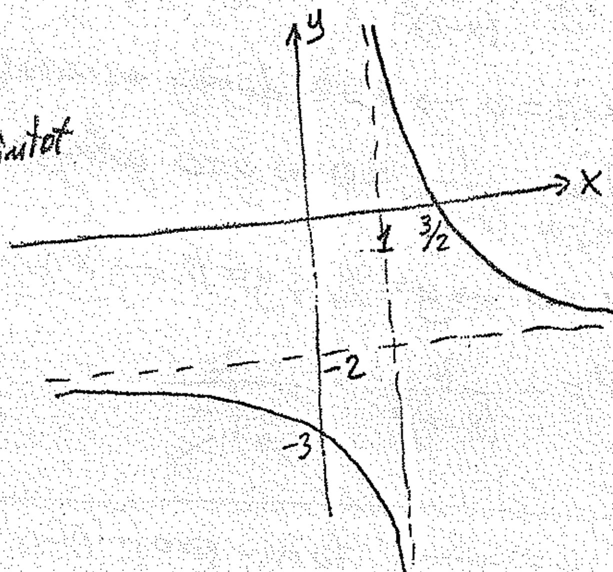
$x=0 \rightarrow y=-3$

$y=0 \rightarrow x=3/2$

$y' = -\dfrac{1}{(x-1)^2} <$ $x \neq 1$ hariç her yerde azalan

$y'' = +\dfrac{2}{(x-1)^3}$

	1	
$-$ ∩		$+$ ∪



Ör. $y = \dfrac{2x-3}{x-1}$ fonk. grafiğini çiz

$x-1=0 \rightarrow x=1$ düşey asimtot

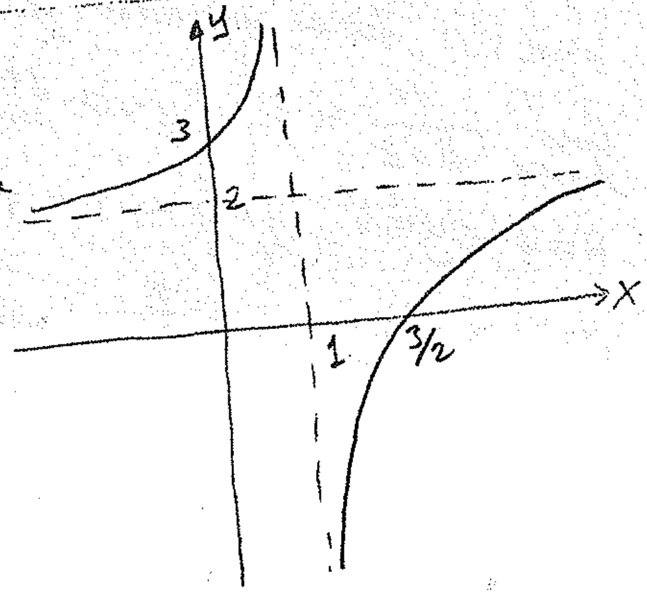
$\lim\limits_{x\to\mp\infty} \dfrac{2x-3}{x-1} = 2 \rightarrow y=2$ yatay asimtot

$x=0 \rightarrow y=3$, $y=0 \rightarrow x=3/2$

$y' = \dfrac{1}{(x-1)^2} > 0$ $x \neq 1$ hariç her yerde artan

$y'' = -\dfrac{2}{(x-1)^3}$

	1	
$+$ ∪		$-$ ∩



Örnek $y = \frac{3}{x+2}$ fonk. grafiğini çiz.

$x+2=0 \rightarrow x=-2$ düşey asimtot

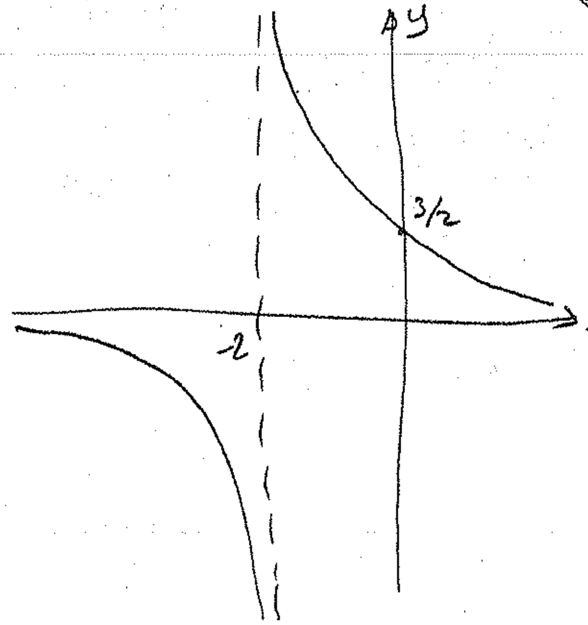
$\lim\limits_{x \to \mp\infty} \frac{3}{x+2} = 0 \rightarrow y=0$ yatay asimtot

$x=0 \rightarrow y = 3/2$, $y=0 \rightarrow x=\mp\infty$

$y' = -\frac{3}{(x+2)^2} < 0$ $x=-2$ hariç her yerde azalan

$y'' = \frac{6}{(x+2)^3}$

-2	
$-$	$+$
$\cap$	$\cup$



Örnek $y = \frac{1}{(x-2)^2}$ fonk grafiğini çiz

$x-2=0 \rightarrow x=2$ düşey asimtot.

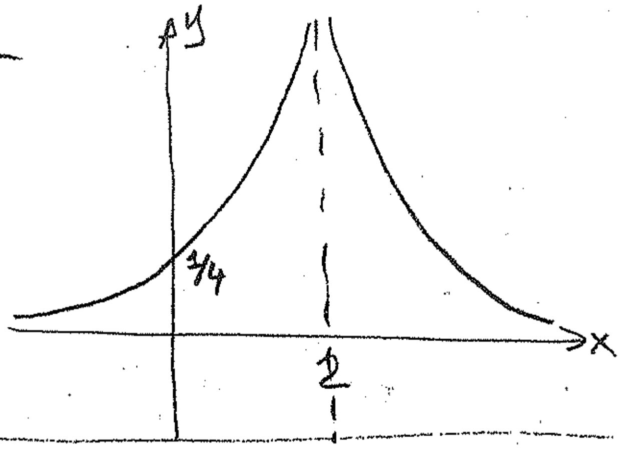
$\lim\limits_{x \to \mp\infty} \frac{1}{(x-2)^2} = 0 \rightarrow y=0$ yatay asimtot

$x=0 \rightarrow y=1/4$, $y=0 \rightarrow x=\mp\infty$

$y' = -\frac{2}{(x-2)^3}$

2	
$+$	$-$
$\nearrow$	$\searrow$

$y'' = \frac{6}{(x-2)^4} > 0$ her yerde çukur



Örnek $y = \frac{1}{x^2+1}$ fonk. grafiğini çiz.

$x^2+1 > 0$ düşey asimtot yok

$\lim\limits_{x \to \mp\infty} \frac{1}{x^2+1} = 0 \rightarrow y=0$ yatay asimtot

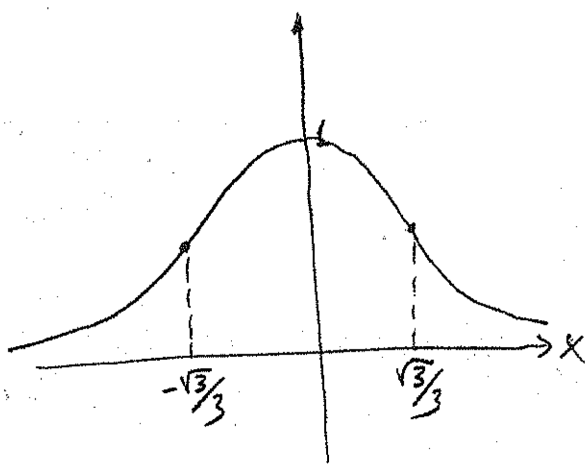
$x=0 \rightarrow y=1$, $y=0 \rightarrow x=\mp\infty$

$y' = -\frac{2x}{(x^2+1)^2}$ $y(0)=1$

0	
$+$	$-$
$\nearrow$	$\searrow$

$y'' = \frac{6x^2-2}{(x^2+1)^3}$

$-\sqrt{3}/3$	$\sqrt{3}/3$	
$+$	$-$	$+$
$\cup$	$\cap$	$\cup$



Ör: $y = \dfrac{4-x^2}{x^2-9}$ fonk. grafiğini çiz.

$x^2-9=0 \rightarrow x=\mp 3$ düşey asimtotlar

$\lim\limits_{x\to\mp\infty} \dfrac{4-x^2}{x^2-9} = -1 \rightarrow y=-1$ yatay asimtot

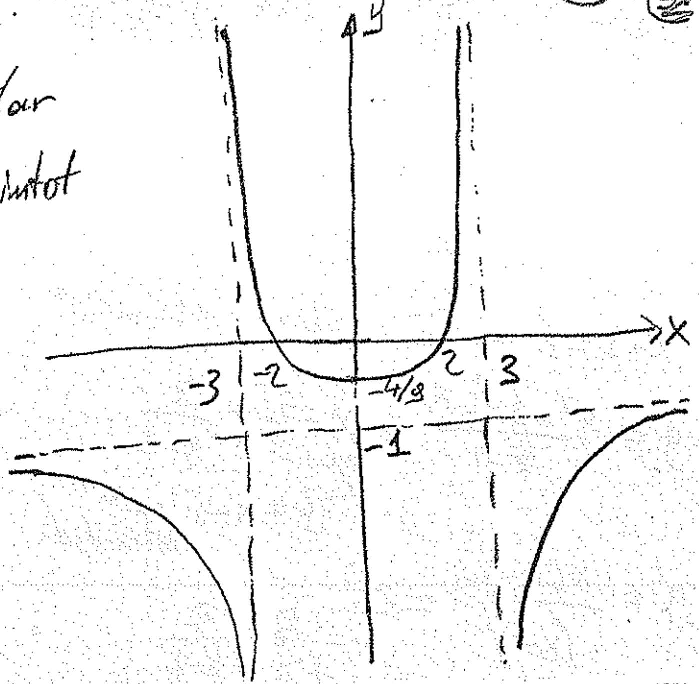
$x=0 \rightarrow y=-4/9$, $y=0 \rightarrow x=\mp 2$

$y' = \dfrac{10x}{(x^2-9)^2}$

	0	
y'	$-$ ↘	$+$ ↗

$y'' = -30\dfrac{x^2+3}{(x^2-9)^3}$

	-3		3	
y''	$-$ ∩	$+$ ∪		$-$ ∩



Ör: $y = \dfrac{3x^2+5}{x^2-4}$ fonk. grafiğini çiz

$x^2-4=0 \rightarrow x=\mp 2$ düşey asimtotlar

$\lim\limits_{x\to\mp\infty} \dfrac{3x^2+5}{x^2-4} = 3 \rightarrow y=3$ yatay asimtot

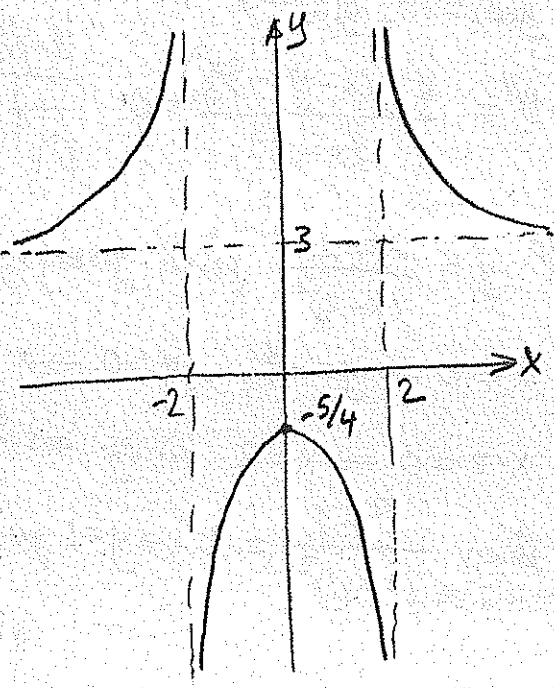
$x=0 \rightarrow y=-5/4$, $y=0 \rightarrow 3x^2+5>0$ çözüm yok

$y' = \dfrac{-34x}{(x^2-4)^2}$

	0	
y'	$+$ ↗	$-$ ↘

$y'' = \dfrac{34(3x^2+4)}{(x^2-4)^3}$

	-2		2	
y''	$+$ ∪	$-$ ∩		$+$ ∪



Ör: $f(x) = \sqrt{x^2-4x+3}$ fonk. grafiğini çiz

$x^2-4x+3=0$

$(x-1)(x-3)=0$

$x=1, x=3$

	1		3	
	$+$	$-$		$+$
	tanımlı	tanımsız		tanımlı

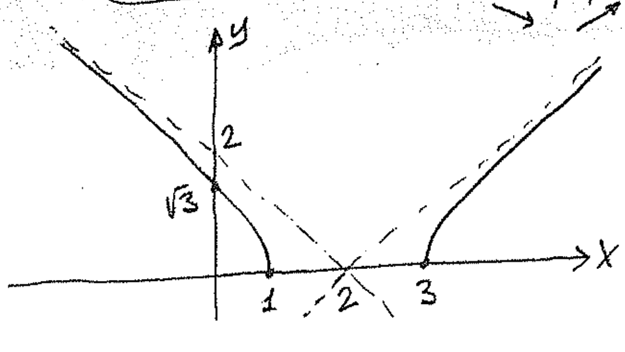
$x=0 \rightarrow y=\sqrt{3}$, $y=0 \Rightarrow x=1, x=3$

$y' = \dfrac{x-2}{\sqrt{x^2-4x+3}}$ tanım aralığında pozitif

	2	
y'	$-$ ↘	$+$ ↗

$f(x) = \sqrt{(x-2)^2-1} = |x-2|\sqrt{1-\dfrac{1}{(x-2)^2}}$

$y=|x-2|$ eğik asimtot



ör $y = \dfrac{x^2-3x+5}{x-1}$ fonk. grafiğini çiz.

$x=0 \to y=-5$

$y=0 \to x^2-3x+5 = (x-\frac{3}{2})^2+\frac{11}{4} > 0$

kök yok

$x-1=0 \to x=1$ düşey asimtot

$y = \dfrac{x^2-3x+5}{x-1} = x-2 + \dfrac{3}{x-1}$

$\to y=x-2$ eğik asimtot

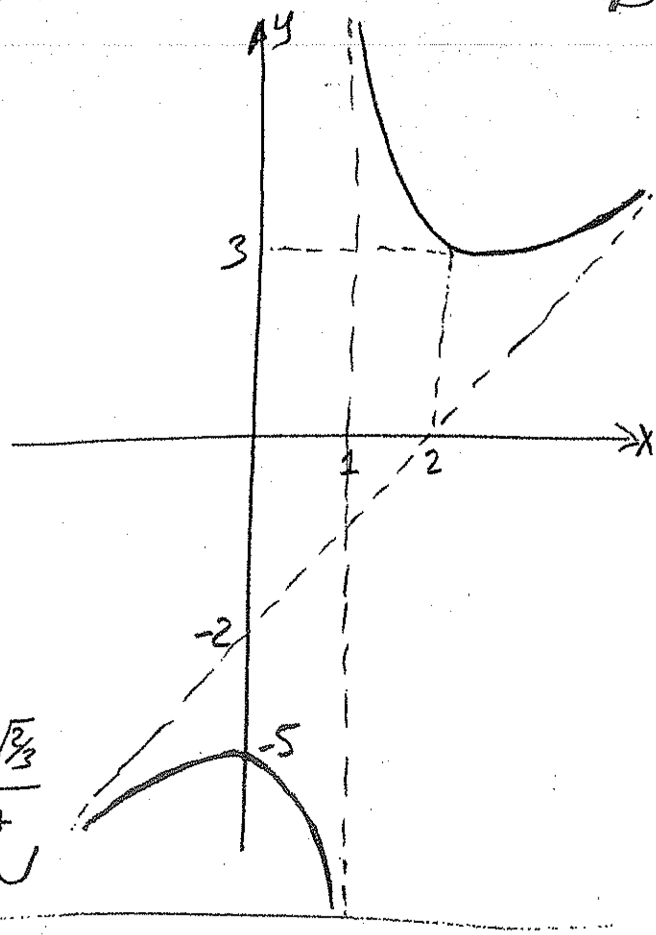
$y' = \dfrac{x(x-2)}{(x-1)^2} = 0$

0	2	
+ ↗	− ↘	+ ↗

$y(0)=-5,\ y(2)=3$

$y'' = \dfrac{6\left((x-1)^2-\frac{2}{3}\right)}{(x-1)^3} = 0$

$1-\sqrt{\frac{2}{3}}$	1	$1+\sqrt{\frac{2}{3}}$	
− ∩	+ ∪	− ∩	+ ∪



ör $y = \dfrac{x^2-1}{(x-2)^2}$ fonk. grafiğini çiz.

$x=0 \to y=-\frac{1}{4},\ y=0 \to x=\mp 1$

$x-2=0 \to x=2$ düşey asimtot.

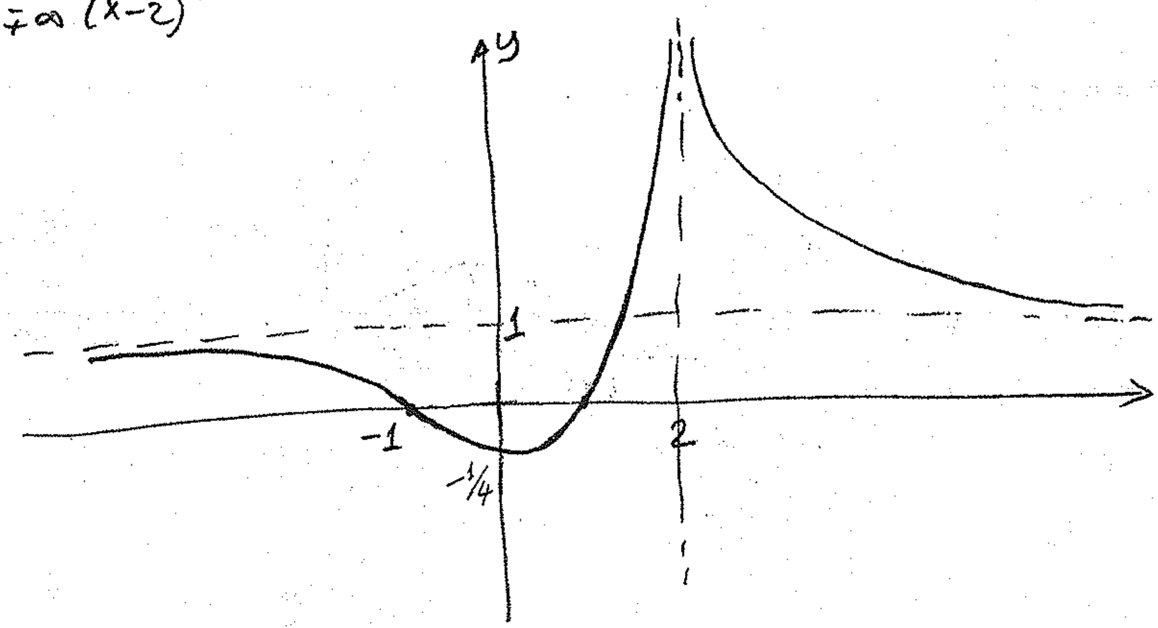
$\lim\limits_{x\to\mp\infty} \dfrac{x^2-1}{(x-2)^2} = 1 \to y=1$ yatay asimtot

$y' = \dfrac{2-4x}{(x-2)^3} = 0$

½	2	
− ↘	+ ↗	− ↘

$y'' = \dfrac{8x+2}{(x-4)^4}$

$-\frac{1}{4}$	
− ∩	+ ∪



Ör $y = 4x^3 - x^4$ fonksiyonunun grafiğini çiz.

$x=0 \rightarrow y=0$

$y=0 \rightarrow 4x^3 - x^4 = 0$

$x^3(4-x)=0$

$x=0,\ x=4$

$y' = 4x^2(3-x) = 0$

$x=0,\ x=3$

	0		3	
+		+		−
↗		↗		↘

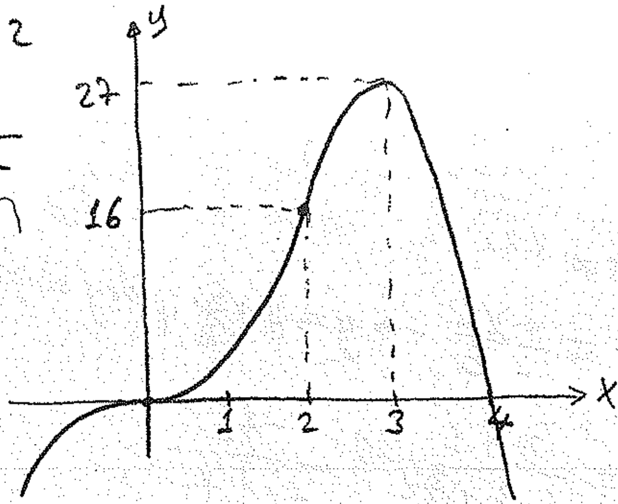
$x=2 \rightarrow y=16$

$x=3 \rightarrow y=27$

$y'' = 12x(2-x) = 0$

$x=0,\ x=2$

	0		2	
−		+		−
∩		∪		∩



Ör $y = x^{4/3} + 4x^{1/3}$ fonk. grafiğini çiziniz.

$x=0 \rightarrow y=0$

$y=0 \rightarrow x^{4/3} + 4x^{1/3} = 0$

$x^{1/3}(x+4)=0$

$x=0,\ x=-4$

$y' = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$

$= \frac{4}{3}x^{-2/3}(x+1) = 0$

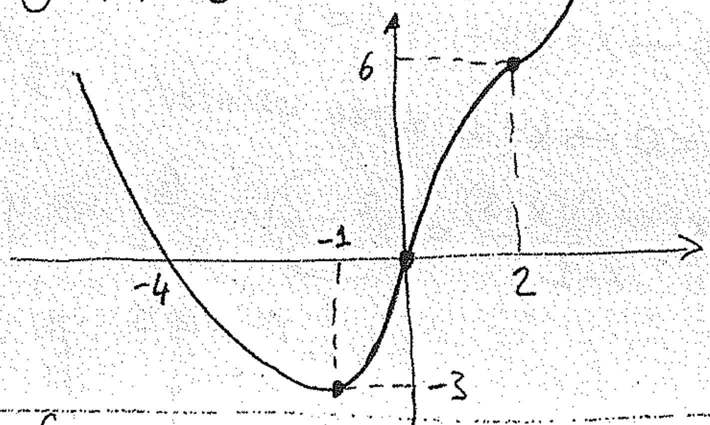
	−1	
−		+
↘		↗

$y'' = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = 0$

	0		2	
+		−		+
∪		∩		∪

$x=-1 \rightarrow y=-3$

$x=2 \rightarrow y=6\sqrt[3]{2}$



Ör $y = x(x-2)e^x$ fonk. grafiğini çiz.

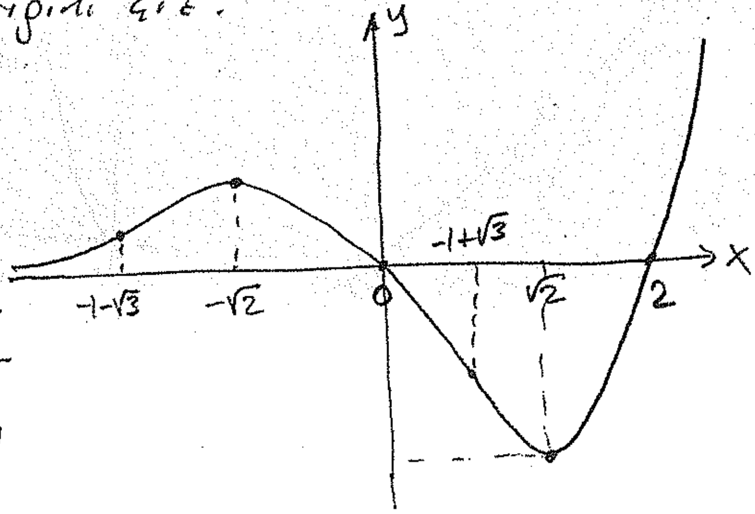
$x=0,\ x=2,\ x=-\infty \rightarrow y=0$

$y' = (x^2-2)e^x = 0 \rightarrow x = \mp\sqrt{2}$

$y'' = (x^2+2x-2)e^x = 0 \rightarrow x = -1 \mp \sqrt{3}$

	$-\sqrt{2}$		$\sqrt{2}$	
+		−		+
↗		↘		↗

	$-1-\sqrt{3}$		$1+\sqrt{3}$	
+		−		+
∪		∩		∪



$y = \dfrac{x-1}{\sqrt{2x^2+1}}$ fonk. grafiğini çiz.

$\sqrt{2x^2+1} > 0$ düşey asimtot yok

$x=0 \to y=-1$

$y=0 \to x=1$

$\lim\limits_{x\to\mp\infty} \dfrac{x-1}{\sqrt{2x^2+1}} = \lim\limits_{x\to\mp\infty} \dfrac{x(1-1/x)}{|x|\sqrt{2+1/x^2}}$

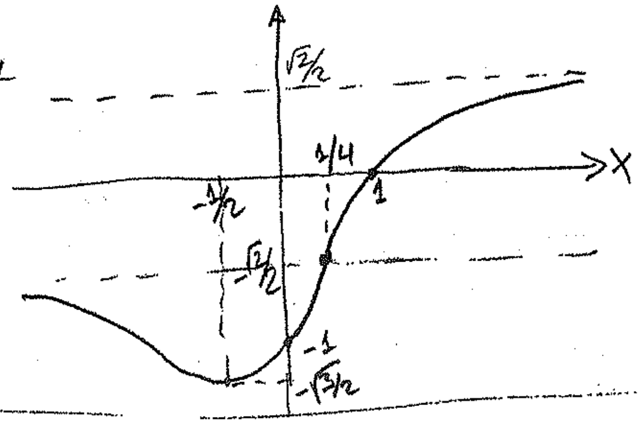
$= \pm\dfrac{\sqrt{2}}{2} \to y = \mp\sqrt{2}/2$ yatay asimtot

$y' = \dfrac{2x+1}{(2x^2+1)^{3/2}} = 0$ $\quad$ $-1/2$: $-$ | $+$ (↘ | ↗)

$y(-1/2) = -\sqrt{3/2}$

$y'' = -2\dfrac{(x+1)(4x-1)}{(2x^2+1)^{5/2}} = 0$

$-1 \quad 1/4$: $-$ | $+$ | $-$ ($\cap$ | $\cup$ | $\cap$) $\quad y(\tfrac{1}{4}) = -\sqrt{2}/2$



$y = \dfrac{x(x+2)(x-3)}{x-1}$ fonk. grafiğini çiz.

$y' = 2x - \dfrac{6}{(x-1)^2}$ $\checkmark$ zor

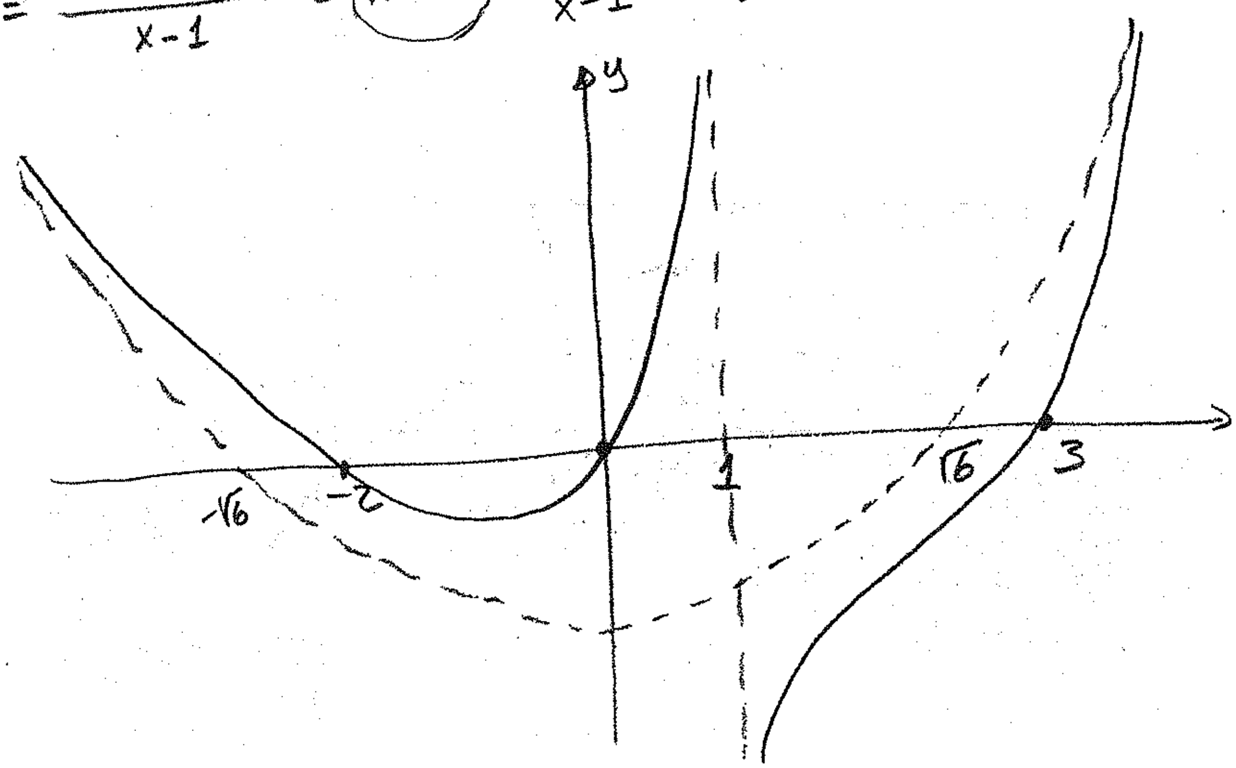
$x=0 \to y=0$

$y=0 \to x=0, x=-2, x=3$

$y'' = 2 + \dfrac{12}{(x-1)^3}$ $\checkmark$ zor

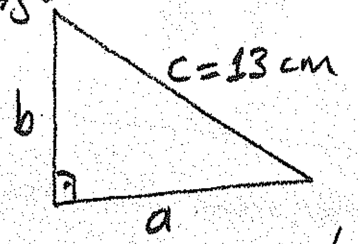
$x-1=0 \to x=1$ düşey asimtot.

$y = \dfrac{x(x+2)(x-3)}{x-1} = (x^2-6) - \dfrac{6}{x-1} \Rightarrow y = x^2-6$ eğri asimtot



Bağlı Oranlar

Ör Hipotenüsü sabit ve 13 cm olan bir dik üçgenin kenarlarından biri 1 cm/sn hızla uzamaktadır. Uzayan kenar 12 cm olduğunda üçgenin alanı hangi hızla değişir?



$\frac{da}{dt} = 1\ cm/sn$ $\quad a = 12\ cm$ iken

$b = \sqrt{(13)^2 - (12)^2} = 5\ cm$

$a^2 + b^2 = 169$

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0 \rightarrow \frac{db}{dt} = -\frac{a}{b}\frac{da}{dt} = -\frac{12}{5}\times 1 = -\frac{12}{5}\ cm/sn$

$A = \frac{1}{2}ab \rightarrow \frac{dA}{dt} = \frac{1}{2}\left(b\frac{da}{dt} + a\frac{db}{dt}\right)$

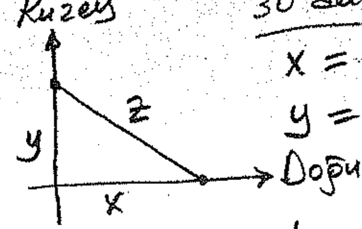
$\left.\frac{dA}{dt}\right|_{a=12} = \frac{1}{2}\left(5\times 1 - 12\times\frac{12}{5}\right) = -11.9\ cm^2/sn$

Ör Küre şeklindeki bir balona dakikada $20\ m^3$ hava pompalanıyor. Balonun yarıçapı 5 m olduğunda yarıçapın değişim oranı nedir?

$V = \frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$

$\left.\frac{dr}{dt}\right|_{r=5} = \frac{1}{4\pi\times 5^2}\times 20 = \frac{1}{4\pi}\ m/dak$

Ör İki kişi aynı anda aynı noktadan başlayarak biri doğuya, diğeri kuzeye doğru gidiyor. Doğuya gidenin saatteki ortalama hızı 8 km, kuzeye gideninki ise 6 km'dir. 30 dakika sonra aralarındaki mesafenin değişim oranı nedir?



30 dakika (1/2 saat) sonra

$x = (8\ km/saat)\cdot\left(\frac{30}{60}\ saat\right) = 4\ km$ $\quad z = \sqrt{x^2+y^2} = 5\ km$

$y = (6\ km/saat)\cdot\left(\frac{30}{60}\ saat\right) = 3\ km$

$\frac{dx}{dt} = 8\ km/saat \quad \frac{dy}{dt} = 6\ km/saat$

$x^2 + y^2 = z^2$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$

$\left.\frac{dz}{dt}\right| = \frac{1}{5}\left(4\times 8 + 3\times 6\right) = 10\ km/saat$

5/ Küre şeklindeki bir balon dakikada 5 m³ hava kaybediyor. Yarıçapı 4 m olduğunda yüzey değişim oranı nedir?

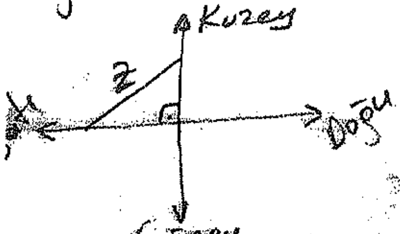
$$V=\frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dt}=4\pi r^2\frac{dr}{dt} \rightarrow \frac{dr}{dt}=\frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$S=4\pi r^2 \rightarrow \frac{dS}{dt}=8\pi r\frac{dr}{dt}=8\pi r\cdot\frac{1}{4\pi r^2}\frac{dV}{dt}=\frac{2}{r}\frac{dV}{dt}$$

$$\left.\frac{dS}{dt}\right|_{r=4}=\frac{2}{4}\times 5=2.5\ m^2/dak.$$

6/ Bir uçak 200 km/saat hızla batıya doğru, diğer uçak ise 1 saat sonra aynı yerden 300 km/saat hızla kuzeye doğru kalkış yapıyor. İlk uçağın kalkışından 2 saat sonra aralarındaki mesafenin değişim hızı nedir?

$$\frac{dx}{dt}=-200\ km/saat \qquad \frac{dy}{dt}=300\ km/saat$$



$$x=-200t \qquad y=300(t-1)$$

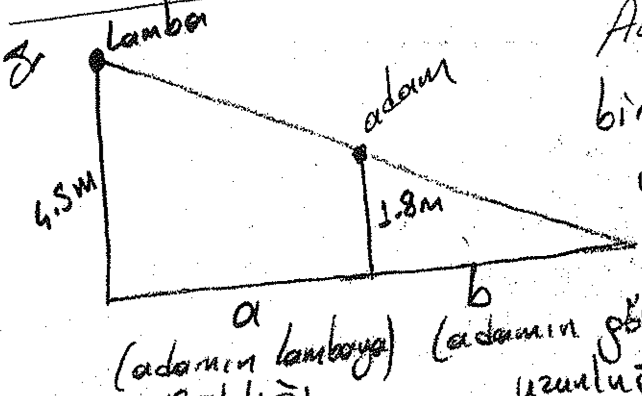
~~2 saat sonra~~

$$x=-400,\ y=300 \rightarrow z=\sqrt{x^2+y^2}=500\ km$$

$$z^2=x^2+y^2 \rightarrow 2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt} \rightarrow \frac{dz}{dt}=\frac{1}{z}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$$

$$\left.\frac{dz}{dt}\right|_{t=2}=\frac{1}{500}\left((-400)\cdot(-200)+(300)(300)\right)=340\ km/saat$$

8/ Adam lambaya doğru 0.75 m/sn'lik bir hızla gidiyor. Adamın lambaya uzaklığı 6 m olduğunda gölgesinin değişim hızı nedir?



$$\frac{b}{1.8}=\frac{a+b}{4.5} \rightarrow b=\frac{2a}{3} \longrightarrow \frac{db}{dt}=\frac{2}{3}\frac{da}{dt}=\frac{2}{3}\times 0.75=0.5\ m/sn$$

lambaya uzaklık ne olursa olsun sabittir.

Minimum-Maksimum Problemleri

Ör Kenar uzunlukları $(3x-7)$ ve $(2x+1)$ olan dikdörtgenin alanı x'in hangi değeri için minimum olur?

$A(x) = (3x-7)(2x+1) = 6x^2 - 11x - 7$ $\quad A''(x) = 12 > 0$ $\cup$ çukur

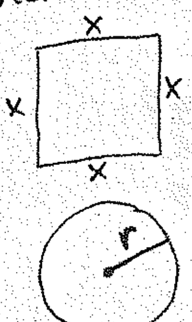
$A'(x) = 12x - 11 = 0 \rightarrow x = 11/12$ için Alan minimum olur.

Ör Kenar uzunlukları $(7-3x)$ ve $(2x+1)$ olan dikdörtgenin alanı x'in hangi değeri için maksimum olur? $\quad A''(x) = -12 < 0$ $\cap$ tümsek

$A(x) = (7-3x)(2x+1) = 7 + 11x - 6x^2$

$A'(x) = 11 - 12x = 0 \rightarrow x = 11/12$ için Alan maksimum olur.

Ör 10 m tel ikiye kesilerek hem kare hem de çember oluşturuluyor. Kare ve çemberin toplam alanı ne olmalı ki minimum olsun?



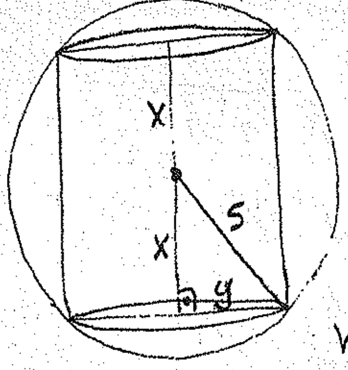
$$4x + 2\pi r = 10 \longrightarrow r = \frac{5-2x}{\pi}$$

$$A = x^2 + \pi r^2 \longrightarrow A(x) = x^2 + \pi\left(\frac{5-2x}{\pi}\right)^2 = x^2 + \frac{1}{\pi}(5-2x)^2$$

$$A'(x) = 2x - \frac{4}{\pi}(5-2x) = 0 \longrightarrow x = \frac{10}{\pi+4}$$

$$x = \frac{10}{\pi+4} \text{ için } A = \left(\frac{10}{\pi+4}\right)^2 + \frac{1}{\pi}\left(5 - \frac{20}{\pi+4}\right)^2 = \frac{25}{\pi+4}$$

Ör Yarıçapı 5 m olan bir kürenin içine yerleştirilen maksimum hacimli dik bir silindirin yüksekliğini bulunuz?



$$x^2 + y^2 = 25 \longrightarrow y^2 = 25 - x^2$$

Silindirin Hacmi = Taban Alanı × Yükseklik

$$V = \pi y^2 \cdot 2x$$

$$V(x) = \pi(25 - x^2) \cdot 2x = 2\pi x(25 - x^2)$$

$$V'(x) = 2\pi\left((25 - x^2) + x(-2x)\right)$$

$$= 2\pi(25 - 3x^2) = 0 \longrightarrow x = \mp 5/\sqrt{3}$$

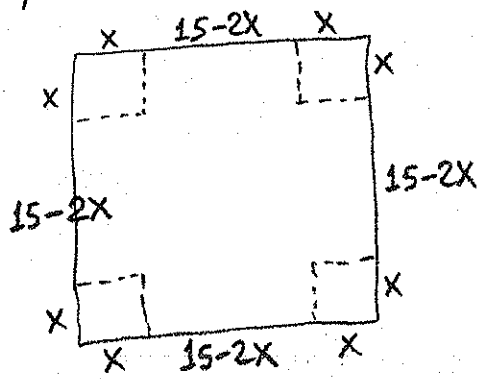
$-5/\sqrt{3}$		$5/\sqrt{3}$	
$-$	$+$		$-$
↘ min	↗	max	↘

$x = 5/\sqrt{3}$ için hacim ~~minimum~~ max

$$V(5/\sqrt{3}) = 2\pi\frac{5}{\sqrt{3}}\left(25 - \frac{25}{3}\right) = \frac{500\pi}{3\sqrt{3}}$$

$$h = 2x = \frac{10}{\sqrt{3}}$$

Ör. Kenar uzunluğu 15 cm olan kare şeklindeki bir kartonun köşelerinden eş kareler kesilerek üstü açık bir kutu yapılıyor. Kutunun hacmi en fazla kaç cm^3 olur?



$$V(x) = x.(15-2x)^2$$

$$V'(x) = (15-2x)^2 + x.2(15-2x).(-2)$$

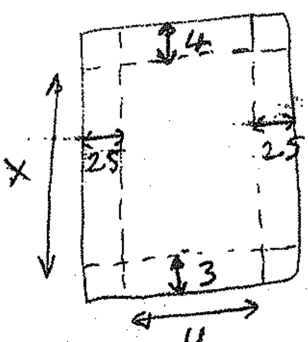
$$= (15-2x)(15-6x) = 0 \to x = 15/2,\ x = 5/2$$

$$\begin{array}{c|c|c} & 5/2 & 15/2 \\ \hline + & - & + \end{array}$$

max min

$$V(5/2) = 250\ cm^3$$

Ör. Bir kağıdın 140 cm^2'lik kısmına yazı yazılacaktır. Üstten 4 cm, alttan 3 cm, soldan ve sağdan 2.5 cm boşluk bırakılmaktadır. Bu kağıdın alanı en az kaç cm^2 olur?



$$S = (x+7)(y+5) \qquad xy = 140 \to y = \frac{140}{x}$$

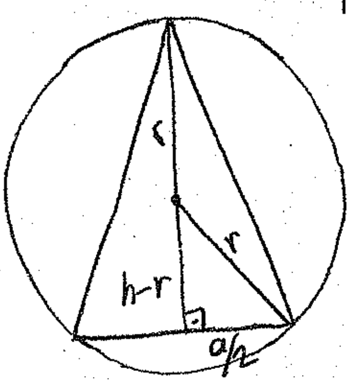
$$\cancel{S(x) = (x+7)}\left(\frac{140}{x} + 5\right) = 175 + 5x + \frac{980}{x}$$

$$S'(x) = 5 - \frac{980}{x^2} = 0 \to x = \cancel{7}\ \cancel{14\ cm}$$

$$x = 14,\ y = 10 \to S = 21 \times 15 = 315\ cm^2$$

Ör. Yarıçapı r cm olan bir çemberin içine çizilebilen bir ikizkenar üçgenin alanı en fazla kaç cm^2 olur?

$$r^2 = (h-r)^2 + \left(\frac{a}{2}\right)^2 \to a = 2\sqrt{2rh-h^2}$$



$$A = \frac{1}{2}ah \to A(h) = h.\sqrt{2rh-h^2}$$

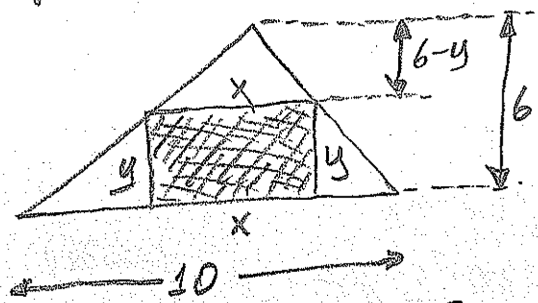
$$A'(h) = \sqrt{2rh-h^2} + h.\frac{2r-2h}{2\sqrt{2rh-h^2}} = 0$$

$$= \frac{h(3r-2h)}{\sqrt{2rh-h^2}} = 0 \to h = 0,\ h = \frac{3r}{2}$$

$h = \frac{3r}{2}$ için alan maksimum

$$h = \frac{3r}{2},\ a = \sqrt{3}r \to A = \frac{1}{2}ah = \frac{3\sqrt{3}}{4}r^2$$

Ör Taban kenar uzunluğu 10 cm, yüksekliği 6 cm olan bir üçgenin içine bir kenarı tabana bitişik bir dikdörtgen çiziliyor. Dikdörtgenin alanı en fazla kaç cm^2 olur.



$$\frac{6-y}{x} = \frac{6}{10} \longrightarrow x = 10 - \frac{5y}{3}$$

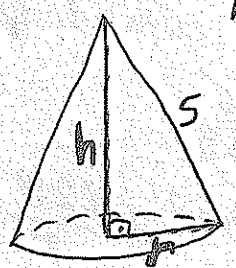
$$A = xy \longrightarrow A(y) = \left(10 - \frac{5y}{3}\right) \cdot y$$

$$A'(y) = 10 - \frac{10}{3}y = 0 \longrightarrow y = 3,\ x = 5$$

$$A''(y) = -10/3 < 0 \ \cap \text{ tümsek}$$

$A = xy = 5 \times 3 = 15\ cm^2$ maksimum alan.

Ör Hipotenüsü 5 m olan bir dik üçgen bir kenarı etrafında döndürülürse oluşan dairesel koninin hacmi en fazla kaç m^3 olur.



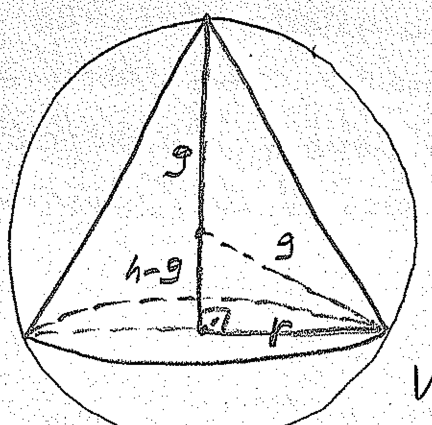
$$h^2 + r^2 = 25 \longrightarrow r^2 = 25 - h^2$$

$$V = \frac{\pi r^2 h}{3} \longrightarrow V(h) = \frac{\pi(25-h^2)h}{3} = \frac{\pi}{3}(25h - h^3)$$

$$V'(h) = \frac{\pi}{3}(25 - 3h^2) = 0 \longrightarrow h = 5/\sqrt{3},\ r^2 = 50/3$$

$$V = \frac{\pi r^2 h}{3} = \frac{\pi}{3}\,\frac{50}{3}\,\frac{5}{\sqrt{3}} = \frac{250\pi}{9\sqrt{3}}\ m^3 \text{ max. hacim}$$

Ör Yarıçapı 9 cm olan bir küre içine yerleştirilen bir dik koninin hacmi en fazla kaç cm^3 olur.



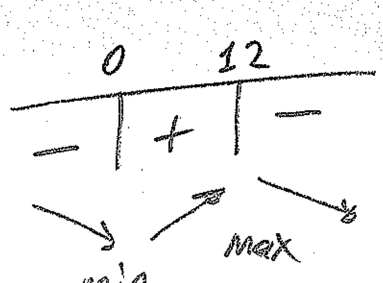
$$r^2 + (h-9)^2 = 81 \longrightarrow r^2 = 18h - h^2$$

$$V = \frac{\pi r^2 h}{3} \text{ koninin hacmi}$$

$$V(h) = \frac{\pi}{3}(18h - h^2).h = \frac{\pi}{3}(18h^2 - h^3)$$

$$V'(h) = \frac{\pi}{3}(36h - 3h^2) = 0 \longrightarrow h = 0,\ h = 12$$

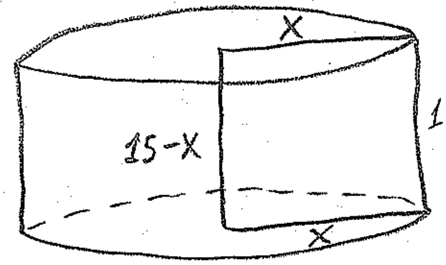
$h = 12$ için maksimum hacim



$$V_{max} = \frac{\pi}{3}(18-h)h^2$$

$$= \frac{\pi}{3}(18-12) \times 12 \times 12 = 288\pi\ cm^3$$

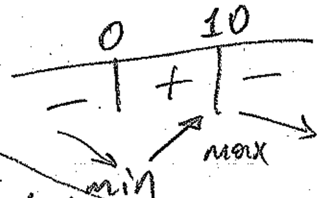
Ör: Çevresi 30 cm olan dikdörtgen şeklindeki bir karton kenarlarından biri etrafında döndürülüyor. Meydana gelen dairesel silindirin hacmi en fazla kaç cm^3 olur.



$$V = \pi x^2 (15-x) = \pi (15x^2 - x^3)$$

$$V' = \pi (30x - 3x^2) = 0$$

$$3\pi x (10-x) = 0$$



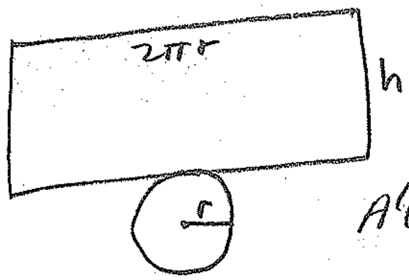
$$V = \pi x^2 (15-x)$$

$$= \pi \times 100 \times 5$$

$$= 500\pi \ cm^3$$

max hacim

Ör: Bir sanayici alüminyum dik dairesel silindir şeklinde üstü açık, 64 cm^3 hacminde kutular yapmaktadır. En az alüminyum kullanılması için yapacağı silindirin taban yarıçapı kaç cm olmalıdır.



$$V = \pi r^2 h = 64 \rightarrow h = \frac{64}{\pi r^2}$$

$$A = \pi r^2 + 2\pi r h \rightarrow A(h) = \pi r^2 + \frac{128}{r}$$

$$A'(r) = 2\pi r - \frac{128}{r^2} = 0 \rightarrow r = \frac{4}{\sqrt[3]{\pi}}$$

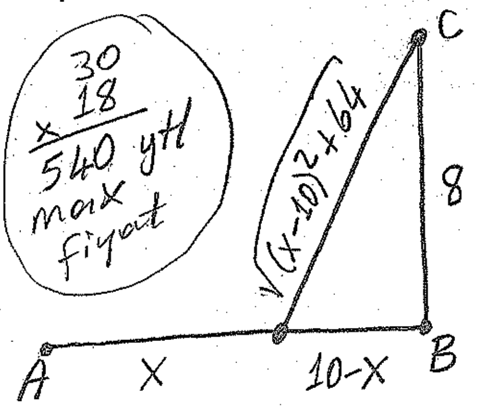
Ör: B köyü A köyünün 10 km doğusunda, C köyü de B köyünün 8 km kuzeyindedir. A ile B, B ile C arasında stabilize yol mevcuttur. A ile C arası asfaltlanacaktır. 1 km stabilize yolun asfaltlanması 30 bin ytl, 1 km yeni yolun açılıp asfaltlanması 50 bin ytl'ye mal olmaktadır. A ile C arasındaki asfalt yol en az kaç ytl'ye yapılır?

$$M(x) = 30x + 50\sqrt{(x-10)^2 + 64}$$

$$M'(x) = 30 + 50 \frac{x-10}{\sqrt{(x-10)^2+64}} = 0$$

$$|x-10| = 6 \longrightarrow x = 4 \ , \ \cancel{x = 16}$$

30
x 18
540 ytl
max fiyat



$$x = 4 \text{ için } M(4) = 120 + 50\sqrt{36+64}$$

$$= 620$$

yanlış

integral

Belirsiz İntegral

$F'(x) = f(x)$ ise $\int f(x)dx = F(x) + C$

$\frac{d}{dx}\int f(x)dx = \left(\int f(x)dx\right)' = f(x)$

$\int k f(x) dx = k \int f(x) dx$

$\int (f(x) \mp g(x)) dx = \int f(x)dx \mp \int g(x)dx$

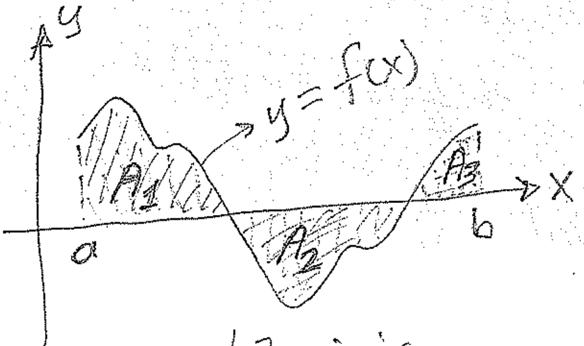
Belirli İntegral

$F'(x) = f(x)$ ise $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a), \; x \in [a,b]$

$\int_a^b (A f(x) \mp B g(x))\,dx = A\int_a^b f(x)\,dx \mp B\int_a^b g(x)\,dx$

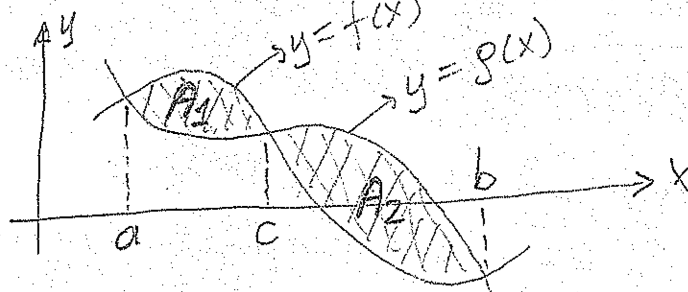
$\int_a^b f(x)dx = -\int_b^a f(x)dx \qquad \int_a^a f(x)dx = 0$

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad a \le c \le b$



$x \in [a,b]$ için

$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$

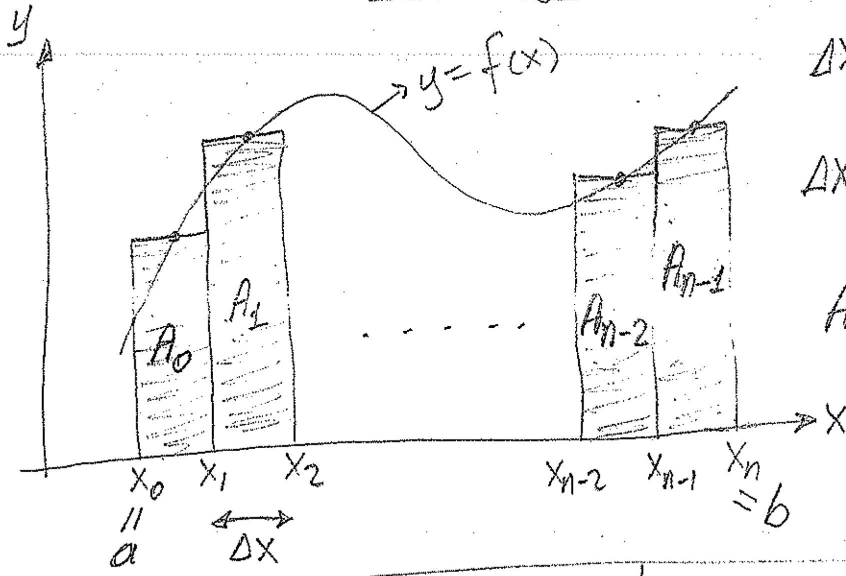


$x \in [a,c]$ için $f(x) \geq g(x)$

$x \in [c,b]$ için $g(x) \geq f(x)$

$\int_a^b (f(x) - g(x))dx = A_1 - A_2$

Dikdörtgenler Kuralı



$$\Delta x = \frac{b-a}{n} = \frac{x_n - x_0}{n}$$

$$\Delta x = x_{i+1} - x_i \ , \quad i = 0, \ldots, n-1$$

$$A_T = \sum_{i=0}^{n-1} A_i \approx \int_a^b f(x)\,dx = I$$

$\Delta x \to 0$ için $n \to \infty$

Bu durumda $A_T = I$

Kural

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \in R \text{ ve } n \neq -1$$

$n=-1$ ise $\int \frac{dx}{x} = \ln|x| + C$

$u = u(x)$ alınırsa $du = u'(x).dx$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad n \in R \text{ ve } n \neq -1$$

$n=-1$ ise $\int \frac{du}{u} = \ln|u| + C$

Ör $\int \frac{dx}{\sqrt{x}} = 2\int \frac{dx}{2\sqrt{x}} = 2\int du = 2u + C = 2\sqrt{x} + C$

$u = \sqrt{x} \quad du = \frac{dx}{2\sqrt{x}}$

Ör $\int \frac{dx}{x^2} = \int x^{-2} dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

Ör $\int \left(4x - \frac{2}{\sqrt[3]{x}} + \frac{5}{x^2}\right) dx = 4\int x\,dx - 2\int x^{-1/3} dx + 5\int x^{-2} dx$

$= 2x^2 - 2\frac{x^{2/3}}{2/3} + 5\frac{x^{-1}}{-1} + C = 2x^2 - 3x^{2/3} - \frac{5}{x} + C$

Ör $\int (x^2+3x+5)^3 . (2x+3).dx = \int u^3 du = \frac{u^4}{4} + C$

$u = x^2+3x+5 \to du = (2x+3)dx \qquad = \frac{1}{4}(x^2+3x+5)^4 + C$

Ör $\int \frac{2x+3}{x^2+3x+5} dx = \int \frac{du}{u} = \ln|u| + C$

$= \ln|x^2+3x+5| + C$

$u = x^2+3x+5$

$du = (2x+3).dx \qquad = \ln(x^2+3x+5) + C$

$x^2+3x+5 > 0$

$(x+\frac{3}{2})^2 + \frac{11}{4} > 0$

Ör

$$I = \int \frac{2x+7}{x^2+5x+6}\,dx = 3\int \frac{dx}{x+2} - \int \frac{dx}{x+3}$$

$$= 3\ln|x+2| - \ln|x+3| + C$$

$$= \ln\left|\frac{(x+2)^3}{x+3}\right| + C$$

$$\frac{2x+7}{x^2+5x+6} = \frac{a}{x+2} + \frac{b}{x+3}$$

$(x+3) \quad (x+2)$

$$= \frac{(a+b)x + (3a+2b)}{(x+2)(x+3)}$$

$$\left.\begin{array}{l} a+b=2 \\ 3a+2b=7 \end{array}\right\} \begin{array}{l} a=3 \\ b=-1 \end{array}$$

Ör

$$I = \int \frac{3x+5}{x^2-2x-3}\,dx$$

$$\frac{3x+5}{(x-3)(x+1)} = \frac{a}{x-3} + \frac{b}{x+1}$$

$(x+1) \quad (x-3)$

$$= \frac{(a+b)x + (a-3b)}{(x-3)(x+1)}$$

$$\left.\begin{array}{l} a+b=3 \\ a-3b=5 \end{array}\right\} \begin{array}{l} a=7/2 \\ b=-1/2 \end{array}$$

$$I = \frac{7}{2}\int \frac{dx}{x-3} - \frac{1}{2}\int \frac{dx}{x+1}$$

$$= \frac{7}{2}\ln|x-3| - \frac{1}{2}\ln|x+1| + C$$

$$= \ln\sqrt{\frac{(x-3)^7}{x+1}} + C$$

Ör

$$I = \int \frac{2x^2+5x+1}{x^2+2x-3}\,dx = \int \left(2 + \frac{x+7}{x^2+2x-3}\right).dx$$

$$= 2\int dx + 2\int \frac{dx}{x-1} - \int \frac{dx}{x+3} = 2x + \ln\frac{(x-1)^2}{|x+3|} + C$$

Ör

$$I = \int \frac{x^3}{1+x^2}\,dx = \int \frac{x^2}{1+x^2}\cdot x\,dx = \int \frac{u-1}{u}\,\frac{du}{2} = \frac{1}{2}\int du - \frac{1}{2}\int \frac{du}{u}$$

$$u = 1+x^2 \rightarrow x^2 = u-1$$

$$du = 2x\,dx$$

$$= \frac{u}{2} - \frac{1}{2}\ln|u| + C$$

$$= \frac{1+x^2}{2} - \frac{1}{2}\ln|1+x^2| + C$$

$$= \frac{x^2}{2} - \ln\sqrt{1+x^2} + C_1$$

Ör

$$I = \int \frac{x\,dx}{\sqrt{x-4}} = \int \frac{(u^2+4).2u\,du}{u}$$

$$u = \sqrt{x-4}$$

$$x = u^2+4$$

$$dx = 2u\,du$$

$$= 2\int u^2\,du + 8\int du = \frac{2}{3}u^3 + 8u + C$$

$$= \frac{2u}{3}(u^2+12) + C = \frac{2}{3}(x+8).\sqrt{x-4} + C$$

Ör $I=\int \frac{\sqrt{x+4}}{x}\,dx=\int \frac{u}{u^2-4}\cdot 2u\,du=\int \frac{2u^2}{u^2-4}\,du$

$u=\sqrt{x+4}$ $\quad =2\int du+8\int \frac{du}{u^2-4}=2\int du+2\int \frac{du}{u-2}-2\int\frac{du}{u+2}$

$x=u^2-4$ $\quad =2u+2\ln|u-2|-2\ln|u+2|+C$

$dx=2u\,du$

$=2u+2\ln\left|\frac{u-2}{u+2}\right|+C=2\sqrt{x+4}+2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|+C$

Ör $I=\int x^3\cdot\sqrt{x^2+6}\cdot dx=\int x^2\cdot\sqrt{x^2+6}\cdot x\,dx$

$u=\sqrt{x^2+6}$ $\quad =\int(u^2-6)\cdot u\cdot u\,du=\int(u^4-6u^2)\,du$

$u^2=x^2+6$ $\quad =\frac{u^5}{5}-2u^3+C$

$2u\,du=2x\,dx$ $\quad =\frac{1}{5}(x^2+6)^{5/2}-2(x^2+6)^{3/2}+C$

$u\,du=x\,dx$

Ör $I=\int x\cdot\sqrt{4x^2+9}\cdot dx=\int u\cdot\frac{u\,du}{4}=\frac{1}{4}\int u^2\,du$

$u=\sqrt{4x^2+9}$ $\quad =\frac{u^3}{12}+C=\frac{1}{12}(4x^2+9)^{3/2}+C$

$u^2=4x^2+9$

$2u\,du=8x\,dx$

$x\,dx=\frac{u\,du}{4}$

Ör $I=\int x^2\cdot\sqrt{x+3}\cdot dx=\int(u^2-3)^2\cdot u\cdot 2u\,du$

$=\int(u^4-6u^2+9)\cdot 2u^2\cdot du$

$u=\sqrt{x+3}$ $\quad =\int(2u^6-12u^4+18u^2)\,du$

$x=u^2-3$

$dx=2u\,du$ $\quad =\frac{2}{7}u^7-\frac{12}{5}u^5+\frac{18}{3}u^3+C$

$=\frac{2}{7}(x+3)^{7/2}-\frac{12}{5}(x+3)^{5/2}+6(x+3)^{3/2}+C$

Ör $I=\int\frac{1+\ln x}{3+x\cdot\ln x}\,dx=\int\frac{du}{u}=\ln|u|+C$

$u=3+x\ln x$ $\quad =\ln|3+x\ln x|+C$

$du=(1+\ln x)\cdot dx$

Üstel fonksiyonların integrali

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C \qquad a \in \mathbb{R}^+,\ a \neq 1$$

$$\int e^u du = e^u + C \qquad \int a^u \cdot du = \frac{a^u}{\ln a} + C$$

Ör $I = \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int 2e^u du = 2e^u + C$

$= 2e^{\sqrt{x}} + C$

$u = \sqrt{x} \to du = \frac{dx}{2\sqrt{x}}$

Ör $I = \int \frac{3dx}{\sqrt{e^x+9}} = \int \frac{3e^x dx}{\sqrt{e^x+9} \cdot e^x} = \int \frac{6u\,du}{u \cdot (u^2-9)} = \int \frac{6du}{u^2-9}$

$u = \sqrt{e^x+9}$ $\qquad = \int \frac{du}{u-3} - \int \frac{du}{u+3} = \ln|u-3| - \ln|u+3| + C$

$u^2 - 9 = e^x$ $\qquad = \ln\left|\frac{u-3}{u+3}\right| + C = \ln \frac{\sqrt{e^x+9} - 3}{\sqrt{e^x+9} + 3} + C$

$2u\,du = e^x dx$

Ör $I = \int \frac{e^x dx}{\sqrt[3]{e^x+8}} = \int \frac{3u^2 du}{u} = 3\int u\,du = \frac{3}{2}u^2 + C$

$= \frac{3}{2}(e^x+8)^{2/3} + C$

$u = \sqrt[3]{e^x+8}$

$e^x = u^3 - 8$

$e^x dx = 3u^2 du$

Ör $I = \int \sqrt{e^x+4}\,.dx$ integralini rasyonelleştirerek çöz.

$u = \sqrt{e^x+4}$

$u^2 - 4 = e^x$

$2u\,du = e^x dx$

$I = \int \sqrt{e^x+4} \cdot \frac{e^x dx}{e^x} = \int u \cdot \frac{2u\,du}{u^2-4}$

$= \int \frac{2u^2}{u^2-4} du = 2\int du + 8\int \frac{du}{u^2-4}$

$I = 2\int du + 2\int \frac{du}{u-2} - 2\int \frac{du}{u+2} = 2u + 2\ln|u-2| - 2\ln|u+2| + C$

$= 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C$

$= 2\sqrt{e^x+4} + 2\ln \frac{\sqrt{e^x+4} - 2}{\sqrt{e^x+4} + 2} + C$

Trigonometrik Fonksiyonların İntegrali

$(\cos x)' = -\sin x \longrightarrow \int \sin x\, dx = -\cos x + c$

$(\sin x)' = \cos x \longrightarrow \int \cos x\, dx = \sin x + c$

$(\tan x)' = \sec^2 x \longrightarrow \int \sec^2 x\, dx = \tan x + c$

$(\cot x)' = -\csc^2 x \longrightarrow \int \csc^2 x\, dx = -\cot x + c$

$(\sec x)' = \sec x.\tan x \longrightarrow \int \sec x.\tan x.dx = \sec x + c$

$(\csc x)' = -\csc x.\cot x \longrightarrow \int \csc x.\cot x.dx = -\csc x + c$

$\int \sin u.du = -\cos u + c$	$\int \sec u.\tan u.du = \sec u + c$
$\int \cos u.du = \sin u + c$	$\int \csc u.\cot u.du = -\csc u + C$
$\int \sec^2 u.du = \tan u + c$	
$\int \csc^2 u.du = -\cot u + c$	

Ör $I = \int \sec x.dx = \int \sec x.\dfrac{\sec x + \tan x}{\sec x + \tan x}.dx = \int \dfrac{du}{u} = \ln|u| + c$

$u = \sec x + \tan x$ $\qquad = \ln|\sec x + \tan x| + C$

$du = \sec x.(\sec x + \tan x).dx$

Ör $I = \int \csc x.dx = \int \csc x.\dfrac{\csc x + \cot x}{\csc x + \cot x}.dx = -\int \dfrac{du}{u} = -\ln|u| + c$

$u = \csc x + \cot x$ $\qquad = -\ln|\csc x + \cot x| + c$

$du = -\csc x.(\csc x + \cot x).dx$

Ör $\int \tan^2 x\, dx = \int (\sec^2 x - 1) dx$

$= \int \sec^2 x dx - \int dx$

$= \tan x - x + c$

Ör $\int \cot^2 x\, dx = \int (\csc^2 x - 1) dx$

$= \int \csc^2 x dx - \int dx$

$= -\cot x - x + c$

ör $\int \tan x.dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{du}{u} = -\ln|u|+c = -\ln|\cos x|+c$

$= \ln|\sec x| + C$

$u=\cos x \rightarrow du = -\sin x.dx$

ör $\int \cot x.dx = \int \frac{\cos x}{\sin x}dx = \int \frac{du}{u} = \ln|u|+c = \ln|\sin x|+c$

$u=\sin x \rightarrow du = \cos x.dx$

ör $\int \cos^2 x.dx = \int \frac{1+\cos 2x}{2}dx = \frac{x}{2} + \frac{\sin 2x}{4} + c$

ör $\int \sin^2 x.dx = \int \frac{1-\cos 2x}{2}dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

ör $\int \cos^2 x.\sin^2 x.dx = \int \frac{1+\cos 2x}{2}.\frac{1-\cos 2x}{2}.dx = \frac{1}{4}\int(1-\cos^2 2x)dx$

$= \frac{1}{4}\int\left(1 - \frac{1+\cos 4x}{2}\right)dx = \frac{1}{8}\int(1-\cos 4x).dx = \frac{x}{8} - \frac{\sin 4x}{32} + C$

ör $\int \sin^4 x.dx = \int\left(\frac{1-\cos 2x}{2}\right)^2.dx = \frac{1}{4}\int(1-2\cos 2x+\cos^2 2x).dx$

$= \frac{1}{4}\int\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right).dx = \frac{3}{8}\int dx - \frac{1}{2}\int\cos 2x dx + \frac{1}{8}\int \cos 4x dx$

$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

ör $I = \int \cos^3 x.dx = \int \cos^2 x.\cos x.dx = \int(1-\sin^2 x).\cos x.dx = \int(1-u^2).du$

$= u - \frac{u^3}{3} + c = \sin x - \frac{1}{3}\sin^3 x + C$

$u=\sin x \rightarrow du = \cos x.dx$

ör $I = \int \sin^5 x.dx = \int \sin^4 x.\sin x.dx = \int(1-\cos^2 x)^2.\sin x.dx$

$u = \cos x$

$du = -\sin x.dx$

$= -\int(1-u^2)^2 du = -\int(u^4-2u^2+1)du$

$= -\frac{u^5}{5} + \frac{2}{3}u^3 - u + C = -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

ör $I = \int \cos^3 x.\sin^2 x.dx = \int \cos^2 x.\sin^2 x.\cos x.dx = \int(1-\sin^2 x).\sin^2 x.\cos x.dx$

$u = \sin x$

$du = \cos x.dx$

$= \int(1-u^2).u^2.du = \int(u^2-u^4)du = \frac{u^3}{3} - \frac{u^5}{5} + C$

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

Or

$I=\int \cos^5 x \cdot \sin^3 x \cdot dx = \int \cos^5 x \cdot \sin^2 x \cdot \sin x \cdot dx$

$u=\cos x \qquad = \int \cos^5 x \cdot (1-\cos^2 x)\cdot \sin x \cdot dx = \int u^5 \cdot (1-u^2)\cdot(-du)$

$du=-\sin x \cdot dx \qquad = \int (u^7-u^5)\,du = \frac{u^8}{8}-\frac{u^6}{6}+C = \frac{\cos^8 x}{8}-\frac{\cos^6 x}{6}+C$

Or

$I=\int \sin^3 x \cdot \cos x \cdot dx = \int u^3 du = \frac{u^4}{4}+C = \frac{1}{4}\sin^4 x + C$

$u=\sin x \rightarrow du=\cos x \cdot dx$

Or

$I=\int \frac{\sin^2 x}{\cos x}dx = \int \frac{1-\cos^2 x}{\cos x}dx = \int \sec x\, dx - \int \cos x\, dx$

$= \ln|\sec x + \tan x| - \sin x + C$

Or

$I=\int \frac{\cos^2 x}{\sin x}dx = \int \frac{1-\sin^2 x}{\sin x}dx = \int \csc x - \int \sin x\, dx$

$= -\ln|\csc x + \cot x| + \cos x + C$

Or

$I=\int \frac{\sin x}{\cos^3 x}dx = -\int \frac{du}{u^3} = -\int u^{-3}du = \frac{1}{2u^2}+C = \frac{1}{2\cos^2 x}+C$

$u=\cos x \rightarrow du=-\sin x\, dx \qquad = \frac{1}{2}\sec^2 x + C$

Or

$I=\int \frac{dx}{\cos^2 x \cdot \sin^2 x} = \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \cdot \sin^2 x}dx = \int \frac{dx}{\sin^2 x} + \int \frac{dx}{\cos^2 x}$

$= \int \csc^2 x\, dx - \int \sec^2 x\, dx = -\cot x + \tan x + C$

Or

$I=\int \frac{\sin^3 x}{\cos^2 x}dx = \int \frac{\sin^2 x}{\cos^2 x}\sin x\, dx = \int \frac{1-\cos^2 x}{\cos^2 x}\sin x\, dx = \int \frac{u^2-1}{u^2}du$

$u=\cos x \qquad = \int du - \int \frac{du}{u^2} = u + \frac{1}{u} + C = \cos x + \sec x + C$

$du=-\sin x\, dx$

Or

$I=\int \frac{\cos^3 x}{\sin^5 x}dx = \int \frac{\cos^2 x}{\sin^5 x}\cos x\, dx = \int \frac{1-\sin^2 x}{\sin^5 x}\cos x\, dx$

$u=\sin x \qquad = \int \frac{1-u^2}{u^5}du = \int (u^{-5}-u^{-3})\,du$

$du=\cos x\, dx$

$= -\frac{1}{4u^4}+\frac{1}{2u^2}+C = \frac{1}{2}\csc^2 x - \frac{1}{4}\csc^4 x + C$

Ör
$$I = \int \frac{\sin x}{\sqrt{1+\cos x}}\,dx = -\int \frac{2u\,du}{u} = -2\int du = -2u + C$$
$$= -2\sqrt{1+\cos x} + C$$
$u = \sqrt{1+\cos x}$

$u^2 - 1 = \cos x \rightarrow 2u\,du = -\sin x.dx$

Ör
$$I = \int (1-3\sin^2 x).\cos x.dx = \int (1-3u^2)du = u - u^3 + C$$
$u = \sin x \rightarrow du = \cos x\,dx$
$$= u(1-u^2) + C = \sin x.\cos^2 x + C$$

Ör
$$I = \int \frac{\sin^3 x}{2+\cos x}dx = \int \frac{\sin^2 x}{2+\cos x}.\sin x.dx = \int \frac{1-\cos^2 x}{2+\cos x}.\sin x\,dx$$
$u = \cos x$

$du = -\sin x.dx$
$$= \int \frac{u^2-1}{u+2}du = \int \left(u - 2 + \frac{3}{u+2}\right)du$$
$$= \frac{u^2}{2} - 2u + 3\ln|u+2| + C$$
$$= \frac{1}{2}\cos^2 x - 2\cos x + 3\ln(2+\cos x) + C$$

Ör
$$I = \int \frac{\sin^4 x}{\cos^2 x}dx = \int \frac{(1-\cos^2 x)^2}{\cos^2 x}dx = \int \frac{1-2\cos^2 x + \cos^4 x}{\cos^2 x}dx$$
$$= \int \sec^2 x\,dx - 2\int dx + \int \cos^2 x\,dx = \tan x - 2x + \int \frac{1+\cos 2x}{2}dx$$
$$= \tan x - 2x + \frac{x}{2} + \frac{\sin 2x}{4} + C = \tan x - \frac{3x}{2} + \frac{\sin 2x}{4} + C$$

Ör
$$I = \int \frac{\sin 2x}{5+\sin^2 x}dx = \int \frac{du}{u} = \ln|u| + C = \ln(5+\sin^2 x) + C$$
$u = 5 + \sin^2 x$

$du = 2\sin x\cos x\,dx$

$= \sin 2x.dx$

Ör
$$I = \int (2x+3).\sin(x^2+3x+5).dx = \int \sin u.du$$
$u = x^2+3x+5$

$du = (2x+3).dx$
$$= -\cos x + C$$
$$= -\cos(x^2+3x+5) + C$$

Ör
$$I = \int \frac{dx}{x.\sin^2(\ln x)} \quad I = \int \frac{du}{\sin^2 u} = \int \csc^2 u.du = -\cot u + C$$
$$= -\cot(\ln x) + C$$
$u = \ln x \rightarrow du = \frac{dx}{x}$

Ör
$$I = \int e^{\sin x + \ln(\cos x)}.dx = \int e^{\sin x}.e^{\ln(\cos x)}.dx$$
$$= \int e^{\sin x}.\cos x.dx = \int e^u du = e^u + C$$
$u = \sin x$

$du = \cos x.dx$
$$= e^{\sin x} + C$$

01) $I=\int e^{\tan x}\cdot\sec^2x\cdot dx=\int e^u du=e^u+c=e^{\tan x}+c$

$u=\tan x\rightarrow du=\sec^2x\,dx$

01) $I=\int\frac{\sec^2x\cdot dx}{\sqrt{3\tan x+5}}=\int\frac{\frac{2udu}{3}}{u}=\frac{2}{3}\int du=\frac{2u}{3}+c$

$=\frac{2}{3}\cdot\sqrt{3\tan x+5}+c$

$u=\sqrt{3\tan x+5}$

$u^2-5=3\tan x$

$2udu=3\sec^2x\,dx$

01) $I=\int\frac{dx}{\tan x\cdot\sqrt{1+\ln(\sin x)}}=\int\frac{2udu}{u}=2\int du=2u+c$

$=2\sqrt{1+\ln(\sin x)}+c$

$u=\sqrt{1+\ln(\sin x)}$

$u^2-1=\ln(\sin x)$

$2udu=\frac{\cos x}{\sin x}dx=\frac{dx}{\tan x}$

01) $I=\int\frac{\sin^3x}{\sqrt{\cos x}}dx$

$u=\sqrt{\cos x}$

$u^2=\cos x$

$2udu=-\sin x\,dx$

$I=\int\frac{\sin^2x}{\sqrt{\cos x}}\sin x\,dx=\int\frac{1-\cos^2x}{\sqrt{\cos x}}\cdot\sin x\,dx$

$=\int\frac{u^4-1}{u}\cdot 2udu=2\int u^4du-2\int du=\frac{2}{5}u^5-2u+c$

$=\frac{2}{5}(\cos x)^{5/2}-2\sqrt{\cos x}+c$

01) $I=\int\sqrt{1+\sin6x}\cdot dx$

$(\cos3x+\sin3x)^2=\underbrace{\cos^23x+\sin^23x}+\underbrace{2\cos3x\sin3x}=1+\sin6x$

$I=\int(\cos3x+\sin3x)\,dx=\frac{\sin3x-\cos3x}{3}+c$

01) $I=\int\sqrt{1-\cos4x}\,dx=\int\sqrt{2\sin^22x}\,dx=\sqrt{2}\int\sin2x\,dx=-\frac{\sqrt{2}}{2}\cos2x+c$

01) $I=\int\sqrt{1+\cos6x}\,dx=\int\sqrt{2\cos^23x}\,dx=\sqrt{2}\int\cos3x\,dx=\frac{\sqrt{2}}{3}\sin3x+c$

01) $I=\int\sqrt{1-\sin8x}\,dx=\int(\cos4x-\sin4x)\,dx=\frac{\sin4x+\cos4x}{4}+c$

$(\cos4x-\sin4x)^2=\cos^24x+\sin^24x-2\cos4x\sin4x$

$=1-\sin8x$

Ör $I=\int \sin 2x \cdot \cos 5x \cdot dx=\int \frac{\sin(2x+5x)+\sin(2x-5x)}{2}dx$

$=\int \frac{\sin 7x-\sin 3x}{2}dx=-\frac{\cos 7x}{14}+\frac{\cos 3x}{6}+C$

Ör $I=\int \cos 3x \cdot \sin x \cdot dx=\int \frac{\sin(3x+x)-\sin(3x-x)}{2}\cdot dx$

$=\int \frac{\sin 4x-\sin 2x}{2}dx=-\frac{\cos 4x}{8}+\frac{\cos 2x}{4}+C$

Ör $I=\int \cos 5x \cdot \cos 3x \cdot dx=\int \frac{\cos(5x+3x)+\cos(5x-3x)}{2}dx$

$=\int \frac{\cos 8x+\cos 2x}{2}dx=\frac{\sin 8x}{16}+\frac{\sin 2x}{4}+C$

Ör $I=\int \sin 5x \cdot \sin 3x \cdot dx=\int \frac{\cos(5x-3x)-\cos(5x+3x)}{2}dx$

$=\int \frac{\cos 2x-\cos 8x}{2}dx=\frac{\sin 2x}{4}-\frac{\sin 8x}{16}+C$

Ör $I=\int \frac{5.\cos x.dx}{\sin^2 x+\sin x-6}=\int \frac{5\,du}{u^2+u-6}=\int \frac{du}{u-2}-\int \frac{du}{u+3}$

$u=\sin x$
$du=\cos x.dx$

$=\ln\left|\frac{u-2}{u+3}\right|+C=\ln\left(\frac{2-\sin x}{3+\sin x}\right)+C$

Ör $I=\int \sec^4 x\,dx=\int \sec^2 x.\sec^2 x.dx=\int (1+\tan^2 x).\sec^2 x.dx$

$u=\tan x$
$du=\sec^2 x\,dx$

$=\int (1+u^2)\,du=u+\frac{u^3}{3}+C$

$=\tan x+\frac{1}{3}\tan^3 x+C$

Ör $I=\int \tan^4 x.dx=\int \tan^2 x.\tan^2 x.dx=\int \tan^2 x.(\sec^2 x-1).dx$

$u=\tan x$
$du=\sec^2 x\,dx$

$=\int \tan^2 x.\sec^2 x.dx-\int \tan^2 x.dx$

$=\int u^2 du-\int (\sec^2 x-1)dx=\frac{u^3}{3}-\int \sec^2 x\,dx+\int dx$

$=\frac{1}{3}\tan^3 x-\tan x+x+C$

Ör

$$I=\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx=\int \frac{2u\,du}{u}=2\int du=2u+C$$

$u=\sqrt{\tan x}$ $\quad = 2\sqrt{\tan x}+C$

$u^2=\tan x$

$2u\,du=\sec^2 x\,dx$

Ör

$$I=\int \tan^3 x\,.dx=\int \tan x.\tan^2 x.dx=\int \tan x.(\sec^2 x-1).dx$$

$$=\int \tan x.\sec^2 x\,dx-\int \tan x\,dx=\int u\,du-\int \frac{\sin x.dx}{\cos x}$$

$$=\int u\,du+\int \frac{dv}{v}=\frac{u^2}{2}+\ln|v|+C$$

$u=\tan x$ $\quad v=\cos x$

$du=\sec^2 x\,dx$ $\quad dv=-\sin x\,dx$

$$=\frac{1}{2}\tan^2 x+\ln|\cos x|+C$$

Ör

$$I=\int \tan^3 x.\sec x.dx=\int \tan^2 x.\sec x.\tan x.dx$$

$$=\int (\sec^2 x-1).\sec x.\tan x.dx$$

$u=\sec x$

$du=\sec x.\tan x.dx$

$$=\int (u^2-1)\,du=\frac{u^3}{3}-u+C$$

$$=\frac{1}{3}\sec^3 x-\sec x+C$$

Ör

$$I=\int \frac{e^x dx}{e^{2x}+3e^x+2}=\int \frac{du}{u^2+3u+2}=\int \frac{du}{u+1}-\int \frac{du}{u+2}$$

$u=e^x \longrightarrow du=e^x dx$

$$=\ln\left|\frac{u+1}{u+2}\right|+C=\ln\left(\frac{1+e^x}{2+e^x}\right)+C$$

Ör

$$I=\int \frac{2-\sqrt{4-x^2}}{x}\,dx=\int \frac{2-\sqrt{4-x^2}}{x^2}\,x\,dx=-\int \frac{2-u}{4-u^2}.u\,du$$

$u=\sqrt{4-x^2}$

$x^2=4-u^2$

$2x\,dx=-2u\,du$

$x\,dx=-u\,du$

$$=2\int \frac{du}{u+2}-\int du=2\ln|u+2|-u+C$$

$$=2\ln\left(2+\sqrt{4-x^2}\right)-\sqrt{4-x^2}+C$$

Ör

$$y'=\frac{(1-2x)^3.\sqrt{3+y^2}}{y} \quad \text{ise } y=?$$

$v=1-2x$

$dv=-2dx$

$$\frac{y\,dy}{\sqrt{3+y^2}}=(1-2x)^3dx \longrightarrow \int \frac{y\,dy}{\sqrt{3+y^2}}=\int (1-2x)^3dx$$

$$\int \frac{u\,du}{u}=-\frac{1}{2}\int v^3dv$$

$u=\sqrt{3+y^2}\longrightarrow u^2=3+y^2$

$\cancel{2}u\,du=\cancel{2}y\,dy$

$$\sqrt{3+y^2}+\frac{(1-2x)^4}{8}=C$$

Hiperbolik Fonksiyonların integrali

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) + \sinh(x) = e^x$$
$$\cosh(x) - \sinh(x) = e^{-x}$$
$$\overline{\cosh^2(x) - \sinh^2(x) = 1}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{\coth(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$

$$\int \sinh(x).dx = \cosh(x) + C$$

$$\int \cosh(x).dx = \sinh(x) + C$$

$$\int \operatorname{sech}^2(x).dx = \tanh(x) + C$$

$$\int \operatorname{csch}^2(x).dx = -\coth(x) + C$$

$$\int \operatorname{sech}(x).\tanh(x).dx = -\operatorname{sech}(x) + C$$

$$\int \operatorname{csch}(x).\coth(x).dx = -\operatorname{csch}(x) + C$$

Ör

$$I = \int \cosh^3(x).dx = \int \cosh^2(x).\cosh(x).dx$$
$$= \int (1 + \sinh^2(x)).\cosh(x).dx$$
$$= \int (1 + u^2)\,du = u + \frac{u^3}{3} + C$$

$u = \sinh(x)$

$du = \cosh(x).dx$

$$= \sinh(x) + \frac{1}{3}\sinh^3(x) + C$$

$u^2+a^2 \longrightarrow u = a.\tan\theta \ , \ du = a.\sec^2\theta.d\theta$

$u^2-a^2 \longrightarrow u = a.\sec\theta \ , \ du = a.\sec\theta.\tan\theta.d\theta$

$a^2-u^2 \longrightarrow u = a\sin\theta \ , \ du = a.\cos\theta.d\theta$

Ör

$$I = \int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{\frac{3}{2}\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}} = \int \frac{\frac{3}{2}\cancel{\cos\theta}\, d\theta}{3\cancel{\cos\theta}}$$

$2x = 3\sin\theta$

$x = \frac{3}{2}\sin\theta$

$dx = \frac{3}{2}\cos\theta\, d\theta$

$\sin\theta = \frac{2x}{3}$

$$= \frac{1}{2}\int d\theta = \frac{\theta}{2} + c$$

$$= \frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + c$$

Ör

$$I = \int \frac{dx}{4x^2+9} = \int \frac{3/2\sec^2\theta\, d\theta}{9\tan^2\theta+9} = \frac{1}{6}\int \frac{\sec^2\theta.d\theta}{\tan^2\theta+1}$$

$2x = 3\tan\theta$

$x = \frac{3}{2}\tan\theta$

$dx = 3/2\sec^2\theta\, d\theta$

$\tan\theta = 2x/3$

$$= \frac{1}{6}\int d\theta = \frac{\theta}{6} + c$$

$$= \frac{1}{6}\arctan\left(\frac{2x}{3}\right) + c$$

Ör

$$I = \int \frac{dx}{x.\sqrt{4x^2-9}} = \int \frac{\cancel{\frac{3}{2}\sec\theta}.\tan\theta.d\theta}{\cancel{\frac{3}{2}\sec\theta}.\sqrt{9\sec^2\theta-9}}$$

$2x = 3\sec\theta$

$x = \frac{3}{2}\sec\theta$

$dx = \frac{3}{2}\sec\theta.\tan\theta.d\theta$

$\sec\theta = 2x/3$

$$= \int \frac{\tan\theta\, d\theta}{3\sqrt{\sec^2\theta-1}} = \frac{1}{3}\int d\theta$$

$$= \frac{\theta}{3} + c = \frac{1}{3}\operatorname{arcsec}\left(\frac{2x}{3}\right) + c$$

$$(\arctan u)' = \frac{u'}{u^2+1}$$

$$(\arcsin u)' = \frac{u'}{\sqrt{1-u^2}}$$

$$(\operatorname{arcsec} x)' = \frac{u'}{|u|.\sqrt{u^2-1}}$$

$\int \frac{dx}{ax^2+bx+c}$ tipi integraller $\quad \Delta = b^2-4ac$

① $\Delta < 0$ ise

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right)+C$$

② $\Delta = 0$ ise

$$\int \frac{dx}{x^2+2x+1} = \int \frac{dx}{(x+1)^2} = -\frac{1}{x+1}+C$$

③ $\Delta > 0$ ise

$$\int \frac{dx}{x^2+3x+2} = \int \left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx = \ln\left|\frac{x+1}{x+2}\right|+C$$

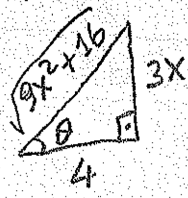
Ör

$$I = \int \frac{dx}{\sqrt{9x^2+16}} = \int \frac{\frac{4}{3}\sec^2\theta\, d\theta}{\sqrt{16\tan^2\theta+16}} = \frac{1}{3}\int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \frac{1}{3}\int \sec\theta . d\theta$$

$3x = 4\tan\theta \longrightarrow \tan\theta = \frac{3x}{4}$

$x = \frac{4}{3}\tan\theta$

$dx = \frac{4}{3}\sec^2\theta\, d\theta$



$$= \frac{1}{3}\ln|\sec\theta+\tan\theta| + C_1$$
$$= \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+16}}{4}+\frac{3x}{4}\right| + C_1$$
$$= \frac{1}{3}\ln\left|3x+\sqrt{9x^2+16}\right| + C$$

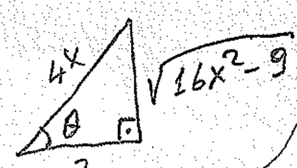
Ör

$$I = \int \frac{dx}{\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}} = \frac{1}{4}\int \sec\theta\, d\theta$$

$4x = 3\sec\theta \longrightarrow \sec\theta = \frac{4x}{3}$

$x = \frac{3}{4}\sec\theta$

$dx = \frac{3}{4}\sec\theta . \tan\theta . d\theta$



$$= \frac{1}{4}\ln|\sec\theta+\tan\theta| + C_1$$
$$= \frac{1}{4}\ln\left|\frac{4x}{3}+\frac{\sqrt{16x^2-9}}{3}\right| + C_1$$
$$= \frac{1}{4}\ln\left|4x+\sqrt{16x^2-9}\right| + C$$

Ör

$$I = \int \frac{x^2-1}{x^2+4}dx = \int dx - 5\int \frac{dx}{x^2+4} = x - \frac{5}{2}\arctan(x/2) + C$$

Ör

$$I = \int \frac{x^2 dx}{\sqrt{9-4x^6}} = \frac{1}{6}\int \frac{du}{\sqrt{9-u^2}} = \frac{1}{6}\int \frac{3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}} = \frac{1}{6}\int d\theta$$

$u = 2x^3$

$du = 6x^2 dx$

$x^2 dx = \frac{du}{6}$

$u = 3\sin\theta$

$du = 3\cos\theta\, d\theta$

$\sin\theta = u/3$

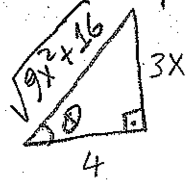
$$= \frac{\theta}{6} + C = \frac{1}{6}\arcsin\left(\frac{u}{3}\right) + C$$
$$= \frac{1}{6}\arcsin\left(\frac{2x^3}{3}\right) + C$$

Ör $I = \int \frac{dx}{x^2 \cdot \sqrt{9x^2+16}} = \int \frac{\frac{4}{3}\sec^2\theta\, d\theta}{\frac{16}{9}\tan^2\theta \cdot \sqrt{16\tan^2\theta + 16}} = \frac{3}{16}\int \frac{\sec\theta\, d\theta}{\tan^2\theta}$

$3x = 4\tan\theta \longrightarrow \tan\theta = \frac{3x}{4}$

$x = \frac{4}{3}\tan\theta$

$dx = \frac{4}{3}\sec^2\theta . d\theta$



$u = \sin\theta \rightarrow du = \cos\theta\, d\theta$

$= \frac{3}{16}\int \frac{\cos\theta\, d\theta}{\sin^2\theta} = \frac{3}{16}\int \frac{du}{u^2} = -\frac{3}{16u} + c$

$= -\frac{3}{16\sin\theta} + c$

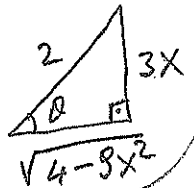
$= -\frac{\sqrt{9x^2+16}}{16x} + C$

Ör $I = \int \frac{dx}{x^2 \cdot \sqrt{4-9x^2}} = \int \frac{\frac{2}{3}\cos\theta\, d\theta}{\frac{4}{9}\sin^2\theta \cdot \sqrt{4 - 4\sin^2\theta}} = \frac{3}{4}\int \frac{d\theta}{\sin^2\theta} = \frac{3}{4}\int \csc^2\theta\, d\theta$

$3x = 2\sin\theta \longrightarrow \sin\theta = \frac{3x}{2}$

$x = \frac{2}{3}\sin\theta$

$dx = \frac{2}{3}\cos\theta\, d\theta$



$= -\frac{3}{4}\cot\theta + C$

$= -\frac{3}{4}\,\frac{\sqrt{4-9x^2}}{3x} + C$

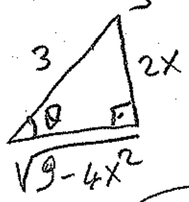
$= -\frac{\sqrt{4-9x^2}}{4x} + C$

Ör $I = \int \frac{x^2\, dx}{\sqrt{9-4x^2}} = \int \frac{\frac{9}{4}\sin^2\theta . \frac{3}{2}\cos\theta\, d\theta}{\sqrt{9 - 9\sin^2\theta}} = \frac{9}{8}\int \sin^2\theta\, d\theta$

$2x = 3\sin\theta \longrightarrow \sin\theta = \frac{2x}{3}$

$x = \frac{3}{2}\sin\theta$

$dx = \frac{3}{2}\cos\theta\, d\theta$



$\sin 2\theta = 2\cos\theta\sin\theta = 2\,\frac{\sqrt{9-4x^2}}{3}\,\frac{2x}{3}$

$= \frac{4x}{9}\sqrt{9-4x^2}$

$= \frac{9}{16}\int (1 - \cos 2\theta)\, d\theta$

$= \frac{9\theta}{16} - \frac{9\sin 2\theta}{32} + C$

$= \frac{9}{16}\arcsin\left(\frac{2x}{3}\right) - \frac{x}{8}\cdot\sqrt{9-4x^2} + C$

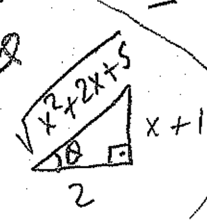
Ör $\int \frac{2x+3}{x^2+2x+5}\, dx = \int \frac{2x+3}{(x+1)^2+4}\, dx = \int \frac{4\tan\theta + 1}{4\tan^2\theta + 4}\, 2\sec^2\theta\, d\theta$

$x + 1 = 2\tan\theta$

$x = 2\tan\theta - 1$

$dx = 2\sec^2\theta\, d\theta$

$\tan\theta = \frac{x+1}{2}$



$= 2\int \tan\theta\, d\theta + \frac{1}{2}\int d\theta$

$= 2\ln|\sec\theta| + \frac{1}{2}\theta + c_1$

$= \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + 2\ln\frac{\sqrt{x^2+2x+5}}{2} + c_1$

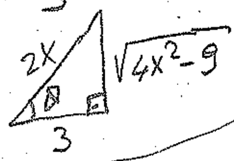
$= \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + \ln(x^2+2x+5) + C$

$$I=\int\frac{\sqrt{4x^2-9}}{x}\,dx=\int\frac{\sqrt{9\sec^2\theta-9}}{\cancel{\tfrac{3}{2}\sec\theta}}\cdot\cancel{\tfrac{3}{2}\sec\theta}\tan\theta\,d\theta$$

$2x=3\sec\theta\longrightarrow\sec\theta=\frac{2x}{3}$

$x=\frac{3}{2}\sec\theta$

$dx=\frac{3}{2}\sec\theta\tan\theta\,d\theta$



$$=3\int\tan^2\theta\,d\theta=3\int(\sec^2\theta-1)\,d\theta$$

$$=3\tan\theta-3\theta+C$$

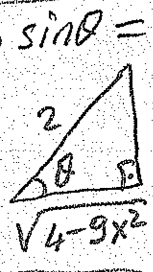
$$=\sqrt{4x^2-9}-3\,\mathrm{arcsec}\left(\frac{2x}{3}\right)+C$$

$$I=\int\frac{\sqrt{4-9x^2}}{x}\,dx=\int\frac{\sqrt{4-4\sin^2\theta}}{\cancel{\tfrac{2}{3}}\sin\theta}\cdot\cancel{\tfrac{2}{3}}\cos\theta\,d\theta=2\int\frac{\cos^2\theta}{\sin\theta}\,d\theta$$

$3x=2\sin\theta\longrightarrow\sin\theta=\frac{3x}{2}$

$x=\frac{2}{3}\sin\theta$

$dx=\frac{2}{3}\cos\theta\,d\theta$



$$=2\int\frac{1-\sin^2\theta}{\sin\theta}\,d\theta=2\int\csc\theta\,d\theta-2\int\sin\theta\,d\theta$$

$$=-2\ln|\csc\theta+\cot\theta|+2\cos\theta+C$$

$$=\sqrt{4-9x^2}-2\ln\left|\frac{2}{3x}+\frac{\sqrt{4-9x^2}}{3x}\right|+C$$

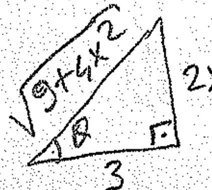
$$=\sqrt{4-9x^2}+2\ln\left|\frac{3x}{2+\sqrt{4-9x^2}}\right|+C$$

$$I=\int\frac{dx}{x\cdot\sqrt{9+4x^2}}$$

$2x=3\tan\theta\longrightarrow\tan\theta=\frac{2x}{3}$

$x=\frac{3}{2}\tan\theta$

$dx=\frac{3}{2}\sec^2\theta\,d\theta$



$$I=\int\frac{\cancel{\tfrac{3}{2}}\sec^2\theta\,d\theta}{\cancel{\tfrac{3}{2}}\tan\theta\cdot\sqrt{9+9\tan^2\theta}}$$

$$=\frac{1}{3}\int\frac{\sec\theta}{\tan\theta}\,d\theta=\frac{1}{3}\int\csc\theta\,d\theta$$

$$I=-\frac{1}{3}\ln|\csc\theta+\cot\theta|+C$$

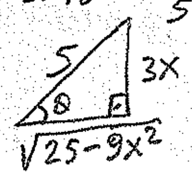
$$=-\frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}}{2x}+\frac{3}{2x}\right|+C=\frac{1}{3}\ln\left|\frac{2x}{3+\sqrt{9+4x^2}}\right|+C$$

$$I=\int\frac{2x+3}{\sqrt{25-9x^2}}\,dx=\int\frac{\frac{10}{3}\sin\theta+3}{\cancel{\sqrt{25-25\sin^2\theta}}}\cdot\frac{5}{3}\cancel{\cos\theta}\,d\theta=\int\left(\frac{10}{9}\sin\theta+1\right)d\theta$$

$3x=5\sin\theta\longrightarrow\sin\theta=\frac{3x}{5}$

$x=\frac{5}{3}\sin\theta$

$dx=\frac{5}{3}\cos\theta\,d\theta$



$$=-\frac{10}{9}\cos\theta+\theta+C$$

$$=\arcsin\left(\frac{3x}{5}\right)-\frac{2}{9}\sqrt{25-9x^2}+C$$

$$I=\int\frac{dx}{x^2-6x+25}=\int\frac{dx}{(x-3)^2+16}=\int\frac{4\sec^2\theta\,d\theta}{16\tan^2\theta+16}=\frac{1}{4}\int d\theta=\frac{\theta}{4}+C$$

$x-3=4\tan\theta$

$dx=4\sec^2\theta\,d\theta$

$$=\frac{1}{4}\arctan\left(\frac{x-3}{4}\right)+C$$

Ör $I=\int \frac{dx}{e^x+e^{-x}}=\int \frac{e^x dx}{e^{2x}+1}=\int \frac{du}{u^2+1}=\arctan u+C$
$=\arctan(e^x)+C$

$u=e^x \rightarrow du=e^x dx$

Ör $I=\int \frac{dx}{\sqrt{e^{2x}-1}}=\int \frac{e^{2x}dx}{e^{2x}.\sqrt{e^{2x}-1}}=\int \frac{u\,du}{(u^2+1).u}=\int \frac{du}{u^2+1}$

$u=\sqrt{e^{2x}-1}$ $\qquad =\arctan u+C$
$u^2+1=e^{2x}$
$2u\,du=2e^{2x}dx$ $\qquad =\arctan\sqrt{e^{2x}-1}+C$
$u\,du=e^{2x}dx$

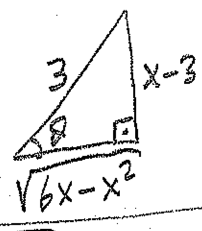
Ör $I=\int \frac{dx}{\sqrt{5+4x-x^2}}=\int \frac{dx}{\sqrt{9-(x-2)^2}}=\int \frac{3\cos\theta d\theta}{\sqrt{9-9\sin^2\theta}}=\int d\theta=\theta+C$
$=\arcsin\left(\frac{x-2}{3}\right)+C$

$x-2=3\sin\theta \longrightarrow \sin\theta=\frac{x-2}{3}$
$dx=3\cos\theta d\theta$

Ör $I=\int \frac{2x-1}{\sqrt{6x-x^2}}dx=\int \frac{2(x-3)+5}{\sqrt{9-(x-3)^2}}.dx=\int \frac{6\sin\theta+5}{\cancel{\sqrt{9-9\sin^2\theta}}}.\cancel{3\cos\theta}\, d\theta$

$=\int(6\sin\theta+5)d\theta=5\theta-6\cos\theta+C$

$x-3=3\sin\theta$
$dx=3\cos\theta d\theta$
$\sin\theta=\frac{x-3}{3}$



$=5\arcsin\left(\frac{x-3}{3}\right)-2.\sqrt{6x-x^2}+C$

Ör $I=\int \frac{1+\sqrt{\cos x}}{\sin x}dx=\int \frac{1+\sqrt{\cos x}}{\sin^2 x}.\sin x dx=\int \frac{1+\sqrt{\cos x}}{1-\cos^2 x}.\sin x dx$

$u=\sqrt{\cos x}$ $\qquad =\int \frac{1+u}{1-u^4}.(-2u).du=\int \frac{2u}{(u-1)(u^2+1)}du$
$u^2=\cos x$
$2u\,du=-\sin x.dx$ $\qquad =\int\left(\frac{1}{u-1}-\frac{1}{2}\frac{2u}{u^2+1}+\frac{1}{u^2+1}\right)du$
$\sin x dx=-2u\,du$

$=\ln|u-1|-\frac{1}{2}\ln(u^2+1)+\arctan u+C$

$=\ln\frac{|u-1|}{\sqrt{u^2+1}}+\arctan u+C=\ln\left(\frac{1-\sqrt{\cos x}}{\sqrt{1+\cos x}}\right)+\arctan\sqrt{\cos x}+C$

Ör $I=\int \frac{5x^2+3}{(2x+1)(x^2+4)}dx$ Kesirlerine ayırarak çöz.

$\frac{5x^2+3}{(2x+1)(x^2+4)}=\frac{a}{2x+1}+\frac{bx+c}{x^2+4}$
$\qquad\qquad (x^2+4) \qquad (2x+1)$

$a=1$ $\quad c=-1$
$b=2$

$$I=\int\frac{dx}{2x+1}+\int\frac{2x-1}{x^2+4}dx=\frac{1}{2}\int\frac{2dx}{2x+1}+\int\frac{2x\,dx}{x^2+4}-\int\frac{dx}{x^2+4}$$
$$=\frac{1}{2}\ln|2x+1|+\ln(x^2+4)-\frac{1}{2}\arctan(x/2)+C$$
$$=\ln\left(\sqrt{2x+1}\cdot(x^2+4)\right)-\frac{1}{2}\arctan(x/2)+C$$

Ör $I=\int\frac{x^3+1}{x(x-1)^3}dx$ kesirlerine ayırarak çöz.

$$\frac{x^3+1}{x(x-1)^3}=\frac{a}{x}+\frac{b}{x-1}+\frac{c}{(x-1)^2}+\frac{d}{(x-1)^3}$$ $a=-1$ $c=1$ $b=2$ $d=2$

$$I=\int\left(-\frac{1}{x}+\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}\right).dx$$
$$=-\ln|x|+2\ln|x-1|-\frac{1}{x-1}-\frac{1}{(x-1)^2}+C=\ln\frac{(x-1)^2}{|x|}-\frac{x}{(x-1)^2}+C$$

Ör $I=\int\frac{x(x+4)}{x^3-8}dx$ Kesirlerine ayırarak çöz

$x^3-a^3=(x-a)(x^2+ax+a^2)$

$$\frac{x(x+4)}{x^3-8}=\frac{a}{x-2}+\frac{bx+c}{x^2+2x+4}=\frac{(a+b)x^2+(2a-2b+c)x+(4a-2c)}{x^3-8}$$

(x^2+2x+4) $(x-2)$

$a+b=1 \longrightarrow a+b=1$
$2a-2b+c=4 \longrightarrow 2a-b=2$
$4a-2c=0\rightarrow c=2a$

$a=1,\ b=0,\ c=2$

$$I=\int\frac{dx}{x-2}+\int\frac{2dx}{x^2+2x+4}=\int\frac{dx}{x-2}+2\int\frac{dx}{(x+1)^2+3}=\ln|x-2|+\frac{2}{\sqrt3}\arctan\left(\frac{x+1}{\sqrt3}\right)+C$$

Ör $I=\int\frac{x\,dx}{x^3+1}$ $\quad\frac{x}{x^3+1}=\frac{a}{x+1}+\frac{bx+c}{x^2-x+1}$ $\quad x^3+a^3=(x+a)(x^2-ax+a^2)$

$a=-1/3$ $b=c=1/3$

$$I=-\frac{1}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{x+1}{x^2-x+1}dx=-\frac{1}{3}\int\frac{dx}{x+1}+\frac{1}{6}\int\frac{2x-1}{x^2-x+1}dx+\frac{1}{2}\int\frac{dx}{x^2-x+1}$$
$$=-\frac{1}{3}\ln|x+1|+\frac{1}{6}\ln|x^2-x+1|+\frac{1}{2}\int\frac{dx}{(x-\frac{1}{2})^2+3/4}$$
$$=\frac{1}{6}\ln\frac{|x^2-x+1|}{(x+1)^2}+\frac{1}{2}\frac{1}{\sqrt{3/4}}\arctan\left(\frac{x-1/2}{\sqrt{3/4}}\right)+C$$
$$=\frac{1}{6}\ln\frac{x^3+1}{(x+1)^3}+\frac{1}{\sqrt3}\arctan\left(\frac{2x-1}{\sqrt3}\right)+C$$

Ör. $I = \int \sqrt{\frac{1+x}{1-x}}\,dx$ rasyonelleştirerek çöz

$$I = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$u = \sqrt{1-x^2}$
$x^2 = 1-u^2 \rightarrow x\,dx = -u\,du$

$$= \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{u\,du}{u} = \arcsin x - \sqrt{1-x^2} + C$$

Binom integraller

$$I = \int x^m \cdot (a+bx^n)^p \cdot dx \text{ şeklindeki integraller.}$$

a) p tamsayı, m ile n'nin paydalarının en küçük ortak katı k ise $x = u^k$ kullanılarak integral çözülür.

b) p tamsayı değil, $\frac{m+1}{n}$ tamsayı, p'nin paydası k ise $a+bx^n = u^k$ kullanılarak integral çözülür.

c) p ve $\frac{m+1}{n}$ tamsayı değil, $p+\frac{m+1}{n}$ tamsayı, p'nin paydası k ise $ax^{-n}+b = u^k$ kullanılarak integral çözülür.

Ör. $I = \int \frac{dx}{\sqrt{x}+2\sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3+2u^2} = 6\int \frac{u^3}{u^2+2}\,du$

$x = u^6 \rightarrow dx = 6u^5\,du$
$\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$

$$= 6\int \left(u^2-2u+4-\frac{8}{u+2}\right)du$$
$$= 2u^3-6u^2+24u-48\ln|u+2| + C$$
$$= 2\sqrt{x}-6\sqrt[3]{x}+24\sqrt[6]{x}-48(2+\sqrt[6]{x}) + C$$

Ör. $I = \int \sqrt[3]{x}\cdot(1+2\sqrt{x})\,dx = \int u^2\cdot(1+2u^3)\cdot 6u^5\,du = 6\int (u^7+2u^{10})\,du$

$x = u^6 \rightarrow dx = 6u^5\,du$
$\sqrt[3]{x} = u^2$, $\sqrt{x} = u^3$, $\sqrt[6]{x} = u$

$$= \frac{6}{8}u^8 + \frac{12}{11}u^{11} + C$$
$$= \frac{3}{4}x^{4/3} + \frac{12}{11}x^{11/6} + C$$

Ör $I = \int \frac{\sqrt{x+2}-5}{\sqrt[3]{x+2}}\,dx = \int \frac{u^3-5}{u^2}\cdot 6u^5\,du = 6\int u^3(u^3-5)\,du$

$x+2 = u^6 \to dx = 6u^5\,du$ $\quad = 6\int (u^6-5u^3)\,du = \frac{6}{7}u^7 - \frac{30}{4}u^4 + C$

$\sqrt{x+2} = u^3, \quad \sqrt[3]{x+2} = u^2$ $\quad = \frac{6}{7}(x+2)^{7/6} - \frac{15}{2}(x+2)^{2/3} + C$

Ör $I = \int \frac{dx}{\sqrt{x}\cdot(4+\sqrt[3]{x})} = \int \frac{6u^5\,du}{u^3(4+u^2)} = 6\int \frac{u^2\,du}{u^2+4} = 6\int\left(1-\frac{4}{u^2+4}\right)du$

$x = u^6 \to dx = 6u^5\,du$ $\quad = 6u - 24\times\frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$

$\sqrt{x} = u^3, \quad \sqrt[3]{x} = u^2$ $\quad = 6\sqrt[6]{x} - 12\arctan\left(\frac{\sqrt[6]{x}}{2}\right) + C$

Ör $I = \int \frac{dx}{x\cdot\sqrt{1-x^2}} = \int \frac{x\,dx}{x^2\cdot\sqrt{1-x^2}} = \int \frac{-u\,du}{(1-u^2)\cdot u} = \int \frac{du}{u^2-1}$

$\quad = \frac{1}{2}\int\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$

$u = \sqrt{1-x^2} \to x^2 = 1-u^2$

$2x\,dx = -2u\,du \to x\,dx = -u\,du$ $\quad = \frac{1}{2}\ln\left(\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right) + C$

Ör $I = \int \frac{dx}{x\cdot\sqrt{9+4x^2}} = \int \frac{x\,dx}{x^2\cdot\sqrt{9+4x^2}} = \int \frac{4x\,dx}{4x^2\cdot\sqrt{9+4x^2}} = \int \frac{u\,du}{(u^2-9)\,u}$

$u = \sqrt{9+4x^2}$ $\quad = \int \frac{du}{u^2-9} = \frac{1}{6}\int\left(\frac{1}{u-3} - \frac{1}{u+3}\right)du$

$4x^2 = u^2-9$

$8x\,dx = 2u\,du$ $\quad = \frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| + C = \frac{1}{6}\ln\left(\frac{3-\sqrt{9+4x^2}}{3+\sqrt{9+4x^2}}\right) + C$

$4x\,dx = u\,du$

Ör $I = \int \frac{dx}{x^3\cdot\sqrt{1+x^2}} = \int \frac{x\,dx}{x^4\cdot\sqrt{1+x^2}} = \int \frac{u\,du}{(u^2-1)^2\cdot u} = \int \frac{du}{(u^2-1)^2}$

$u = \sqrt{1+x^2}$ $\quad = \int \frac{du}{(u-1)^2(u+1)^2}$

$x^2 = u^2-1$

$2x\,dx = 2u\,du$

$x\,dx = u\,du$

$\frac{1}{(u-1)^2(u+1)^2} = \frac{a}{u-1} + \frac{b}{(u-1)^2} + \frac{c}{u+1} + \frac{d}{(u+1)^2} = -\frac{1}{4}\int\frac{du}{u-1} + \frac{1}{4}\int\frac{du}{(u-1)^2} + \frac{1}{4}\int\frac{du}{u+1} + \frac{1}{4}\int\frac{du}{(u+1)^2}$

$= -\frac{1}{4}\ln|u-1| - \frac{1}{4}\frac{1}{u-1} + \frac{1}{4}\ln|u+1| - \frac{1}{4}\frac{1}{u+1}$

$= \frac{1}{4}\ln\left|\frac{u+1}{u-1}\right| - \frac{1}{2}\frac{u}{u^2-1} + C = \frac{1}{4}\ln\left(\frac{1+\sqrt{1+x^2}}{1-\sqrt{1+x^2}}\right) - \frac{1}{2}\frac{\sqrt{1+x^2}}{x^2} + C$

б) $I = \int \frac{x^5}{\sqrt{1+x^3}}\,dx = \int \frac{x^3}{3\sqrt{1+x^3}}\, 3x^2 dx = \int \frac{u^2-1}{3u}\, 2u\,du$

$u = \sqrt{1+x^3}$

$x^3 = u^2 - 1$

$3x^2 dx = 2u\,du$

$= \frac{2}{3}\int (u^2-1)\,du = \frac{2u^3}{9} - \frac{2u}{3} + C = \frac{2u}{9}(u^2-3) + C$

$= \frac{2}{9}\sqrt{1+x^3}\cdot(x^3-2) + C$

б) $I = \int \frac{\sqrt[3]{5+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \int \frac{u}{(u^3-5)^2}\cdot 12u^2(u^3-5)^3\,du$

$u = \sqrt[3]{5+\sqrt[4]{x}}$

$x = (u^3-5)^4 \to \sqrt{x} = (u^3-5)^2$

$dx = 4(u^3-5)^3\cdot 3u^2 du$

$= 12u^2(u^3-5)^3\,du$

$= 12\int u^3(u^3-5)\,du$

$= 12\int (u^6 - 5u^3)\,du$

$= \frac{12}{7}u^7 - \frac{60}{4}u^4 + C$

$= \frac{12}{7}(5+\sqrt[4]{x})^{7/3} - 15(5+\sqrt[4]{x})^{4/3} + C$

б) $I = \int \frac{\sqrt{25+\sqrt[3]{x}}}{x^{2/3}}\,dx = \int \frac{u}{\cancel{(u^2-25)^2}}\, 6u\,\cancel{(u^2-25)^2}\,du = 6\int u^2 du$

$= 2u^3 + C$

$u = \sqrt{25+\sqrt[3]{x}}$

$x = (u^2-25)^3 \to x^{2/3} = (u^2-25)^2$

$= 2(25+\sqrt[3]{x})^{3/2} + C$

$dx = 3(u^2-25)^2\cdot 2u\,du$

б) $I = \int \sqrt{\frac{x}{1-x^3}}\,dx = \int \frac{\sqrt{x}}{\sqrt{1-x^3}}\,dx = \int \frac{\sqrt{x/x^3}\,dx}{\sqrt{\frac{1-x^3}{x^3}}} = \int \frac{dx}{x\cdot\sqrt{x^{-3}-1}}$

$u = \sqrt{x^{-3}-1}$

$x^{-3} = u^2+1$

$-3x^{-4}dx = 2u\,du$

$x^{-4}dx = -\frac{2}{3}u\,du$

$= \int \frac{x^{-4}dx}{x^{-3}\cdot\sqrt{x^{-3}-1}} = \int \frac{-\frac{2}{3}u\,du}{(u^2+1)\cdot u} = -\frac{2}{3}\int \frac{du}{u^2+1}$

$= -\frac{2}{3}\arctan u + C = -\frac{2}{3}\arctan\sqrt{\frac{1-x^3}{x^3}} + C$

$I = \int \frac{dx}{x^4\cdot\sqrt{1+x^2}}$

б) $I = \int \frac{dx}{x^2\cdot\sqrt{1+x^2}} = \int \frac{dx}{x^3\sqrt{\frac{1+x^2}{x^2}}} = \int \frac{x^{-3}dx}{\sqrt{x^{-2}+1}} = \int \frac{-u\,du}{u} = -\int du$

$= -u + C$

$u = \sqrt{x^{-2}+1}$

$x^{-2} = u^2 - 1$

$-2x^{-3}dx = 2u\,du \to x^{-3}dx = -u\,du$

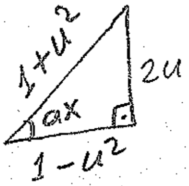
$= -\sqrt{x^{-2}+1} + C = -\frac{\sqrt{1+x^2}}{x} + C$

Yarım Açı Formülleri

$$u = \tan\left(\frac{ax}{2}\right) \qquad \tan(ax) = \frac{2\tan\left(\frac{ax}{2}\right)}{1-\tan^2\left(\frac{ax}{2}\right)} = \frac{2u}{1-u^2}$$

$$x = \frac{2}{a}\arctan u$$

$$dx = \frac{2}{a}\,\frac{du}{1+u^2}$$



$$\cos(ax) = \frac{1-u^2}{1+u^2} \qquad \sin(ax) = \frac{2u}{1+u^2}$$

Ör
$$I = \int \frac{4+3\sin 2x}{(1+\cos 2x)\cdot\sin 2x}\cdot dx = \int \frac{4+3\,\frac{2u}{1+u^2}}{\left(1+\frac{1-u^2}{1+u^2}\right)\cdot\frac{2u}{1+u^2}}\cdot\frac{du}{1+u^2}$$

$$= \int \frac{4u^2+6u+4}{4u}\,du = \int\left(u+\frac{3}{2}+\frac{1}{u}\right).du = \frac{u^2}{2}+\frac{3u}{2}+\ln|u|+C$$

$$= \frac{1}{2}\tan^2 x + \frac{3}{2}\tan x + \ln|\tan x| + C$$

Ör
$$I = \int \frac{dx}{5+3\cos 2x} = \int \frac{\frac{du}{1+u^2}}{5+3\,\frac{1-u^2}{1+u^2}} = \int \frac{du}{8+2u^2} = \frac{1}{2}\int\frac{du}{u^2+4}$$

$$= \frac{1}{4}\arctan\left(\frac{u}{2}\right)+C = \frac{1}{4}\arctan\left(\frac{\tan x}{2}\right)+C$$

Ör
$$I = \int \frac{dx}{1+\cos 2x} = \int \frac{\frac{du}{1+u^2}}{1+\frac{1-u^2}{1+u^2}} = \int\frac{du}{2} = \frac{u}{2}+C = \frac{1}{2}\tan x + C$$

Ör
$$I = \int \frac{dx}{1+\sin 2x} = \int \frac{\frac{du}{1+u^2}}{1+\frac{2u}{1+u^2}} = \int \frac{du}{u^2+2u+1} = \int \frac{du}{(u+1)^2}$$

$$= -\frac{1}{u+1}+C = -\frac{1}{1+\tan x}+C = \frac{-\cos x}{\cos x+\sin x}+C$$

Ör
$$I = \int \frac{1-\sin 2x}{1+\cos 2x}\,dx = \int \frac{1-\frac{2u}{1+u^2}}{1+\frac{1-u^2}{1+u^2}}\,\frac{du}{1+u^2} = \frac{1}{2}\int\frac{u^2-2u+1}{u^2+1}\,du$$

$$= \frac{1}{2}\int du - \frac{1}{2}\int\frac{2u\,du}{u^2+1} = \frac{u}{2}-\frac{1}{2}\ln|u^2+1|+C = \frac{\tan x}{2}-\ln|\sec x|+C$$

Ör
$$I = \int\frac{dx}{(1-\cos 2x)^2} = \int \frac{\frac{du}{1+u^2}}{\left(1-\frac{1-u^2}{1+u^2}\right)^2} = \int\frac{u^2+1}{4u^4}\,du = \frac{1}{4}\int u^{-2}du+\frac{1}{4}\int u^{-4}du$$

$$= -\frac{1}{4u}-\frac{1}{12u^3}+C = -\frac{1}{4}\cot x-\frac{1}{12}\cot^3 x+C$$

Or $I=\int\frac{\tan 2x\cdot dx}{1+\cos 2x}=\int\frac{\frac{2u}{1-u^2}\frac{du}{1+u^2}}{1+\frac{1-u^2}{1+u^2}}=\int\frac{u\,du}{1-u^2}$ $\quad w=1-u^2$, $dw=-2u\,du$, $u\,du=-\frac{dw}{2}$

$=\int\frac{-dw/2}{w}=-\frac{1}{2}\int\frac{dw}{w}=-\frac{1}{2}\ln|w|+C=-\frac{1}{2}\ln|1-u^2|+C$

$=-\frac{1}{2}\ln|1-\tan^2x|+C=\frac{1}{2}\ln\left|\frac{1}{1-\tan^2x}\right|+C$

Or $I=\int\frac{dx}{\sin 2x+\cos 2x+1}=\int\frac{\frac{du}{1+u^2}}{\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}+1}=\int\frac{du}{2u+1-u^2+1+u^2}$

$=\frac{1}{2}\int\frac{du}{u+1}=\frac{1}{2}\ln|u+1|+C=\frac{1}{2}\ln|1+\tan x|+C$

Or $I=\int\frac{dx}{3\sin x+2\cos x+2}=\int\frac{\frac{2du}{1+u^2}}{3\frac{2u}{1+u^2}+2\frac{1-u^2}{1+u^2}+2}=\int\frac{du}{3u+2}$

$=\frac{1}{3}\ln|3u+2|+C=\frac{1}{3}\ln|2+3\tan(x/2)|+C$

Or $I=\int\frac{dx}{1+\sin x}=\int\frac{\frac{2du}{1+u^2}}{1+\frac{2u}{1+u^2}}=\int\frac{2du}{u^2+2u+1}=2\int\frac{du}{(u+1)^2}$

$=-\frac{2}{u+1}+C=-\frac{2}{1+\tan(x/2)}+C=\tan x-\sec x+C$

Or $I=\int\sec x\cdot dx=\int\frac{dx}{\cos x}=\int\frac{\frac{2du}{1+u^2}}{\frac{1-u^2}{1+u^2}}=\int\frac{-2}{u^2-1}du$

$=\int\frac{du}{u+1}-\int\frac{du}{u-1}+C=\ln|u+1|-\ln|u-1|+C$

$=\ln\left|\frac{u+1}{u-1}\right|+C=\ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right|+C=\ln|\sec x+\tan x|+C$

Or $I=\int\frac{1+\cos x}{1-\sin x}dx=\int\frac{1+\frac{1-u^2}{1+u^2}}{1-\frac{2u}{1+u^2}}\cdot\frac{2du}{1+u^2}=\int\frac{4du}{(u^2+1)(u-1)^2}$ $\quad a=2$, $b=0$, $c=-2$, $d=2$

$=\int\frac{2u\,du}{u^2+1}-2\int\frac{du}{u-1}+2\int\frac{du}{(u-1)^2}=\ln(u^2+1)-2\ln|u-1|-\frac{2}{u-1}+C$

$=\ln\frac{u^2+1}{(u-1)^2}-\frac{2}{u-1}+C=\ln\frac{1+\tan^2(x/2)}{(1-\tan(x/2))^2}+\frac{2}{1-\tan(x/2)}+C$

Kısmi İntegral

$$(uv)' = u'v + uv' \longrightarrow \frac{d}{dx}(uv) = \frac{du}{dx}.v + u.\frac{dv}{dx}$$

$$d(uv) = v.du + u.dv \longrightarrow u\,dv = d(uv) - v.du$$

$$\boxed{\int u.dv = u.v - \int v.du}$$

Ör: $I = \int x.e^x.dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x dx$

$u = x \quad dv = e^x dx \qquad = xe^x - e^x + C = (x-1).e^x + C$

$du = dx \quad v = e^x$

Ör: $I = \int (2-3x).e^{-x}.dx = \int u\,dv = uv - \int v\,du = (3x-2)e^{-x} - 3\int e^{-x}dx$

$u = 2-3x \quad dv = e^{-x}dx \qquad = (3x-2)e^{-x} + 3e^{-x} + C$

$du = -3dx \quad v = -e^{-x} \qquad = (3x+1)e^{-x} + C$

Ör: $I = \int \ln x.dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx$

$u = \ln x \quad dv = dx \qquad = x\ln x - x + C = x(\ln x - 1) + C$

$du = \frac{dx}{x} \quad v = x$

Ör: $I = \int x.\ln x.dx = \int u\,dv = uv - \int v\,du = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{dx}{x}$

$u = \ln x \quad dv = x\,dx \qquad = \frac{x^2\ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$

$du = \frac{dx}{x} \quad v = x^2/2 \qquad = \frac{x^2}{4}(2\ln x - 1) + C$

Ör: $I = \int x.\sin x.dx = \int u\,dv = uv - \int v\,du = -x\cos x + \int \cos x.dx$

$u = x \quad dv = \sin x.dx \qquad = -x\cos x + \sin x + C$

$du = dx \quad v = -\cos x$

Ör: $I = \int x.\cos x.dx = \int u\,dv = uv - \int v\,du = x\sin x - \int \sin x\,dx$

$u = x \quad dv = \cos x.dx \qquad = x\sin x + \cos x + C$

$du = dx \quad v = \sin x$

Ör: $I = \int x.\sec^2 x.dx = \int u\,dv = uv - \int v\,du = x\tan x - \int \tan x.dx$

$= x.\tan x - \int \frac{\sin x}{\cos x}dx \qquad w = \cos x \quad dw = -\sin x\,dx$

$u = x \quad dv = \sec^2 x.dx$

$du = dx \quad v = \tan x \qquad = x\tan x + \int \frac{dw}{w}$

$= x\tan x + \ln|\cos x| + C$

Ör $I = \int \arcsin x . dx = \int u\,dv = uv - \int v\,du = x.\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$

$u = \arcsin x \qquad dv = dx \qquad = x.\arcsin x + \int \frac{w\,dw}{w}$

$du = \frac{dx}{\sqrt{1-x^2}} \qquad v = x \qquad = x.\arcsin x + w + C$

$w = \sqrt{1-x^2} \longrightarrow x^2 = 1 - w^2 \qquad = x.\arcsin x + \sqrt{1-x^2} + C$

$2x\,dx = -2w\,dw \rightarrow x\,dx = -w\,dw$

Ör $I = \int \arctan x . dx = \int u\,dv = uv - \int v\,du = x.\arctan x - \int \frac{x\,dx}{1+x^2}$

$u = \arctan x \qquad dv = dx \qquad = x.\arctan x - \frac{1}{2}\int \frac{dw}{w}$

$du = \frac{dx}{1+x^2} \qquad v = x \qquad = x.\arctan x - \frac{1}{2}\ln|w| + C$

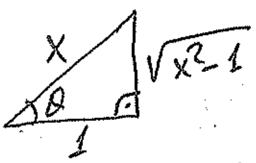
$w = 1+x^2 \rightarrow dw = 2x\,dx \qquad = x.\arctan x - \ln\sqrt{1+x^2} + C$

Ör $I = \int \operatorname{arcsec} x . dx = \int u\,dv = uv - \int v\,du = x.\operatorname{arcsec} x - \underbrace{\int \frac{dx}{\sqrt{x^2-1}}}_{I_1}$

$u = \operatorname{arcsec} x \qquad dv = dx$

$du = \frac{dx}{x.\sqrt{x^2-1}} \qquad v = x \qquad I_1 = \int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta \tan\theta\,d\theta}{\sqrt{\sec^2\theta - 1}} = \int \sec\theta\,d\theta$

$x = \sec\theta \rightarrow dx = \sec\theta \tan\theta\,d\theta \qquad = \ln|\sec\theta + \tan\theta| + C_1$



$\tan\theta = \sqrt{x^2-1} \qquad = \ln|x + \sqrt{x^2-1}| + C_1$

$I = x.\operatorname{arcsec} x - \ln|x + \sqrt{x^2-1}| + C$

Ör $I = \int \arctan\sqrt{x} . dx = \int u\,dv = uv - \int v\,du$

$= x.\arctan\sqrt{x} - \frac{1}{2}\underbrace{\int \frac{\sqrt{x}}{1+x}dx}_{I_1}$

$u = \arctan\sqrt{x}$

$du = \frac{\frac{dx}{2\sqrt{x}}}{1+(\sqrt{x})^2} = \frac{dx}{2\sqrt{x}\,(x+1)} \qquad I_1 = \int \frac{\sqrt{x}}{1+x}dx = \int \frac{w}{1+w^2}2w\,dw$

$dv = dx \longrightarrow v = x \qquad = \int \frac{2w^2}{w^2+1}dw = 2\int dw - 2\int \frac{dw}{w^2+1}$

$w = \sqrt{x} \longrightarrow x = w^2$

$dx = 2w\,dw \qquad = 2w - 2\arctan(w) + C_1$

$= 2\sqrt{x} - 2\arctan\sqrt{x} + C_1$

$I = x.\arctan\sqrt{x} - \frac{1}{2}\left(2\sqrt{x} - 2\arctan\sqrt{x} + C_1\right) = (x+1)\arctan\sqrt{x} - \sqrt{x} + C$

Ör $I=\int x.\arcsin x .dx=\int u dv = uv-\int v du = \frac{x^2}{2}\arcsin x - \frac{1}{2}\underbrace{\int \frac{x^2 dx}{\sqrt{1-x^2}}}_{I_1}$

$u=\arcsin x \quad dv = x dx \qquad x=\sin\theta$

$du=\frac{dx}{\sqrt{1-x^2}} \quad v = x^2/2 \qquad dx=\cos\theta d\theta$

$\cos\theta=\sqrt{1-x^2}$

$I_1=\int \frac{x^2 dx}{\sqrt{1-x^2}} = \int \frac{\sin^2\theta . \cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \int \sin^2\theta . d\theta = \int \frac{1-\cos 2\theta}{2} d\theta$

$=\frac{\theta}{2}-\frac{\sin 2\theta}{4}+c_1 = \frac{\theta}{2}-\frac{\cos\theta\sin\theta}{2}+c_1=\frac{1}{2}\arcsin x - \frac{x\sqrt{1-x^2}}{2}+c_1$

$I=\frac{x^2}{2}\arcsin x - \frac{1}{2}\left(\frac{1}{2}\arcsin x - \frac{x.\sqrt{1-x^2}}{2}+c_1\right)$

$=\frac{2x^2-1}{4}.\arcsin x + \frac{x.\sqrt{1-x^2}}{4}+C$

Ör $I=\int \frac{x.\arcsin x}{\sqrt{1-x^2}}dx = \int u dv = uv - \int v du$

$= -\sqrt{1-x^2}.\arcsin x + \int dx$

$u=\arcsin x \quad dv=\frac{x dx}{\sqrt{1-x^2}}$

$= x-\sqrt{1-x^2}.\arcsin x + C$

$du=\frac{dx}{\sqrt{1-x^2}} \quad V=-\sqrt{1-x^2}$

Ör $I=\int \frac{x.\sin x.dx}{\cos^2 x} = \int u dv = uv-\int v du = x\sec x - \int \sec x dx$

$= x.\sec x - \ln|\sec x + \tan x| + c$

$u=x \quad dv=\frac{\sin x}{\cos^2 x}dx$

$du=dx \quad V=\sec x$

Ör $I=\int x^2.\sin x .dx = \int u dv = uv-\int v du = -x^2\cos x + 2\underbrace{\int x\cos x dx}_{I_1}$

$u=x^2 \quad dv=\sin x dx \qquad u=x \quad dv=\cos x dx$

$du=2x dx \quad v=-\cos x \qquad du=dx \quad v=\sin x$

$I_1=\int x\cos x dx = \int u dv = uv-\int v du = x\sin x - \int \sin x dx$

$= x\sin x + \cos x + c_1$

$I=-x^2\cos x + 2(x\sin x + \cos x + c_1) = (2-x^2)\cos x + 2x\sin x + C$

Ör $I=\int \frac{\ln x}{\sqrt{x}}dx = \int u dv = uv-\int v du = 2\sqrt{x}.\ln x - \int 2\sqrt{x}\frac{dx}{x}$

$=2\sqrt{x}.\ln x - 2\int\frac{dx}{\sqrt{x}} = 2\sqrt{x}\ln x - 4\sqrt{x}+C$

$u=\ln x \quad dv=\frac{dx}{\sqrt{x}}$

$du=\frac{dx}{x} \quad V=2\sqrt{x} \qquad = 2\sqrt{x}(\ln x - 2)+C$

Ör $I = \int \frac{xe^x}{(x+1)^2}dx = \int u\,dv = uv - \int v\,du = -\frac{x}{x+1}e^x + \int e^x dx$

$u = xe^x$ $\quad dv = \frac{dx}{(x+1)^2}$ $\quad = -\frac{x}{x+1}e^x + e^x + c$

$du = (x+1)e^x dx$ $\quad v = -\frac{1}{x+1}$ $\quad = \frac{e^x}{x+1} + c$

Ör $I = \int \sqrt{x^2+5}\,dx = \int u\,dv = uv - \int v\,du = x.\sqrt{x^2+5} - \int \frac{x^2\,dx}{\sqrt{x^2+5}}$

$u = \sqrt{x^2+5}$ $\quad dv = dx$ $\quad = x.\sqrt{x^2+5} - \underbrace{\int \sqrt{x^2+5}\,dx}_{I} + 5\underbrace{\int \frac{dx}{\sqrt{x^2+5}}}_{I_1}$

$du = \frac{x\,dx}{\sqrt{x^2+5}}$ $\quad v = x$

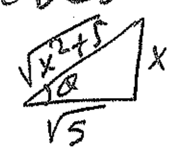
$$I = \frac{x.\sqrt{x^2+5}}{2} + \frac{5}{2}I_1$$

$I_1 = \int \frac{dx}{\sqrt{x^2+5}} = \int \frac{\sqrt{5}\sec^2\theta\,d\theta}{\sqrt{5\tan^2\theta + 5}} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1$

$x = \sqrt{5}\tan\theta$

$dx = \sqrt{5}\sec^2\theta\,d\theta$ $\quad = \ln\left|\frac{x+\sqrt{x^2+5}}{\sqrt{5}}\right| + C_1$

$\tan\theta = \frac{x}{\sqrt{5}}$



$= \ln|x + \sqrt{x^2+5}| + C_2$

$$I = \frac{x}{2}\cdot\sqrt{x^2+5} + \frac{5}{2}\ln|x+\sqrt{x^2+5}| + C$$

Ör $I_1 = \int e^x.\cos x.dx \longrightarrow I_1 = \frac{1}{2}e^x(\cos x + \sin x)$

$I_2 = \int e^x.\sin x.dx \longrightarrow I_2 = \frac{1}{2}e^x(\sin x - \cos x)$

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Ör $I_1 = \int \cos(\ln x).dx \longrightarrow I_1 = \frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right)$

$I_2 = \int \sin(\ln x).dx \longrightarrow I_2 = \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right)$

I_1 için

$u = \cos(\ln x)$

$du = -\frac{1}{x}\sin(\ln x)$

$dv = dx$

$v = x$

I_2 için

$u = \sin(\ln x)$

$du = \frac{1}{x}\cos(\ln x)$

$dv = dx$

$v = x$

Belirli integral

$F'(x) = f(x)$ için $\int_a^b f(x).dx = F(x)\Big|_a^b = F(b)-F(a)$, $x \in [a,b]$

$x \in [a,b]$ için $f(x) \geq g(x)$ ise $\int_a^b (f(x)-g(x)).dx \geq 0$ olur.

$x \in [a,b]$ için $\left|\int_a^b f(x).dx\right| \leq \int_a^b |f(x)|.dx$ olur.

$f(x)$ tek fonksiyon ise $\int_{-a}^{a} f(x)dx = 0$ $f(-x) = -f(x)$ tek fonk.

$f(x)$ çift fonksiyon ise $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ $f(-x) = f(x)$ çift fonk.

Ortalama Değer

$$f_{ort} = \frac{1}{b-a}\int_a^b f(x).dx \quad , \quad x \in [a,b]$$

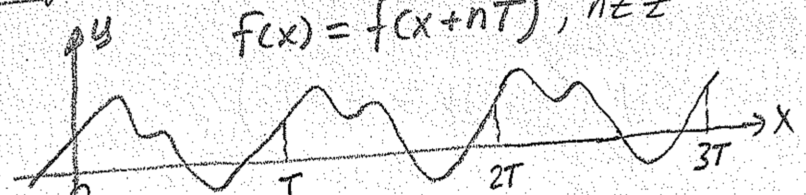
Mutlak Değer

$$K = \int_a^b |f(x)|.dx \text{ ise } -K \leq \int_a^b f(x)dx \leq +K$$

Peryodik Fonksiyon

$f(x) = f(x+nT)$, $n \in Z$

$$\int_a^b f(x).dx = \int_{a+nT}^{b+nT} f(x).dx$$



$$\int_0^T f(x)dx = \int_\alpha^{\alpha+T} f(x)dx$$

Ör

$$I = \int_0^{\sqrt{2}/3} \frac{dx}{\sqrt{4-9x^2}} = \int_0^{\pi/4} \frac{\frac{2}{3}\cos\theta . d\theta}{\sqrt{4-4\sin^2\theta}} = \frac{1}{3}\int_0^{\pi/4} d\theta = \frac{\theta}{3}\Big|_0^{\pi/4} = \frac{\pi/4 - 0}{3} = \pi/12$$

$3x = 2\sin\theta \longrightarrow \sin\theta = 3x/2$

$x = \frac{2}{3}\sin\theta \qquad \theta = \arcsin(3x/2)$

$dx = 2/3 \cos\theta d\theta$

Ör

$$I = \int_3^6 \frac{\sqrt{x^2-9}}{x}dx = \int_0^{\pi/3} \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta} 3\sec\theta\tan\theta d\theta$$

$$= 3\int_0^{\pi/3} \tan^2\theta d\theta = 3\int_0^{\pi/3}(\sec^2\theta - 1)d\theta = 3(\tan\theta - \theta)\Big|_0^{\pi/3}$$

$$= 3\left((\tan\pi/3 - \pi/3) - (\tan 0 - 0)\right)$$

$$= 3(\sqrt{3} - \pi/3) = 3\sqrt{3} - \pi$$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta d\theta$

$\sec\theta = x/3 \longrightarrow \cos\theta = 3/x$

Ör $$I=\int_0^{\pi/4}\sin^2\theta\,d\theta=\int_0^{\pi/4}\frac{1-\cos 2\theta}{2}\,d\theta=\left(\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)\Big|_0^{\pi/4}=\frac{\pi-2}{8}$$

Ör $$I=\int_0^{\pi/2}\frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta=\int_{\sqrt2}^{1}\frac{-2u\,du}{u}=2\int_1^{\sqrt2}du=2u\Big|_1^{\sqrt2}=2(\sqrt2-1)$$

$u=\sqrt{1+\cos\theta}\rightarrow u^2-1=\cos\theta$

$2u\,du=-\sin\theta\,d\theta$

Ör $$I=\int_0^{\ln 9}\frac{e^x\,dx}{\sqrt[3]{e^x-1}}\quad ?=?$$

$u=\sqrt[3]{e^x-1}\longrightarrow u^3+1=e^x$

$3u^2du=e^x dx$

$$I=\int_0^2\frac{3u^2du}{u}=3\int_0^2u\,du=\frac{3u^2}{2}\Big|_0^2=6$$

Ör $$I=\int_0^1\arcsin x.\,dx=x.\arcsin x\Big|_0^1-\int_0^1\frac{x\,dx}{\sqrt{1-x^2}}=\arcsin(1)-\int_1^0\frac{-w\,dw}{w}$$

$u=\arcsin x \quad dv=dx \quad w=\sqrt{1-x^2}\rightarrow x^2=1-w^2$

$du=\frac{dx}{\sqrt{1-x^2}} \quad v=x \quad 2x\,dx=-2w\,dw$

$x\,dx=-w\,dw$

$$=\frac{\pi}{2}-\int_0^1dw=\frac{\pi}{2}-w\Big|_0^1=\frac{\pi}{2}-1$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)}f(t)\,dt=f(u(x)).u'(x)-f(v(x)).v'(x)$$

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$$\frac{d}{dx}\int_{v(x)}^{u(x)}f(t)\,dt=\frac{d}{dx}\left(F(t)\Big|_{v(x)}^{u(x)}\right)=\frac{d}{dx}\Big(F(u(x))-F(v(x))\Big)$$

$$=F'(u(x))-F'(v(x))=f(u(x)).u'(x)-f(v(x)).v'(x)$$

$$\frac{d}{dx}\int_a^{u(x)}f(t).dt=f(u(x)).u'(x) \qquad \frac{d}{dx}\int_a^{x}f(t).dt=f(x)$$

Ör $$p(x)=\frac{d}{dx}\int_x^{x^2+1}(2t+3)\,dt \text{ ise } p(1)=?$$

1. yol

$$p(x)=\frac{d}{dx}\left((t^2+3t)\Big|_x^{x^2+1}\right)=\frac{d}{dx}\Big((x^2+1)^2+3(x^2+1)-(x^2+3x)\Big)$$

$$=2(x^2+1)2x+6x-2x-3=4x^3+8x-3\longrightarrow p(1)=9$$

2. yol

$$p(x)=(x^2+1)'.\big(2(x^2+1)+3\big)-(x)'(2x+3)=2x.(2x^2+5)-(2x+3)$$

$$=4x^3+8x-3\longrightarrow p(1)=9$$

Ör $p(x) = \frac{d}{dx}\int_{x^2}^{\ln x} \frac{e^t}{t}dt = (\ln x)'\left(\frac{e^{\ln x}}{\ln x}\right) - (x^2)'\left(\frac{e^{x^2}}{x^2}\right)$

$= \frac{1}{x}\,\frac{x}{\ln x} - 2x\,\frac{e^{x^2}}{x^2} = \frac{1}{\ln x} - \frac{2e^{x^2}}{x}$

Ör $p(x) = \frac{d}{dx}\int_{x+2}^{2x+1} \sqrt{t^2-2t+5}\,dx = (2x+1)'.\sqrt{(2x+1)^2-2(2x+1)+5} - (x+2)'.\sqrt{(x+2)^2-2(x+2)+5}$

$= 2.\sqrt{4x^2+4x+1-4x-2+5} - \sqrt{x^2+4x+4-2x-4+5} = 4.\sqrt{x^2+1} - \sqrt{x^2+2x+5}$

Ör $p(x) = \frac{d}{dx}\int_{\cos x}^{\sin x} t^2dt$ ise $p(\pi/3) = ?$

$p(x) = (\sin x)'.\sin^2 x - (\cos x)'.\cos^2 x = \sin^2 x.\cos x + \cos^2 x.\sin x$

$p(\pi/3) = \sin^2(\pi/3).\cos(\pi/3) + \cos^2(\pi/3)\sin(\pi/3) = \left(\frac{\sqrt3}{2}\right)^2.\frac{1}{2} + \left(\frac{1}{2}\right)^2.\left(\frac{\sqrt3}{2}\right) = \frac{3+\sqrt3}{8}$

Ör $I = \int_2^5 \frac{dx}{\sqrt{5+4x-x^2}} = \int_2^5 \frac{dx}{\sqrt{9-(x-2)^2}} = \arcsin\left(\frac{x-2}{3}\right)\Big|_2^5 = \arcsin(1) - \arcsin(0) = \pi/2$

Ör $f(x) = 3x^2-2x+1$ fonk. ortalama değerini $[-2,3]$ aralığı için hesapla.

$f_{ort} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{3+2}\int_{-2}^{3}(3x^2-2x+1)dx = \frac{1}{5}(x^3-x^2+x)\Big|_{-2}^{3}$

$= \frac{1}{5}\left((27-9+3)-(-8-4-2)\right) = 7$

Ör $f(x) = \sqrt[3]{x}$ fonk. ortalama değerini $[8,27]$ aralığı için hesapla.

$f_{ort} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{27-8}\int_8^{27} x^{1/3}dx = \frac{1}{19}\,\frac{x^{4/3}}{4/3}\Big|_8^{27} = \frac{3}{76}\left(27^{4/3} - 8^{4/3}\right) = \frac{195}{76}$

Ör $f'(x) = \frac{\cos x}{x}$ ve $f(\pi/2) = a$, $f(3\pi/2) = b$ ise $I = \int_{\pi/2}^{3\pi/2} f(x)dx$ hesapla.

$u = f(x)$

$du = f'(x)dx$

$dv = dx$

$v = x$

$I = \int u dv = uv - \int v du$

$I = x f(x)\Big|_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} x f'(x)dx = \frac{3\pi}{2}f(\frac{3\pi}{2}) - \frac{\pi}{2}f(\frac{\pi}{2}) - \int_{\pi/2}^{3\pi/2}\cos x\,dx$

$= \frac{3\pi}{2}b - \frac{\pi}{2}a - \sin x\Big|_{\pi/2}^{3\pi/2} = \frac{\pi}{2}(3b-a) - (\sin\frac{3\pi}{2} - \sin\frac{\pi}{2})$

$= \frac{\pi}{2}(3b-a) - (-1-1) = 2 + \frac{\pi}{2}(3b-a)$

Ör)

$$f(x)=\begin{cases}2x & x<2\\ 5 & 2\le x<3\\ 1/x & x\geq 3\end{cases} \quad \text{ise} \quad I=\int_0^5 f(x)\,dx$$

$$I=\int_0^5 f(x)dx=\int_0^2 2x\,dx+\int_2^3 5\,dx+\int_3^5 \frac{dx}{x}=(x^2)\Big|_0^2+(5x)\Big|_2^3+(\ln|x|)\Big|_3^5$$

$$=(4-0)+(15-10)+(\ln 5-\ln 3)=9+\ln(5/3)$$

Ör)

$$f(x)=\begin{cases}3x^2-2x+5 & x<2\\ 2x+7 & x\geq 2\end{cases} \quad \text{ise} \quad I=\int_1^3 f(x)\,dx$$

$$I=\int_1^3 f(x)dx=\int_1^2(3x^2-2x+5)dx+\int_2^3(2x+7)dx$$

$$=(x^3-x^2+5x)\Big|_1^2+(x^2+7x)\Big|_2^3=(8-4+10)-(1-1+5)+(9+21)-(4+14)$$

$$=14-5+30-18=21$$

Ör)

$$I=\int_{-3}^5|x^2-2x-3|dx=\int_{-3}^{-1}(x^2-2x-3)dx-\int_{-1}^{3}(x^2-2x-3)dx+\int_3^5(x^2-2x-3)dx$$

$x^2-2x-3=0$

$(x+1)(x-3)=0$

$$\begin{array}{c|c|c} & -1 & 3 & \\ + & - & + \end{array}$$

$$=\left(\frac{x^3}{3}-x^2-3x\right)\Big|_{-3}^{-1}-\left(\frac{x^3}{3}-x^2-3x\right)\Big|_{-1}^{3}+\left(\frac{x^3}{3}-x^2-3x\right)\Big|_3^5$$

$$=\left(-\tfrac{1}{3}-1+3\right)-(-9-9+9)-(9-9-9)+\left(-\tfrac{1}{3}-1+3\right)+\left(\tfrac{125}{3}-25-15\right)-(9-9-9)$$

$$=-\tfrac{1}{3}+2+9+9-\tfrac{1}{3}+2+\tfrac{125}{3}-40+9=32$$

Ör) $f(x)=|x+1|+|2-x|$ ise $I=\int_{-2}^3 f(x)\,dx$

$x+1=0$ $\quad$ $2-x=0$

$x=-1$ $\quad$ $x=2$

$$\begin{array}{c|c} & -1 \\ - & + \end{array} \qquad \begin{array}{c|c} & 2 \\ + & - \end{array}$$

$x<-1$ için $f(x)=-(x+1)+(2-x)$
$=1-2x$

$-1\le x<2$ için $f(x)=(x+1)+(2-x)$
$=3$

$x\geq 2$ için $f(x)=(x+1)-(2-x)$
$=2x-1$

$$I=\int_{-2}^3 f(x)dx=\int_{-2}^{-1}(1-2x)dx+\int_{-1}^2 3\,dx+\int_2^3(2x-1)dx$$

$$=(x-x^2)\Big|_{-2}^{-1}+(3x)\Big|_{-1}^{2}+(x^2-x)\Big|_2^3$$

$$=(-1-1)-(-2-4)+(6)-(-3)+(9-3)-(4-2)$$

$$=-2+6+6+3+6-2$$

$$=17$$

Genelleştirilmiş İntegraller

$x \in [a,\infty)$ için $f(x)$ sürekli ise

$$\int_a^{\infty} f(x).dx = \lim_{u\to\infty} \int_a^{u} f(x).dx$$

$x \in (-\infty, a]$ için $f(x)$ sürekli ise

$$\int_{-\infty}^{a} f(x).dx = \lim_{u\to -\infty} \int_u^{a} f(x).dx$$

$x \in R$ için $f(x)$ sürekli ise $\int_{-\infty}^{\infty} f(x)dx = \lim_{u\to\infty} \int_{-u}^{u} f(x)dx$

$x \in R$ için $f(x)$ sürekli ve tek fonksiyon ise

$$\int_{-\infty}^{\infty} f(x)dx = 0$$

$x \in R$ için $f(x)$ sürekli ve çift fonksiyon ise

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx = 2\lim_{u\to\infty} \int_0^{u} f(x)dx$$

$x \in [a,b)$ için $f(x)$ sürekli ve $\lim_{x\to b^-} f(x) = \mp\infty$ ise $\int_a^b f(x)dx = \lim_{u\to b^-} \int_a^u f(x)dx$

$x \in (a,b]$ için $f(x)$ sürekli ve $\lim_{x\to a^+} f(x) = \mp\infty$ ise $\int_a^b f(x)dx = \lim_{u\to a^+} \int_u^b f(x)dx$

Yakınsaklık ve Iraksaklık

$x \in [a,b]$ için $f(x) \geq g(x) \geq 0$ ise

$\int_a^b f(x)dx$ yakınsak ise $\int_a^b g(x)dx$ de yakınsaktır.

$\int_a^b g(x)dx$ ıraksak ise $\int_a^b f(x)dx$ de ıraksaktır

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$\int_a^b |f(x)|.dx$ yakınsak ise $\int_a^b f(x).dx$ mutlak yakınsaktır.

$\int_a^b |f(x)|.dx$ ıraksak fakat $\int_a^b f(x)dx$ yakınsak ise

bu yakınsaklığa koşullu yakınsaklık denir.

Ör) $I = \int_{-\infty}^{\infty} \frac{dx}{x^2+9}$ integrali yakınsak mıdır?

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+9} = 2\int_{0}^{\infty} \frac{dx}{x^2+9} = 2 \lim_{u\to\infty} \int_{0}^{u} \frac{dx}{x^2+9} = 2 \lim_{u\to\infty} \left(\frac{1}{3} \arctan(x/3) \Big|_0^u \right)$$

$$= \frac{2}{3} \lim_{u\to\infty} (\arctan u - \arctan 0) = \frac{2}{3}(\arctan(\infty) - \arctan(0)) = \frac{\pi}{3}$$ yakınsak

Ör) $I = \int_{1}^{\infty} \frac{dx}{x^3} = \lim_{u\to\infty} \int_{1}^{u} \frac{dx}{x^3} = \lim_{u\to\infty} \left(-\frac{1}{2x^2} \Big|_1^u \right) = \lim_{u\to\infty} \left(-\frac{1}{2u^2} + \frac{1}{2} \right) = 1/2$ yakınsak

Ör) ① $\int_0^e \ln x\,dx$ ve ② $\int_e^\infty \ln x\,dx$ integrallerini yakınsaklık bakımından incele

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx$$

$u = \ln x \quad dv = dx$

$du = dx/x \quad v = x$

$$= x\ln x - x + c$$
$$= x(\ln x - 1) + c$$

① $$\int_0^e \ln x\,dx = \lim_{u\to 0^+} \int_u^e \ln x\,dx = \lim_{u\to 0^+} \left(x(\ln x - 1) \Big|_u^e \right) = \lim_{u\to 0^+} u.(1 - \ln u)$$

$$= \lim_{u\to 0^+} \frac{1-\ln u}{1/u} = \lim_{u\to 0^+} \frac{-1/u}{-1/u^2} = \lim_{u\to 0^+}(u) = 0 < \infty$$ yakınsak

② $$\int_e^\infty \ln x\,dx = \lim_{u\to\infty} \int_e^u \ln x\,dx = \lim_{u\to\infty} \left(x(\ln x - 1) \Big|_e^u \right) = \lim_{u\to\infty} u.(\ln u - 1)$$

$$= \infty$$ Iraksak.

Ör) $I = \int_0^\infty e^{-x}dx$ integrali yakınsak. Göster. Sonuç $= 1$

Ör) $I = \int_{-1}^{1} \frac{dx}{x^2}$ integrali ıraksak. Göster. Sonuç $= \infty$

Ör) $I = \int_0^3 \frac{dx}{(x-1)^2}$ integrali ıraksak. Göster.

$I = I_1 + I_2$

$I_1 = \int_0^1 \frac{dx}{(x-1)^2} = \infty$, $I_2 = \int_1^3 \frac{dx}{(x-1)^2} = \infty$

$x=1$ için ikiye ayırmak gerekiyor. Yoksa yakınsak çıkar.

Ör $I=\int_{-\infty}^{\infty}\frac{dx}{x}$ integrali yakınsak mıdır?

$x\in\mathbb{R}$ için $f(x)=\frac{1}{x}$ tek fonksiyon olduğundan $\int_{-\infty}^{\infty}\frac{dx}{x}=0<\infty$ yakınsak

Ör $I=\int_{-\infty}^{\infty}\frac{dx}{e^x+4e^{-x}}$ integrali yakınsak mıdır?

$$\int\frac{dx}{e^x+4e^{-x}}=\int\frac{e^x dx}{e^{2x}+4}=\int\frac{du}{u^2+4}=\frac{1}{2}\arctan\left(\frac{u}{2}\right)+C$$
$$=\frac{1}{2}\arctan\left(\frac{e^x}{2}\right)+C$$

$u=e^x\rightarrow du=e^x dx$

$$I=\int_{-\infty}^{\infty}\frac{dx}{e^x+4e^{-x}}=\lim_{u\to\infty}\int_{-u}^{u}\frac{dx}{e^x+4e^{-x}}=\lim_{u\to\infty}\left(\frac{1}{2}\arctan\left(\frac{e^x}{2}\right)\Big|_{-u}^{u}\right)$$
$$=\frac{1}{2}\lim_{u\to\infty}\left(\arctan\left(\frac{e^u}{2}\right)-\arctan\left(\frac{e^{-u}}{2}\right)\right)=\frac{1}{2}\left(\arctan(\infty)-\arctan(0)\right)$$
$$=\frac{1}{2}\left(\pi/2-0\right)=\pi/4<\infty$$ olduğundan integral yakınsak.

Ör $I=\int_{0}^{1}\frac{dx}{2\sqrt{x}}$ integrali yakınsak mıdır?

$$I=\lim_{u\to0^+}\int_{u}^{1}\frac{dx}{2\sqrt{x}}=\lim_{u\to0^+}\left(\sqrt{x}\Big|_{u}^{1}\right)=\lim_{u\to0^+}(1-\sqrt{u})=1<\infty$$ yakınsak

Ör $I=\int_{0}^{\pi/4}\frac{\sec^2x\,dx}{\sqrt{3+\tan x}}$ integrali yakınsak mıdır?

$u=\sqrt{3+\tan x}\rightarrow u^2-3=\tan x\rightarrow 2u\,du=\sec^2x\,dx$

$$\int\frac{\sec^2x\,dx}{\sqrt{3+\tan x}}=\int\frac{2u\,du}{u}=2\int du=2u+C=2.\sqrt{3+\tan x}+C$$

$$I=\int_{0}^{\pi/4}\frac{\sec^2x\,dx}{\sqrt{3+\tan x}}=2.\sqrt{3+\tan x}\Big|_{0}^{\pi/4}=2\sqrt{3+\tan\pi/4}-2\sqrt{3+\tan0}=4-2\sqrt{3}<\infty$$ yakınsak

Ör $I=\int_{e}^{\infty}\frac{dx}{\ln x}$ integralinin yakınsak olup olmadığını kıyaslama testi kullanarak göster.

$x\in[e,\infty)$ için $\ln x\le x<\infty\longrightarrow 0<\frac{1}{x}\le\frac{1}{\ln x}$

$$\int_{e}^{\infty}\frac{dx}{x}=\lim_{u\to\infty}\int_{e}^{u}\frac{dx}{x}=\lim_{u\to\infty}\left(\ln|x|\Big|_{e}^{u}\right)=\lim_{u\to\infty}(\ln u-1)=\infty$$ ıraksak

$\int_{e}^{\infty}\frac{dx}{x}$ ıraksak olduğundan $\int_{e}^{\infty}\frac{dx}{\ln x}$ de ıraksaktır.

Ör. $I=\int_0^\infty e^{-x}\cos x\,dx$ integralinin yakınsak olup olmadığını kıyaslama testi kullanarak göster.

$$|\cos x|\leq 1 \longrightarrow e^{-x}|\cos x|\leq e^{-x}\longrightarrow |e^{-x}\cos x|\leq e^{-x}$$

$$\int_0^\infty e^{-x}dx=\lim_{u\to\infty}\int_0^u e^{-x}dx=\lim_{u\to\infty}\left(-e^{-x}\Big|_0^u\right)=\lim_{u\to\infty}(-e^{-u}+1)=\underset{\text{yakınsak}}{1<\infty}$$

$\int_0^\infty e^{-x}dx$ yakınsak olduğundan $\int_0^\infty |e^{-x}\cos x|dx$ de yakınsaktır.

$\int_0^\infty e^{-x}\cos x\,dx$ ise mutlak yakınsaktır.

Ör. $I=\int_2^\infty \frac{\cos x}{x^2}dx$ integralinin yakınsak olup olmadığını kıyas. testi ile göster.

$$|\cos x|\leq 1 \longrightarrow \frac{1}{x^2}|\cos x|\leq \frac{1}{x^2}\longrightarrow \left|\frac{\cos x}{x^2}\right|\leq\frac{1}{x^2}$$

$$\int_2^\infty\frac{dx}{x^2}=\lim_{u\to\infty}\int_2^u\frac{dx}{x^2}=\lim_{u\to\infty}\left(-\frac{1}{x}\Big|_2^u\right)=\lim_{u\to\infty}\left(-\frac{1}{u}+\frac{1}{2}\right)=\underset{\text{yakınsak}}{\frac{1}{2}<\infty}$$

$\int_2^\infty\frac{dx}{x^2}$ yakınsak olduğundan $\int_2^\infty\left|\frac{\cos x}{x^2}\right|dx$ de yakınsaktır.

$\int_2^\infty\frac{\cos x}{x^2}dx$ ise mutlak yakınsaktır.

Ör. $I=\int_1^\infty\frac{dx}{\sqrt{x^4+3}}$ integrali yakınsak mı?

$$x^4+3>x^4\longrightarrow\sqrt{x^4+3}>x^2\longrightarrow\frac{1}{\sqrt{x^4+3}}<\frac{1}{x^2}$$

$$\int_1^\infty\frac{dx}{x^2}=\lim_{u\to\infty}\int_1^u\frac{dx}{x^2}=\lim_{u\to\infty}\left(-\frac{1}{x}\Big|_1^u\right)=\lim_{u\to\infty}\left(-\frac{1}{u}+1\right)=\underset{\text{yakınsak}}{1<\infty}$$

$\int_1^\infty\frac{dx}{x^2}$ integrali yakınsak olduğundan

$\int_1^\infty\frac{dx}{\sqrt{x^4+3}}$ de yakınsaktır.

Alan Hesabı

x eksenine göre alan

$x \in [x_1, x_2]$ için $f(x) \geqslant g(x)$ ise

$$A = \int_{x_1}^{x_2} (f(x) - g(x)) \cdot dx$$

y eksenine göre alan

$y \in [y_1, y_2]$ için $f(y) \geqslant g(y)$ ise

$$A = \int_{y_1}^{y_2} (f(y) - g(y)) \cdot dy$$

Ör $y = x+3$ doğrusu ile $y = x^2+1$ eğrisi arasında kalan bölgenin alanı?

$y = y$

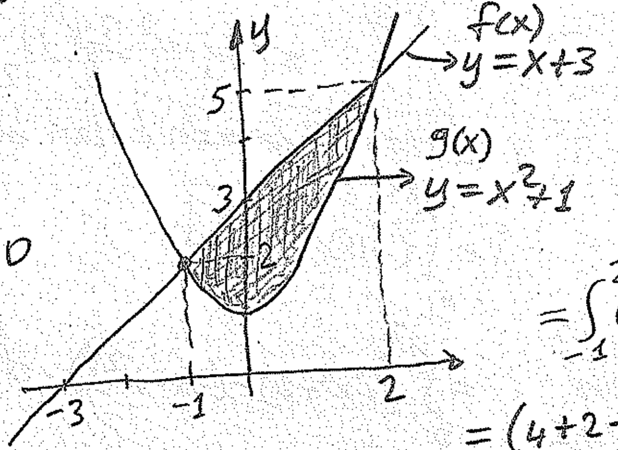
$x+3 = x^2+1$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x=-1 \quad x=2$

$y=2 \quad y=5$



$$A = \int_{x_1}^{x_2} (f(x) - g(x)) \cdot dx$$

$$= \int_{-1}^{2} ((x+3) - (x^2+1)) \cdot dx$$

$$= \int_{-1}^{2} (2 + x - x^2) \cdot dx = \left(2x + \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-1}^{2}$$

$$= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = 9/2$$

Ör $x = y+2$ doğrusu ile $x = 4-y^2$ eğrisi arasında kalan bölgenin alanı?

$x = x$

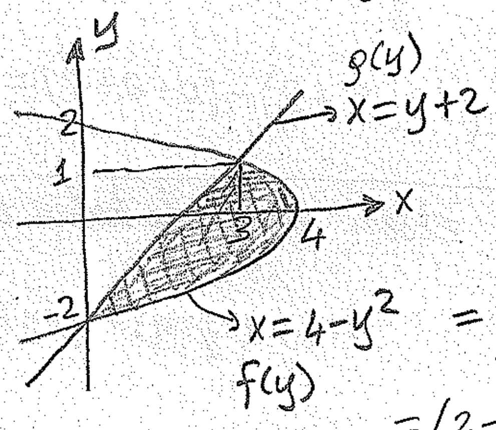
$y+2 = 4-y^2$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y=-2 \quad y=1$

$x=0 \quad x=3$



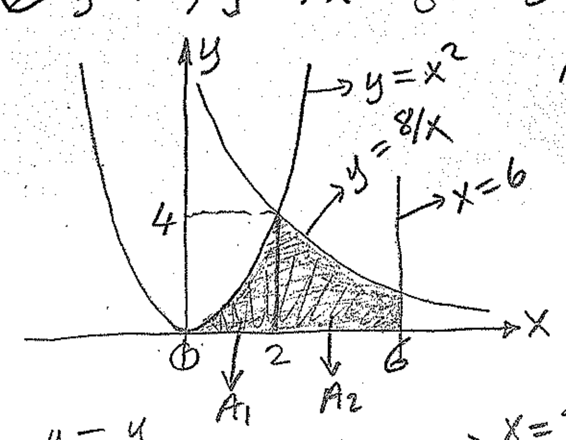
$$A = \int_{y_1}^{y_2} (f(y) - g(y)) \cdot dy$$

$$= \int_{-2}^{1} ((4-y^2) - (y+2)) \cdot dy$$

$$= \int_{-2}^{1} (2 - y - y^2) dy = \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = 9/2$$

Ör $y = x^2$, $y = 8/x$ eğrileriyle $x=6$, $y=0$ doğruları arasında kalan bölge?



$y = y$

$x^2 = 8/x \longrightarrow x^3 = 8 \longrightarrow \begin{matrix} x=2 \\ y=4 \end{matrix}$

$$A = \int_0^2 (x^2 - 0) dx + \int_2^6 \left(\frac{8}{x} - 0\right) dx$$

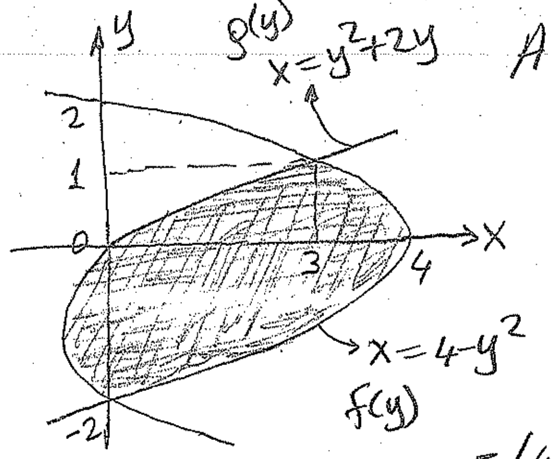
$$= \int_0^2 x^2 dx + 8 \int_2^6 \frac{dx}{x} = \frac{x^3}{3}\Big|_0^2 + 8 \ln|x| \Big|_2^6$$

$$= \frac{8-0}{3} + 8(\ln 6 - \ln 2)$$

$$= \frac{8}{3} + 8 \ln 3$$

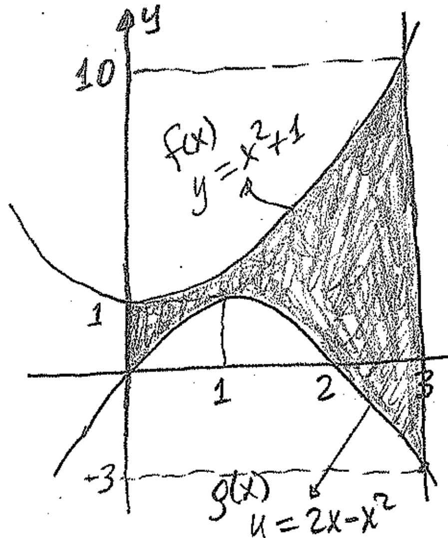
Ör) $x = 4 - y^2$, $x = y^2 + 2y$ eğrileri arasında kalan bölgenin alanı?

$$x = x$$
$$4 - y^2 = y^2 + 2y$$
$$2y^2 + 2y - 4 = 0$$
$$y^2 + y - 2 = 0$$
$$(y+2)(y-1) = 0$$
$$y = -2 \quad y = 1$$
$$x = 0 \quad x = 3$$



$$A = \int_{y_1}^{y_2} (f(y) - g(y)) \cdot dy$$
$$= \int_{-2}^{1} \left((4 - y^2) - (y^2 + 2y) \right) dy$$
$$= \int_{-2}^{1} (4 - 2y - 2y^2)\, dy$$
$$= \left(4y - y^2 - \frac{2y^3}{3}\right) \Big|_{-2}^{1} = 9$$

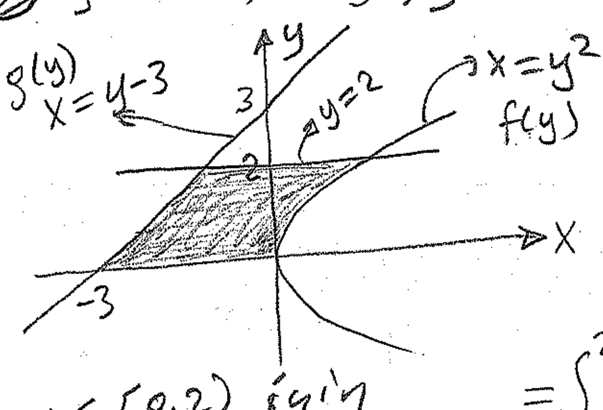
Ör) $y = x^2 + 1$, $y = 2x - x^2$, $x = 3$, y ekseni arasında kalan bölgenin alanı?



$x \in [0,3]$ için $y = x^2 + 1$ üstte

$$A = \int_{x_1}^{x_2} (f(x) - g(x))\, dx$$
$$= \int_{0}^{3} \left((x^2 + 1) - (2x - x^2) \right) \cdot dx$$
$$= \int_{0}^{3} (2x^2 - 2x + 1)\, dx = \left(\frac{2x^3}{3} - x^2 + x \right) \Big|_{0}^{3}$$
$$= (18 - 9 + 3) - (0 - 0 + 0) = 12$$

Ör) $y = x + 3$, $x = y^2$, $y = 2$, x ekseni arasında kalan bölgenin alanı?



$y \in [0,2)$ için
$x = y^2$ üstte
$x = y - 3$ altta

$$A = \int_{y_1}^{y_2} (f(y) - g(y)) \cdot dy$$
$$= \int_{0}^{2} \left((y^2) - (y - 3) \right) dy$$
$$= \int_{0}^{2} (y^2 - y + 3)\, dy$$
$$= \left(\frac{y^3}{3} - \frac{y^2}{2} + 3y \right) \Big|_{0}^{2}$$
$$= \left(\frac{8}{3} - 2 + 6 \right) - (0 - 0 + 0) = \frac{20}{3}$$

Ör $y=x^2$, $y=\sqrt{8x}$ eğrileri arasında kalan bölgenin alanı ?

$y=y$

$x^2=\sqrt{8x}$

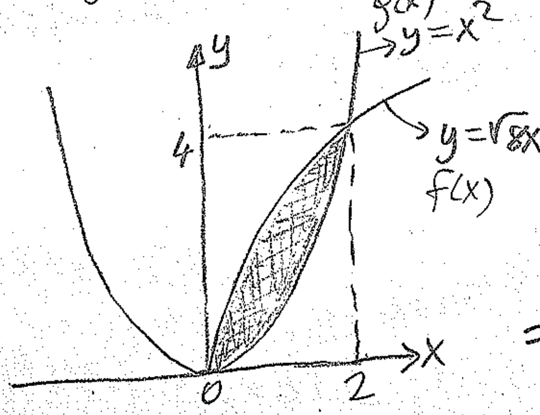
$x^4=8x$

$x^4-8x=0$

$x(x^3-8)=0$

$x=0 \quad x=2$

$y=0 \quad y=4$



$x\in[0,2]$

$$A=\int_{x_1}^{x_2}(f(x)-g(x)).dx$$

$$=\int_0^2(\sqrt{8x}-x^2).dx$$

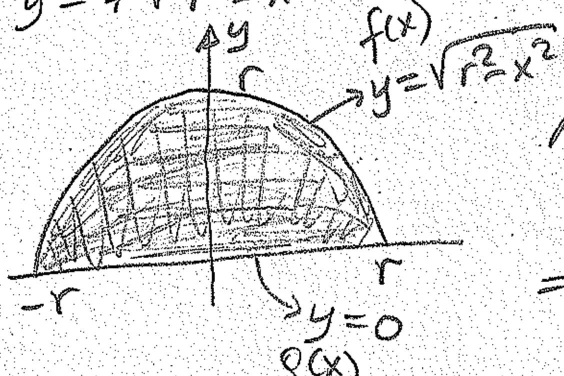
$$=\left(\sqrt{8}\,\frac{x^{3/2}}{3/2}-\frac{x^3}{3}\right)\Big|_0^2$$

$$=\frac{16}{3}-\frac{8}{3}=8/3$$

Ör Yarım Dairenin Alanı

$x^2+y^2=r^2$

$y=\mp\sqrt{r^2-x^2}$



$$A=\int_{x_1}(f(x)-g(x))dx=\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

$x=r\sin\theta \rightarrow dx=r\cos\theta\,d\theta$

$\sin\theta=x/r \rightarrow \theta=\arcsin(x/r)$

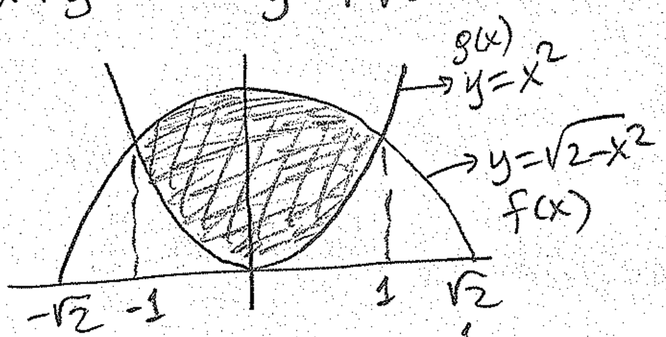
$$A=\int_{-\pi/2}^{\pi/2}\sqrt{r^2-r^2\sin^2\theta}\,.\,r\cos\theta\,d\theta$$

$$=r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta=\frac{r^2}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)d\theta$$

$$=\frac{r^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_{-\pi/2}^{\pi/2}=\frac{\pi r^2}{2}$$

Ör $y=x^2$ parabolünün $x^2+y^2=2$ çemberinden ayırdığı bölgenin alanı?

$x^2+y^2=2 \rightarrow y=\mp\sqrt{2-x^2}$



$x\in[-1,1]$ için

$y=\sqrt{2-x^2}$ üstte

$y=y$

$x^2=\sqrt{2-x^2}$

$x^4+x-2=0$

$\underbrace{(x^2+2)}_{+}(x^2-1)=0$

$x=\mp 1$

$$A=\int_{x_1}^{x_2}(f(x)-g(x))\,dx=\int_{-1}^{1}(\sqrt{2-x^2}-x^2)\,dx=\int_{-1}^{1}\sqrt{2-x^2}\,dx-\frac{x^3}{3}\Big|_{-1}^{1}$$

$$=-\frac{2}{3}+\int_{-\pi/4}^{\pi/4}\sqrt{2-2\sin^2\theta}\,.\,\sqrt{2}\cos\theta\,d\theta=-\frac{2}{3}+\int_{-\pi/4}^{\pi/4}2\cos^2\theta\,d\theta=-\frac{2}{3}+\int_{-\pi/4}^{\pi/4}(1+\cos 2\theta)d\theta$$

$x=\sqrt{2}\sin\theta \rightarrow dx=\sqrt{2}\cos\theta\,d\theta$

$$=-\frac{2}{3}+\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_{-\pi/4}^{\pi/4}=\frac{3\pi+2}{6}$$

Ör $x=2y^2$, $x=y^2+4$ eğrileri arasında kalan bölgenin alanı?

$x = x$

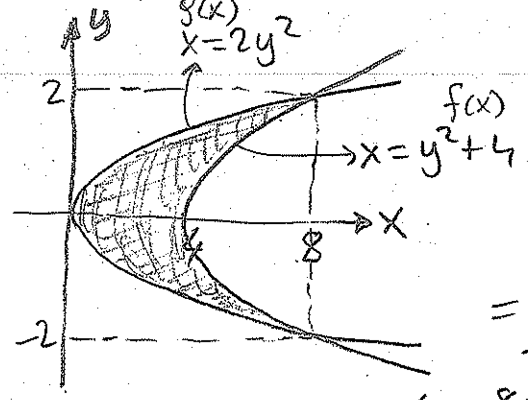
$2y^2 = y^2+4$

$y^2-4=0$

$y=\mp 2 \rightarrow x=8$

$y\in[-2,2]$ için

$x=y^2+4$ üstte



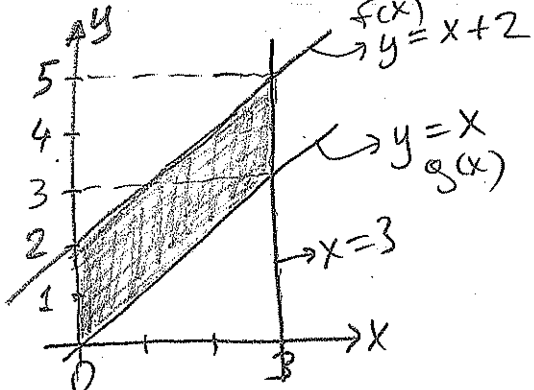
$$A=\int_{y_1}^{y_2}(f(y)-g(y))\,dy$$

$$=\int_{-2}^{2}\left((y^2+4)-(2y^2)\right)dy$$

$$=\int_{-2}^{2}(4-y^2)dy=\left(4y-\frac{y^3}{3}\right)\Big|_{-2}^{2}$$

$$=\left(8-\tfrac{8}{3}\right)-\left(-8+\tfrac{8}{3}\right)=\frac{32}{3}$$

Ör $y=x+2$, $y=x$, $x=0$, $x=3$ fonksiyonları arasında kalan bölge?

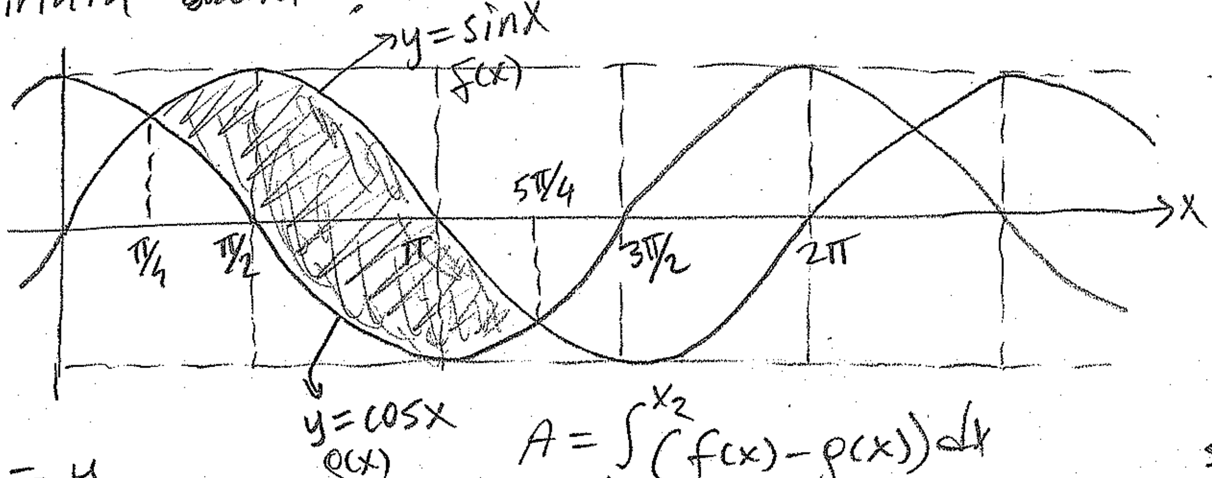


$x\in[0,3]$ için $y=x+2$ üstte

$$A=\int_{x_1}^{x_2}(f(x)-g(x))dx=\int_0^3\left((x+2)-(x)\right)dx$$

$$=\int_0^3 2dx = 2x\Big|_0^3 = 6$$

Ör $y=\sin x$, $y=\cos x$ eğrileri arasında kalan bölgelerden birinin alanı?



$$A=\int_{x_1}^{x_2}(f(x)-g(x))dx$$

$y=y$

$\sin x=\cos x$

$x=\pi/4$ ve $5\pi/4$

$x\in[\pi/4, 5\pi/4]$ için

$y=\sin x$ üstte

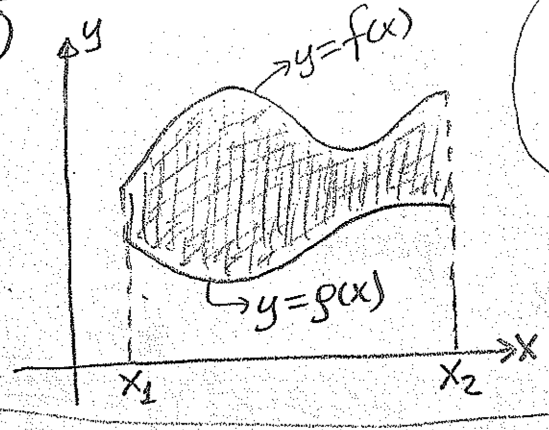
$$A=\int_{\pi/4}^{5\pi/4}(\sin x-\cos x)dx=(-\cos x-\sin x)\Big|_{\pi/4}^{5\pi/4}$$

$$=-\left(\cos\tfrac{5\pi}{4}+\sin\tfrac{5\pi}{4}\right)+\left(\cos\tfrac{\pi}{4}+\sin\tfrac{\pi}{4}\right)$$

$$=\frac{\sqrt2}{2}+\frac{\sqrt2}{2}+\frac{\sqrt2}{2}+\frac{\sqrt2}{2}=2\sqrt2$$

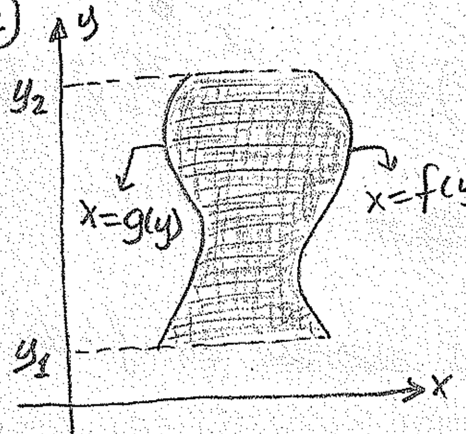
Hacim Hesapları

$$V=\int_{x_1}^{x_2} A(x)\cdot dx \qquad V=\int_{y_1}^{y_2} A(y)\cdot dy \qquad V=\int_{z_1}^{z_2} A(z)\cdot dz$$

① 

x ekseni etrafında döndürülürse
$x\in[x_1,x_2]$ için $f(x)\geq g(x)\geq 0$
$$V=\pi\int_{x_1}^{x_2}\left(f^2(x)-g^2(x)\right)\cdot dx$$

y ekseni etrafında döndürülürse
$x\in[x_1,x_2]$ için $f(x)\geq g(x)$ ve $x_2\geq x_1\geq 0$
$$V=2\pi\int_{x_1}^{x_2} x\cdot\left(f(x)-g(x)\right)\cdot dx$$

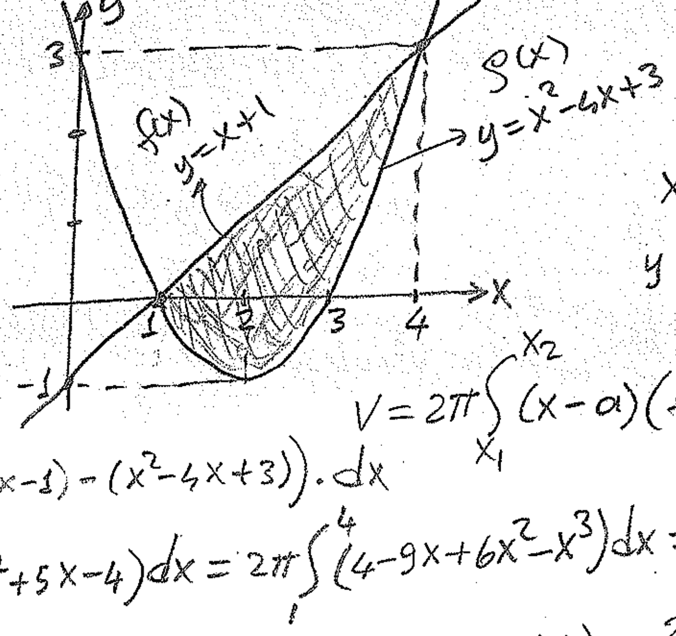
② 

x ekseni etrafında döndürülürse
$y\in[y_1,y_2]$ için $f(y)\geq g(y)$ ve $y_2\geq y_1\geq 0$
$$V=2\pi\int_{y_1}^{y_2} y\cdot\left(f(y)-g(y)\right)\cdot dy$$

y ekseni etrafında döndürülürse
$y\in[y_1,y_2]$ için $f(y)\geq g(y)\geq 0$
$$V=\pi\int_{y_1}^{y_2}\left(f^2(y)-g^2(y)\right)\cdot dy$$

Ör: $y=x^2-4x+3$ eğrisiyle $y=x-1$ doğrusu arasında kalan bölge $x=1$ doğrusu etrafında döndürülürse oluşan hacim?

$y=y$
$x^2-4x+3=x-1$
$x^2-5x+4=0$
$(x-1)(x-4)=0$
$x=1 \quad x=4$
$y=0 \quad y=3$



$y=x^2-4x+3$
$=(x-1)(x-3)$

$x\in[1,4]$ için
$y=x+1$ doğrusu üstte

$$V=2\pi\int_{x_1}^{x_2}(x-a)\left(f(x)-g(x)\right)\cdot dx$$

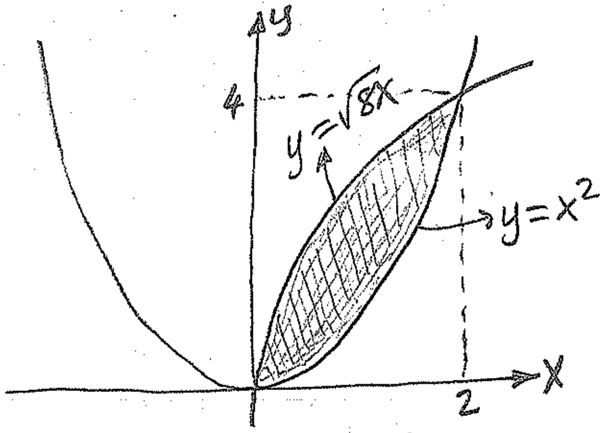
$$V=2\pi\int_1^4(x-1)\left((x-1)-(x^2-4x+3)\right)\cdot dx$$

$$=2\pi\int_1^4(x-1)(-x^2+5x-4)dx=2\pi\int_1^4(4-9x+6x^2-x^3)dx=2\pi\left(4x-\frac{9x^2}{2}+2x^3-\frac{x^4}{4}\right)\Big|_1^4$$

$$=2\pi\left((16-72+128-64)-\left(4-\tfrac{9}{2}+2-\tfrac{1}{4}\right)\right)=\frac{27}{2}\pi$$

Ör $y=x^2$, $y=\sqrt{8x}$ eğrileri arasında kalan bölge
a) x ekseni etrafında döndürülürse oluşan hacim?
b) y ekseni etrafında döndürülürse oluşan hacim?

$y=y$
$x^2=\sqrt{8x}$
$x^4=8x$
$x^4-8x=0$
$x(x^3-8)=0$
$x=0 \quad x=2$
$y=0 \quad y=4$



$x\in[0,2]$ için
$y=\sqrt{8x}$ üstte
$f(x)=\sqrt{8x}$
$g(x)=x^2$

a) $V=\pi\int_{x_1}^{x_2}(f^2(x)-g^2(x)).dx=\pi\int_0^2(8x-x^4)dx=\pi\left(4x^2-\frac{x^5}{5}\right)\Big|_0^2=\frac{48\pi}{5}$

b) $V=2\pi\int_{x_1}^{x_2}x.(f(x)-g(x)).dx=2\pi\int_0^2 x.(\sqrt{8x}-x^2).dx=2\pi\int_0^2(\sqrt{8}x^{3/2}-x^3).dx$

$=2\pi\left(\sqrt{8}\frac{x^{5/2}}{5/2}-\frac{x^4}{4}\right)\Big|_0^2=2\pi\left(\frac{32}{5}-4\right)=\frac{24}{5}\pi$

Ör $x=y^2-2y-3$, $x=-y^2+4y-3$ eğrileri $x=1$ doğrusu etrafında döndürülürse oluşan hacim.

$x=x$
$y^2-2y-3=-y^2+4y-3$
$2y^2-6y=0$
$2y(y-3)=0$
$y=0 \quad y=3$
$x=-3 \quad x=0$

$x=y^2-2y-3=(y-3)(y+1)\longrightarrow g(y)$
$x=-y^2+4y-3=-(y-1)(y-3)\longrightarrow f(y)$

$V=\pi\int_{y_1}^{y_2}\left((a-g(y))^2-(a-f(y))^2\right).dy$

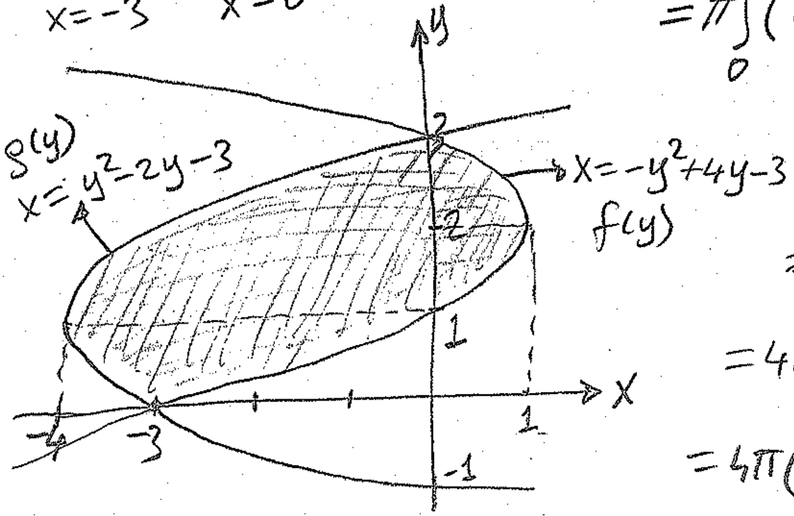
$=\pi\int_0^3\left((4+2y-y^2)^2-(4-4y+y^2)^2\right).dy$

$=\pi\int_0^3(8-2y)(6y-2y^2)dy$

$=4\pi\int_0^3(12y-7y^2+y^3)dy$

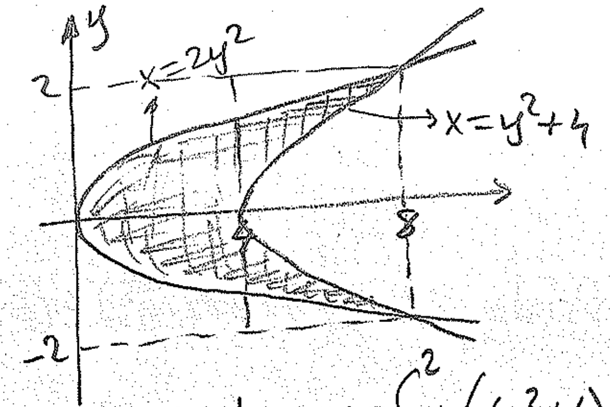
$=4\pi\left(6y^2-\frac{7}{3}y^3+\frac{y^4}{4}\right)\Big|_0^3$

$=4\pi\left(54-63+\frac{81}{4}\right)=45\pi$



Ör $x=2y^2$, $x=y^2+4$ eğrileri arasında kalan bölge
a) x ekseni etrafında döndürülürse oluşan hacim
b) y ekseni etrafında döndürülürse oluşan hacim

$x=x$
$2y^2=y^2+4$
$y^2=4$
$y=\mp 2 \to x=8$



$y\in[-2,2]$ için
$f(y)=y^2+4$
$g(y)=2y^2$

a) $V=2\pi\int_{y_1}^{y_2} y\,(f(y)-g(y)).dy = 2\pi\int_0^2 y\left((y^2+4)-(2y^2)\right).dy$

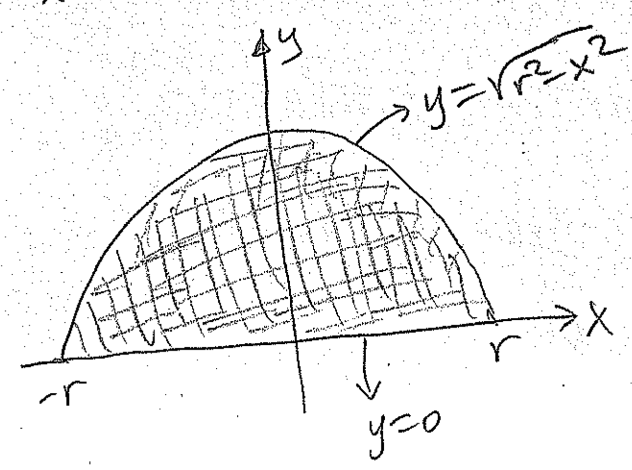
$=2\pi\int_0^2 (4y-y^3)dy = 2\pi\left(2y^2-\frac{y^4}{4}\right)\Big|_0^2 = 2\pi(8-4)=8\pi$

b) $V=\pi\int_{y_1}^{y_2}\left(f^2(y)-g^2(y)\right).dy = \pi\int_{-2}^{2}\left((y^2+4)^2-(2y^2)^2\right).dy$

$=\pi\int_{-2}^{2}(16+8y^2-3y^4)dy = \pi\left(16y+\frac{8}{3}y^3-\frac{3}{5}y^5\right)\Big|_{-2}^{2}$

$=\pi\left(\left(32+\frac{64}{3}-\frac{96}{5}\right)-\left(-32-\frac{64}{3}+\frac{96}{5}\right)\right)=2\pi\left(32+\frac{32}{15}\right)=\frac{1024}{15}\pi$

Ör $y=\sqrt{r^2-x^2}$ eğrisiyle $y=0$ doğrusu arasında kalan bölge x ekseni etrafında döndürülürse oluşan hacim (Küre)?



$f(x)=\sqrt{r^2-x^2}$ $\quad x\in[-r,r]$
$g(x)=0$

$V=\pi\int_{x_1}^{x_2}\left(f^2(x)-g^2(x)\right).dx$

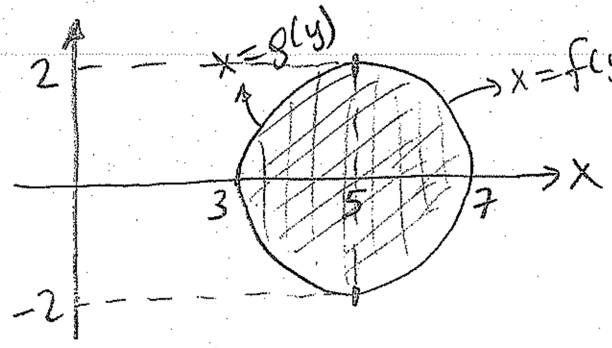
$=\pi\int_{-r}^{r}(r^2-x^2).dx$

$=\pi\left(r^2x-\frac{x^3}{3}\right)\Big|_{-r}^{r}$

$=\pi\left(\left(r^3-\frac{r^3}{3}\right)-\left(-r^3+\frac{r^3}{3}\right)\right)$

$=\pi\left(2r^3-\frac{2r^3}{3}\right)=\frac{4\pi r^3}{3}$

(8) $(x-5)^2+y^2=4$ dairesi y ekseni etr. dön. oluşan hacim? (124)



$(x-5)^2+y^2=4$

$x=5\mp\sqrt{4-y^2}$

$f(y)=5+\sqrt{4-y^2}$ $\quad y\in[-2,2]$ aralığı

$g(y)=5-\sqrt{4-y^2}$

$$V=\pi\int_{y_1}^{y_2}\left(f^2(y)-g^2(y)\right).dy=\pi\int_{-2}^{2}\left((5+\sqrt{4-y^2})^2-(5-\sqrt{4-y^2})^2\right).dy$$

$$=\pi\int_{-2}^{2}\left((25+10\sqrt{4-y^2}+4-y^2)-(25-10\sqrt{4-y^2}+4-y^2)\right).dy$$

$$=20\pi\int_{-2}^{2}\sqrt{4-y^2}\,dy=20\pi\int_{-\pi/2}^{\pi/2}\sqrt{4-4\sin^2\theta}\,.\,2\cos\theta\,d\theta$$

$y=2\sin\theta$

$dy=2\cos\theta\,d\theta$

$\sin\theta=y/2$

$$=80\pi\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta=40\pi\int_{-\pi/2}^{\pi/2}(1+\cos2\theta)\,d\theta$$

$$=40\pi\left(\theta+\frac{\sin2\theta}{2}\right)\Big|_{-\pi/2}^{\pi/2}=40\pi\left((\pi/2+0)-(-\pi/2+0)\right)=40\pi^2$$

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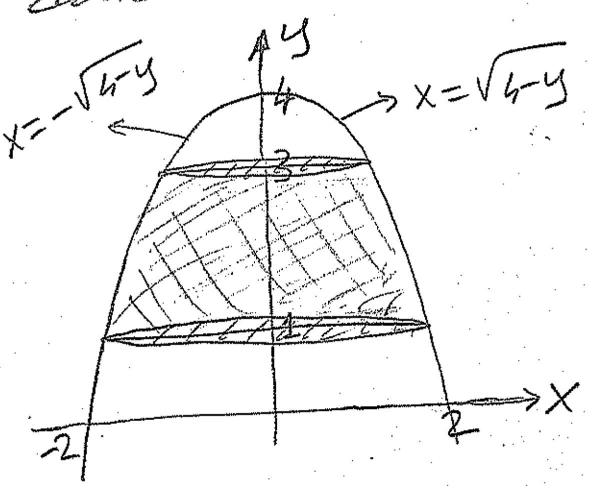
(9) $y=4-x^2$ eğrisi $y\in[1,3]$ aralığı için y ekseni etrafında döndürülürse oluşan hacim?



$$V=\pi\int_{y_1}^{y_2}\left(f^2(y)-g^2(y)\right).dy$$

$$=\pi\int_{1}^{3}(4-y).dy=\pi\left(4y-y^2/2\right)\Big|_1^3$$

$$=\pi\left((12-9/2)-(4-1/2)\right)$$

$$=4\pi$$

$y=4-x^2$

$x^2=4-y$

$x=\mp\sqrt{4-y}$

$x=0\rightarrow y=4$

$y=0\rightarrow x=\mp2$

$y\in[1,3]$ için

$f(y)=\sqrt{4-y}$, $g(y)=0$

(125)

Ör) $y=9-x^2$ eğrisi ile $y=5$ doğrusu arasında kalan bölge $y=3$ doğrusu etrafında döndürülürse oluşan hacim.

$y=y$

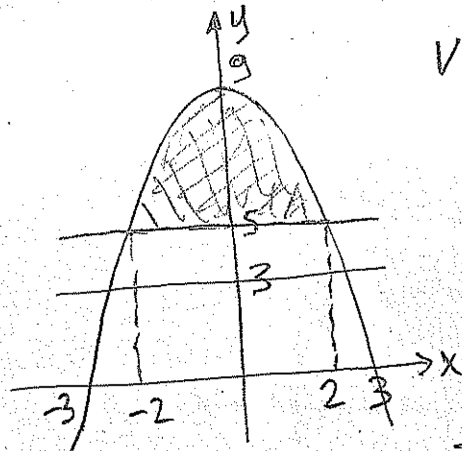
$9-x^2=5$

$x^2=4$

$x=\mp 2 \to y=5$

$f(x)=9-x^2$

$g(x)=5$



$$V=\pi\int_{x_1}^{x_2}\left(\left(f(x)-a\right)^2-\left(g(x)-a\right)^2\right).dx$$

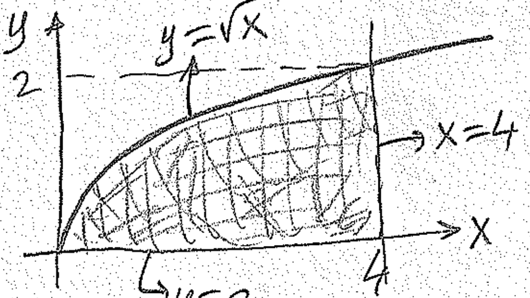
$$=\pi\int_{-2}^{2}\left((9-x^2-3)^2-(5-3)^2\right).dx$$

$$=\pi\int_{-2}^{2}(32-12x^2+x^4).dx$$

$$=\pi\left(32x-4x^3+\frac{x^5}{5}\right)\Big|_{-2}^{2}=\frac{96}{5}\pi$$

Ör) $y=\sqrt{x}$, $y=0$, $x=4$ fonksiyonları arasında kalan bölge

a) x ekseni b) y ekseni etr. dön. oluşan hacimler?



$x\in[0,4]$ için
$f(x)=\sqrt{x}$
$g(x)=0$

$y\in[0,2]$ için
$f(y)=4$
$g(y)=y^2$

a) $$V=\pi\int_{x_1}^{x_2}\left(f^2(x)-g^2(x)\right)dx=\pi\int_0^4\left((\sqrt{x})^2-(0)^2\right)dx=\pi\int_0^4 x\,dx=\frac{\pi x^2}{2}\Big|_0^4=8\pi$$

diğer yol

$$V=2\pi\int_{y_1}^{y_2}y\left(f(y)-g(y)\right).dy=2\pi\int_0^2 y(4-y^2)dy=2\pi\int_0^2(4y-y^3)dy$$

$$=2\pi\left(2y^2-\frac{y^4}{4}\right)\Big|_0^2=2\pi\left((8-4)-(0-0)\right)=8\pi$$

b) $$V=2\pi\int_{x_1}^{x_2}x\left(f(x)-g(x)\right)dx=2\pi\int_0^4 x(\sqrt{x}-0)dx=2\pi\int_0^4 x^{3/2}dx$$

$$=2\pi\frac{x^{5/2}}{5/2}\Big|_0^4=\frac{4\pi}{5}\left(4^{5/2}-0^{5/2}\right)=\frac{128}{5}\pi$$

diğer yol

$$V=\pi\int_{y_1}^{y_2}\left(f^2(y)-g^2(y)\right)dy=\pi\int_0^2(16-y^4)dy$$

$$=\pi\left(16y-\frac{y^5}{5}\right)=\pi\left(\left(32-\frac{32}{5}\right)-(0-0)\right)=\frac{128}{5}\pi$$

(126)

Ör) $y=x+3$ doğrusuyla $y=x^2+1$ eğrisi arasında kalan bölge
a) x ekseni c) $x=-1$ doğrusu etraflarında döndürülürse
b) $y=1$ doğrusu d) $x=2$ doğrusu oluşan hacimler?

Cevap

$y=y$

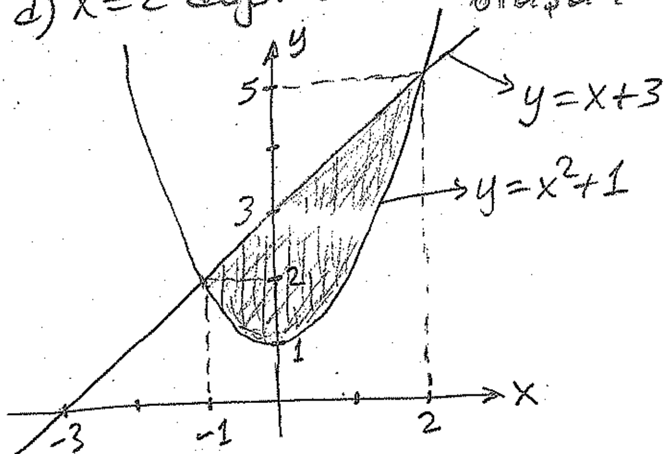
$x+3=x^2+1$

$x^2-x-2=0$

$(x+1)(x-2)=0$

$x=-1 \quad x=2$

$y=2 \quad y=5$



$x\in[-1,2]$ için

$f(x)=x+3$

$g(x)=x^2+1$

a) $V=\pi\int_{x_1}^{x_2}\left(f^2(x)-g^2(x)\right).dx=\pi\int_{-1}^{2}\left((x+3)^2-(x^2+1)^2\right).dx$

$=\pi\int_{-1}^{2}(8+6x-x^2-x^4).dx=\pi\left(8x+3x^2-\frac{x^3}{3}-\frac{x^5}{5}\right)\Big|_{-1}^{2}$

$=\pi\left(\left(16+12-\frac{8}{3}-\frac{32}{5}\right)-\left(-8+3+\frac{1}{3}+\frac{1}{5}\right)\right)=\frac{117}{5}\pi$

b) $V=\pi\int_{x_1}^{x_2}\left((f(x)-b)^2-(g(x)-b)^2\right).dx=\pi\int_{-1}^{2}\left((x+2)^2-(x^2)^2\right).dx$

$=\pi\int_{-1}^{2}(4+4x+x^2-x^4)dx=\pi\left(4x+2x^2+\frac{x^3}{3}-\frac{x^5}{5}\right)\Big|_{-1}^{2}$

$=\pi\left(\left(8+8+\frac{8}{3}-\frac{32}{5}\right)-\left(-4+2-\frac{1}{3}+\frac{1}{5}\right)\right)=\frac{72}{5}\pi$

c) $V=2\pi\int_{x_1}^{x_2}(x-a)(f(x)-g(x))dx=2\pi\int_{-1}^{2}(x+1)\left((x+3)-(x^2+1)\right)dx$

$=2\pi\int_{-1}^{2}(2+3x-x^3)dx=2\pi\left(2x+\frac{3x^2}{2}-\frac{x^4}{4}\right)\Big|_{-1}^{2}$

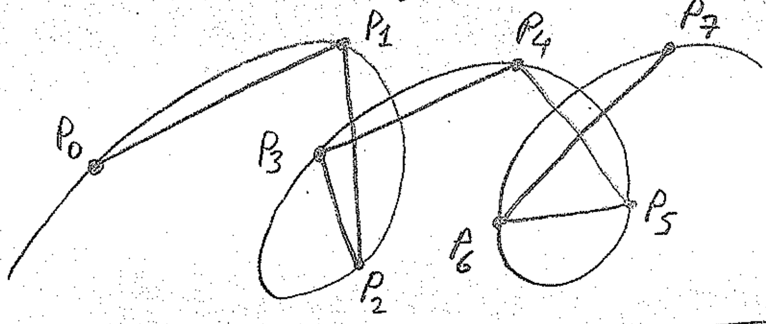
$=2\pi\left((4+6-4)-\left(-2+\frac{3}{2}-\frac{1}{4}\right)\right)=\frac{27}{2}\pi$

d) $V=2\pi\int_{x_1}^{x_2}(a-x)(f(x)-g(x))dx=2\pi\int_{-1}^{2}(2-x)\left((x+3)-(x^2+1)\right)dx$

$=2\pi\int_{-1}^{2}(2-x)(2+x-x^2)dx=2\pi\int_{-1}^{2}(4-3x^2+x^3).dx$

$=2\pi\left(4x-x^3+\frac{x^4}{4}\right)\Big|_{-1}^{2}=2\pi\left((8-8+4)-\left(-4+1+\frac{1}{4}\right)\right)=\frac{27}{2}\pi$

Eğri Uzunluğu



$x = x(t)$, $y = y(t)$ $\quad t_1 < t < t_2$

$P_i = (x(t_i), y(t_i))$

$i = 0, 1, 2, \ldots, n$

$$|P_{i-1}P_i| = \sqrt{(x(t_i) - x(t_{i-1}))^2 + ((y(t_i) - y(t_{i-1}))^2} \quad \text{Doğru uzunluğu}$$

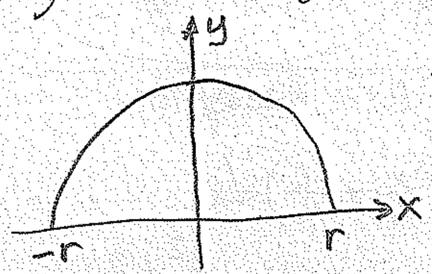
$$\text{Doğruların Toplam Uzunluğu} = \sum_{i=1}^{n} |P_{i-1}P_i|$$

$n \to \infty$ için $\Sigma \to \int$ dönüşür.

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$L = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

Ör $y = \sqrt{r^2 - x^2}$ eğrisinin uzunluğunu $x \in [-r, r]$ için hesapla.



$$y = \sqrt{r^2-x^2} \longrightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$$

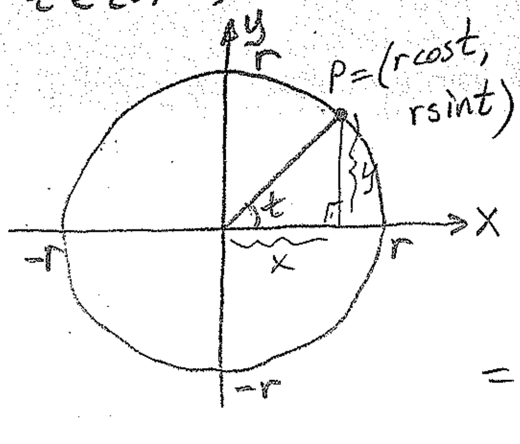
$$x = r\sin\theta \longrightarrow \sin\theta = x/r$$

$$dx = r\cos\theta\, d\theta$$

$$L = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_{-r}^{r} \sqrt{1+\frac{x^2}{r^2-x^2}}\, dx = \int_{-r}^{r} \frac{r\, dx}{\sqrt{r^2-x^2}} = \int_{-\pi/2}^{\pi/2} \frac{r \cdot r\cos\theta\, d\theta}{\sqrt{r^2 - r^2\sin^2\theta}}$$

$$= r\int_{-\pi/2}^{\pi/2} d\theta = r\theta\Big|_{-\pi/2}^{\pi/2} = r\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \pi r$$

Ör $x = r\cos t$, $y = r\sin t$ denklemleriyle verilen eğrinin uzunluğunu $t \in [0, 2\pi]$ için hesapla. (Daire çevresi)



$$\frac{dx}{dt} = -r\sin t \qquad \frac{dy}{dt} = r\cos t$$

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = r\int_0^{2\pi} dt$$

$$= rt\Big|_0^{2\pi} = 2\pi r$$

Ör) $x = 2e^{2t}$, $y = e^{3t}$ denk. verilen eğrinin uzun. $t \in (-\infty, 0]$ için hesapla.

$\frac{dx}{dt} = 4e^{2t}$, $\frac{dy}{dt} = 3e^{3t}$ $\qquad L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

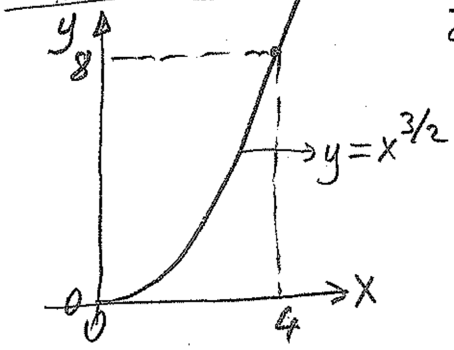
$y^2 = x^3/8$
$x \in [0,2]$
$L = ?$

$$L = \int_{-\infty}^{0} \sqrt{16e^{4t} + 9e^{6t}}\,.dt = \int_{-\infty}^{0} \sqrt{16 + 9e^{2t}}\,.e^{2t}.dt = \int_{4}^{5} u.\frac{u\,du}{9}$$

$u = \sqrt{16 + 9e^{2t}}$
$u^2 - 16 = 9e^{2t}$
$2u\,du = 18e^{2t}dt$
$e^{2t}dt = \frac{u\,du}{9}$

$$= \frac{1}{9}\int_{4}^{5} u^2 du = \frac{u^3}{27}\Big|_4^5 = \frac{125 - 64}{27} = \frac{61}{27}$$

Ör) $y = x^{3/2}$ eğrisinin uzunluğunu $x \in [0,4]$ için hesapla.



$\frac{dy}{dx} = \frac{3\sqrt{x}}{2}$ $\qquad L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$L = \int_0^4 \sqrt{1 + \frac{9x}{4}}\, dx$$

$$= \int_1^{\sqrt{10}} u.\frac{8u\,du}{9} = \frac{8}{9}\int_1^{\sqrt{10}} u^2 du$$

$$= \frac{8}{27}u^3\Big|_1^{\sqrt{10}} = \frac{8}{27}\left(\sqrt{1000} - 1\right) \cong 9$$

$u = \sqrt{1 + \frac{9x}{4}}$
$u^2 - 1 = \frac{9}{4}x$
$2u\,du = \frac{9}{4}dx$
$dx = \frac{8u\,du}{9}$

Ör) $y = \frac{x^2}{8} - \ln x$ eğrisinin uzunluğunu $x \in [1, e]$ için hesapla.

$\frac{dy}{dx} = \frac{x}{4} - \frac{1}{x}$ $\qquad L = \int_1^e \sqrt{1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2}\, dx = \int_1^e \left(\frac{x}{4} + \frac{1}{x}\right) dx$

$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.dx$ $\qquad = \left(\frac{x^2}{8} + \ln|x|\right)\Big|_1^e = \left(\frac{e^2}{8} + \ln e\right) - \left(\frac{1}{8} + \ln 1\right) = \frac{e^2 + 7}{8}$

Ör) $x = \frac{y^3}{6} + \frac{1}{2y}$ eğrisinin uzunluğunu $y \in [1,3]$ için hesapla

$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$ $\qquad L = \int_1^3 \sqrt{1 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2}\, dy = \int_1^3 \left(\frac{y^2}{2} + \frac{1}{2y^2}\right) dy$

$L = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}.dy$ $\qquad = \left(\frac{y^3}{6} - \frac{1}{2y}\right)\Big|_1^3 = \left(\frac{9}{2} - \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{14}{3}$

Ör) $x = \cos t + t\sin t$, $y = \sin t - t\cos t$ denk. ver. eğr. uzu. $0 \le t \le 6$

$\frac{dx}{dt} = t\cos t$ $\qquad L = \int_0^6 \sqrt{t^2\cos^2 t + t^2\sin^2 t}.dt = \int_0^6 t\,dt = \frac{t^2}{2}\Big|_0^6$

$\frac{dy}{dt} = t\sin t$ $\qquad = \frac{36 - 0}{2} = 18$

$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Ör: $y=\frac{x^4}{8}+\frac{1}{4x^2}$ eğrisinin uzunluğunu $x\in[1,2]$ için hesapla.

$\frac{dy}{dx}=\frac{x^3}{2}-\frac{1}{2x^3}$ $\quad L=\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$L=\int_1^2\sqrt{1+\left(\frac{x^3}{2}-\frac{1}{2x^3}\right)^2}\,dx=\int_1^2\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)dx=\left(\frac{x^4}{8}-\frac{1}{4x^2}\right)\Big|_1^2$$

$$=\left(2-\frac{1}{16}\right)-\left(\frac{1}{8}-\frac{1}{4}\right)=2-\frac{1}{16}-\frac{1}{8}+\frac{1}{4}=\frac{33}{16}$$

Ör: $x=\frac{1}{3}y^{3/2}-\sqrt{y}$ eğrisinin uzunluğunu $y\in[0,9]$ için hesapla.

$\frac{dx}{dy}=\frac{\sqrt{y}}{2}-\frac{1}{2\sqrt{y}}$ $\quad L=\int_0^9\sqrt{1+\left(\frac{\sqrt{y}}{2}-\frac{1}{2\sqrt{y}}\right)^2}\,dy=\int_0^9\left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)dy$

$L=\int_{y_1}^{y_2}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ $\quad =\left(\frac{1}{3}y^{3/2}+\sqrt{y}\right)\Big|_0^9=9+3=12$

Ör: $y=2x^{3/2}+3$ eğrisinin uzunluğunu $x\in[0,7]$ için hesapla.

$\frac{dy}{dx}=3\sqrt{x}$ $\quad L=\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^7\sqrt{1+9x}\,.dx$

$u=\sqrt{1+9x}$ $\quad =\int_1^8 u.\frac{2u\,du}{9}=\frac{2}{9}\int_1^8 u^2du=\frac{2u^3}{27}\Big|_1^8=\frac{2}{27}(512-1)=\frac{1022}{27}$

$u^2-1=9x$

$dx=\frac{2u\,du}{9}$

Ör: $x=2t-\sin(2t)$, $y=2\sin^2(t)$ denk. ver. eğrinin uzunluğunu $t\in[0,\pi]$ için hesapla.

$\frac{dx}{dt}=2-2\cos(2t)$ $\quad L=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$\frac{dy}{dt}=4\sin t.\cos t$

$=2\sin(2t)$ $\quad L=\int_0^\pi\sqrt{(2-2\cos 2t)^2+(2\sin 2t)^2}\,.dt$

$$L=\int_0^\pi\sqrt{4-8\cos 2t+4\cos^2 2t+4\sin^2 2t}\,.dt$$

$$=\int_0^\pi\sqrt{8-8\cos 2t}\,dt=\int_0^\pi\sqrt{8(1-\cos 2t)}\,dt=\int_0^\pi\sqrt{16.\sin^2 t}\,.dt$$

$$=4\int_0^\pi\sin t\,dt=-4\cos t\Big|_0^\pi=-4(\cos\pi-\cos 0)$$

$$=-4(-1-1)=8$$

Yüzey Alanı Hesabı

$x = x(t)$, $y = y(t)$, $t_1 \le t \le t_2$

$$S = 2\pi \int_{t_1}^{t_2} y \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

Eğrinin x ekseni etrafında döndürülmesiyle oluşan yüzey alanı

$$S = 2\pi \int_{t_1}^{t_2} x \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

Eğrinin y ekseni etrafında döndürülmesiyle oluşan yüzey alanı

$y = f(x)$, $x \in [x_1, x_2]$

$$S = 2\pi \int_{x_1}^{x_2} y \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

Eğrinin x ekseni etrafında döndürülmesiyle oluşan yüzey alanı

$$S = 2\pi \int_{x_1}^{x_2} x \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

Eğrinin y ekseni etrafında döndürülmesiyle oluşan yüzey alanı

$x = f(y)$, $y \in [y_1, y_2]$

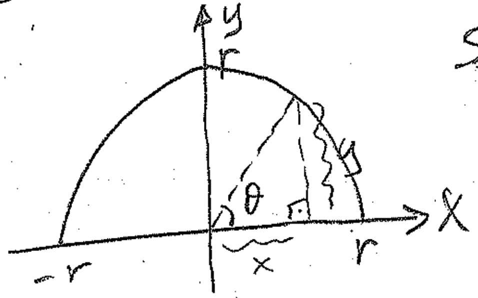
$$S = 2\pi \int_{y_1}^{y_2} y \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Eğrinin x ekseni etrafında döndürülmesiyle oluşan yüzey alanı

$$S = 2\pi \int_{y_1}^{y_2} x \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$$

Eğrinin y ekseni etrafında döndürülmesiyle oluşan yüzey alanı

Ör. $x = r\cos\theta$, $y = r\sin\theta$ eğrisi $\theta \in [0, \pi]$ için x ekseni etrafı. döndürülürse oluşan yüzey alanı?



$$S = 2\pi \int_{\theta_1}^{\theta_2} y \cdot \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\pi \int_0^{\pi} r\sin\theta \cdot \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2} \cdot d\theta$$

$$= 2\pi r^2 \int_0^{\pi} \sin\theta\, d\theta = -2\pi r^2 \cos\theta \Big|_0^{\pi}$$

$$= -2\pi r^2 (\cos\pi - \cos 0) = 4\pi r^2$$

$\frac{dx}{d\theta} = -r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta$

Ör) $y=\sqrt{x}$ eğrisinin x ekseni etrafında döndürülmesiyle oluşan yüzeyin alanını $x\in[2,6]$ için hesapla.

$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$ $\quad S=2\pi\int_{x_1}^{x_2} y\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = 2\pi\int_2^6 \sqrt{x}\cdot\sqrt{1+\frac{1}{4x}}\cdot dx$

$u=\sqrt{4x+1}$ $\quad =\pi\int_2^6\sqrt{4x+1}\cdot dx=\pi\int_3^5 u\cdot\frac{udu}{2}=\frac{\pi}{2}\int_3^5 u^2du=\frac{\pi}{6}u^3\Big|_3^5$

$u^2-1=4x$

$2udu=4dx$ $\quad =\frac{\pi}{6}(125-27)=\frac{49}{3}\pi$

$dx=\frac{udu}{2}$

Ör) $x=\frac{4}{3}y^{3/2}+2$ eğrisinin x ekseni etrafında döndürülmesiyle oluşan yüzeyin alanını $y\in[0,2]$ için hesapla.

$\frac{dx}{dy}=2\sqrt{y}$ $\quad S=2\pi\int_{y_1}^{y_2} y\cdot\sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy=2\pi\int_0^2 y\cdot\sqrt{1+4y}\cdot dy$

$u=\sqrt{1+4y}$ $\quad =2\pi\int_1^3\frac{u^2-1}{4}\cdot u\cdot\frac{udu}{2}=\frac{\pi}{4}\int_1^3(u^2-1)u^2du=\frac{\pi}{4}\int_1^3(u^4-u^2)du$

$u^2-1=4y$

$2udu=4dy$ $\quad =\frac{\pi}{4}\left(\frac{u^5}{5}-\frac{u^3}{3}\right)\Big|_1^3=\frac{\pi}{4}\left(\left(\frac{243}{5}-9\right)-\left(\frac{1}{5}-\frac{1}{3}\right)\right)=\frac{149}{15}\pi$

$dy=\frac{udu}{2}$

Ör) $y=\frac{1}{3}x^{3/2}-\sqrt{x}$ eğrisinin y ekseni etrafında döndürülmesiyle oluşan yüzeyin alanını $x\in[0,4]$ için hesapla.

$\frac{dy}{dx}=\frac{\sqrt{x}}{2}-\frac{1}{2\sqrt{x}}$ $\quad S=2\pi\int_{x_1}^{x_2} x\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx=2\pi\int_0^4 x\cdot\sqrt{1+\left(\frac{\sqrt{x}}{2}-\frac{1}{2\sqrt{x}}\right)^2}\cdot dx$

$=2\pi\int_0^4 x\left(\frac{\sqrt{x}}{2}+\frac{1}{2\sqrt{x}}\right)dx=\pi\int_0^4\left(x^{3/2}+x^{1/2}\right)dx$

$=\pi\left(\frac{x^{5/2}}{5/2}+\frac{x^{3/2}}{3/2}\right)\Big|_0^4=\pi\left(\left(\frac{64}{5}+\frac{16}{3}\right)-(0+0)\right)=\frac{272}{15}\pi$

Ör) $y=\frac{x^2}{8}-\ln x$ eğrisi y eks. etr. dön. oluşan yüzeyin alanını $x\in[0,6]$ için hesapla.

$\frac{dy}{dx}=\frac{x}{4}-\frac{1}{x}$ $\quad S=2\pi\int_{x_1}^{x_2} x\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$

$S=2\pi\int_0^6 x\cdot\sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\cdot dx=2\pi\int_0^6 x\cdot\left(\frac{x}{4}+\frac{1}{x}\right)\cdot dx$

$=2\pi\int_0^6\left(\frac{x^2}{4}+1\right)dx=2\pi\left(\frac{x^3}{12}+x\right)\Big|_0^6=2\pi(18+6)=48\pi$

Ör) $y=e^{-x}$ eğrisi x ekseni etr. dön. oluşan yüzeyin alanını $x\in[0,\infty)$ için hesapla.

$\int\sqrt{1+u^2}\,du=\frac{u\cdot\sqrt{1+u^2}+\ln\left(u+\sqrt{1+u^2}\right)}{2}+C$ kullan

çöz.

Ör) $y=\sqrt{r^2-x^2}$ eğrisi $x\in[-r,r]$ için x eks. etr. dön. oluşan yüzeyin alanı.

$$y=\sqrt{r^2-x^2} \longrightarrow \frac{dy}{dx}=\frac{-x}{\sqrt{r^2-x^2}} \qquad S=2\pi\int_{x_1}^{x_2} y.\sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx$$

$$S=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}.\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}.dx=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}.\sqrt{1+\frac{x^2}{r^2-x^2}}.dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2+x^2}\,dx=2\pi r\int_{-r}^{r}dx=2\pi r\,x\Big|_{-r}^{r}=2\pi r(r+r)=4\pi r^2$$

Ör) $y=\frac{x^3}{3}+\frac{1}{4x}$ eğrisi y eks. etr. dön. oluşan yüzeyin alanını $x\in[1,2]$ için hesapla.

$$\frac{dy}{dx}=x^2-\frac{1}{4x^2} \qquad S=2\pi\int_{x_1}^{x_2} x.\sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx$$

$$S=2\pi\int_1^2 x.\sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}.dx=2\pi\int_1^2 x.\left(x^2+\frac{1}{4x^2}\right).dx$$

$$=2\pi\int_1^2\left(x^3+\frac{1}{4x}\right).dx=2\pi\left(\frac{x^4}{4}+\frac{1}{4}\ln|x|\right)\Big|_1^2=2\pi\left(\left(4+\frac{\ln 2}{4}\right)-\left(\frac{1}{4}+0\right)\right)=\frac{15+\ln 2}{2}\pi$$

Ör) $x=2e^{-t}$, $y=e^{-2t}$ denklemleriyle verilen eğri y ekseni etrafında döndürülürse $t\in[0,\infty)$ için oluşan yüzeyin alanı.

$$x=2e^{-t}\longrightarrow\frac{dx}{dt}=-2e^{-t} \qquad y=e^{-2t}\longrightarrow\frac{dy}{dt}=-2e^{-2t}$$

$$S=2\pi\int_{t_1}^{t_2} x.\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}.dt=2\pi\int_0^\infty 2e^{-t}.\sqrt{(-2e^{-t})^2+(-2e^{-2t})^2}.dt$$

$$=4\pi\int_0^\infty e^{-t}.\sqrt{4e^{-2t}+4e^{-4t}}\,dt=8\pi\int_0^\infty e^{-t}.\sqrt{e^{-2t}(1+e^{-2t})}.dt$$

$$=8\pi\int_0^\infty\sqrt{1+e^{-2t}}.\,e^{-2t}dt=8\pi\int_{\sqrt2}^{1}u.(-u\,du)$$

$u=\sqrt{1+e^{-2t}}$

$u^2-1=e^{-2t}$

$2u\,du=-2e^{-2t}dt$

$e^{-2t}dt=-u\,du$

$t=0\longrightarrow u=\sqrt2$

$t=\infty\longrightarrow u=1$

$$=8\pi\int_1^{\sqrt2}u^2du=\frac{8\pi}{3}u^3\Big|_1^{\sqrt2}$$

$$=\frac{8\pi}{3}(2\sqrt2-1)$$

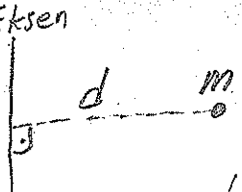
Moment ve Ağırlık Merkezi

σ : Yoğunluk

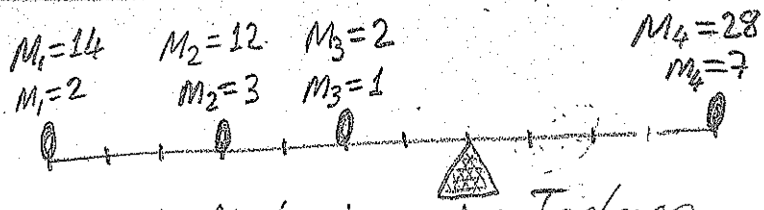
d : Uzunluk

m : Kütle

M : Moment



$M = m.d$



Ağırlık Merkezine göre Toplam Moment sıfır olur.

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n tane nokta için ağırlık merkezi

$$(\bar{x},\bar{y},\bar{z}) = \frac{\text{Toplam Moment}}{\text{Toplam Kütle}} = \frac{\sum_{i=1}^{n} m_i \cdot (x_i, y_i, z_i)}{\sum_{i=1}^{n} m_i}$$

$$m = \int_C \sigma(x)dx \quad \text{veya} \quad m = \iint_R \sigma(x,y)\,dA \quad \text{veya} \quad m = \iiint_T \sigma(x,y,z).dV$$

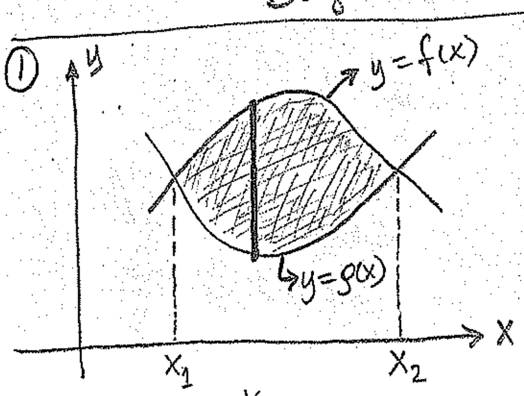
Bölge iki boyutlu ve yoğunluk her yerde aynı ($\sigma = 1$) kabul edilecek. Dolayısıyla kütle yerine alan kullanılacak.

$$A = \iint_R dA = \int_{x_1}^{x_2}\int_{y_1=g(x)}^{y_2=f(x)} dy\,dx \qquad A = \iint_R dA = \int_{y_1}^{y_2}\int_{x_1=g(y)}^{x_2=f(y)} dx\,dy$$

$$\bar{x} = \frac{\text{Bölgenin } y \text{ eksenine göre moment}}{\text{Bölgenin Alanı}} = \frac{\int_{x_1}^{x_2} x.c(x).dx}{A}$$

$$\bar{y} = \frac{\text{Bölgenin } x \text{ eksenine göre moment}}{\text{Bölgenin Alanı}} = \frac{\int_{y_1}^{y_2} y.c(y).dy}{A}$$

1



Kesitin x eksenine göre momenti

$$= \int_{g(x)}^{f(x)} y.dy = \left.\frac{y^2}{2}\right|_{g(x)}^{f(x)} = \frac{1}{2}\left(f^2(x) - g^2(x)\right)$$

Kesitin y eksenine göre momenti

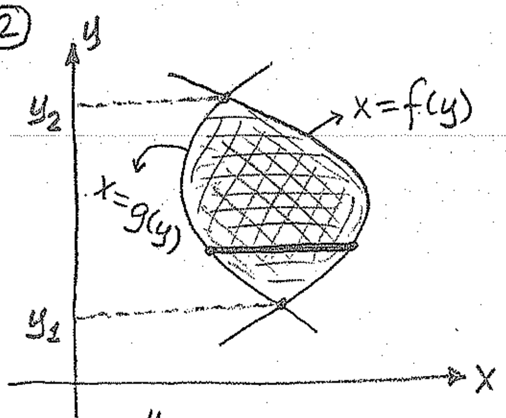
$$= x.(f(x) - g(x))$$

$$A = \int_{x_1}^{x_2} (f(x) - g(x)).dx$$

$$M_x = \frac{1}{2}\int_{x_1}^{x_2} (f^2(x) - g^2(x)).dx$$

$$M_y = \int_{x_1}^{x_2} x.(f(x) - g(x)).dx \qquad \bar{x} = \frac{M_y}{A} \qquad \bar{y} = \frac{M_x}{A}$$

②



Kesitin x eksenine göre momenti

$= y \cdot (f(y) - g(y))$

Kesitin y eksenine göre momenti

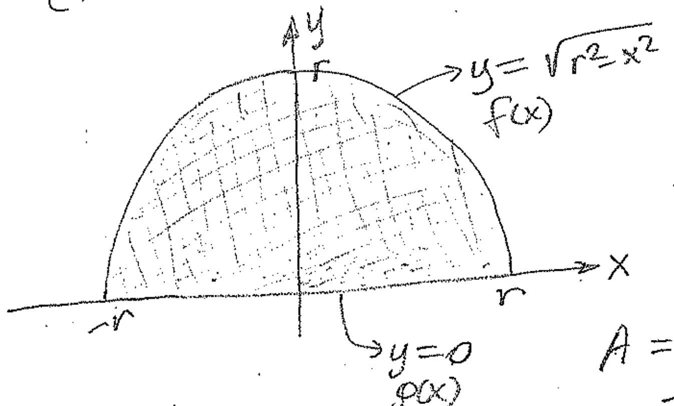
$$= \int_{g(y)}^{f(y)} x\,dx = \frac{x^2}{2}\Big|_{g(y)}^{f(y)} = \frac{1}{2}\cdot\left(f^2(y) - g^2(y)\right)$$

$$M_x = \int_{y_1}^{y_2} y\cdot(f(y) - g(y))\cdot dy \qquad A = \int_{y_1}^{y_2} (f(y) - g(y))\cdot dy$$

$$M_y = \frac{1}{2}\int_{y_1}^{y_2} (f^2(y) - g^2(y))\cdot dy \qquad \bar{x} = \frac{M_y}{A} \qquad \bar{y} = \frac{M_x}{A}$$

Ör $y = \sqrt{r^2 - x^2}$, $y = 0$ fonksiyonları arasında kalan bölgenin ağırlık merkezi

(Yarım Daire) (Yoğunluk her yerde aynı)



$$A = \int_{x_1}^{x_2} (f(x) - g(x))\,dx = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

$$x = r\sin\theta \longrightarrow dx = \cos\theta\, d\theta$$

$$\sin\theta = x/r \longrightarrow \theta = \arcsin(x/r)$$

$$A = \int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2\sin^2\theta}\cdot r\cos\theta\, d\theta = r^2\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{r^2}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta = \frac{r^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{-\pi/2}^{\pi/2} = \frac{\pi r^2}{2}$$

$$M_x = \frac{1}{2}\int_{x_1}^{x_2} (f^2(x) - g^2(x))\,dx = \frac{1}{2}\int_{-r}^{r} (r^2 - x^2)\,dx = \frac{1}{2}\left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r} = \frac{2r^3}{3}$$

$$M_y = \int_{x_1}^{x_2} x\,(f(x) - g(x))\,dx = \int_{-r}^{r} x\cdot\sqrt{r^2 - x^2}\,dx$$

$$= -\int_0^0 u^2\,du = -\frac{u^3}{3}\Big|_0^0 = 0$$

$u = \sqrt{r^2 - x^2}$

$u^2 = r^2 - x^2$

$2u\,du = -2x\,dx$

$x\,dx = -u\,du$

$x = \mp r \to u = 0$

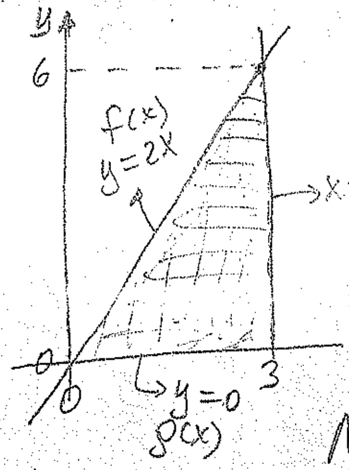
$$\bar{x} = \frac{M_y}{A} = \frac{0}{\pi r^2/2} = 0$$

$$\bar{y} = \frac{M_x}{A} = \frac{2r^3/3}{\pi r^2/2} = \frac{4r}{3\pi}$$

$(\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi}\right)$

noktası ağırlık merkezi

9) $y=2x,\ y=0,\ x=3$ fonksiyonları arasında kalan bölgenin ağırlık merkezi



$$A=\int_{x_1}^{x_2}(f(x)-g(x))\,dx=\int_0^3 2x\,dx=x^2\Big|_0^3=9$$

$$M_x=\frac{1}{2}\int_{x_1}^{x_2}(f^2(x)-g^2(x))\,dx=\frac{1}{2}\int_0^3(2x)^2\,dx=2\int_0^3 x^2\,dx$$

$$=\frac{2}{3}x^3\Big|_0^3=\frac{2}{3}(27-0)=18$$

$$M_y=\int_{x_1}^{x_2}x(f(x)-g(x))\,dx=2\int_0^3 x^2=\frac{2}{3}x^3\Big|_0^3=\frac{2}{3}(27-0)=18$$

$$\bar{x}=\frac{M_y}{A}=\frac{18}{9}=2 \qquad \bar{y}=\frac{M_x}{A}=\frac{18}{9}=2$$

(2,2) noktası ağırlık merkezi

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9) $x=y^2-4y+3,\ x=3-y$ fonk. arasında kalan bölgenin ağırlık merkezi.

$x=x$

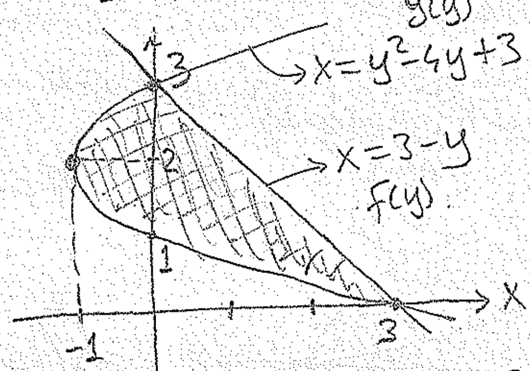
$y^2-4y+3=3-y$

$y^2-3y=0$

$y(y-3)=0$

$y=0 \quad y=3$

$x=3 \quad x=0$



$$A=\int_{y_1}^{y_2}(f(y)-g(y))\,dy$$

$$=\int_0^3\big((3-y)-(y^2-4y+3)\big)\,dy$$

$$=\int_0^3(3y-y^2)\,dy=\left(\frac{3y^2}{2}-\frac{y^3}{3}\right)\Big|_0^3=\frac{9}{2}$$

$$M_x=\int_{y_1}^{y_2}y(f(y)-g(y))\,dy=\int_0^3 y\big((3-y)-(y^2-4y+3)\big)\,dy=\int_0^3 y(3y-y^2)\,dy$$

$$=\int_0^3(3y^2-y^3)\,dy=\left(y^3-\frac{y^4}{4}\right)\Big|_0^3=\frac{27}{4}$$

$$M_y=\frac{1}{2}\int_{y_1}^{y_2}(f^2(y)-g^2(y))\,dy=\frac{1}{2}\int_0^3\big((3-y)^2-(y^2-4y+3)^2\big)\,dy$$

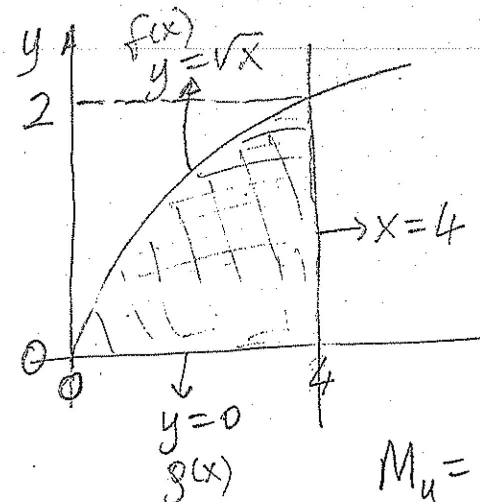
$$=\frac{1}{2}\int_0^3(y^2-5y+6)(3y-y^2)\,dy=\frac{1}{2}\int_0^3(18y-21y^2+8y^3-y^4)\,dy$$

$$=\left(\frac{9y^2}{2}-\frac{7}{2}y^3+y^4-\frac{y^5}{10}\right)\Big|_0^3=\frac{81}{2}-\frac{189}{2}+81-\frac{243}{10}=27-\frac{243}{10}=\frac{27}{10}$$

$$\bar{x}=\frac{M_y}{A}=\frac{27/10}{9/2}=0.6 \qquad \bar{y}=\frac{M_x}{A}=\frac{27/4}{9/2}=1.5$$

$(\bar{x},\bar{y})=(0.6,\ 1.5)$ noktası ağırlık merkezi

60) $y=\sqrt{x}$, $y=0$, $x=4$ fonk. arasında kalan bölgenin ağırlık merkezi



$$A=\int_{x_1}^{x_2}(f(x)-g(x))dx=\int_0^4\sqrt{x}\,dx=\frac{x^{3/2}}{3/2}\Big|_0^4=\frac{16}{3}$$

$$M_x=\frac{1}{2}\int_{x_1}^{x_2}(f^2(x)-g^2(x))dx=\frac{1}{2}\int_0^4 x\,dx$$

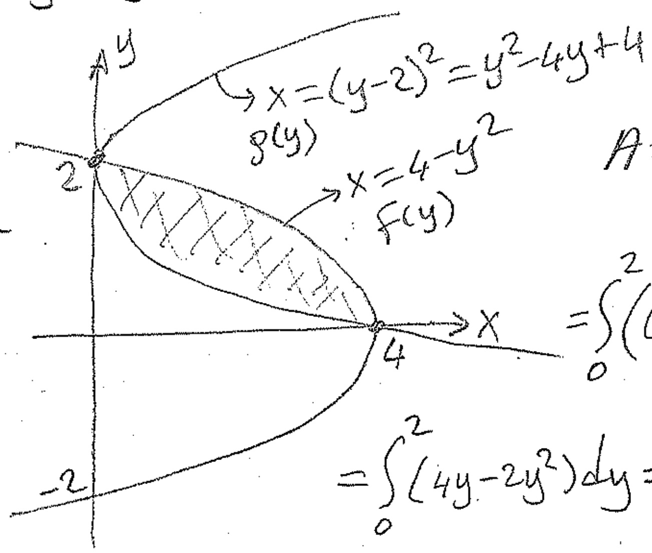
$$=\frac{x^2}{4}\Big|_0^4=4$$

$$M_y=\int_{x_1}^{x_2}x(f(x)-g(x))dx=\int_0^4 x^{3/2}dx=\frac{x^{5/2}}{5/2}\Big|_0^4=\frac{64}{5}$$

$$\bar{x}=\frac{M_y}{A}=\frac{64/5}{16/3}=\frac{12}{5}=2.4 \qquad \bar{y}=\frac{M_x}{A}=\frac{4}{16/3}=\frac{3}{4}=0.75$$

61) $x=(y-2)^2$, $x=4-y^2$ eğrileri arasında kalan bölgenin ağırlık merkezi

$$x=x$$
$$(y-2)^2=4-y^2$$
$$y^2-4y+4=4-y^2$$
$$2y^2-4y=0$$
$$2y(y-2)=0$$
$$y=0 \quad y=2$$
$$x=4 \quad x=0$$



$$A=\int_{y_1}^{y_2}(f(y)-g(y))dy$$

$$=\int_0^2((4-y^2)-(y^2-4y+4))dy$$

$$=\int_0^2(4y-2y^2)dy=\left(2y^2-\frac{2}{3}y^3\right)\Big|_0^2=8/3$$

$$M_x=\int_{y_1}^{y_2}y(f(y)-g(y))dy=\int_0^2 y((4-y^2)-(y^2-4y+4))dy$$

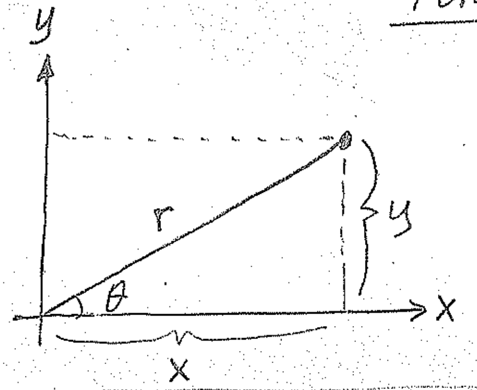
$$=\int_0^2 y(4y-2y^2)dy=\int_0^2(4y^2-2y^3)dy=\left(\frac{4}{3}y^3-\frac{y^4}{2}\right)\Big|_0^2=8/3$$

$$M_y=\frac{1}{2}\int_{y_1}^{y_2}(f^2(y)-g^2(y))dy=\frac{1}{2}\int_0^2((4-y^2)^2-(y^2-4y+4)^2)dy$$

$$=\frac{1}{2}\int_0^2(8-4y)(4y-2y^2)dy=4\int_0^2(y^3-4y^2+4y)dy=\left(y^4-\frac{16}{3}y^3+8y^2\right)\Big|_0^2=\frac{16}{3}$$

$$\bar{x}=\frac{M_y}{A}=\frac{16/3}{8/3}=2 \qquad \bar{y}=\frac{M_x}{A}=\frac{8/3}{8/3}=1$$

Polar Koordinat

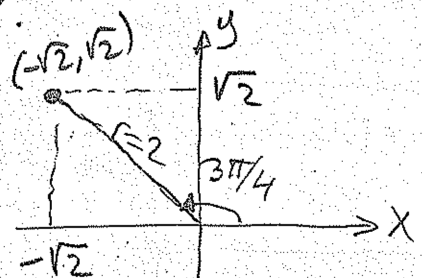


$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad \tan\theta = y/x$$

$$\theta = \arctan(y/x)$$

$$r = \sqrt{x^2+y^2}$$

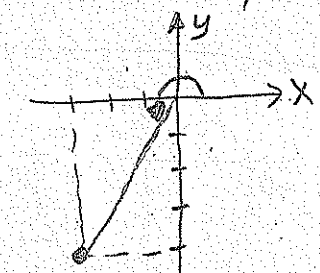
Ör) $P = (2, 3\pi/4)$ polar koordinatta gösterim ise noktanın kartezyen koordinatta gösterimini bul.



$$x = r\cos\theta = 2\cos(3\pi/4) = 2.\left(-\sqrt{2}/2\right) = -\sqrt{2}$$

$$y = r\sin\theta = 2\sin(3\pi/4) = 2.\left(\sqrt{2}/2\right) = \sqrt{2}$$

Ör) $P = (-3, -4)$ Kartezyen koordinatta gösterim ise noktanın polar koordinatta gösterimini bul.



$$r = \sqrt{x^2+y^2} = \sqrt{(-3)^2+(-4)^2} = 5$$

$$\theta = \arctan(y/x) = \arctan(4/3) \begin{cases} \cancel{\approx 53.1} \\ \approx 180+53.1 \approx 233.1 \end{cases}$$

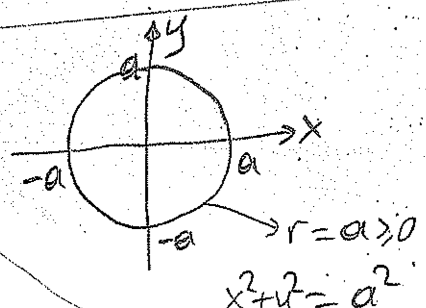
$r = f(\theta)$ → Polar koordinatta bir fonksiyonun gösterimi

$$x = r\cos\theta = f(\theta).\cos\theta$$

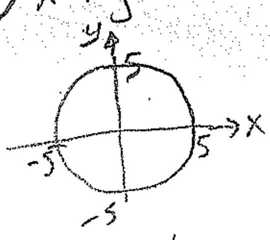
$$y = r\sin\theta = f(\theta).\sin\theta$$

eğrinin her hangi bir noktadaki eğimi

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(f(\theta).\sin\theta)}{\frac{d}{d\theta}(f(\theta).\cos\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$



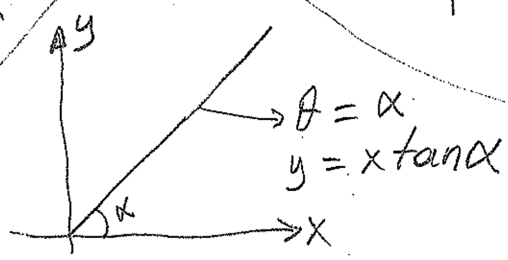
Ör) $x^2+y^2 = 25$ veya $r = 5$ ise $\frac{dy}{dx} = ?$

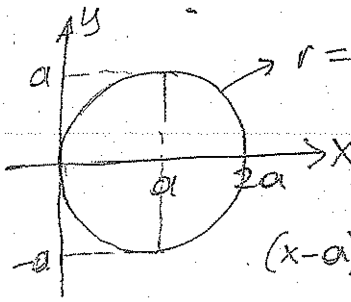


$$x = r\cos\theta = 5\cos\theta$$

$$y = r\sin\theta = 5\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{5\cos\theta}{-5\sin\theta} = -\frac{x}{y}$$

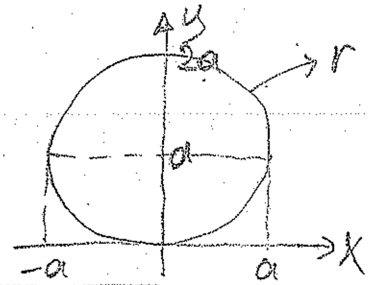




$r = 2a\cos\theta$

$a \geq 0$

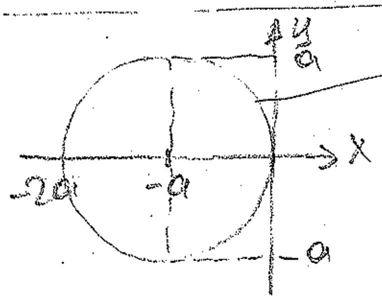
$(x-a)^2 + y^2 = a^2$



$r = 2a\sin\theta$

$a \geq 0$

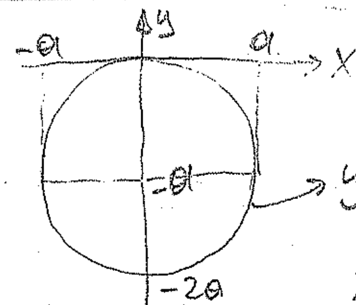
$x^2 + (y-a)^2 = a^2$



$r = -2a\cos\theta$

$a \geq 0$

$(x+a)^2 + y^2 = a^2$



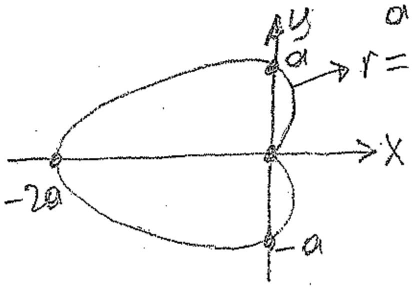
$y = -2a\sin\theta$

$a \geq 0$

$x^2 + (y+a)^2 = a^2$

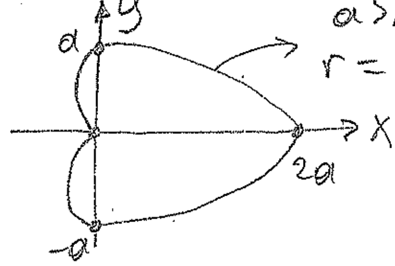
$a \geq 0$

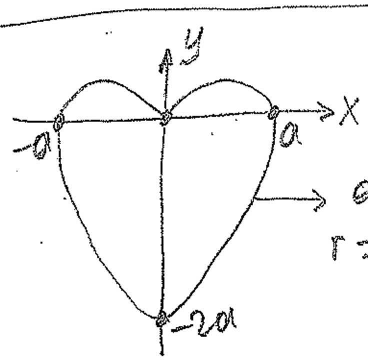
$r = a(1-\cos\theta)$



$a \geq 0$

$r = a(1+\cos\theta)$



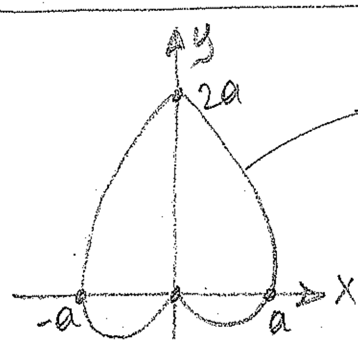


$a \geq 0$

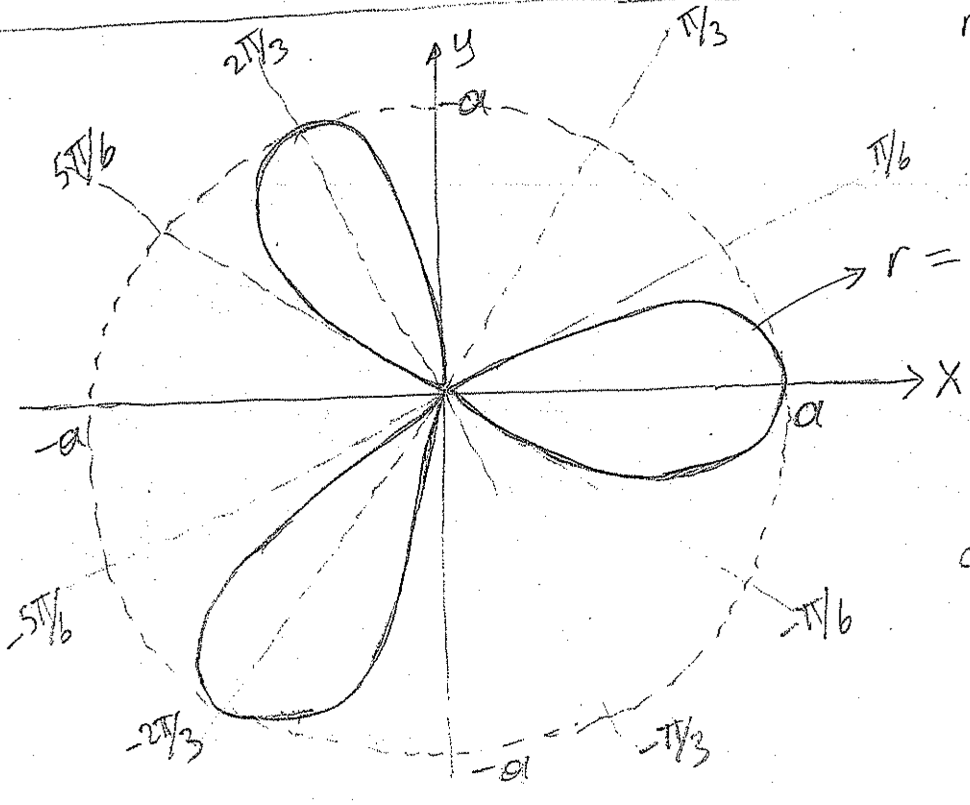
$r = a(1-\sin\theta)$

$a \geq 0$

$r = a(1+\sin\theta)$



$r = a\cos(n\theta)$ ↳ yaprak sayısı



üç eşit yaprak

$r = a\cos 3\theta$

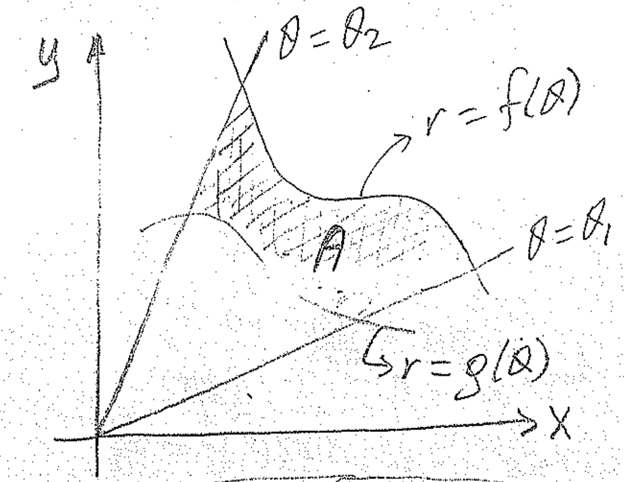
$a \geq 0$

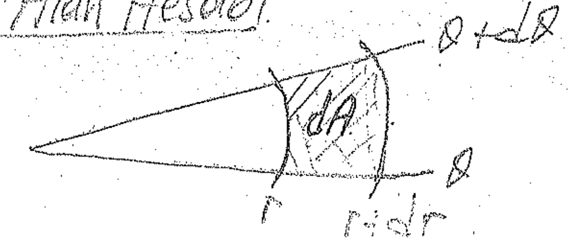
şekli π/6 derece döndürürsek

$r = a\sin 3\theta$

elde edilir.

Polar Koordinatta Alan Hesabı

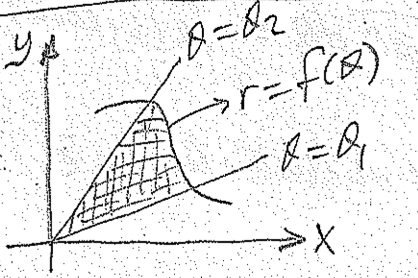




$$dA=(dr).(rd\theta)=rdrd\theta$$

$$A=\int_{\theta_1}^{\theta_2}\int_{r=g(\theta)}^{r=f(\theta)} r\,dr\,d\theta$$

$$A=\frac{1}{2}\int_{\theta_1}^{\theta_2}\left(f^2(\theta)-g^2(\theta)\right).d\theta$$ alan hesabı



$$A=\frac{1}{2}\int_{\theta_1}^{\theta_2} f^2(\theta)\,d\theta$$
$$=\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$$ alan hesabı

Polar Koordinatta Eğri Uzunluğu

$r=f(\theta)\,,\quad \theta\in[\theta_1,\theta_2]$

$x=r\cos\theta=f(\theta)\cos\theta$

$y=r\sin\theta=f(\theta)\sin\theta$

$$L=\int_{\theta_1}^{\theta_2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$\frac{dx}{d\theta}=f'(\theta)\cos\theta-f(\theta)\sin\theta$$

$$\frac{dy}{d\theta}=f'(\theta)\sin\theta+f(\theta)\cos\theta$$

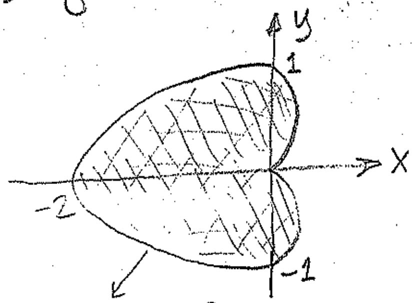
$$L=\int_{\theta_1}^{\theta_2}\sqrt{\left(f'(\theta)\cos\theta-f(\theta)\sin\theta\right)^2+\left(f'(\theta)\sin\theta+f(\theta)\cos\theta\right)^2}\,d\theta$$

Sonuç olarak

eğri uzunluğu

$$L=\int_{\theta_1}^{\theta_2}\sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Ör: $r = 1 - \cos\theta$ eğrisi veriliyor.
a) eğrinin içinde kalan bölgenin alanı.
b) eğrinin uzunluğu. $\theta \in [0, 2\pi]$.



$\frac{dr}{d\theta} = \sin\theta$

a) $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2 d\theta$

$= \frac{1}{2}\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$

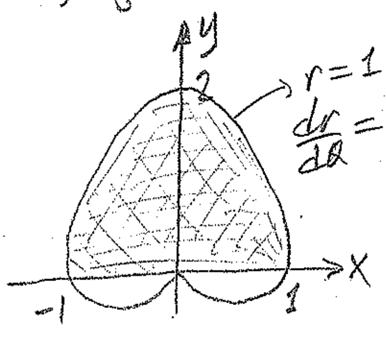
$= \int_0^{2\pi}\left(\frac{3}{4} - \cos\theta + \frac{\cos 2\theta}{4}\right)d\theta = \left(\frac{3\theta}{4} - \sin\theta + \frac{\sin 2\theta}{8}\right)\Big|_0^{2\pi} = \frac{3\pi}{2}$

b) $L = \int_{\theta_1}^{\theta_2}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi}\sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\,d\theta$

$= \int_0^{2\pi}\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta = \int_0^{2\pi}\sqrt{2 - 2\cos\theta}\,d\theta = \int_0^{2\pi}\sqrt{2(1-\cos\theta)}\,d\theta$

$= \int_0^{2\pi}\sqrt{4\sin^2\frac{\theta}{2}}\,d\theta = 2\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta = -4\cos\frac{\theta}{2}\Big|_0^{2\pi} = -4(\cos\pi - \cos 0) = 8$

Ör: $r = 1 + \sin\theta$ eğrisi veriliyor. $\theta \in [0, \pi]$ için
a) eğrinin taradığı alan. b) eğrinin uzunluğu.



$\frac{dr}{d\theta} = \cos\theta$

$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta = \frac{1}{2}\int_0^{\pi}(1+\sin\theta)^2 d\theta$

$= \frac{1}{2}\int_0^{\pi}(1 + 2\sin\theta + \sin^2\theta)d\theta$

$= \frac{1}{2}\int_0^{\pi}\left(1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right)d\theta$

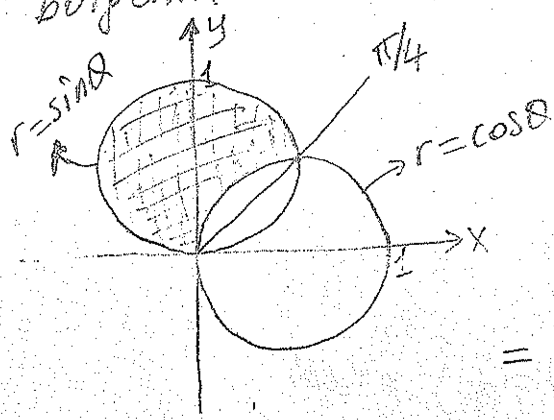
$= \int_0^{\pi}\left(\frac{3}{4} + \sin\theta - \frac{\cos 2\theta}{4}\right)d\theta = \left(\frac{3\theta}{4} - \cos\theta - \frac{\sin 2\theta}{8}\right)\Big|_0^{\pi} = 2 + \frac{3\pi}{4}$

b) $L = \int_{\theta_1}^{\theta_2}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{\pi}\sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\,d\theta$

$= \int_0^{\pi}\sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\,d\theta = \int_0^{\pi}\sqrt{2(1+\sin\theta)}\,d\theta$

$= \sqrt{2}\int_0^{\pi}\sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\,d\theta = \sqrt{2}\int_0^{\pi}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)d\theta = 2\sqrt{2}\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)\Big|_0^{\pi} = 4\sqrt{2}$

Ör. $r=\sin\theta$ eğrisinin içinde ve $r=\cos\theta$ eğrisinin dışında kalan bölgenin alanı.



$$r=r$$
$$\sin\theta=\cos\theta \longrightarrow \tan\theta=1 \rightarrow \theta=\pi/4$$

$$A_1=\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta=\frac{1}{2}\int_{\pi/4}^{\pi}\sin^2\theta\,d\theta$$

$$=\frac{1}{4}\int_{\pi/4}^{\pi}(1-\cos2\theta)\,d\theta=\frac{1}{4}\left(\theta-\frac{\sin2\theta}{2}\right)\Big|_{\pi/4}^{\pi}$$

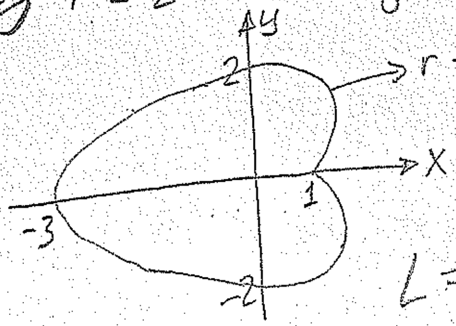
$$=\frac{1}{4}\left((\pi-0)-\left(\pi/4-1/2\right)\right)=\frac{3\pi+2}{16}$$

$$A_2=\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta=\frac{1}{2}\int_{\pi/4}^{\pi/2}\cos^2\theta\,d\theta=\frac{1}{4}\int_{\pi/4}^{\pi/2}(1+\cos2\theta)\,d\theta$$

$$=\frac{1}{4}\left(\theta+\frac{\sin2\theta}{2}\right)\Big|_{\pi/4}^{\pi/2}=\frac{1}{4}\left(\left(\frac{\pi}{2}+0\right)-\left(\pi/4+\frac{1}{2}\right)\right)=\frac{\pi-2}{16}$$

$$A=A_1-A_2=\frac{3\pi+2}{16}-\frac{\pi-2}{16}=\frac{\pi+2}{8}$$

Ör. $r=2-\cos\theta$ eğrisinin uzunluğunu ve içinde kalan bölgenin alanını



$r=2-\cos\theta$

$$\frac{dr}{d\theta}=\sin\theta \qquad L=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$L=\int_0^{2\pi}\sqrt{(2-\cos\theta)^2+(\sin\theta)^2}\,d\theta$$

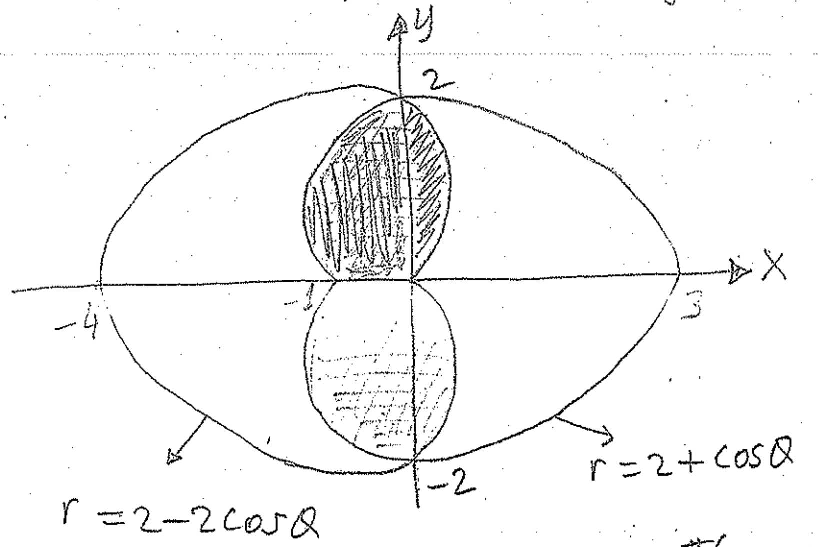
$$=\int_0^{2\pi}\sqrt{4-4\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta=\int_0^{2\pi}\sqrt{5-4\cos\theta}\,d\theta \rightarrow \text{Çöz}$$

$$A=\frac{1}{2}\int_0^{2\pi}r^2\,d\theta=\frac{1}{2}\int_0^{2\pi}(2-\cos\theta)^2\,d\theta=\frac{1}{2}\int_0^{2\pi}(4-4\cos\theta+\cos^2\theta)\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(4-4\cos\theta+\frac{1+\cos2\theta}{2}\right)d\theta=\int_0^{2\pi}\left(\frac{9}{4}-2\cos\theta+\frac{\cos2\theta}{4}\right)d\theta$$

$$=\left(\frac{9\theta}{4}-2\sin\theta+\frac{\sin2\theta}{4}\right)\Big|_0^{2\pi}=\frac{9\pi}{2}$$

Ör. $r = 2-2\cos\theta$, $r = 2+\cos\theta$ eğrileri arasında kalan bölgenin alanı.



$$r = r$$
$$2-2\cos\theta = 2+\cos\theta$$
$$3\cos\theta = 0$$
$$\cos\theta = 0$$
$$\theta = \mp \pi/2$$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$$

$$A_1 = \frac{1}{2}\int_0^{\pi/2}(2-2\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{\pi/2}(4-8\cos\theta+4\cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(4-8\cos\theta+4\,\frac{1+\cos 2\theta}{2}\right)d\theta = \int_0^{\pi/2}(3-4\cos\theta+\cos 2\theta)\,d\theta$$

$$= \left(3\theta-4\sin\theta+\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = \frac{3\pi}{2}-4$$

$$A_2 = \frac{1}{2}\int_{\pi/2}^{\pi}(2+\cos\theta)^2 d\theta = \frac{1}{2}\int_{\pi/2}^{\pi}(4+4\cos\theta+\cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi}\left(4+4\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta = \int_{\pi/2}^{\pi}\left(\frac{9}{4}+2\cos\theta+\frac{\cos 2\theta}{4}\right)d\theta$$

$$= \left(\frac{9\theta}{4}+2\sin\theta+\frac{\sin 2\theta}{8}\right)\Big|_{\pi/2}^{\pi} = \left(\frac{9\pi}{4}+0+0\right)-\left(\frac{9\pi}{8}+2+0\right) = \frac{9\pi}{8}-2$$

$$A = 2(A_1+A_2) = 2\left(\left(\frac{3\pi}{2}-4\right)+\left(\frac{9\pi}{8}-2\right)\right) = \frac{21\pi}{4}-12$$

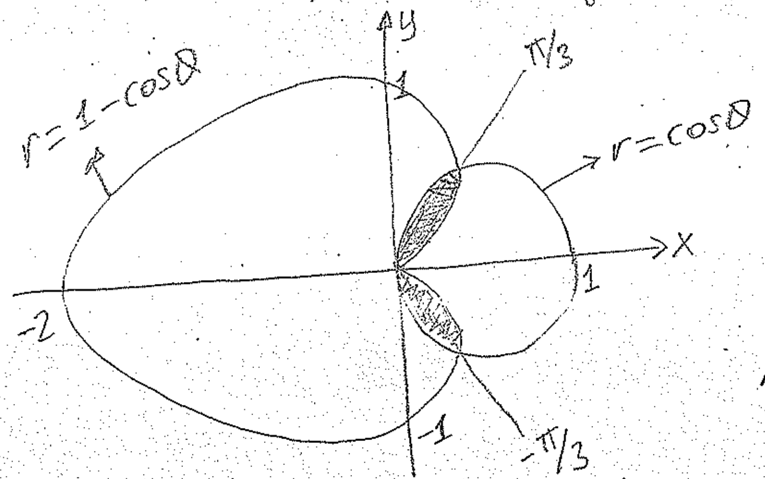
Ör. $r = e^{3\theta/4}$, $\theta\in[0, \ln 16]$ için eğri uzunluğu.

$$r = e^{3\theta/4} \longrightarrow \frac{dr}{d\theta} = \frac{3}{4}e^{3\theta/4} \qquad L = \int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$L = \int_0^{\ln 16}\sqrt{e^{6\theta/4}+\frac{9}{16}e^{6\theta/4}}\,d\theta = \int_0^{\ln 16}\sqrt{\frac{25}{16}e^{6\theta/4}}\,d\theta = \frac{5}{4}\int_0^{\ln 16} e^{3\theta/4}\,d\theta$$

$$= \frac{5}{4}\,\frac{e^{3\theta/4}}{3/4}\Big|_0^{\ln 16} = \frac{5}{3}\left(e^{\frac{3}{4}\ln 16}-e^0\right) = \frac{5}{3}\left(e^{\ln 8}-1\right) = \frac{35}{3}$$

Ör: $r = 1-\cos\theta$, $r=\cos\theta$ eğrileri arasında kalan bölgenin alanı



$$r=r$$
$$1-\cos\theta = \cos\theta$$
$$\cos\theta = 1/2$$
$$\theta = \mp \pi/3$$

$$A=\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

$$A_1 = \frac{1}{2}\int_0^{\pi/3}(1-\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{\pi/3}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta = \int_0^{\pi/3}\left(\frac{3}{4}-\cos\theta+\frac{\cos 2\theta}{4}\right)d\theta$$

$$= \left(\frac{3\theta}{4}-\sin\theta+\frac{\sin 2\theta}{8}\right)\Big|_0^{\pi/3} = \frac{\pi}{4}-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}/2}{8} = \frac{\pi}{4}-\frac{7}{16}\sqrt{3}$$

$$A_2 = \frac{1}{2}\int_{\pi/3}^{\pi/2}\cos^2\theta\, d\theta = \frac{1}{4}\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)d\theta = \left(\frac{\theta}{4}+\frac{\sin 2\theta}{8}\right)\Big|_{\pi/3}^{\pi/2}$$

$$= \left(\frac{\pi}{8}+0\right)-\left(\frac{\pi}{12}+\frac{\sqrt{3}/2}{8}\right) = \frac{\pi}{24}-\frac{\sqrt{3}}{16}$$

$$A = 2(A_1+A_2) = 2\left(\left(\frac{\pi}{4}-\frac{7\sqrt{3}}{16}\right)+\left(\frac{\pi}{24}-\frac{\sqrt{3}}{16}\right)\right) = \frac{7\pi}{12}-\sqrt{3}$$

Ör: $r=\cos\theta+\sin\theta$, $\theta\in[0,\pi]$ eğri uzunluğu

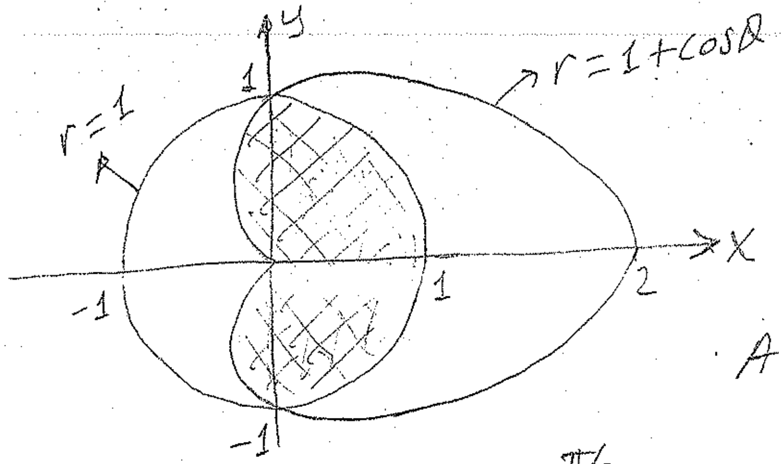
$$r=\cos\theta+\sin\theta \longrightarrow \frac{dr}{d\theta} = -\sin\theta+\cos\theta$$

$$L=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{\pi}\sqrt{(\cos\theta+\sin\theta)^2+(-\sin\theta+\cos\theta)^2}\,d\theta$$

$$= \int_0^{\pi}\sqrt{\cos^2\theta+\sin^2\theta+\cancel{2\cos\theta\sin\theta}+\sin^2\theta+\cos^2\theta-\cancel{2\sin\theta\cos\theta}}\,d\theta$$

$$= \sqrt{2}\int_0^{\pi}d\theta = \sqrt{2}\,\theta\Big|_0^{\pi} = \sqrt{2}(\pi-0) = \sqrt{2}\,\pi$$

Ör: $r = 1 + \cos\theta$, $r = 1$ eğrileri arasında kalan bölgenin alanı (144)



$$r = r$$
$$1 = 1 + \cos\theta$$
$$\cos\theta = 0$$
$$\theta = \mp \pi/2$$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

$$A_1 = \frac{1}{2}\int_0^{\pi/2} (1)^2 d\theta = \frac{\theta}{2}\Big|_0^{\pi/2} = \pi/4$$

$$A_2 = \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2 d\theta = \frac{1}{2}\int_{\pi/2}^{\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta = \int_{\pi/2}^{\pi}\left(\frac{3}{4} + \cos\theta + \frac{\cos 2\theta}{4}\right)d\theta$$

$$= \left(\frac{3\theta}{4} + \sin\theta + \frac{\sin 2\theta}{8}\right)\Big|_{\pi/2}^{\pi} = \left(\frac{3\pi}{4} + 0 + 0\right) - \left(\frac{3\pi}{8} + 1 + 0\right) = \frac{3\pi}{8} - 1$$

$$A = 2(A_1 + A_2) = 2\left(\left(\frac{\pi}{4}\right) + \left(\frac{3\pi}{8} - 1\right)\right) = \frac{5\pi}{4} - 2$$

Ör: $r = \sec\theta$, $\theta \in [-\pi/4, \pi/4]$ eğri uzunluğu

$$r = \sec\theta \longrightarrow \frac{dr}{d\theta} = \sec\theta\tan\theta \qquad L = \int_{\theta_1}^{\theta_2}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$L = \int_{-\pi/4}^{\pi/4}\sqrt{(\sec\theta)^2 + (\sec\theta\tan\theta)^2}\,d\theta = \int_{-\pi/4}^{\pi/4}\sqrt{\sec^2\theta + \sec^2\theta\tan^2\theta}\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\sqrt{\sec^2\theta.(1+\tan^2\theta)}\,d\theta = \int_{-\pi/4}^{\pi/4}\sqrt{\sec^4\theta}\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\sec^2\theta\,d\theta = \tan\theta\Big|_{-\pi/4}^{\pi/4} = \tan(\pi/4) - \tan(-\pi/4) = 1 - (-1) = 2$$

Seriler

(a_n) sembolü diziyi, $\sum_{n=1}^{\infty} a_n$ sembolü ise seriyi ifade eder.

$S_n = \sum_{k=1}^{n} a_k$ ise serinin ilk n teriminin toplamıdır. Buna kısmi toplam denir. (S_n) dizisine kısmi toplamlar dizisi adı verilir.

Kısmi toplamlar dizisi yakınsak ise seri yakınsaktır.

Kısmi toplamlar dizisinin limitine serinin toplamı denir.

Yakınsak olmayan serilere ıraksak seriler adı verilir.

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$ Serinin n.kısmi toplamı

$$S = \lim_{n\to\infty} S_n < \infty$$ ise seri yakınsaktır.

Teoremler

① $\sum_{n=1}^{\infty} a_n$ serisi yakınsak ise $\lim_{n\to\infty} a_n = 0$ olmalıdır.

② $\lim_{n\to\infty} a_n = 0$ olması, $\sum_{n=1}^{\infty} a_n$ serisinin yakınsak olmasını gerektirmez.

Mesela, $\sum_{n=1}^{\infty} \frac{1}{n}$ ve $\sum_{n=1}^{\infty} \ln(1+1/n)$ serileri ıraksaktır.

③ $\lim_{n\to\infty} a_n \neq 0$ ise $\sum_{n=1}^{\infty} a_n$ serisi ıraksaktır.

④ $\sum_{n=1}^{\infty} a_n = L$ ve $\sum_{n=1}^{\infty} b_n = M$ serileri yakınsak ise

$\sum_{n=1}^{\infty} (a_n \mp b_n) = L \mp M$ ve $\sum_{n=1}^{\infty} c.a_n = cL$ serileri de yakınsaktır.

$$R_n = \sum_{k=n+1}^{\infty} a_k = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$ toplamına

kalan terimi adı verilir. Yakınsak bir $\sum_{n=1}^{\infty} a_n$ serisinin kalan teriminin limiti 0 dır. $\left(\lim_{n\to\infty} R_n = 0\right)$

Aritmetik Seri

(a_n) aritmetik dizi ise $\sum_{n=1}^{\infty} a_n$ aritmetik seridir.

Ardışık iki terim arasındaki fark sabittir.

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = r \quad : \text{Ortak fark}$$

$$a_n = a_1 + (n-1).r = a_p + (n-p).r$$

$$r = \frac{a_n - a_1}{n-1} = \frac{a_n - a_p}{n-p} \qquad a_n = \frac{a_{n-p} + a_{n+p}}{2}$$

$$S_n = \sum_{k=1}^{n} a_k = n . \frac{a_1 + a_n}{2}$$ ilk n teriminin toplamı

Geometrik Seri

(a_n) geometrik dizi ise $\sum_{n=1}^{\infty} a_n$ geometrik seridir.

Ardışık iki terimi arasındaki oran sabittir.

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_n}{a_{n-1}} = r \quad : \text{Ortak çarpan}$$

$$a_n = a_1 . r^{n-1} = a_p . r^{n-p}$$

$$r = \sqrt[n-1]{\frac{a_n}{a_1}} = \sqrt[n-p]{\frac{a_n}{a_p}} \qquad a_n = \sqrt{a_{n-p} \cdot a_{n+p}}$$

$$S = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_1 r^{n-1} = \sum_{n=1}^{\infty} a\, r^{n-1} = \sum_{n=0}^{\infty} a . r^n$$

$$S_n = \sum_{k=0}^{n-1} a . r^k = a + ar + ar^2 + \cdots + ar^{n-1}$$ ilk n terimin toplamı.

$$r S_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

$$S_n - r S_n = a - ar^n = a(1 - r^n) \Rightarrow S_n = a . \frac{1 - r^n}{1 - r}, \quad r \neq 1$$

$$S = \frac{a}{1-r}, \quad |r| < 1$$

$|r| < 1$ ise yakınsak

$|r| > 1$ ise ıraksak

Harmonik Seriler

$\sum_{n=1}^{\infty} \frac{1}{an+b}$, $a \neq 0$ şeklindeki serilerdir.

$\lim_{n\to\infty} a_n = 0$ olmasına rağmen harmonik seriler ıraksaktır.

$a = 3/2$, $b = -1/2 \longrightarrow \sum_{n=1}^{\infty} \frac{2}{3n-1}$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ serisi $p>1$ ise yakınsak, $p \leq 1$ ise ıraksaktır.

$p=1$ için $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonik seriye dönüşür.

İntegral Testi

$a_n = f(n)$, $n \in N^+$ olsun

$x \geq 1$ için $f(x) \geq 0$ ve azalan bir fonk. ise integral testi kullanılır.

$\int_1^{\infty} f(x)dx$ integrali yakınsak ise $\sum_{n=1}^{\infty} a_n$ serisi de yakınsaktır.

integral ıraksak ise seri de ıraksaktır.

Kıyaslama Testi

$a_n \geq 0$, $b_n \geq 0$ $n \in N^+$ olsun. $a_n \geq b_n$ için

1. $\sum_{n=1}^{\infty} a_n$ serisi yakınsak ise $\sum_{n=1}^{\infty} b_n$ serisi de yakınsaktır.
2. $\sum_{n=1}^{\infty} b_n$ serisi ıraksak ise $\sum_{n=1}^{\infty} a_n$ serisi de ıraksaktır.

Limit Kıyas Testi

$a_n \geq 0$, $b_n \geq 0$ $n \in N^+$ olsun. $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ için

1. L pozitif bir sabit ise her ikisi birden yakınsak veya her ikisi birden ıraksaktır.
2. $L = 0$ ve $\sum_{n=1}^{\infty} b_n$ yakınsak ise $\sum_{n=1}^{\infty} a_n$ de yakınsaktır.
3. $L = \infty$ ve $\sum_{n=1}^{\infty} b_n$ ıraksak ise $\sum_{n=1}^{\infty} a_n$ de ıraksaktır.

Mutlak Yakınsaklık

$\sum_{n=1}^{\infty} |a_n|$ serisi yakınsak ise $\sum_{n=1}^{\infty} a_n$ serisi mutlak yakınsaktır.

Koşullu yakınsaklık

$\sum_{n=1}^{\infty} |a_n|$ serisi ıraksak fakat $\sum_{n=1}^{\infty} a_n$ serisi yakınsak ise bu yakınsaklığa koşullu yakınsaklık denir.

Oran Testi

$a_n \geq 0$, $n \in N^+$ için $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ olsun.

$\sum_{n=1}^{\infty} a_n$ serisi $L<1$ ise yakınsak, $L>1$ ise ıraksaktır.

$L=1$ için bir şey söylenemez.

Kök Testi

$a_n \geq 0$, $n \in N^+$ için $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ olsun.

$\sum_{n=1}^{\infty} a_n$ serisi $L<1$ ise yakınsak, $L>1$ ise ıraksaktır.

$L=1$ ise bir şey söylenemez.

Alterne Seriler

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ tipi serilerdir.

$n \geq 1$ için $0 < a_{n+1} \leq a_n$ ve $\lim_{n\to\infty} a_n = 0$ ise

$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$ serisi yakınsaktır.

Ör $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$ serisi yakınsak mıdır?

$$S_n = \sum_{k=1}^{n} \ln(1+\frac{1}{k}) = \sum_{k=1}^{n} \ln(\frac{k+1}{k}) = \sum_{k=1}^{n} (\ln(k+1) - \ln(k))$$

$$= (\cancel{\ln 2} - \ln 1) + (\cancel{\ln 3} - \cancel{\ln 2}) + (\ln 4 - \cancel{\ln 3}) + \cdots + (\ln(n+1) - \cancel{\ln(n)})$$

$$= \ln(n+1) - \ln(1) = \ln(n+1)$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \ln(\infty) = \infty \text{ ıraksaktır.}$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln(1+1/n) = \ln(1) = 0$ fakat yakınsak değil.

Ör $0.\overline{27} = 3/11$ olduğunu göster.

$$0.\overline{27} = 0.272727\ldots = 0.27 + 0.0027 + 0.000027 + \cdots$$

$$= 27 \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n = \frac{27}{100} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n = \frac{27}{100} \frac{1}{1-\frac{1}{100}} = \frac{27}{99} = \frac{3}{11}$$

Ör: $\sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)}$ serisi yakınsak mıdır?

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{(k+3)(k+4)} = \sum_{k=1}^{n}\left(\frac{1}{k+3} - \frac{1}{k+4}\right)$$

$$= \left(\frac{1}{4} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n+3}} - \frac{1}{n+4}\right) = \frac{1}{4} - \frac{1}{n+4}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{4} - \frac{1}{n+4}\right) = \frac{1}{4} < \infty$$ Yakınsak.

Ör: $\sum_{n=1}^{\infty} \frac{4n-1}{5n+3}$ serisi yakınsak mıdır?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n-1}{5n+3} = \lim_{n\to\infty} \frac{4 - 1/n}{5 + 3/n} = \frac{4}{5} \neq 0$$ ıraksak.

Ör: $\sum_{n=1}^{\infty} \frac{2}{n(n+1)(n+2)}$ serisi yakınsak mıdır?

$$a_n = \frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} = \left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$

$$= \left(\left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)\right) - \left(\left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right)\right)$$

$$= \left(1 - \frac{1}{n+1}\right) - \left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$S = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right) = \frac{1}{2} < \infty$$ Yakınsak

Ör: $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$ serisi yakınsak mıdır?

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{2}{4n^2-1} = \sum_{k=1}^{n}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \left(1 - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \cdots + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)$$

$$= 1 - \frac{1}{2n+1}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = 1 < \infty$$ Yakınsak

Ör $\sum_{n=0}^{\infty} \frac{3}{7^n}$ serisi yakınsak mıdır?

$$\sum_{n=0}^{\infty} \frac{3}{7^n} = 3\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n = 3\,\frac{1}{1-1/7} = \frac{7}{2} < \infty \text{ Yakınsak.}$$

Ör $\sum_{n=0}^{\infty} \frac{4^n-3^n}{12^n}$ serisi yakınsak mıdır?

$$\sum_{n=0}^{\infty} \frac{4^n-3^n}{3^n.4^n} = \sum_{n=0}^{\infty}\left(\frac{1}{3^n}-\frac{1}{4^n}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

$$= \frac{1}{1-1/3} - \frac{1}{1-1/4} = \underset{(3)}{\frac{3}{2}} - \underset{(2)}{\frac{4}{3}} = \frac{1}{6} < \infty \text{ Yakınsak.}$$

Ör $\sum_{n=1}^{\infty} \frac{1}{n}$ serisi ıraksak mıdır? integral testi

$x \geq 1$ için $f(x) = \frac{1}{x} \geq 0$ ve azalan bir fonksiyon

$$\int_1^{\infty} f(x)dx = \int_1^{\infty}\frac{dx}{x} = \lim_{u\to\infty}\int_1^u \frac{dx}{x} = \lim_{u\to\infty}\left(\ln|x|\Big|_1^u\right) = \lim_{u\to\infty}\left(\ln|u| - \ln|1|\right)$$

$$= \lim_{u\to\infty} \ln|u| = \ln\infty = \infty \text{ ıraksak.}$$

integral ıraksak olduğundan $\sum_{n=1}^{\infty}\frac{1}{n}$ serisi de ıraksak.

Ör $\sum_{n=2}^{\infty} \frac{1}{n.\ln n}$ serisi ıraksak mıdır? integral test.

$x \geq 2$ için $f(x) = \frac{1}{x.\ln x} \geq 0$ ve azalan bir fonksiyon

$$\int f(x)dx = \int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\ln x| + C$$

$u = \ln x \rightarrow du = \frac{dx}{x}$

$$\int_2^{\infty} f(x)dx = \int_2^{\infty}\frac{dx}{x\ln x} = \lim_{u\to\infty}\int_2^u \frac{dx}{x\ln x} = \lim_{u\to\infty}\left(\ln|\ln x|\Big|_2^u\right)$$

$$= \lim_{u\to\infty}\left(\ln|\ln u| - \ln|\ln 2|\right) = \ln|\ln(\infty)| - \ln|\ln(2)| = \infty \text{ ıraksak}$$

integral ıraksak olduğundan $\sum_{n=2}^{\infty}\frac{1}{n.\ln n}$ serisi de ıraksaktır.

Ör) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ serisi yakınsak mıdır? integral test.

$x \geq 2$ için $f(x) = \frac{1}{x(\ln x)^2} \geq 0$ ve azalan bir fonksiyon.

$$\int f(x)dx = \int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

$u = \ln x \rightarrow du = dx/x$

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{u\to\infty} \int_2^{u} \frac{dx}{x(\ln x)^2} = \lim_{u\to\infty} \left(-\frac{1}{\ln x}\Big|_2^u\right)$$

$$= \lim_{u\to\infty} \left(-\frac{1}{\ln u} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} < \infty \text{ yakınsak}$$

integral yakınsak olduğundan $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ serisi de yakınsak.

Ör) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ serisi yakınsak mıdır? integral test.

$x \geq 1$ için $f(x) = \frac{1}{x^2} \geq 0$ ve azalan bir fonksiyon

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{dx}{x^2} = \lim_{u\to\infty} \int_1^{u} \frac{dx}{x^2} = \lim_{u\to\infty} \left(-\frac{1}{x}\Big|_1^u\right)$$

$$= \lim_{u\to\infty} \left(-\frac{1}{u} + 1\right) = 1 < \infty \text{ yakınsak}$$

integral yakınsak olduğundan $\sum_{n=1}^{\infty} \frac{1}{n^2}$ serisi de yakınsak.

Ör) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ serisi yakınsak mıdır? integral test.

$x \geq 1$ için $f(x) = \frac{1}{x^2+1} \geq 0$ ve azalan bir fonksiyon.

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{dx}{x^2+1} \lim_{u\to\infty} \int_1^{u} \frac{dx}{x^2+1} = \lim_{u\to\infty} \left(\arctan x\Big|_1^u\right)$$

$$= \lim_{u\to\infty} (\arctan u - \arctan 1) = \arctan(\infty) - \arctan(1)$$

$$= \pi/2 - \pi/4 = \pi/4 < \infty \text{ yakınsak}$$

integral yakınsak olduğundan $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ serisi de yakınsak.

Ör: $\sum_{n=1}^{\infty} \frac{n}{n^3+4}$ serisi yakınsak mıdır? Karşılaştırma test.

$n \geq 1$ için $\frac{n}{n^3+4} \leq \frac{n}{n^3} = \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ serisinin yakınsak olduğunu önceki örnekte gösterdik.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ yakınsak olduğundan $\sum_{n=1}^{\infty} \frac{n}{n^3+4}$ de yakınsak.

Ör: $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ serisi ıraksak mıdır? Karşılaştırma test.

$n \geq 2$ için $\ln(n) < n$ olduğundan $\frac{1}{\ln(n)} > \frac{1}{n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ serisinin ıraksak olduğunu önceki örnekte gösterdik.

$\sum_{n=2}^{\infty} \frac{1}{n}$ ıraksak olduğundan $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ de ıraksaktır.

Ör: $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n(n+1)}$ serisi yakınsak mıdır? Karşılaştırma test.

$\sin^2(n) \leq 1$ olduğundan

$$\frac{\sin^2(n)}{n(n+1)} \leq \frac{1}{n(n+1)} \quad n \geq 1 \text{ için}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \text{ serisini inceleyelim.}$$

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \dots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 < \infty \text{ yakınsak}$$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ yakınsak olduğundan $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n(n+1)}$ de yakınsak

Ör: $\sum_{n=1}^{\infty} \frac{\ln(n+2)}{n}$ serisi ıraksak mıdır? Karşılaştırma test.

$n \geq 1$ için $\ln(n+2) > 1$ olduğundan $\frac{\ln(n+2)}{n} > \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ serisinin ıraksak olduğunu önceki örnekte gösterdik.

$\sum_{n=1}^{\infty} \frac{1}{n}$ ıraksak olduğundan $\sum_{n=1}^{\infty} \frac{\ln(n+2)}{n}$ de ıraksak.

Ör: $\sum_{n=1}^{\infty} \frac{3}{2n^2+5n-1}$ serisi yakınsak mıdır? Limit kıyas testi

$a_n = \frac{3}{2n^2+5n-1}$, $b_n = \frac{1}{n^2}$ alalım

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3}{2n^2+5n-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^2}{2n^2+5n-1}$$ her iki tarafı n^2 ile böl

$$= \lim_{n\to\infty} \frac{3}{2+\frac{5}{n}-1/n^2} = \frac{3}{2} = L$$

L pozitif bir sabit ve $\sum_{n=1}^{\infty} \frac{1}{n^2}$ yakınsak olduğundan

$\sum_{n=1}^{\infty} \frac{3}{2n^2+5n-1}$ serisi de yakınsak.

Ör: $\sum_{n=1}^{\infty} \frac{2n+3}{\sqrt{n^3+5n}}$ serisi ıraksak mıdır? Limit kıyas test.

$a_n = \frac{2n+3}{\sqrt{n^3+5n}}$, $b_n = \frac{1}{n}$ alalım.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n+3}{\sqrt{n^3+5n}}}{1/n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{n^3+5n}}$$ her iki tarafı n^2 ile böl.

$$= \lim_{n\to\infty} \frac{2+3/n}{\sqrt{\frac{1}{n}+5/n^3}} = \frac{2+0}{0} = \infty = L$$

$L=\infty$ ve $\sum_{n=1}^{\infty} \frac{1}{n}$ serisi ıraksak olduğundan

$\sum_{n=1}^{\infty} \frac{2n+3}{\sqrt{n^3+5n}}$ serisi de ıraksak.

Teorem

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

İspat

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln n^{1/n}} = e^{\lim_{n\to\infty} \frac{\ln n}{n}}$$

$$= e^{\lim_{n\to\infty} \frac{1/n}{1}} = e^{\lim_{n\to\infty} \frac{1}{n}} = e^0 = 1$$

Ör $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$ serisi ıraksak mıdır? Kök test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^4}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^4} = 2 = L$$

$L > 1$ olduğundan $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$ serisi ıraksaktır.

Ör $\sum_{n=1}^{\infty} \frac{n}{3^n}$ serisi yakınsak mıdır? Kök test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{3^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{3} = \frac{1}{3} = L$$

$L < 1$ olduğundan $\sum_{n=1}^{\infty} \frac{n}{3^n}$ serisi yakınsak

Ör $\sum_{n=1}^{\infty} \left(\frac{3}{2} + \frac{5}{n}\right)^n$ serisi yakınsak mıdır? Kök test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{2} + \frac{5}{n}\right)^n} = \lim_{n\to\infty} \left(\frac{3}{2} + \frac{5}{n}\right) = 3/2 = L$$

$L > 1$ olduğundan $\sum_{n=1}^{\infty} \left(\frac{3}{2} + \frac{5}{n}\right)^n$ serisi ıraksak.

Ör $\sum_{n=0}^{\infty} \frac{1}{n!}$ serisi yakınsak mıdır? Oran test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0 = L$$

$L < 1$ olduğundan $\sum_{n=0}^{\infty} \frac{1}{n!}$ serisi yakınsaktır.

Ör $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ serisi mutlak yakınsak mıdır?

$$a_n = \left| \frac{(-1)^n}{n^2} \right| = \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ yakınsak olduğundan $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ mutlak yakınsak.

Ör $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ serisi yakınsak mıdır?

$a_n = \frac{1}{n}$ alınırsa a_n azalan ve $\lim_{n\to\infty} a_n = 0$ olduğundan $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ serisi yakınsak. Normalde $\sum_{n=1}^{\infty} \frac{1}{n}$ serisi ıraksak olduğundan $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ koşullu yakınsak.

Kuvvet Serileri

$\sum_{n=0}^{\infty} c_n (x-a)^n$ veya $\sum_{n=0}^{\infty} a_n x^n$ şeklinde gösterilen serilerdir.

Bütün fonksiyonlar kuvvet serisine dönüştürülebilir.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \xrightarrow{\text{türev}} f'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$$

$$\xrightarrow{\text{integral}} \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \; ; \; \lvert x\rvert<1$	$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot x^{2n+1} \; , \; \lvert x\rvert<1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \; , \; x \in R$	$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} \; , \; x \in R$
$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \; , \; \lvert x\rvert<1$	$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} \; , \; x \in R$

Ör $\sum_{n=0}^{\infty} \frac{x^n}{3^n \cdot (n+1)^2}$ kuvvet serisinin yakınsaklık yarıçapı.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{3^{n+1} \cdot (n+2)^2}}{\frac{x^n}{3^n \cdot (n+1)^2}} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^2 \frac{|x|}{3} = \frac{|x|}{3} < 1 \rightarrow |x|<3$$

$r = 3$ Yakınsaklık yarıçapı

Ör $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 7^n}$ kuvvet serisinin yakınsaklık yarıçapı ve aralığı.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-5)^{n+1}}{(n+1)\, 7^{n+1}}}{\frac{(x-5)^n}{n \cdot 7^n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \frac{|x-5|}{7} = \frac{|x-5|}{7} < 1 \rightarrow |x-5|<7$$

$y.y = 7 \; , \; y.a = (-2, 12)$

Ör $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ kuvvet serisinin yakınsaklık yarıçapı ve aralığı.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \quad y.y = \infty \; ; \; y.a \in R$$

Ör $\sum_{n=0}^{\infty} \left(\frac{4}{5-x^2}\right)^n$ kuvvet serisinin yakınsaklık aralığı.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\left(\frac{4}{5-x^2}\right)^{n+1}}{\left(\frac{4}{5-x^2}\right)^n} \right| = \lim_{n\to\infty} \frac{4}{|5-x^2|} < 1 \longrightarrow |x^2-5| > 4$$

$x^2 - 5 > 4$
$x^2 > 9$
$Ç_1 = (-\infty, -3) \cup (3, \infty)$

$x^2 - 5 < -4$
$x^2 < 1$
$Ç_2 = (-1, 1)$

$Ç = Ç_1 \cup Ç_2$
$= (-\infty, -3) \cup (-1, 1) \cup (3, \infty)$

Ör. $\sum_{n=0}^{\infty}\left(\frac{x^2-5}{4}\right)^n$ kuvvet serisinin yakınsaklık aralığını bul.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{x^2-5}{4}\right)^{n+1}}{\left(\frac{x^2-5}{4}\right)^n}\right| = \lim_{n\to\infty}\frac{|x^2-5|}{4} < 1 \longrightarrow |x^2-5| < 4$$

$$-4 < x^2-5 < 4 \longrightarrow 1 < x^2 < 9 \begin{cases} \rightarrow x^2 < 9 \rightarrow G_1 = (-3,3) \\ \rightarrow x^2 > 1 \rightarrow G_2 = (-\infty,-1)\cup(1,\infty) \end{cases}$$

$G = G_1 \cap G_2$

$G = (-3,-1)\cup(1,3)$

Ör. $\sum_{n=1}^{\infty}\frac{n}{3^n}$ serisinin toplamı.

$|x|<1$ için $\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$ olduğundan $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ olur.

$$\sum_{n=1}^{\infty}\frac{n}{3^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^n = \frac{1}{3}\sum_{n=1}^{\infty} n\cdot\left(\frac{1}{3}\right)^{n-1} = \frac{1}{3}\,\frac{1}{(1-1/3)^2} = \frac{3}{4}$$

Ör. $\sum_{n=1}^{\infty}\frac{1}{n.5^n}$ serisinin toplamı.

$|x|<1$ için $\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$ olduğundan $\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = \int_0^x \frac{dt}{1-t} = -\ln|1-x| = \sum_{n=1}^{\infty}\frac{x^n}{n}$

$$\sum_{n=1}^{\infty}\frac{1}{n.5^n} = \sum_{n=1}^{\infty}\frac{(1/5)^n}{n} = -\ln|1-1/5| = -\ln(4/5) = \ln(5/4)$$

Ör. $f(x) = \frac{1}{(1-x)^2}$ fonk. kuvvet serisi şeklinde ifade et.

$|x|<1$ olduğundan $f(x) = \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty}x^n\right)'$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n, \quad |x|<1 \qquad 1+2x+3x^2+4x^3+\dots \quad |x|<1$$

$$f(x) = \sum_{n=1}^{\infty} n.x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n = 1+2x+3x^2+4x^3+\dots$$

Ör. $f(x) = \frac{1}{5+3x}$ fonk. kuvvet serisi şeklinde ifade et.

$$\left|\frac{3x}{5}\right| < 1 \rightarrow |x| < 5/3$$

$$f(x) = \frac{1}{5+3x} = \frac{1}{5}\,\frac{1}{1-\left(-\frac{3x}{5}\right)} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{3x}{5}\right)^n$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{3^n.x^n}{5^n} = \sum_{n=0}^{\infty}\frac{(-1)^n.3^n.x^n}{5^{n+1}}, \quad |x|<5/3$$

Ör. $f(x) = \arctan x$ fonk. kuvvet serisi şeklinde ifade et

$|x|<1$ için $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$ olduğundan

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n.x^{2n} \quad |x|<1$$

$$f(x) = \int_0^x\frac{dt}{1+t^2} = \arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}, \quad |x|<1$$