Title

Author

1 Derivation

Given a signal $\vec{s}$ with parameters $\vec{\theta}$:

$$\vec{\theta} = \{m_1, m_2, \chi, \rho\} \tag{1}$$

where m_1 and m_2 are the masses, χ is the effective spin, and ρ is the nominal signal-to-noise-ratio.

Probability that the signal $\vec{s}$ is recovered in a chirp mass bin i:

$$P(\text{bin } i|\vec{s}(\vec{\theta})) = \sum_{k \text{ templates in bin } i} P(\vec{s} \text{ recovered with } \vec{t}_k|\vec{s}(\vec{\theta})) \tag{2}$$

where $P(\vec{s} \text{ recovered with } \vec{t}_k|\vec{s}(\vec{\theta}))$ is the probability that template $\vec{t}_k$ is the best match for the signal and noise. $\vec{t}_k$ is the kth template that lies on the unit sphere, and $|\vec{t}_k| = 1$.

$$\vec{d} = \vec{n} + \vec{s} \tag{3}$$

$$\rho_{obs,k}\vec{t}_k = \vec{n} + \rho\vec{t}_j \tag{4}$$

Here, we assume that that signal $\vec{s}$ is described by one of the templates in the bank. $\rho_{obs,k}$ is the observed signal-to-noise ratio.

Isolating for $\vec{n}$, we can solve for the squared magnitude $|\vec{n}|^2$:

$$\vec{n} = \rho_{obs,k}\vec{t}_k - \rho\vec{t}_j \tag{5}$$

$$|\vec{n}|^2 = (\rho_{obs,k}\vec{t}_k - \rho\vec{t}_j) \cdot (\rho_{obs,k}\vec{t}_k - \rho\vec{t}_j) \tag{6}$$

$$= \rho^2 + \rho_{obs,k}^2 - 2\rho_{obs,k}\rho(\vec{t}_j \cdot \vec{t}_k) \tag{7}$$

$\vec{n}$ is Gaussian distributed with constant variance of 1, so the probability density function for N dimensions is given as

$$f(\vec{n}) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}|\vec{n}|^2} \tag{8}$$

We want to find the probability that $\vec{d} = \rho\vec{t}_j + \vec{n}$ lies inside the conic volume of some template $\vec{t}_k$ with solid angle $\Delta\Omega$ (assuming $\Delta\Omega$ not constant and $N = 2$, go here for calculations: 3). If $\vec{d}$ is inside the conic volume, it means that it is recoverable by that template $\vec{t}_k$.

$$P(\vec{s} \text{ recovered with } \vec{t}_k|\vec{s}(\vec{\theta})) = \frac{\int_0^\infty \frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}|\vec{n}|^2} \rho_{obs,k}^{N-1} d\rho_{obs,k} \Delta\Omega}{\sum_{\vec{t}_{k'}} \int_0^\infty \frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}|\vec{n}|^2} \rho_{obs,k'}^{N-1} d\rho_{obs,k'} \Delta\Omega} \tag{9}$$

Let's deal with the numerator in Equation 9 first. Moving constant terms outside the integral and substituting in Equation 7, we get

$$= \frac{\Delta\Omega}{(2\pi)^{N/2}} \int_0^\infty e^{-\frac{1}{2}(\rho^2 + \rho_{obs,k}^2 - 2\rho_{obs,k}\rho(\vec{t}_j \cdot \vec{t}_k))} \rho_{obs,k}^{N-1} d\rho_{obs,k} \tag{10}$$

$$= \frac{\Delta\Omega}{(2\pi)^{N/2}} e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j \cdot \vec{t}_k)^2)} \int_0^\infty e^{-\frac{1}{2}(\rho_{obs,k} - \rho(\vec{t}_j \cdot \vec{t}_k))^2} \rho_{obs,k}^{N-1} d\rho_{obs,k} \tag{11}$$

Let $x = \rho_{obs,k} - \rho(\vec{t}_j \cdot \vec{t}_k)$,

$$= \frac{\Delta\Omega}{(2\pi)^{N/2}} e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j \cdot \vec{t}_k)^2)} \int_{-\rho(\vec{t}_j \cdot \vec{t}_k)}^\infty e^{-\frac{1}{2}x^2} (x + \rho(\vec{t}_j \cdot \vec{t}_k))^{N-1} dx \tag{12}$$

$$= \frac{\Delta\Omega}{(2\pi)^{N/2}} e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j \cdot \vec{t}_k)^2)} \int_{-\rho(\vec{t}_j \cdot \vec{t}_k)}^\infty e^{-\frac{1}{2}x^2} \sum_{n=0}^{N-1} \binom{N-1}{n} x^n (\rho(\vec{t}_j \cdot \vec{t}_k))^{N-1-n} dx \tag{13}$$

$$= \frac{\Delta\Omega(\rho(\vec{t}_j \cdot \vec{t}_k))^{N-1}}{(2\pi)^{N/2} e^{\frac{1}{2}\rho^2(1-(\vec{t}_j \cdot \vec{t}_k)^2)}} \sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t}_j \cdot \vec{t}_k))^{-n} \int_{-\rho(\vec{t}_j \cdot \vec{t}_k)}^\infty e^{-\frac{1}{2}x^2} x^n dx \tag{14}$$

(What if we switched exponents when taking the binomial distribution of $(x + \rho(\vec{t}_j \cdot \vec{t}_k))^{N-1}$? Go here for alternate calculations: 4.) We can do the same for the denominator, and cancel out the constants $\Delta\Omega$, ρ^{N-1}, and $(2\pi)^{-N/2}$. And so Equation 9 becomes the following:

$$= \frac{(\vec{t_j} \cdot \vec{t_k})^{N-1} e^{-\frac{1}{2}\rho^2(1-(\vec{t_j}\cdot\vec{t_k})^2)} \sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t_j} \cdot \vec{t_k}))^{-n} \int_{-\rho(\vec{t_j}\cdot\vec{t_k})}^{\infty} e^{-\frac{1}{2}x^2} x^n dx}{\sum_{\vec{t}_{k'}} (\vec{t_j} \cdot \vec{t_{k'}})^{N-1} e^{-\frac{1}{2}\rho^2(1-(\vec{t_j}\cdot\vec{t_{k'}})^2)} \sum_{n'=0}^{N-1} \binom{N-1}{n'} (\rho(\vec{t_j} \cdot \vec{t_{k'}}))^{-n'} \int_{-\rho(\vec{t_j}\cdot\vec{t_{k'}})}^{\infty} e^{-\frac{1}{2}x^2} x^{n'} dx} \tag{15}$$

We argue that for very large N, the dominant term in the denominator is in the case $\vec{t_k} = \vec{t_j}$, such that $(\vec{t_j} \cdot \vec{t_k}) = 1$. We also know the indefinite integral $\int e^{ax^r} x^{b-1} = -(1/r)x^b(-ax^r)^{-b/r}\Gamma(b/r, -ax^r)$ (link here). Therefore, this becomes:

$$\approx \frac{(\vec{t_j} \cdot \vec{t_k})^{N-1} e^{-\frac{1}{2}\rho^2(1-(\vec{t_j}\cdot\vec{t_k})^2)} \sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t_j} \cdot \vec{t_k}))^{-n} \int_{\rho(\vec{t_j}\cdot\vec{t_k})}^{\infty} e^{-\frac{1}{2}x^2} x^n dx}{(1)^{N-1} e^{-\frac{1}{2}\rho^2(1-1^2)} \sum_{n'=0}^{N-1} \binom{N-1}{n'} (\rho(1))^{-n'} \int_{\rho(1)}^{\infty} e^{-\frac{1}{2}x^2} x^{n'} dx} \tag{16}$$

$$= \frac{(\vec{t_j} \cdot \vec{t_k})^{N-1} e^{-\frac{1}{2}\rho^2(1-(\vec{t_j}\cdot\vec{t_k})^2)} \sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t_j} \cdot \vec{t_k}))^{-n} \int_{\rho(\vec{t_j}\cdot\vec{t_k})}^{\infty} e^{-\frac{1}{2}x^2} x^n dx}{e^0 \sum_{n'=0}^{N-1} \binom{N-1}{n'} \rho^{-n'} \int_{\rho}^{\infty} e^{-\frac{1}{2}x^2} x^{n'} dx} \tag{17}$$

$$= (\vec{t_j} \cdot \vec{t_k})^{N-1} e^{-\frac{1}{2}\rho^2(1-(\vec{t_j}\cdot\vec{t_k})^2)} \frac{\sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t_j} \cdot \vec{t_k}))^{-n} \Big[-(2)^{\frac{1}{2}(n-1)} \Gamma\Big(\frac{n+1}{2}, \frac{1}{2}x^2\Big) \Big]_{\rho(\vec{t_j}\cdot\vec{t_k})}^{\infty}}{\sum_{n'=0}^{N-1} \binom{N-1}{n'} \rho^{-n'} \Big[-(2)^{\frac{1}{2}(n'-1)} \Gamma\Big(\frac{n'+1}{2}, \frac{1}{2}x^2\Big) \Big]_{\rho}^{\infty}} \tag{18}$$

$$= (\vec{t_j} \cdot \vec{t_k})^{N-1} e^{-\frac{1}{2}\rho^2(1-(\vec{t_j}\cdot\vec{t_k})^2)} \frac{\sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t_j} \cdot \vec{t_k}))^{-n} \Big[2^{\frac{n}{2}} \Gamma\Big(\frac{n+1}{2}, \frac{1}{2}(\rho(\vec{t_j} \cdot \vec{t_k}))^2\Big) \Big]}{\sum_{n'=0}^{N-1} \binom{N-1}{n'} \rho^{-n'} \Big[2^{\frac{n'}{2}} \Gamma\Big(\frac{n'+1}{2}, \frac{1}{2}\rho^2\Big) \Big]} \tag{19}$$

We can expand $\binom{N-1}{n}$ in terms of Gamma functions: $\binom{N-1}{n} = \frac{(N-1)!}{n!(N-n-1)!} = \frac{\Gamma(N)}{\Gamma(n+1)\Gamma(N-n)}$ to get:

$$= \frac{(\vec{t}_j \cdot \vec{t}_k)^{N-1}}{e^{\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)}} \frac{\sum_{n=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n+1)\Gamma(N-n)} (\rho(\vec{t}_j \cdot \vec{t}_k))^{-n} \left[2^{\frac{n}{2}} \Gamma\left(\frac{n+1}{2}, \frac{1}{2}(\rho(\vec{t}_j \cdot \vec{t}_k))^2\right)\right]}{\sum_{n'=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n'+1)\Gamma(N-n')} \rho^{-n'} \left[2^{\frac{n'}{2}} \Gamma\left(\frac{n'+1}{2}, \frac{1}{2}\rho^2\right)\right]} \tag{20}$$

$$= \frac{(\vec{t}_j \cdot \vec{t}_k)^{N-1}}{e^{\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)}} \frac{\sum_{n=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n+1)\Gamma(N-n)} \left(\frac{\sqrt{2}}{\rho(\vec{t}_j\cdot\vec{t}_k)}\right)^n \Gamma\left(\frac{n+1}{2}, \frac{1}{2}(\rho(\vec{t}_j \cdot \vec{t}_k))^2\right)}{\sum_{n'=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n'+1)\Gamma(N-n')} \left(\frac{\sqrt{2}}{\rho}\right)^{n'} \Gamma\left(\frac{n'+1}{2}, \frac{1}{2}\rho^2\right)} \tag{21}$$

$$\tag{22}$$

In the limit of $n \to \infty$, $\Gamma(a, x) \to \Gamma(a)$. This approximation is good enough when $a > 50$. Therefore:

$$\approx \frac{(\vec{t}_j \cdot \vec{t}_k)^{N-1}}{e^{\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)}} \frac{\sum_{n=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n+1)\Gamma(N-n)} \left(\frac{\sqrt{2}}{\rho(\vec{t}_j\cdot\vec{t}_k)}\right)^n \Gamma\left(\frac{n+1}{2}\right)}{\sum_{n'=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n'+1)\Gamma(N-n')} \left(\frac{\sqrt{2}}{\rho}\right)^{n'} \Gamma\left(\frac{n'+1}{2}\right)} \tag{23}$$

$$= \frac{(\vec{t}_j \cdot \vec{t}_k)^{N-1}}{e^{\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)}} \frac{\sum_{n=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n+1)\Gamma(N-n)} \left(\frac{\sqrt{2}}{\rho(\vec{t}_j\cdot\vec{t}_k)}\right)^n \Gamma\left(\frac{n+1}{2}\right)}{\sum_{n'=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n'+1)\Gamma(N-n')} \left(\frac{\sqrt{2}}{\rho}\right)^{n'} \Gamma\left(\frac{n'+1}{2}\right)} \tag{24}$$

2 Computing $P(\vec{s}$ recovered with $\vec{t}_k|\vec{s}(\vec{\theta}))$ in Python

Rather than compute the entire summation term in both the numerator and denominator, we want to find the values of n which contribute the most to the summation term. We cannot compute this directly because Python gives overflow errors. First, we can locate the peak of the individual summation term by locating the peak of the logarithm of the individual summation term.

$$= \frac{(\vec{t}_j \cdot \vec{t}_k)^{N-1}}{e^{\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)}} \frac{\sum_{n=0}^{N-1} X_n}{\sum_{n'=0}^{N-1} Y_{n'}} \tag{25}$$

$$= \frac{(\vec{t}_j \cdot \vec{t}_k)^{N-1}}{e^{\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)}} \frac{\sum_{n=0}^{N-1} [\exp(\ln(X_0)), \exp(\ln(X_1)), ..., \exp(\ln(X_{N-1}))]}{\sum_{n'=0}^{N-1} [\exp(\ln(Y_0)), \exp(\ln(Y_1)), ..., \exp(\ln(Y_{N-1}))]} \tag{26}$$

Let $\max[\ln(...)] = \ln(X_\alpha), \ln(Y_\beta)$. Subtract $\ln$(max) from each $\ln(...)$ term to normalize the function. Then, take the exponential of each $\ln(...)$ to get back the original, albeit normalized terms.

$$\begin{aligned}
&\rightarrow \frac{[\ln(X_0), \ln(X_1), ..., \ln(X_{N-1})]}{[\ln(Y_0), \ln(Y_1), ..., \ln(Y_{N-1})]} && (27)\\
&= \frac{[\ln(X_0) - \ln(X_\alpha), \ln(X_1) - \ln(X_\alpha), ..., \ln(X_{N-1}) - \ln(X_\alpha)]}{[\ln(Y_0) - \ln(Y_\beta), \ln(Y_1) - \ln(Y_\beta), ..., \ln(Y_{N-1}) - \ln(Y_\beta)]} && (28)\\
&= \frac{[\ln(X_0/X_\alpha), \ln(X_1/X_\alpha), ..., \ln(X_{N-1}/X_\alpha))]}{[\ln(Y_0/Y_\beta), \ln(Y_1/Y_\beta), ..., \ln(Y_{N-1}/Y_\beta)]} && (29)\\
&\rightarrow \frac{[\exp(\ln(X_0/X_\alpha)), \exp(\ln(X_1/X_\alpha)), ..., \exp(\ln(X_{N-1}/X_\alpha))]}{[\exp(\ln(Y_0/Y_\beta)), \exp(\ln(Y_1/Y_\beta)), ..., \exp(\ln(Y_{N-1}/Y_\beta))]} && (30)\\
&= \frac{[X_0/X_\alpha, X_1/X_\alpha, ..., X_{N-1}/X_\alpha]}{[Y_0/Y_\beta, Y_1/Y_\beta, ..., Y_{N-1}/Y_\beta]} && (31)\\
&\rightarrow \frac{\frac{1}{X_\alpha}[X_0, X_1, ..., X_{N-1}]}{\frac{1}{Y_\beta}[Y_0, Y_1, ..., Y_{N-1}]} \times \frac{X_\alpha}{Y_\beta} && (32)
\end{aligned}$$

In the final equality, we multiply the term by X_α/Y_β to get back the correct value. Python is able to calculate this term to $N < 1 \times 10^7$ without overflow errors, which is more than we need ($\sim$ 10 seconds). For $N = 1 \times 10^6$, Python is able to compute the entire summation in $\sim$ 2 seconds.

Can use this special value of $\Gamma(n, z) = (n-1)! e^{-z} \sum_{m=0}^{n-1} \frac{z^m}{m!}$ (link here and here).

3 Calculating with angle

Here, we work out the probability where $|\vec{n}|$ depends on θ, which is the angle of the cone of $\vec{t}_k$. We calculate this for $N = 2$. Using cosine law,

$$|\vec{n}|^2 = \rho^2 + \rho_{obs,k}^2 - 2\rho\rho_{obs,k}\cos\theta. \tag{33}$$

$$\begin{aligned}
&\rightarrow \int_{\theta_1}^{\theta_2} \int_0^\infty \frac{1}{2\pi} e^{-\frac{1}{2}(\rho^2 + \rho_{obs,k}^2 - 2\rho\rho_{obs,k}\cos\theta)} \rho_{obs,k} d\rho_{obs,k} d\theta && (34)\\
&= \int_0^\infty \frac{1}{2\pi} e^{-\frac{1}{2}(\rho^2 + \rho_{obs,k}^2)} \rho_{obs,k} \left[\int_{\theta_1}^{\theta_2} e^{\rho\rho_{obs,k}\cos\theta} d\theta \right] d\rho_{obs,k} && (35)
\end{aligned}$$

To integrate $\int f(\theta)d\theta$, we can expand the exponential term into an infinite sum (because there is no straightforward analytic solution). Also use the definite integral $\int_0^\infty e^{-Ax^2}x^n dx = \frac{1}{2}\Gamma(\frac{n+1}{2})A^{-(n+1)/2}$ (link here).

$$
\begin{aligned}
&= \int_0^\infty \frac{1}{2\pi}e^{-\frac{1}{2}(\rho^2+\rho_{obs,k}^2)}\rho_{obs,k}\Big[\int_{\theta_1}^{\theta_2}\sum_{a=0}^{\infty}\frac{(\rho\rho_{obs,k}\cos\theta)^a}{a!}d\theta\Big]d\rho_{obs,k} && (36)\\
&= \int_0^\infty \frac{1}{2\pi}e^{-\frac{1}{2}(\rho^2+\rho_{obs,k}^2)}\rho_{obs,k}\Big[\int_{\theta_1}^{\theta_2}(1+\rho\rho_{obs,k}\cos\theta+\frac{1}{2}\rho^2\rho_{obs,k}^2\cos^2\theta+\frac{1}{6}\rho^3\rho_{obs,k}^3\cos^3\theta && (37)\\
&\quad +\frac{1}{24}\rho^4\rho_{obs,k}^4\cos^4\theta+\frac{1}{120}\rho^5\rho_{obs,k}^5\cos^5\theta+\frac{1}{720}\rho^6\rho_{obs,k}^6\cos^6\theta+...)d\theta\Big]d\rho_{obs,k}\\
&\approx \int_0^\infty \frac{1}{2\pi}e^{-\frac{1}{2}(\rho^2+\rho_{obs,k}^2)}\rho_{obs,k}\Big[\Delta\theta+\rho\rho_{obs,k}\sin\theta|_{\theta_1}^{\theta_2}+\frac{1}{2}\rho^2\rho_{obs,k}^2\frac{1}{2}(\Delta\theta+\sin\theta\cos\theta|_{\theta_1}^{\theta_2}) && (38)\\
&\quad +\frac{1}{6}\rho^3\rho_{obs,k}^3\frac{1}{12}(9\sin\theta+\sin 3\theta)|_{\theta_1}^{\theta_2}+\frac{1}{24}\rho^4\rho_{obs,k}^4\frac{1}{32}(12\Delta\theta+8\sin 2\theta+\sin 4\theta)|_{\theta_1}^{\theta_2}\\
&\quad +\frac{1}{120}\rho^5\rho_{obs,k}^5(\frac{5}{8}\sin\theta+\frac{5}{48}\sin 3\theta+\frac{1}{80}\sin 5\theta)|_{\theta_1}^{\theta_2}\\
&\quad +\frac{1}{720}\rho^6\rho_{obs,k}^6\frac{1}{192}(60\Delta\theta+45\sin 2\theta+9\sin 4\theta+\sin 6\theta)|_{\theta_1}^{\theta_2}\Big]d\rho_{obs,k}\\
&=\frac{1}{2\pi}e^{-\frac{1}{2}\rho^2}\Big[\Gamma(1)\Delta\theta+\rho(1/2)^{-1/2}\Gamma(3/2)\sin\theta|_{\theta_1}^{\theta_2}+\rho^2(1/2)\Gamma(2)(\Delta\theta+\sin\theta\cos\theta|_{\theta_1}^{\theta_2}) && (39)\\
&\quad +\frac{1}{6}\rho^3(1/2)^{-3/2}\frac{1}{12}\Gamma(5/2)(9\sin\theta+\sin 3\theta)|_{\theta_1}^{\theta_2}\\
&\quad +\rho^4\frac{1}{24}\frac{1}{32}(1/2)^{-2}\Gamma(3)(12\Delta\theta+8\sin 2\theta+\sin 4\theta)|_{\theta_1}^{\theta_2}\\
&\quad +\rho^5\frac{1}{120}(1/2)^{-5/2}\Gamma(7/2)(\frac{5}{8}\sin\theta+\frac{5}{48}\sin 3\theta+\frac{1}{80}\sin 5\theta)|_{\theta_1}^{\theta_2}\\
&\quad +\rho^6\frac{1}{720}\frac{1}{192}(1/2)^{-3}\Gamma(4)(60\Delta\theta+45\sin 2\theta+9\sin 4\theta+\sin 6\theta)|_{\theta_1}^{\theta_2}\Big] && (40)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2\pi}e^{-\frac{1}{2}\rho^2}\Big[\Delta\theta+\rho(1/2)^{-1/2}\frac{\sqrt{\pi}}{2}\sin\theta|_{\theta_1}^{\theta_2}+\rho^2(1/2)(\Delta\theta+\sin\theta\cos\theta|_{\theta_1}^{\theta_2}) \qquad (41)\\
&+\frac{1}{6}\rho^3(1/2)^{-3/2}\frac{1}{12}\frac{3\sqrt{\pi}}{4}(9\sin\theta+\sin3\theta)|_{\theta_1}^{\theta_2}\\
&+\rho^4\frac{1}{24}\frac{1}{32}(1/2)^{-3}(12\Delta\theta+8\sin2\theta+\sin4\theta)|_{\theta_1}^{\theta_2}\\
&+\rho^5\frac{1}{120}(1/2)^{-5/2}\frac{15\sqrt{\pi}}{8}(\frac{5}{8}\sin\theta+\frac{5}{48}\sin3\theta+\frac{1}{80}\sin5\theta)|_{\theta_1}^{\theta_2}\\
&+\rho^6\frac{1}{120}\frac{1}{192}(1/2)^{-3}(60\Delta\theta+45\sin2\theta+9\sin4\theta+\sin6\theta)|_{\theta_1}^{\theta_2}\Big]\\
=&\frac{1}{2\pi}e^{-\frac{1}{2}\rho^2}\Big[\Delta\theta+\rho\sqrt{\frac{\pi}{2}}\sin\theta|_{\theta_1}^{\theta_2}+\rho^2\frac{1}{2}(\Delta\theta+\frac{1}{2}\sin2\theta|_{\theta_1}^{\theta_2}) \qquad (42)\\
&+\rho^3\frac{\sqrt{2\pi}}{48}(9\sin\theta+\sin3\theta)|_{\theta_1}^{\theta_2}+\rho^4\frac{1}{96}(12\Delta\theta+8\sin2\theta+\sin4\theta)|_{\theta_1}^{\theta_2}\\
&+\rho^5\frac{\sqrt{2\pi}}{128}(5\sin\theta+\frac{5}{6}\sin3\theta+\frac{1}{10}\sin5\theta)|_{\theta_1}^{\theta_2}\\
&+\rho^6\frac{1}{2880}(60\Delta\theta+45\sin2\theta+9\sin4\theta+\sin6\theta)|_{\theta_1}^{\theta_2}\Big]\\
=&\frac{1}{2\pi}e^{-\frac{1}{2}\rho^2}\Big[\Delta\theta\Big(1+\rho^2\frac{1}{2}+\rho^4\frac{1}{8}+\rho^6\frac{1}{48}\Big)+\sin\theta|_{\theta_1}^{\theta_2}(\rho\sqrt{\frac{\pi}{2}}+\rho^3\frac{3\sqrt{2\pi}}{16}+\rho^5\frac{5\sqrt{2\pi}}{128}) \qquad (43)\\
&+\sin2\theta|_{\theta_1}^{\theta_2}(\rho^2\frac{1}{4}+\rho^4\frac{1}{12}+\rho^6\frac{1}{64})+\sin3\theta|_{\theta_1}^{\theta_2}(\rho^3\frac{\sqrt{2\pi}}{48}+\rho^5\frac{5\sqrt{2\pi}}{768})\\
&+\sin4\theta|_{\theta_1}^{\theta_2}(\rho^4\frac{1}{96}+\rho^6\frac{1}{320})+\sin5\theta|_{\theta_1}^{\theta_2}(\rho^5\frac{\sqrt{2\pi}}{1280})+\sin6\theta|_{\theta_1}^{\theta_2}(\rho^6\frac{1}{2880})\Big]
\end{aligned}
$$

In the end, we concluded that doing the $\Delta\Omega$ approximation is fine because we don't pick up additional terms of ρ and the answer doesn't change when we increase N.

Numerator only:

$$\int_{\delta\theta=-\frac{1}{2}\Delta\theta}^{\delta\theta=+\frac{1}{2}\Delta\theta} e^{\rho\rho_{obs,k}\cos(\theta+\delta\theta)}\delta\theta \approx \Delta\theta+O(\Delta\theta^2)+... \qquad (44)$$

4 Alternate binomial expansion calculation

$$= \frac{\Delta\Omega}{(2\pi)^{N/2}} e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)} \int_{-\rho(\vec{t}_j\cdot\vec{t}_k)}^{\infty} e^{-\frac{1}{2}x^2}(x+\rho(\vec{t}_j\cdot\vec{t}_k))^{N-1}dx \tag{45}$$

$$= \frac{\Delta\Omega}{(2\pi)^{N/2}} e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)} \int_{-\rho(\vec{t}_j\cdot\vec{t}_k)}^{\infty} e^{-\frac{1}{2}x^2} \sum_{n=0}^{N-1} \binom{N-1}{n} x^{N-n-1}(\rho(\vec{t}_j\cdot\vec{t}_k))^n dx \tag{46}$$

$$= \frac{\Delta\Omega}{(2\pi)^{N/2} e^{\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)}} \sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t}_j\cdot\vec{t}_k))^n \int_{-\rho(\vec{t}_j\cdot\vec{t}_k)}^{\infty} e^{-\frac{1}{2}x^2} x^{N-n-1} dx \tag{47}$$

$$= \frac{\Delta\Omega}{(2\pi)^{N/2}} e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)} \sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t}_j\cdot\vec{t}_k))^n \Big[-\frac{1}{2}\Big(\frac{1}{2}\Big)^{-(N-n)/2} \Gamma\Big(\frac{N-n}{2}, \frac{x^2}{2}\Big)\Big]_{-\rho(\vec{t}_j\cdot\vec{t}_k)}^{\infty} \tag{48}$$

$$= \frac{\Delta\Omega}{(2\pi)^{N/2}} e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)} \sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t}_j\cdot\vec{t}_k))^n \Big[\frac{1}{2}\Big(\frac{1}{2}\Big)^{-(N-n)/2} \Gamma\Big(\frac{N-n}{2}, \frac{(\rho(\vec{t}_j\cdot\vec{t}_k))^2}{2}\Big)\Big] \tag{49}$$

Putting it together, numerator and denominator (where denominator only keeps $(\vec{t}_j\cdot\vec{t}_k) = 1$) and cancelling out constants:

$$= \frac{e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)} \sum_{n=0}^{N-1} \binom{N-1}{n} (\rho(\vec{t}_j\cdot\vec{t}_k))^n \left[\frac{1}{2}\left(\frac{1}{2}\right)^{-(N-n)/2} \Gamma\left(\frac{N-n}{2}, \frac{(\rho(\vec{t}_j\cdot\vec{t}_k))^2}{2}\right)\right]}{\sum_{n=0}^{N-1} \binom{N-1}{n} \rho^n \left[\frac{1}{2}\left(\frac{1}{2}\right)^{-(N-n)/2} \Gamma\left(\frac{N-n}{2}, \frac{\rho^2}{2}\right)\right]} \tag{50}$$

$$= \frac{e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)} \sum_{n=0}^{N-1} \binom{N-1}{n} \left(\frac{\rho(\vec{t}_j\cdot\vec{t}_k)}{\sqrt{2}}\right)^n \Gamma\left(\frac{N-n}{2}, \frac{(\rho(\vec{t}_j\cdot\vec{t}_k))^2}{2}\right)}{\sum_{n=0}^{N-1} \binom{N-1}{n} \left(\frac{\rho}{\sqrt{2}}\right)^n \Gamma\left(\frac{N-n}{2}, \frac{\rho^2}{2}\right)} \tag{51}$$

$$= \frac{e^{-\frac{1}{2}\rho^2(1-(\vec{t}_j\cdot\vec{t}_k)^2)} \sum_{n=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n+1)\Gamma(N-n)} \left(\frac{\rho(\vec{t}_j\cdot\vec{t}_k)}{\sqrt{2}}\right)^n \Gamma\left(\frac{N-n}{2}, \frac{(\rho(\vec{t}_j\cdot\vec{t}_k))^2}{2}\right)}{\sum_{n=0}^{N-1} \frac{\Gamma(N)}{\Gamma(n+1)\Gamma(N-n)} \left(\frac{\rho}{\sqrt{2}}\right)^n \Gamma\left(\frac{N-n}{2}, \frac{\rho^2}{2}\right)} \tag{52}$$

(53)