Runge-Kutta-Chebyshev Methods

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Outline

- Preliminaries (Stability functions, stiffness)
- How to design and implement the Runge-Kutta-Chebyshev (RKC) methods.
- Numerical experiments

Preliminaries

Stability function

The function $R(z)$ is called the **stability function** of the method if it can be interpreted as the numerical solution after one step for

$$y' = \lambda y, \quad y_0 = 1, \quad z = h\lambda$$

The set

$$S = \{z \in \mathbb{C} : |R(z)| \leq 1\}$$

is called the **stability domain** of the method.

Order of method

Theorem. If the Runge-Kutta method is of order p, then

$$R(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^p}{p!} + O(z^{p+1})$$

Preliminaries

A **stiff equation** is a differential equation for which certain numerical methods for solving the equation are numerically *unstable*, unless the step size is taken to be extremely small.

Stiffness of ODEs

Traditionally, a linear stiff system of size n was defined by $Re(\lambda_i) < 0, \quad 1 \leq i \leq n$ with

$$\max_{1 \leq i \leq n} |Re(\lambda_i)| \gg \min_{1 \leq i \leq n} |Re(\lambda_i)|$$

The **stiffness ratio** R provided a measure of stiffness:

$$\frac{\max_{1 \leq i \leq n} |Re(\lambda_i)|}{\min_{1 \leq i \leq n} |Re(\lambda_i)|}$$

λ_i are the eigenvalues of the Jacobian of the system.

Semi-discretizated heat problem

$$u_t = u_{xx} \qquad \text{(PDE)}$$

Or it can be rewrite by approximation

$$U'(t) = \frac{1}{h^2} tridiag(1, -2, 1) U(t) \qquad \text{(ODEs)}$$

The largest eigenvalue of $\frac{1}{h^2} tridiag(1, -2, 1)$ is

$$\frac{4}{h^2} \cos^2\left(\frac{\pi h}{2}\right) \approx \frac{-4}{h^2}$$

RKC methods

Let us consider the following stiff systems of ODEs:

$$u'(t) = f(t, u(t)), \quad 0 < t \leq T, \quad u(0) = u_0 \text{ is given}$$

The Runge-Kutta-Chebyshev(RKC) is an $s-$stage Runge-Kutta(RK) method designed for explicit integration of **modestly** stiff systems of ODEs. It satisfies two conditions:

1. The eigenvalue spectrum of the Jacobian matrix $\partial f(t, u)/\partial U$ should lie in a narrow strip along the **negative axis** of the complex plane.
2. The Jacobian matrix should not digress too much from a **normal matrix**.

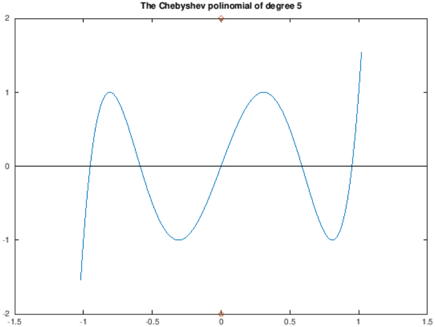
RKC methods

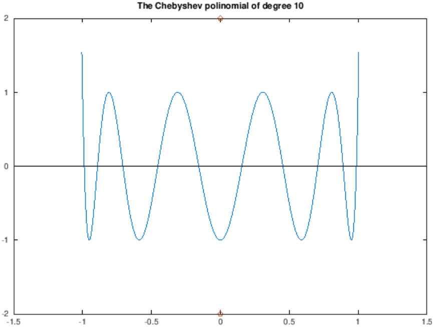
Chebyshev polynomials

$$T_0(x) = 1,\ T_1(x) = x,\ T_s(x) = 2xT_{s-1}(x) - T_{s-2}(x)$$

$$T_s(x) = \cos(s \arccos x) \quad \text{if } x \in [-1, 1]$$

$$T_s(x) = \cosh(s \cosh^{-1} x) \quad \text{if } x \notin [-1, 1]$$





RKC methods

Theorem

For any explicit, consistent Runge-Kutta method we have stable interval is not exceed $2s^2$. The optimal stability polynomial is the shifted Chebyshev polynomial of the first kind

$$R_s(z) = T_s\left(1 + \frac{z}{s^2}\right)$$

RKC methods

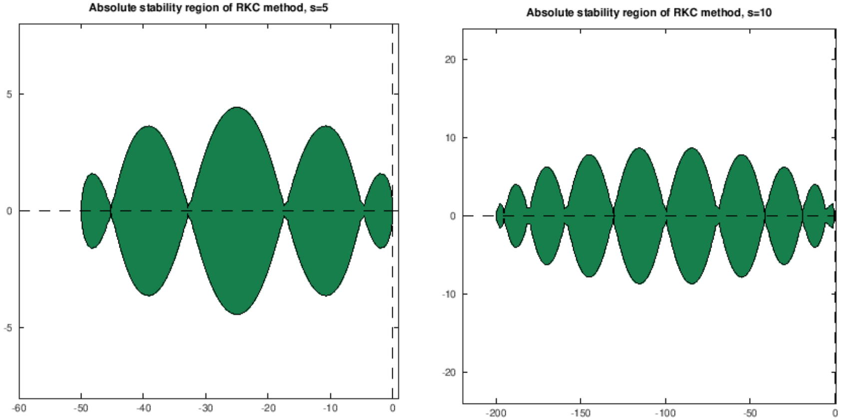


Figure.The absolute stability region of the second-order RKC methods with $s = 5$(left) and $s = 10$(right)

RKC methods

Damped first-order RKC methods

$$R_s(z) = \frac{T_s(w_0 + w_1 z)}{T_s(w_0)} \quad , \quad w_1 = \frac{T_s(w_0)}{T_s'(w_0)} \quad , w_0 > 1$$

Choose $w_0 = 1 + \frac{\epsilon}{s^2}$

RKC methods

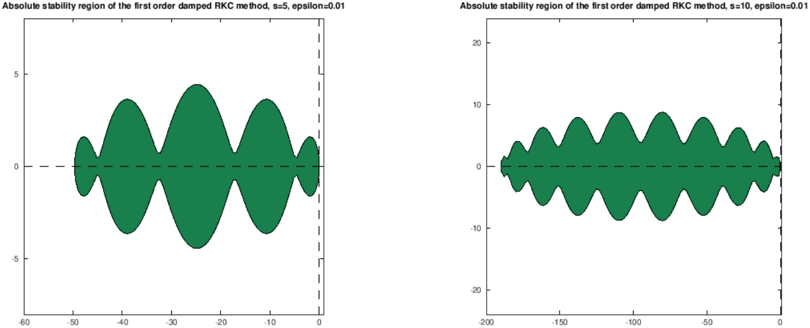


Figure.The absolute stability region of the damped first-order RKC methods

RKC methods

Second-order RKC methods (undamped)

- Stabil polynomials

$$B_s(z) = \frac{2}{3} + \frac{1}{3s^2} + \left(\frac{1}{3} - \frac{1}{3s^2}\right) T_s\left(1 + \frac{3z}{s^2-1}\right)$$

  or

$$\frac{2}{2-z} - \frac{z}{2-z} T_s\left(\cos\frac{\pi}{2} + \frac{z}{2}\left(1 - \cos\frac{\pi}{s}\right)\right)$$

- In general, the optimal bound for second-order RKC methods is approximately $0.82s^2$.

Second-order RKC methods (damped)

- Stabil polynomials

$$B_s(z) = 1 + \frac{T''s(w_0)}{(T_s'(w_0))^2}(T_s(w_0 + w_1 z) - T_s(w_0)) \quad , \quad w_1 = \frac{T_s'(w_0)}{T_s''(w_0)}$$

- If we are expected that the interior of the stability interval get 5% damping, we need to choose the damping parameter $\epsilon \approx 0.15$. The stability boundary will reduced 2% compared to undamped case.

RKC methods

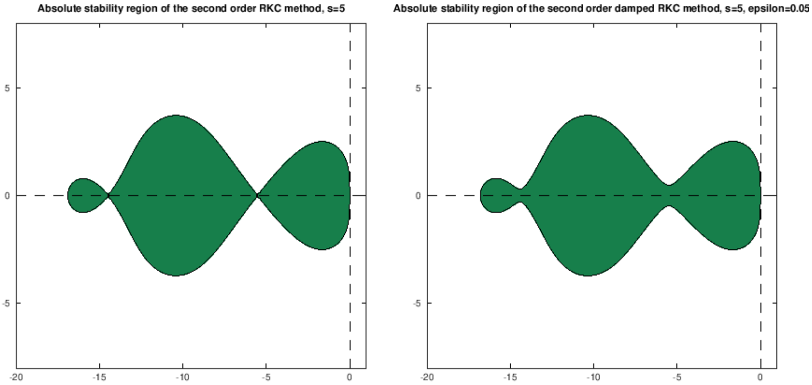


Figure. The absolute stability region of the second-order RKC methods with $s = 5$, undamped (left) and damped (right)

RKC methods

General kernel representation

$$
\begin{aligned}
y_{n0} &= y_n \\
y_{n1} &= y_n + h\mu_1 f(t_n + c_0 h, y_{n0})) \\
&\vdots \\
y_{nj} &= (1 - \mu_i - \nu_j) y_n + \mu_j y_{n,j-1} + \nu_j y_{n,j-2} \\
&\quad + h\left(\tilde{\mu_j} f(t_n + c_{j-1} h, y_{n,j-1}) + \tilde{\gamma}_j f(t_n + c_0 h, y_{n0})\right) \\
&\quad \text{where } j = 2, \cdots, s \\
y_{n+1} &= y_{ns}
\end{aligned}
$$

where

$$\tilde{\mu_1} = b_1 w_1, \quad \mu_j = \frac{2 b_j w_0}{b_{j-1}}, \quad \nu_j = \frac{-b_j}{b_{j-2}}, \quad \tilde{\mu_j} = \frac{2 b_j w_1}{b_{j-1}}, \quad \tilde{\gamma}_j = -a_{j-1} \tilde{\mu_j}$$

RKC methods

First-order RKC methods

- $$b_j = \frac{1}{T_j(w_0)}$$
- $$c_j = \frac{T_s(w_0)\,T_j'(w_0)}{T_s'(w_0)\,T_j(w_0)}$$

Second-order RKC methods

- $$b_j = \frac{T_j''(w_0)}{(T_j'(w_0))^2}$$
- $$c_j = \frac{T_s'(w_0)\,T_j''(w_0)}{T_s''(w_0)\,T_j'(w_0)}$$

Numerical Experiments

Test problem

Let us consider the initial value problem

$$x''(t) = -\frac{1}{4}x(t), \quad t \in [0, 4\pi]$$

with the initial conditions: $x(0) = 0; x'(0) = 1$.
The exact solution for this problem is $x(t) = \cos(\frac{x}{2})$

$N = 2^i$	$i = 10$	$i = 11$	$i = 12$	$i = 13$	$i = 14$
Error (RKC5)	1.3692e-2	6.8231e-3	3.4059e-3	1.7015e-3	8.5039e-4
Error (RKC10)	1.3480e-2	6.7179e-3	3.3534e-3	1.6753e-3	8.3731e-4
Error (EE)	2.0450e-2	1.0174e-2	5.0747e-3	2.5342e-3	1.2663e-3

Table: Errors for explicit 1st-order RKC method ($s = 5, 10$) and EE

Numerical Experiments

$N = 2^i$	$i = 10$	$i = 11$	$i = 12$	$i = 13$	$i = 14$
Error (RKC5)	1.7356e-5	4.3377e-6	1.0843e-6	2.7104e-7	6.7758e-8
Error (RKC10)	1.5229e-5	3.8060e-6	9.5136e-7	2.3782e-7	5.9453e-8
Error (RK2)	3.7001e-5	9.2462e-6	2.3111e-6	5.7770e-7	1.4442e-7

Table: Errors for explicit 2nd-order RKC method ($s = 5, 10$) and RK2

Numerical Experiments

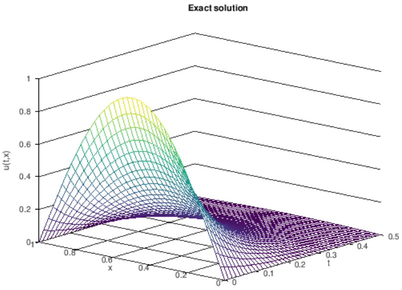
Semi-discretizated heat problem

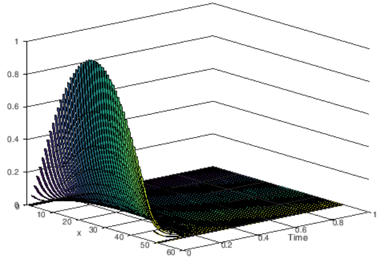
Let us consider semi-discretizated heat problem with perturbation:

$$u_t = u_{xx} + u, \quad x \in [0, 1], \quad t \in [0, 1/2]$$

with initial condition $u(0, x) = \sin(\pi x)$ and boundary conditions $u(t, 0) = u(t, 1) = 0$.
The exact solution is $u(t, x) = e^{(1-\pi^2)t} \sin(\pi x)$

Numerical Experiments





Numerical Experiments

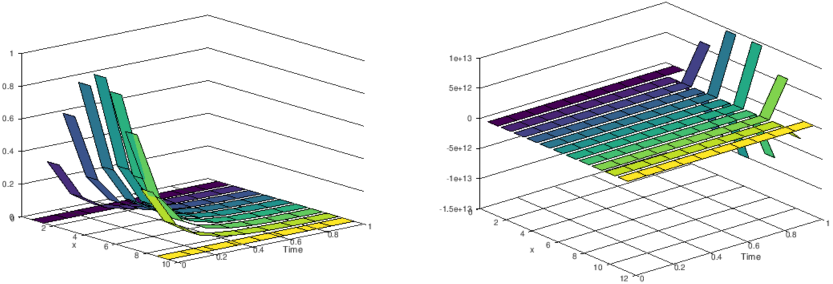


Figure.Numerical solution solved by RKC methods, $s = 10, N = 8$(left), $s = 10, N = 10$(right)

Numerical Experiments

The number of stage and interval	Error (inf norm)
$s = 10, N = 8$	0.67001
$s = 15, N = 18$	0.38906
$s = 19, N = 30$	0.25595
$s = 25, N = 50$	0.16549
$s = 30, N = 70$	0.11901
$s = 40, N = 125$	0.068492
$s = 50, N = 196$	0.044244

Table: Errors between exact solution and numerical solution using different RKC multi-stage methods

Diffusion problem

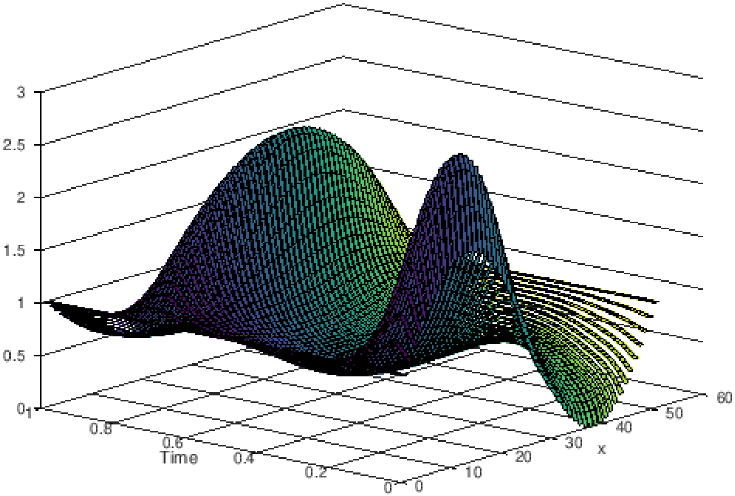
The diffusion problem is the following

$$\frac{\partial u}{\partial t} = A + u^2 v - (B+1)u + \alpha \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial v}{\partial t} = Bu - u^2 v + \alpha \frac{\partial^2 v}{\partial x^2}$$
$$0 \le x \le 1, \quad 0 \le t \le 10$$

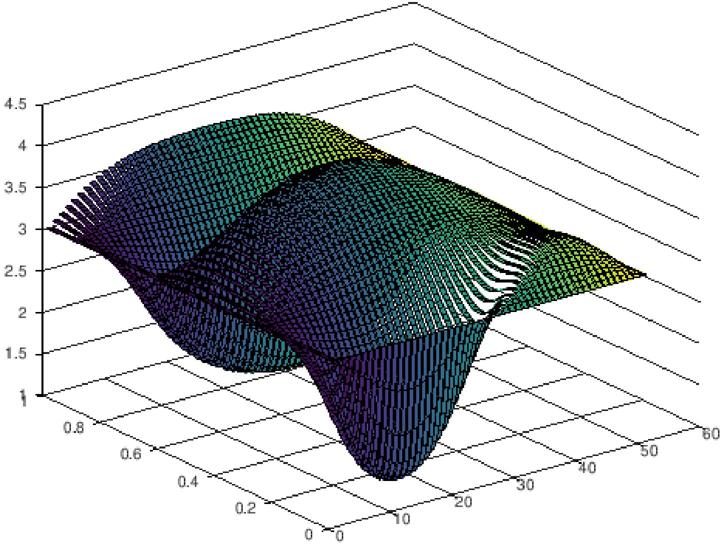
where $A = 1, B = 3, \alpha = 1/50$ and boundary conditions

$$u(0,t) = u(1,t) = 1, \quad v(0,t) = v(1,t) = 3$$
$$u(x,0) = 1 + \sin(2\pi x), \quad v(x,0) = 3$$

Numerical Experiments



Numerical Experiments



THANK YOU FOR YOUR ATTENTION!