ECE 374, ZJUI, Spring 2023 **HW1 Solutions** February 25, 2023

First Last*, 3xxxxxxxxx, First Last, 3xxxxxxxxx, and First Last, 3xxxxxxxxx

Q1.A solution

According to the text, $f(i)$ is the depth of the first i characters of the string s. And $f(j)$ is the depth of the first j characters of the string s.

As $f(i) = f(j)$, $d(S_{\leq i}) = d(S_{\leq j})$, which means that

$$\#a(S_{\leq i}) - \#b(S_{\leq i}) = \#a(S_{\leq j}) - \#b(S_{\leq j})$$

.

By simplifying this equation, we get $\#a(S_{\leq j}) - \#a(S_{\leq i}) - (\#b(S_{\leq j}) - \#b(S_{\leq i})) = 0$.

It can be seen that $\#a(S_{\leq j}) - \#a(S_{\leq i})$ is the number of a in the string constructed by index $i+1$ to j in string s, $\#b(S_{\leq j}) - \#b(S_{\leq i})$ is the number of b in the string constructed by index $i+1$ to j in string s.

So, $s_{i+1}s_{i+2}\ldots s_j$ is a weakly balanced string.

Q1.B solution

The string is balanced, so the number of a must be equal to the number of b in this string. The base case is $s = \epsilon$ or $s \in \sum^*$.

So, (i) is proved.

As $x, y \in \sum^*$, by definition, we have $d(xy) = d(x) + d(y)$. As $d(x) = d(y) = 0$, $d(xy) = 0$. So $s = xy$ is also balanced, (ii) is proved.

Assume $s_1 = a$, $s_2 = b$. By definition, $d(s_1) = 1$, $d(s_2) = -1$. Since we also have $d(xy) = d(x) + d(y)$, we can get $d(axb) = d(s_1) + d(x) + d(s_2) = 1 + d(x) - 1$. As x is a balanced string, $d(x) = 0$. So, $d(axb) = 0$, $s = axb$ is also balanced, (iii) is proved.

There are no other cases.

Q1.C solution

There are two scenarios. As s is balanced, its length n must be even.

The first scenario is the string s is in the form of "$aaa\ldots bbb\ldots$ ", where the number of a in the front is equal to the number of b behind. And the number of "a"sor"b"s must larger than 1.

For this case, we can get that $d_{max} = \frac{1}{2}n$.

Let $y = \frac{1}{2}n - \sqrt{n}$, by taking the derivative of it, we get $y' = \frac{1}{2} - \frac{1}{2\sqrt{n}}$.

When $n = 1$, $y' = 0$, which means y is decreasing when $n < 1$, and increasing when $n \geq 1$.

Because $n \geq 4$, when $n = 4$, $y = 0$. So, $y \geq 0$, which means in this scenario the maximum depth of s is $\geq \sqrt{n}$.

The other scenario is the remaining cases. The base case is $s = ab$. By the definition, it can be broken into 1 substring, that is $m = 1$. And $\sqrt{n} - 1 = \sqrt{2} - 1 < 1 = m$.

In terms of this scenario, according to the question, it can be inferred that the first letter must be a, and the last letter must be b. Also, let $f(i) = d(s_{\leq} i)$, then $f(0) = 0, f(1) = 1, \ldots, f(n-1) = 1, f(n) = 0$.

Q1.D solution

Q2 solution

We use induction to solve this problem.

Base case: Let n = 2. There are only two people in the world and if there's only one tribe with 2 people, no lambs are sacrificed. If there are two tribes each with 1 person, then one tribe will lose and one lamb is sacrificed. In both cases. The number of lambs sacrificed is smaller than $nlog_2n = 2$

Let n = 3, Then then number of lambs sacrificed can be 0 or 1, both of which is smaller than $3log_23$.

Inductive hypothesis: For any $n > 1$, at most $nlog_2n$ lambs got sacrificed.

Induction step: Let W be a world of n people, Assume inductive hypothesis holds for all worlds with number of people greater than 2 and less than n.

Then in the world with n-1 people, we have number of lambs sacrificed $\leq (n-1)log_2(n-1)$. By adding one more person, this world will have n people. If the tribe this new-added person is in wins all the fight, then the number of lambs sacrificed doesn't change. If the the tribe this new-added person is in loses, there will be one more lamb sacrificed. In this case, we have

$$\begin{aligned} Number\ of\ lambs\ sacrificed\ in\ world\ W &\leq (n-1)log_2(n-1)+1 \\ &= nlog_2(n-1)+1-log_2(n-1) \\ &\leq nlog_2(n-1) \\ &< nlog_2(n) \end{aligned}$$

In both cases, the number of lambs sacrificed in world W is less than $nlog_2(n)$

So during this process, at most $nlog_2n$ lambs got sacrificed.

Q3.A solution

Q3.B solution