Instructions

- please check HW1-3 for detailed instructions (they remain the same)

10 (100 PTS.) This is all wrong.

baab a

10.A. (30 PTS.) Let $\Sigma = \{a, b\}$. For a word $w = w_1w_2 \dots w_n \in \Sigma^*$, let $w_o = w_1w_3w_5 \dots w_{2\lceil n/2 \rceil - 1}$ be the string formed by the odd characters of w. Prove that the following language is not regular by providing a fooling set. Your fooling set needs to infinite, and you need also to prove that it is a valid fooling set. The language is $L = \left\{ ww_o \;\middle|\; w \in \Sigma^+ \right\}$.

10.B. (30 PTS.) Provide a counter-example for the following claim (if you need to prove that a specific language is regular [or not], please do so):

Claim: Consider two languages L and L'. If L and L' are not regular, and $L \cup L'$ is regular, then $L \cap L'$ is regular.

10.C. (40 PTS.) Suppose you are given three languages L_1, L_2, L_3, such that:

- $L_1 \cup L_2 \cup L_3$ is not regular.
- For all $i \neq j$: $L_i \setminus L_j$ is regular.

Prove that $L_1 \cap L_2 \cap L_3$ is not regular. (Hint: Use closure properties of regular languages.)
(Not for submission: Can you come up with an example of such languages?)

11 (100 PTS.) Grammarticus.

For (A) and (C) below, describe a context free grammar for the following languages. Clearly explain how they work and the role of each non-terminal. Unclear grammars will receive little to no credit.

11.A. (40 PTS.) $L = \left\{ a^i b^j c^k d^\ell \;\middle|\; i, j, k, \ell \geq 0 \text{ and } j + \ell = i + k \right\}$.
Hint for this question would be posted on piazza question thread.

11.B. (30 PTS.) Let $\Sigma = \{a, b\}$. Consider the language

$$L_B = \{ z \in \Sigma^* \mid \text{ for any prefix } y \text{ of } z \text{ we have } \#_a(y) \geq \#_b(y) \}.$$

Prove that any $w \in L_B$, can be written as $w = w_1 \cdots w_m$, such that $w_i = a$ or w_i is a balanced string, for all i. A string $s \in \{a.b\}^*$ is ***balanced*** if $\#_a(s) = \#_b(s)$.

(One can also prove a stronger version, where in addition each w_i is strongly balanced [i.e., $w_i \in L_B$].)

11.C. (30 PTS.) Describe a grammar for the language L_B defined above, using the property you proved in **(11.B.)** (you can use the stronger version without proving it). **Prove** the correctness of your grammar.

12 (100 PTS.) The pain never ends.

12.A. (50 PTS.) Let $\Sigma = \{a, b\}$. A string $s \in \Sigma^*$ is a ***palindrome*** if $s = s^R$. For a prespecified integer $k \geq 0$, a string $s \in \Sigma^*$ is ***k-close*** to being a palindrome, if there is a string $w \in \Sigma^*$ that is a palindrome, and one recover w from s by a sequence of (***at most***) k operations. Each such operation is either inserting one character or deleting a character. Thus $ababaaab$ is 2-close to a palindrome since

$$ababaaab \rightarrow babaaab \rightarrow baaaab.$$

Similarly, the string $ab^2a^2b^5a^5b^4a^3b^2a$ is 2-close to being a palindrome since

$$ab^2a^2b^5a^5b^4a^3b^2a \rightarrow ab^2\underline{a}^3b^5a^5b^4a^3b^2a \rightarrow ab^2a^3\underline{b}^4a^5b^4a^3b^2a.$$

Let L_k be the language of all strings that are k-close to being a palindrome. Give a CFG for L_3. Argue why your solution is correct.

12.B. (50 PTS.) Let $\Sigma = \{a, b\}$. Prove that if $L \subseteq \Sigma^*$ is context-free language then

$$\mathrm{subsequence}(L) = \{x \in \Sigma^* \mid \exists y \in L, x \text{ is a subsequence of } y\}$$

is a context-free language.