Notes on the Voxel Intensity Distribution

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1 Voxel volume

The comoving lenght corresponding to an angular separation $\Delta\theta$, for a given redshift z, is given by

$$D = r(z)\Delta\theta = \Delta\theta \int_0^z \frac{cdz'}{H(z')}, \tag{1}$$

where $r(z)$ is the comoving distance travelled by light emitted from redshift z to us.

The comoving radial distance corresponding to a small redshift interval $\Delta z = z_1 - z_2 = \nu_0/\nu_1^{\text{obs}} - \nu_0/\nu_2^{\text{obs}} \approx (1+z)^2\Delta\nu^{\text{obs}}/\nu_0$, where $z_1 > z_2$, is given by

$$\Delta r = \int_{z_2}^{z_1} \frac{cdz}{H(z)} \approx \frac{c\Delta z}{H(z)} \approx \frac{c}{H(z)} \frac{(1+z)^2\Delta\nu^{\text{obs}}}{\nu_0}, \tag{2}$$

for small Δz.

So the volume of a voxel at redshift z, with sides of angular size $\Delta\theta$ and depth Δz, is given approximately by

$$\begin{aligned} V_{\text{vox}} &\approx D^2\Delta r \approx r(z)^2\Delta\theta^2 \frac{c}{H(z)} \frac{(1+z)^2\Delta\nu^{\text{obs}}}{\nu_0} \\ &= \Delta\theta^2 \frac{(1+z)^2\Delta\nu^{\text{obs}}}{\nu_0 E(z)} \frac{c^3}{H_0^3} \left(\int_0^z \frac{dz'}{E(z')}\right)^2, \end{aligned} \tag{3}$$

where $E(z) \equiv H(z)/H_0$.

2 Fourier transform conventions

These conventions are for 1D, but are easily generalized to arbitrary dimension. We here use the convention where the map and the fourier coefficients have the same dimensions:

$$\tilde{f}(k) = \frac{1}{L}\int dx f(x) e^{-ikx}, \tag{4}$$

$$f(x) = L\int \frac{dk}{2\pi}\tilde{f}(k)e^{ikx}. \tag{5}$$

The discrete versions of these are defined such that $f_n = f(x_n)$ and $\tilde{f}_l = f(k_l)$:

$$\tilde{f}_l = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-i \frac{2\pi}{N} nl}, \tag{6}$$

$$f_n = \sum_{l=0}^{N-1} \tilde{f}_l e^{i \frac{2\pi}{N} nl}, \tag{7}$$

where $k_l = \frac{2\pi}{L} l$ and $x_n = \frac{L}{N} n$. Note that the upper half (the exact number depends on if N is even or odd) and higher actually mirror negative freqencies. So the largest physical frequency is $f = \frac{N}{2}\frac{1}{L}$ or $k = \frac{N}{2}\frac{2\pi}{L}$, where $\frac{N}{2}$ is understood as integer division.

The (continous) power spectrum, $P(k)$, is defined as:

$$P(k) = L\langle|\tilde{f}(k)|^2\rangle. \tag{8}$$

While the discrete one is given by:

$$P_{k_l} = \langle|\tilde{f}_l|^2\rangle, \tag{9}$$

so that $P_{k_l} = \frac{P(k_l)}{L}$.

3 Window function and power spectrum

We define the pixel (voxel) window function, $W(x)$, as follows:

$$W(x) = \begin{cases} 0 & , \quad \text{outside pixel} \\ \frac{L}{\Delta x_{\text{vox}}} & , \quad \text{inside pixel} \end{cases} \tag{10}$$

where Δx_{vox} is the length of the voxel.

This ensures the normalization condition is fulfilled:

$$\frac{1}{L} \int dx\, W(x) = 1. \tag{11}$$

The pixel variance of a Gaussian random field with power spectrum $P(k)$ is then given by:

$$\sigma_G^2 = \int \frac{dk}{2\pi} P(k)|\tilde{W}(k)|^2. \tag{12}$$

The discrete versions of these equations are

$$\sum_n W_n = 1, \tag{13}$$

and

$$\sigma_G^2 = \sum_l P_{k_l}|\tilde{W}_{k_l}|^2. \tag{14}$$

4 Noise levels

We use the radiometer equation to estimate the voxel noise level:

$$\sigma_T = \frac{T_{\rm sys}}{\sqrt{\tau \Delta\nu}} = \frac{T_{\rm sys}}{\sqrt{\frac{\tau_{\rm tot} e_{\rm obs} N_{\rm feeds}}{N_{\rm pixels}} \Delta\nu}}, \tag{15}$$

where σ_T^2 is the variance of the voxel noise, $T_{\rm sys}$ is the system temperature, τ is the observation time per pixel, $\tau_{\rm tot}$ is the total time, $e_{\rm obs}$ is the observation efficiency, $N_{\rm feeds}$ is the number of feeds, $N_{\rm pixels}$ is the number of pixels, and $\Delta\nu$ is the frequency resolution.

For the COMAP experiment we expect a system temperature of about $T_{\rm sys} \approx$ 35 K. We are also considering using 1024 frequency channels, which would give a frequency resolution of about $\Delta\nu = \frac{8\,\mathrm{GHz}}{1024} \approx 7.8\,\mathrm{MHz}$.

For a grid of 25×25 pixels, for two years of observation time, with 19 feeds and assuming a 35% observation efficiency we get:

$$\sigma_T = \frac{35\,\mathrm{K}}{\sqrt{2 \times 365 \times 24 \times 3600 \times 0.35 \times 19/(25 \times 25) \times 7.8 \times 10^6}} \approx 15.3\,\mu K, \tag{16}$$