# Hamilton’s Principle of Stationarity

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Classical Dynamics - Part IB Physics B

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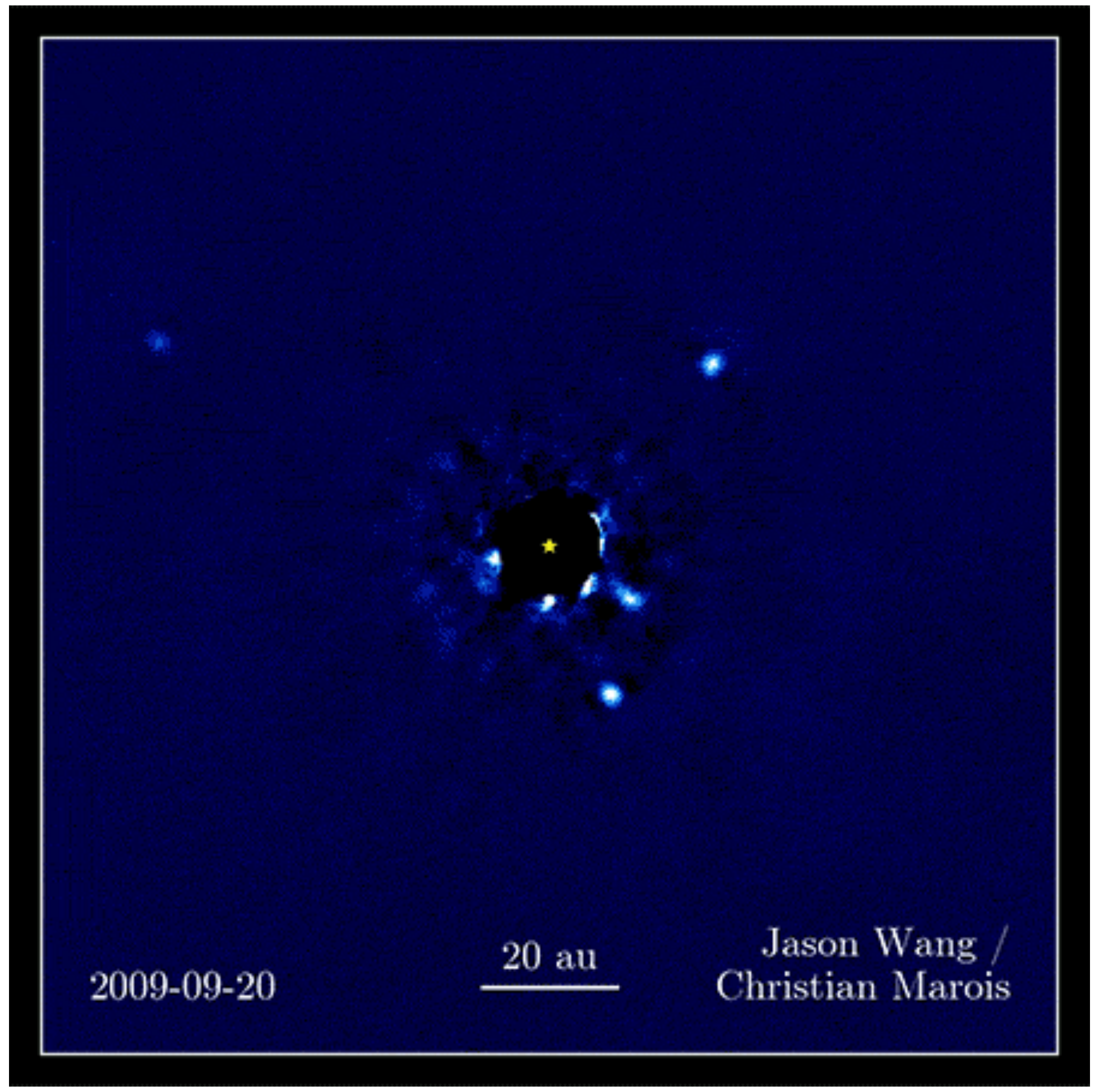
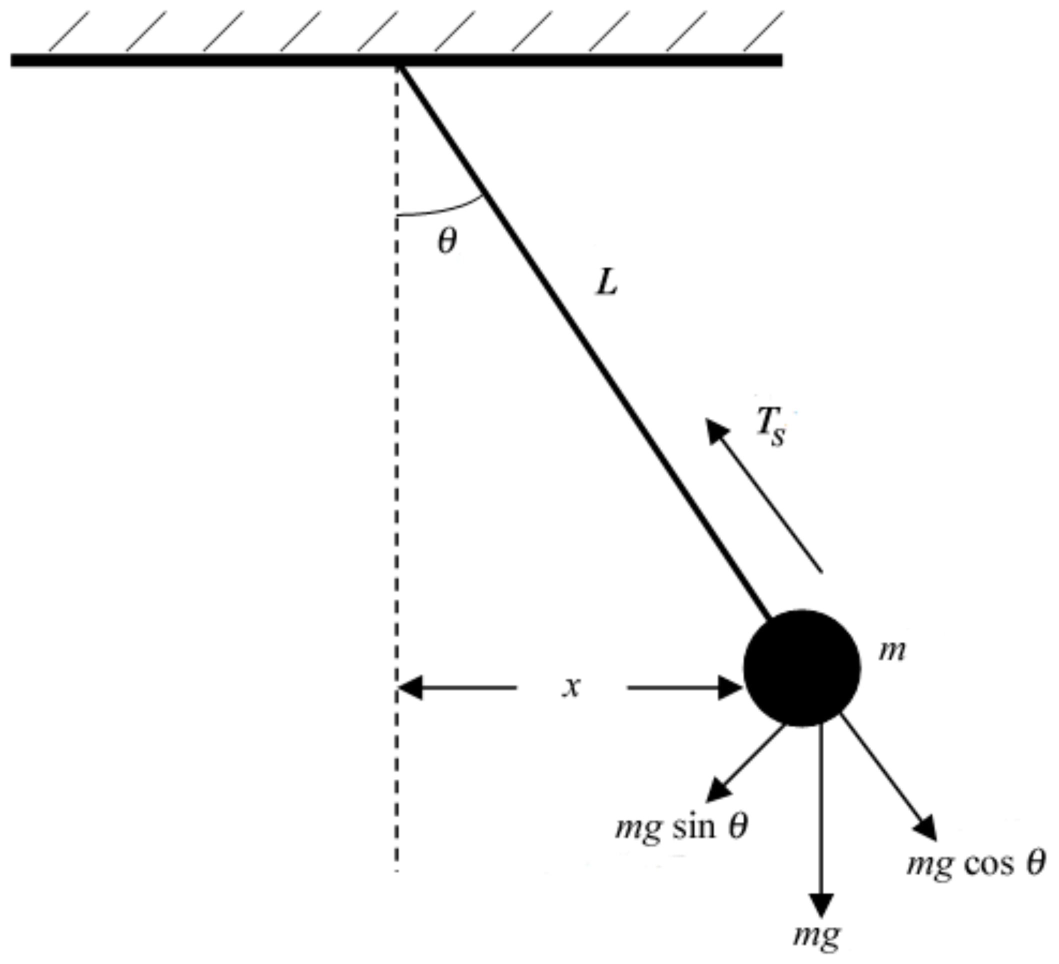
**Resources: bit.ly/4m4StgI**

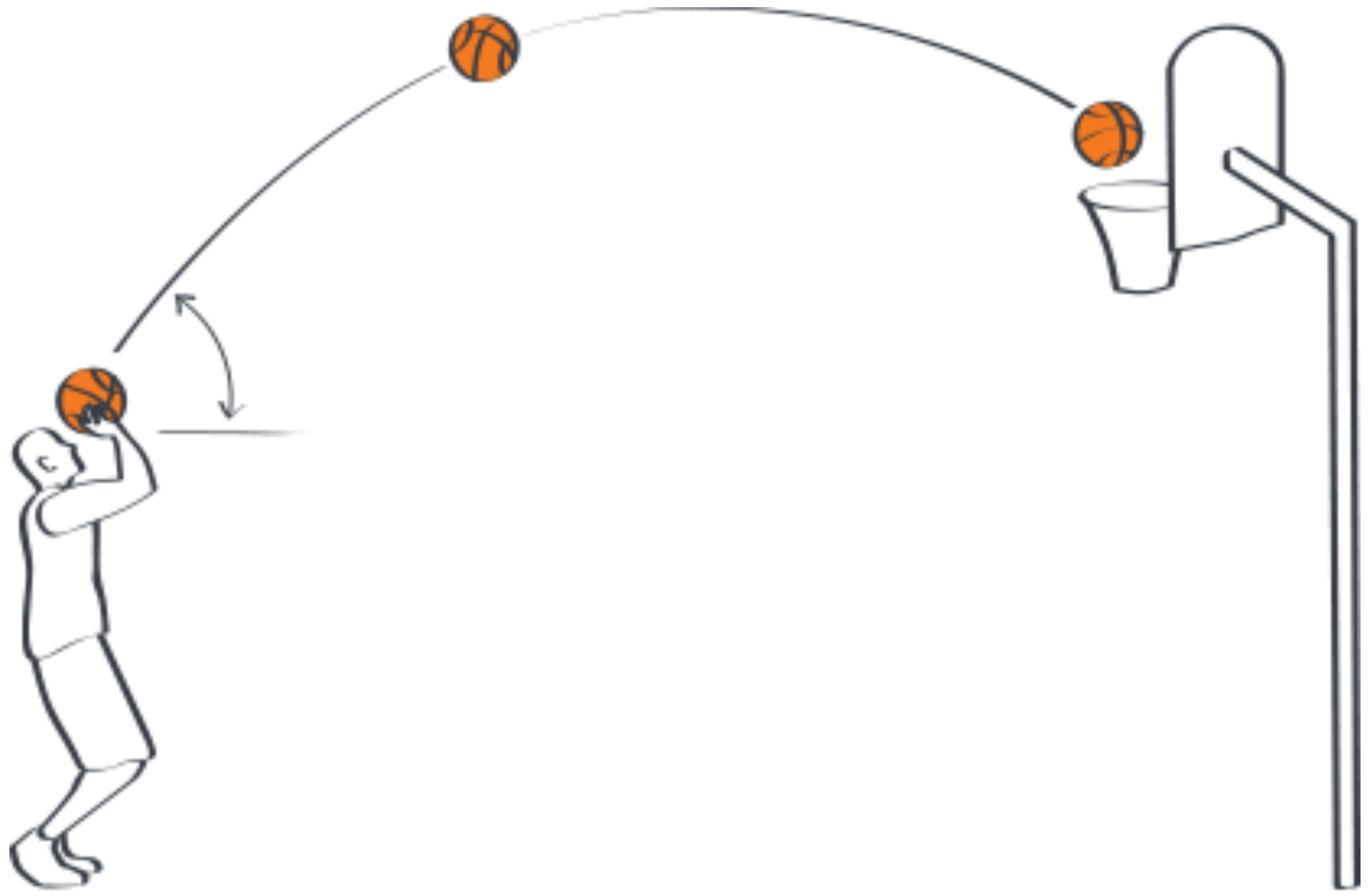
# Understanding motion in complex systems




**Exoplanet orbits**
**Four Hot Jupiter planets orbiting HR 8799**




**Motion of a pendulum**



**Parabolic motion of projectiles**

# Understanding motion in complex systems

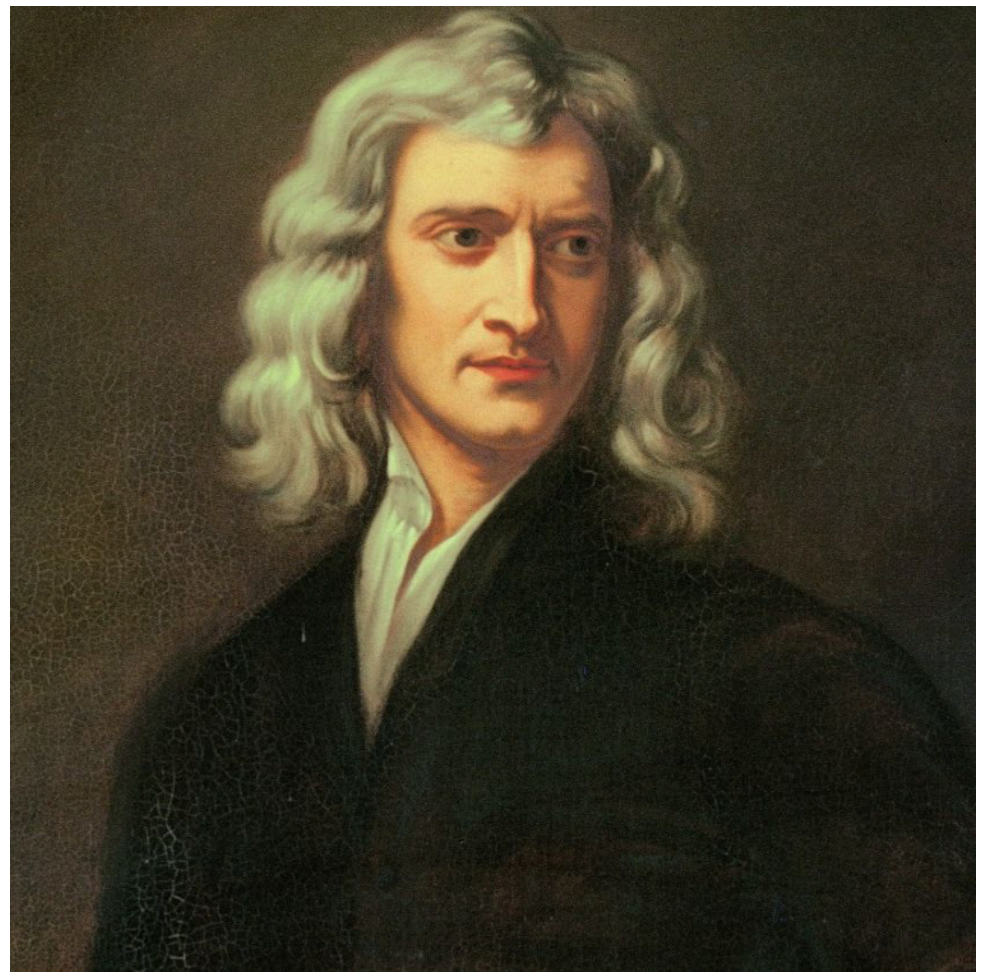
Kepler took observations from Tycho Brahe and derived three laws of orbital mechanics (1609 - 1619)

Newton uses Kepler's laws to illustrate his inverse square law for gravitation (1687)

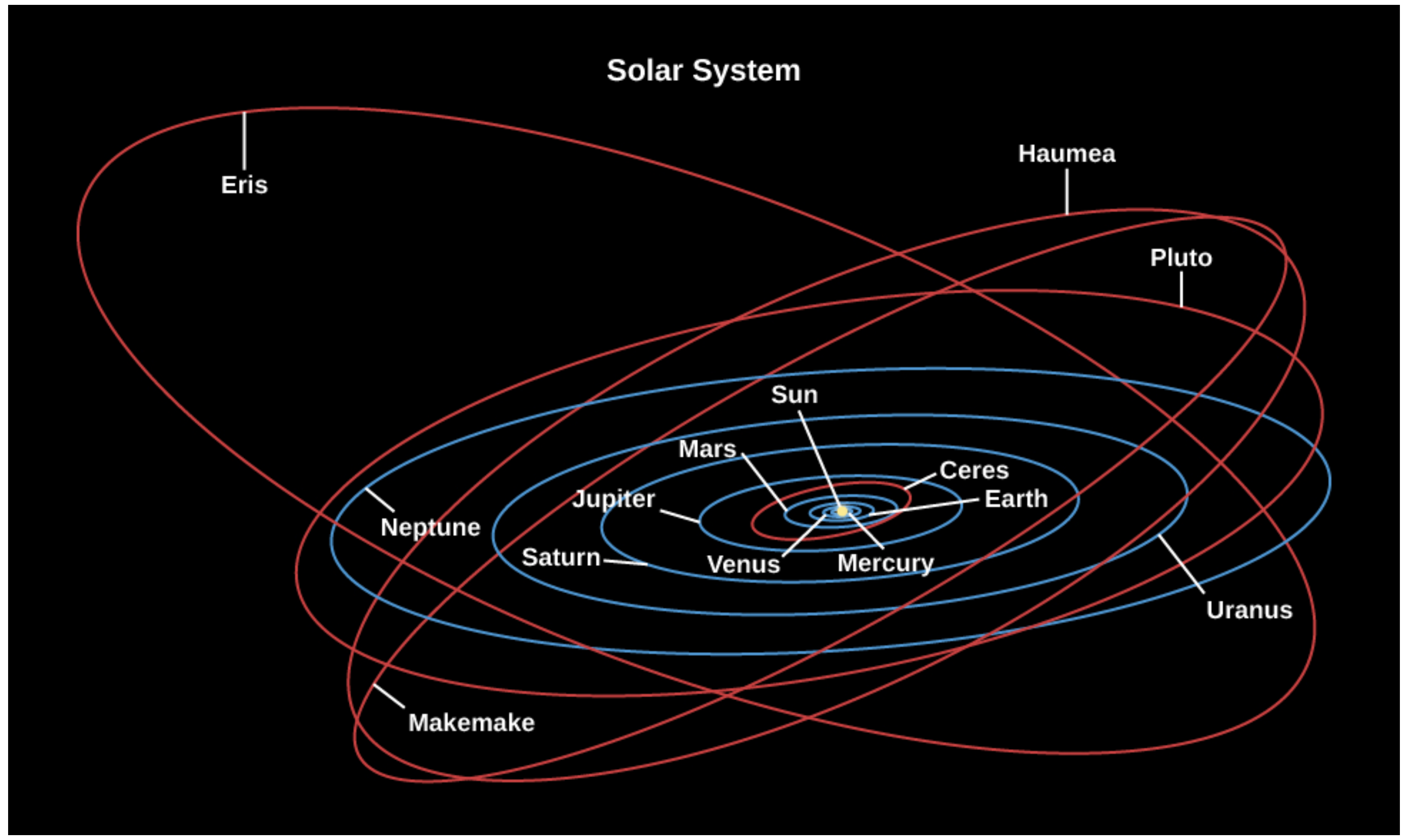
Hamilton's established a deeper fundamental description of mechanical systems (1834) that explains Newton's law of gravitation and a whole lot more!

Tycho Brahe

Johannes Kepler



Isaac Newton

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# Lagrangian

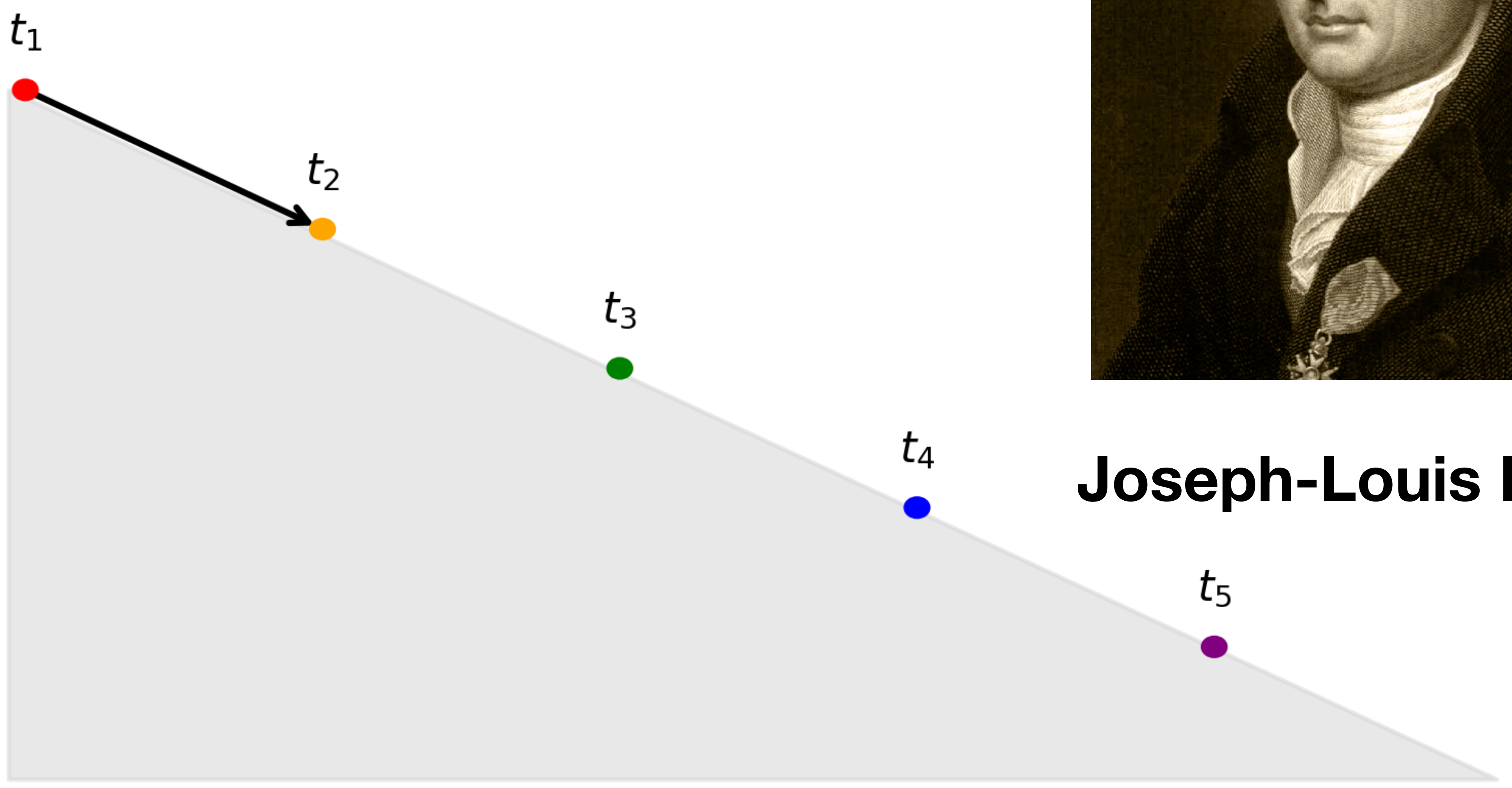
The Lagrangian is given by

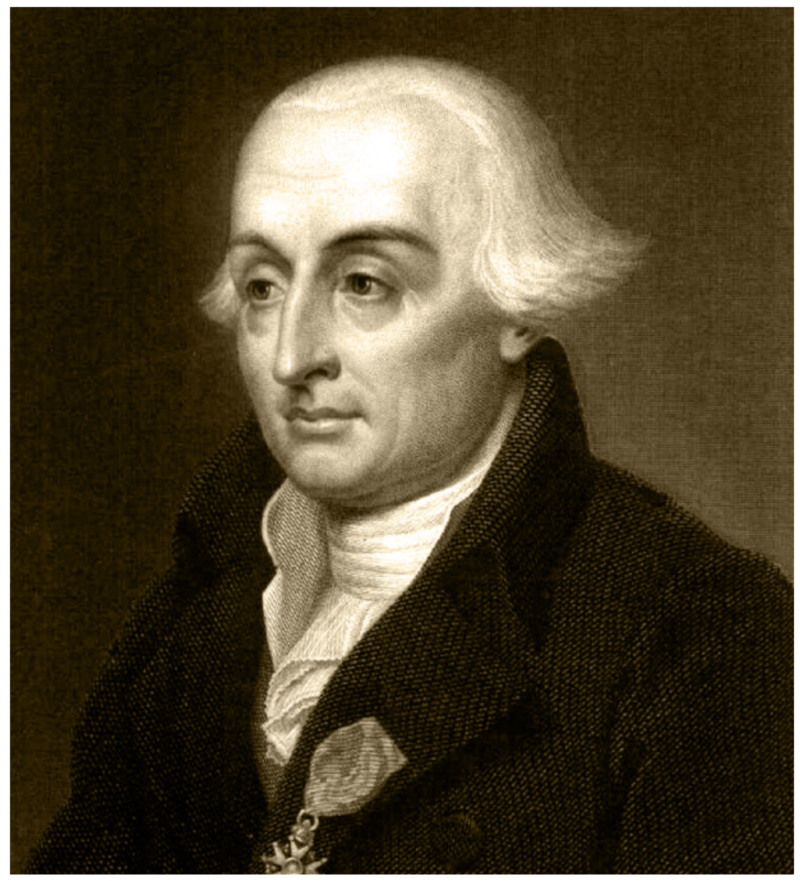
$$L(x, \dot{x}, t) = T(\dot{x}, t) - V(x, t)$$

where $T$ is kinetic energy and $V$ is a potential

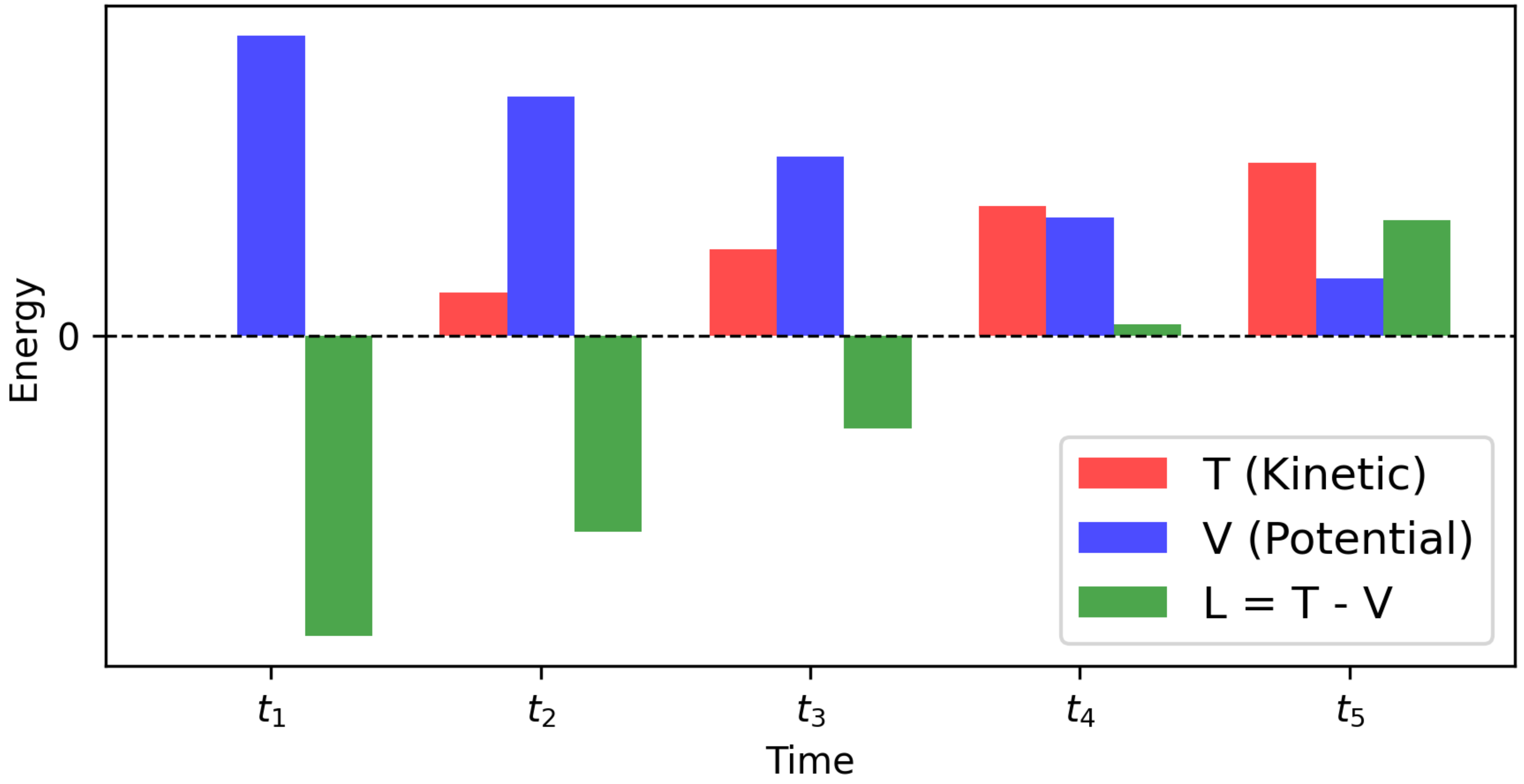
Encodes the dynamics of a system

Move from thinking about forces to motion in an energy landscape



**Joseph-Louis Lagrange**



# Action

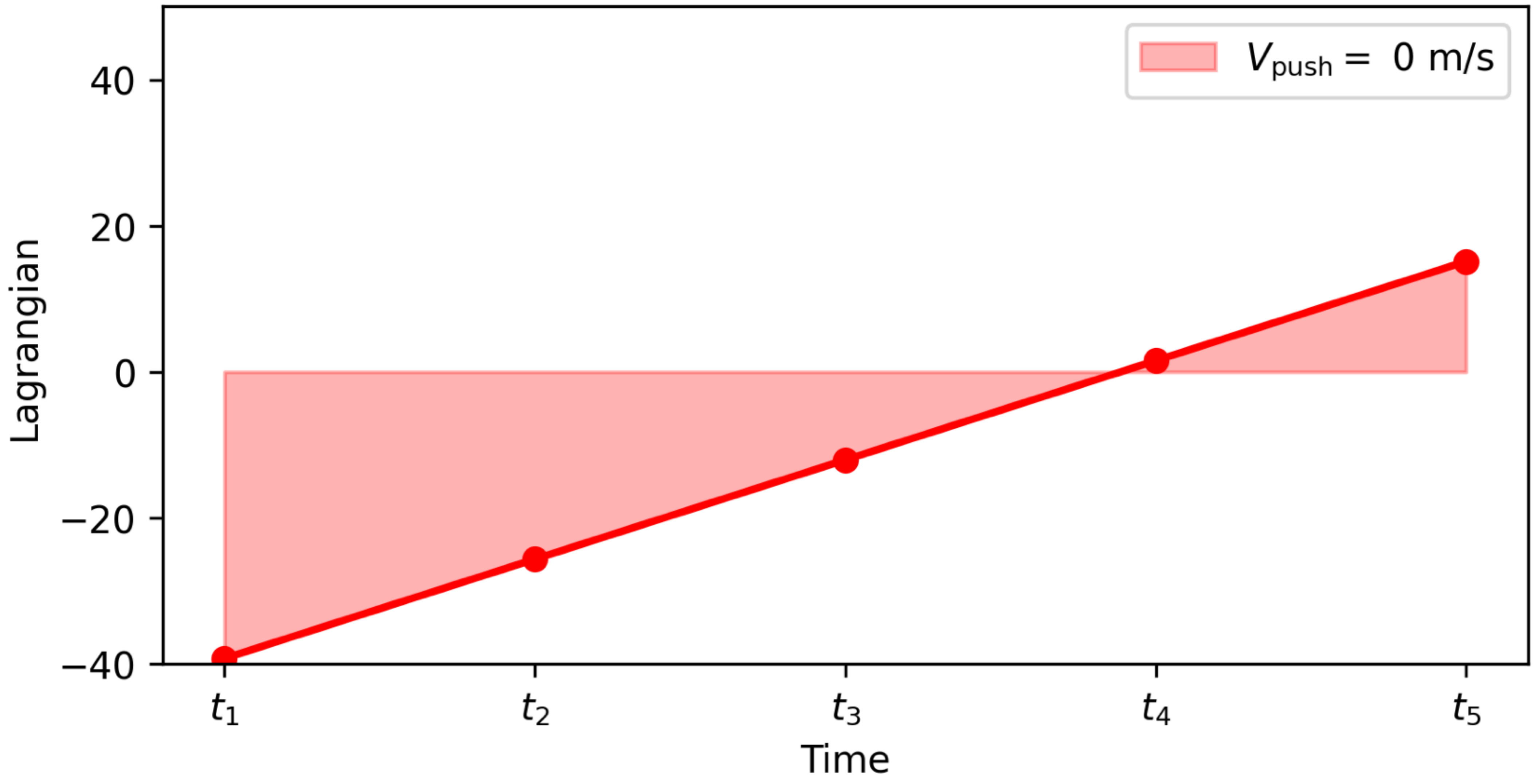
The action for a system is the time integrated Lagrangian

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt$$

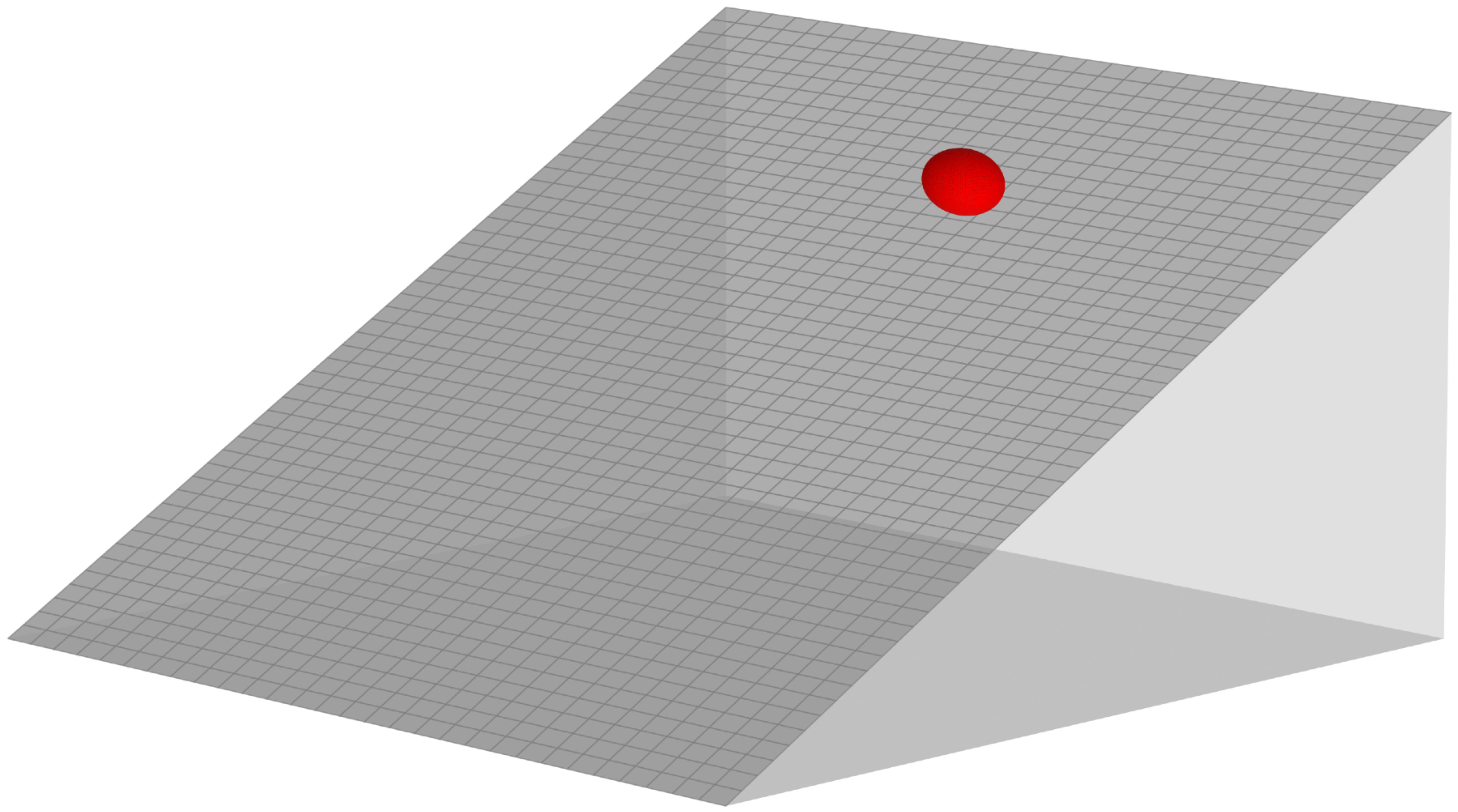
Quantifies the balance between kinetic and potential energy in a system over time

The fundamental object of Hamilton's Principle of Stationarity

William Rowan Hamilton

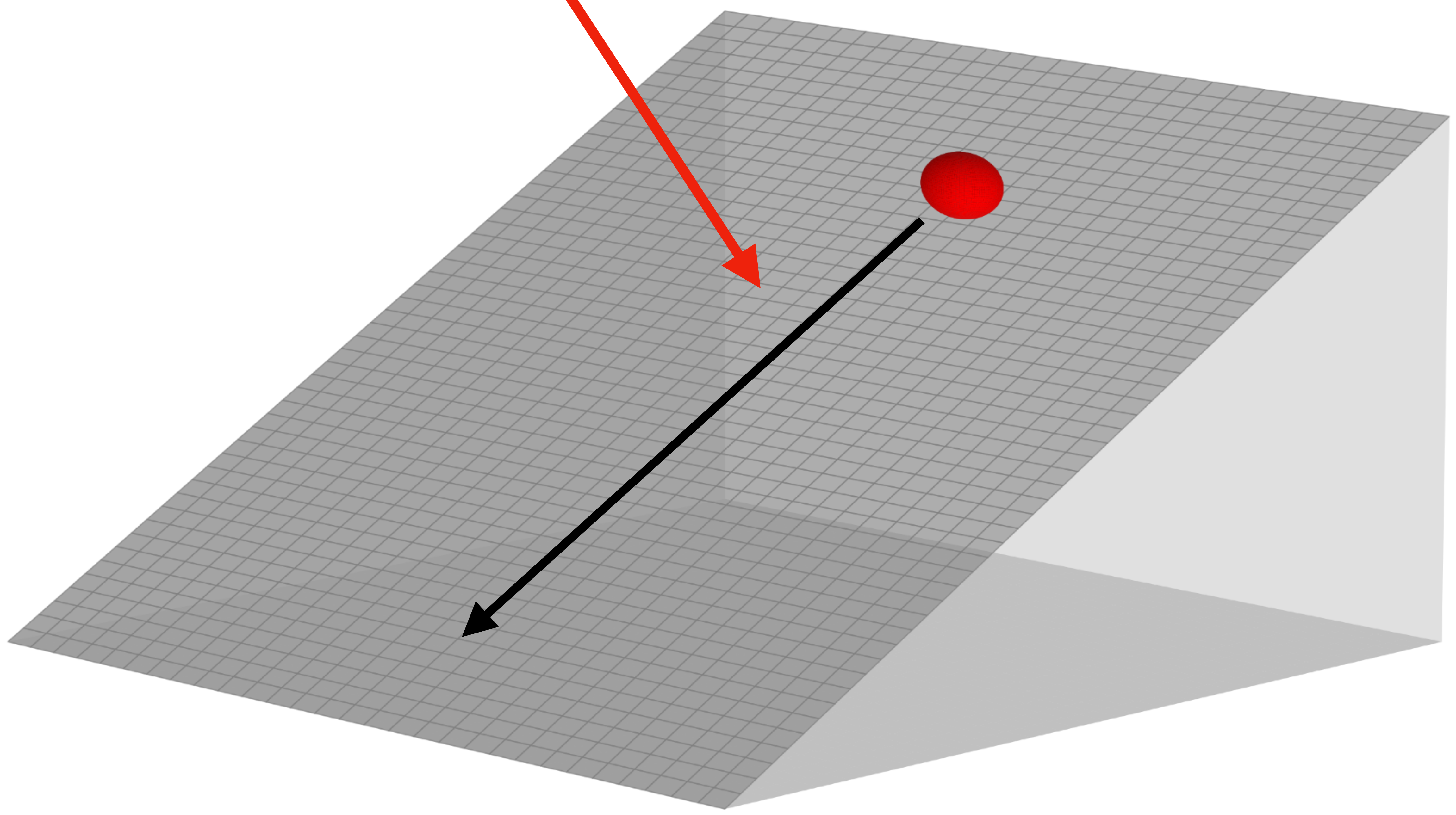



# Rolling Ball



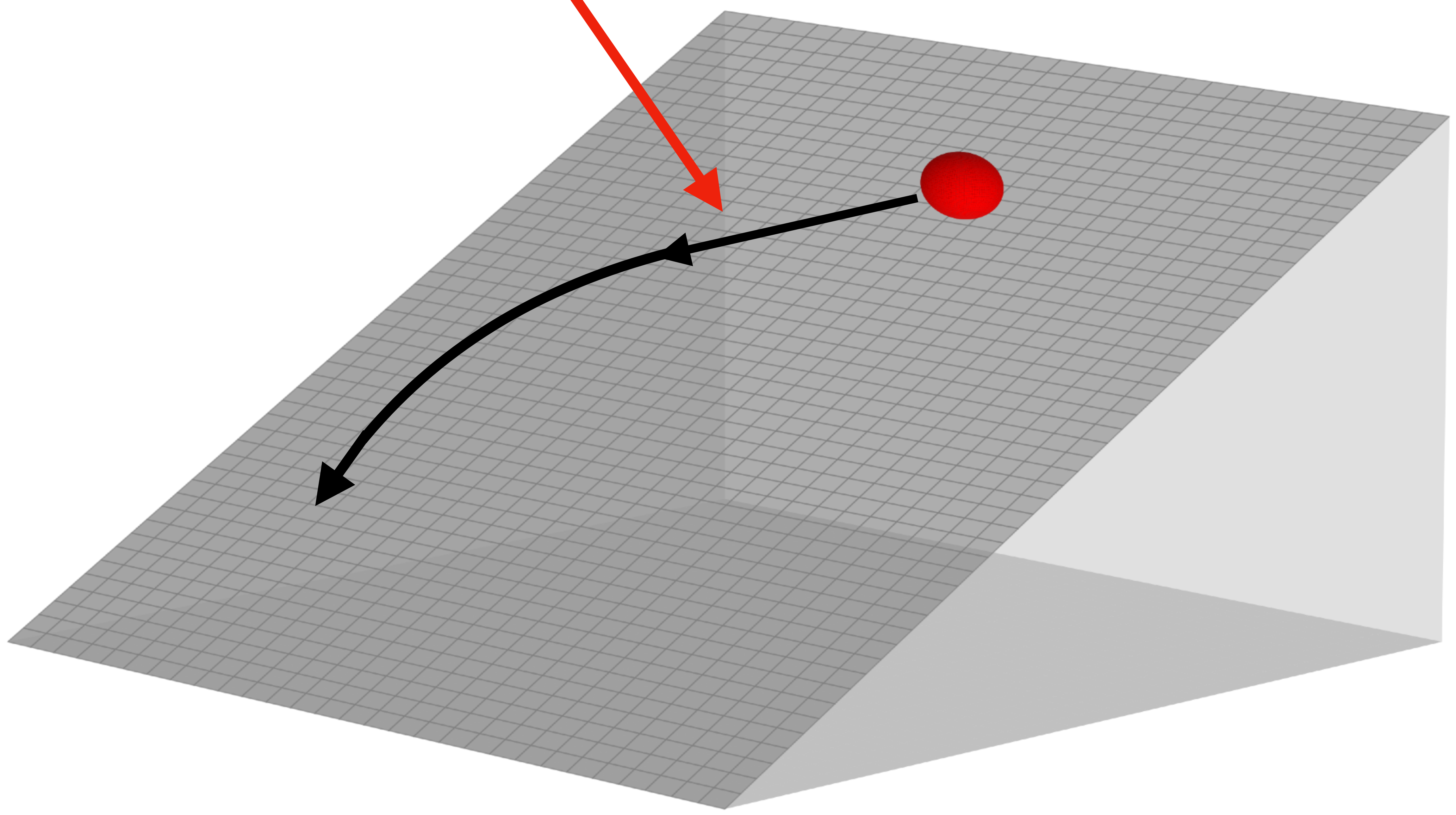
# Rolling Ball

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt$$



# Rolling Ball

$$S'[x'(t)] = \int_{t_0}^{t_1} L'(x', \dot{x}', t)dt$$



# Hamilton's Principle of Stationarity

**"The action of a path followed by a particle is insensitive to first order perturbations in the path."**

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt$$

# Hamilton's Principle of Stationarity

$$x(t) \to x(t) + \epsilon\eta(t) \qquad \eta(t_0) = \eta(t_1) = 0$$

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$$S[x(t) + \epsilon\eta(t)] \approx \ldots$$

## Hamilton's Principle of Stationarity

$$x(t) \to x(t) + \epsilon\eta(t) \qquad \eta(t_0) = \eta(t_1) = 0$$

$$S[x(t) + \epsilon\eta(t)] \approx S[x(t)] + \left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} \epsilon + \ldots$$

# Hamilton's Principle of Stationarity

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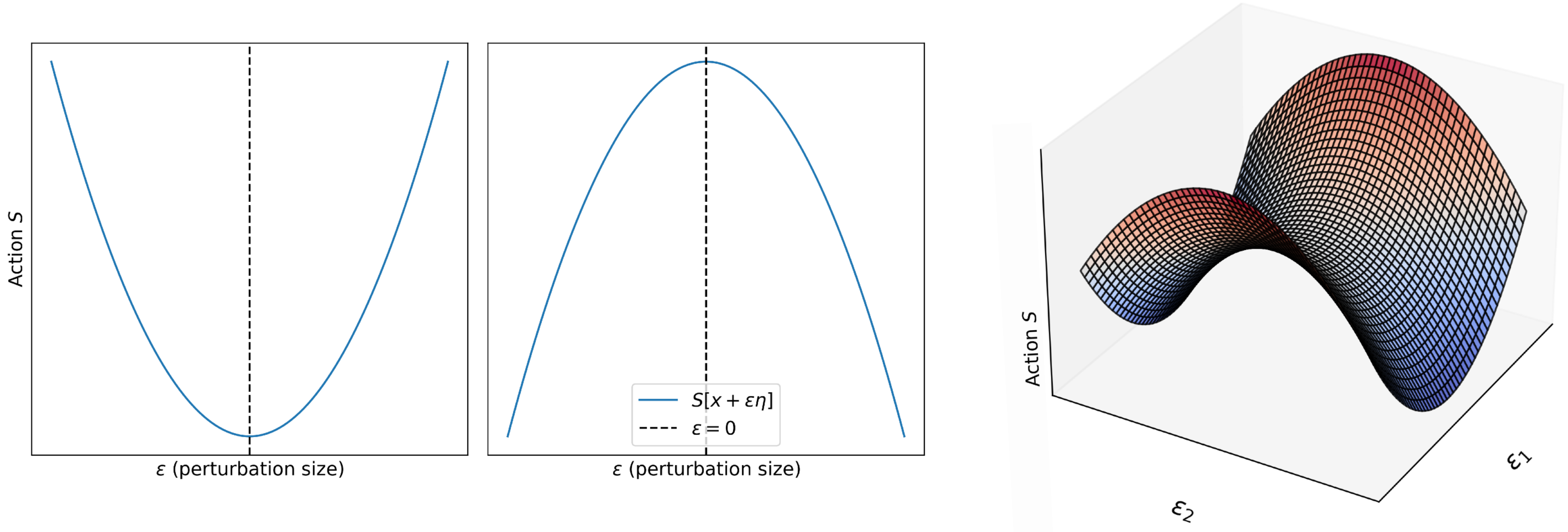
$$S[x(t) + \epsilon\eta(t)] \approx S[x(t)] + \left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} \epsilon + \ldots$$

Hamilton's Principle requires $\left.\dfrac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = 0$

for small $\epsilon$ where the Taylor expansion is valid

# Hamilton’s Principle of Stationarity

Hamilton’s Principle requires $\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = 0$



# From Hamilton to Euler-Lagrange

$$S[x(t) + \epsilon\eta(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)dt$$

# From Hamilton to Euler-Lagrange

$$S[x(t) + \epsilon\eta(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)dt$$

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \frac{\delta L}{\delta x}\frac{\delta(x + \epsilon\eta)}{\delta \epsilon} + \frac{\delta L}{\delta \dot{x}}\frac{\delta(\dot{x} + \epsilon\dot{\eta})}{\delta \epsilon}dt = 0$$

# From Hamilton to Euler-Lagrange

$$S[x(t)+\epsilon\eta(t)] = \int_{t_0}^{t_1} L(x+\epsilon\eta, \dot{x}+\epsilon\dot{\eta}, t)dt$$

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \frac{\delta L}{\delta x}\frac{\delta(x+\epsilon\eta)}{\delta\epsilon} + \frac{\delta L}{\delta \dot{x}}\frac{\delta(\dot{x}+\epsilon\dot{\eta})}{\delta\epsilon} dt = 0$$

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \left(\frac{\delta L}{\delta x} - \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right)\right)\ \eta\ dt = 0.$$

# From Hamilton to Euler-Lagrange

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \left( \frac{\delta L}{\delta x} - \frac{\delta}{\delta t}\left( \frac{\delta L}{\delta \dot{x}} \right) \right) \eta \ dt = 0.$$

**From this we can derive the Euler-Lagrange equation**

$$\boxed{\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0}$$

# Euler-Lagrange

Hamilton's Principle
Classical Dynamics

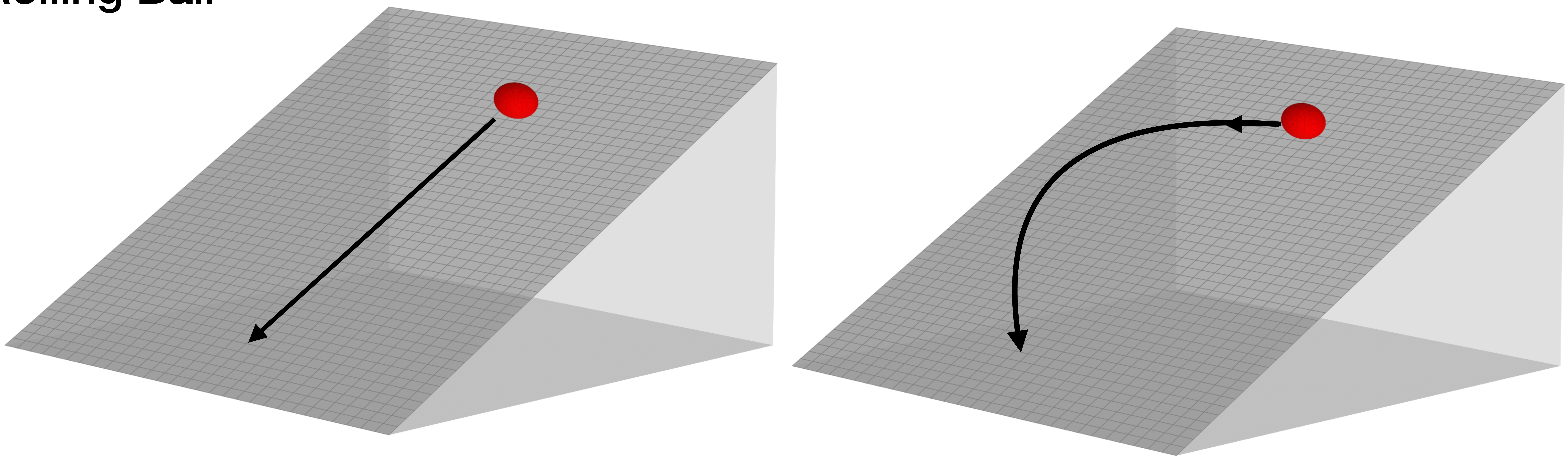
Dr Harry Bevins

July 2025

## 1 Introduction

As physicists, we are deeply interested in how complex systems evolve over time. Historically, we have used observations to inform our understanding of the forces acting on objects in systems. A classic example is the study of orbital mechanics. Between 1609 and 1619 Kepler outlined his three laws of orbital motion which he inferred from observations taken by Tycho Brahe of the motion of

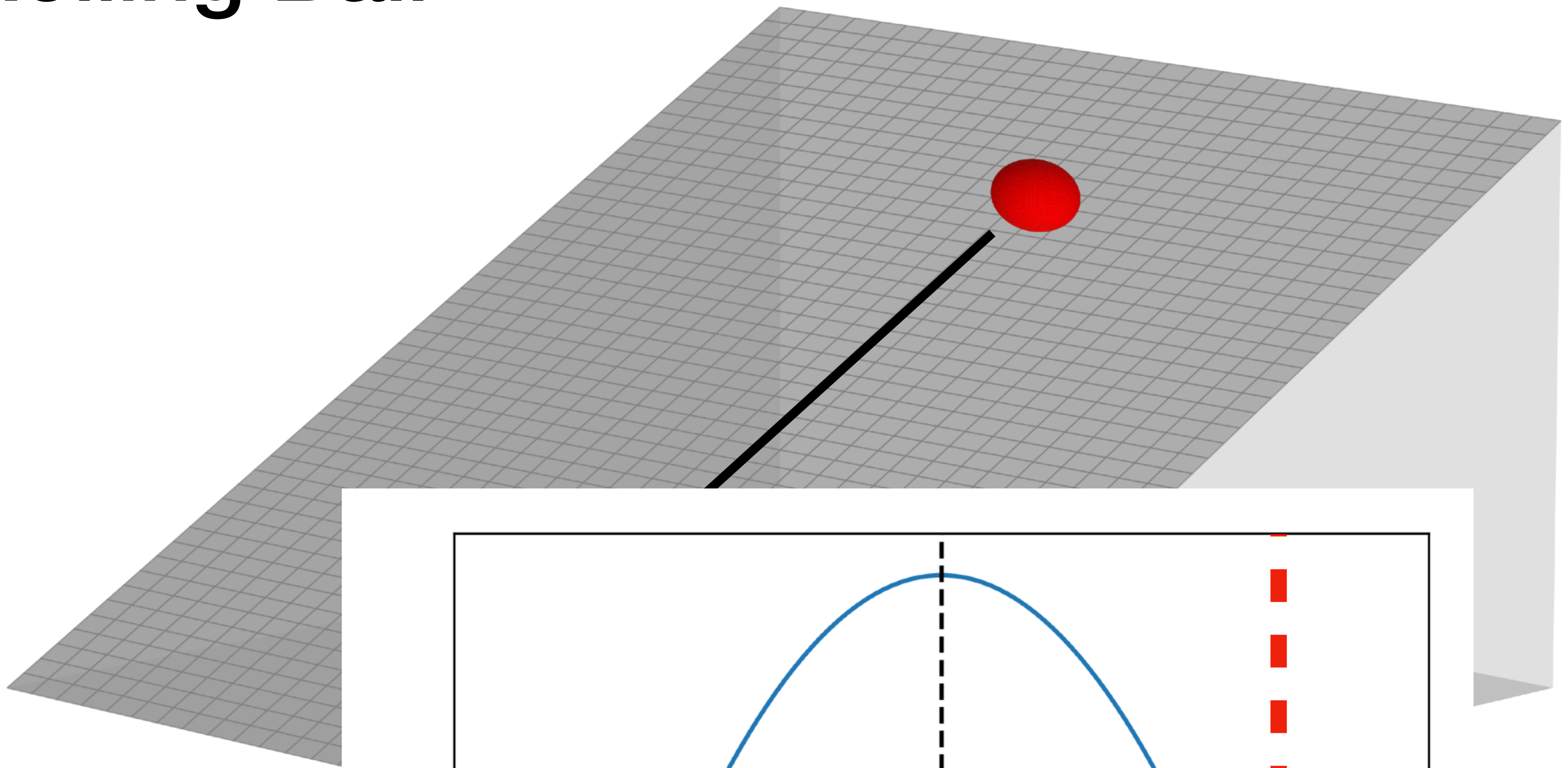
# Rolling Ball

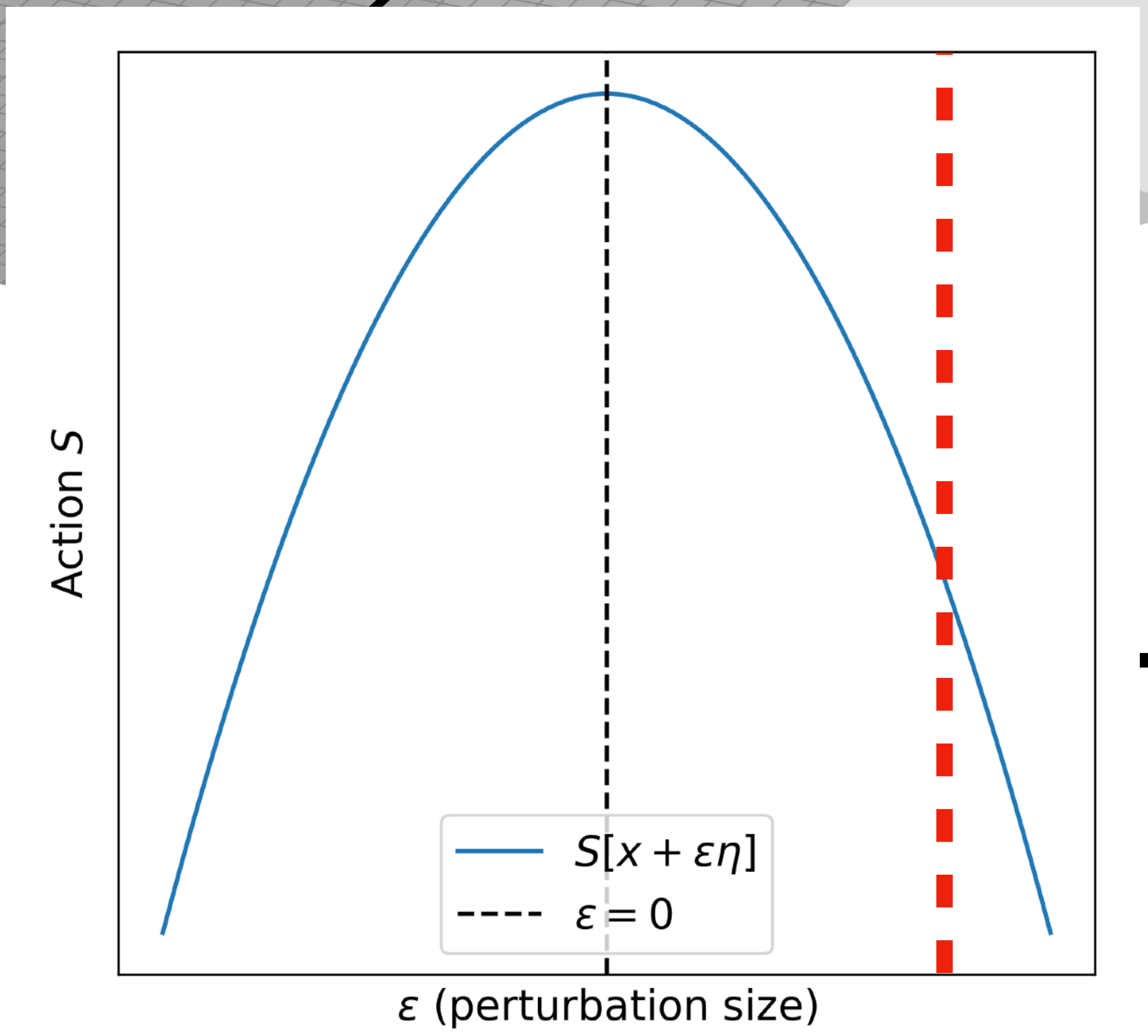


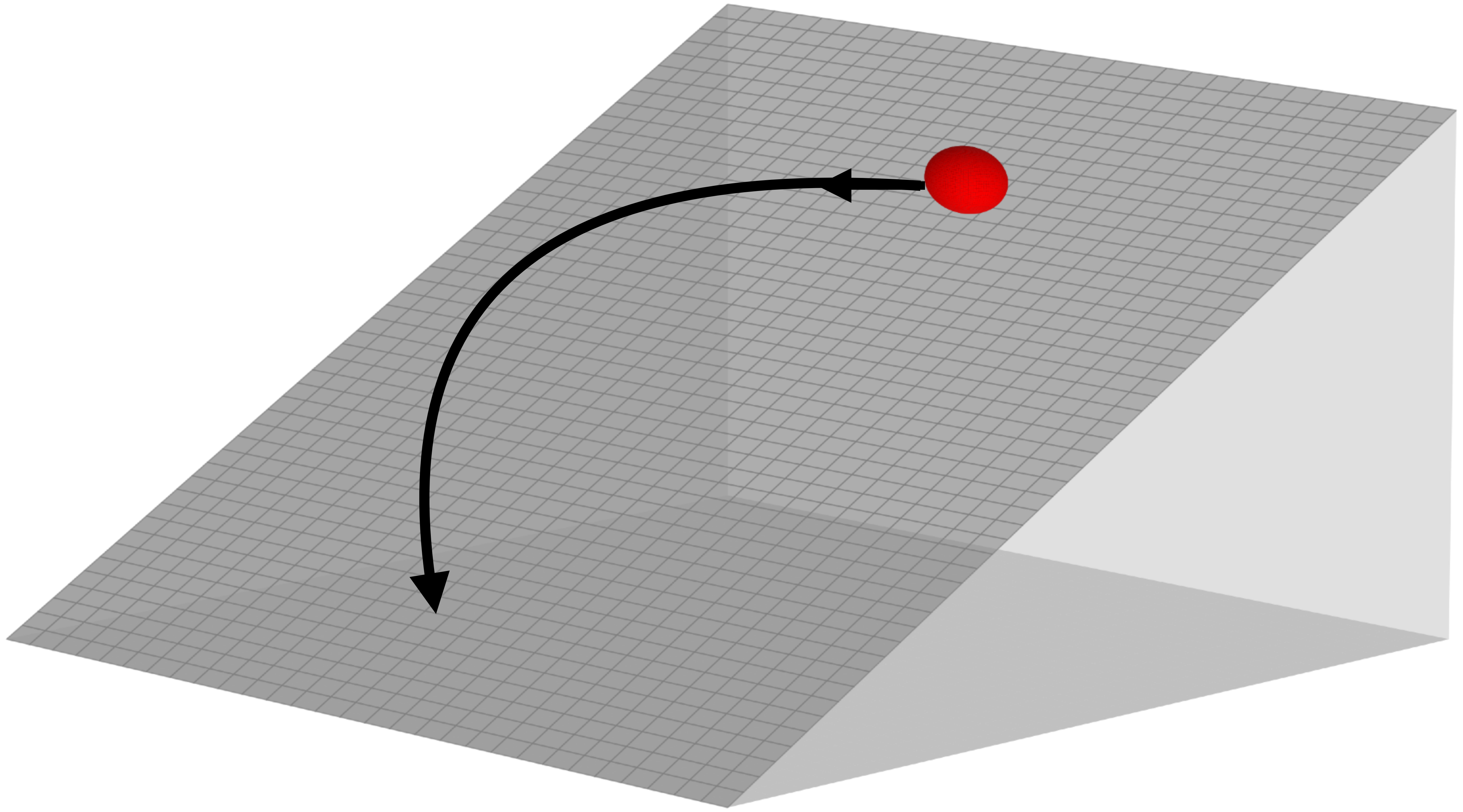
$x(t)$ $\longrightarrow$ $x(t) + \epsilon\eta(t)$ big $|\epsilon|$

big $|\Delta S|$

# Rolling Ball





$x(t) + \epsilon\eta(t)$ big $|\epsilon|$

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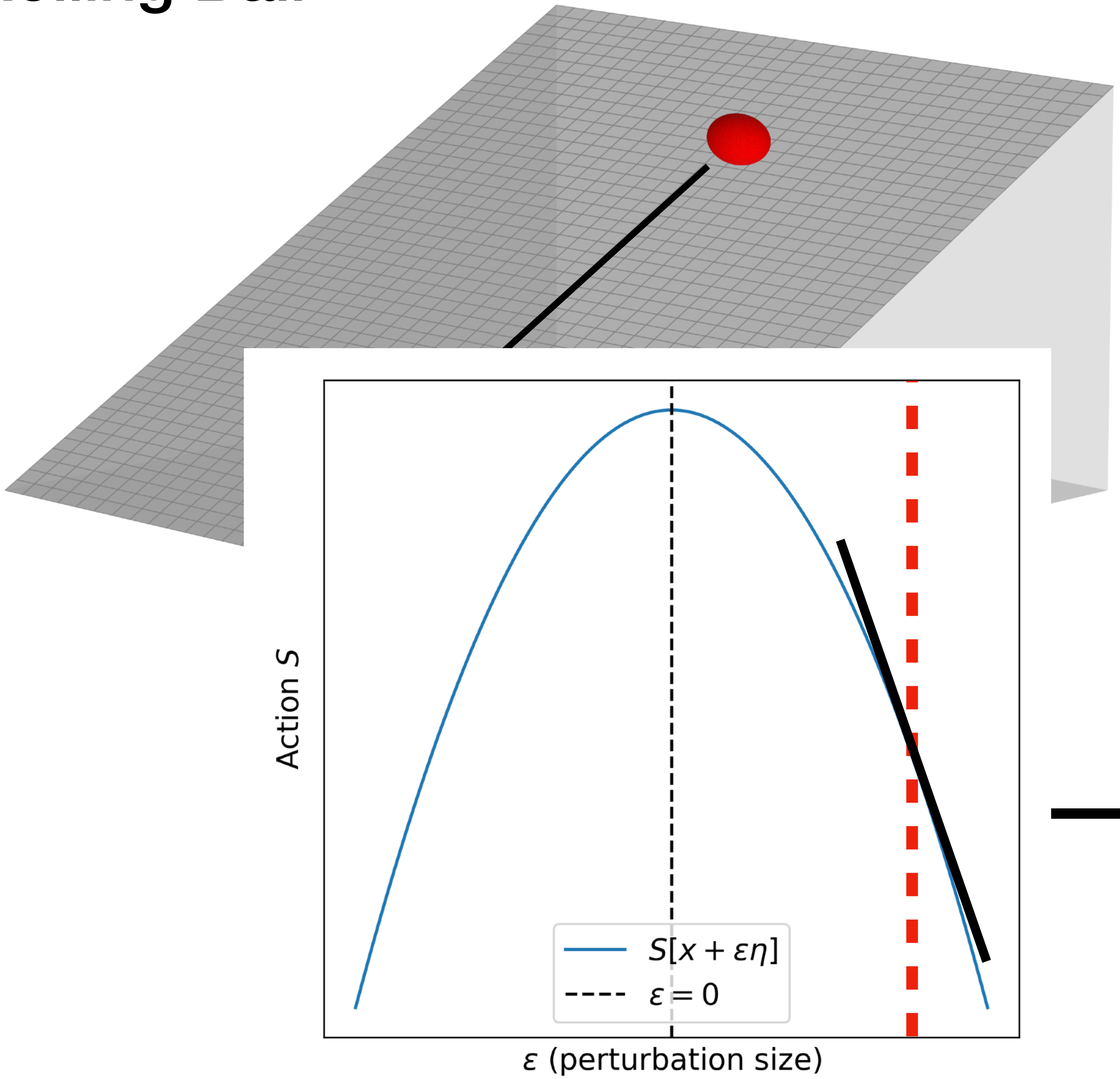
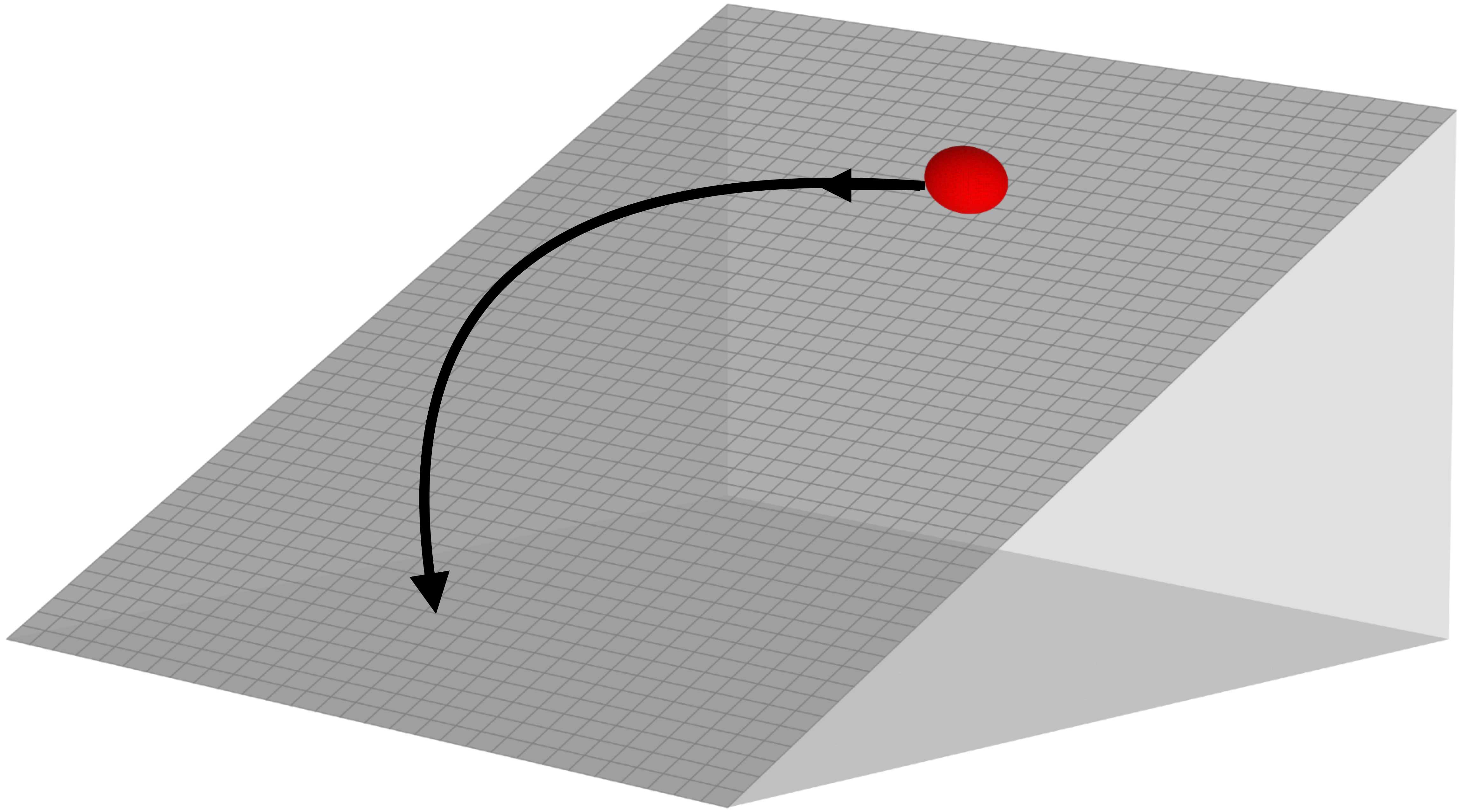


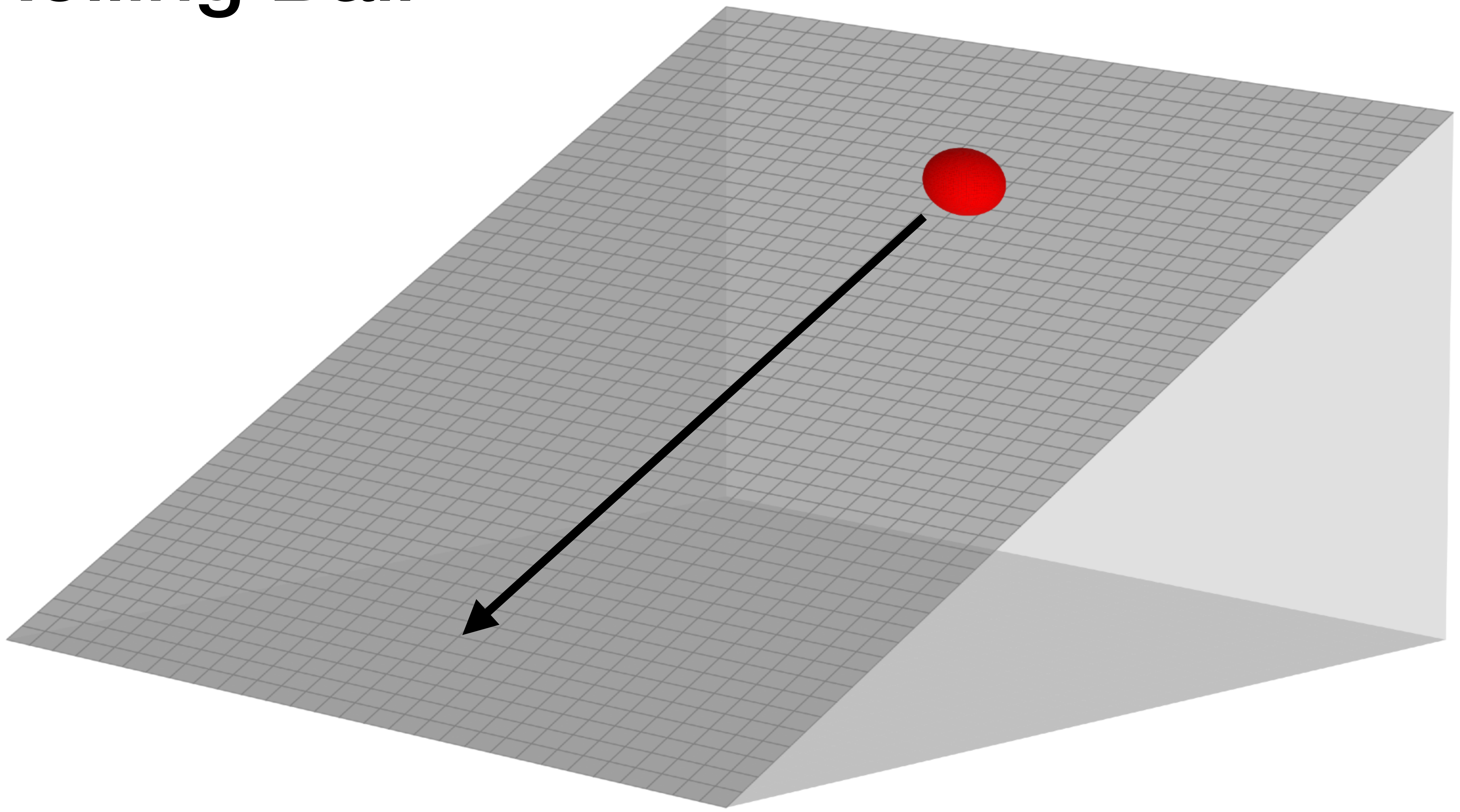



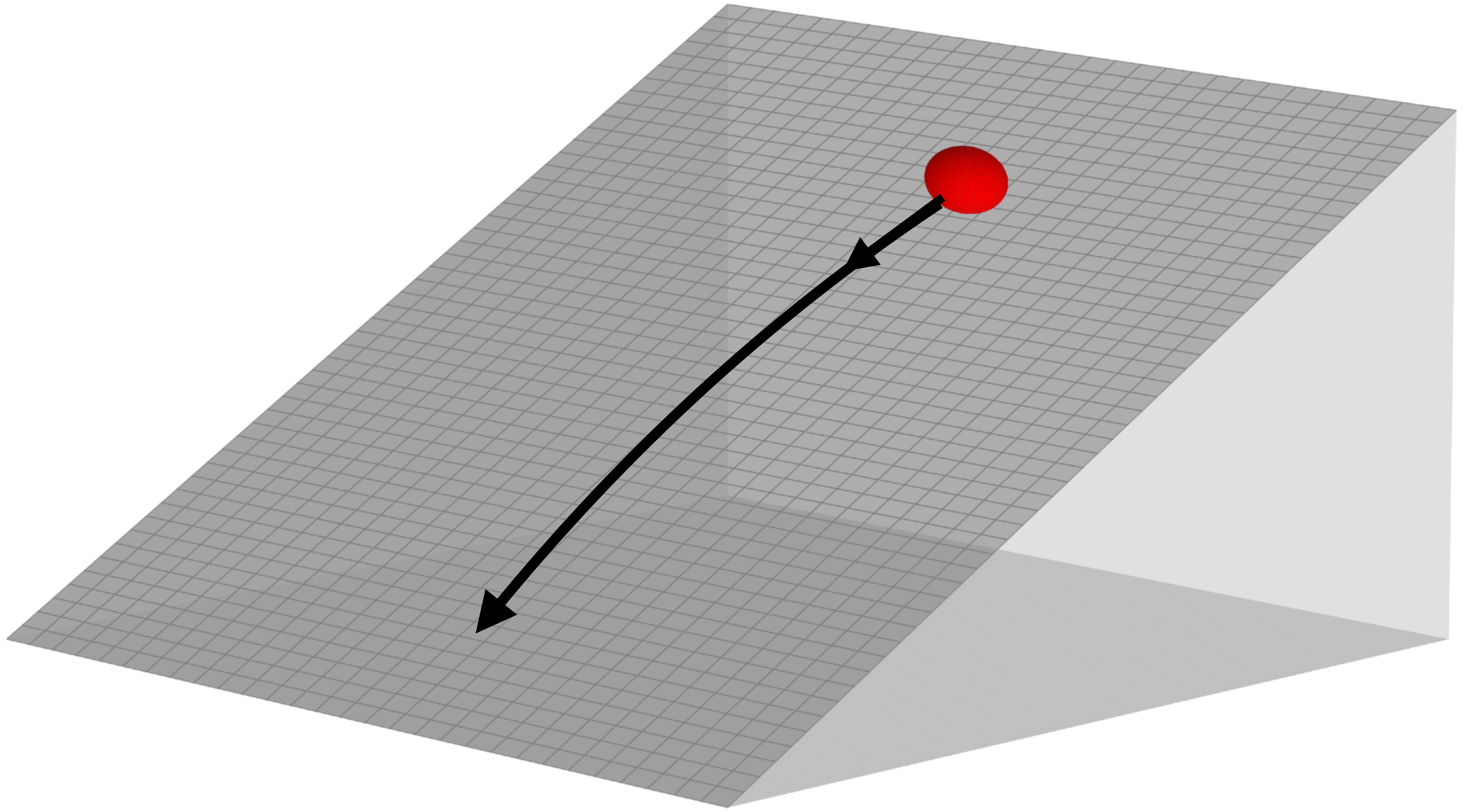
$$x(t) + \epsilon\eta(t)$$

big $|\epsilon|$

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=\epsilon'} \neq 0 \qquad \frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} \neq 0$$

## Rolling Ball

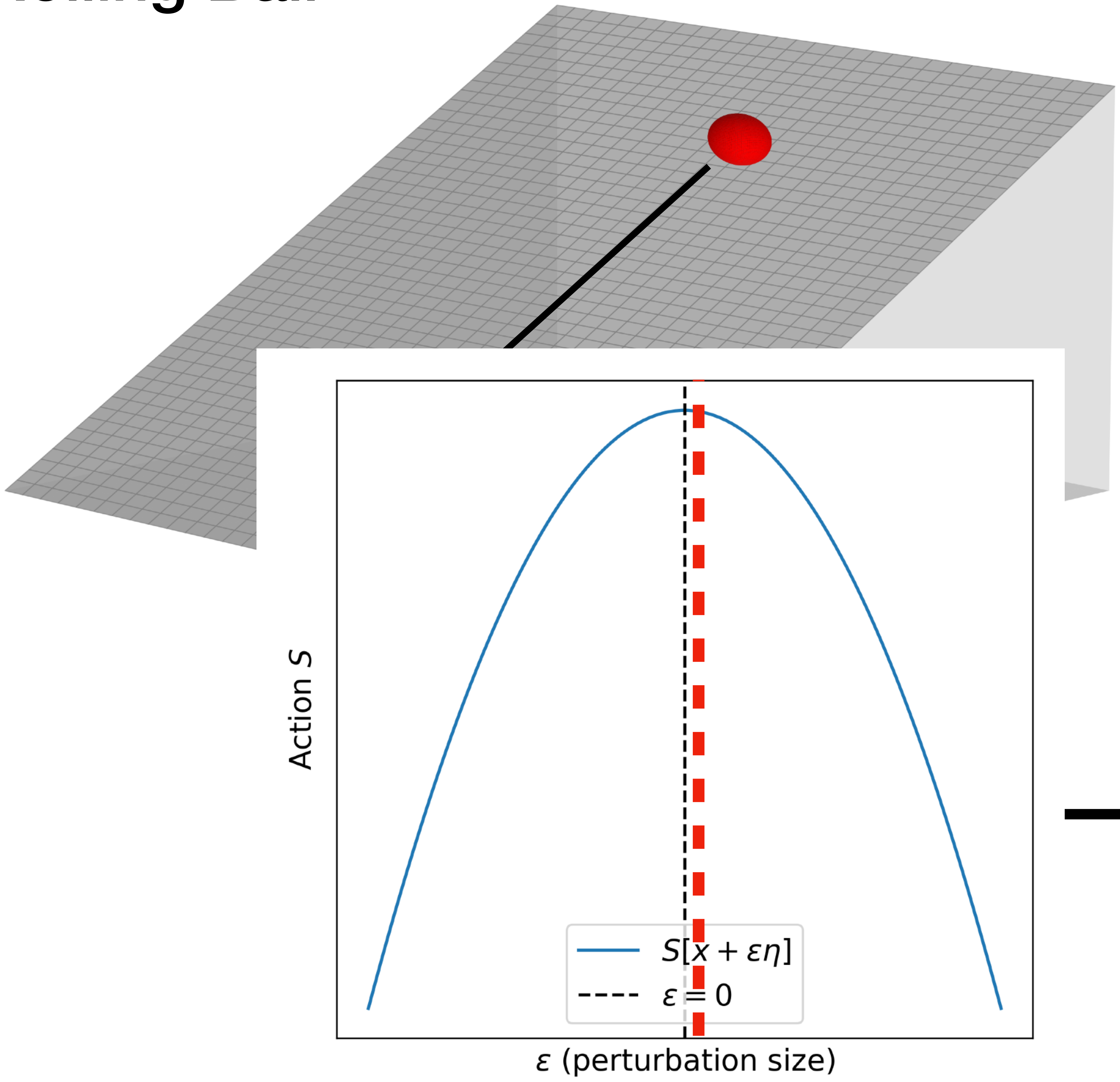


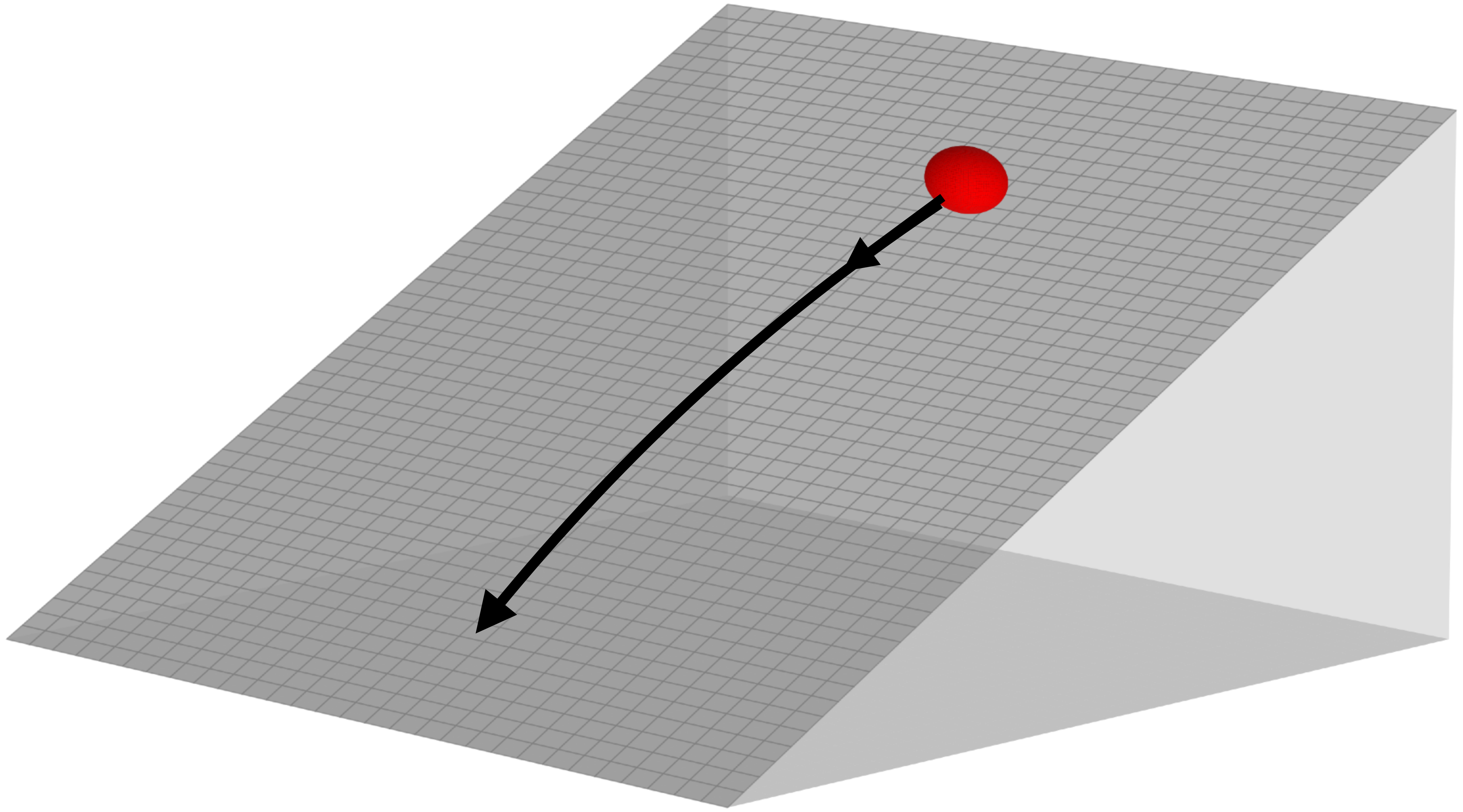


$x(t)$ $\longrightarrow$ $x(t) + \epsilon\eta(t)$ small $|\epsilon|$

small $|\Delta S|$

# Rolling Ball

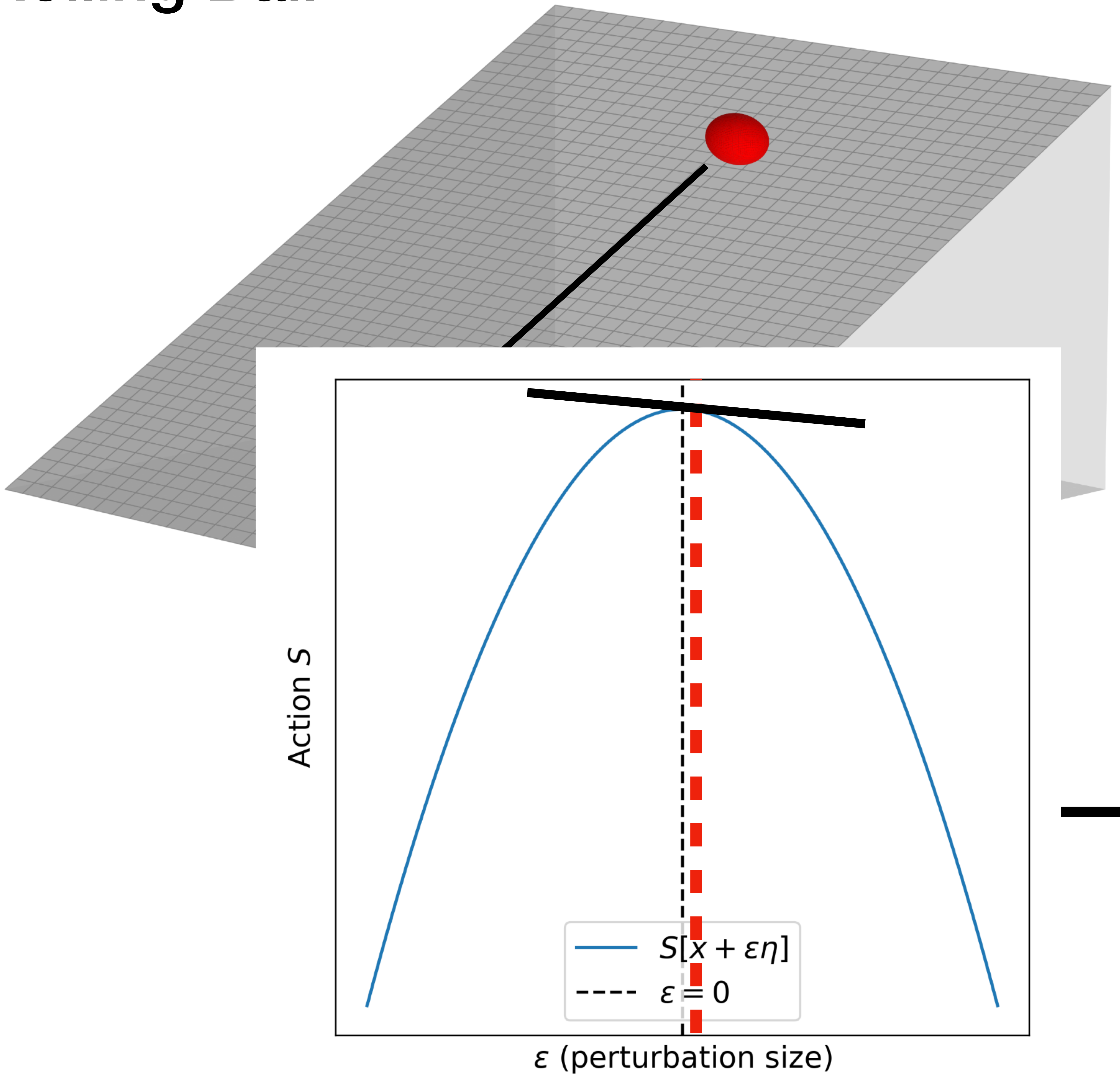


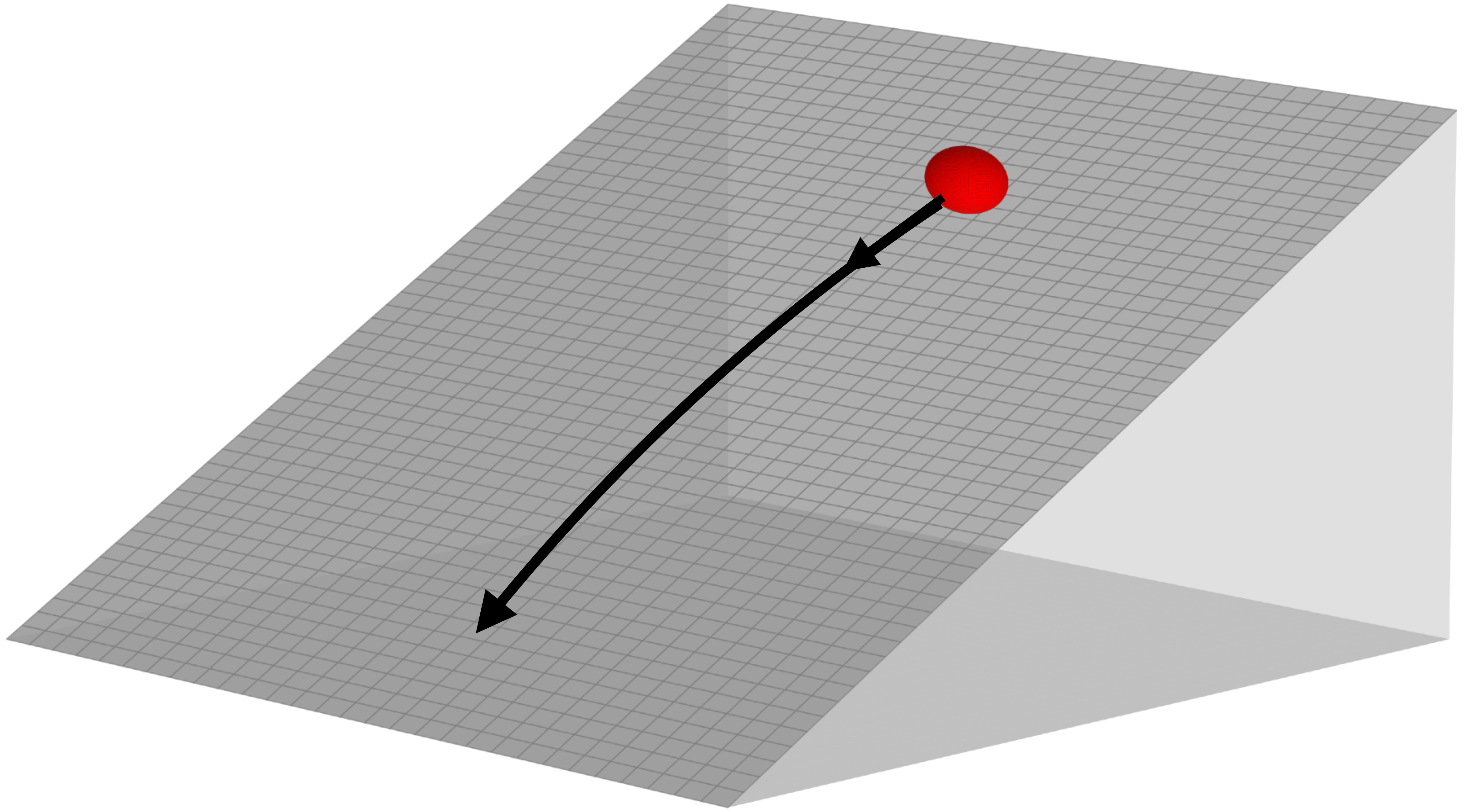



$$x(t) + \epsilon\eta(t) \qquad \text{small } |\epsilon|$$

$$\text{small } |\Delta S|$$

# Rolling Ball



$$x(t) + \epsilon\eta(t) \qquad \text{small } \epsilon$$

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=\epsilon'} \neq 0 \qquad \frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} \neq 0$$

# Euler-Lagrange

The point here is that we don't know the true path taken by our particle

We are postulating that the true path satisfies $\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = 0$

Using this to define the Euler-Lagrange equation and then deriving the true path by solving it for the dynamics

$$\boxed{\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0}$$

# Hamilton's Principle in Action

Lets think about a particle in free space i.e.

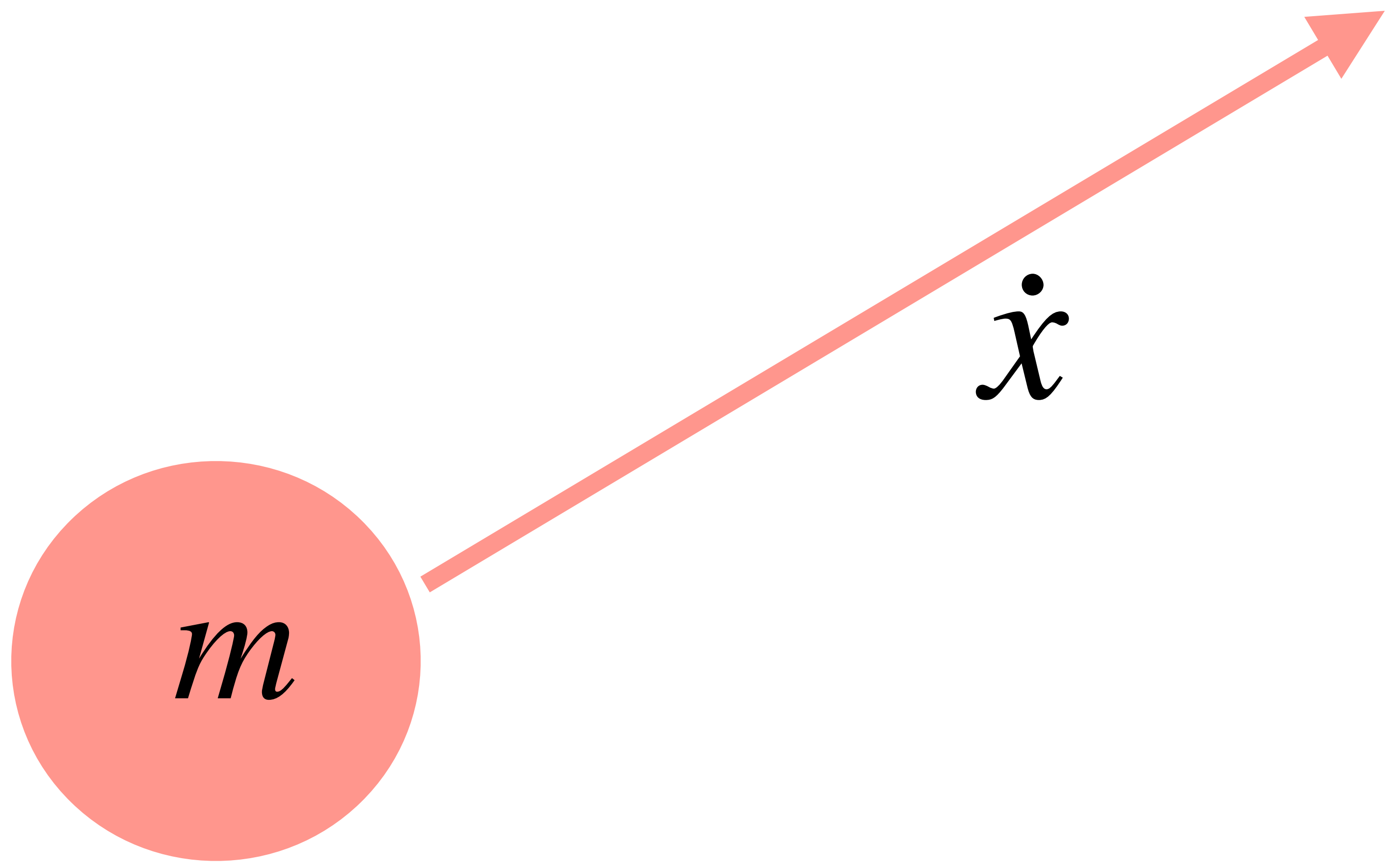
$$V(x, t) = 0$$

Our particle has some velocity $\dot{x}$ meaning it has kinetic energy

$$T = \frac{1}{2}m\dot{x}^2$$

So it's Lagrangian is given by

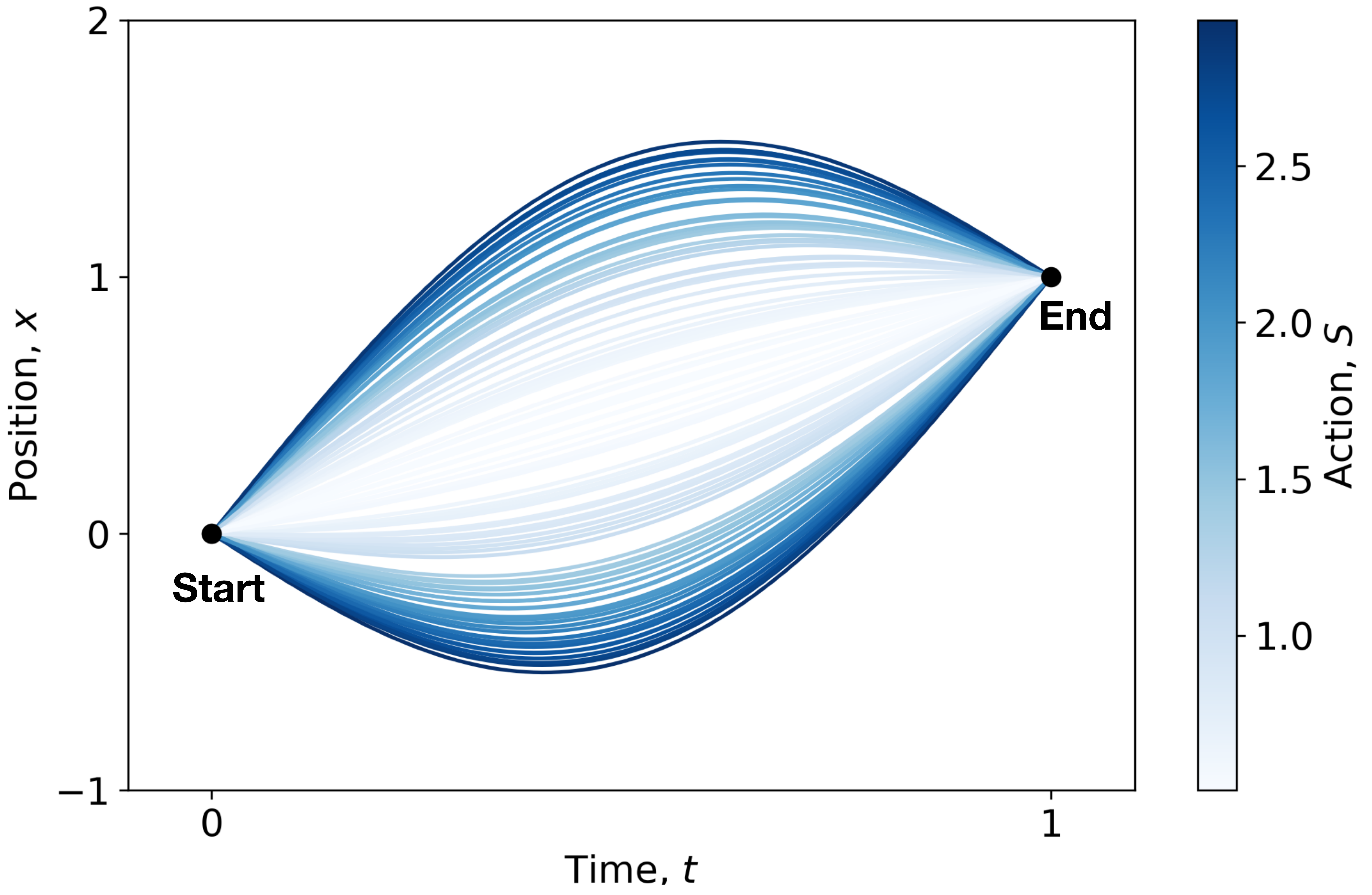
$$L = T = \frac{1}{2}m\dot{x}^2$$



# Hamilton's Principle in Action

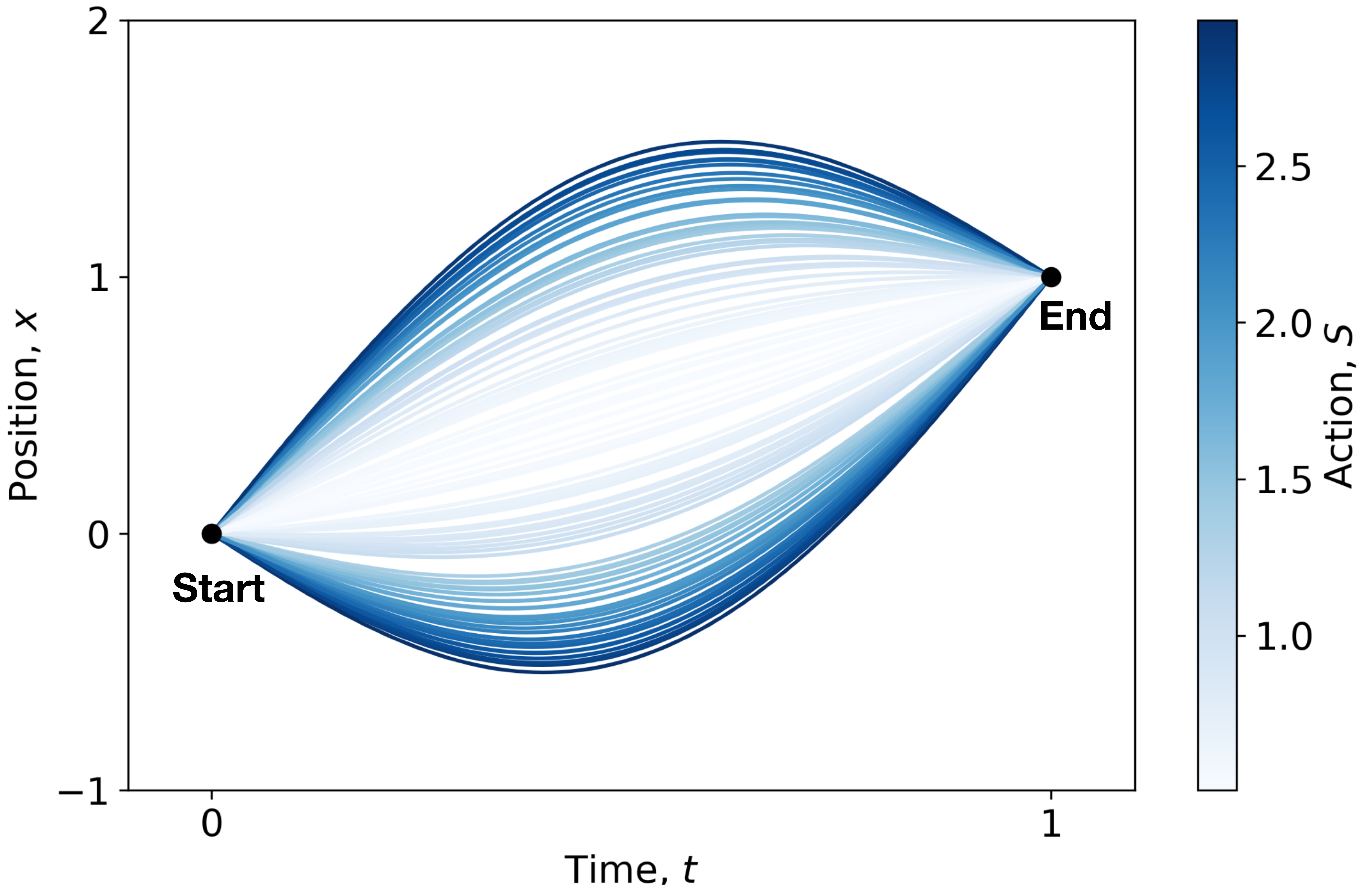
So the action is given by

$$S[x(t)] = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \frac{1}{2} m\dot{x}^2 dt$$



# Hamilton's Principle in Action

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2}m\dot{x}^2$$



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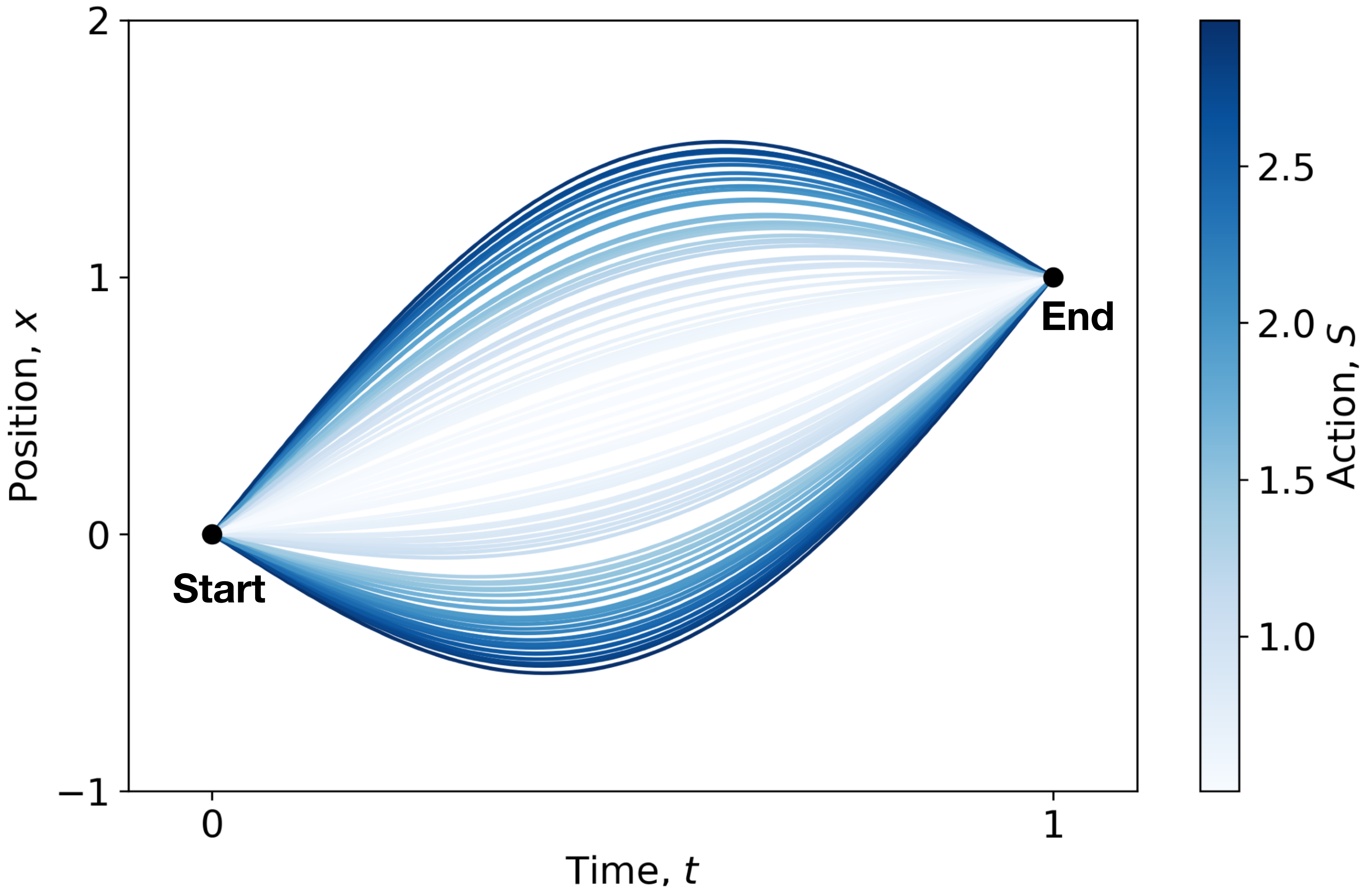
$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) - \frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2\right) = 0$$



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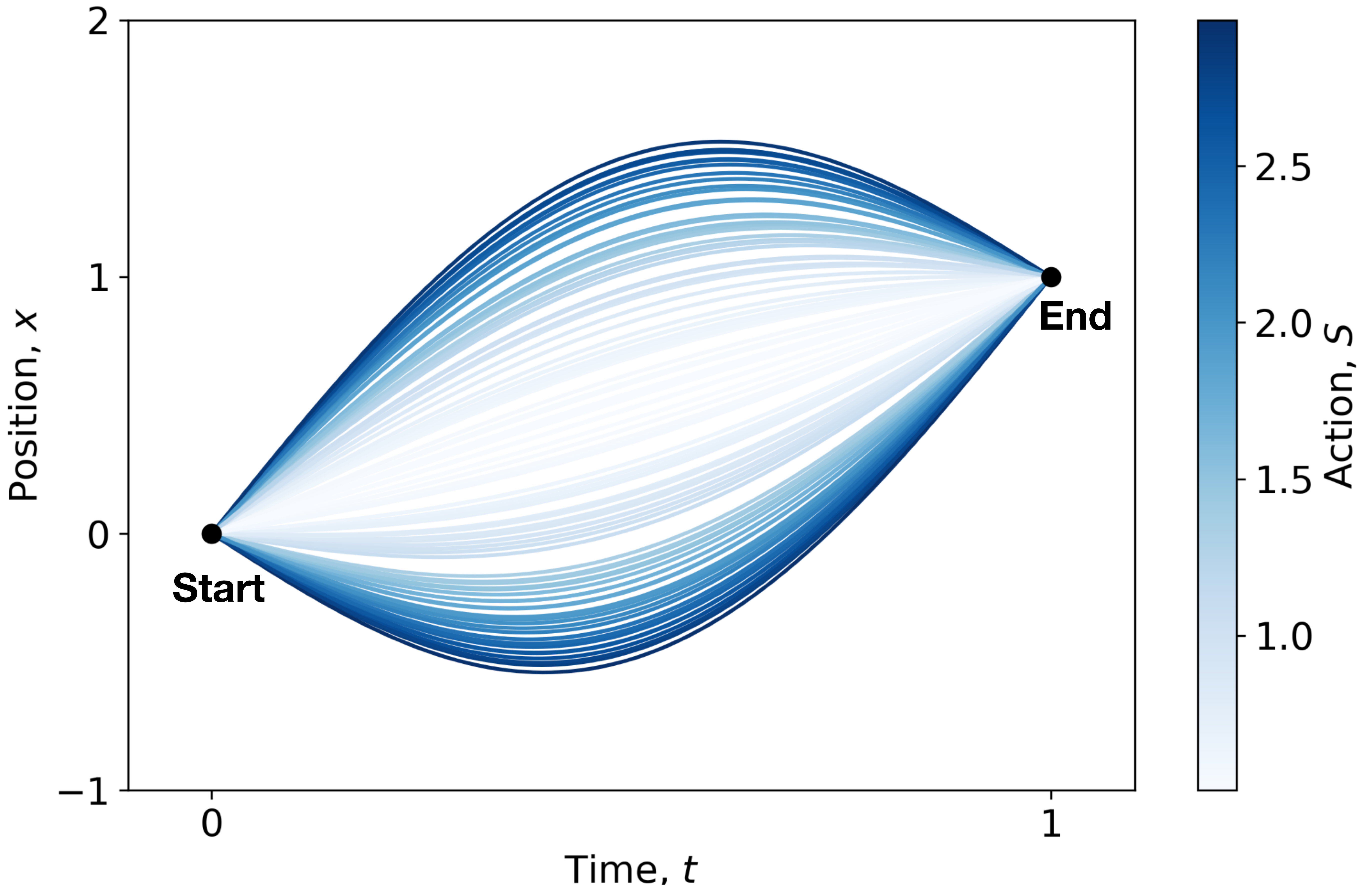
$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2}m\dot{x}^2$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) - \cancel{\frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2\right)} = 0$$

$$\frac{\delta}{\delta t}(m\dot{x}) = 0$$



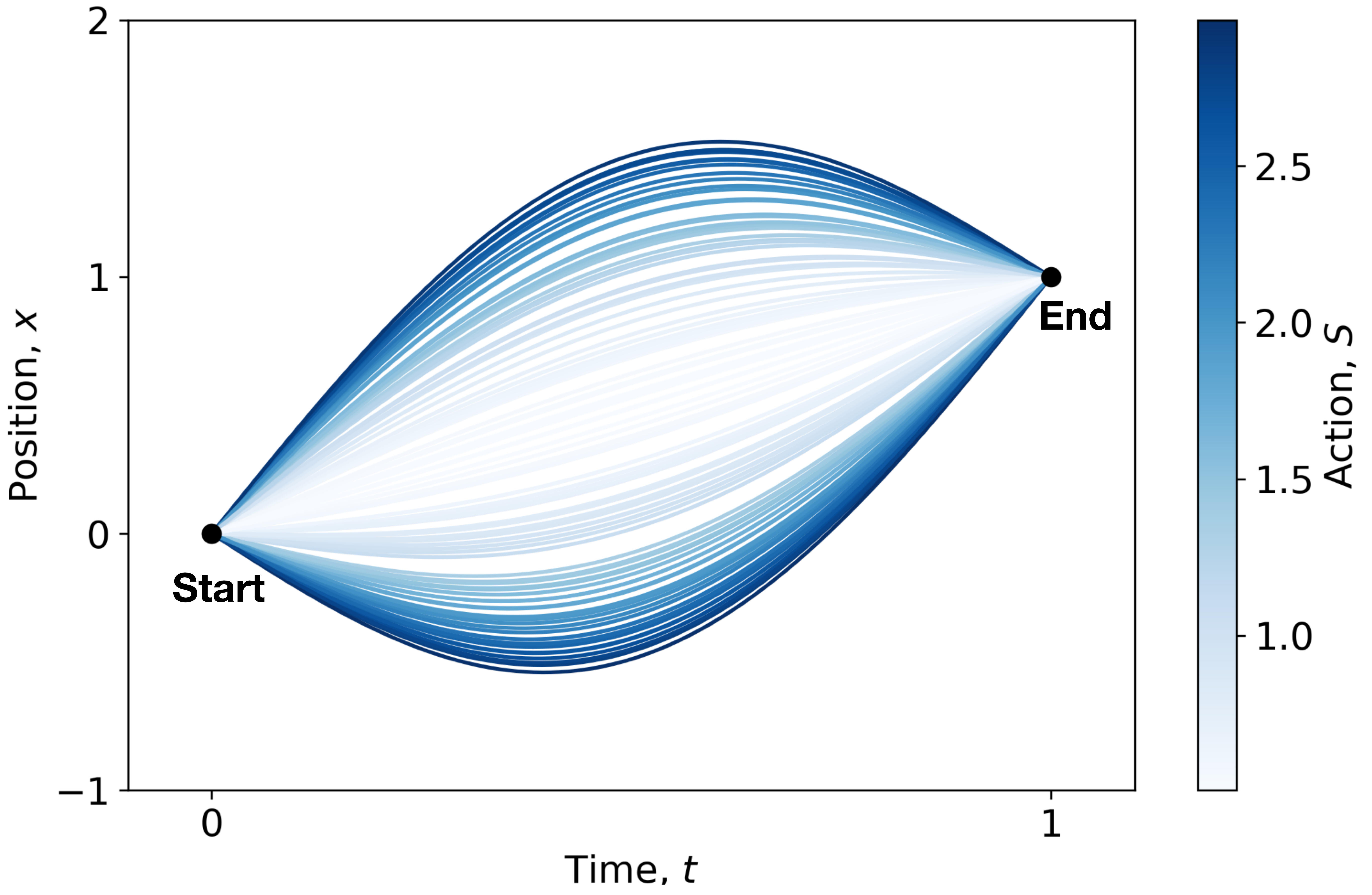
# Hamilton's Principle in Action

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2}m\dot{x}^2$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) - \frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2\right) = 0$$

$$\frac{\delta}{\delta t}(m\dot{x}) = 0$$

$$m\ddot{x} = 0$$



# Hamilton’s Principle in Action

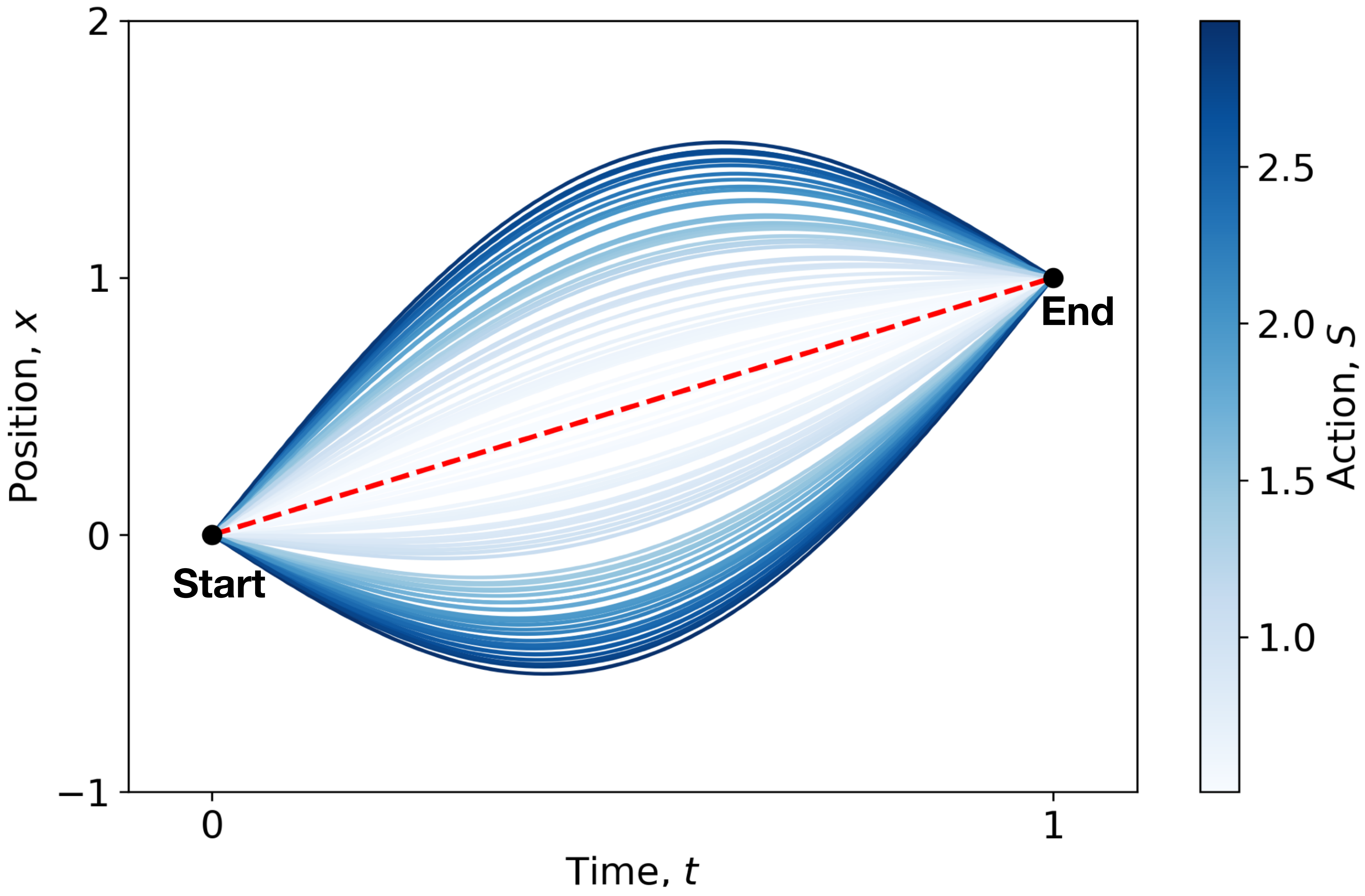
$$m\ddot{x} = 0$$

$$\xrightarrow{\text{Integrate w.r.t. t}}$$

$$\dot{x} = \text{const} = v_0$$

$$\xrightarrow{\text{Integrate w.r.t. t}}$$

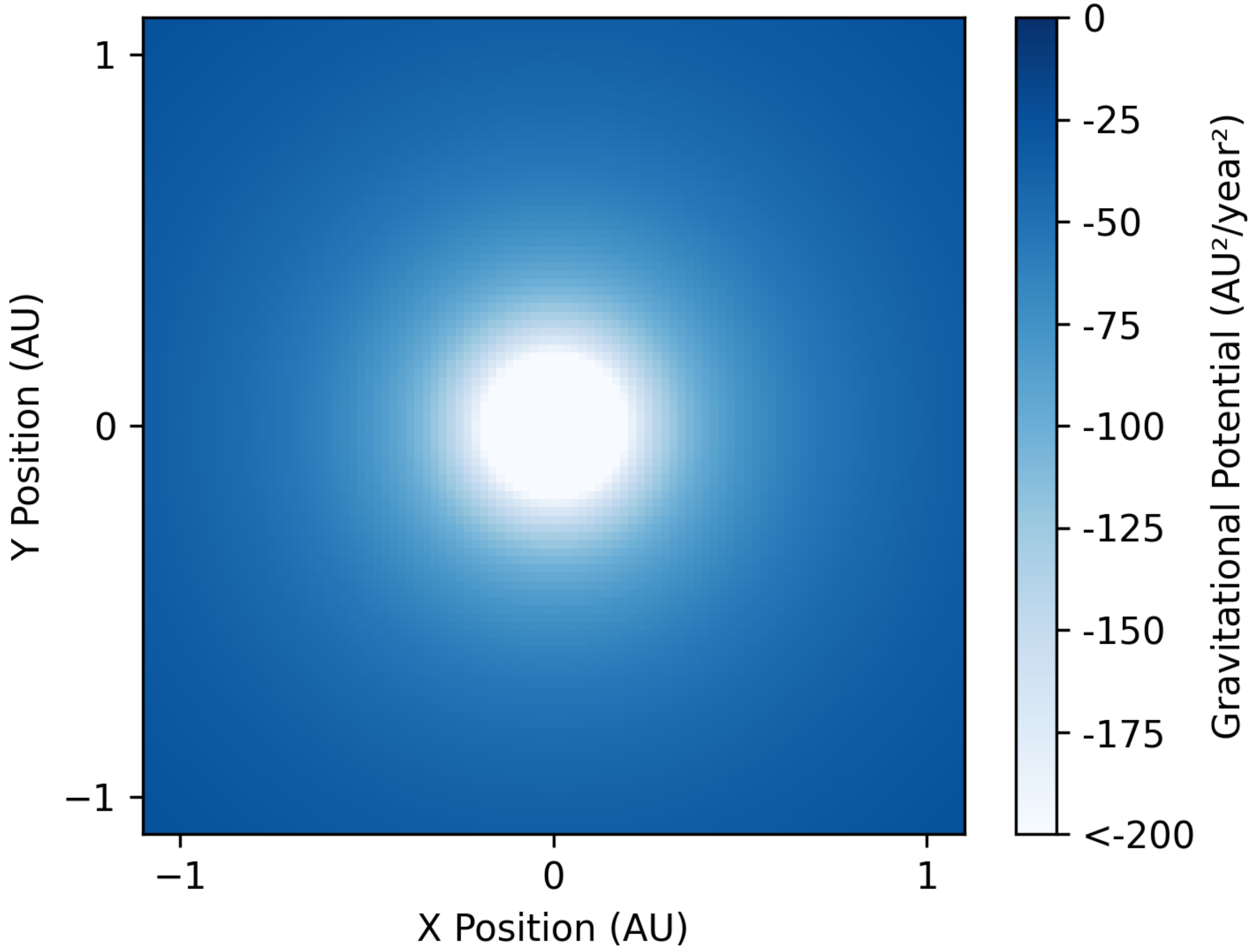
$$x = v_0 t + c$$



# Newton's Law of Gravitation

**How about if we place our particle in a gravitational potential?**

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$



# Newton's Law of Gravitation

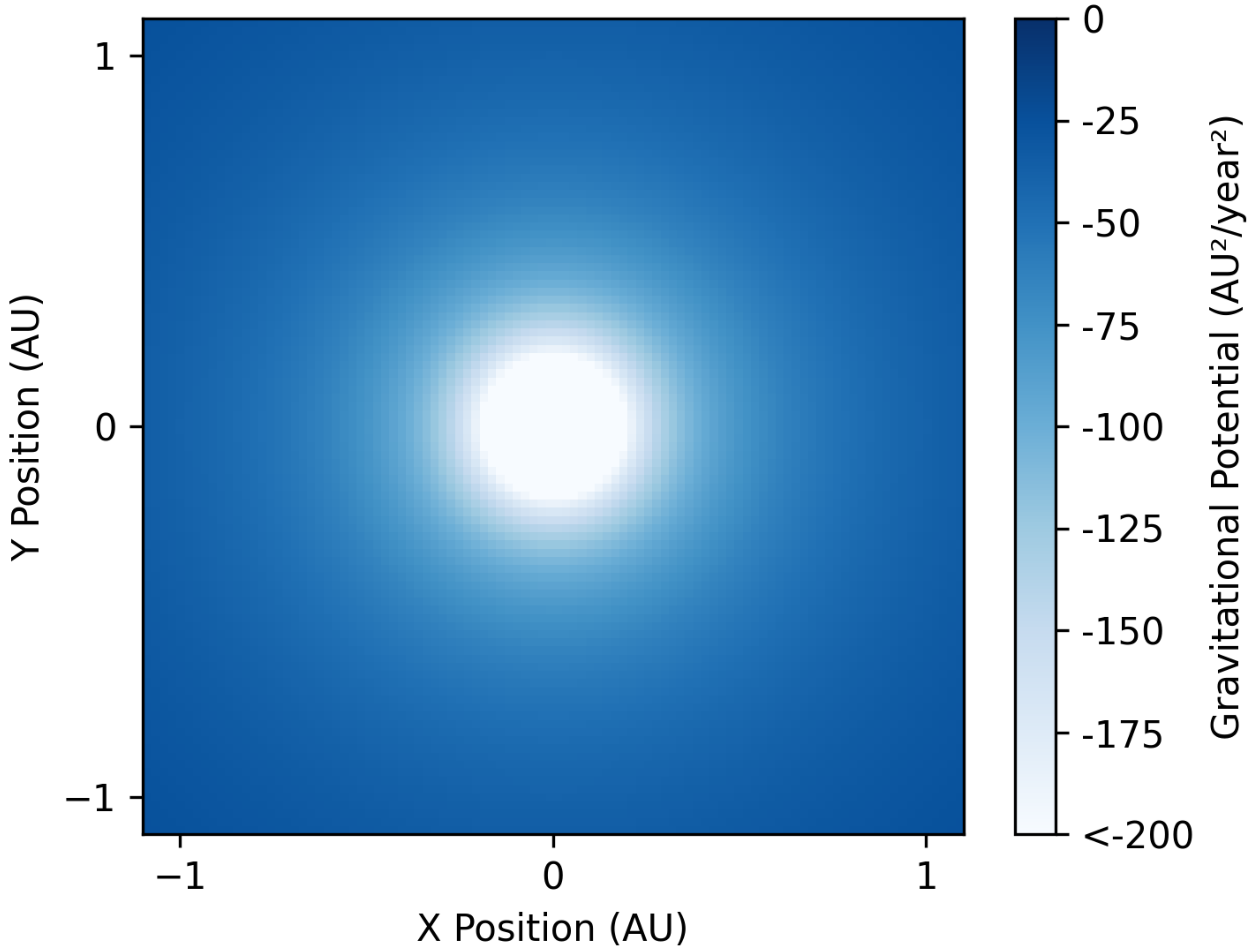
$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{r}} - \frac{\delta L}{\delta r} = 0$$

$$\frac{\delta L}{\delta \dot{r}} = m\dot{r}$$

$$\frac{\delta L}{\delta r} = -\frac{GMm}{r^2}$$

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{r}}\right) = m\ddot{r}$$



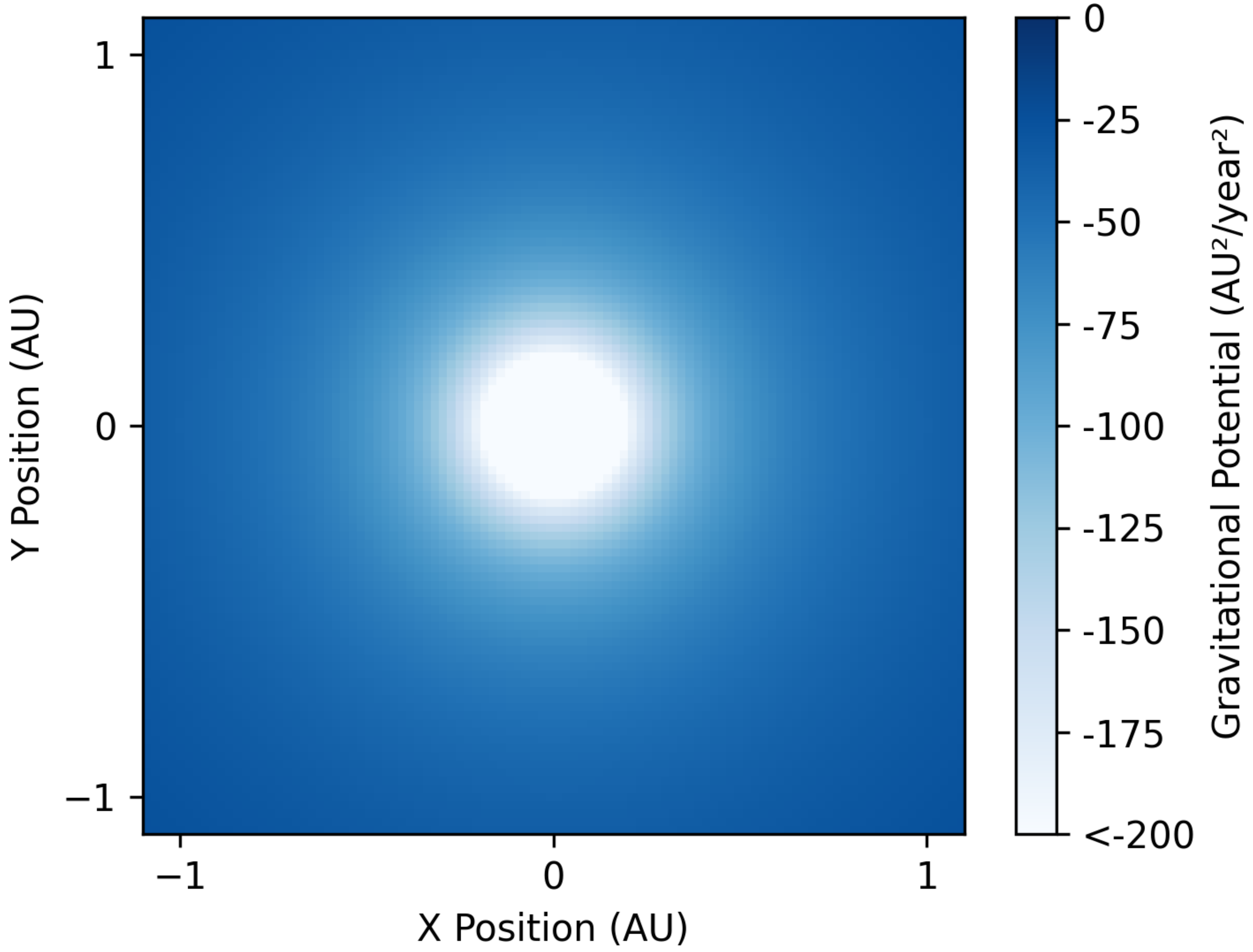
# Newton's Law of Gravitation

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{r}} - \frac{\delta L}{\delta r} = 0$$

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{r}}\right) = m\ddot{r} \qquad \frac{\delta L}{\delta r} = -\frac{GMm}{r^2}$$

$$m\ddot{r} - \frac{GMm}{r^2} = 0 \longrightarrow F = m\ddot{r} = \frac{GMm}{r^2}$$

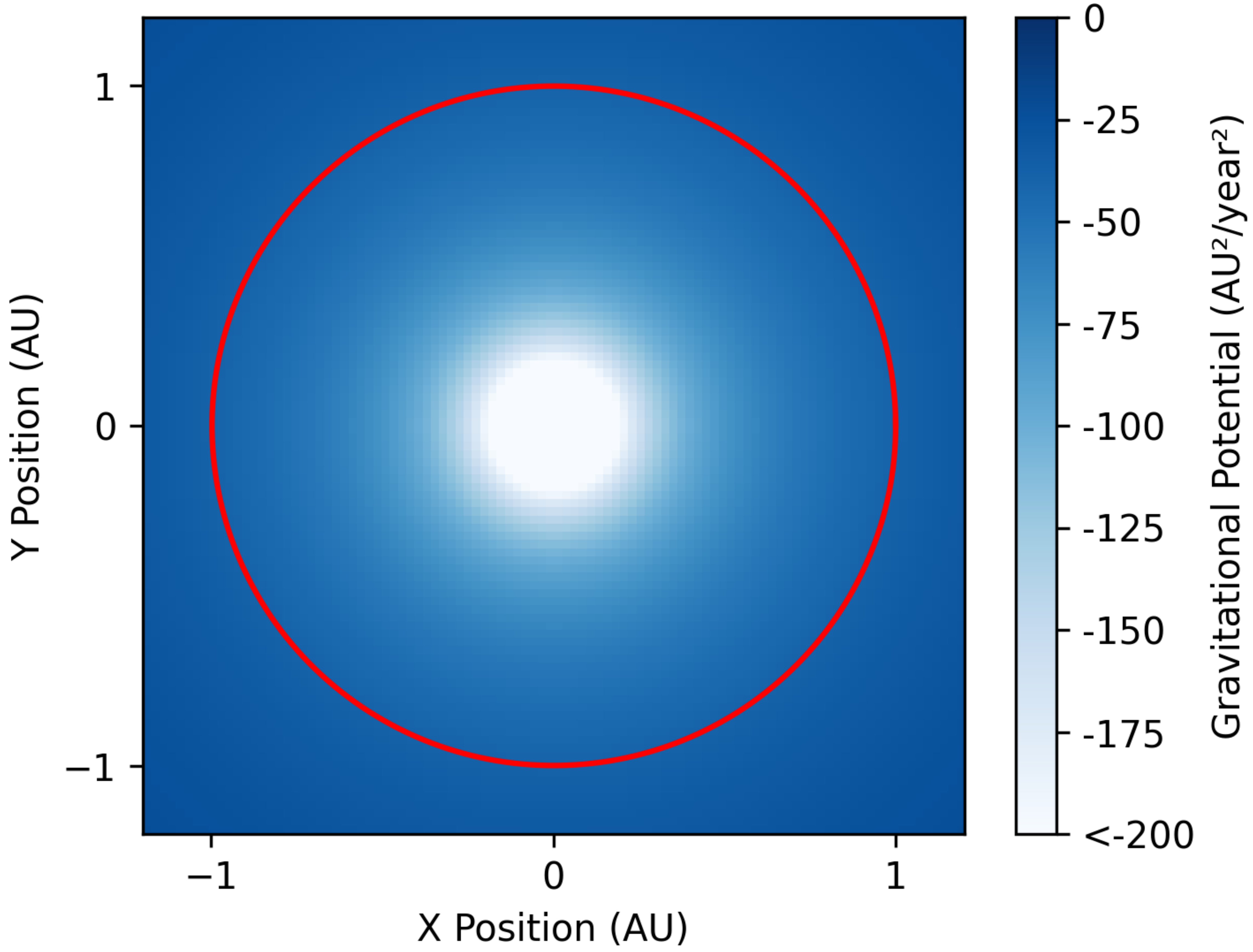


# Newton's Law of Gravitation

$$m\ddot{r} - \frac{GMm}{r^2} = 0 \longrightarrow F = m\ddot{r} = \frac{GMm}{r^2}$$

**By enforcing Hamilton's Principle we recover the Euler-Lagrange equation and using this we can understand orbital motion**

$$\ddot{r} = \frac{GM}{r^2} \xrightarrow{\text{integrate w.r.t. t}} \textbf{Orbital Motion}$$

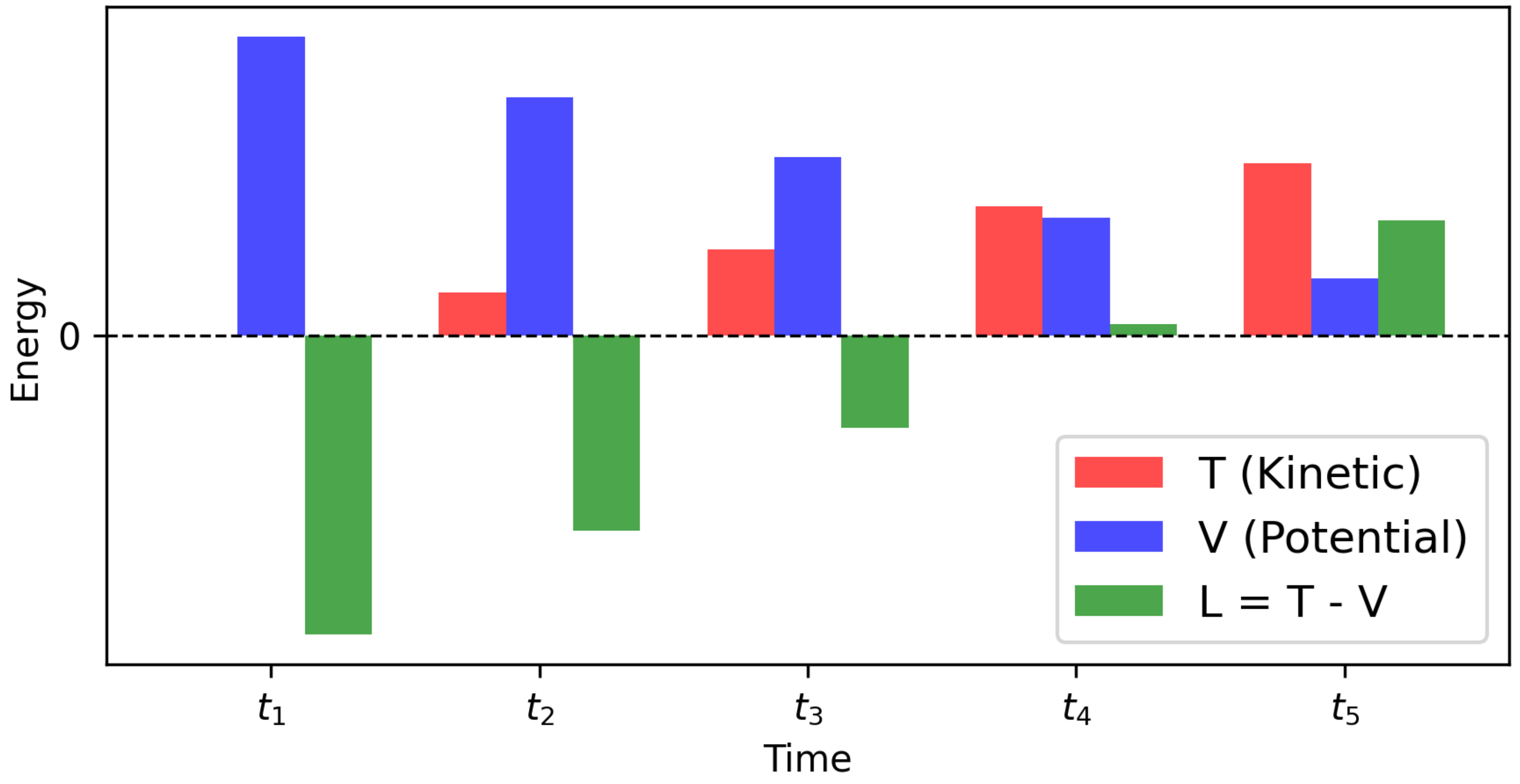


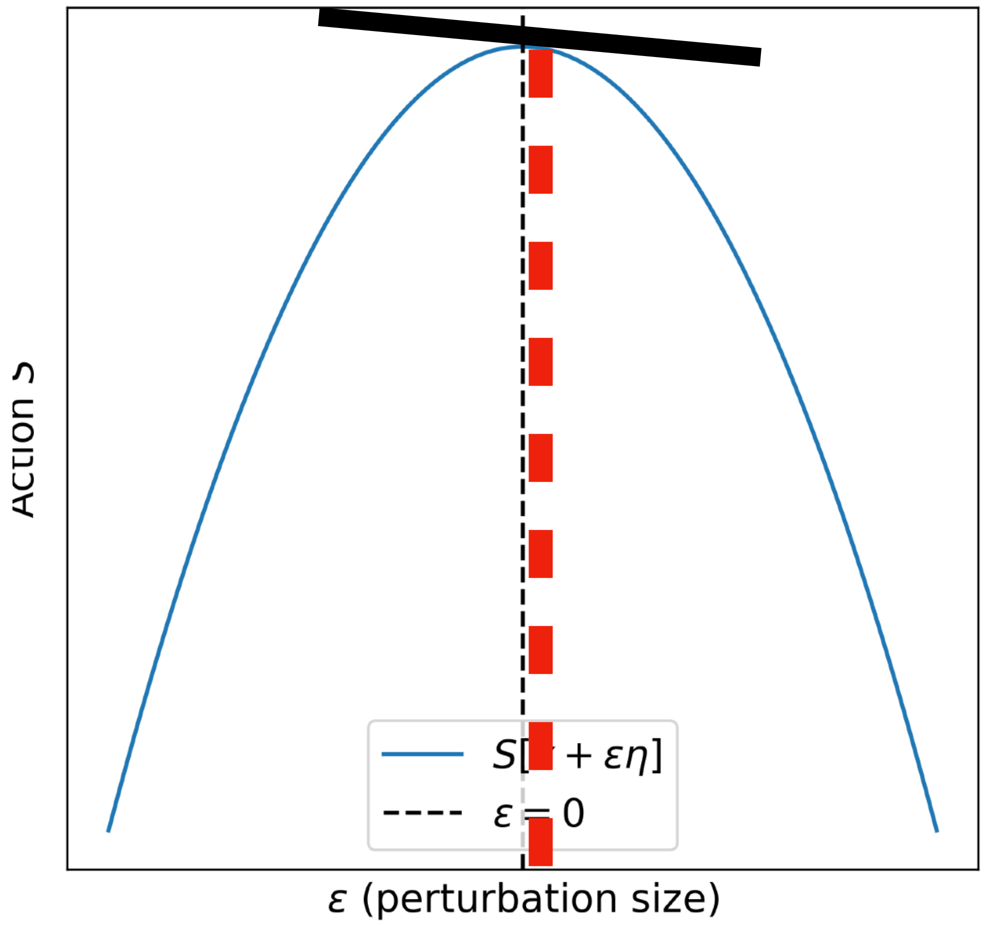
# Conclusions

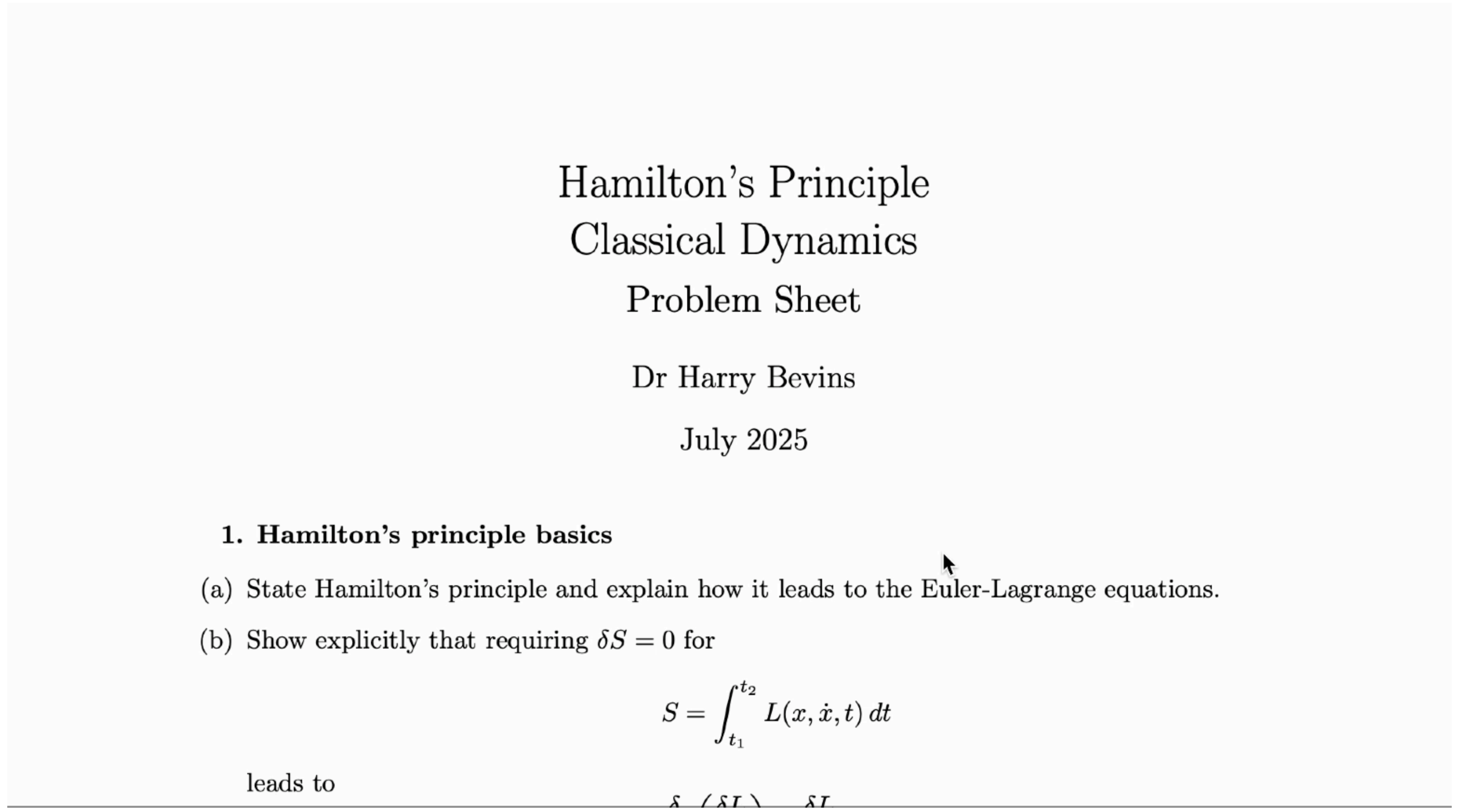
**Hamilton's principle is really a postulate that leads to the Euler-Lagrange equation**

$$\frac{\delta S}{\delta \epsilon}\bigg|_{\epsilon=0} = 0 \longrightarrow \frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0$$

**Here we derived Newtons Laws of motion but Hamilton's Principle is the gateway to modern physics**



Hamilton's Principle
Classical Dynamics
Problem Sheet

Dr Harry Bevins

July 2025

**1. Hamilton's principle basics**

(a) State Hamilton's principle and explain how it leads to the Euler-Lagrange equations.

(b) Show explicitly that requiring $\delta S = 0$ for

$$S = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$$

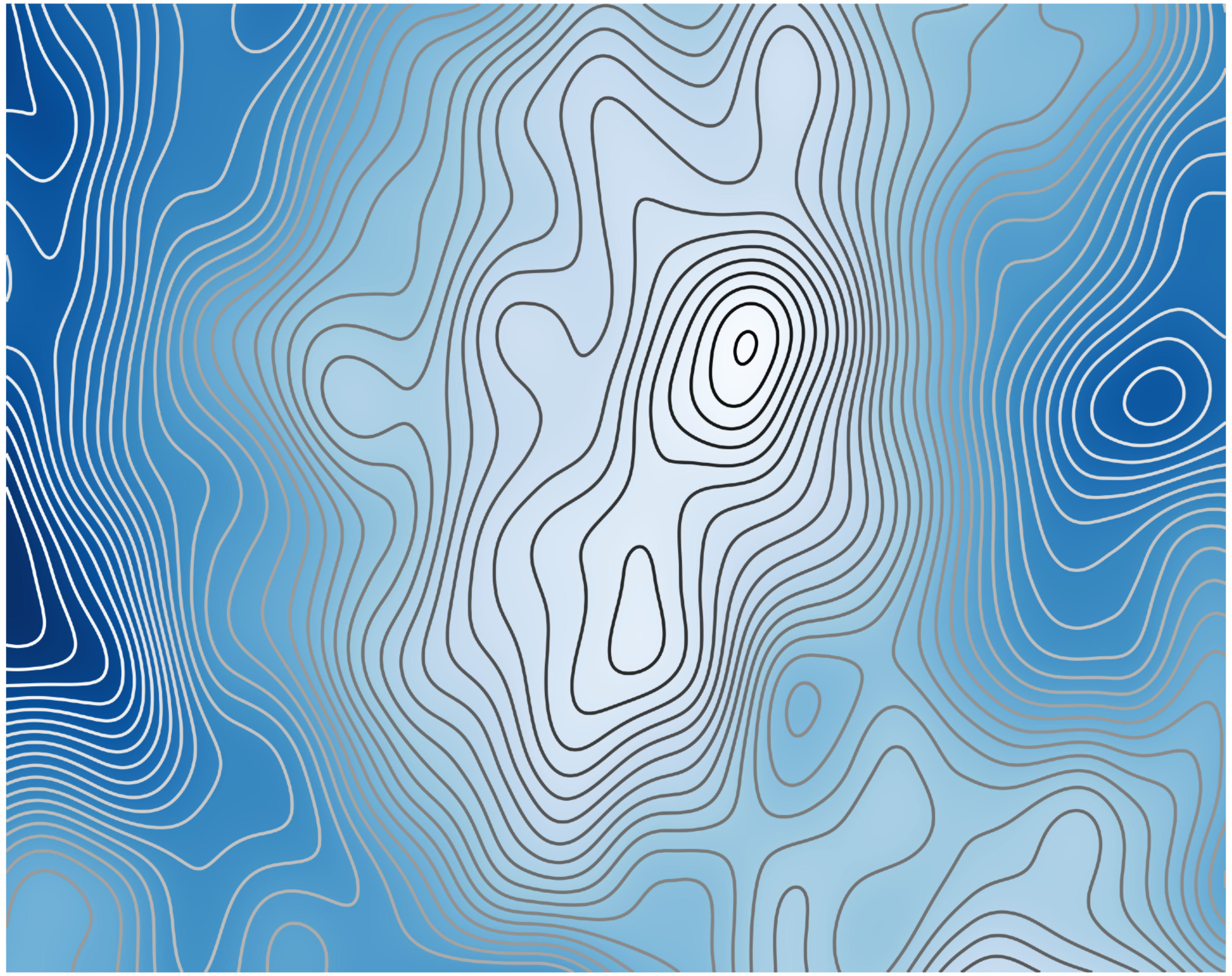
leads to

# Additional Slides

# Arbitrary Potential

Let's place our particle in a non-zero potential

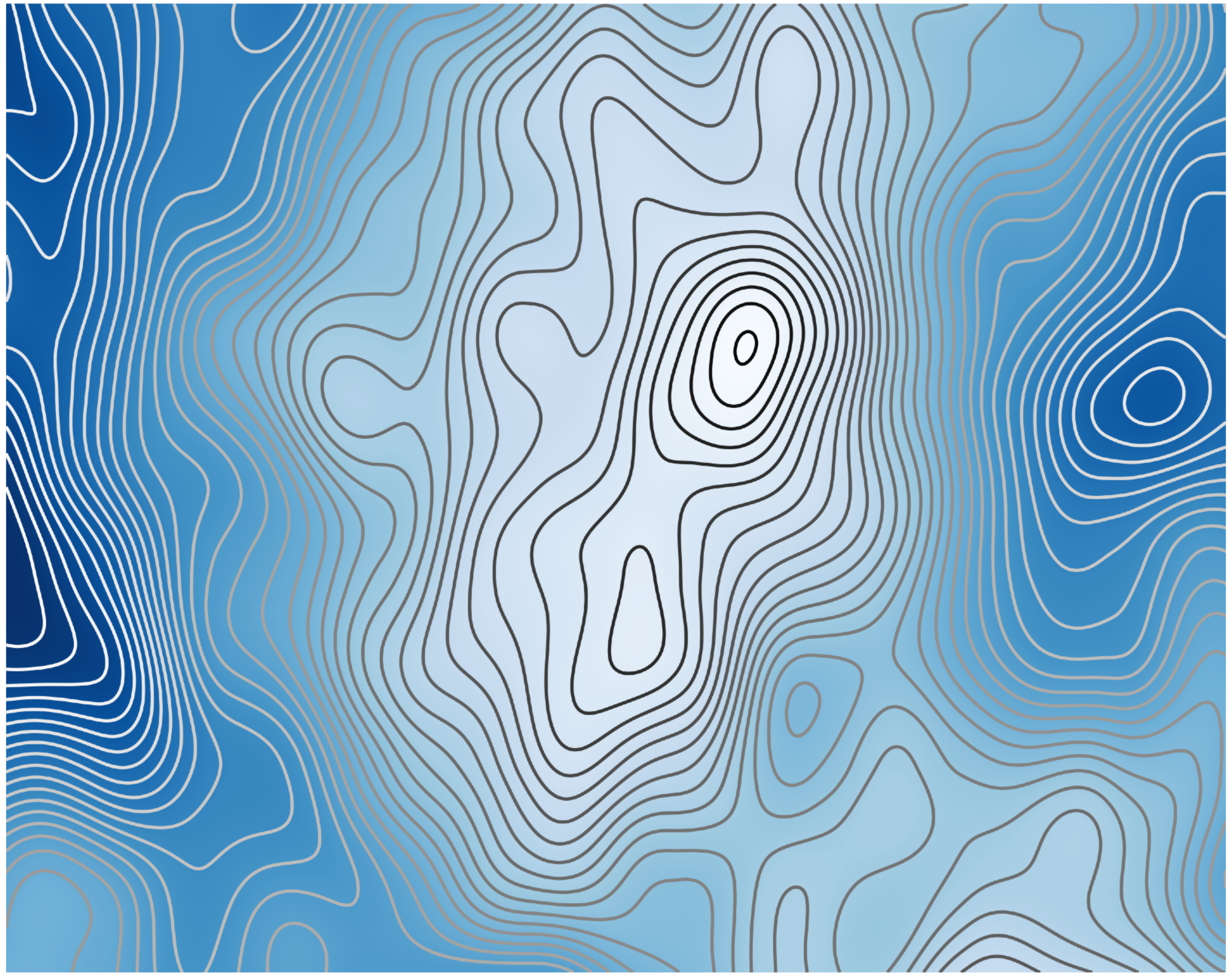
$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$



# Arbitrary Potential

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x) \qquad \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right) - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = m\ddot{x}$$

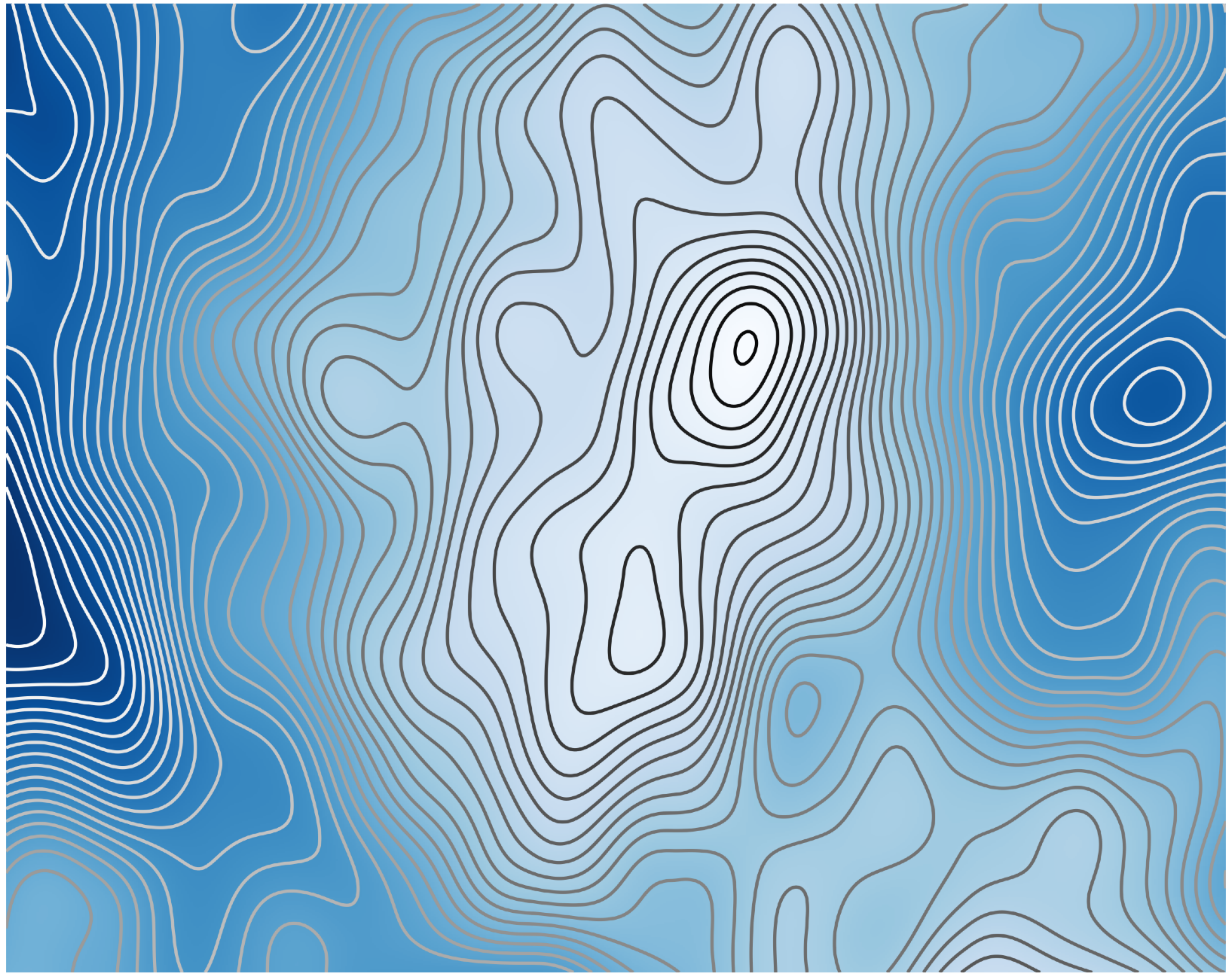


# Arbitrary Potential

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x) \qquad \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right) - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = m\ddot{x}$$

$$-\frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \frac{\delta V}{\delta x}$$

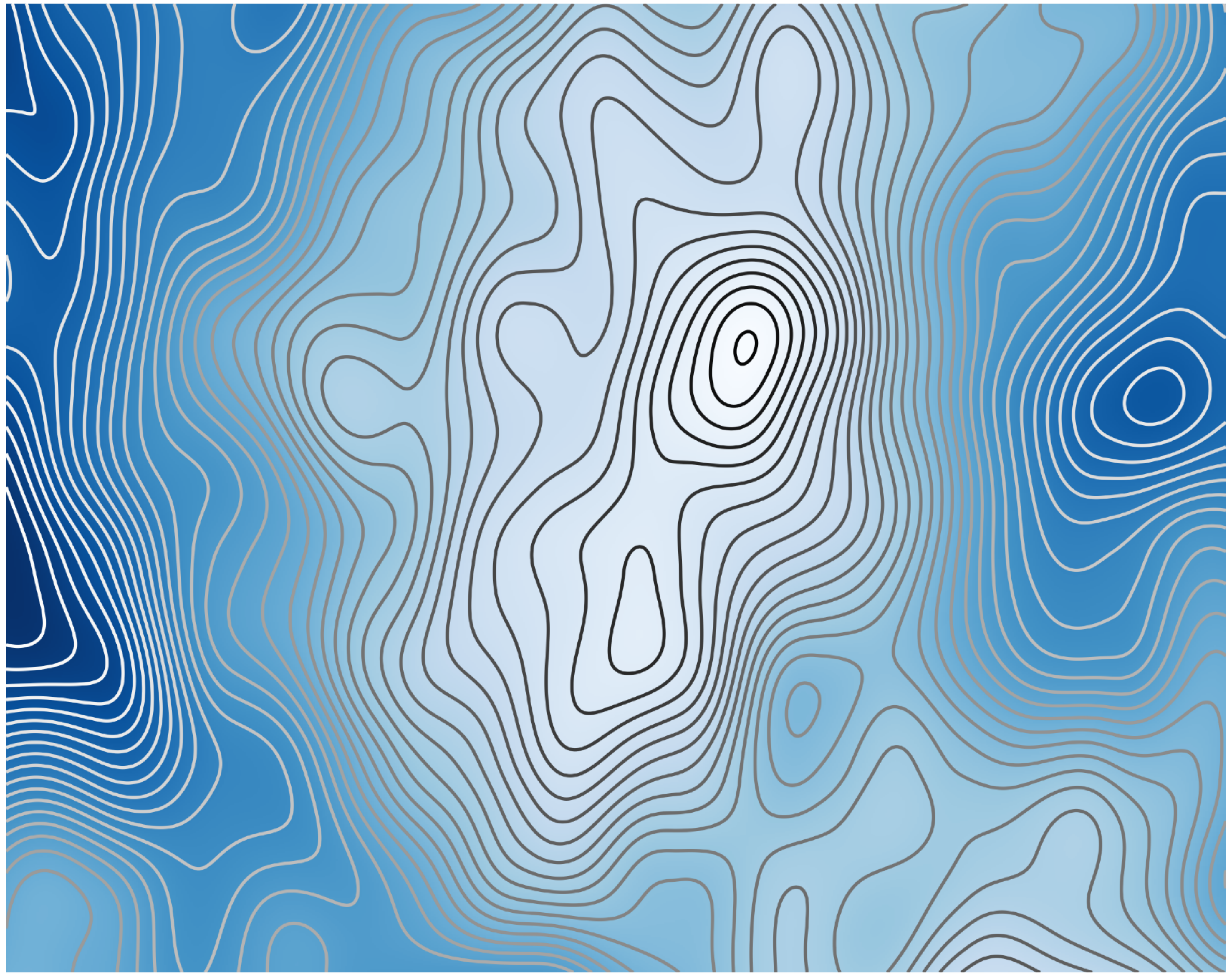


# Arbitrary Potential

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right) - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = m\ddot{x}$$

$$-\frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \frac{\delta V}{\delta x}$$



$$m\ddot{x} + \frac{\delta V}{\delta x} = 0 \longrightarrow m\ddot{x} = -\frac{\delta V}{\delta x}$$

# Newton's Second Law

$$m\ddot{x} + \frac{\delta V}{\delta x} = 0 \longrightarrow m\ddot{x} = -\frac{\delta V}{\delta x}$$

$$F = m\ddot{x} = -\frac{\delta V}{\delta x}$$