

Hamilton's Principle of Stationarity

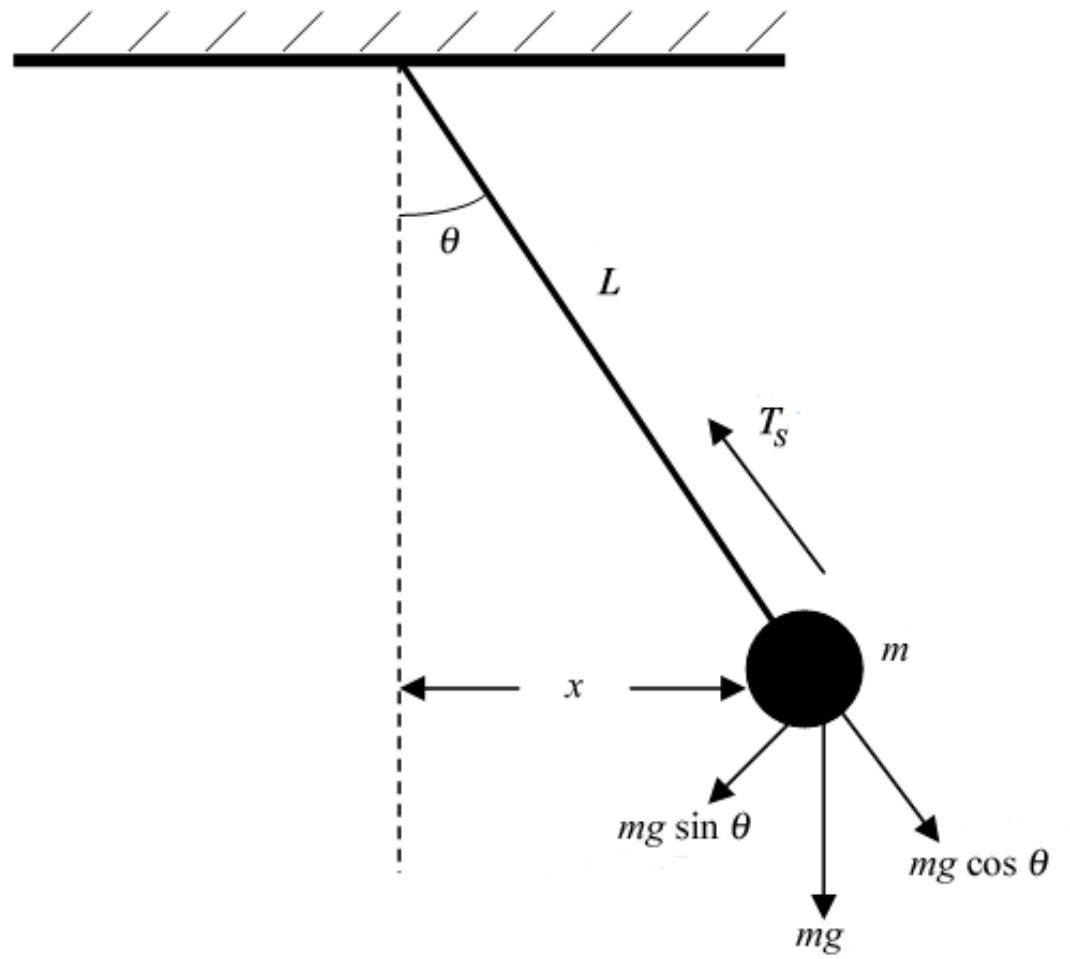
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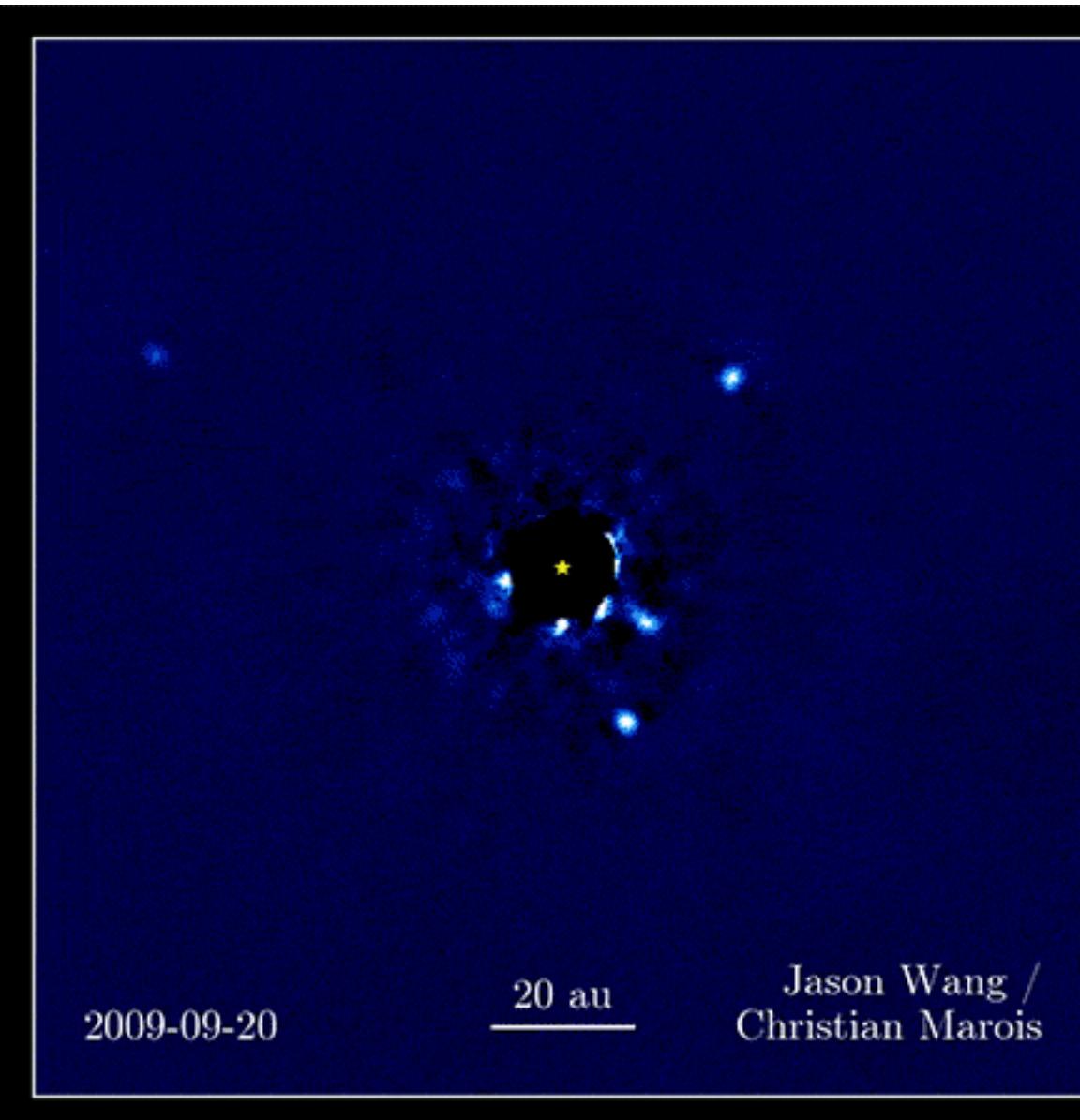
Classical Dynamics - Part IB Physics B

Resources: bit.ly/4m4Stgl

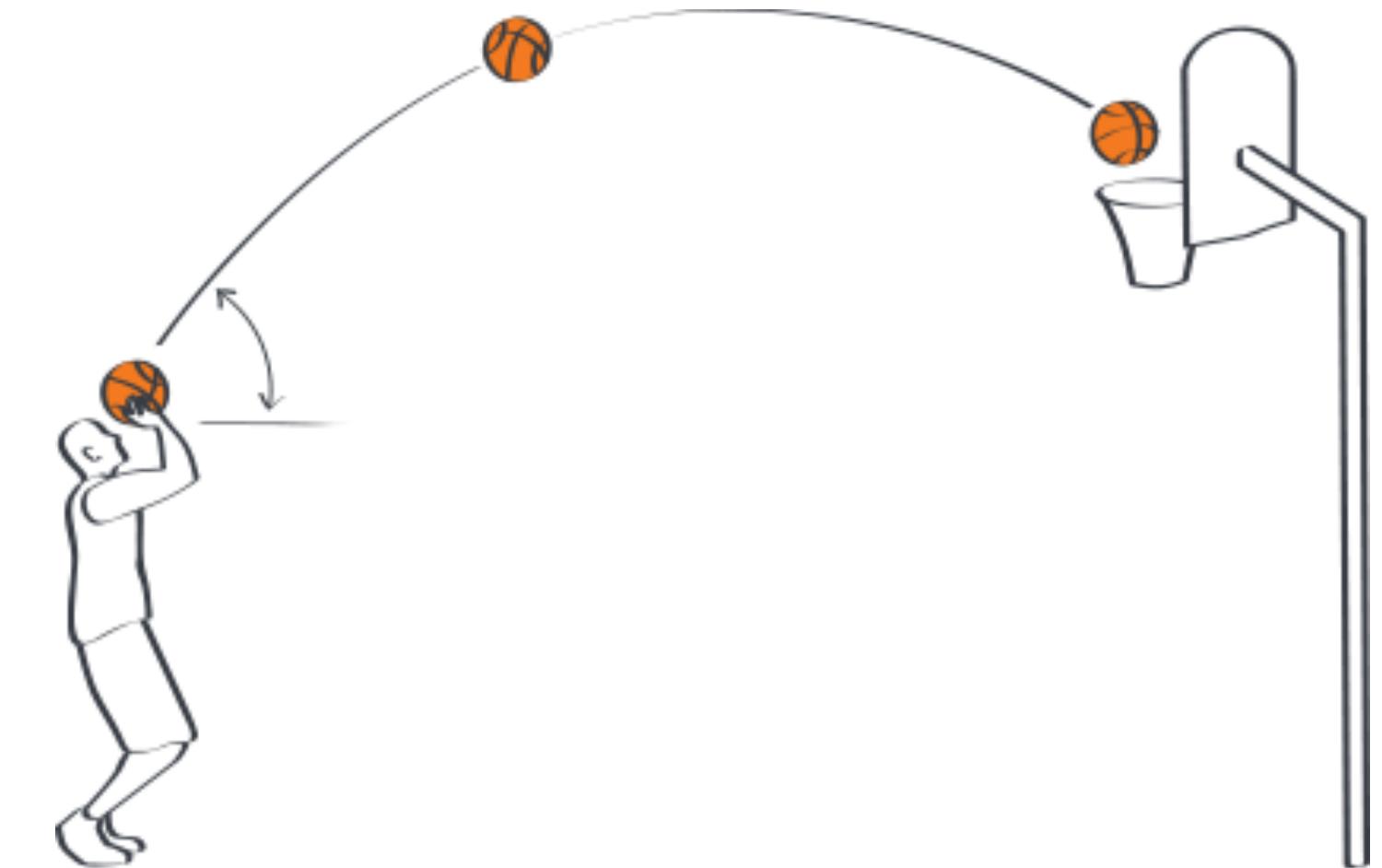
Understanding motion in complex systems



Motion of a pendulum



**Exoplanet orbits
Four Hot Jupiter planets
orbiting HR 8799**



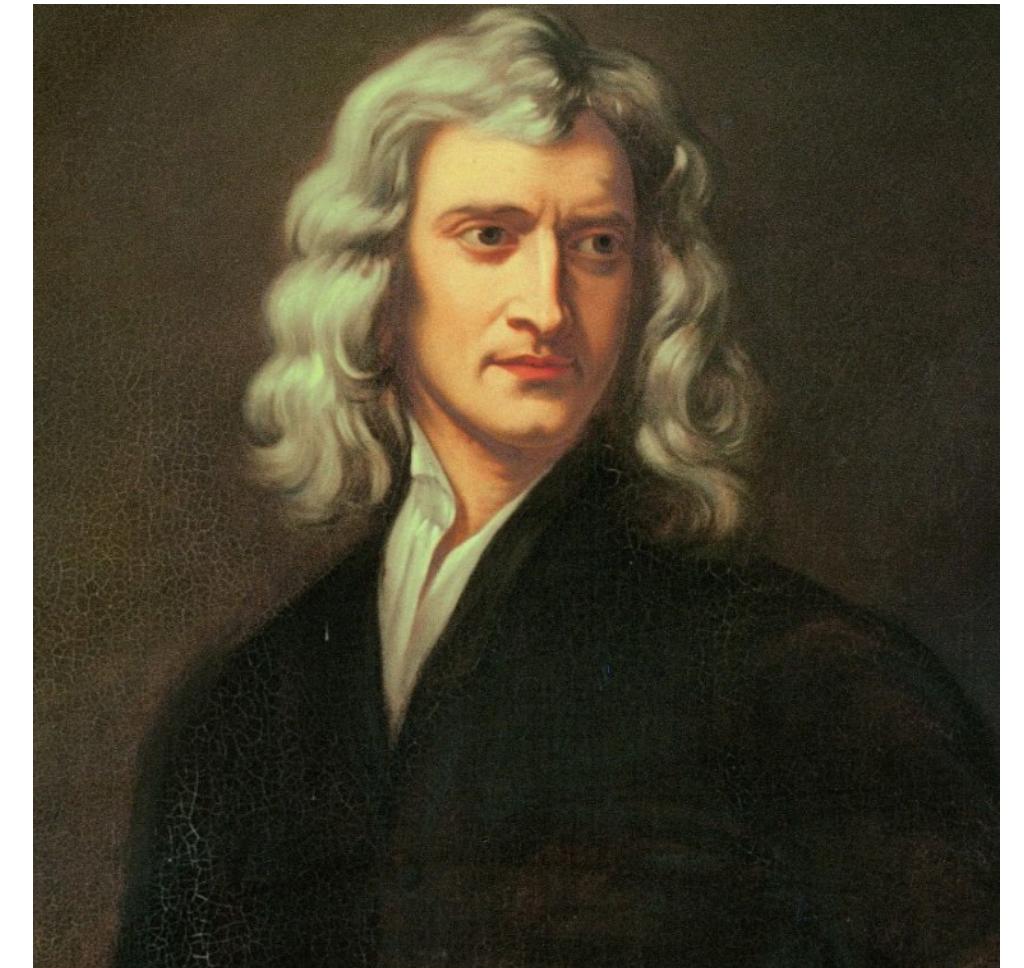
Parabolic motion of projectiles

Understanding motion in complex systems

Kepler took observations from Tycho Brahe and derived three laws of orbital mechanics (1609 - 1619)



Newton uses Kepler's laws to illustrate his inverse square law for gravitation (1687)

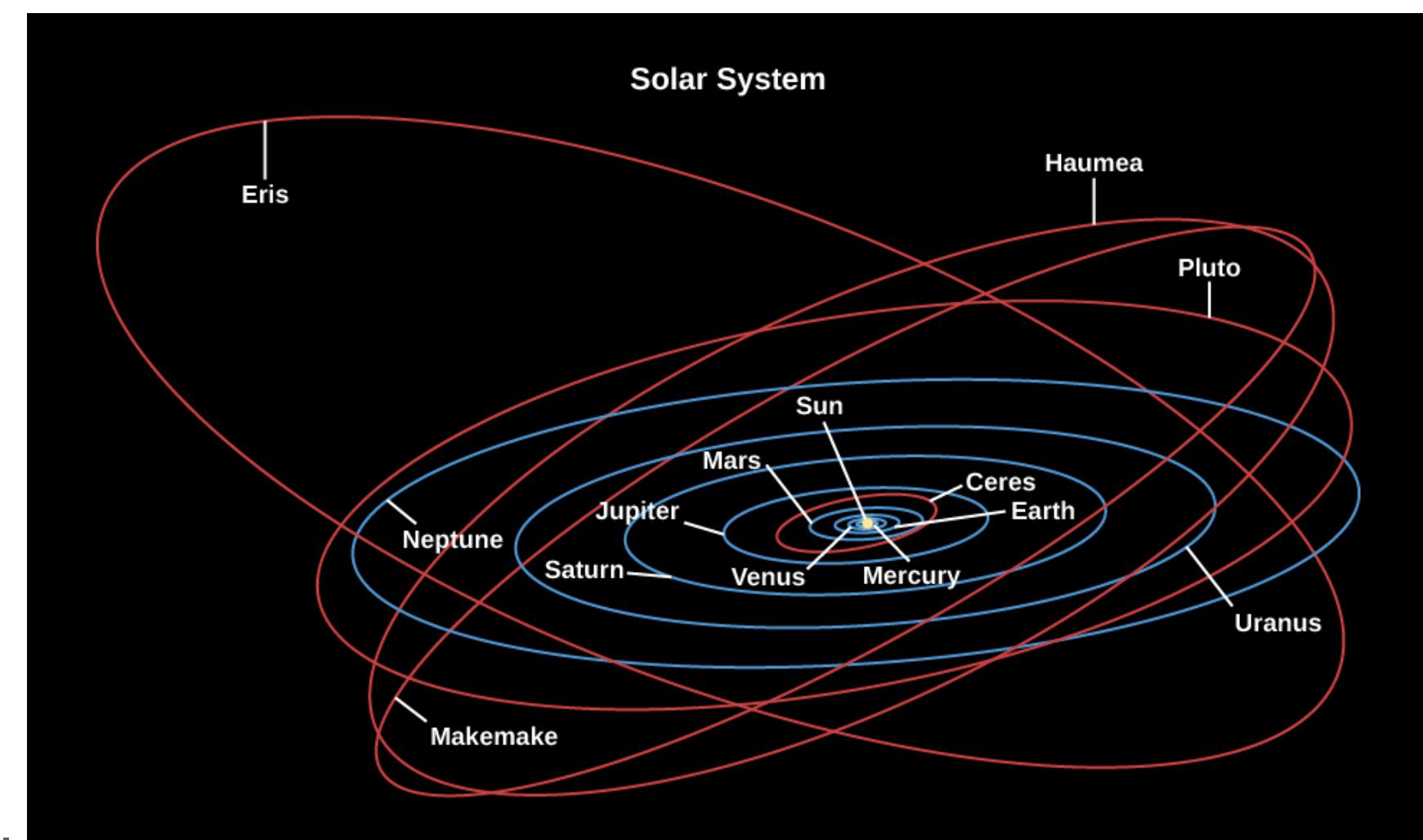


Tycho Brahe

Johannes Kepler

Isaac Newton

Hamilton's established a deeper fundamental description of mechanical systems (1834) that explains Newton's law of gravitation and a whole lot more!



Lagrangian

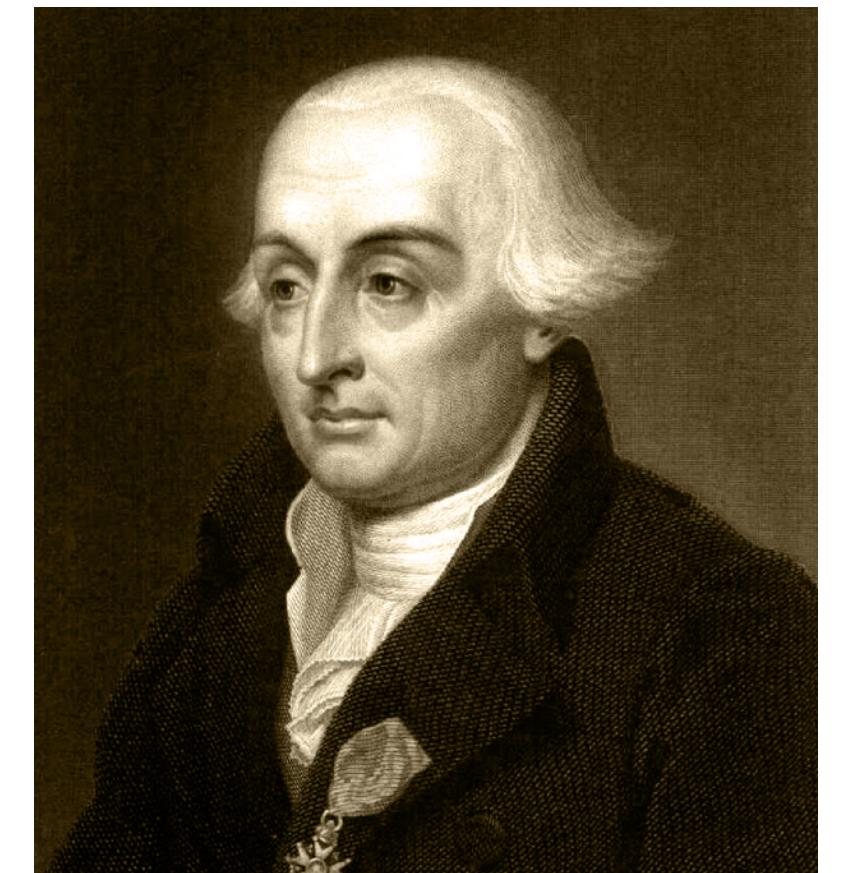
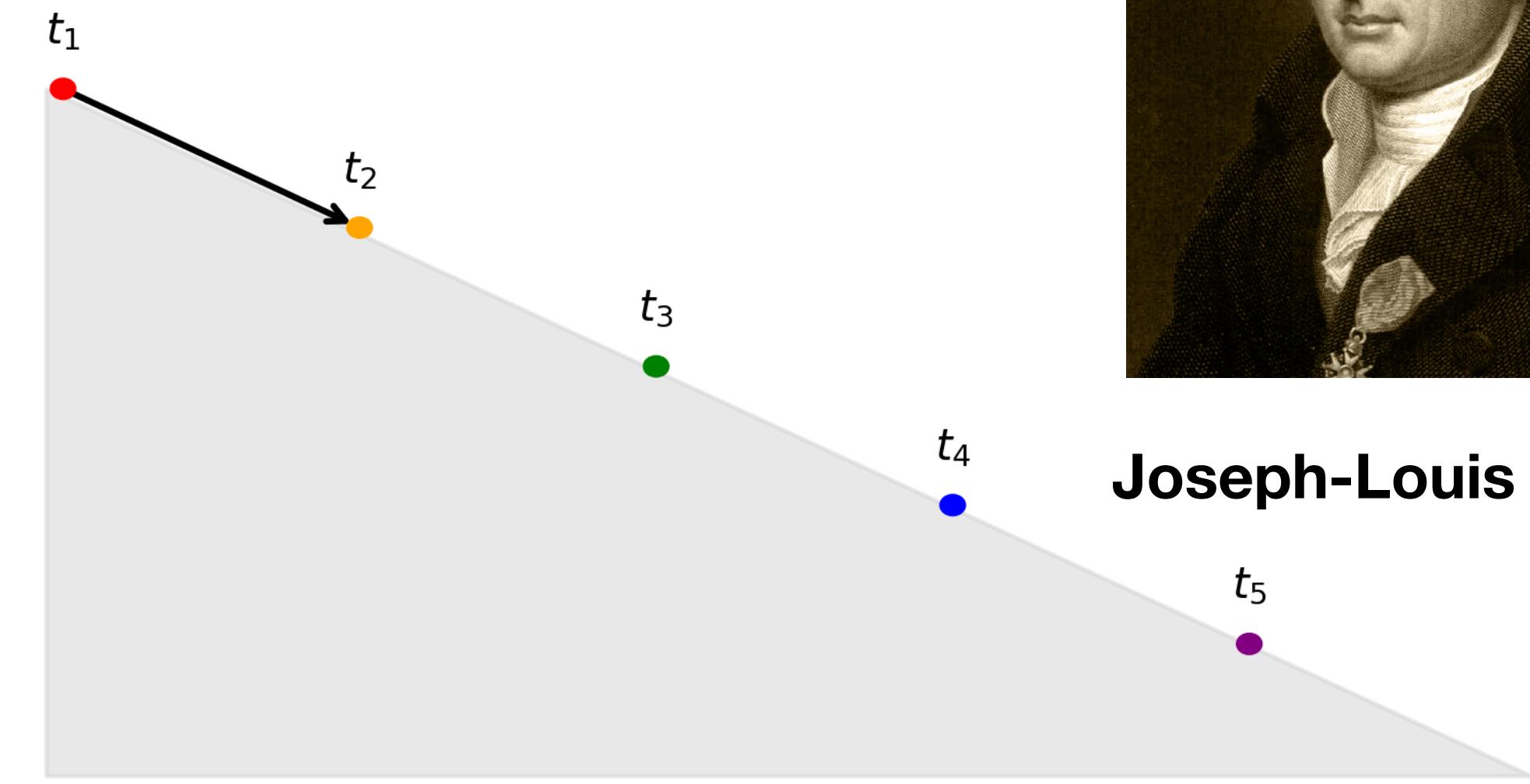
The Lagrangian is given by

$$L(x, \dot{x}, t) = T(\dot{x}, t) - V(x)$$

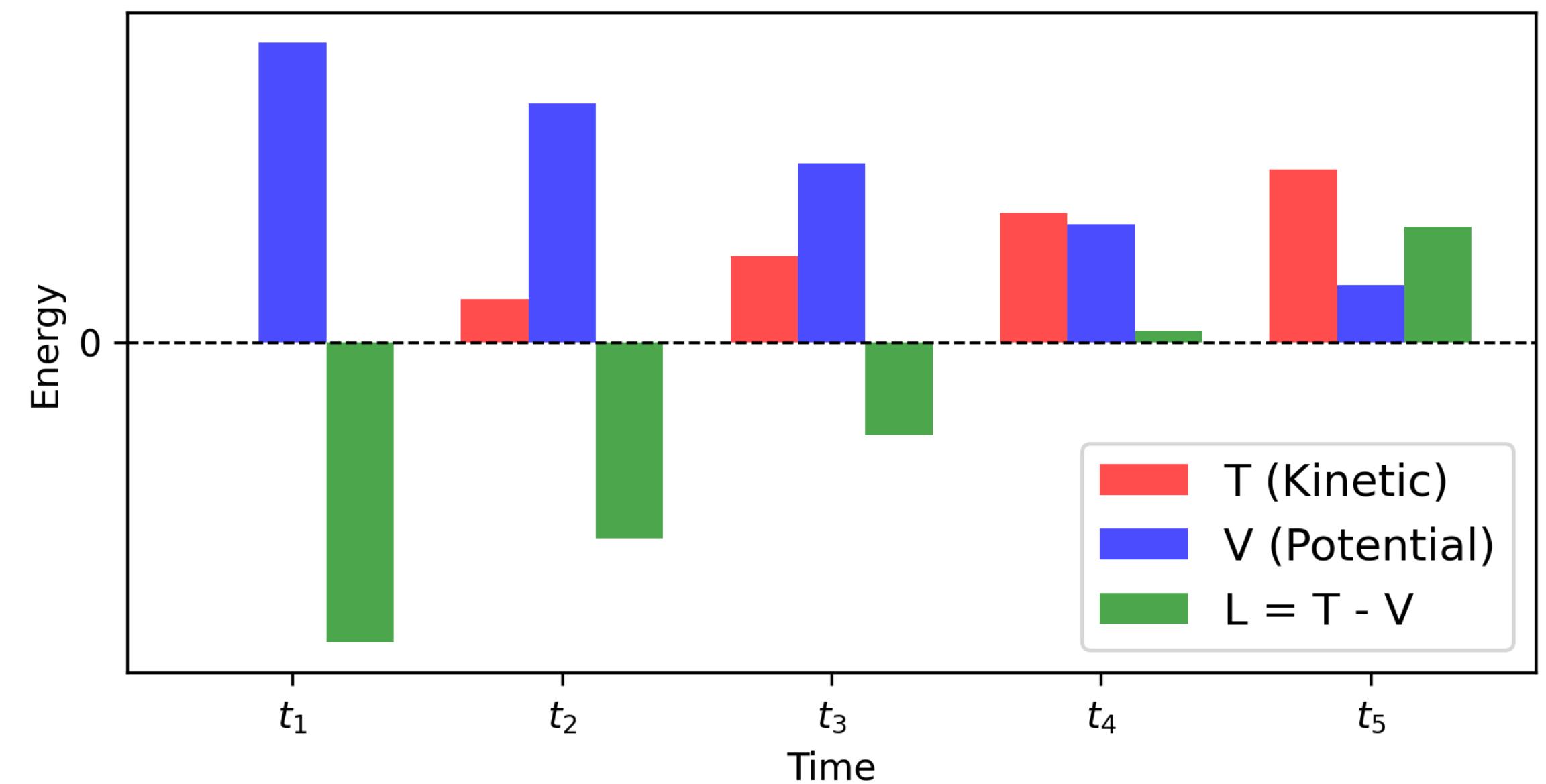
where T is kinetic energy and V is a potential

Encodes the dynamics of a system

Move from thinking about forces to motion in an energy landscape



Joseph-Louis Lagrange



Action

The fundamental object of Hamilton's Principle of Stationarity

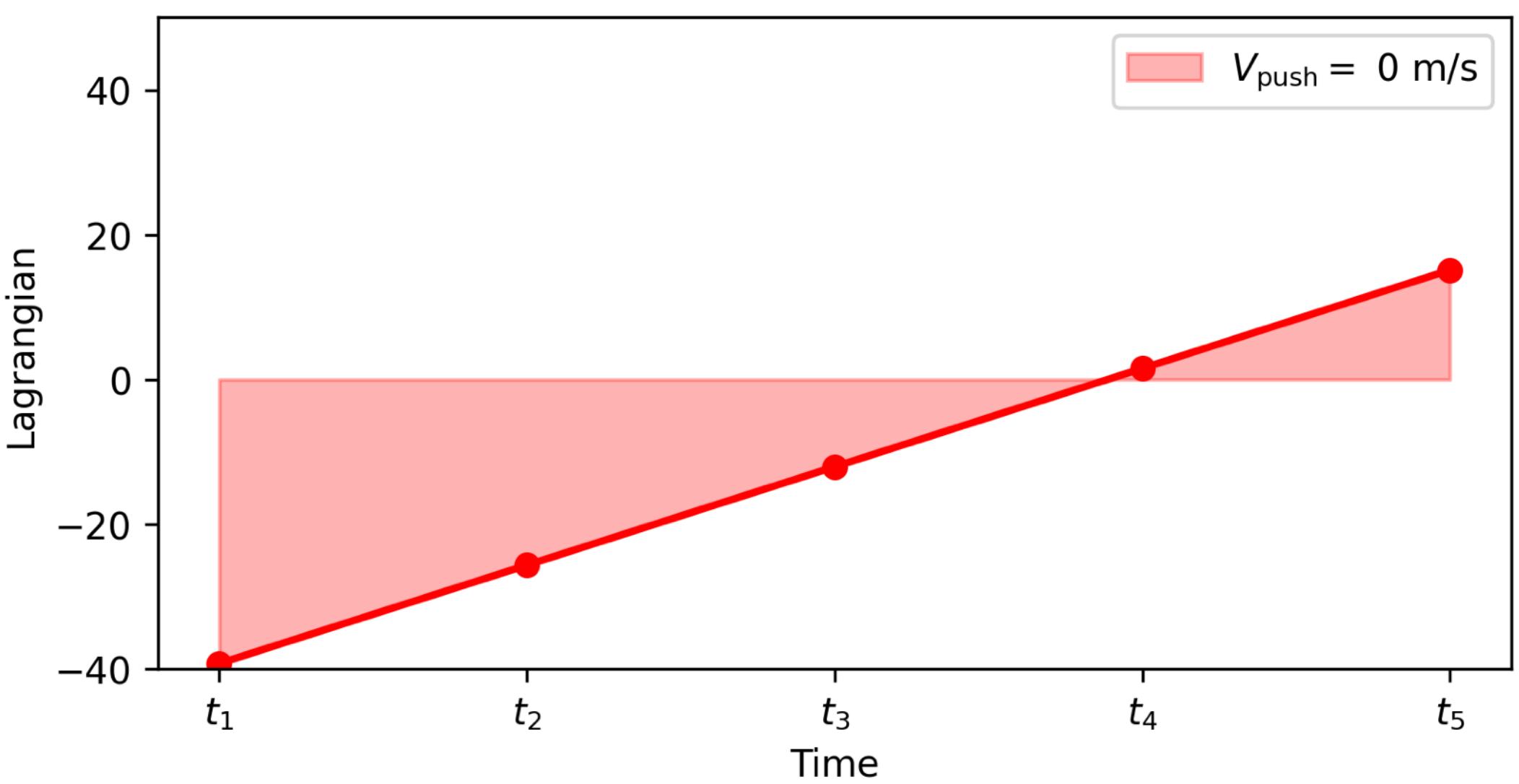
The action for a system is the time integrated Lagrangian

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t) dt$$

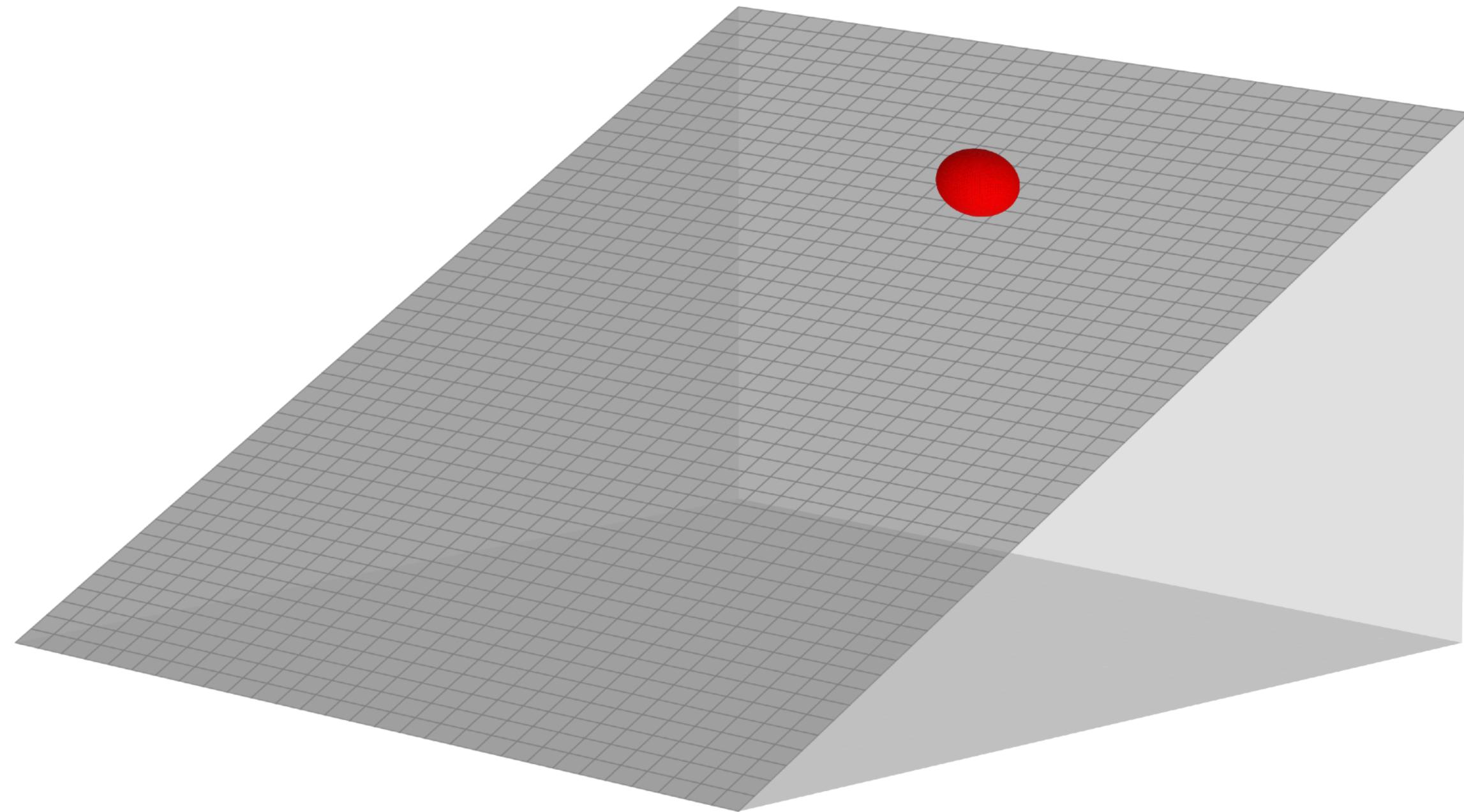
Quantifies the balance between kinetic and potential energy in a system over time



William Rowan Hamilton

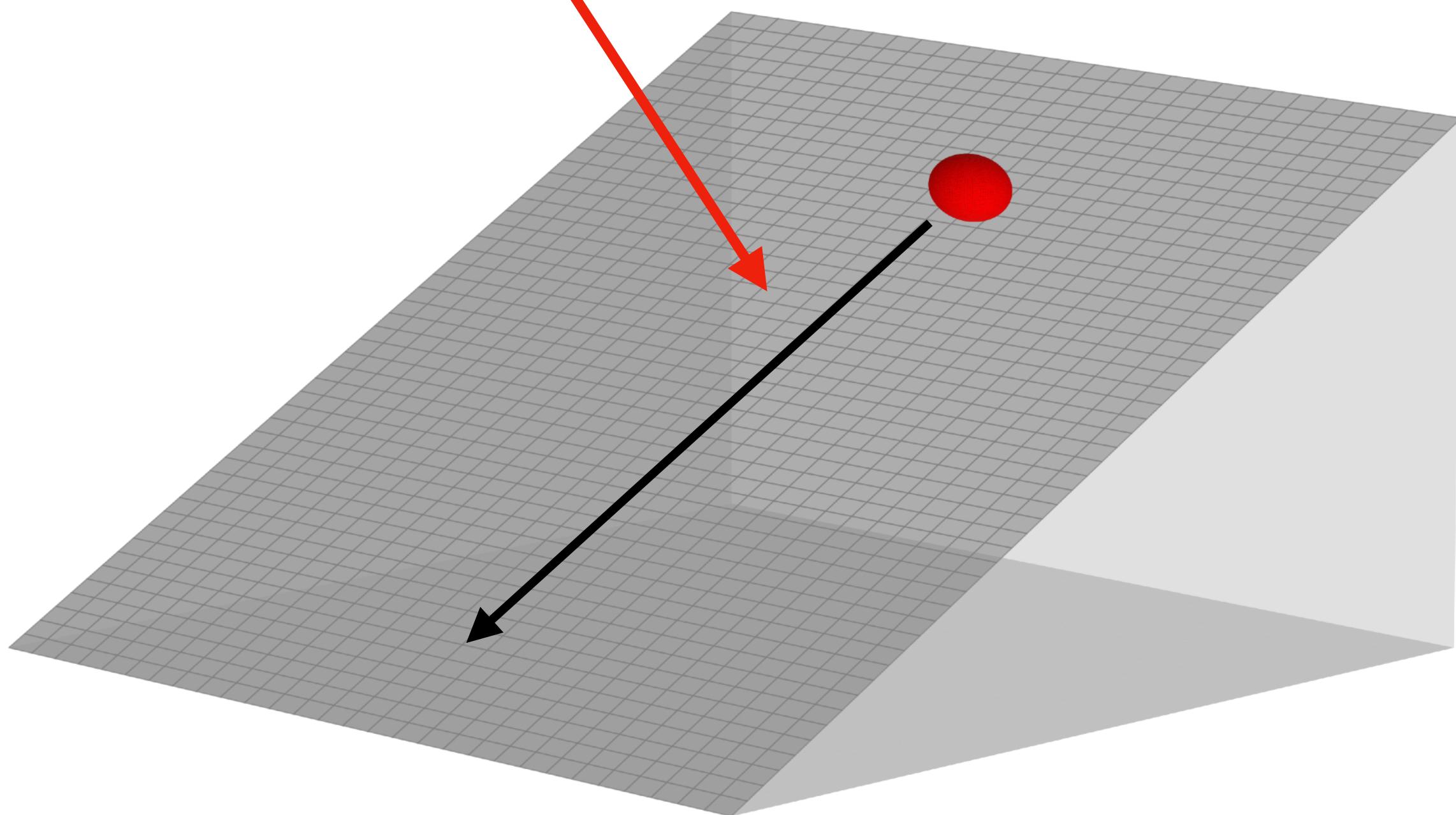


Rolling Ball



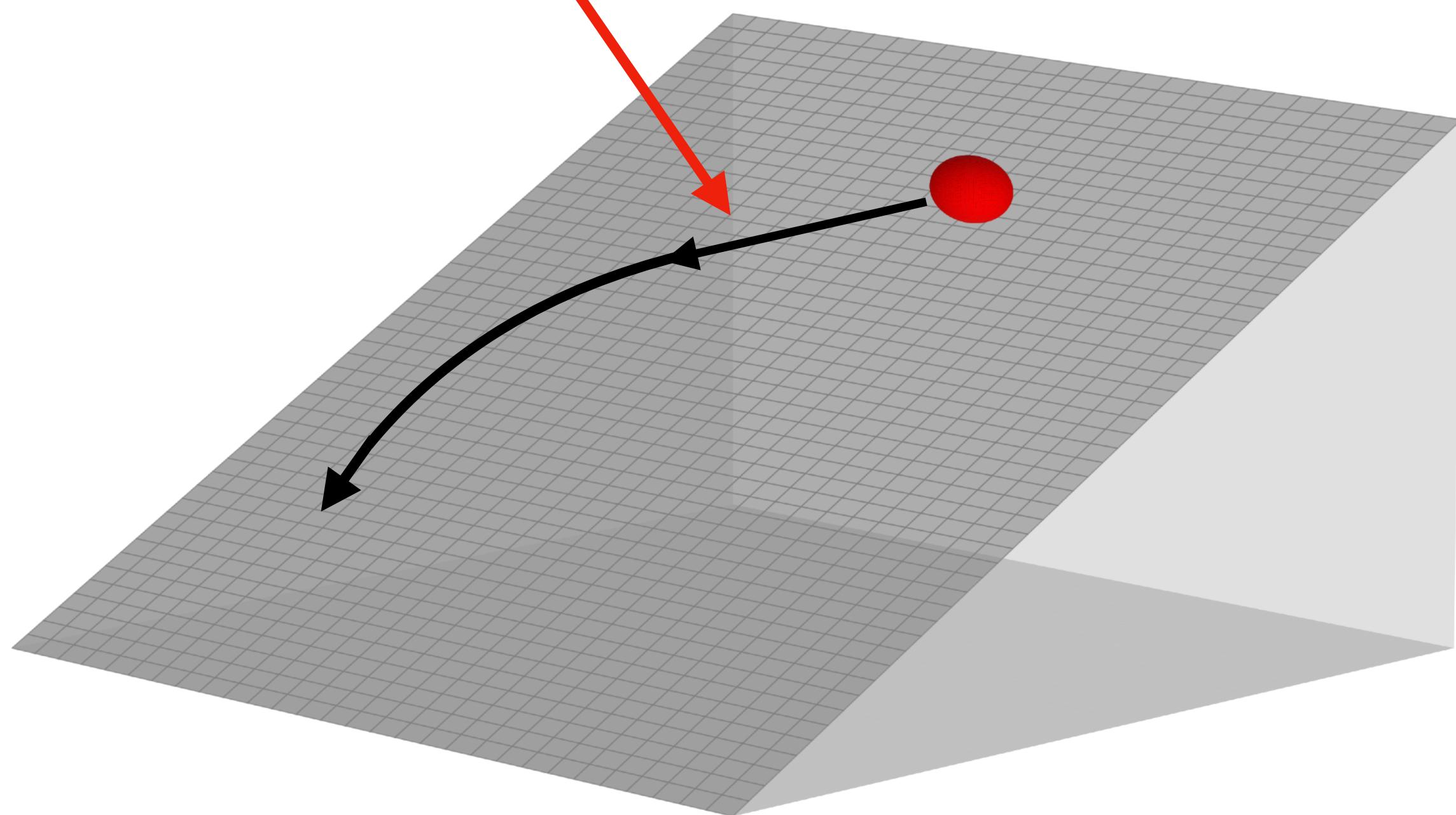
Rolling Ball

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t) dt$$



Rolling Ball

$$S'[x'(t)] = \int_{t_0}^{t_1} L'(x', \dot{x}', t) dt$$



Hamilton's Principle of Stationarity

“The action of a path followed by a particle moving in a potential is insensitive to first order perturbations in the path.”

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t) dt$$

From Hamilton to Euler-Lagrange

$$x(t) \rightarrow x(t) + \epsilon\eta(t)$$

From Hamilton to Euler-Lagrange

$$x(t) \rightarrow x(t) + \epsilon\eta(t)$$

$$S[x(t) + \epsilon\eta(t)] \approx \dots$$

From Hamilton to Euler-Lagrange

$$x(t) \rightarrow x(t) + \epsilon \eta(t)$$

$$S[x(t) + \epsilon \eta(t)] \approx S[x(t)] + \frac{\delta S}{\delta \epsilon} \Big|_{\epsilon=0} \epsilon + \dots$$

From Hamilton to Euler-Lagrange

$$x(t) \rightarrow x(t) + \epsilon\eta(t)$$

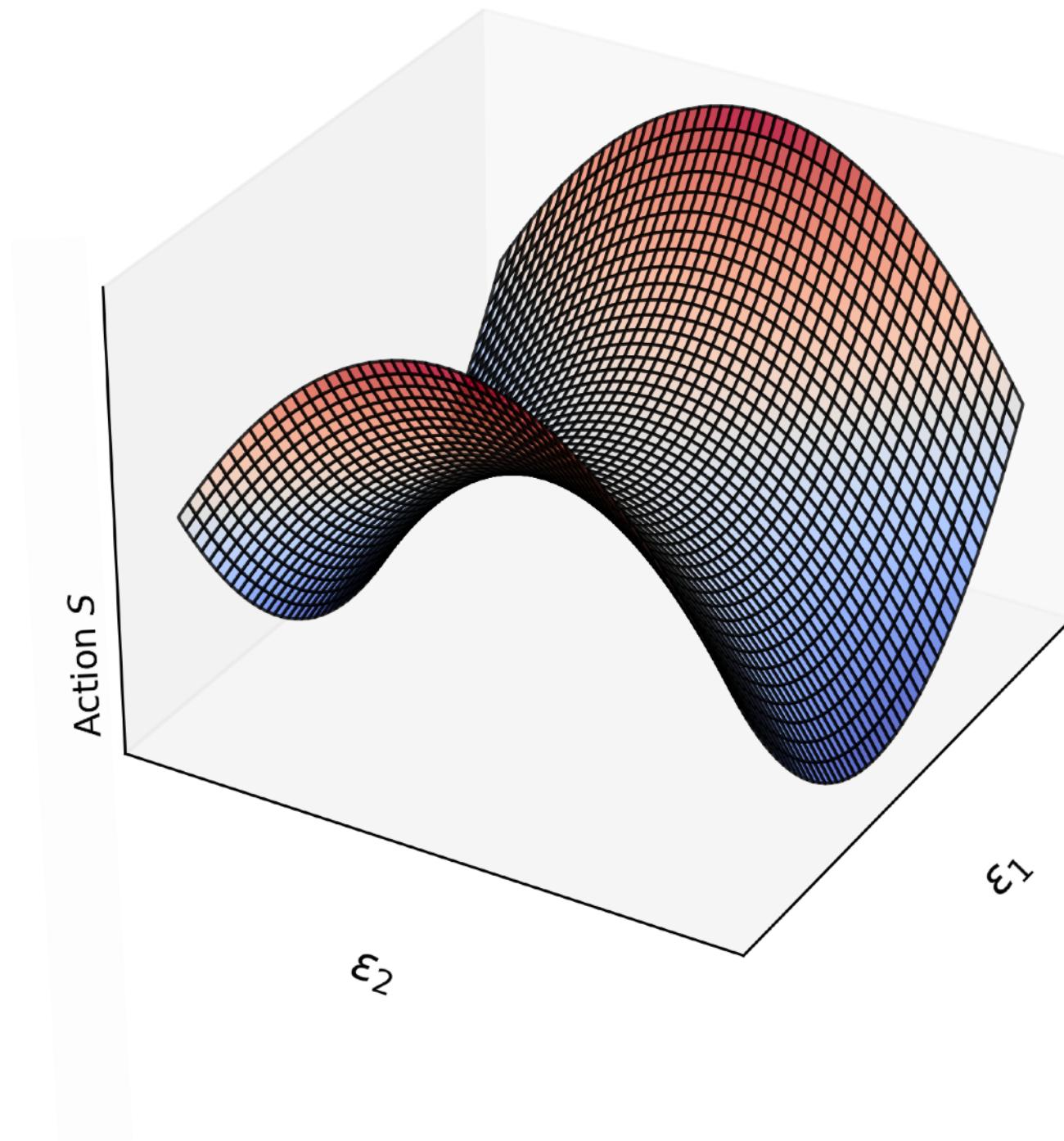
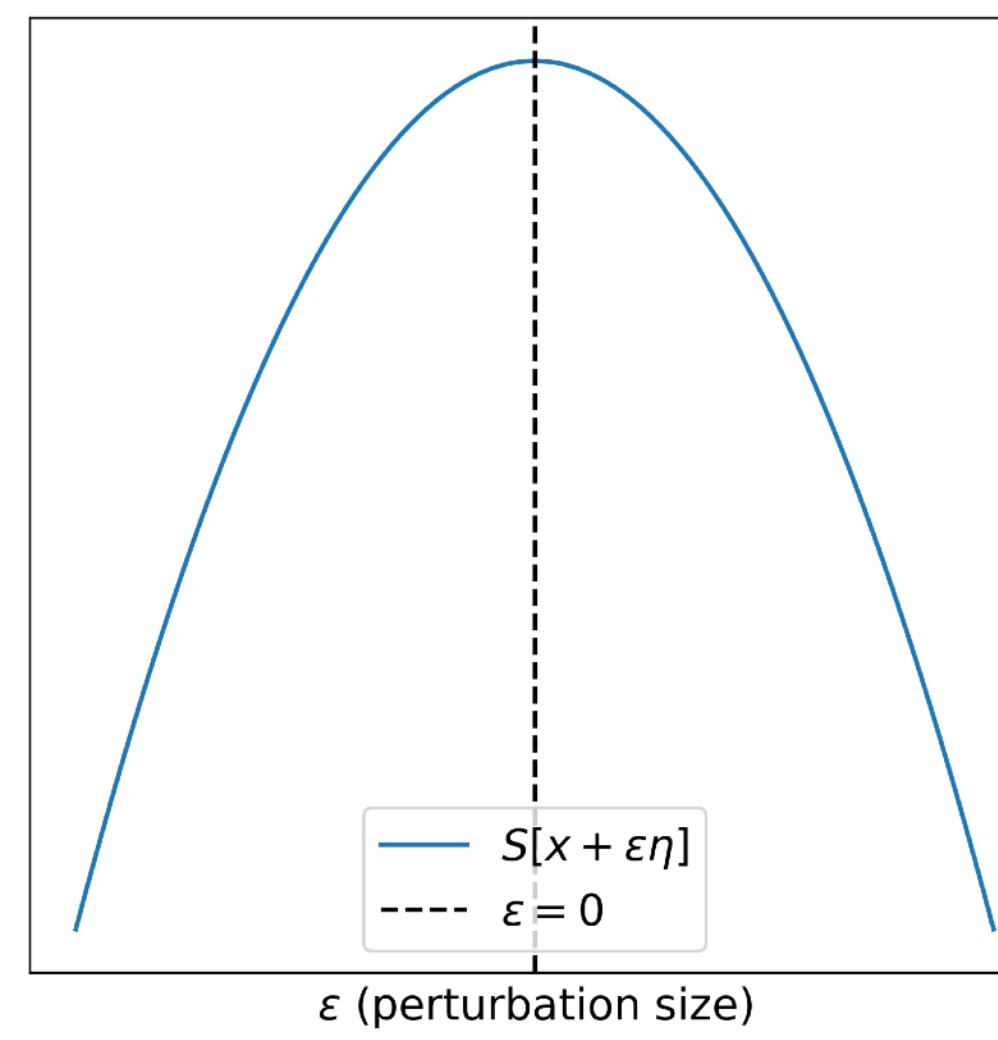
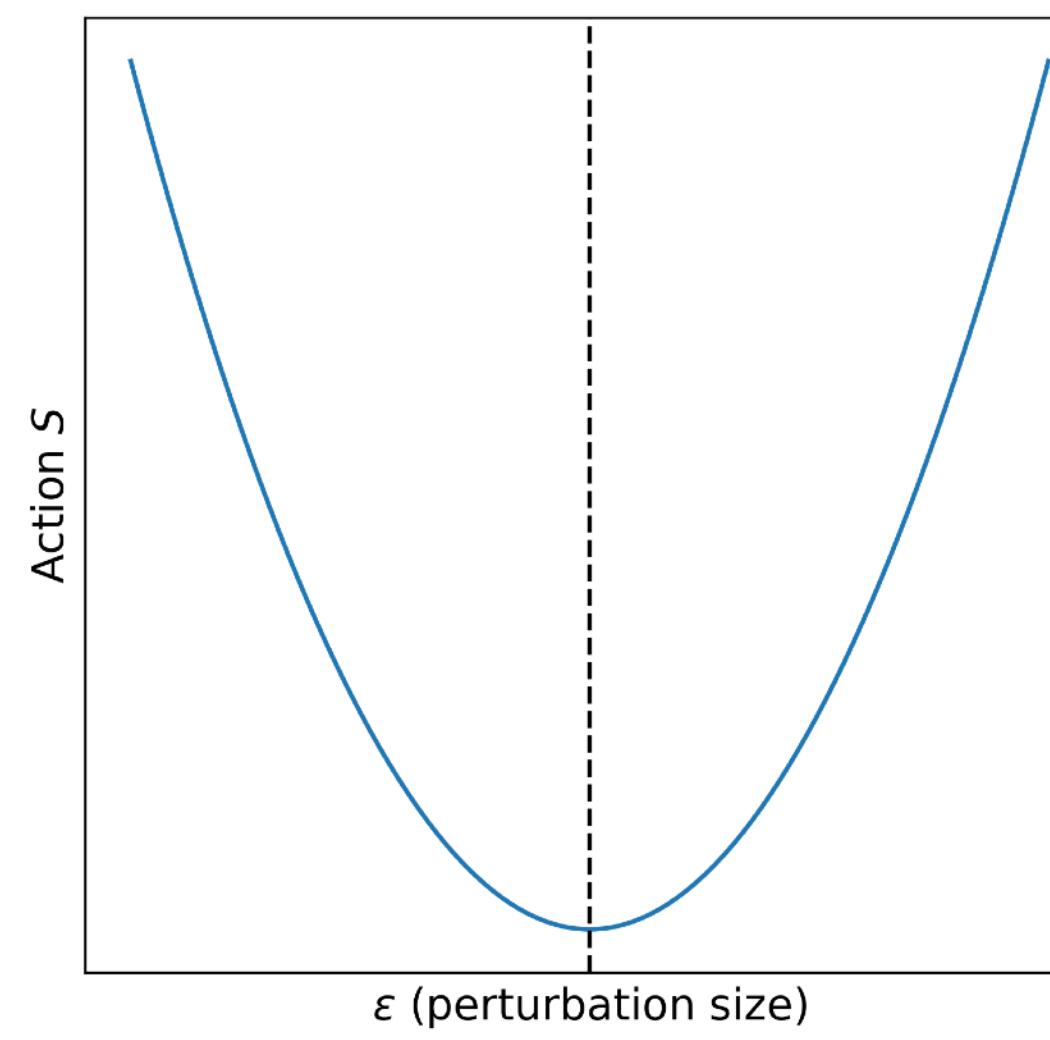
$$S[x(t) + \epsilon\eta(t)] \approx S[x(t)] + \frac{\delta S}{\delta \epsilon} \Big|_{\epsilon=0} \epsilon + \dots$$

Hamilton's Principle requires $\frac{\delta S}{\delta \epsilon} \Big|_{\epsilon=0} = 0$

“The action of a path followed by a particle moving in a potential is insensitive to first order perturbations in the path.”

From Hamilton to Euler-Lagrange

Hamilton's Principle requires $\left. \frac{\delta S}{\delta \epsilon} \right|_{\epsilon=0} = 0$



From Hamilton to Euler-Lagrange

$$S[x(t) + \epsilon\eta(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t) dt$$

From Hamilton to Euler-Lagrange

$$S[x(t) + \epsilon\eta(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t) dt$$

$$\frac{\delta S}{\delta \epsilon} \Big|_{\epsilon=0} = \int_{t_0}^{t_1} \frac{\delta L}{\delta x} \frac{\delta(x + \epsilon\eta)}{\delta \epsilon} + \frac{\delta L}{\delta \dot{x}} \frac{\delta(\dot{x} + \epsilon\dot{\eta})}{\delta \epsilon} dt = 0$$

From Hamilton to Euler-Lagrange

$$S[x(t) + \epsilon\eta(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t) dt$$

$$\frac{\delta S}{\delta \epsilon} \Big|_{\epsilon=0} = \int_{t_0}^{t_1} \frac{\delta L}{\delta x} \frac{\delta(x + \epsilon\eta)}{\delta \epsilon} + \frac{\delta L}{\delta \dot{x}} \frac{\delta(\dot{x} + \epsilon\dot{\eta})}{\delta \epsilon} dt = 0$$

$$\frac{dS}{\delta \epsilon} \Big|_{\epsilon=0} = \int_{t_0}^{t_1} \left(\frac{\delta L}{\delta x} - \frac{\delta}{\delta t} \left(\frac{\delta L}{\delta \dot{x}} \right) \right) \color{red}{\eta(t)} dt = 0.$$

From Hamilton to Euler-Lagrange

$$\frac{dS}{\delta\epsilon} \Big|_{\epsilon=0} = \int_{t_0}^{t_1} \left(\frac{\delta L}{\delta x} - \frac{\delta}{\delta t} \left(\frac{\delta L}{\delta \dot{x}} \right) \right) \eta(t) dt = 0.$$

From this we can derive the Euler-Lagrange equation

$$\boxed{\frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0}$$

Euler-Lagrange

Hamilton's Principle Classical Dynamics

Dr Harry Bevins

July 2025

1 Introduction

As physicists, we are deeply interested in how complex systems evolve over time. Historically, we have used observations to inform our understanding of the forces acting on objects in systems. A classic example is the study of orbital mechanics. Between 1609 and 1619 Kepler outlined his three laws of orbital motion which he inferred from observations taken by Tycho Brahe of the motion of



Euler-Lagrange

$$\frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0$$

Hamilton's Principle leads us to the Euler-Lagrange equation and its this that describes the dynamics of a system.

Using Euler-Lagrange we can derive all of Newton's Laws and much more!

Hamilton's Principle in Action

Lets think about a particle in free space i.e.

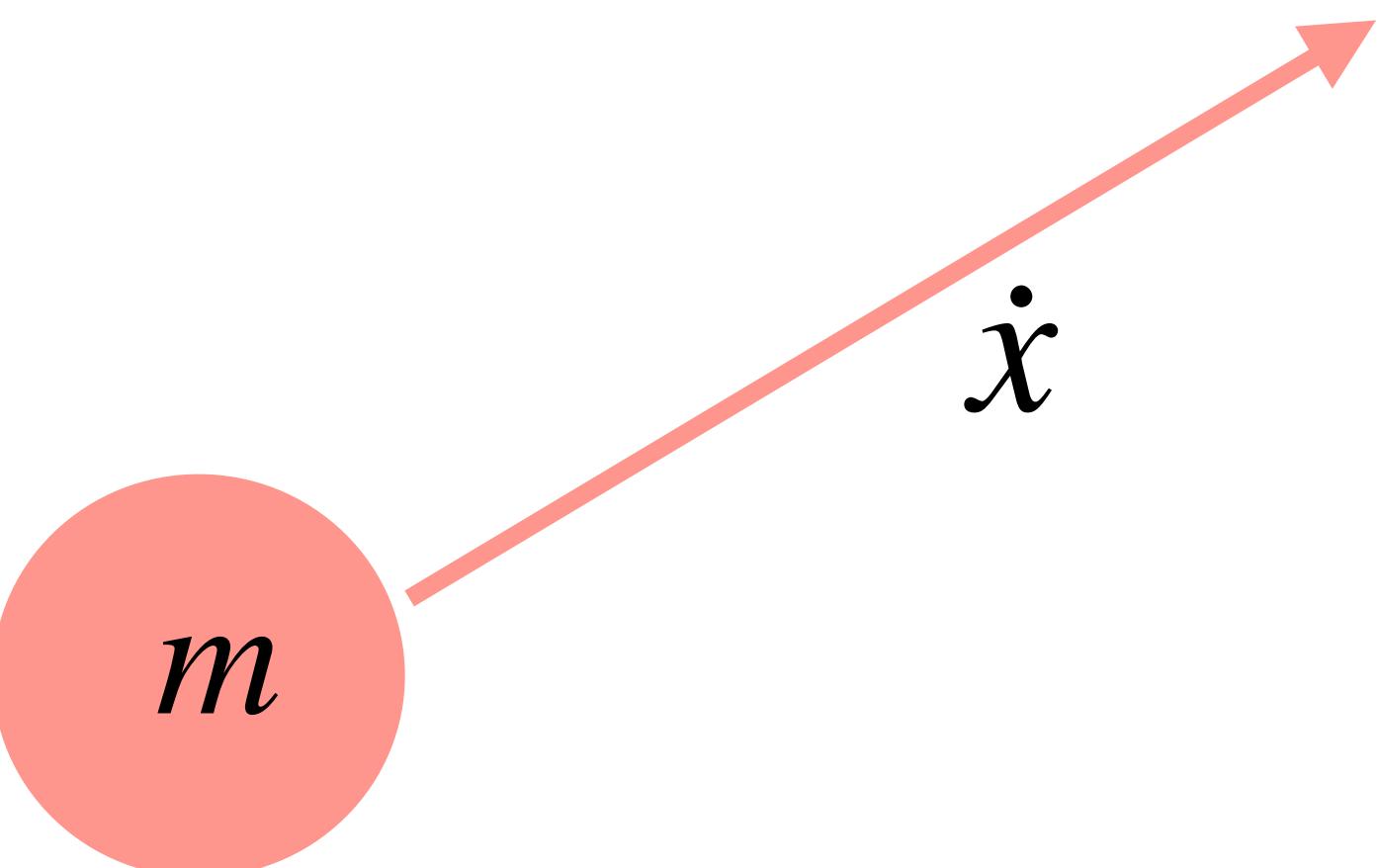
$$V(x, t) = 0$$

Our particle has some velocity \dot{x} meaning it has kinetic energy

$$T = \frac{1}{2}m\dot{x}^2$$

So it's Lagrangian is given by

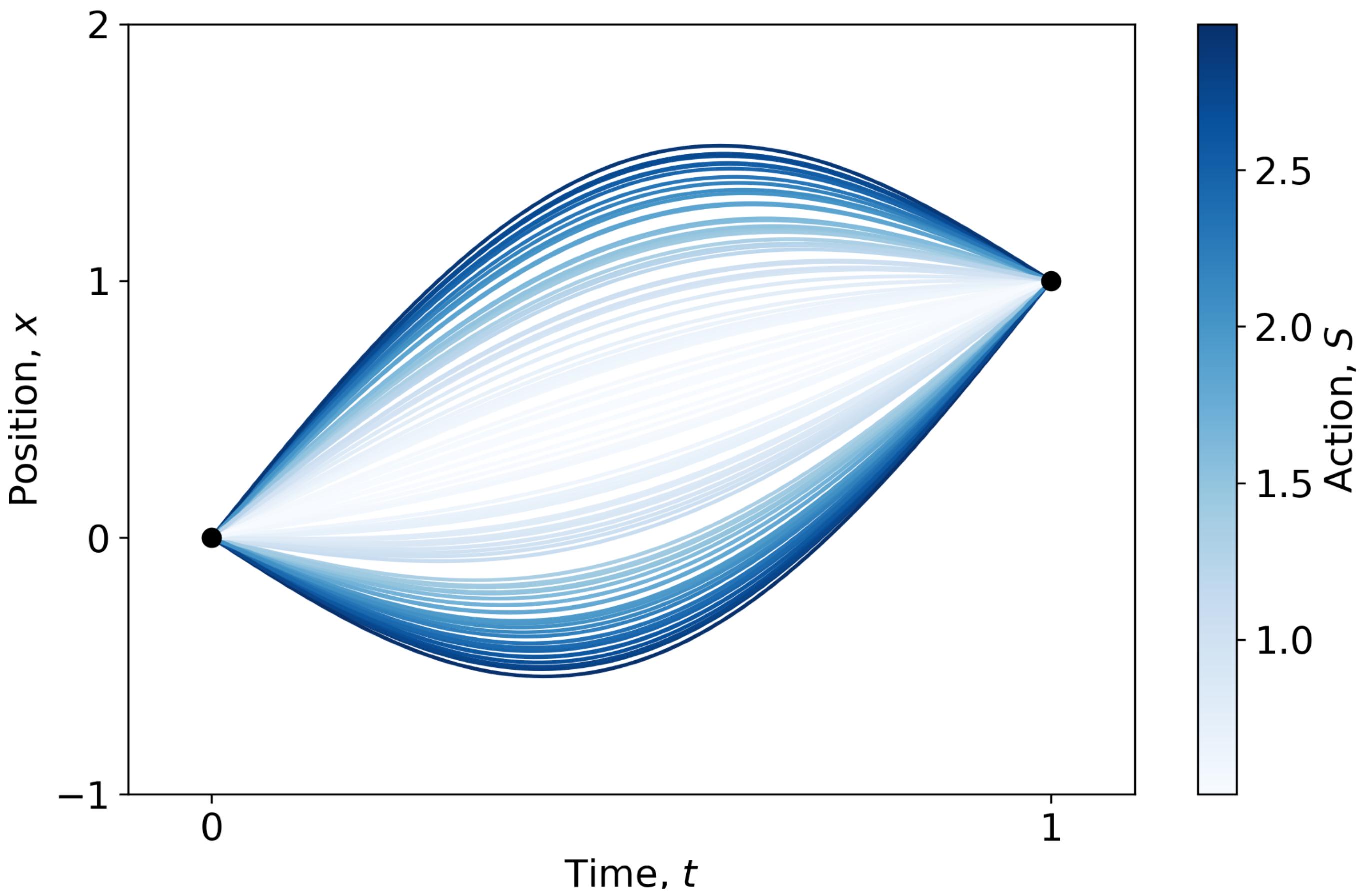
$$L = T = \frac{1}{2}m\dot{x}^2$$



Hamilton's Principle in Action

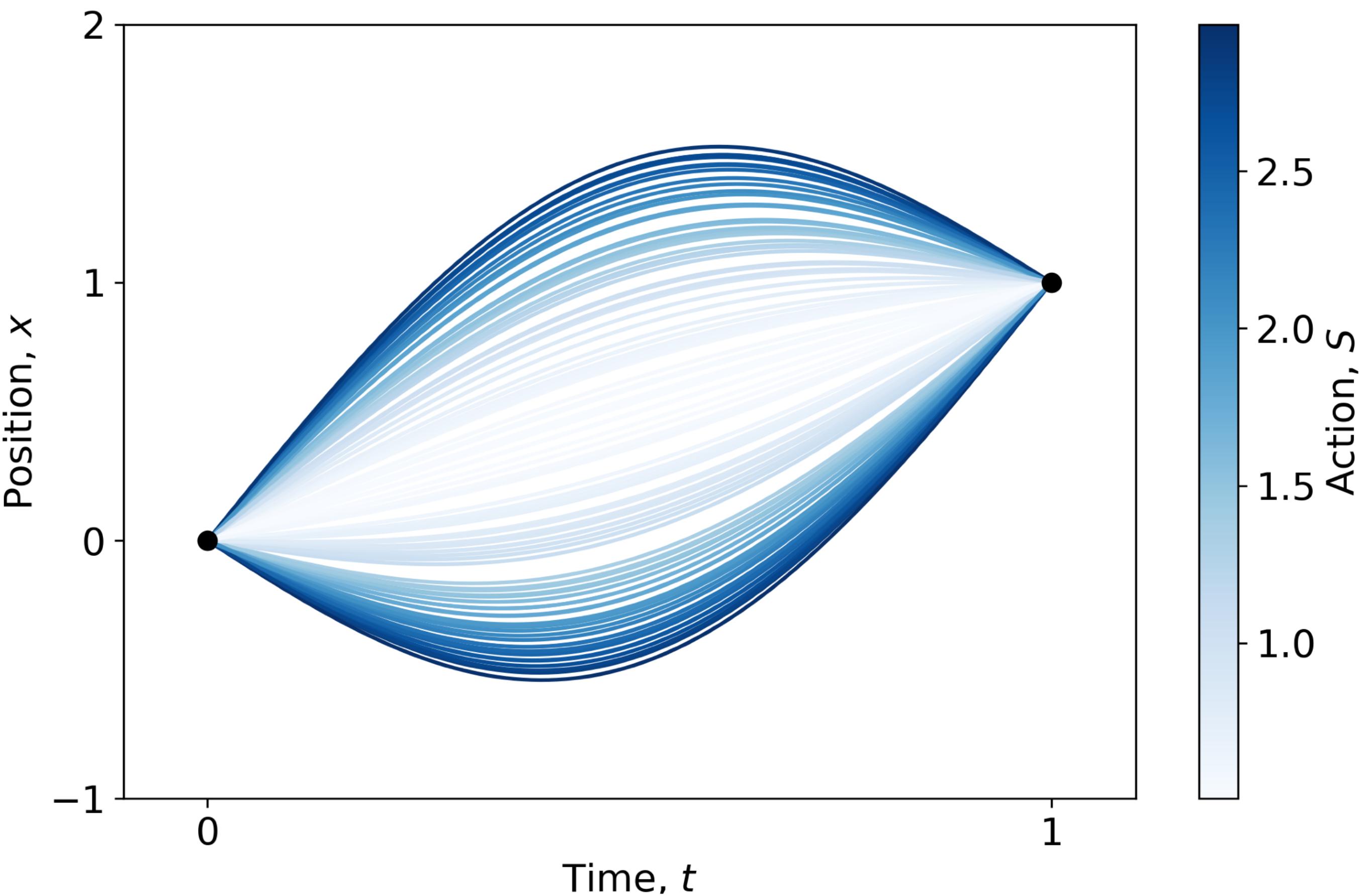
So the action is given by

$$S = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \frac{1}{2} m \dot{x}^2 dt$$



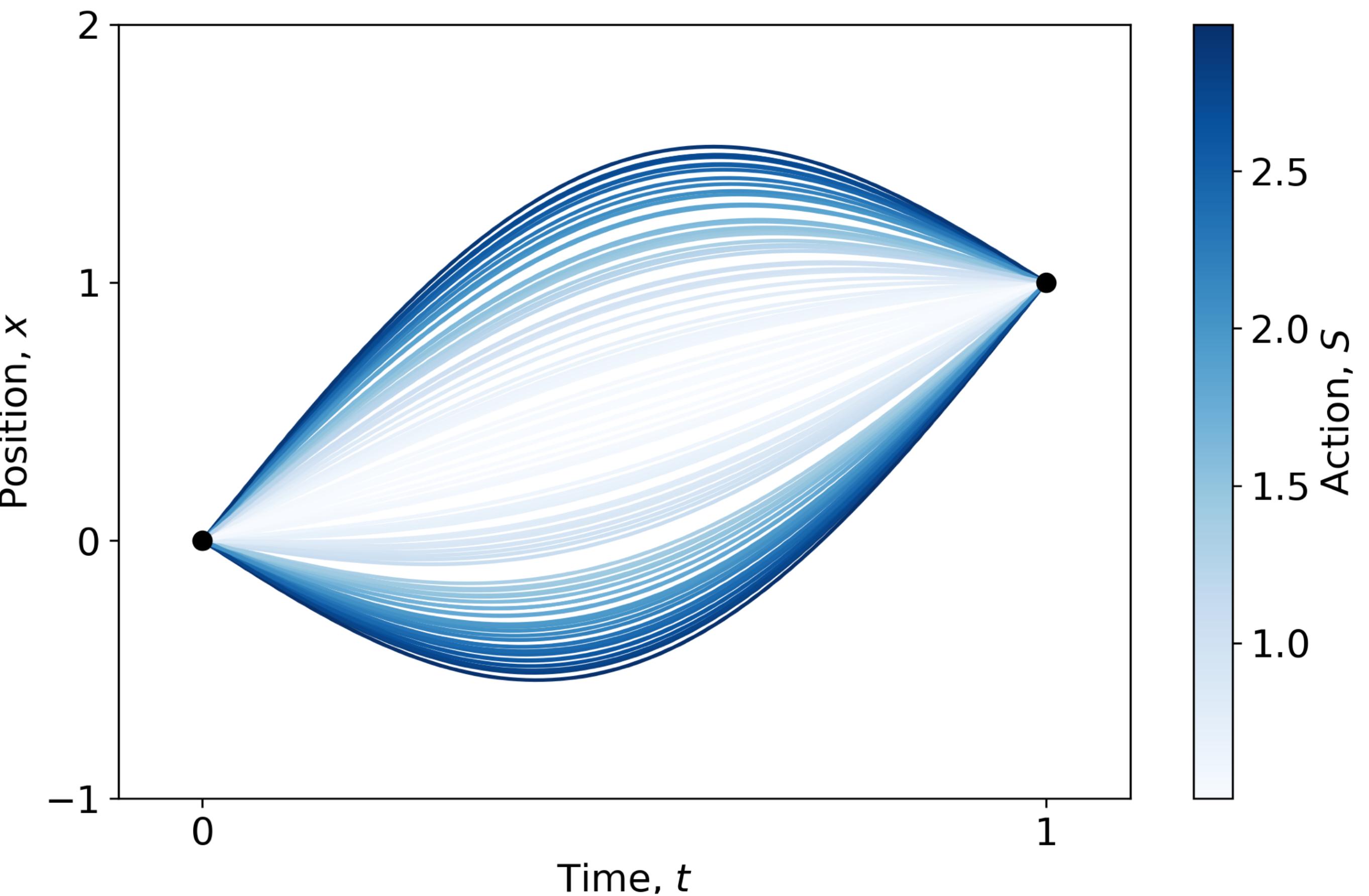
Hamilton's Principle in Action

$$\frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2} m \dot{x}^2$$



Hamilton's Principle in Action

$$\frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2} m \dot{x}^2$$
$$\frac{\delta}{\delta t} \frac{\delta}{\delta \dot{x}} \left(\frac{1}{2} m \dot{x}^2 \right) - \frac{\delta}{\delta x} \left(\frac{1}{2} m \dot{x}^2 \right) = 0$$

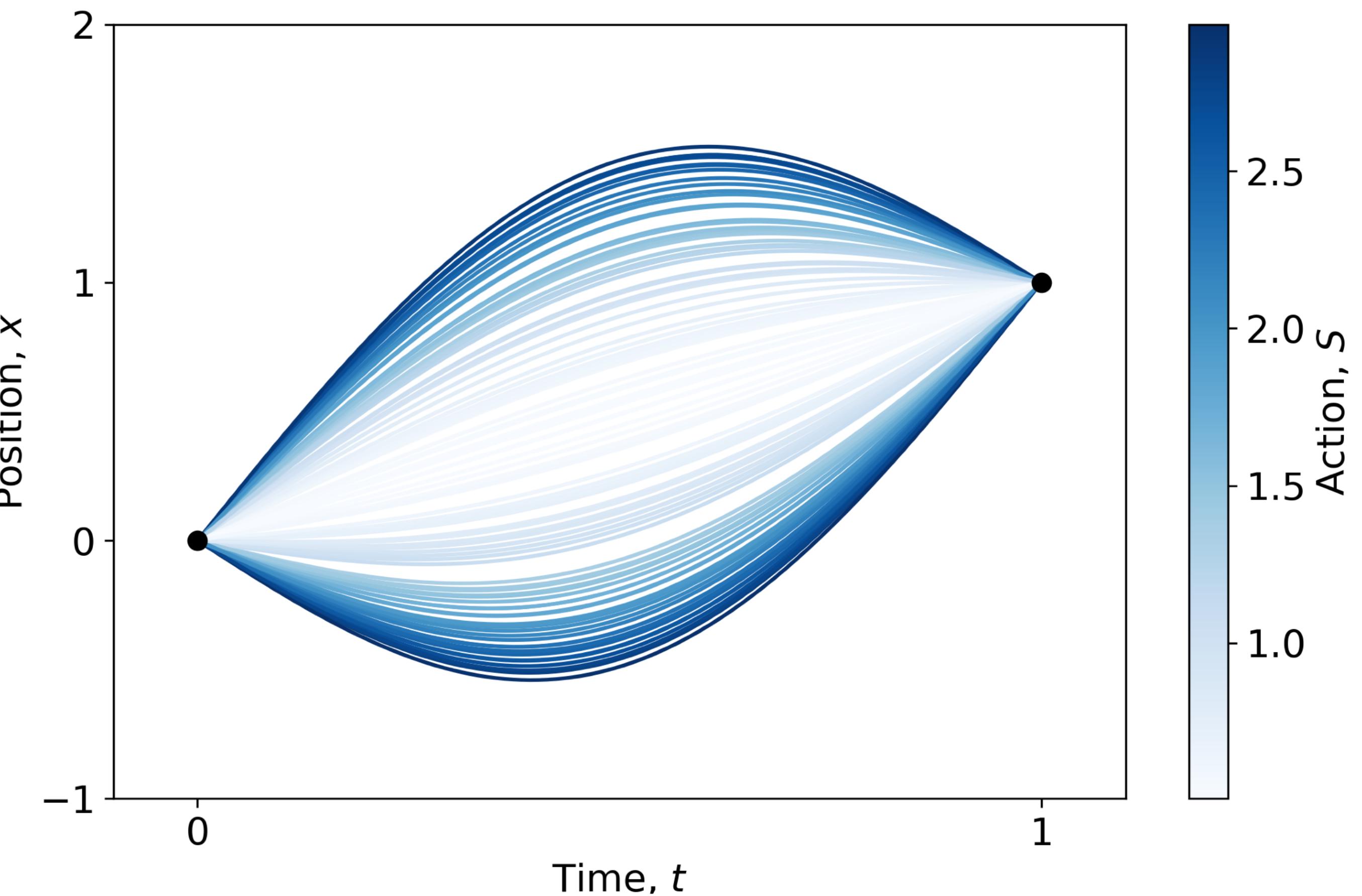


Hamilton's Principle in Action

$$\frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2} m \dot{x}^2$$

$$\frac{\delta}{\delta t} \frac{\delta}{\delta \dot{x}} \left(\frac{1}{2} m \dot{x}^2 \right) - \frac{\delta}{\delta x} \left(\frac{1}{2} m \dot{x}^2 \right) = 0$$

$$\frac{\delta}{\delta t} (m \dot{x}) = 0$$



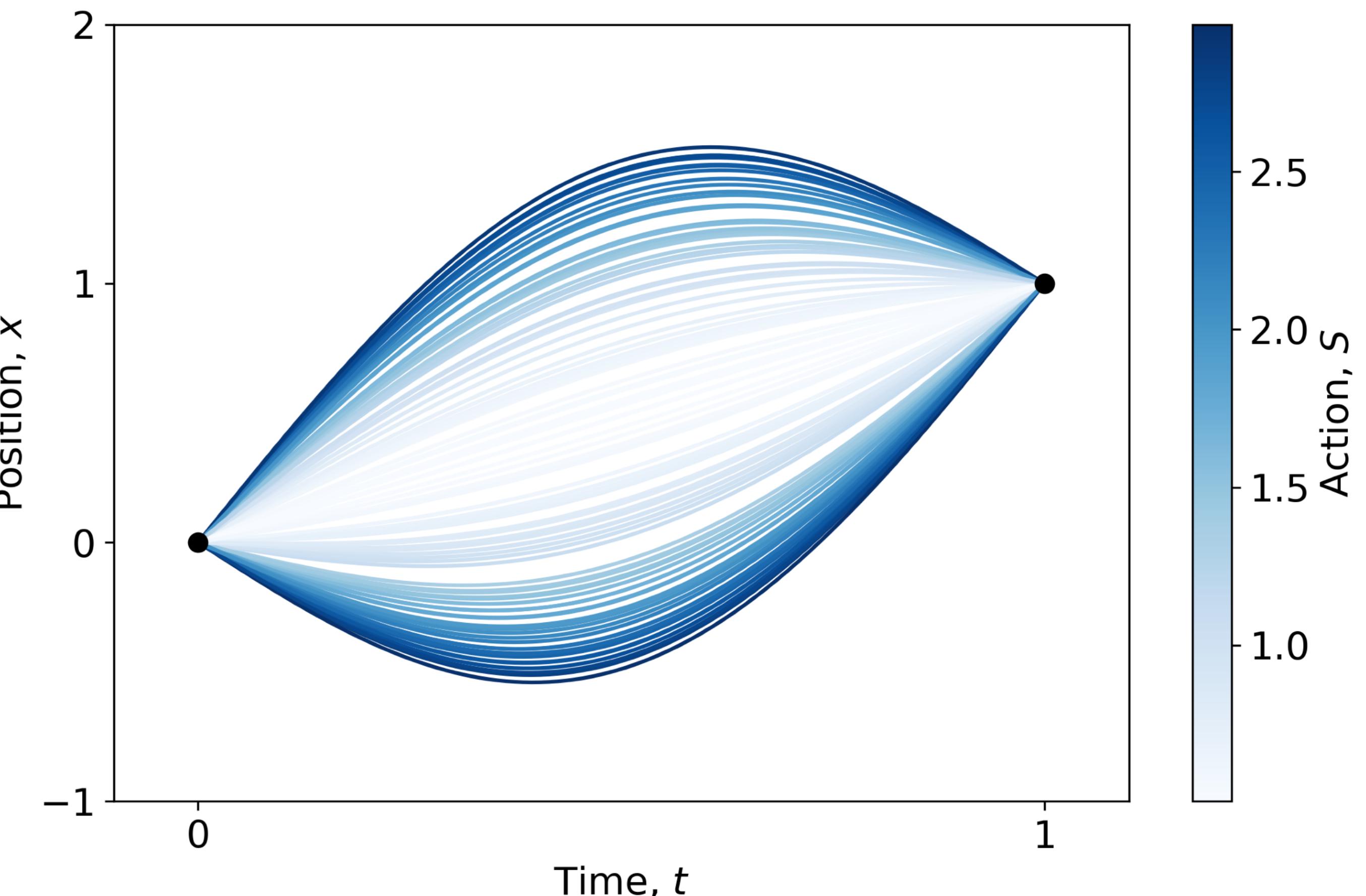
Hamilton's Principle in Action

$$\frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2} m \dot{x}^2$$

$$\frac{\delta}{\delta t} \frac{\delta}{\delta \dot{x}} \left(\frac{1}{2} m \dot{x}^2 \right) - \frac{\delta}{\delta x} \left(\frac{1}{2} m \dot{x}^2 \right) = 0$$

$$\frac{\delta}{\delta t} (m \dot{x}) = 0$$

$$m \ddot{x} = 0$$



Hamilton's Principle in Action

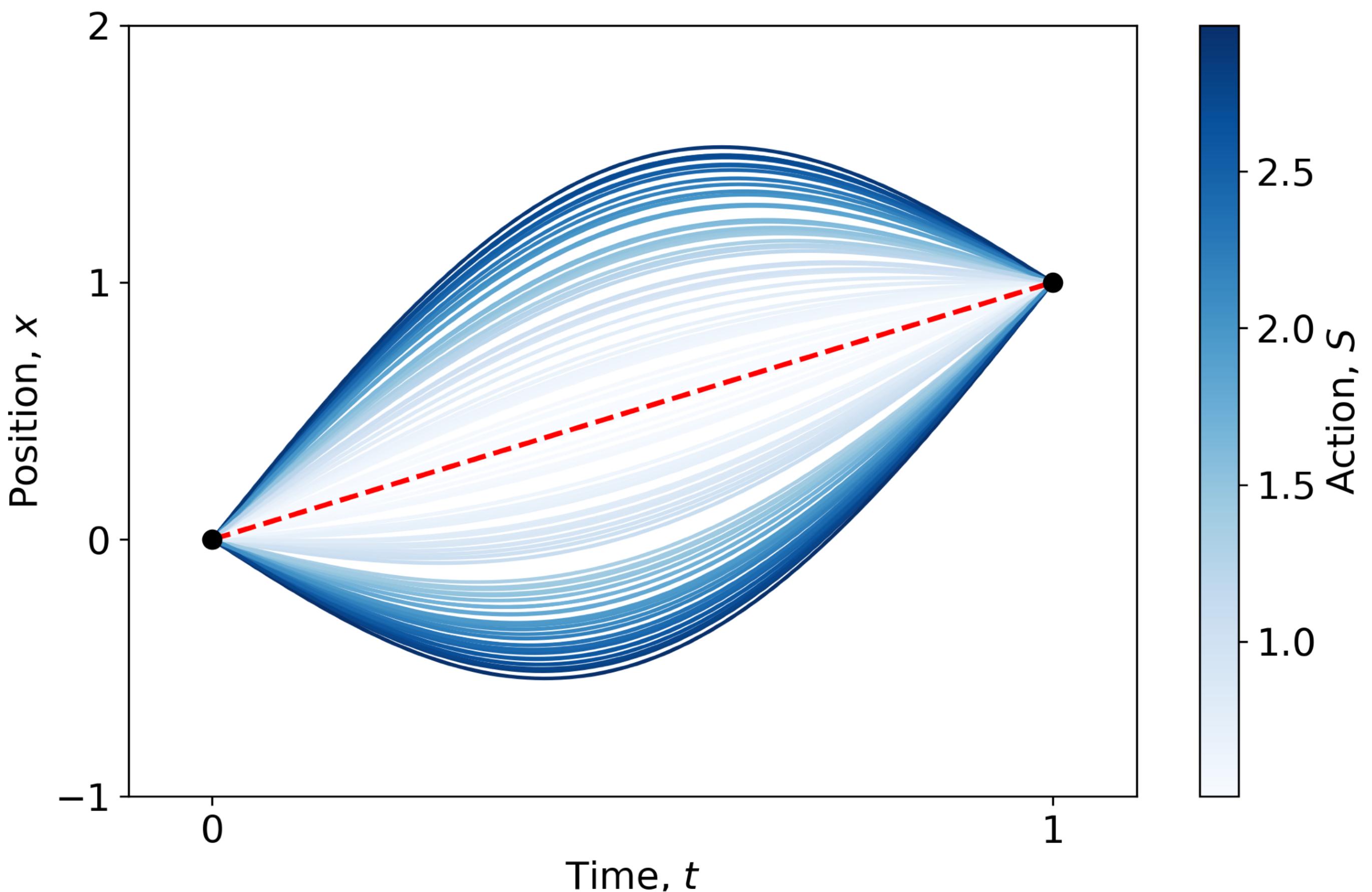
$$m\ddot{x} = 0$$

Integrate w.r.t. t
→

$$\dot{x} = \text{const} = v_0$$

Integrate w.r.t. t
→

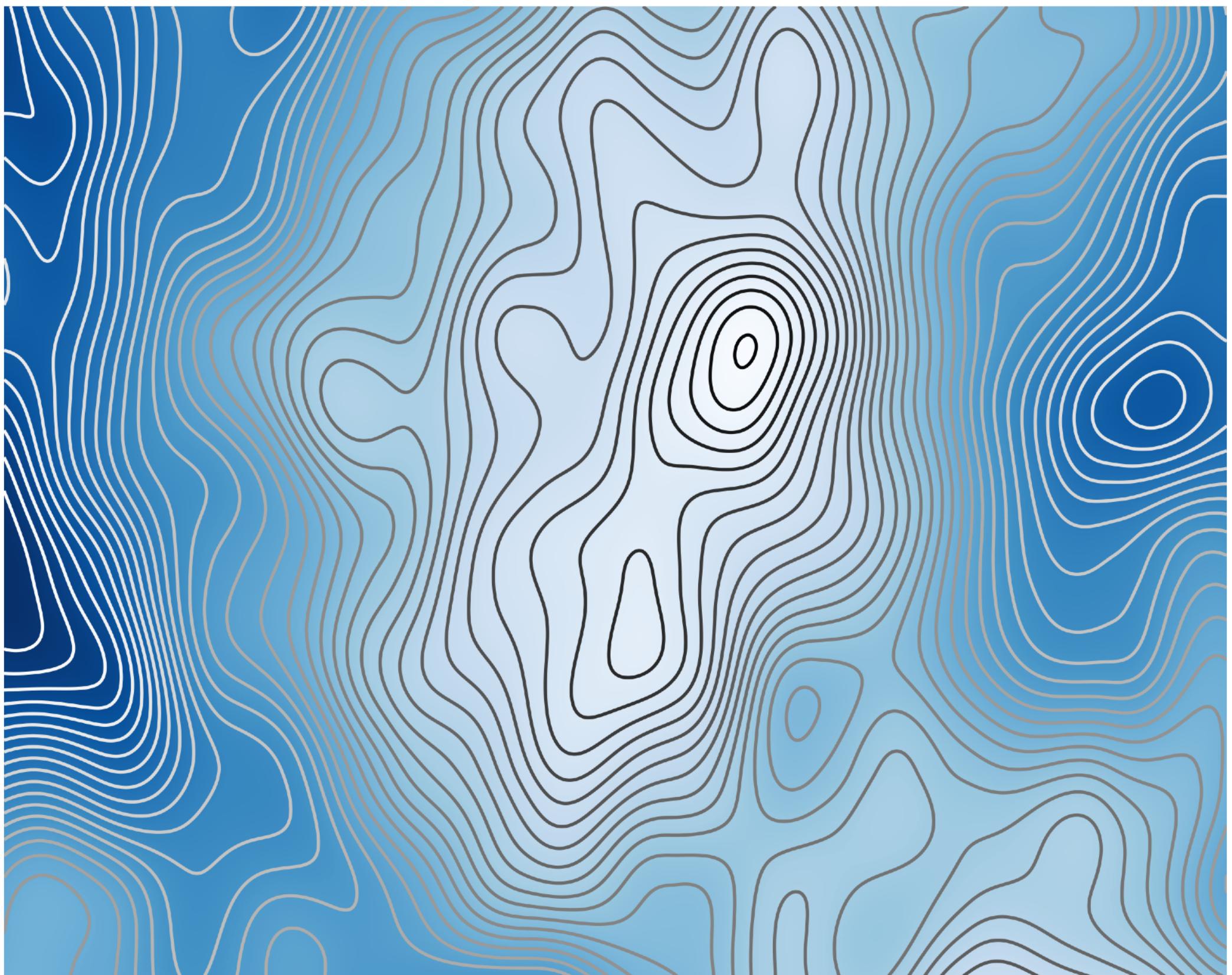
$$x = v_0 t + c$$



Arbitrary Potential

Let's place our particle in a non-zero potential

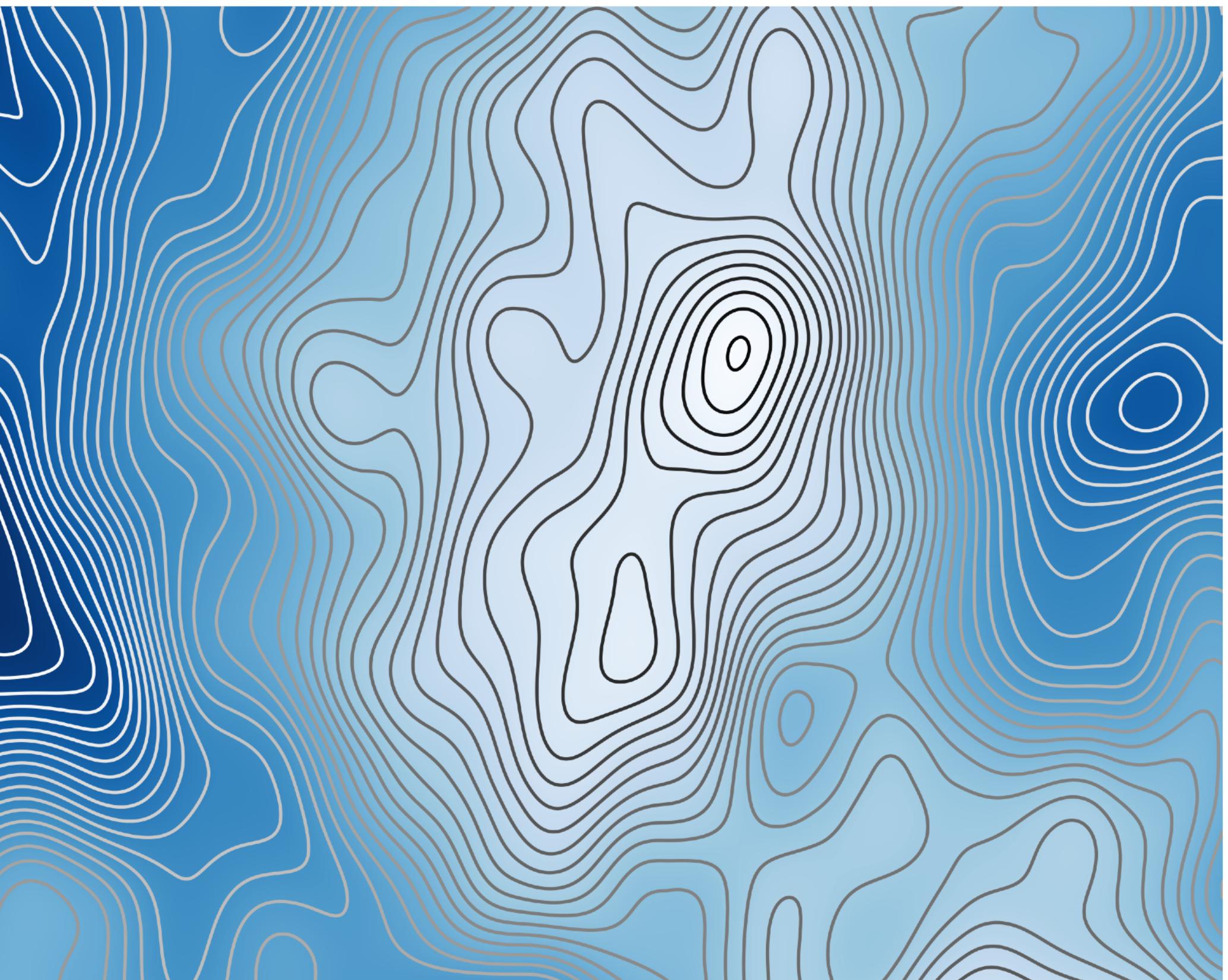
$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$



Arbitrary Potential

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x) \quad \frac{\delta}{\delta t} \left(\frac{\delta L}{\delta \dot{x}} \right) - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t} \frac{\delta}{\delta \dot{x}} \left(\frac{1}{2}m\dot{x}^2 - V(x) \right) = m\ddot{x}$$

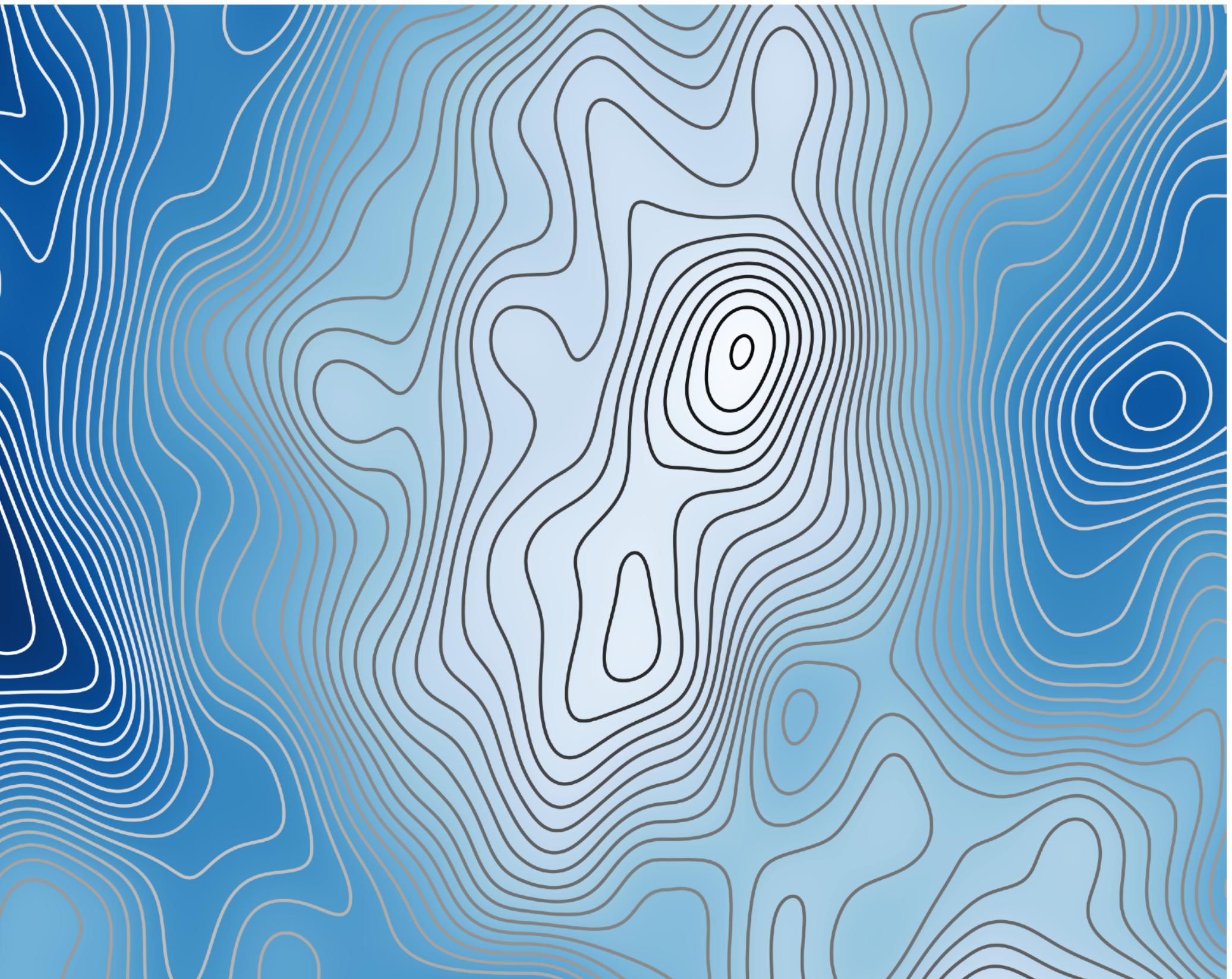


Arbitrary Potential

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x) \quad \frac{\delta}{\delta t} \left(\frac{\delta L}{\delta \dot{x}} \right) - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t} \frac{\delta}{\delta \dot{x}} \left(\frac{1}{2}m\dot{x}^2 - V(x) \right) = m\ddot{x}$$

$$-\frac{\delta}{\delta x} \left(\frac{1}{2}m\dot{x}^2 - V(x) \right) = \frac{\delta V}{\delta x}$$

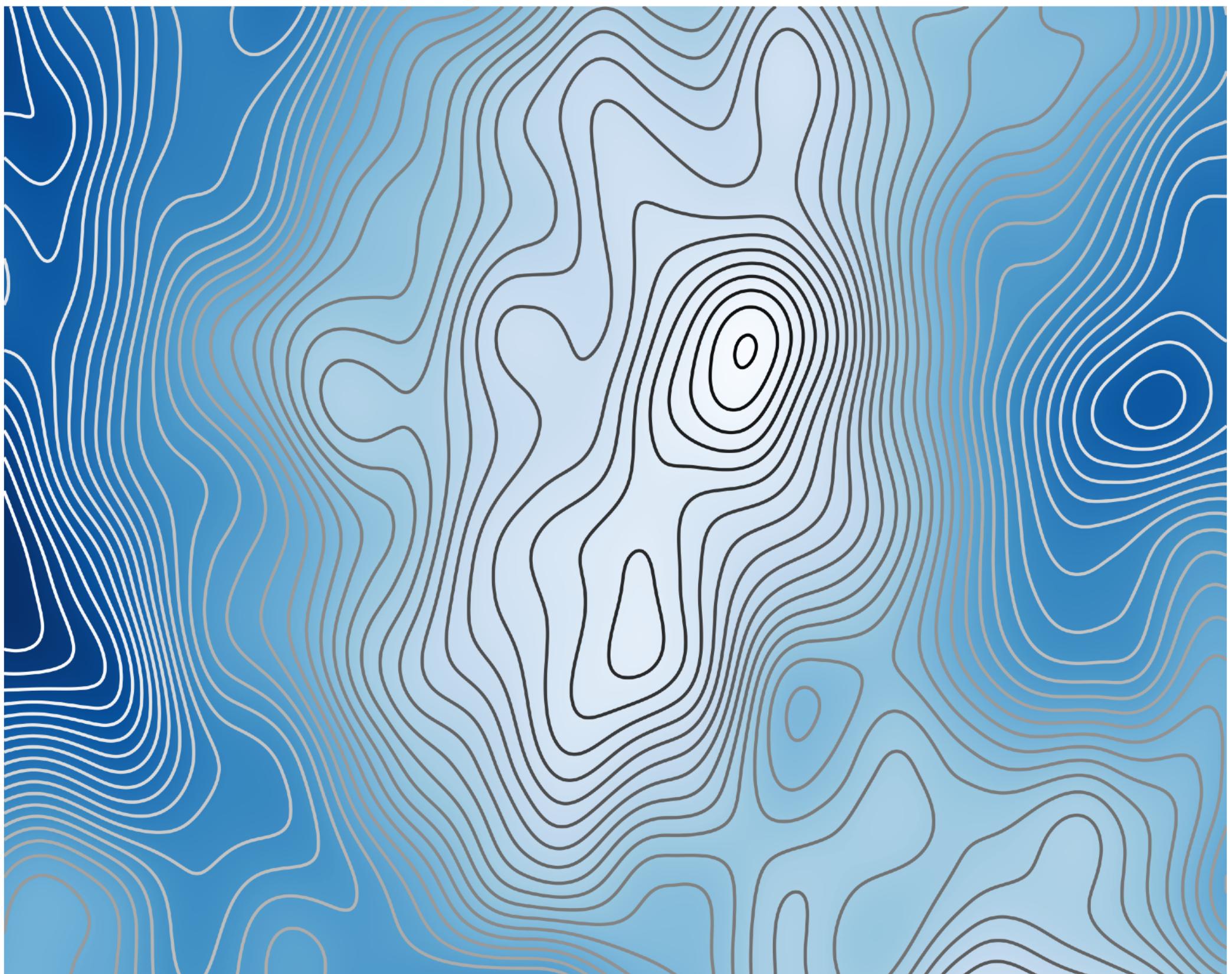


Arbitrary Potential

$$\frac{\delta}{\delta t} \left(\frac{\delta L}{\delta \dot{x}} \right) - \frac{\delta L}{\delta x} = 0$$
$$\frac{\delta}{\delta t} \frac{\delta}{\delta \dot{x}} \left(\frac{1}{2} m \dot{x}^2 - V(x) \right) = m \ddot{x}$$

$$-\frac{\delta}{\delta x} \left(\frac{1}{2} m \dot{x}^2 - V(x) \right) = \frac{\delta V}{\delta x}$$

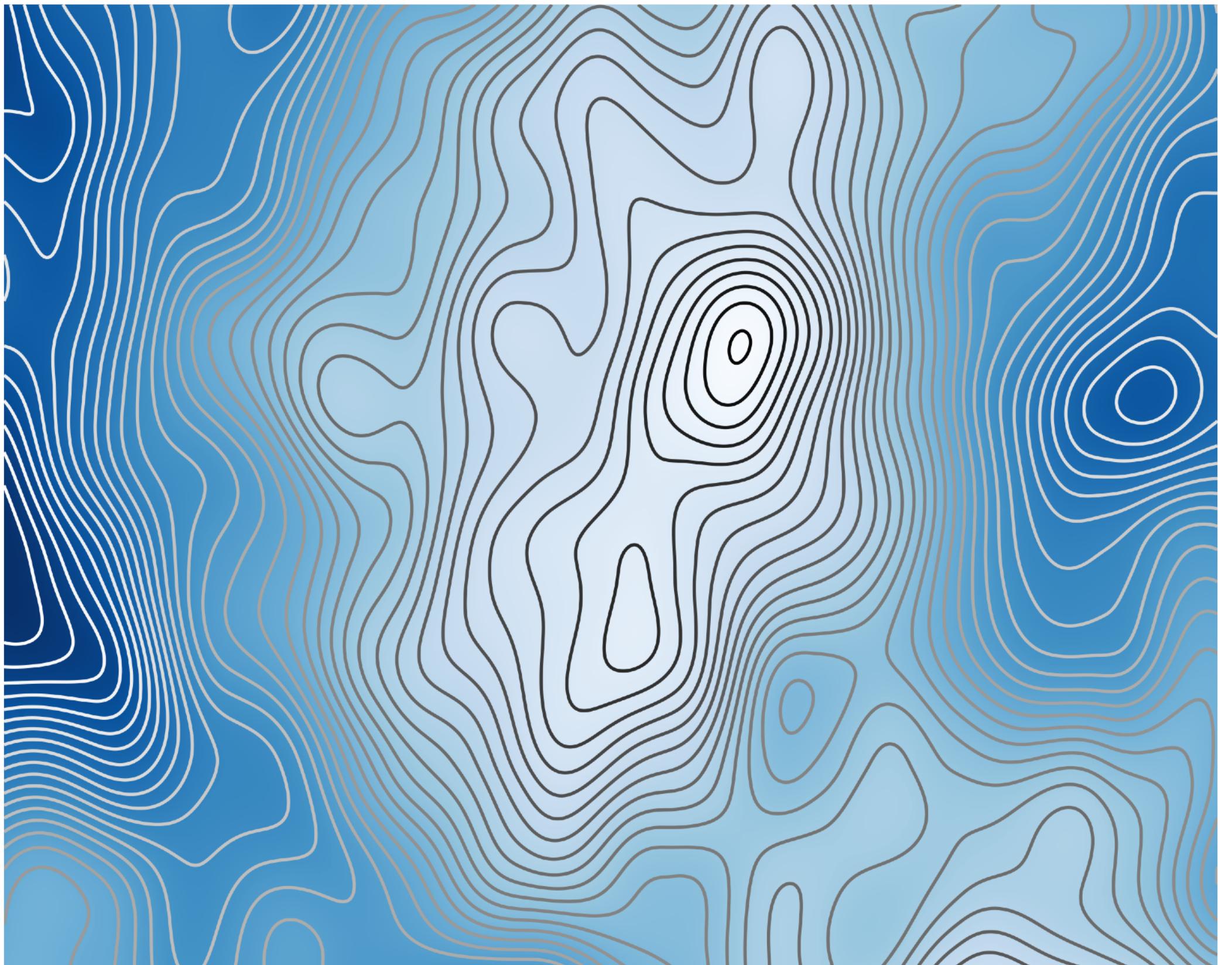
$$m \ddot{x} + \frac{\delta V}{\delta x} = 0 \longrightarrow m \ddot{x} = - \frac{\delta V}{\delta x}$$



Newton's Second Law

$$m\ddot{x} + \frac{\delta V}{\delta x} = 0 \longrightarrow m\ddot{x} = -\frac{\delta V}{\delta x}$$

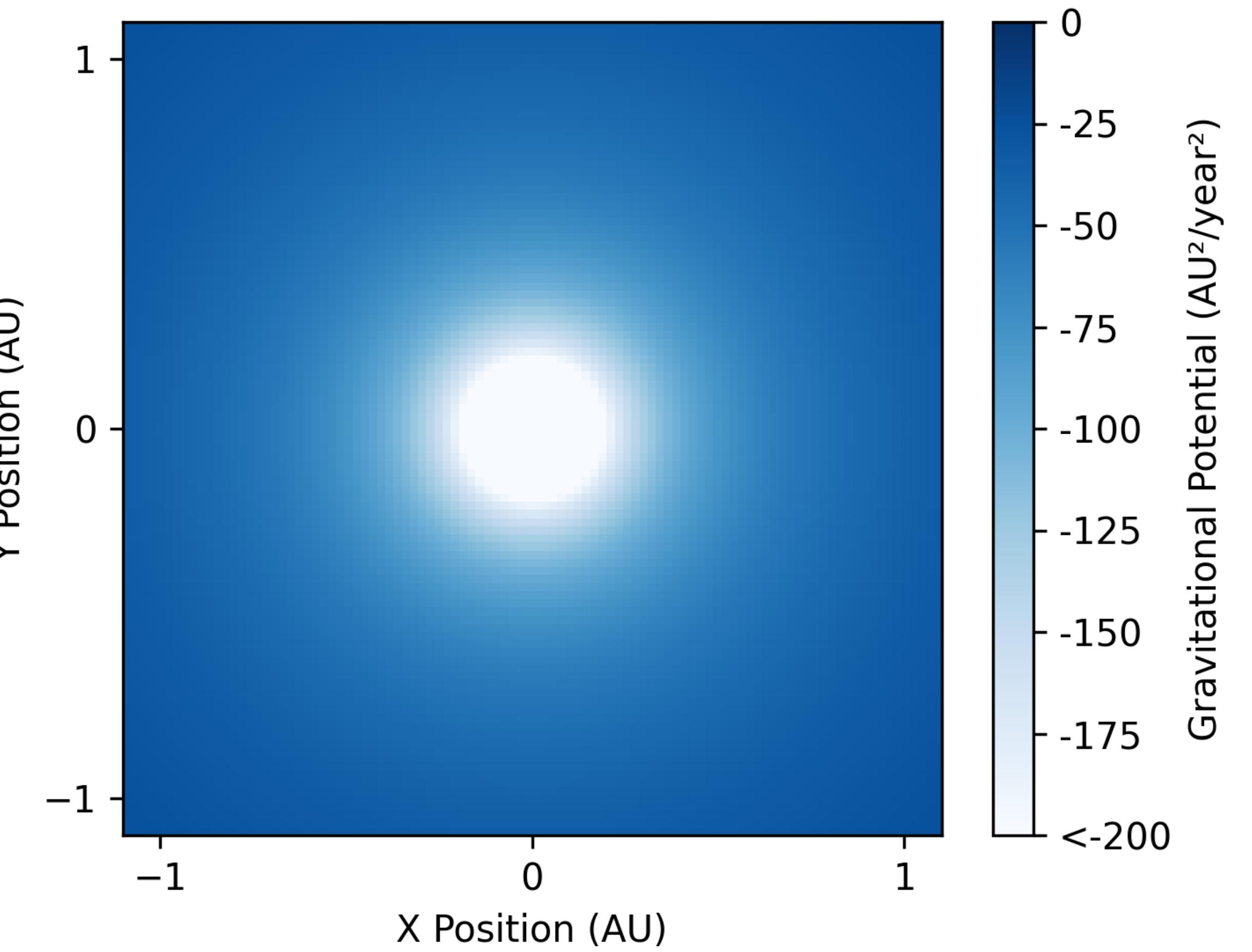
$$F = m\ddot{x} = -\frac{\delta V}{\delta x}$$



Newton's Law of Gravitation

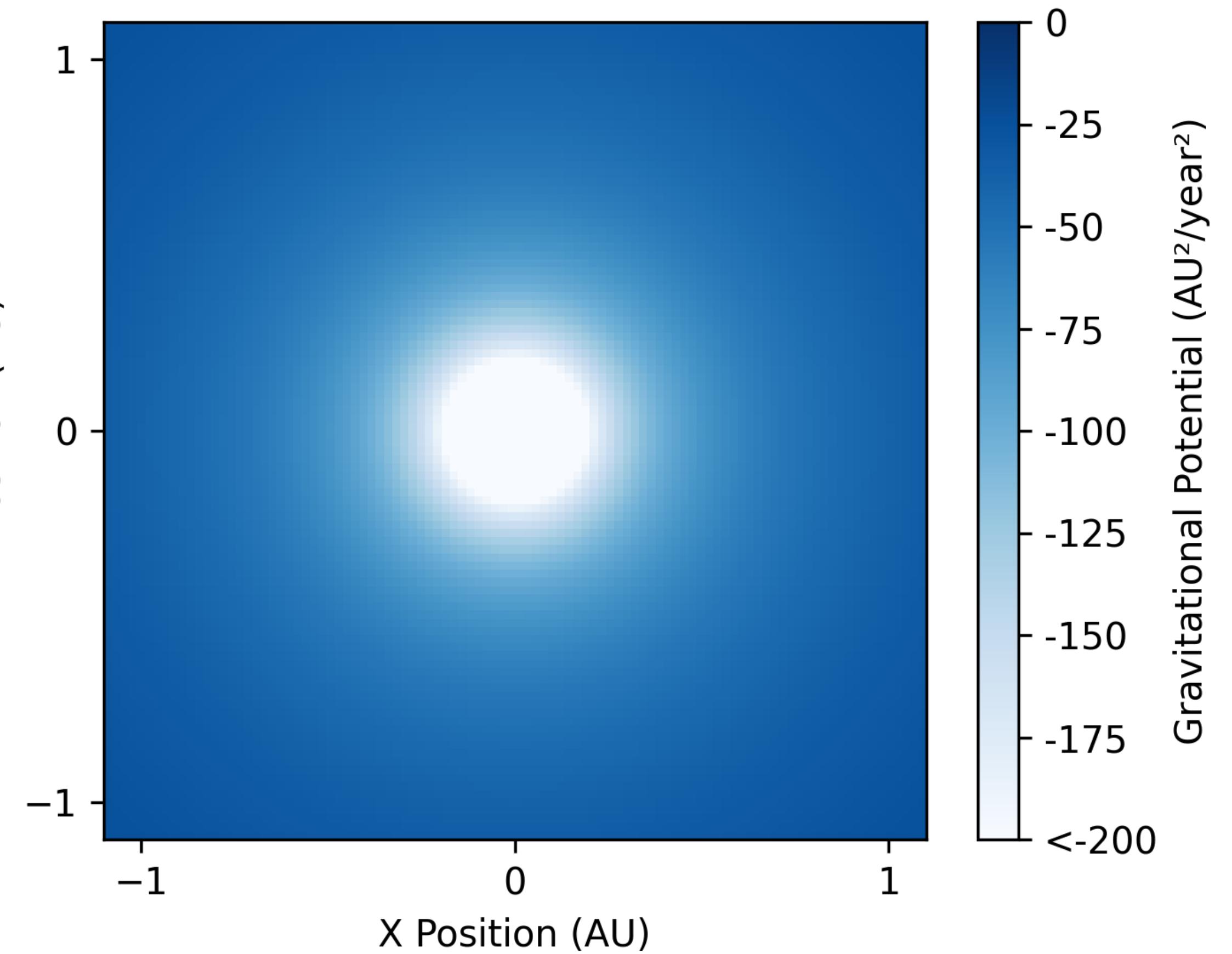
How about if we place our particle in a gravitational potential?

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$



Newton's Law of Gravitation

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$
$$\frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{r}} - \frac{\delta L}{\delta r} = 0$$
$$\frac{\delta L}{\delta \dot{r}} = m\dot{r}$$
$$\frac{\delta L}{\delta r} = -\frac{GMm}{r^2}$$
$$\frac{\delta}{\delta t} \left(\frac{\delta L}{\delta \dot{r}} \right) = m\ddot{r}$$



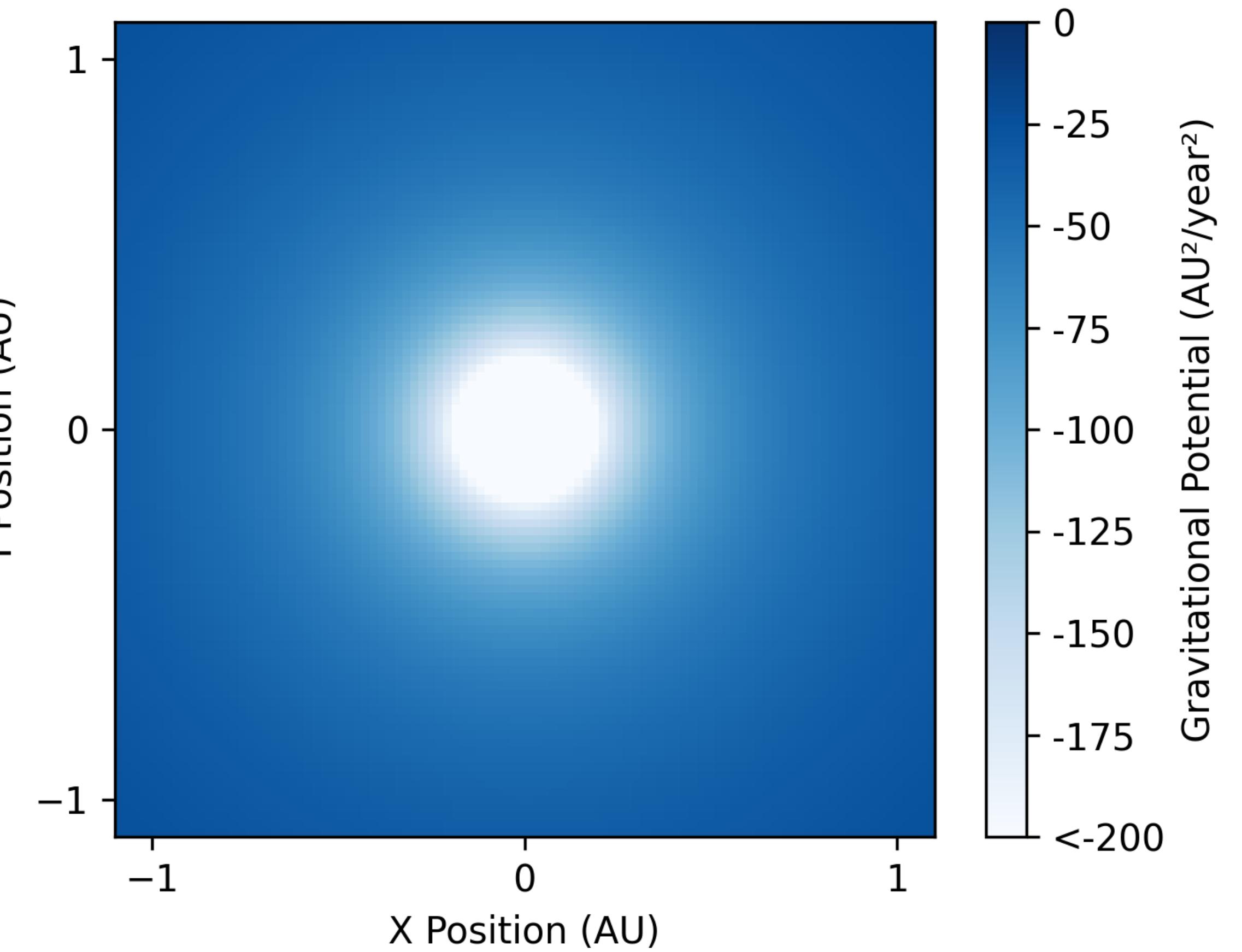
Newton's Law of Gravitation

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$

$$\frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{r}} - \frac{\delta L}{\delta r} = 0$$

$$\frac{\delta}{\delta t} \left(\frac{\delta L}{\delta \dot{r}} \right) = m\ddot{r}$$
$$\frac{\delta L}{\delta r} = -\frac{GMm}{r^2}$$

$$m\ddot{r} - \frac{GMm}{r^2} = 0 \rightarrow F = m\ddot{r} = \frac{GMm}{r^2}$$

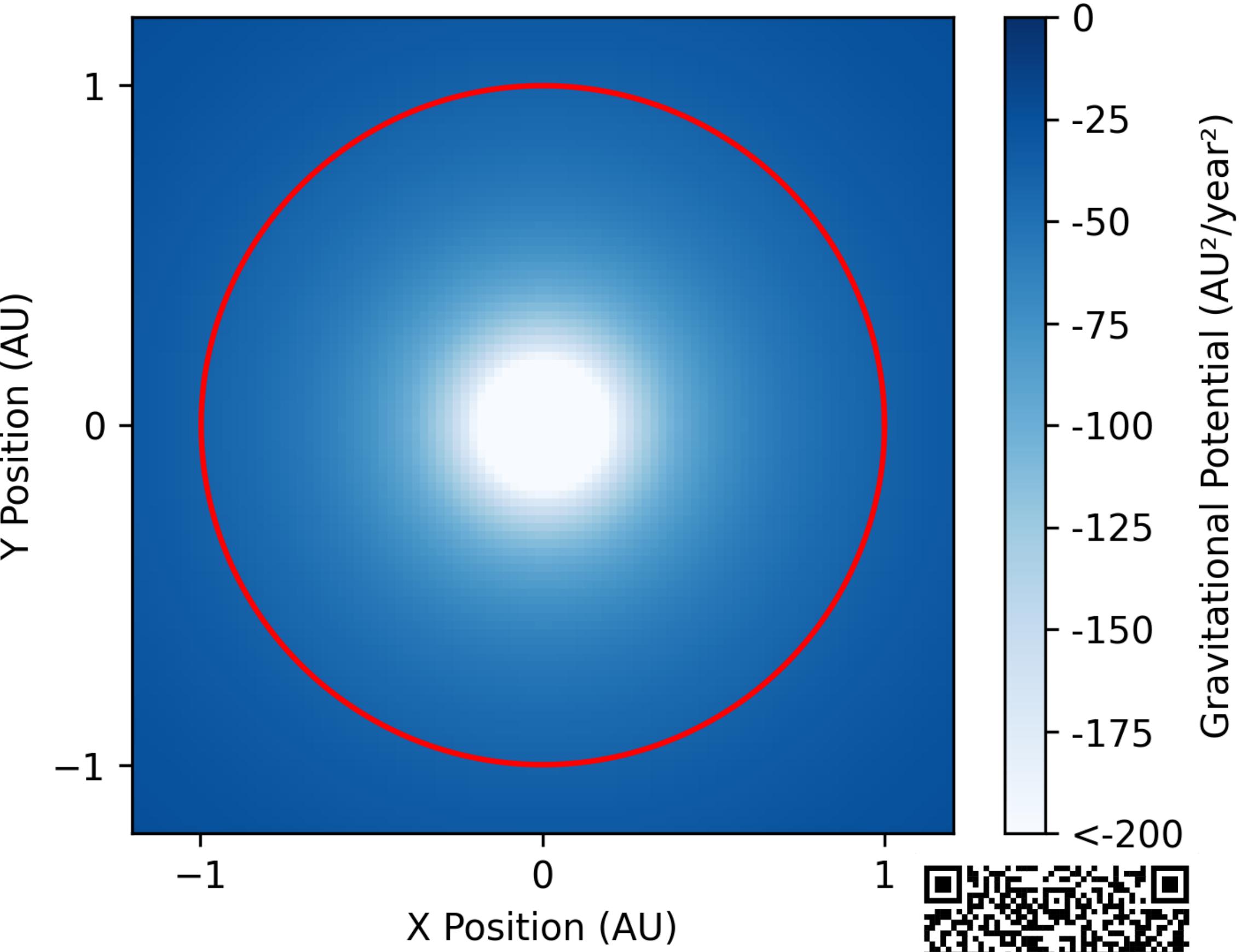


Newton's Law of Gravitation

$$m\ddot{r} - \frac{GMm}{r^2} = 0 \longrightarrow F = m\ddot{r} = \frac{GMm}{r^2}$$

By enforcing Hamilton's Principle we recover the Euler-Lagrange equation and using this we can understand orbital motion

$$\ddot{r} = \frac{GM}{r^2} \xrightarrow{\text{integrate w.r.t. } t} \text{Orbital Motion}$$



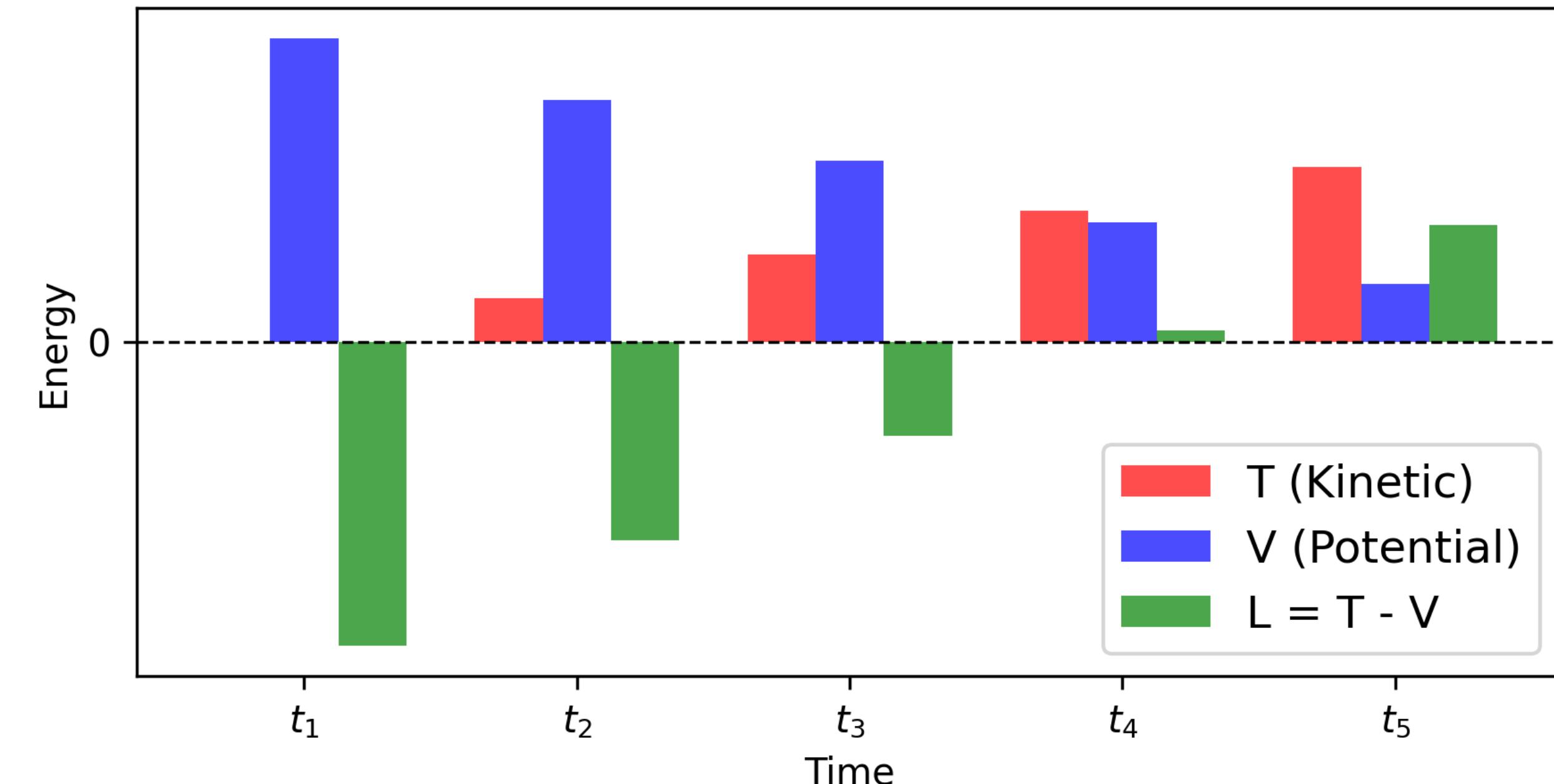
Conclusions

You can do lots of exciting things with Hamilton's Principle and the Euler-Lagrange equation

$$\frac{\delta S}{\delta \epsilon} \Big|_{\epsilon=0} = 0 \rightarrow \frac{\delta}{\delta t} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0$$

Here we derived Newtons Laws of motion but Hamilton's Principle is the gateway to modern physics

“The action of a path followed by a particle moving in a potential is insensitive to first order perturbations in the path.”



Hamilton's Principle
Classical Dynamics

Problem Sheet

Dr Harry Bevins

July 2025

1. Hamilton's principle basics

- State Hamilton's principle and explain how it leads to the Euler-Lagrange equation
- Show explicitly that requiring $\delta S = 0$ for

$$S = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt$$

leads to

