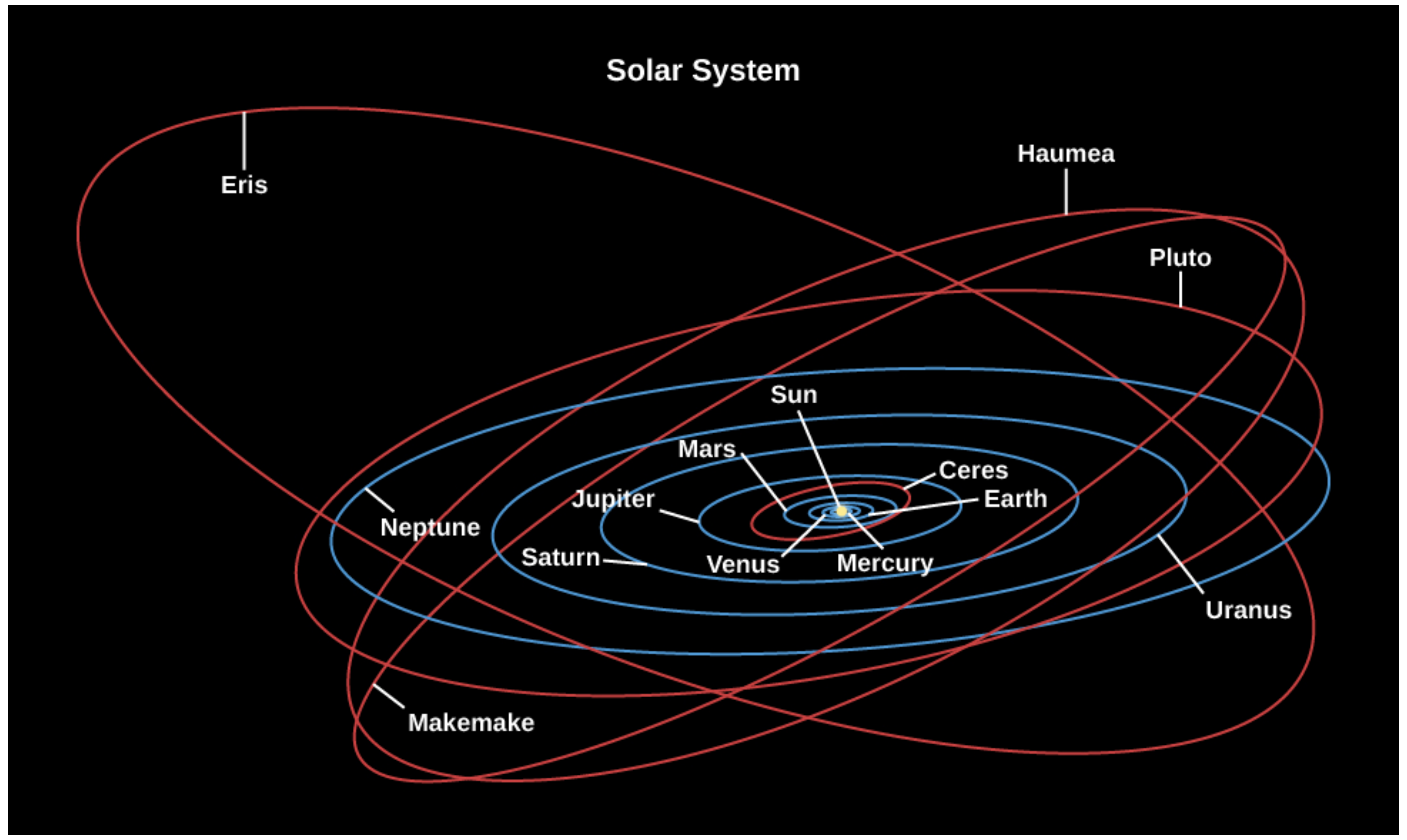
Hamilton’s Principle of Stationarity

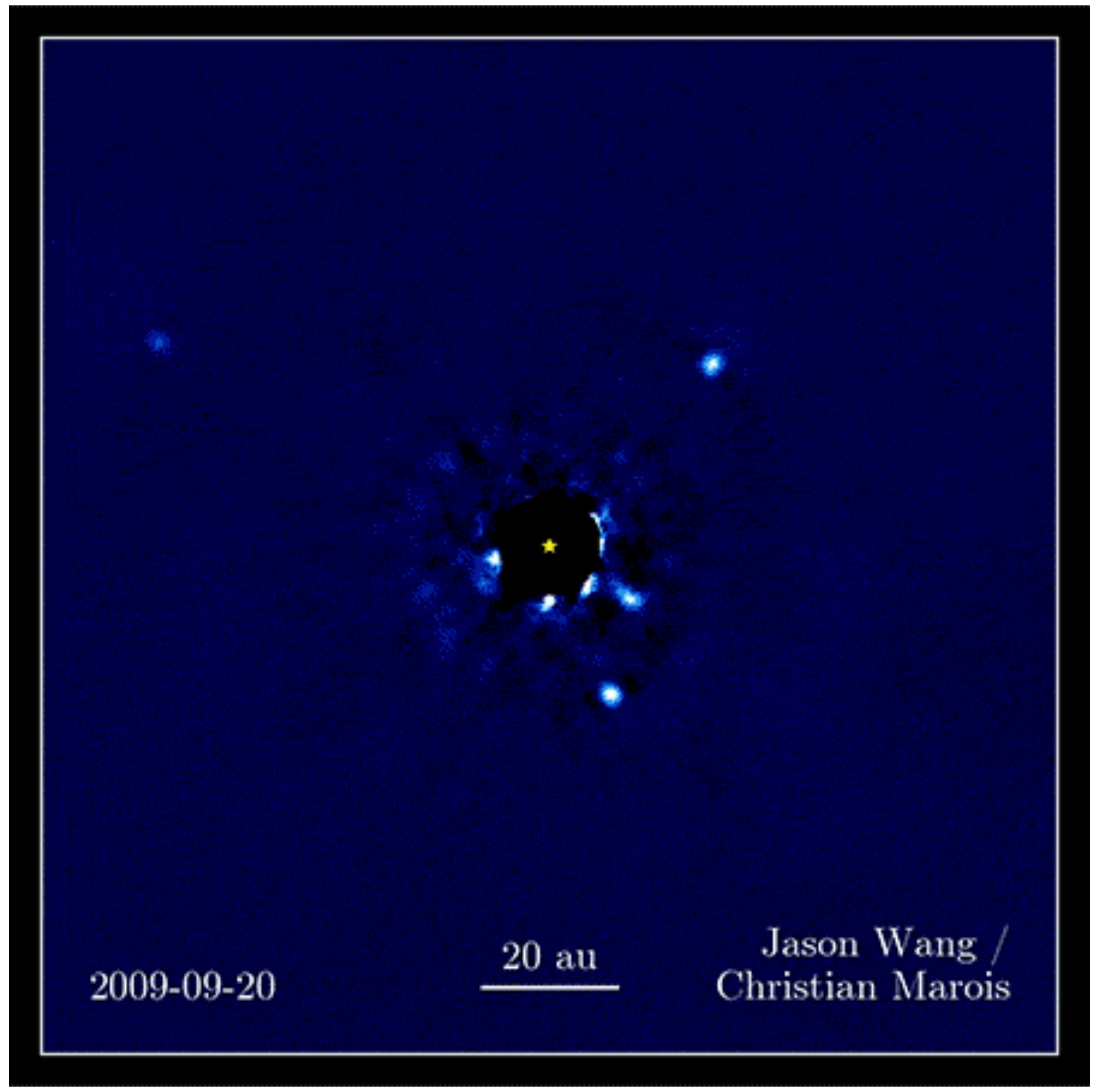
Dr Harry Bevins
htjb2@cam.ac.uk
Classical Dynamics - Part IB Physics B

Resources: bit.ly/4m4StgI

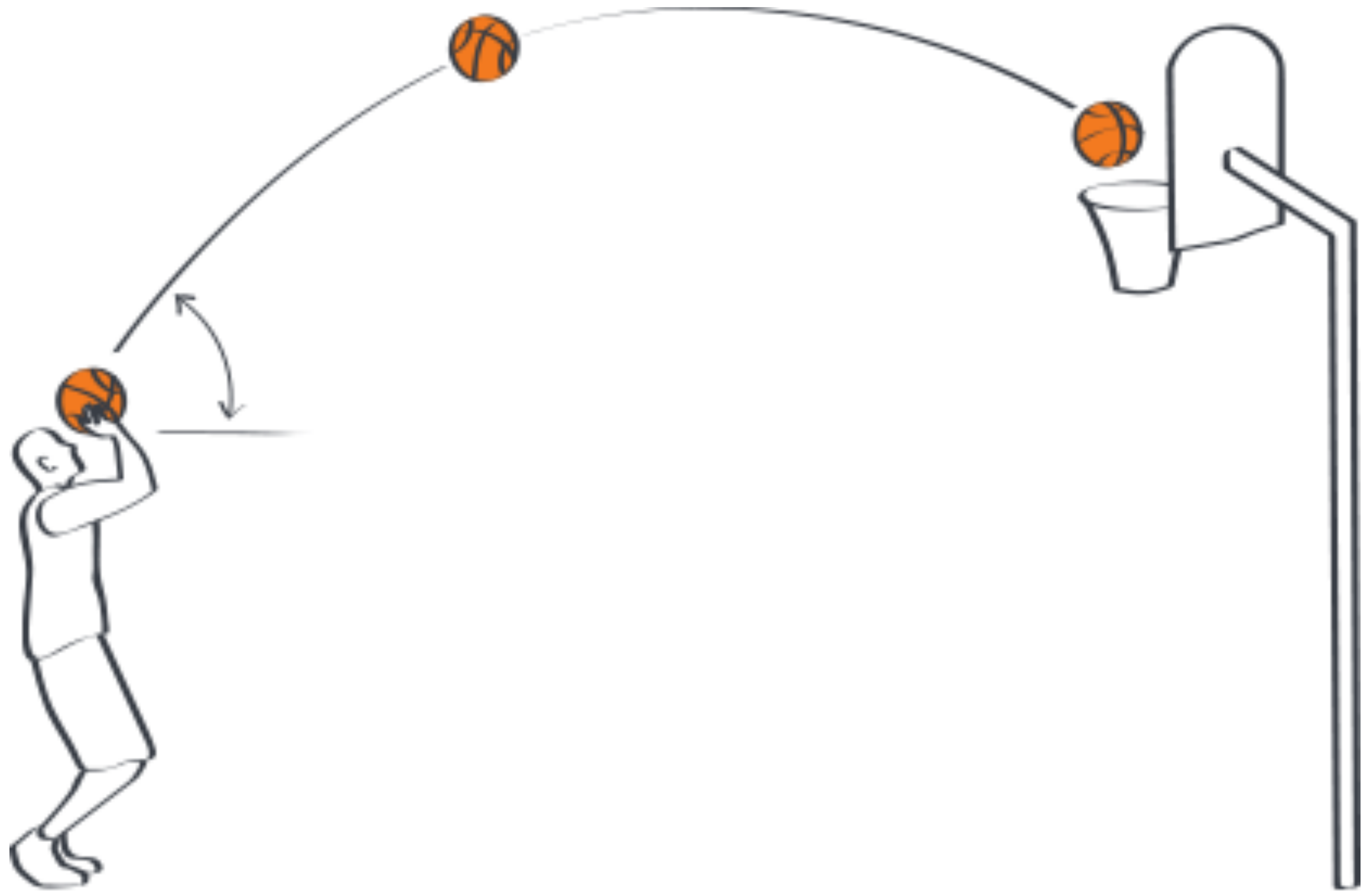
Understanding motion in complex systems



Motion of planets and dwarf planets in the solar system



Exoplanet orbits
Four Hot Jupiter planets orbiting HR 8799



Parabolic motion of projectiles

Understanding motion in complex systems

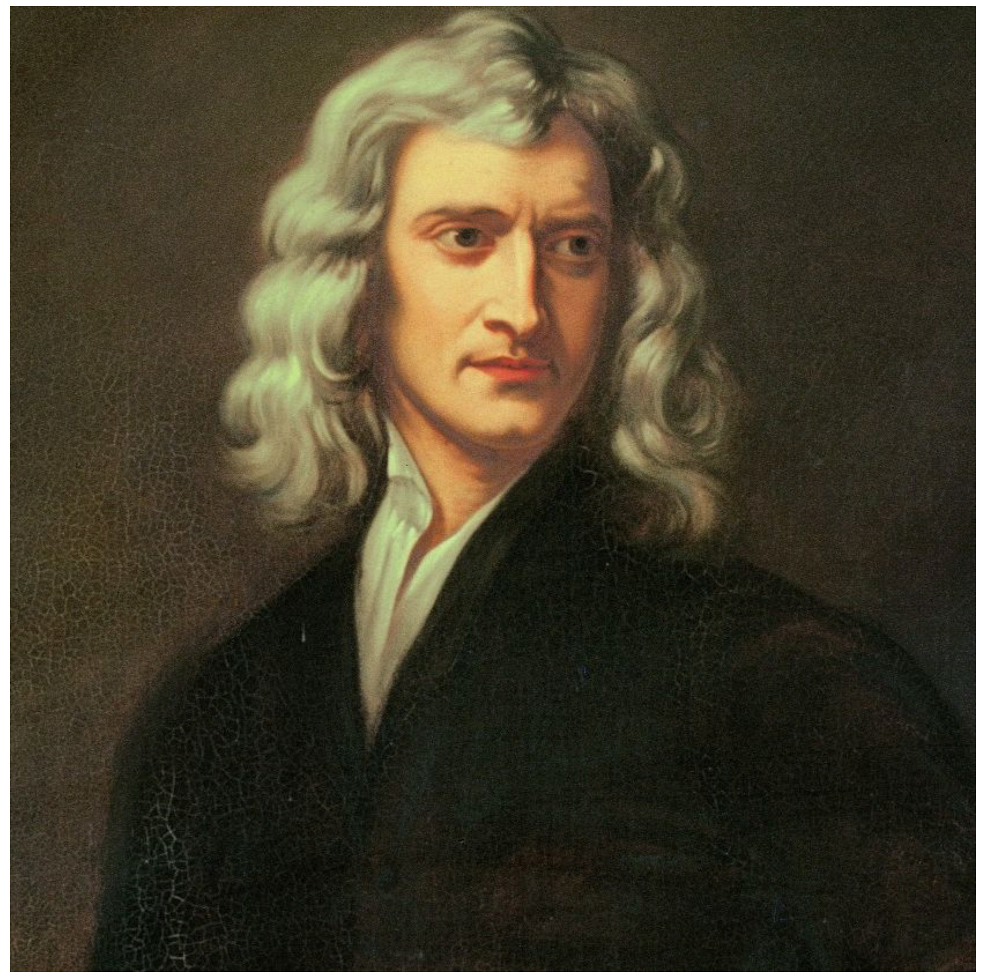
Kepler took observations from Tycho Brahe and derived three laws of orbital mechanics (1609 - 1619)

Newton uses Kepler's laws to illustrate his inverse square law for gravitation (1687)

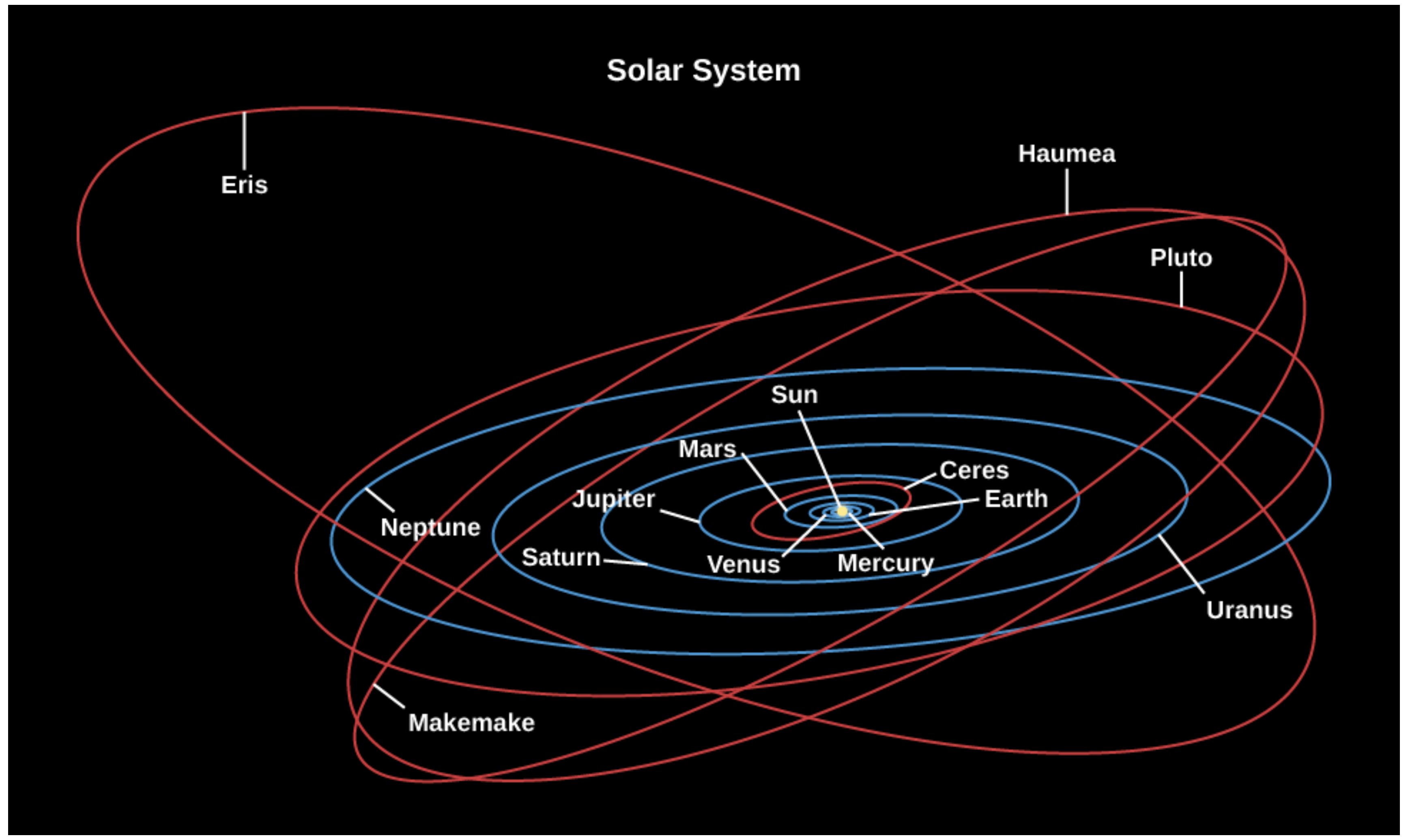
Hamilton's established a deeper fundamental description of mechanical systems (1834) that explains Newton's law of gravitation and a whole lot more!

Tycho Brahe

Johannes Kepler



Isaac Newton



bit.ly/4m4StgI

Lagrangian

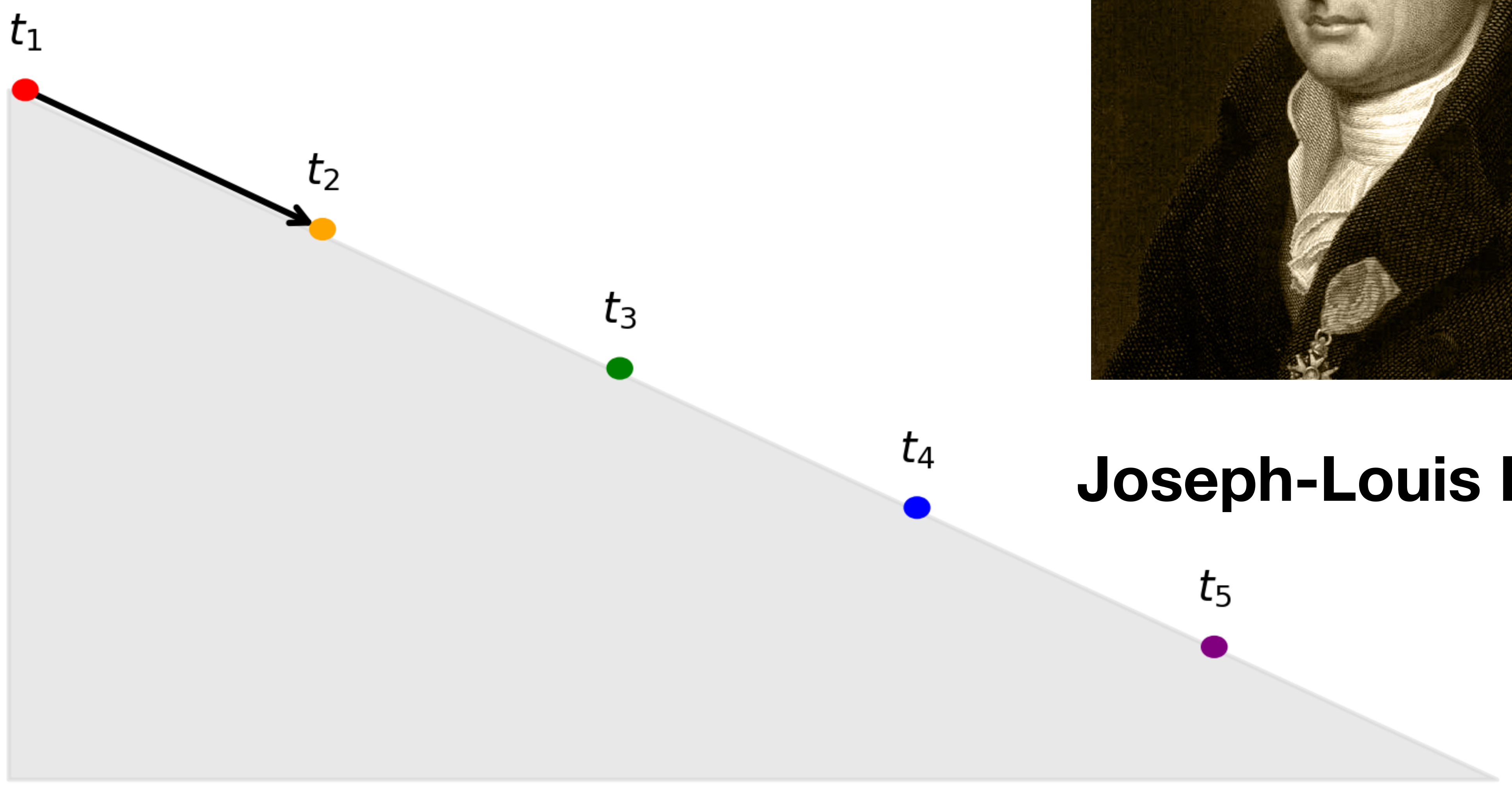
The Lagrangian is given by

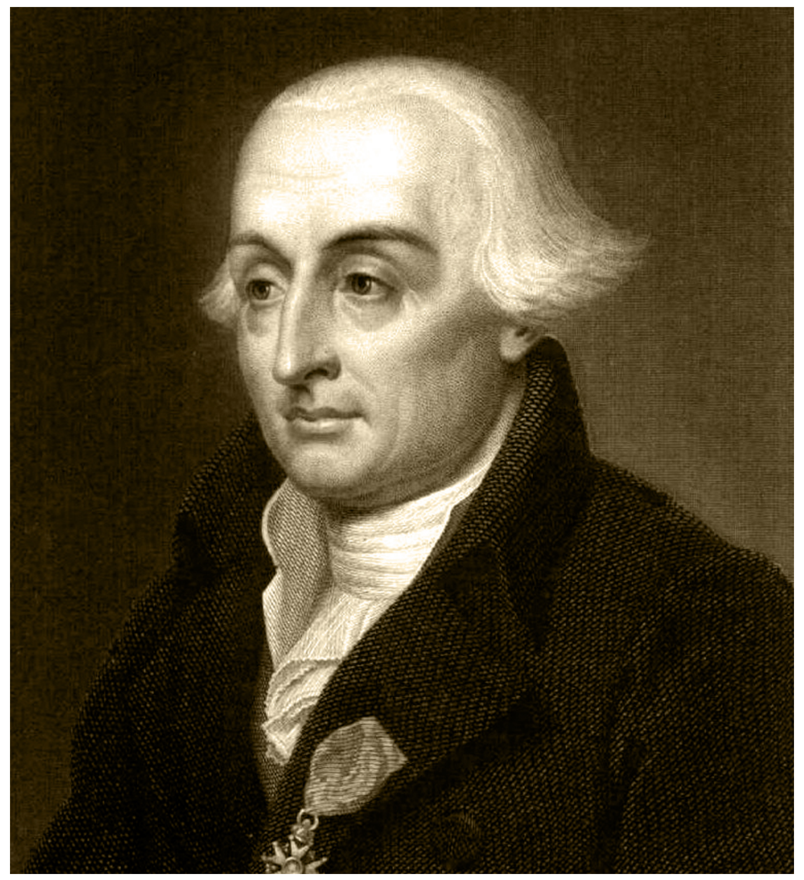
$$L(x, \dot{x}, t) = T(\dot{x}, t) - V(x)$$

where T is kinetic energy and V is a potential

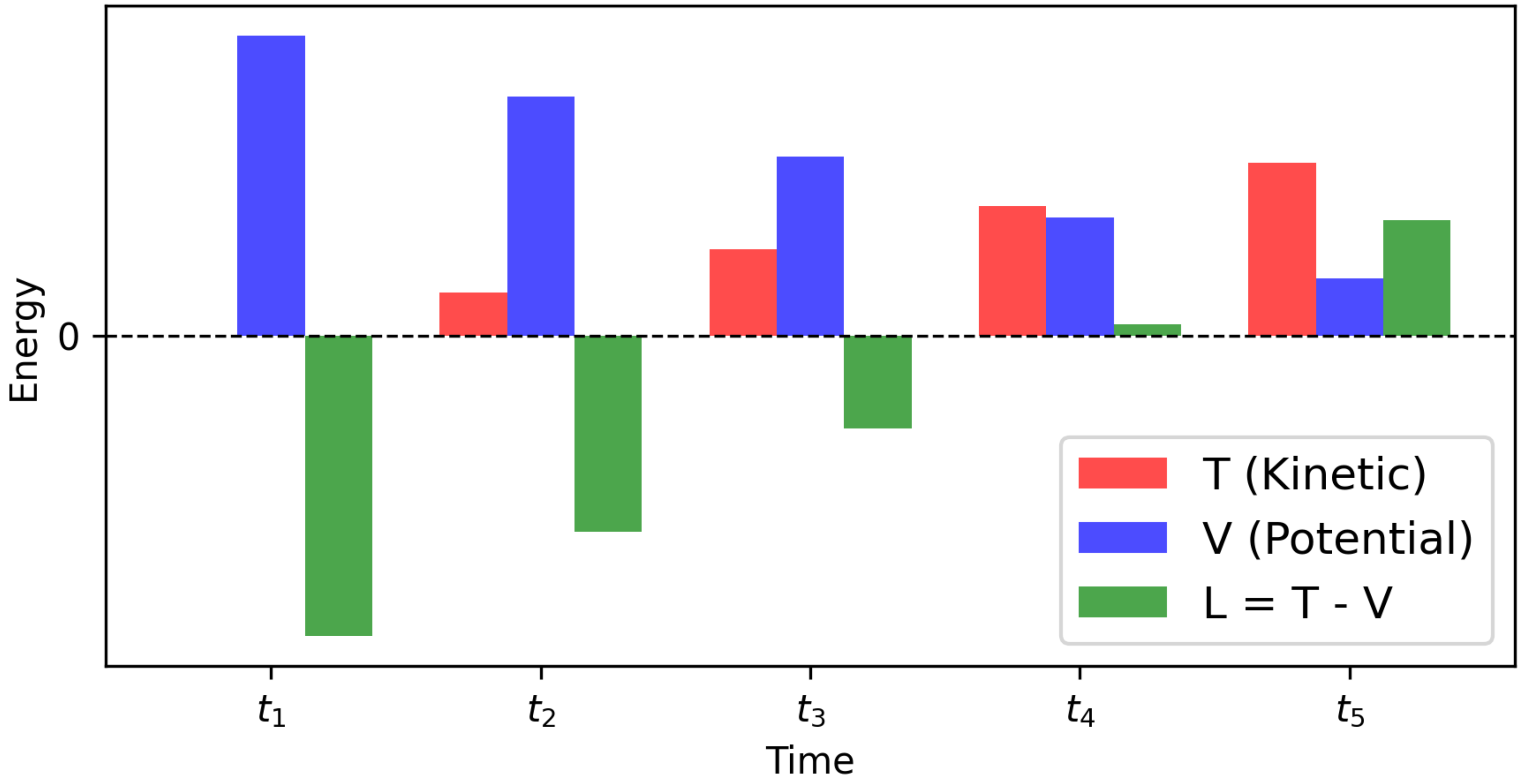
Encodes the dynamics of a system

Move from thinking about forces to motion in an energy landscape





Joseph-Louis Lagrange



Action

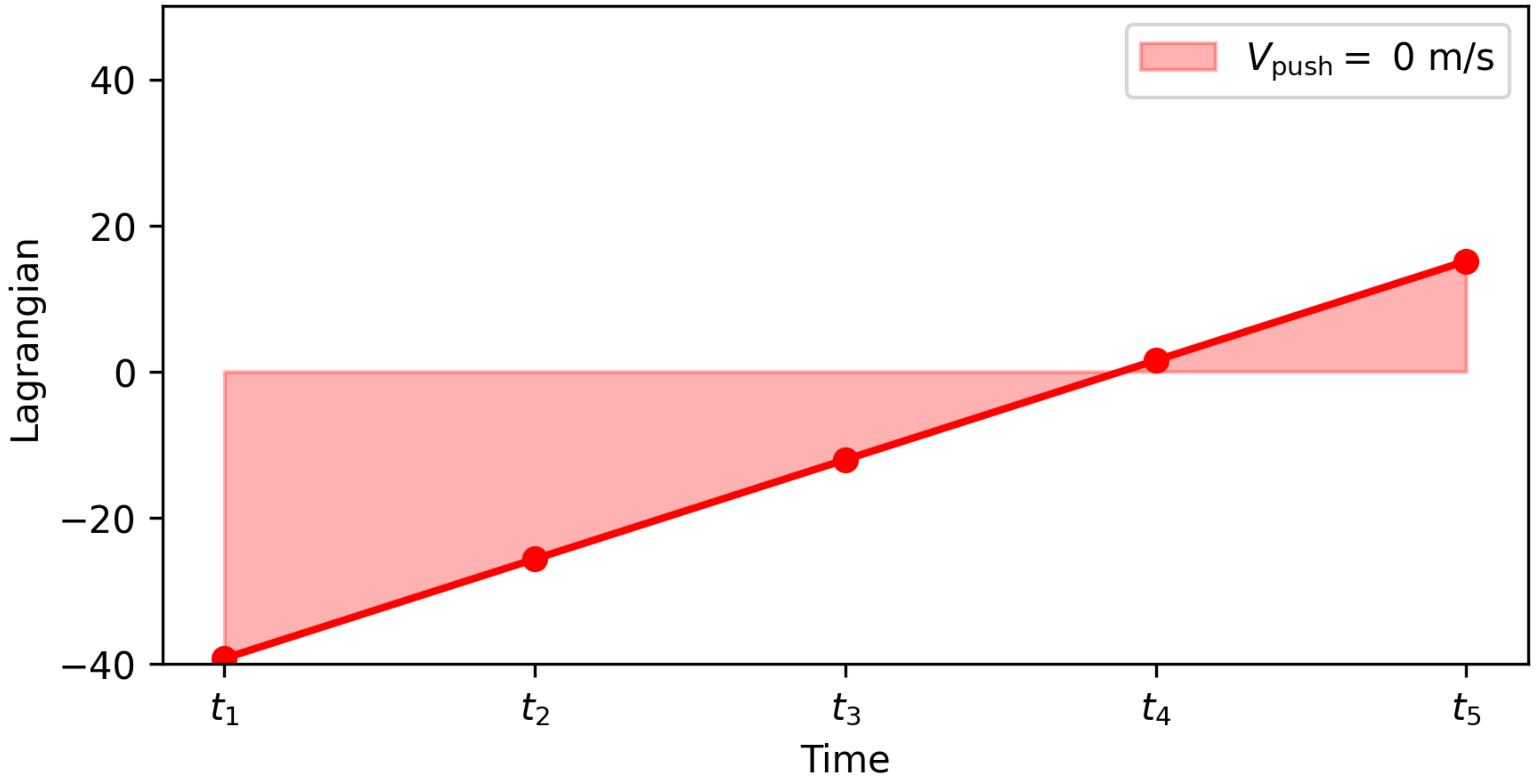
The action for a system is the time integrated Lagrangian

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt$$

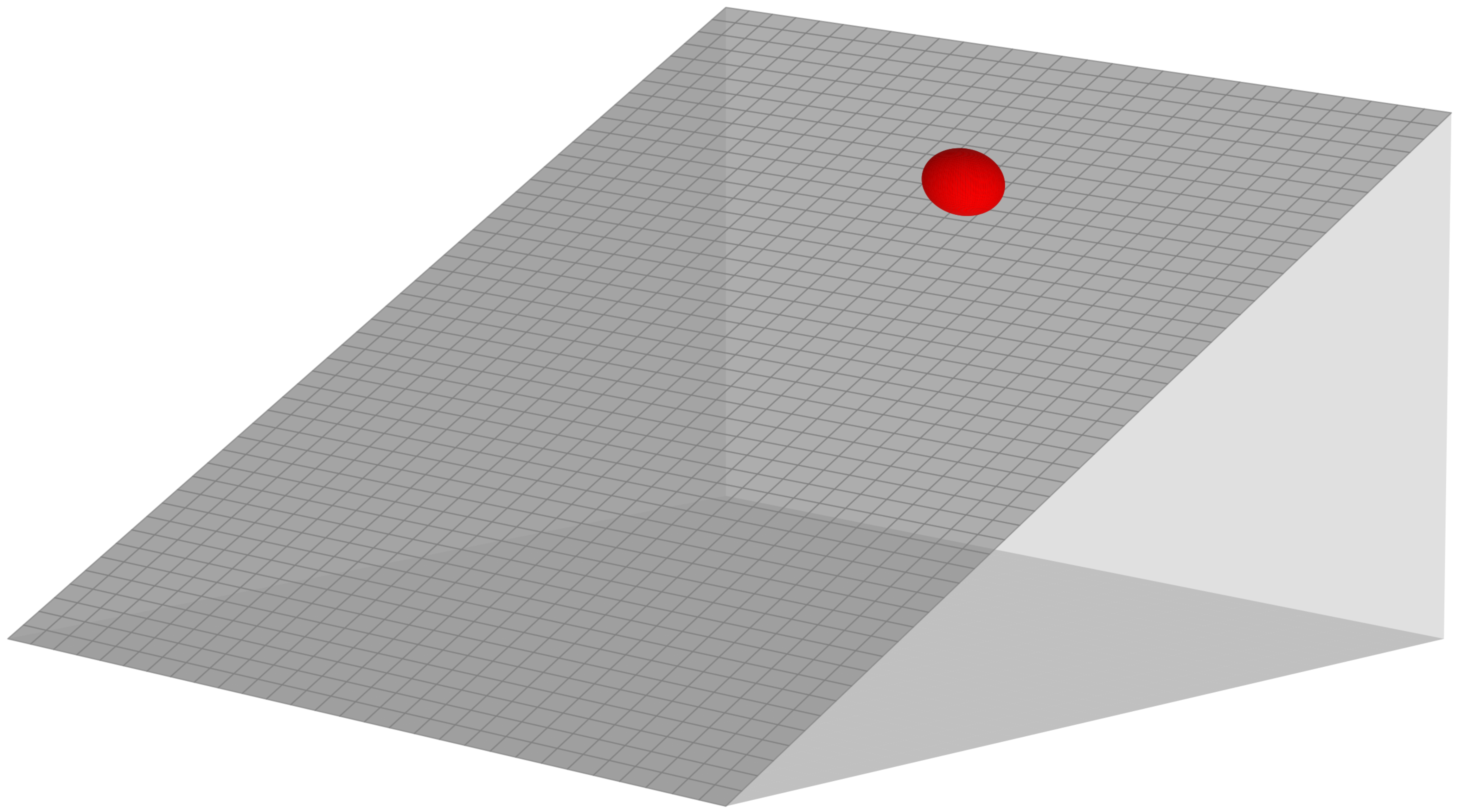
Quantifies the balance between kinetic and potential energy in a system over time

The fundamental object of Hamilton's Principle of Stationarity

William Rowan Hamilton

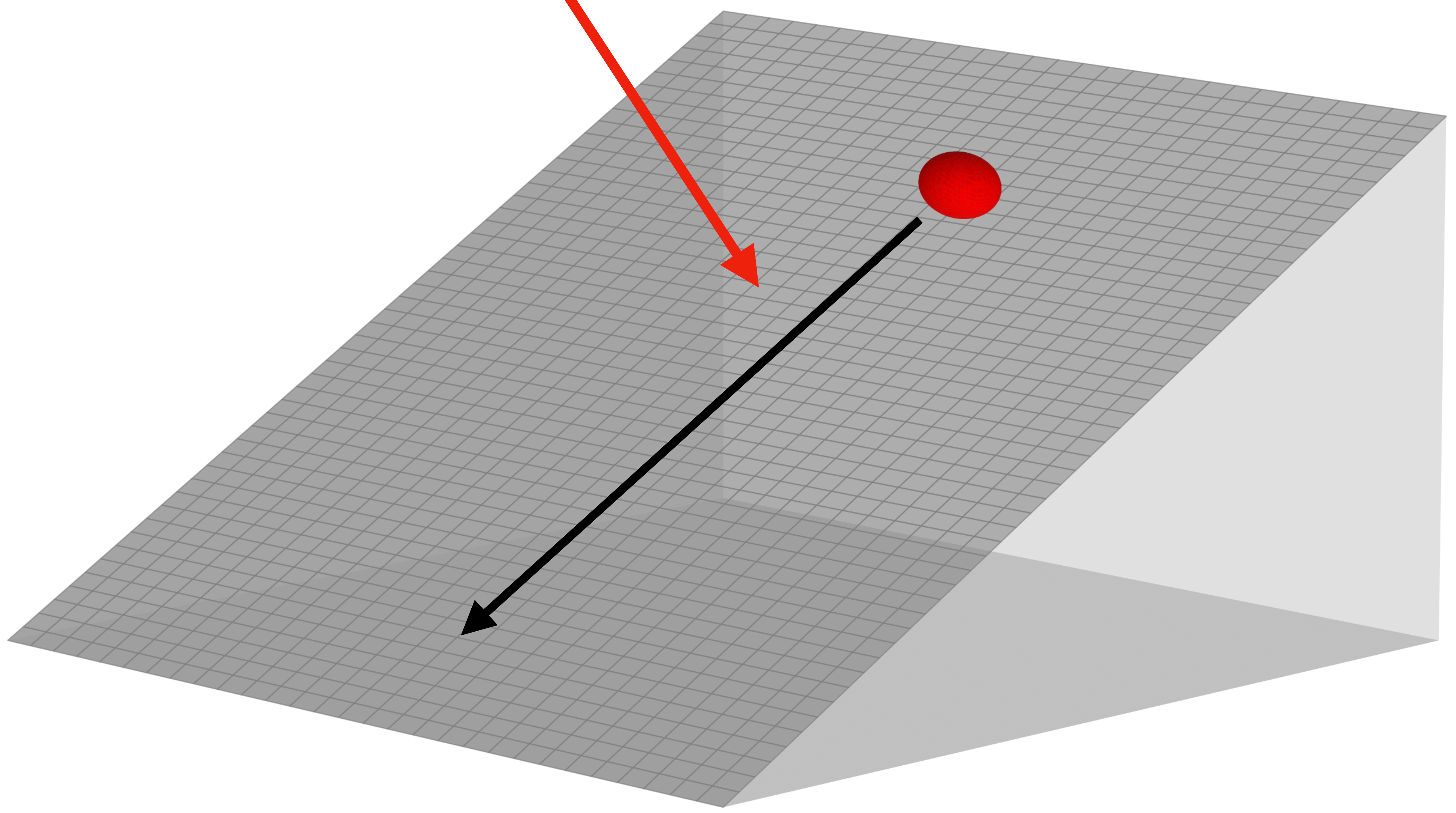


Rolling Ball



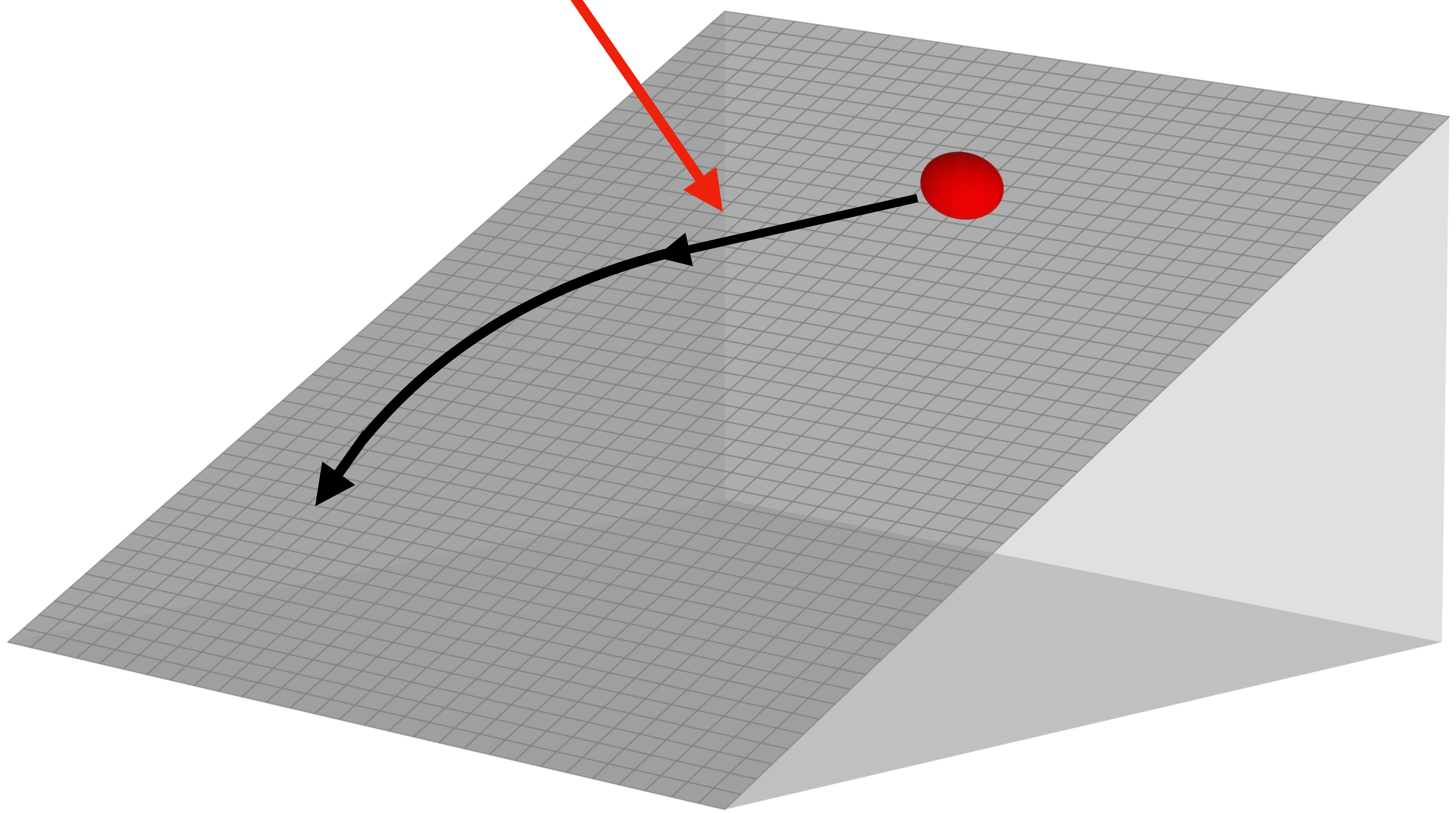
Rolling Ball

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt$$



Rolling Ball

$$S'[x'(t)] = \int_{t_0}^{t_1} L'(x', \dot{x}, t)dt$$



Hamilton’s Principle

“The action of a path followed by a particle moving in a potential is insensitive to first order perturbations in the path.”

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t) dt$$

From Hamilton to Euler-Lagrange

$$x(t) \to x(t) + \epsilon\eta(t)$$

From Hamilton to Euler-Lagrange

$$x(t) \to x(t) + \epsilon\eta(t)$$

$$S[x(t) + \epsilon\eta(t)] \approx S[x(t)] + \left.\frac{\delta S}{\delta\epsilon}\right|_{\epsilon=0} \epsilon + \ldots$$

From Hamilton to Euler-Lagrange

$$x(t) \to x(t) + \epsilon\eta(t)$$

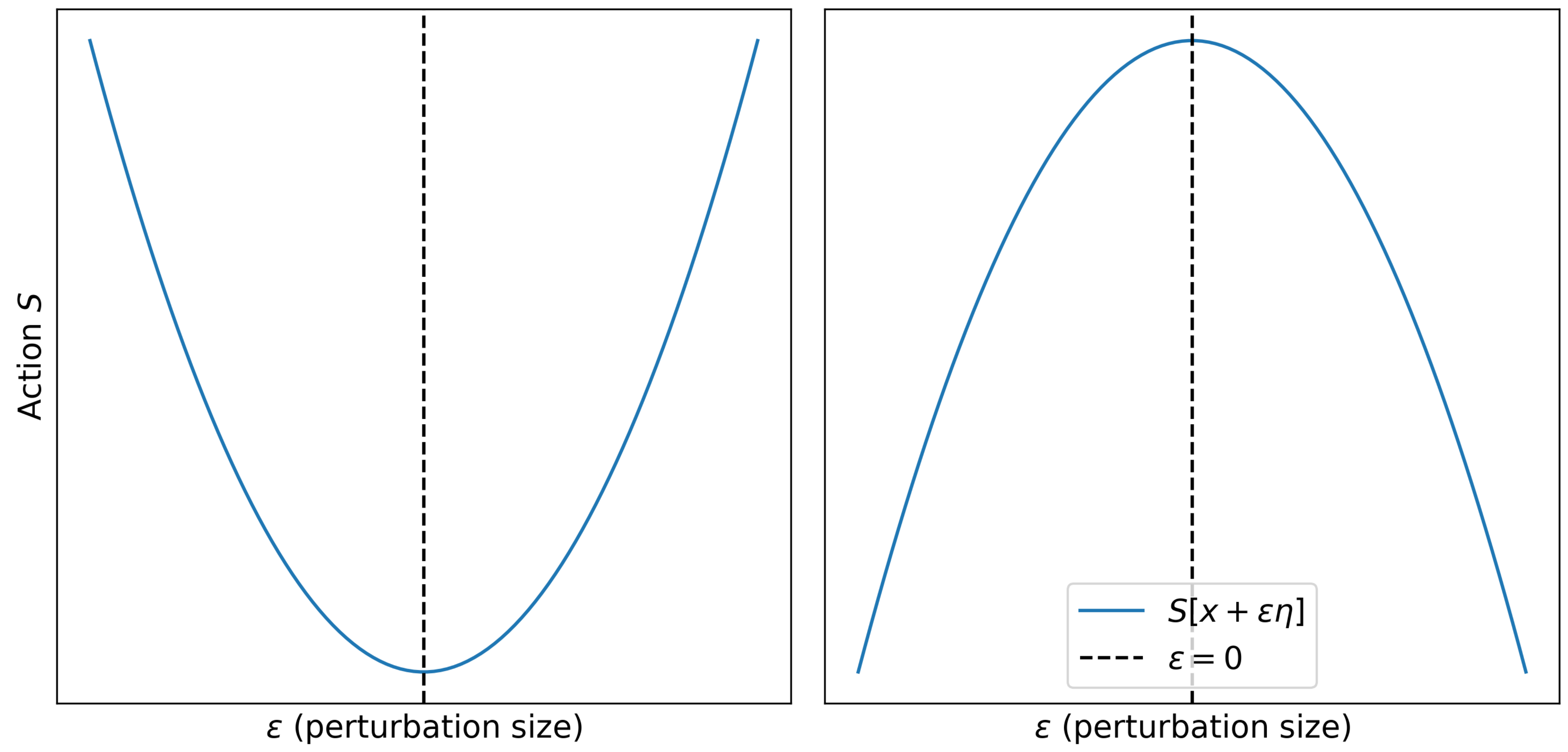
$$S[x(t) + \epsilon\eta(t)] \approx S[x(t)] + \left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} \epsilon + \dots$$

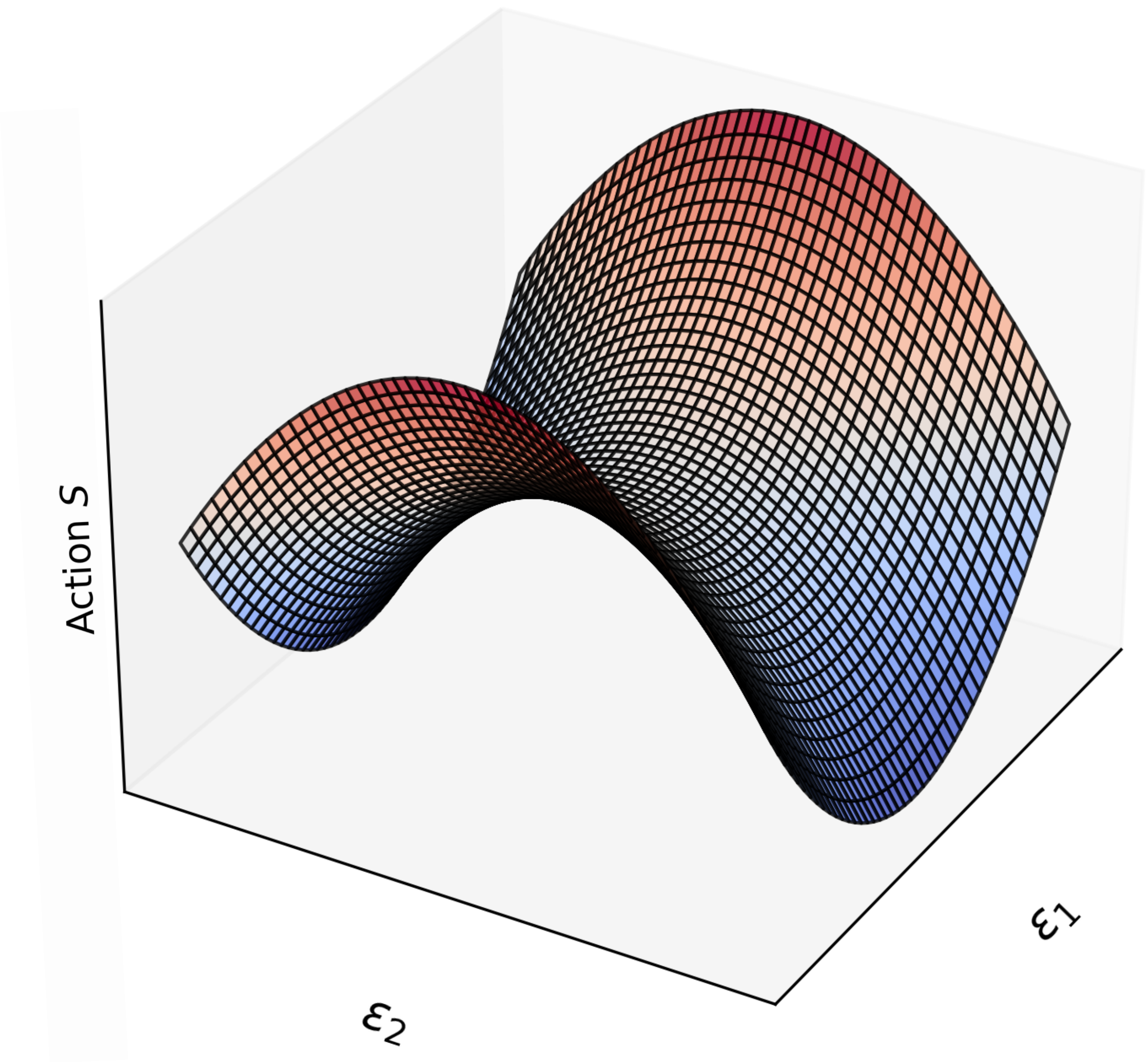
Hamilton's Principle requires $\left.\dfrac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = 0$

"The action of a path followed by a particle moving in a potential is insensitive to first order perturbations in the path."

From Hamilton to Euler-Lagrange

Hamilton's Principle requires $\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = 0$





From Hamilton to Euler-Lagrange

$$S[x(t) + \epsilon\eta(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)dt$$

From Hamilton to Euler-Lagrange

$$S[x(t) + \epsilon\eta(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)dt$$

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \frac{\delta L}{\delta x}\frac{\delta(x + \epsilon\eta)}{\delta \epsilon} + \frac{\delta L}{\delta \dot{x}}\frac{\delta(\dot{x} + \epsilon\dot{\eta})}{\delta \epsilon} dt = 0$$

From Hamilton to Euler-Lagrange

$$S[x(t) + \epsilon\eta(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)dt$$

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \frac{\delta L}{\delta x}\frac{\delta(x + \epsilon\eta)}{\delta\epsilon} + \frac{\delta L}{\delta \dot{x}}\frac{\delta(\dot{x} + \epsilon\dot{\eta})}{\delta\epsilon} dt = 0$$

$$\left.\frac{dS}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \left(\frac{\delta L}{\delta x} - \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right)\right)\eta dt = 0.$$

From Hamilton to Euler-Lagrange

$$\frac{dS}{\delta\epsilon}\bigg|_{\epsilon=0} = \int_{t_0}^{t_1} \left(\frac{\delta L}{\delta x} - \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right)\right)\eta dt = 0.$$

From this we can derive the Euler-Lagrange equation

$$\boxed{\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0}$$

Euler-Lagrange

Hamilton's Principle
Classical Dynamics

Dr Harry Bevins

July 2025

1 Introduction

In this lecture we will introduce Hamilton's Principle, the Lagrangian and the Action. We will use Hamilton's Principle to derive Newton's first and second laws of motion, as well as his law of gravity.

Hamilton's Principle states that

Euler-Lagrange

$$\boxed{\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0}$$

Hamilton's Principle leads us to the Euler-Lagrange equation and its this that describes the dynamics of a system.

Using Euler-Lagrange we can derive all of Newton's Laws and much more!

Hamilton's Principle in Action

Lets think about a particle in free space i.e.

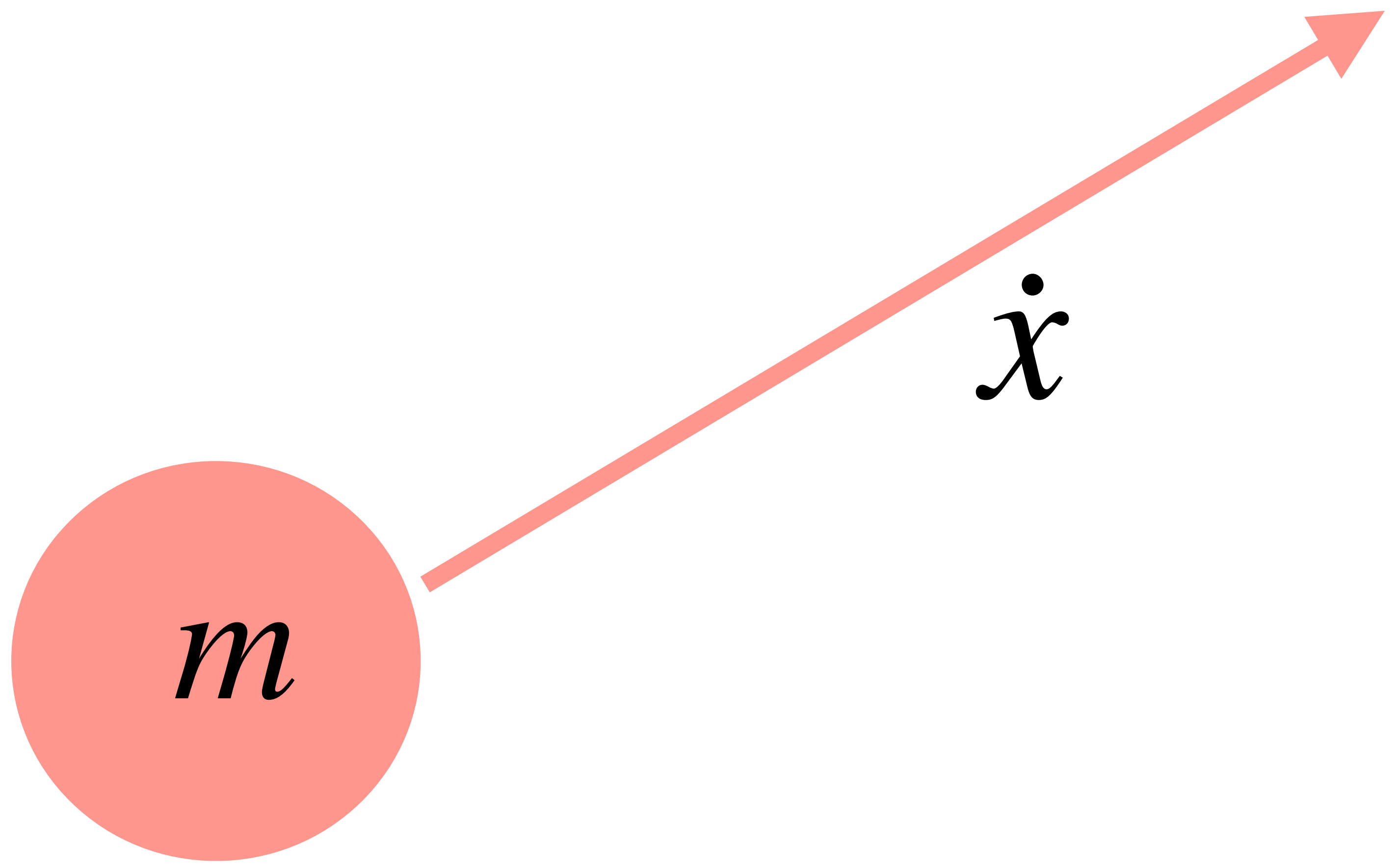
$$V(x, t) = 0$$

Our particle has some velocity $\dot{x}$ meaning it has kinetic energy

$$T = \frac{1}{2}m\dot{x}^2$$

So it's Lagrangian is given by

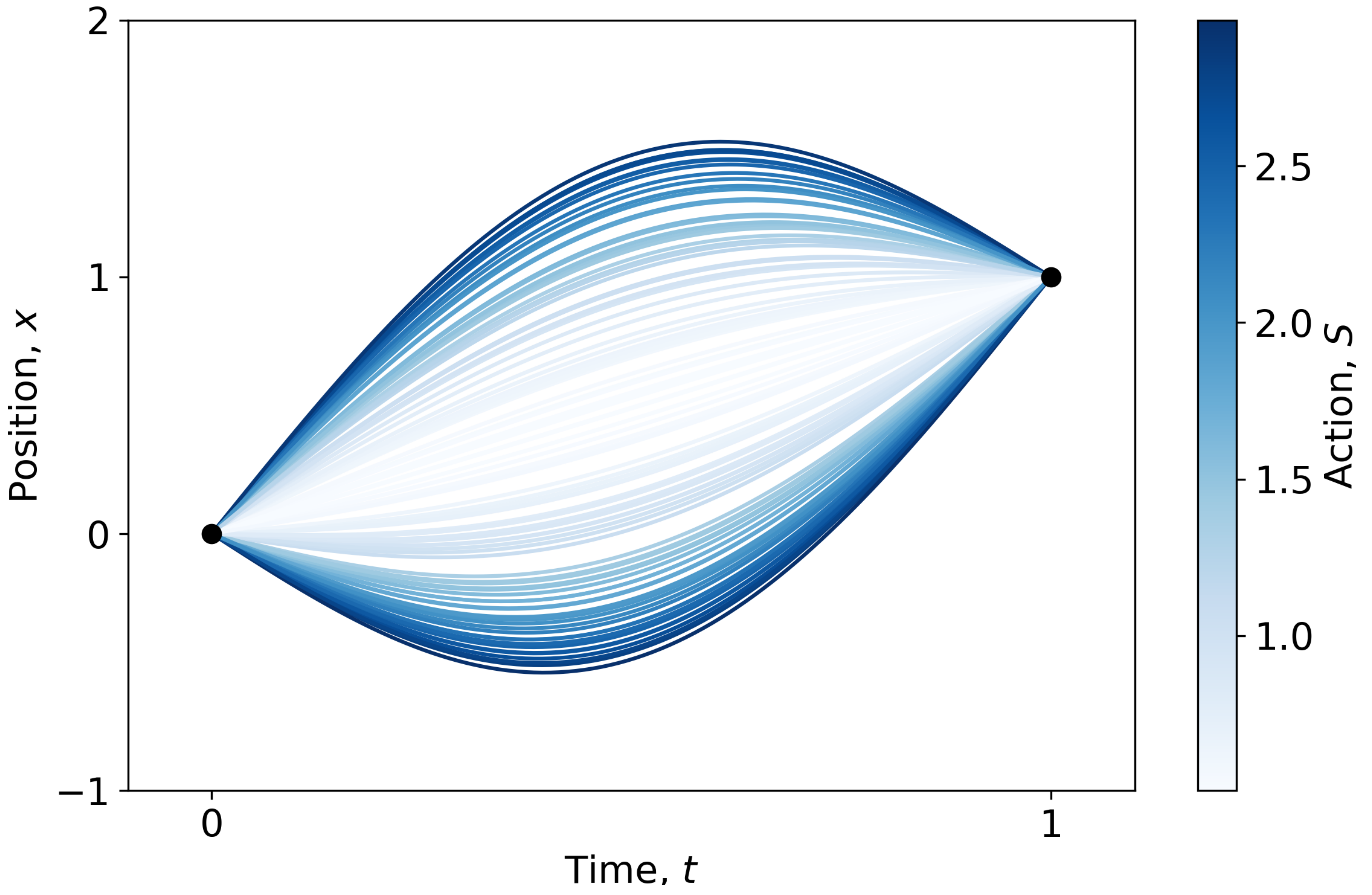
$$L = T = \frac{1}{2}m\dot{x}^2$$



Hamilton's Principle in Action

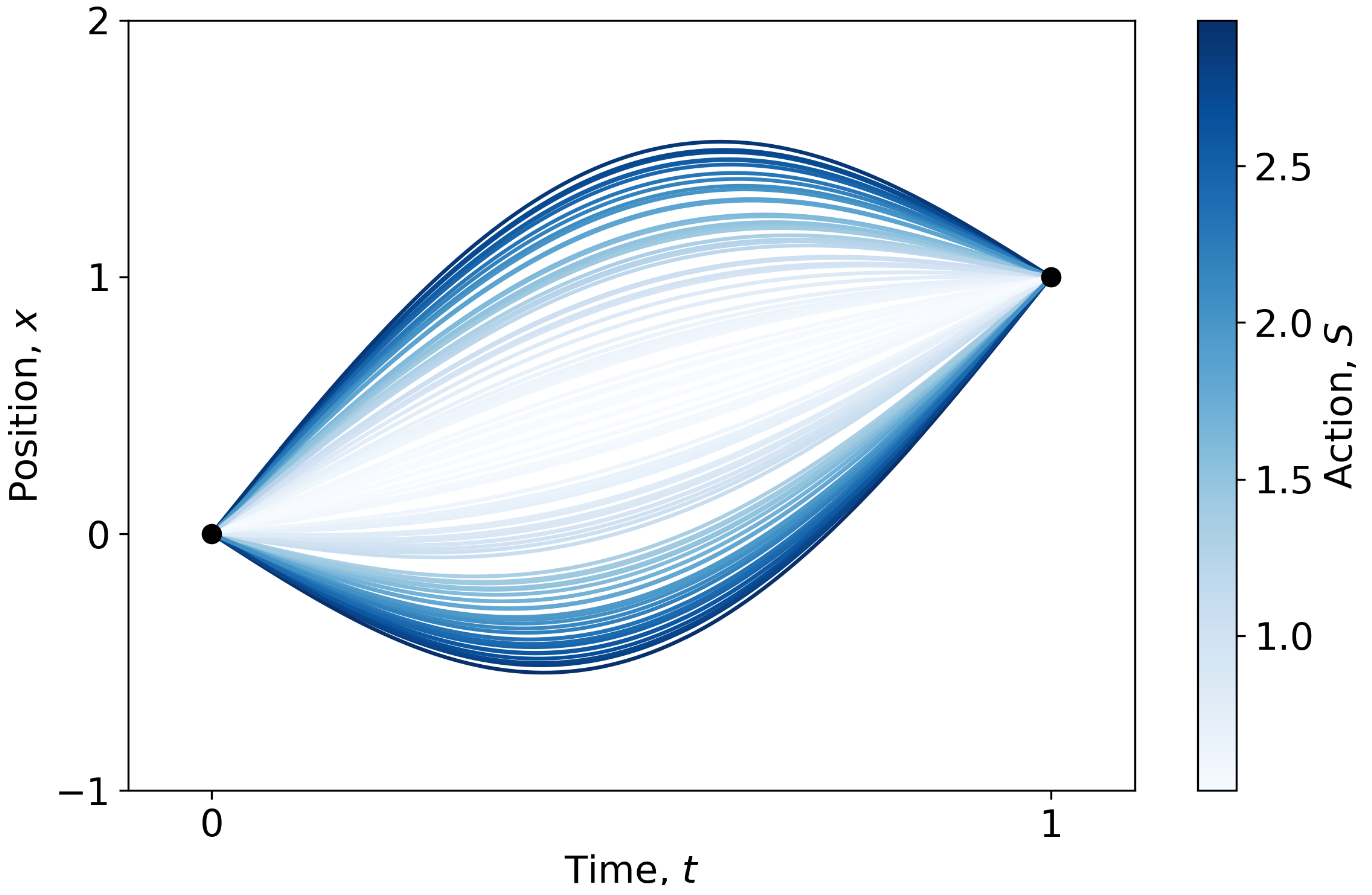
So the action is given by

$$S = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \frac{1}{2} m \dot{x}^2 dt$$



Hamilton's Principle in Action

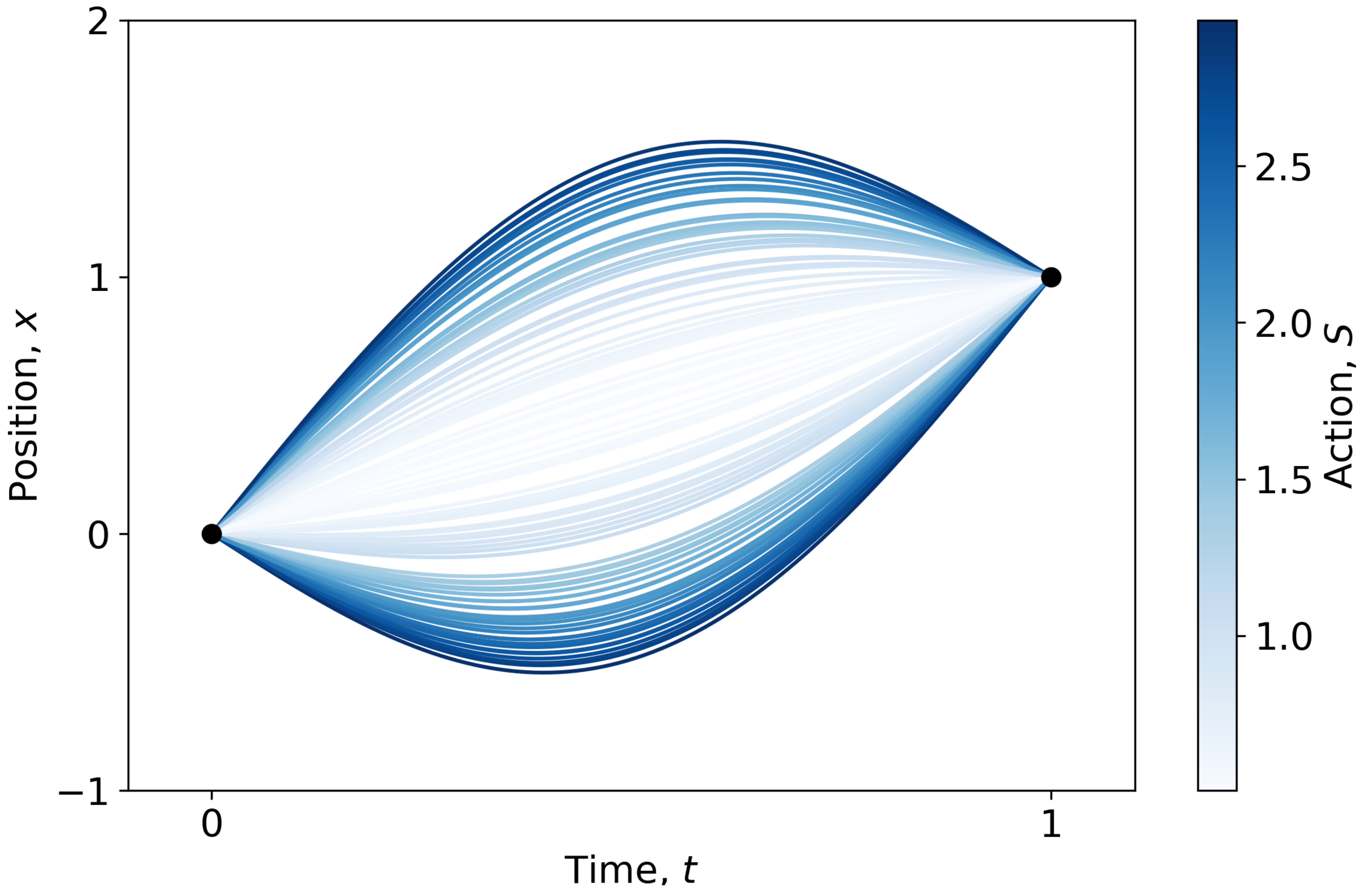
$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2}m\dot{x}^2$$



Hamilton's Principle in Action

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2}m\dot{x}^2$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) - \frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2\right) = 0$$

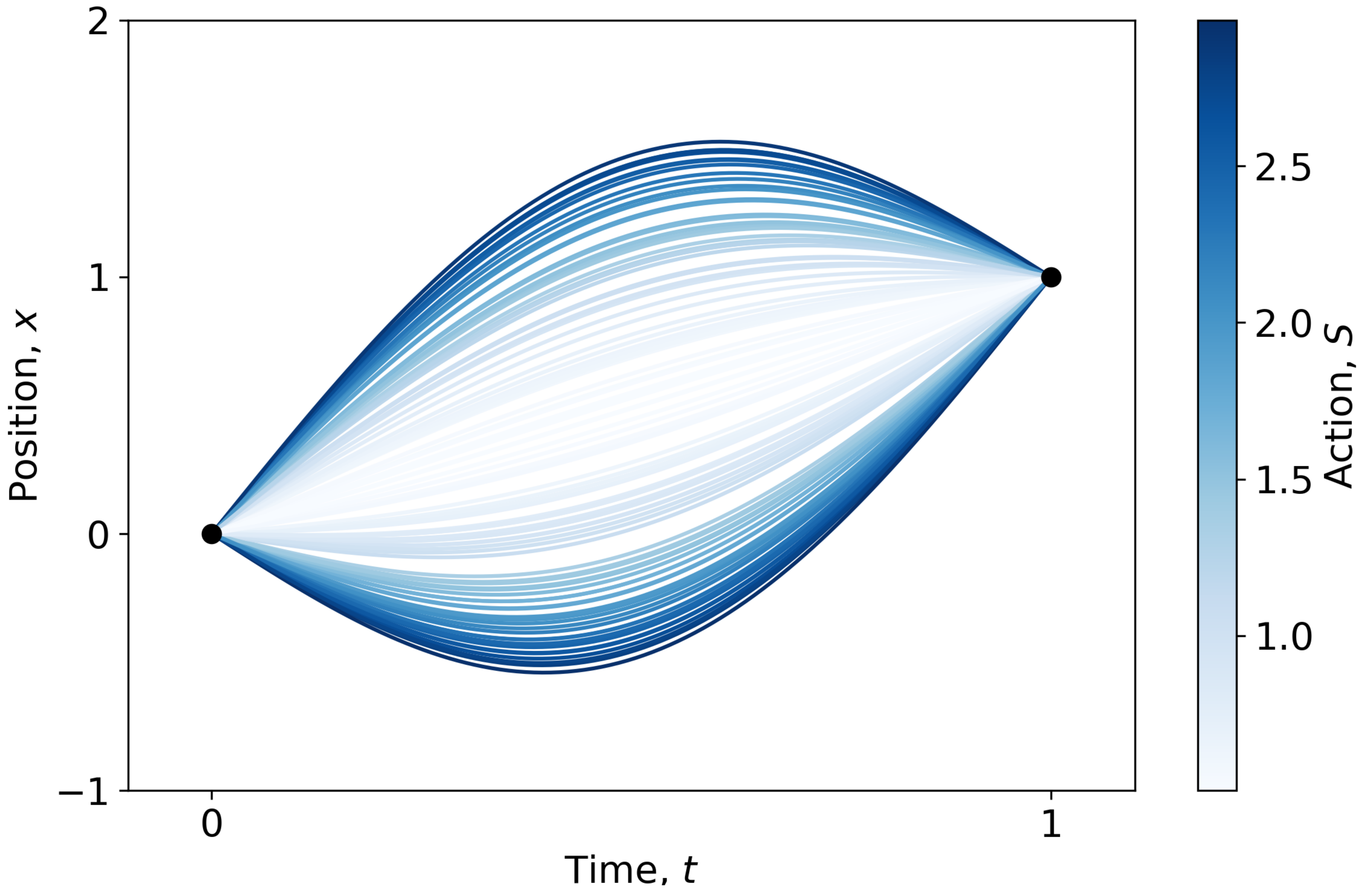


Hamilton's Principle in Action

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2}m\dot{x}^2$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) - \cancel{\frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2\right)} = 0$$

$$\frac{\delta}{\delta t}(m\dot{x}) = 0$$



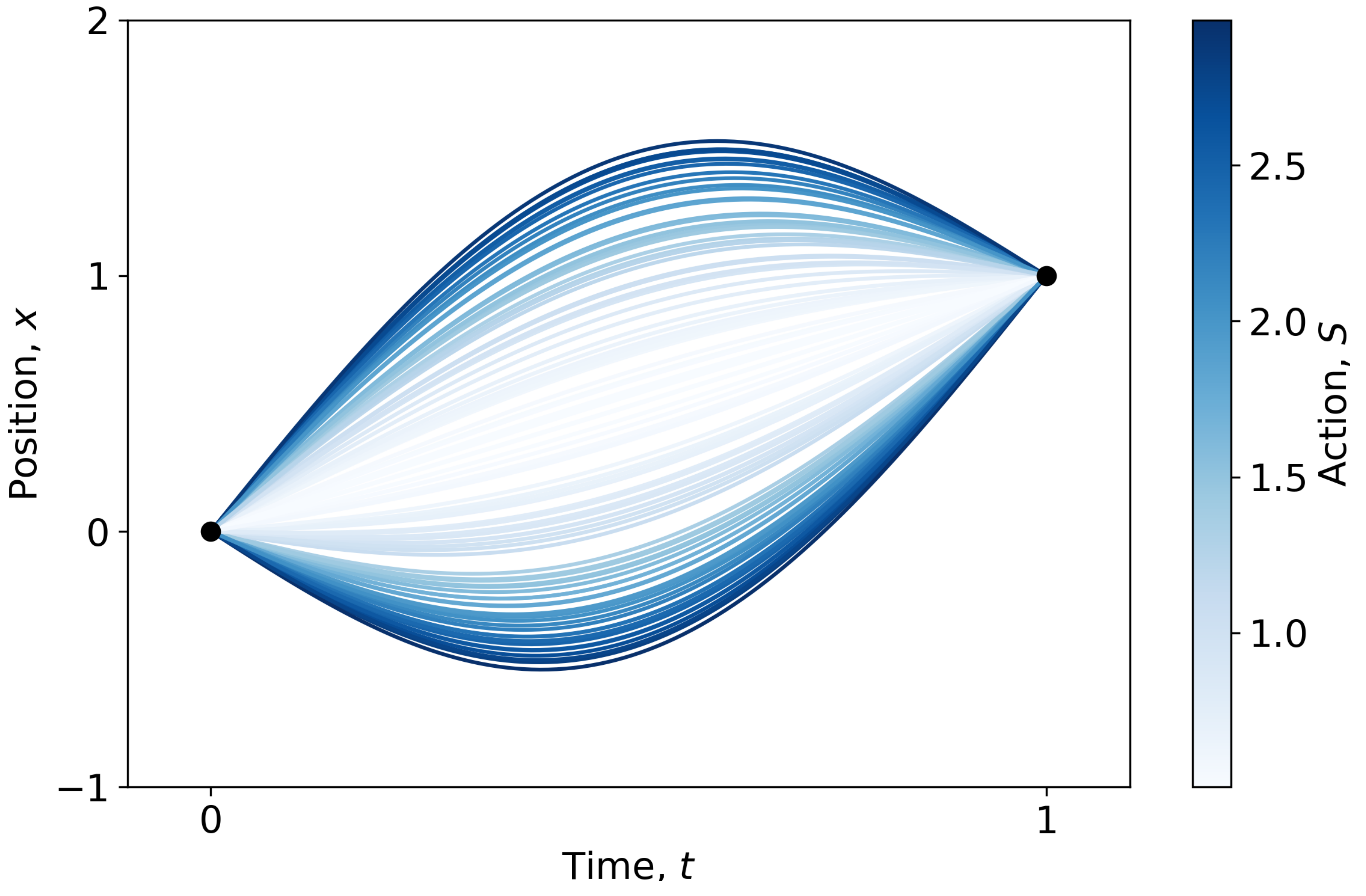
Hamilton's Principle in Action

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0 \text{ and } L = \frac{1}{2}m\dot{x}^2$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) - \frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2\right) = 0$$

$$\frac{\delta}{\delta t}(m\dot{x}) = 0$$

$$m\ddot{x} = 0$$



Hamilton's Principle in Action

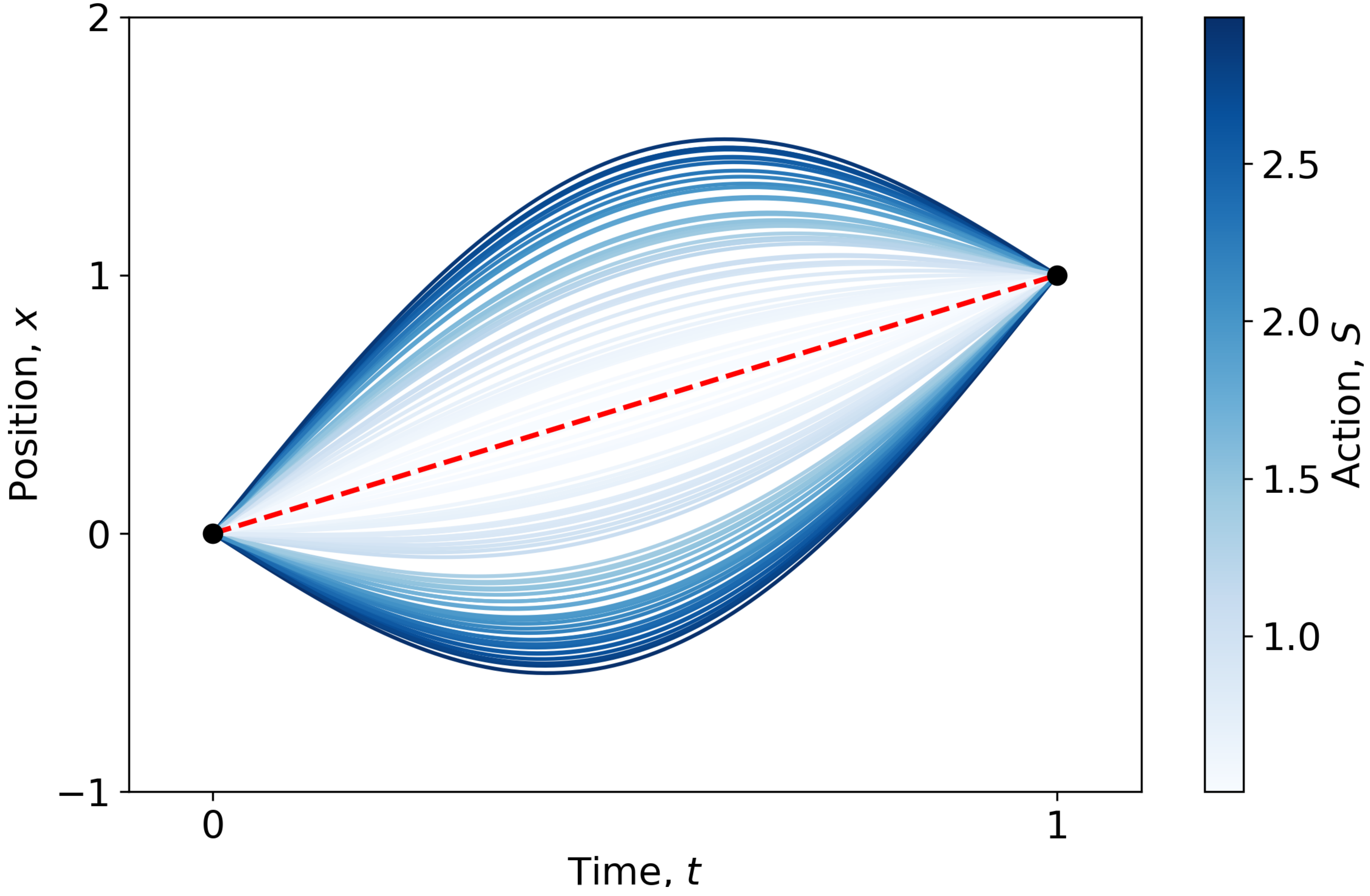
$$m\ddot{x} = 0$$

$$\xrightarrow{\text{Integrate w.r.t. t}}$$

$$\dot{x} = \text{const} = v_0$$

$$\xrightarrow{\text{Integrate w.r.t. t}}$$

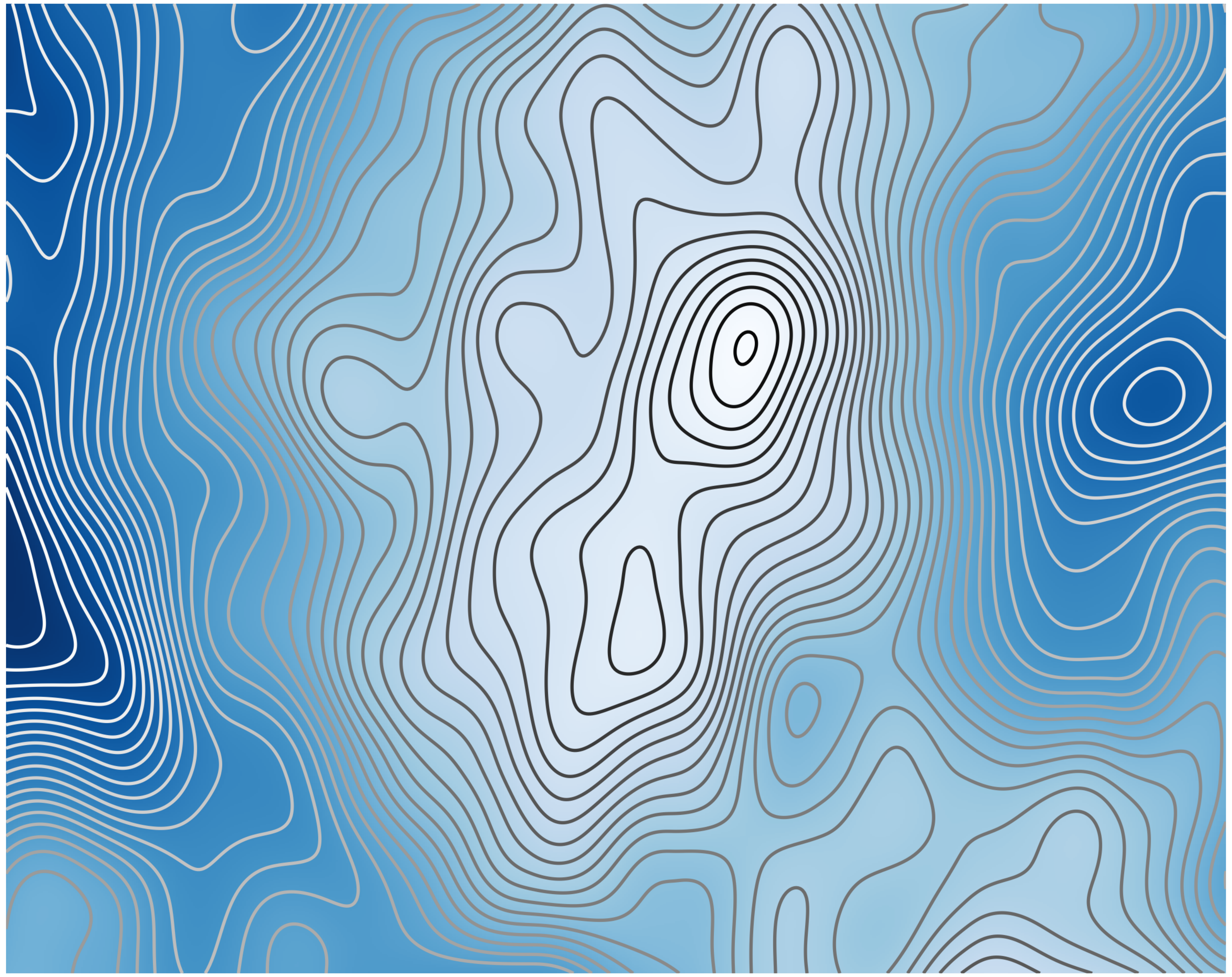
$$x = v_0 t + c$$



Arbitrary Potential

Let's place our particle in a non-zero potential

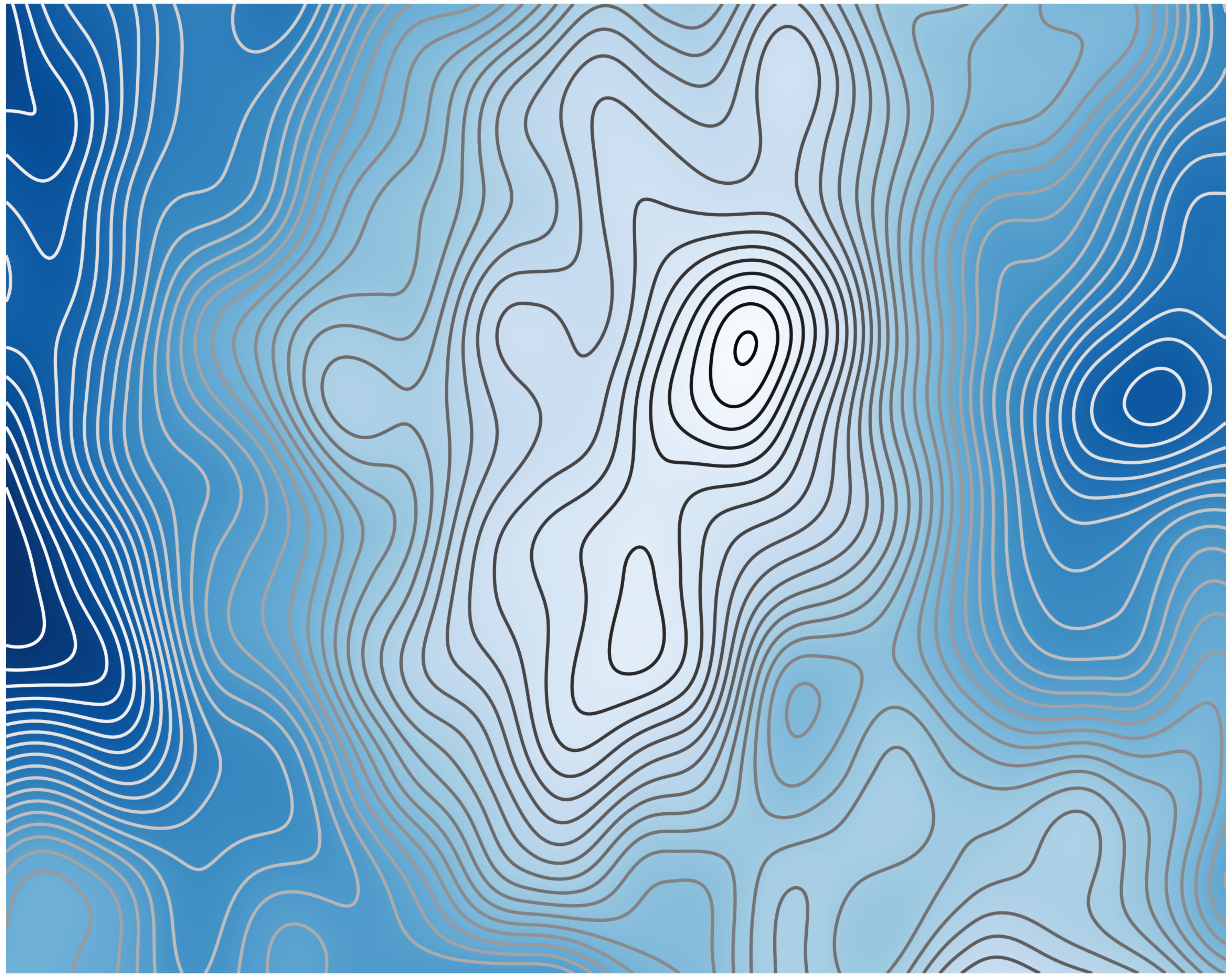
$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$



Arbitrary Potential

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x) \qquad \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right) - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = m\ddot{x}$$

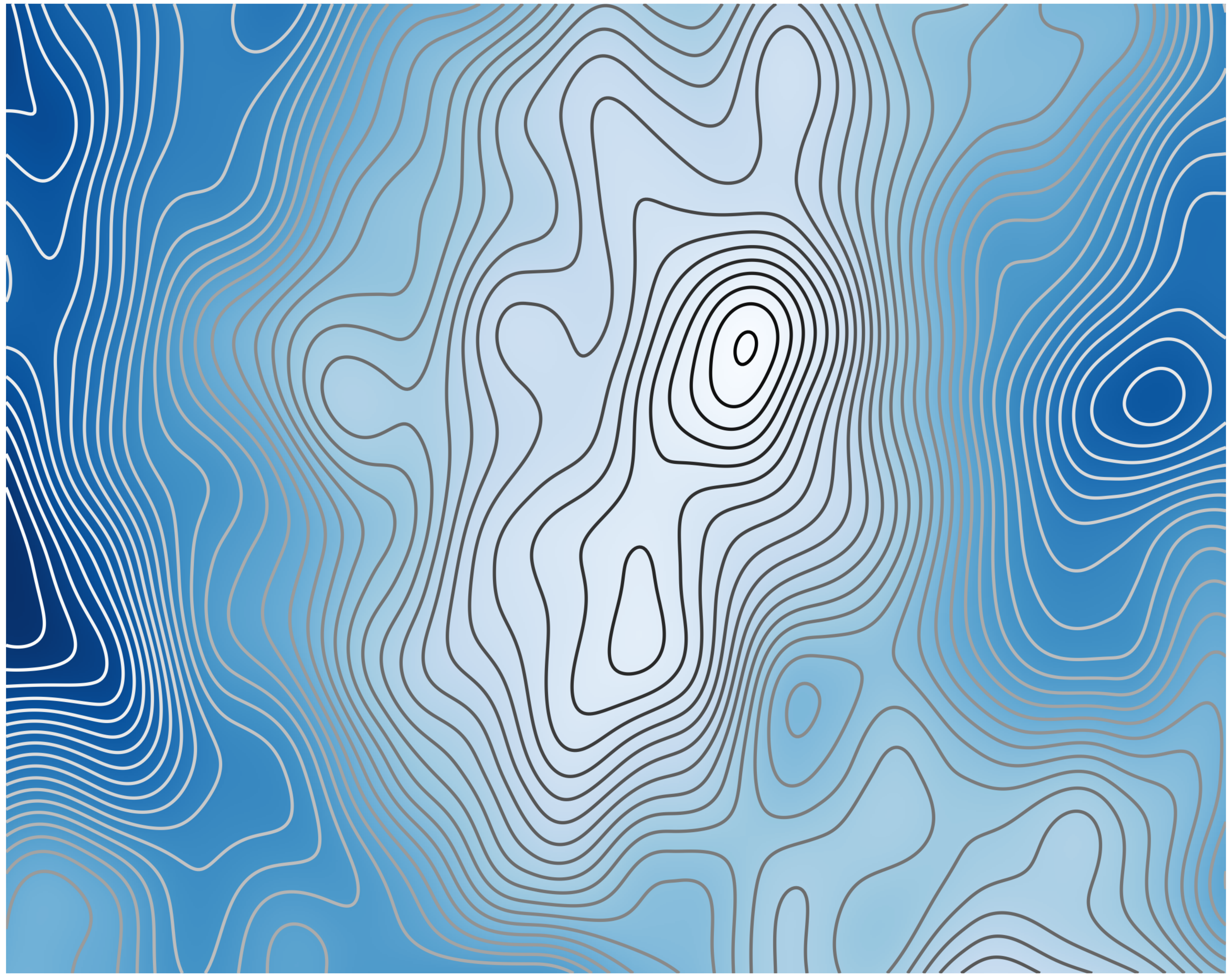


Arbitrary Potential

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x) \qquad \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right) - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = m\ddot{x}$$

$$-\frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \frac{\delta V}{\delta x}$$



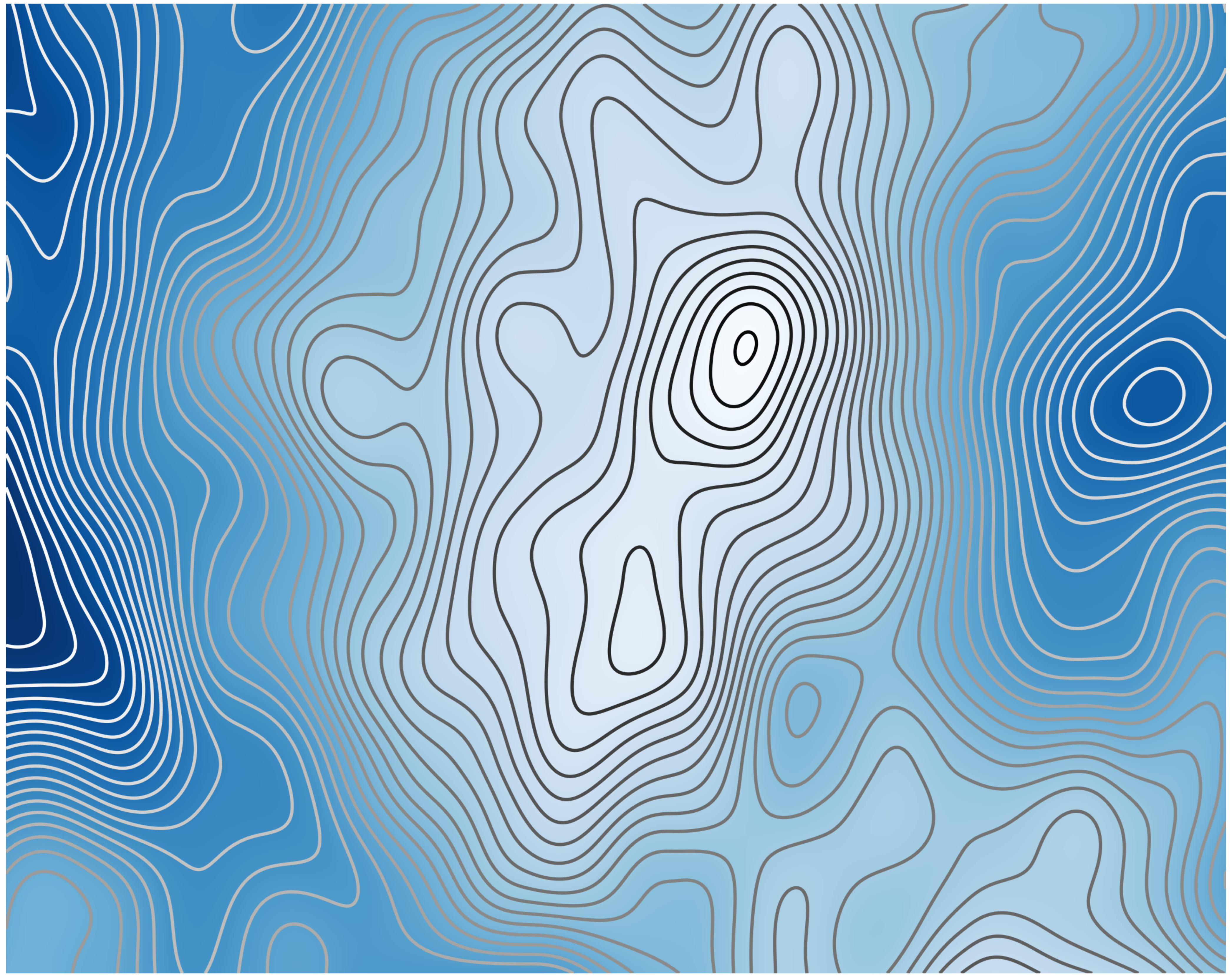
Arbitrary Potential

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right) - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = m\ddot{x}$$

$$-\frac{\delta}{\delta x}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \frac{\delta V}{\delta x}$$

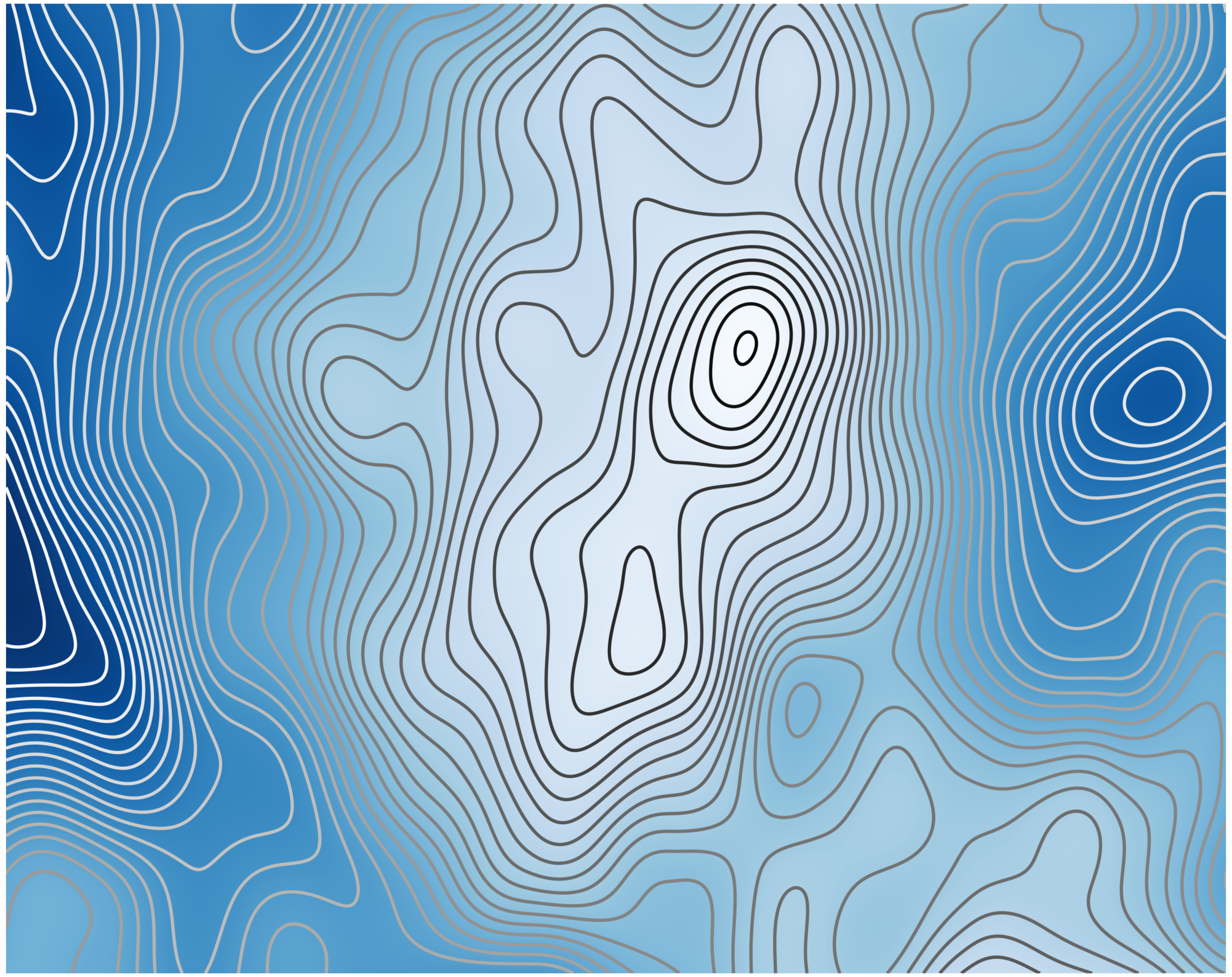
$$m\ddot{x} + \frac{\delta V}{\delta x} = 0 \longrightarrow m\ddot{x} = -\frac{\delta V}{\delta x}$$



Newton's Second Law

$$m\ddot{x} + \frac{\delta V}{\delta x} = 0 \longrightarrow m\ddot{x} = -\frac{\delta V}{\delta x}$$

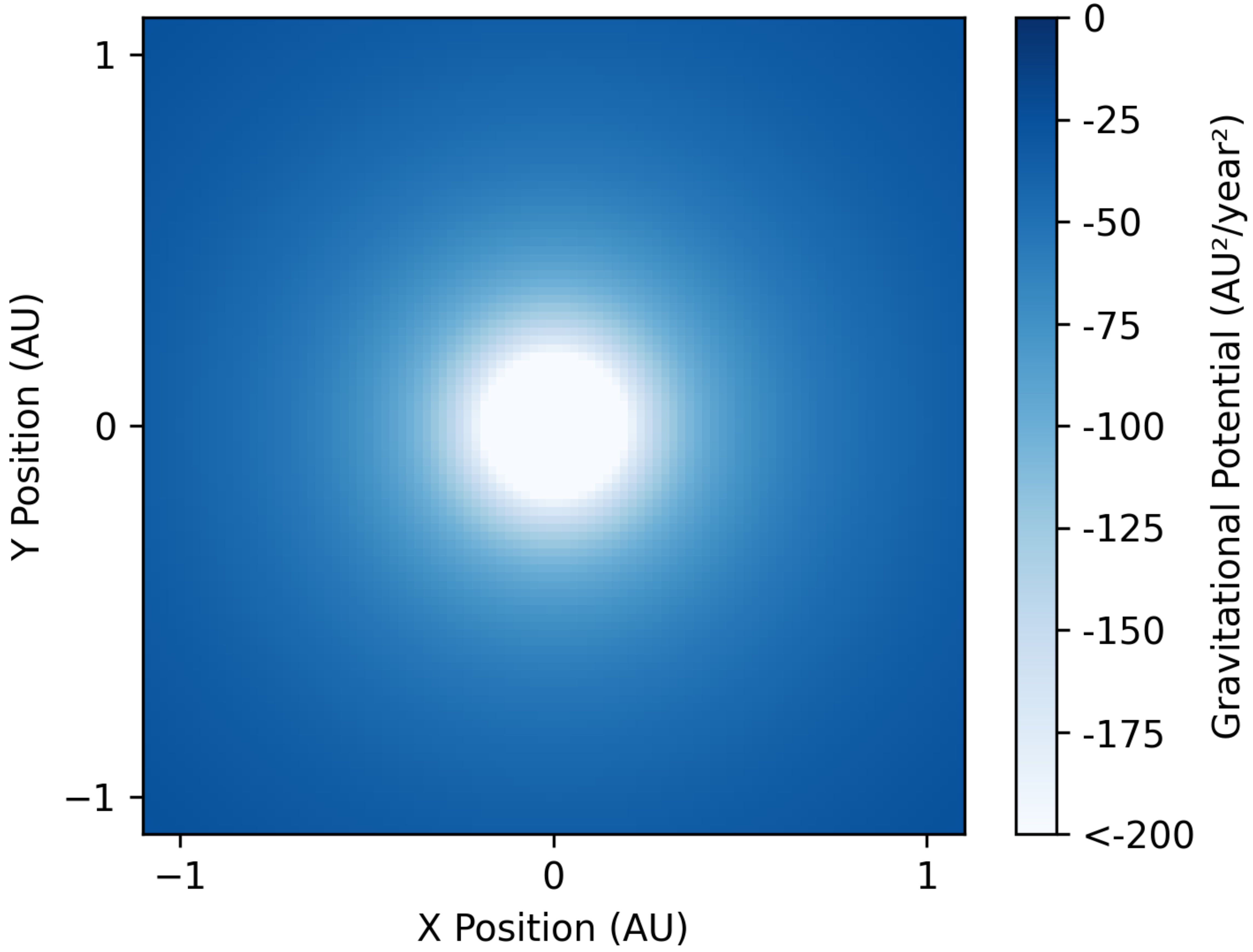
$$F = m\ddot{x} = -\frac{\delta V}{\delta x}$$



Newton's Law of Gravitation

How about if we place our particle in a gravitational potential?

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$



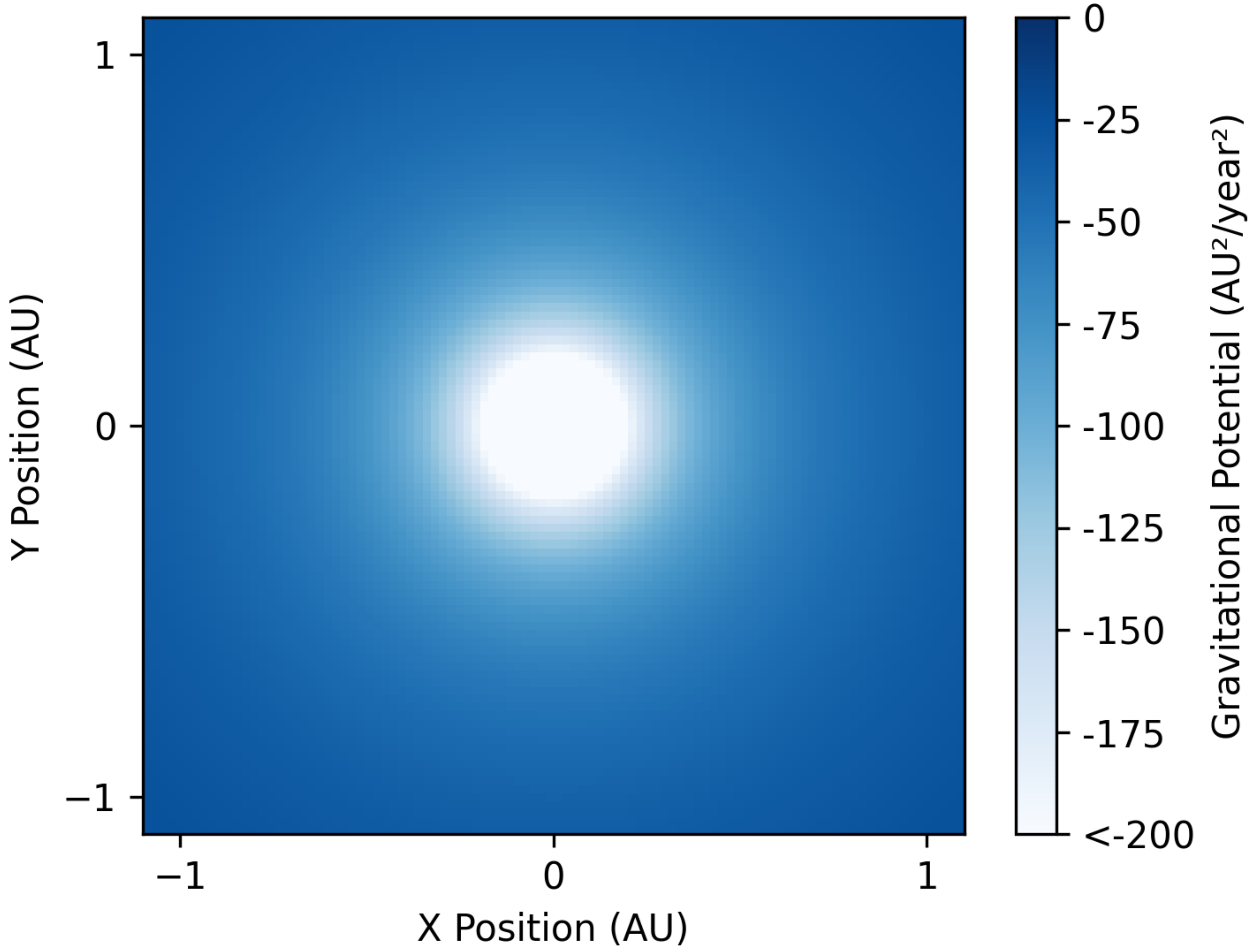
Newton's Law of Gravitation

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta L}{\delta \dot{r}} = m\dot{r} \qquad \frac{\delta L}{\delta r} = -\frac{GMm}{r^2}$$

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{r}}\right) = m\ddot{r}$$



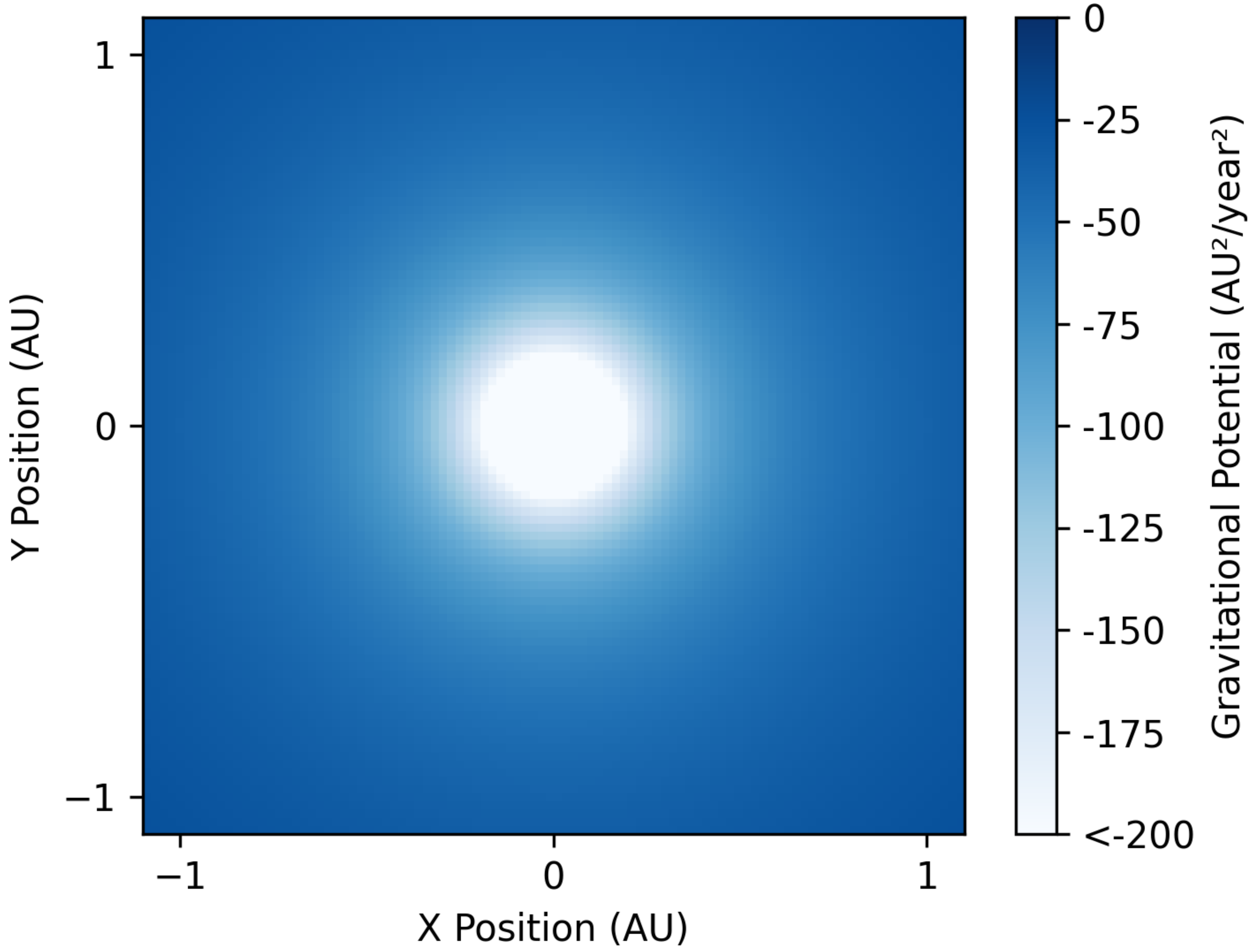
Newton's Law of Gravitation

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r}$$

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0$$

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{r}}\right) = m\ddot{r} \qquad \frac{\delta L}{\delta r} = -\frac{GMm}{r^2}$$

$$m\ddot{r} - \frac{GMm}{r^2} = 0 \longrightarrow F = m\ddot{r} = \frac{GMm}{r^2}$$

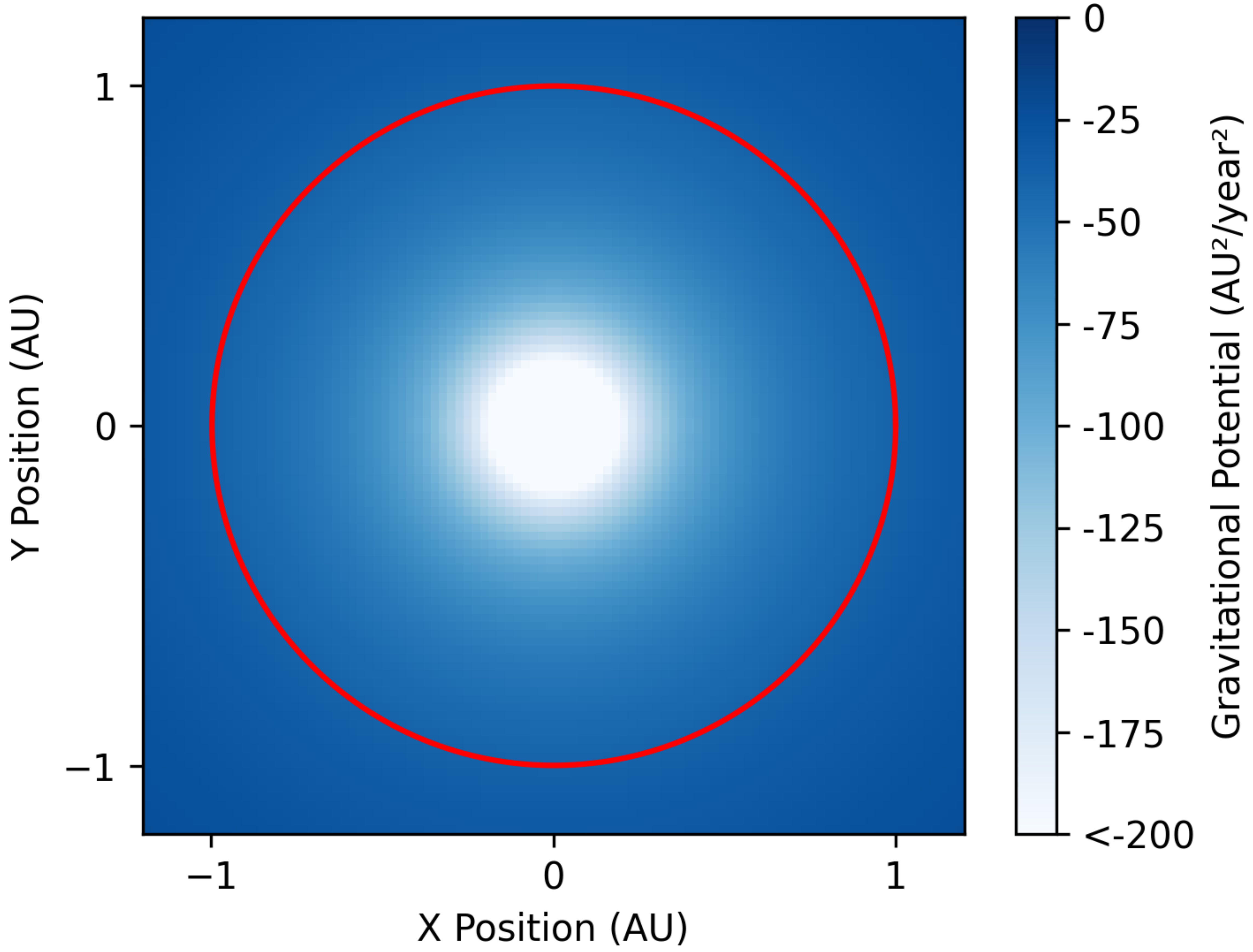


Newton's Law of Gravitation

$$m\ddot{r} - \frac{GMm}{r^2} = 0 \longrightarrow F = m\ddot{r} = \frac{GMm}{r^2}$$

By enforcing Hamilton's Principle we recover the Euler-Lagrange equation and using this we can understand orbital motion

$$\ddot{r} = \frac{GM}{r^2} \xrightarrow{\text{integrate w.r.t. t}} \text{Orbital Motion}$$



Conclusions

You can do lots of exciting things with Hamilton's Principle and the Euler-Lagrange equation

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = 0 \longrightarrow \frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} - 0$$

Here we derived Newtons Laws of motion but Hamilton's Principle is the gateway to modern physics

"The action of a path followed by a particle moving in a potential is insensitive to first order perturbations in the path."