Hamilton's Principle
Classical Dynamics

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1 Introduction

As physicists, we are deeply interested in how complex systems evolve over time. Historically, we have used observations to inform our understanding of the forces acting on objects in systems. A classic example is the study of orbital mechanics. Between 1609 and 1619 Kepler outlined his three laws of orbital motion which he inferred from observations taken by Tycho Brahe of the motion of objects in our solar system. In 1687 Newton used Kepler's laws to illustrate his inverse square law for gravitation.

Up until now, you have mostly thought about how systems evolve using the forces acting on the objects, but we can also describe motion by thinking about the relationship between an object's kinetic energy and the potential it sits in. In 1834 Hamilton established a deeper fundamental description of motion based on energy landscape that can be used to derive Newton's laws and much more.

In this lecture we will introduce Hamilton's Principle, the Lagrangian and the Action. We will use Hamilton's Principle to derive Newton's first and second laws of motion, as well as his law of gravity. Hamilton's Principle states that

> *The action of a path followed by a particle moving in a potential is insensitive to first order perturbations in the path.*

To understand Hamilton's principle, we first need to think about the Lagrangian and its relation to the action, which we define in the following section.

2 The Lagrangian and the Action

The action captures the cumulative balance between kinetic and potential energy along a path and effectively quantifies the total "cost" of travelling between two points in time and space. One can think of the Lagrangian as the instantaneous cost of motion in a potential field,

$$L = T - V, \tag{1}$$

and the action as the total cost accumulated over time,

$$S = \int_{t_0}^{t_1} L \, dt. \tag{2}$$

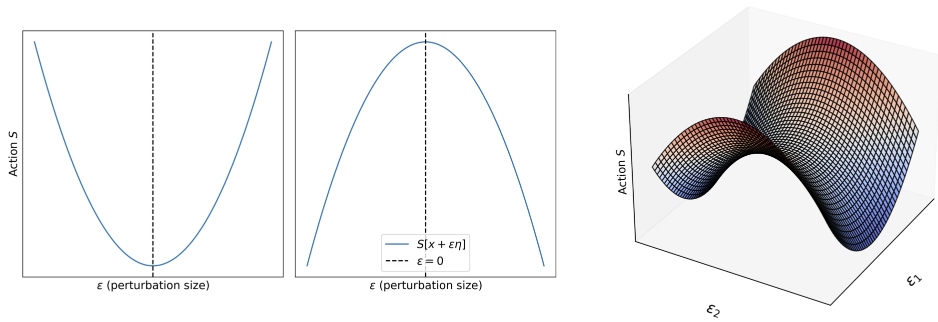

Figure 1: Stationarity can occur at a minimum, maximum or saddle point in the action and Hamilton's principle says that the path taken by a particle in a potential is given by a stationary action.

Hamilton's Principle tells us that the path taken by a particle in a potential extremises the action. The path is not always the one of least energy or least action, but is the one for which small variations ϵ lead to no first-order change in the action

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = 0, \tag{3}$$

in other words the action is stationary. Importantly, stationary here does not necessarily mean minimal; the action could be at a minimum, maximum, or saddle point as shown in Fig. 1.

3 The Euler-Lagrange equation

We can use the Lagrangian, the Action and Hamilton's principle to derive the Euler-Lagrange equation, which allows one to investigate the dynamics of different systems. In this section we will derive the Euler-Lagrange equation and in the following section demonstrate how it can help us understand how systems evolve over time.

If we start with the definition of the action of a particle

$$S[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt, \tag{4}$$

and consider what happens when we vary the path taken by our particle such that $x(t) \rightarrow x(t)+\epsilon\eta(t)$ where ϵ is small and $\eta(t_0) = \eta(t_1) = 0$ (i.e. any variation vanishes at the end points of the wiggles) then

$$S[x(t)] = \int_{t_0}^{t_1} L(x + \epsilon\eta, \dot{x} + \epsilon\dot{\eta}, t)dt. \tag{5}$$

The condition of stationarity implied by Hamilton's principle suggests

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = 0, \tag{6}$$

and

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \frac{\delta L}{\delta x}\frac{\delta(x+\epsilon\eta)}{\delta\epsilon} + \frac{dL}{\delta \dot{x}}\frac{\delta(\dot{x}+\epsilon\dot{\eta})}{\delta\epsilon} dt = 0, \tag{7}$$

which can be written as

$$\left.\frac{\delta S}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \frac{\delta L}{\delta x}\eta + \frac{\delta L}{\delta \dot{x}}\dot{\eta} dt = 0. \tag{8}$$

Looking now at the second term in the integrand we can use integration by parts

$$\int u dv = uv - \int v du \tag{9}$$

to say

$$\begin{aligned} u &= \frac{\delta L}{\delta \dot{x}} \\ dv &= \dot{\eta} dt \\ du &= \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right) dt \\ v &= \eta \end{aligned} \tag{10}$$

$$\int_{t_0}^{t_1} \frac{\delta L}{\delta \dot{x}}\dot{\eta} dt = \left[\frac{\delta L}{\delta \dot{x}}\eta\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right)\eta dt.$$

The first term in the equation above disappears because $\eta(t_0) = \eta(t_1) = 0$ and so substituting the last term back into Eq. 11 we end up with

$$\left.\frac{dS}{\delta \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} \left(\frac{\delta L}{\delta x} - \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right)\right)\eta dt = 0. \tag{11}$$

Since η an arbitrary perturbation in the path the only way for this to be 0 is for

$$\boxed{\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0.} \tag{12}$$

This is known as the Euler-Lagrange equation and from it, we can learn how systems evolve over time. In the following section, we will derive Newton's Laws with Hamilton's principle.

4 Examples

4.1 Free Particle

For a free particle moving with some kinetic energy in $V(x,t) = 0$ the Lagrangian is given by

$$L = \frac{1}{2}m\dot{x}^2 \tag{13}$$

which using the Euler-Lagrange equation implies

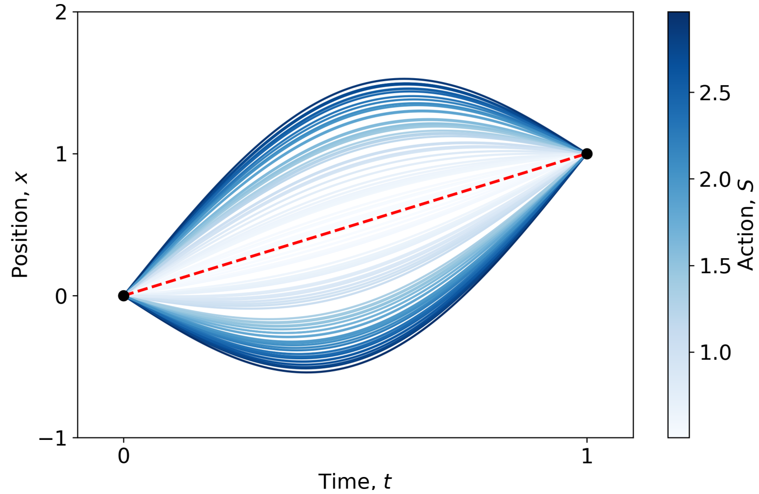

Figure 2: The graph shows the possible paths taken by a free particle ($V(x) = 0$) between two points in space and time. The colour bar shows the action and the path that extremises the action (in this case minimises) is shown as a red dashed line.

$$\begin{aligned} \frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}(\frac{1}{2}m\dot{x}^2) - \frac{\delta}{\delta x}(\frac{1}{2}m\dot{x}^2) &= 0 \\ \frac{\delta}{\delta t}(m\dot{x}) &= 0 \\ m\ddot{x} &= 0 \end{aligned} \tag{14}$$

This is Newton's first law of motion, i.e. that a particle in motion or at rest will stay in motion or at rest unless it is acted upon by an external force. In other words if we set a particle off in motion with velocity $\dot{x}$ at time t_0 it will follow a straight path towards x_1 at t_1. This is shown in Fig. 2 and can be seen if we solve the differential equation above

$$\begin{aligned} m\ddot{x} = 0 &\xrightarrow{\text{Divide m}} \ddot{x} = 0 \\ \xrightarrow{\text{integrate w.r.t. t}} &\ \dot{x} = \text{const} = v_0 \\ \xrightarrow{\text{integrate w.r.t. t}} &\ x = v_0 t' + c. \end{aligned} \tag{15}$$

Since we know that $x(t_0) = x_0$ and $x(t_1) = x_1$ then we can solve for $v_0 = \frac{x_1 - x_0}{t_1 - t_0}$, c is just the starting position x_0 and t' is the time since t_0 so $t' = (t - t_0)$ and

$$x = \frac{x_1 - x_0}{t_1 - t_0}(t - t_0) + x_0. \tag{16}$$

This is illustrated in the python notebook here:`https://github.com/htjb/Talks/tree/master/Lectures/Hamiltons_Principle/free-particle.ipynb`.

4.2 Newton's Second Law

Newton's second law of motion states that the force acting on an object is equal to its mass times its acceleration squared

$$F = ma. \tag{17}$$

It can be derived using the Euler-Lagrange equation. We define the Lagrangian as

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{18}$$

where V is some arbitrary potential. Remembering that the Euler-Lagrange equation is given by

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0, \tag{19}$$

we have

$$\begin{aligned} \frac{\delta}{\delta t}\frac{\delta}{\delta \dot{x}}(\frac{1}{2}m\dot{x}^2 - V(x)) - \frac{\delta}{\delta x}(\frac{1}{2}m\dot{x}^2 - V(x)) &= 0 \\ \frac{\delta}{\delta t}m\dot{x} + \frac{\delta V}{\delta x} &= 0 \\ m\ddot{x} &= -\frac{\delta V}{\delta x} \\ m\ddot{x} &= F \end{aligned} \tag{20}$$

Thus we see then that the path that extremises the action satisfies Newton's second law.

4.3 Newton's Law of Gravity

In this final example we will show that you can derive Newton's Law of Gravitation with the Euler-Lagrange equation. For a particle travelling in a gravitational field the Lagrangian is given by

$$L = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r}, \tag{21}$$

where r is the position of our particle with mass m relative to the attracting body with mass M and G is Newton's gravitational constant. Note that r is a radial vector. Substituting this into the Euler-Lagrange equation

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{r}} - \frac{\delta L}{\delta r} = 0, \tag{22}$$

we have

$$\begin{aligned} \frac{\delta}{\delta t}\frac{\delta}{\delta \dot{r}}(\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r}) - \frac{\delta}{\delta r}(\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r}) &= 0 \\ \frac{\delta}{\delta t}m\dot{r} - \frac{GMm}{r^2} &= 0 \\ m\ddot{r} &= \frac{GMm}{r^2}. \end{aligned} \tag{23}$$

In other words, the acceleration experienced by a particle in a gravitational field is $\ddot{r} \propto 1/r^2$, giving us Newton's inverse square law of gravitation. This result is remarkable: we've derived one of the most fundamental laws of physics purely from the principle that nature extremizes the action.

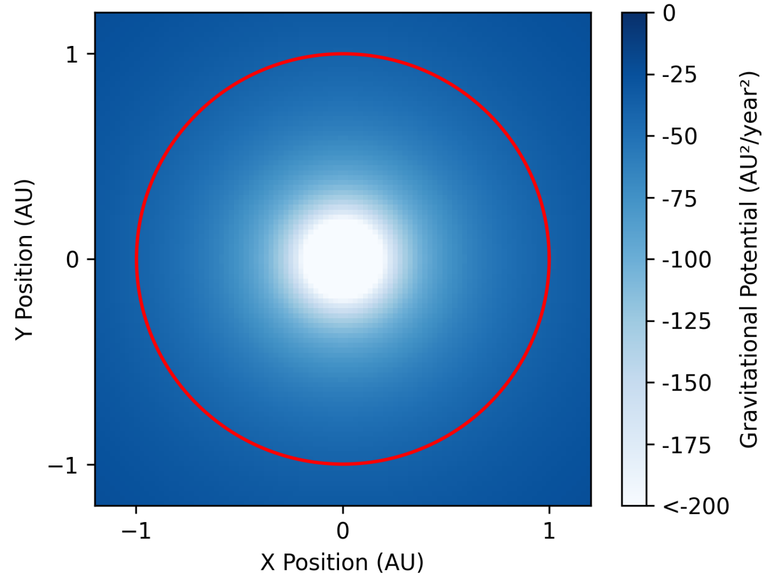

Figure 3: The orbital motion of an object of mass m around another of mass M can be derived with Hamilton's principle. The orbit shown here is the approximate orbit of the Earth around the Sun. You can change the masses of the objects, initial velocity and position in the python notebook (`https://github.com/htjb/Talks/tree/master/Lectures/Hamiltons_Principle/gravity.ipynb`) and see how Hamilton's principle gives rise to elliptical orbits, circular orbits and deflection of objects under gravity.

The solutions to this equation describe all possible orbital motions - from the elliptical orbits of planets to the parabolic trajectories of comets. By integrating equation (24), we can determine the complete motion of any object in a gravitational field, as illustrated in Fig. 3 and demonstrated in the accompanying Python notebook here:`https://github.com/htjb/Talks/tree/master/Lectures/Hamiltons_Principle/gravity.ipynb`.

This example beautifully demonstrates the power of Hamilton's principle: a single, elegant variational principle unifies mechanics and reveals the deep connection between geometry (the path through space-time) and dynamics (the forces and accelerations we observe).

5 The outlook

Hamilton's principle is a powerful tool from which you can derive a huge range of physical phenomena. Hamilton's principle and the Euler-Lagrange equation are the gateway to modern physics, and you will encounter the Euler-Lagrange equation a lot during your studies and beyond. Provided you can write down a Lagrangian or define your action you can do lots of exciting things like derive Maxwell's equations for electromagnetism, Einstein's equations of general relativity and derive the fundamental equations of quantum field theory.