Unit 05 – Proofs

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Outline

Axioms

Direct Proof

Indirect Proof

Proof by Contradiction

Axioms

Definition

Axioms are basic propositions or propositional formulas which are true (hold) from the beginning. They are normally accepted without any proof.

Example

- An equivalence relation R is reflexive, symmetric, and transitive.
- The composition $\wedge$ is associative, commutative, and distributive.

Proof Rules

Proof rules determine how you can make new true formulas from already existing true formulas.

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- $(A \Rightarrow C) \wedge (C \Rightarrow B) \vdash (A \Rightarrow B)$

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3. If you own a longhair cat your couch will be (sooner or later) contaminated with hairs

Remark

A simplified but famous version of the direct proof is the so-called *Modus Ponens*: $A \wedge (A \Rightarrow B) \vdash B$

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Prove the rule via truth table.

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- Assume that a is odd and b is odd
- Show under this assumption that $a \cdot b$ must be odd

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Example

- Prove that a^2 is even $\Rightarrow$ a is even
- Assume that a^2 is even
- Assume for the sake of contradiction that a is odd
- Then you can conclude that a^2 is odd, which is in contradiction to our first assumption

Proof by Contradiction

A Specific Variant

- Sometimes the proposition to be proven is **not** of the form $A \Rightarrow B$
- Instead only a proposition S is to be proven
- This is a short form of $K \Rightarrow S$, where K is the already proven knowledge or the axioms about the underlying universe of discourse
- In this case it is to proof $K \wedge \neg S \Rightarrow \mathbf{F}$
- In short: In order to proof S we assume $\neg S$ and show that this leads to "absurdity"

Proof by Contradiction

A Specific Variant

Example

- $\neg(\exists n \in \mathbb{N}, \forall i \in \mathbb{N} : i \leq n)$ aka "There is no greatest natural number"
- We assume $\exists n \in \mathbb{N}, \forall i \in \mathbb{N} : i \leq n$ aka "There is a greatest natural number"
- Lets take a new natural number $k = n + 1$ which exists since $n \in \mathbb{N}$ and adding 1 to n is again a natural number (since $\mathbb{N}$ is closed under $+$).
- Now we have a new natural number $k > n$ which contradicts our assumption
- Therefore the initial proposition is proven