Unit 05 – Proofs

Peter Bauer

Outline

Axioms

Definition

Axioms are basic propositions or propositional formulas which are true (hold) from the beginning. They are normally accepted without any proof.

Example

- An equivalence relation R is reflexive, symmetric, and transitive.
- The composition $\wedge$ is associative, commutative, and distributive.

Proof Rules

Proof rules determine how you can make new true formulas from already existing true formulas.

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Remark

A simplified but famous version of the direct proof is the so-called *Modus Ponens*: $A \wedge (A \Rightarrow B) \vdash B$

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Prove the rule via truth table.

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- Show under this assumption that $a \cdot b$ must be odd

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Example

- Prove that a^2 is even $\Rightarrow$ a is even
- Assume that a^2 is even
- Assume for the sake of contradiction that a is odd
- Then you can conclude that a^2 is odd, which is in contradiction to our first assumption

Proof by Contradiction

A Specific Variant

- Sometimes the proposition to be proven is **not** of the form $A \Rightarrow B$
- Instead only a proposition S is to be proven
- This is a short form of $K \Rightarrow S$, where K is the already proven knowledge or the axioms about the underlying universe of discourse
- In this case it is to proof $K \wedge \neg S \Rightarrow \mathbf{F}$
- In short: In order to proof S we assume $\neg S$ and show that this leads to "absurdity"

Proof by Contradiction

A Specific Variant

Example

- $\neg(\exists n \in \mathbb{N}, \forall i \in \mathbb{N} : i \leq n)$ aka "There is no greatest natural number"
- We assume $\exists n \in \mathbb{N}, \forall i \in \mathbb{N} : i \leq n$ aka "There is a greatest natural number"
- Lets take a new natural number $k = n + 1$ which exists since $n \in \mathbb{N}$ and adding 1 to n is again a natural number (since $\mathbb{N}$ is closed under $+$).
- Now we have a new natural number $k > n$ which contradicts our assumption
- Therefore the initial proposition is proven