Unit 03 – Minimizing Formulas

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Outline

Propositional Logic Proofs

Disjunctive Normal Form (DNF)

Minimization of Propositional Logic Formulas

- Karnaugh Map
- Quine-McCluskey Algorithm

Minimizing

- Given: a propositional formula P
- Wanted: a propositional formula Q
- where $P \equiv Q$
- Q should be minimal according to some criterion (e.g., by number of variables, length of formula, ...)
- We "go" from P to Q by tiny steps which are already proven
- These small steps are the well known properties of the compositions

Proofs

- Given: two propositional formulas P and Q
- Wanted: a “way” from P to Q
- in order to show that $P \equiv Q$
- The “way” is again gone by tiny steps which are already proven

Summary and Motivation for Training

- Minimizing and proofing require the same skills
- Manipulate propositional formulas by using our well-known properties
- Two birds with one stone!
- So lets start doing some training

A First Example

We want to proof that $C \vee \neg(B \wedge C) \equiv \mathbf{T}$

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\mathbf{T} \vee \neg B &\equiv && \text{[Taut.]} && (5)\\
\mathbf{T} & \quad \square && && (6)
\end{aligned}
$$

A Second Example

Your Turn

Proof that $(P \vee \neg Q) \Rightarrow Q \equiv (\neg P \vee Q) \wedge Q$
Bring the lines below in the right order and write down the properties used.
Hint: Here a property $A \Rightarrow B \equiv \neg A \vee B$ is used. We call it the Implication Elimination Rule (IER).

$$(P \vee \neg Q) \Rightarrow Q \equiv \tag{1}$$

$$(\neg P \vee Q) \wedge Q \quad \square \tag{2}$$

$$(\neg P \vee Q) \wedge (Q \vee Q) \equiv \tag{3}$$

$$(\neg P \wedge Q) \vee Q \equiv \tag{4}$$

$$\neg(P \vee \neg Q) \vee Q \equiv \tag{5}$$

A Third Example

Again Your Turn

Proof that $A \Rightarrow B \equiv A \wedge \neg B \Rightarrow \mathbf{F}$

$$
\begin{aligned}
A \Rightarrow B &\equiv && [\mathit{IER}] && (1)\\
\neg A \;\underline{\quad}\;\underline{\quad} &\equiv && [\mathit{DeMorgan}] && (2)\\
\underline{\quad}\;(\neg\neg\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;) &\equiv && [\mathit{DoubleNegation}] && (3)\\
\underline{\quad}\;(\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;) &\equiv && [\mathit{Tautology}] && (4)\\
\underline{\quad}\;(\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;)\underline{\quad}\;\mathbf{F} &\equiv && [\mathit{IER}] && (5)\\
\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\; \Rightarrow \underline{\quad} & \;\;\square && && (6)
\end{aligned}
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- DNF is a special form of propositional formulas
- Has only propositional variables, $\neg$, $\wedge$ and $\vee$
- Every propositional formula can be transformed into DNF easily
- Is a basis for several methods for automated formula minimization

Disjunctive Normal Form

Definition

A propositional logic formula of the form

$$
\begin{aligned}
(A_{11} \wedge A_{12} \wedge \ldots \wedge A_{1n}) \vee \\
(A_{21} \wedge A_{22} \wedge \ldots \wedge A_{2n}) \vee \\
\ldots \\
(A_{m1} \wedge A_{m2} \wedge \ldots \wedge A_{mn})
\end{aligned}
$$

is called a *formula in disjunctive normal form* (DNF).

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$$Q_j = \left(\begin{pmatrix} A_1 \\ \text{or} \\ \neg A_1 \end{pmatrix} \wedge \begin{pmatrix} A_2 \\ \text{or} \\ \neg A_2 \end{pmatrix} \wedge \ldots \wedge \begin{pmatrix} A_n \\ \text{or} \\ \neg A_n \end{pmatrix} \right) .$$

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- Choose A_i for the i^{th} variable if the truth table holds a t and $\neg A_i$ otherwise.
- The complete formula P' is realized by assembling all the sub-formulas $Q_1, Q_2, \ldots, Q_m$ as follows:

$$P' = Q_1 \vee Q_2 \vee \ldots \vee Q_m$$

Example

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Consider the formula $P(A, B) = A \Rightarrow B$. The truth table would be as follows:

A	B	$P(A, B)$	$Q_j(A, B)$
t	t		

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t	t	t	$A \wedge B$
t	f		

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t	f	f	

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t	f	f	—
f	t	t	$\neg A \wedge B$
f	f	t	$\neg A \wedge \neg B$

Therefore, P could be rewritten in disjunctive normal form as $P'(A, B) = (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$.

Minimizing Formulas – Motivation

- Easier to understand
- Save money when constructing integrated circuits

Karnaugh Map

Consider the following formula in DNF

$$P'(A, B) = (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B).$$

In the two dimensional case we construct a diagram in the following way:

	$\neg A$	A
$\neg B$		
B		

Karnaugh Map

Consider the following formula in DNF

$$P'(A,B) = \boxed{(A \wedge B)} \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B).$$

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B		

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$\neg B$		
B		$\boxed{1}$

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B		1

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$\neg B$		
B	1	1

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$$P'(A, B) = (A \wedge B) \vee (\neg A \wedge B) \vee \boxed{(\neg A \wedge \neg B)}.$$

In the two dimensional case we construct a diagram in the following way:

	$\neg A$	A
$\neg B$		
B	1	1

Karnaugh Map

Consider the following formula in DNF

$$P'(A,B) = (A \land B) \lor (\neg A \land B) \lor \boxed{(\neg A \land \neg B)}.$$

In the two dimensional case we construct a diagram in the following way:

	$\neg A$	A
$\neg B$	1	
B	1	1

Karnaugh Map

Consider the following formula in DNF

$$P'(A,B) = (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B).$$

In the two dimensional case we construct a diagram in the following way:

	$\neg A$	A
$\neg B$	1	
B	1	1

Now we try to find boxes of size 2 or 4 or 8, etc. These boxes should be as large as possible. The number of boxes should be as small as possible. The boxes may overlap.

Karnaugh Map

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$\neg B$	1	
B	1	1

$\neg A$

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In the two dimensional case we construct a diagram in the following way:

	$\neg A$	A
$\neg B$	1	
B	1	1

$\neg A \vee B$

Karnaugh Map – Another Example

Example

$$
\begin{aligned}
&(\neg A \wedge B \wedge C \wedge \neg D) \vee (A \wedge \neg B \wedge \neg C \wedge \neg D) && \vee (A \wedge \neg B \wedge \neg C \wedge D) \vee \\
&\;(A \wedge \neg B \wedge C \wedge D) \vee (A \wedge \neg B \wedge C \wedge \neg D) && \vee (A \wedge B \wedge \neg C \wedge \neg D) \vee \\
&\;(A \wedge B \wedge \neg C \wedge D) \vee (A \wedge B \wedge C \wedge \neg D)
\end{aligned}
$$

Another Example – Sample Solution

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	A	A	$\neg A$	$\neg A$	
B		1			D
B	1	1		1	$\neg D$
$\neg B$	1	1			$\neg D$
$\neg B$	1	1			D
	C	$\neg C$	$\neg C$	C	

$$(A \wedge \neg C) \vee (A \wedge \neg B) \vee (B \wedge C \wedge \neg D)$$

Karnaugh Map for Three Variables

	A	A	$\neg A$	$\neg A$	
B					
$\neg B$					
	C	$\neg C$	$\neg C$	C	

Minterm and Literal

Definition

Given a propositional logic formula $P = Q_1 \vee Q_2 \vee \ldots \vee Q_m$, where all $Q_j (1 \leq j \leq m)$ are of the form

$$Q_j = \left(\begin{pmatrix} A_1 \\ \text{or} \\ \neg A_1 \end{pmatrix} \wedge \begin{pmatrix} A_2 \\ \text{or} \\ \neg A_2 \end{pmatrix} \wedge \ldots \wedge \begin{pmatrix} A_n \\ \text{or} \\ \neg A_n \end{pmatrix} \right)$$

we call Q_j a *minterm*. The A_i or $\neg A_i$ $(1 \leq i \leq n)$ are called *literals*.

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Remark

Sometimes we also call a row of a truth table which result is true a minterm.

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Remark

In case of a multiple appearance of a literal A_i in a Q_j, Q_j is called a product term.

Implicant

Definition

Given a propositional logic formula $P = Q_1 \vee \ldots \vee Q_m$ with each minterm Q_i having n literals $A_1, A_2, \ldots, A_n$. Furthermore we have a product term

$$I = \left(\begin{pmatrix} A_1 \\ \text{or} \\ \neg A_1 \end{pmatrix} \wedge \begin{pmatrix} A_2 \\ \text{or} \\ \neg A_2 \end{pmatrix} \wedge \ldots \wedge \begin{pmatrix} A_n \\ \text{or} \\ \neg A_n \end{pmatrix} \right).$$

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I is called a *prime implicant* of P if $I \Rightarrow P$, i.e., that whenever the evaluation of I with a specific set of truth values for $A_1, A_2, \ldots, A_n$ yields true, the evaluation of P with the same set of truth values also yields true.

Example and Remark

Consider the following propositional logic formula P:

$$P \quad = \quad (A_1 \wedge A_2) \vee (A_2 \wedge A_3) \vee A_4$$

This formula is implied by the formulas $A_1 \wedge A_2$, $A_2 \wedge A_3$, A_4, $A_1 \wedge A_2 \wedge A_3$, $A_1 \wedge A2_{\wedge} A_4$ and many others. These are the implicants of P.

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Remark

Implicants are sometimes called "coverings" of one or more minterms.

Prime Implicant

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Example

Let $P = (A_1 \wedge A_2) \vee (A_2 \wedge A_3) \vee A_4$ be a propositional logic formula. The prime implicants of P would be $A_1 \wedge A_2$, $A_2 \wedge A_3$, and A_4 would be the prime implicants of P.

Prime Implicant and Minimal Formulas

Definition

A disjunction of prime implicants covering a complete formula P is called a minimal formula of P.

Remark

There may exist more than one minimal formula.

Example

Given the following truth table M:

x_1	x_2	x_3	y
f	f	f	f
f	f	t	f
f	t	f	f
f	t	t	t
t	f	f	t
t	f	t	t
t	t	f	f
t	t	t	t

Example continued

- $$\begin{array}{ccccc} \neg A_1 & \wedge & A_2 & \wedge & A_3 \\ A_1 & \wedge & \neg A_2 & \wedge & \neg A_3 \\ A_1 & \wedge & \neg A_2 & \wedge & A_3 \\ A_1 & \wedge & A_2 & \wedge & A_3 \end{array}$$
  are the minterms of M.

Example continued

- $$\begin{array}{ccccc} \neg A_1 & \wedge & A_2 & \wedge & A_3 \\ A_1 & \wedge & \neg A_2 & \wedge & \neg A_3 \\ A_1 & \wedge & \neg A_2 & \wedge & A_3 \\ A_1 & \wedge & A_2 & \wedge & A_3 \end{array}$$
  are the minterms of M.
- $A_1 \wedge \neg A_2$ and $A_2 \wedge A_3$ are prime implicants of M.

Example continued

- $$\begin{array}{ccccc} \neg A_1 & \wedge & A_2 & \wedge & A_3 \\ A_1 & \wedge & \neg A_2 & \wedge & \neg A_3 \\ A_1 & \wedge & \neg A_2 & \wedge & A_3 \\ A_1 & \wedge & A_2 & \wedge & A_3 \end{array}$$
  are the minterms of M.
- $A_1 \wedge \neg A_2$ and $A_2 \wedge A_3$ are prime implicants of M.
- $A_1 \wedge \neg A_2 \vee A_2 \wedge A_3$ is a minimal formula representing M.

Quine-McCluskey Algorithm

- An algorithm to find minimal formulas
- Finds all prime implicants of a given formula S
- Finds a combination of prime implicants which covers S
- Example as given in class