Unit 05 – Proofs

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Outline

Axioms

Definition

Axioms are basic propositions or propositional formulas which are true (hold) from the beginning. They are normally accepted without any proof.

Example

- An equivalence relation R is reflexive, symmetric, and transitive.
- The composition $\wedge$ is associative, commutative, and distributive.

Proof Rules

Proof rules determine how you can make new true formulas from already existing true formulas.

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- $(A \Rightarrow C) \wedge (C \Rightarrow B) \vdash (A \Rightarrow B)$

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Remark

A simplified but famous version of the direct proof is the so-called *Modus Ponens*: $A \wedge (A \Rightarrow B) \vdash B$

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Prove the rule via truth table.

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- Show under this assumption that $a \cdot b$ must be odd

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Example

- Prove that a^2 is even $\Rightarrow$ a is even
- Assume that a^2 is even
- Assume for the sake of contradiction that a is odd
- Then you can conclude that a^2 is odd, which is in contradiction to our first assumption