Predicate Logic

Peter Bauer

Outline

Motivation

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- Propositions were considered to be "atomic"
- To express propositions more concretely we need more sophisticated terms
- Examples
  - "x is a prime number."
  - "4 is an even number."
  - "This shark is 20 ft. long."

Variable

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- Variables are assigned specific values out of the universe
- Names of variables are combinations of characters, numbers and other symbols.

Constant

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Definition

Constants are names of concrete objects. These can be concrete values (e.g., π, 17, 42), function constants (e.g., $+$, cos, $\sqrt{\ }$) or predicate constants (e.g., $\geq, \in$).

Term

Definition

Constants and variables are *terms*. Syntactic structures of the form $f(t_1, t_2, \ldots, t_n)$ are *terms*, if f is a function constant with arity n and $t_1, t_2, \ldots, t_n$ are terms.

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Remark

Constant values like π, 17, or "Vergissmeinnicht" can be interpreted as terms. These are terms with a functions constant of arity 0.

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Syntactic structures of the form $P(t_1, t_2, \ldots, t_n)$ are *atomic formulas*, if P is a predicate constant with arity n and $t_1, t_2, \ldots t_n$ are terms.

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- In the sequel, we will use upper case letters like $P, Q, R, \ldots$ to denote predicate constants.
- $P, Q, R, \ldots$ are always predicate constants and are *not* predicate variables (first order predicate logic)

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- $P(x) \wedge Q(x) =$ "x studies LOAL and is a first grader".

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- Now consider the formula $7 \geq 3$.
- Furthermore, consider the formula "for all $a \in \mathbb{N}$ holds: $a \geq 3$"

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- For this purpose we have *Quantifiers* in mathematical logic.
- Quantifiers bind variables in formulas of predicate logic.

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Remark

In the sequel we will use the notations $\forall x : P(x)$ and $\exists x : P(x)$.

Free and Bound Variables

- Variables which are referred to by a quantifier are called *bound variables*.
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- x and y are both free variables.
- $\exists x : x$ has birthday in the same month as y.
- x is a bound variable and y is a free variable.

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- The third remains a formula of predicate logic.

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- Further consider the formula "x attends the 3rd grade".
- This formula has one free variable x.
- "$\forall x : x$ attends the 3rd grade" would only be true if all students of this school would attend the 3rd grade.
- "$\exists x : x$ attends the 3rd grade" is true, since there exists at least one student at this school who attends the 3rd grade.

Summary

We know two possibilities to turn a formula of predicate logic into a proposition

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2. Binding the free variables with quantifiers.

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1. Assignment of constants out of the universe to each free variable.
2. Binding the free variables with quantifiers.
3. Formulas with no free variables are propositions and are sometimes called *closed formulas*.
4. Only these have a truth value.

Relationships Between Negation and Quantifiers

Theorem

Let P be a unary predicate constant and $x \in D$ a variable. Then the following relations hold:

$$\neg \exists x : P(x) \equiv \forall x : \neg P(x)$$
$$\neg \forall x : P(x) \equiv \exists x : \neg P(x)$$

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Example

$$\neg\forall x \in \mathbb{N} : \text{isPrime}(x) \Rightarrow \text{odd}(x)$$
$$\exists x \in \mathbb{N} : \text{isPrime}(x) \land \neg\text{odd}(x)$$