Unit 02 – Propositional Logic – Basics

Peter Bauer

Outline

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- Example: "The circuit is electrically conducting."

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- Expresses that we are not interested in a specific proposition but a variable that can hold arbitrary propositions
- Simplified we can understand propositional variables as variables that hold either the value t or f
- Instead of "Let A be an arbitrary proposition" we can say "Let A be a propositional variable"

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A *proposition* is a “linguistic item” for which it makes sense to ask whether it is “true” or “false”. The terms “true” and “false” are called the *truth values* of a proposition.

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- Propositions are mainly denoted by upper case letters A, B, C, ...
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- We are only interested in the structure of propositions and not in its content.

Examples

- $x \leq 3$

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- In German: Wenn Anastasia den Most holt, wird es bald etwas zu trinken geben, und das Abendessen wird beginnen, vorausgesetzt, dass Bartholomäus das Brot schon gebacken hat.

Basic Compositions

Definition

Let A and B be two propositions. Then the truth values of the propositions $A \wedge B$ ("A and B"), $A \vee B$ ("A or B), and $\neg A$ ("not A") are given by the following truth table:

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Remark

$\wedge$ is called *conjunction*, $\vee$ is called *disjunction* and $\neg$ is called *negation*.

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- To reduce the number of necessary parentheses we use the precedence rules $\neg$ has higher precedence than $\wedge$ higher than $\vee$ higher than $\Rightarrow$ higher than $\Leftrightarrow$
- The syntax of the compositions has to be considered, e.g., $\neg \wedge AB$ is NO proposition

Propositional Formula

Definition

Given a set of propositional variables or propositions $A, B, C, \ldots$. If we combine these propositions by means of the compositions given above, the result of this compositions is called a *propositional formula*. If we assign true or false propositions to each of the propositional variables in a propositional formula we get a concrete proposition.

Equivalence and Implication of Propositional Formulas

Definition

Let $P = P(A, B, C, \ldots)$ and $Q = Q(A, B, C, \ldots)$ be propositional formulas then

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Remark

Very often we loosely interchange the symbols $\equiv$ and $\Leftrightarrow$ as well as $\vdash$ and $\Rightarrow$

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Theorem

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$$\begin{aligned} A \wedge (B \wedge C) &\equiv (A \wedge B) \wedge C \equiv A \wedge B \wedge C \\ A \vee (B \vee C) &\equiv (A \vee B) \vee C \equiv A \vee B \vee C \end{aligned}$$

Distributivity, De-Morgan

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2. *De-Morgan*

$$\begin{aligned} \neg(A \wedge B) &\equiv \neg A \vee \neg B \\ \neg(A \vee B) &\equiv \neg A \wedge \neg B \end{aligned}$$

Absorption, Idempotence, and Double Negation

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3. *Double Negation*

$$\neg(\neg A) \equiv A$$

Tautology and Contradiction

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Example

It is easy to show that $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$ is a tautology.

Laws with Contradictions and Tautologies

Theorem

For a proposition A holds:

$$\begin{aligned} A \wedge \mathbf{T} \equiv A, &\quad A \vee \mathbf{T} \equiv \mathbf{T} \\ A \wedge \mathbf{F} \equiv \mathbf{F}, &\quad A \vee \mathbf{F} \equiv A \\ A \wedge \neg A \equiv \mathbf{F}, &\quad A \vee \neg A \equiv \mathbf{T} \end{aligned}$$