Formal Languages

Matthias Braun

Today's Plan

- What's a formal language?

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- How can we create our own formal language?

Formal Languages

The fuzzy explanation

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- Formal languages are a generalization of programming languages
- A formal language has rules that define which sentences (think programs) are allowed
- This allows us to create (programming) languages!

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- Languages are sets of strings

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- Vertical bar | means choice ("or")
- How many different strings (sentences) does the language generated from this grammar have?

Grammars, formalized

- A grammar consists of **productions rules**:
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- The notation we use for our grammars is called **extended Backus-Naur form** (EBNF)

Formal Languages

The formal explanation

- A formal language is defined by an alphabet and a grammar
- A formal language is the set of all strings generated from the grammar

Test your knowledge

- List all the terminals from the previous grammar
- How many production rules does the grammar have?
- In EBNF, what's the difference between the comma and the vertical bar?

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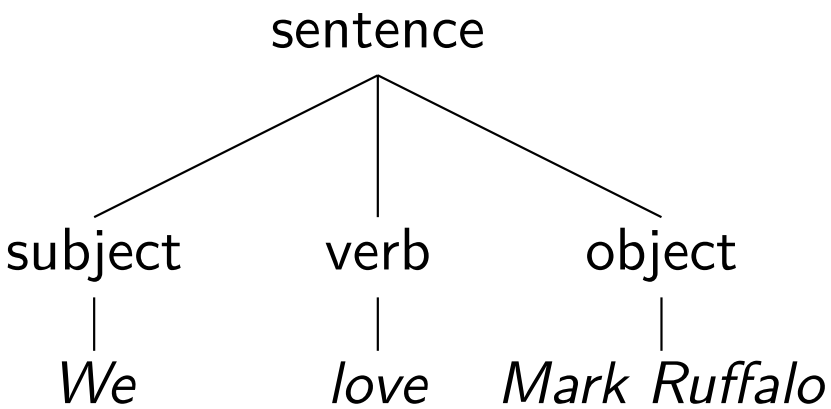

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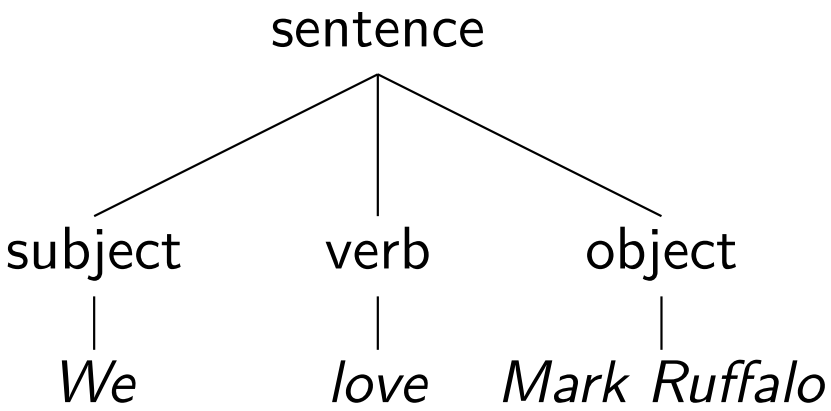

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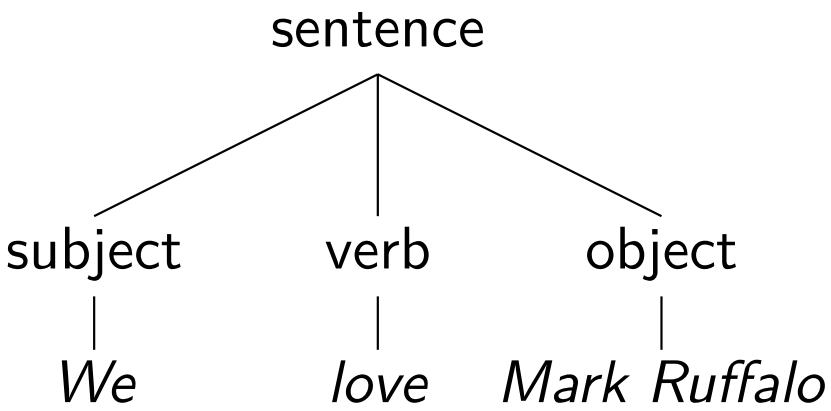

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- Each interior node is a non-terminal production rule

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- `sentence = bit, (sentence | "");`

Your turn

- Write a grammar for a language with *exactly* these three strings:
  {abc, cba, bac}
- A grammar for a language with an *infinite* number of strings like these:
  {a, aa, aaa, aaaa, aaaaa, ...}
- A grammar for a language with an *infinite* number of strings like these:
  {a, b, aba, ababa, aaaa, ...}

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- We want to create bit palindromes:
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- Let's generate the example palindromes above by replacing `pal` step-by-step

Grammar for Addition and Subtraction

- Some valid expressions:
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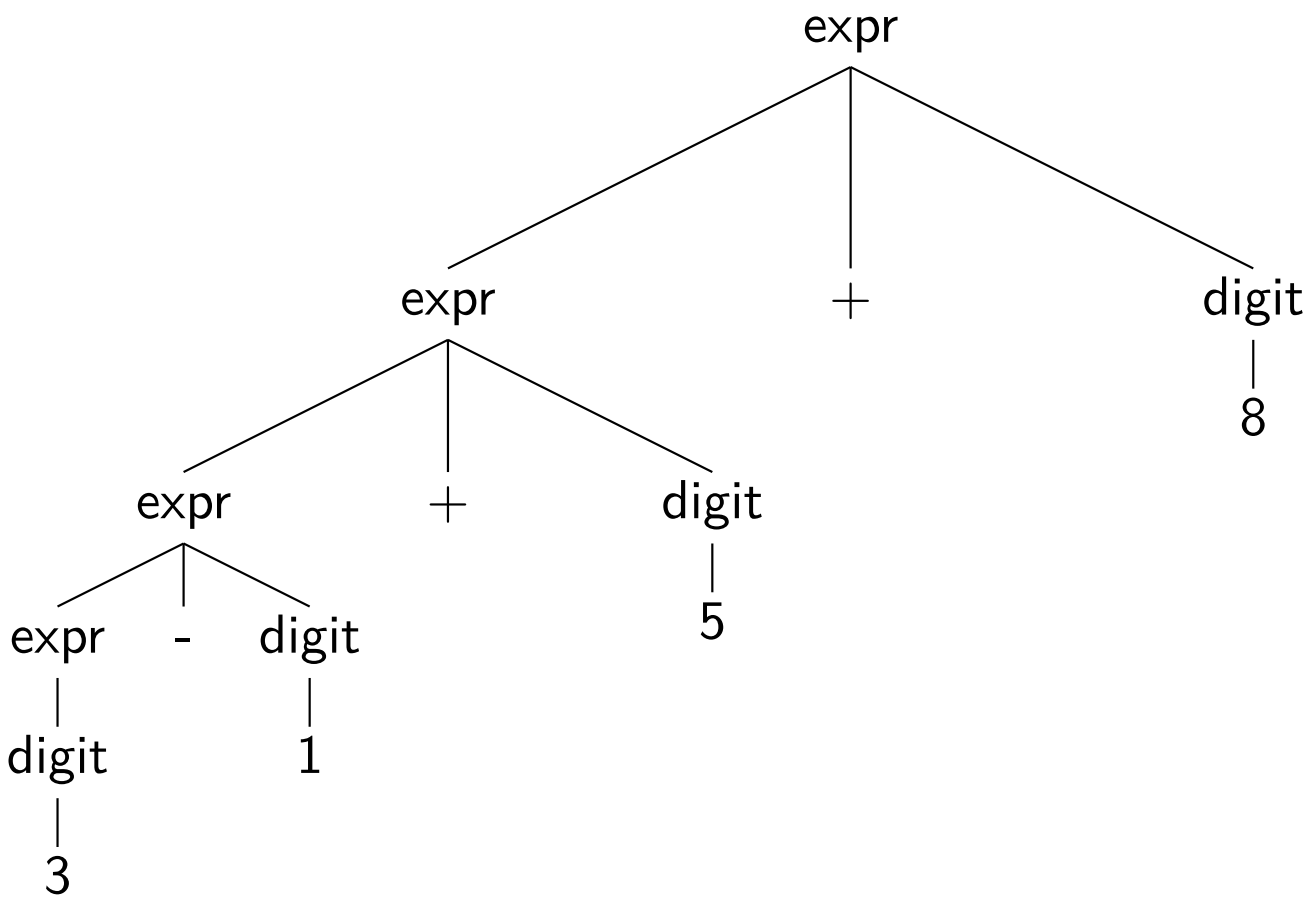
- Production rules for the recursive cases:

```
3 expr = expr, " + ", digit;
4 expr = expr, " - ", digit;
```

Grammar for Addition and Subtraction

Parse Tree for "3 - 1 + 5 + 8"

Notice how this tree grows down towards the *left* because the recursive part, `expr`, is on the left-hand side of `expr, " + ", digit;` and `expr, " - ", digit;`



Your turn

- Generate the following strings from the previous grammar and draw the parse tree. Make a note of the production rule you use in each step:
  - "1"
  - "1 + 2"
  - "8 + 9 - 0"
- Write down a grammar for the language of propositional logic (you know: $\vee$, $\wedge$, $\neg$, parentheses). You can limit the number of different variables in this language to a, b, and c. The language should contain strings like these:
  - "a"
  - "$a \wedge b$"
  - "$a \vee \neg b$"
  - "$a \vee \neg(b \wedge c)$"
  - "$a \vee \neg a$"
- Hint: You can do this with *two* production rules

Your Turn

- Create a grammar that generates integer numbers larger than 99.
- Create a grammar that generates constants in a programming language. The naming convention for constants is that a constant:
  - starts with an upper-case letter
  - only contains upper-case letters and underscores (no digits)
  - has at least one letter
- Let's say that a password should contain at least one lower-case letter and a digit. The minimum password length is two, the maximum is infinity. Create a grammar that generates such passwords. Some valid example passwords:
  - "a0"
  - "9z"
  - "z12ab90zkl"
  - "hunter2"