

Unit 03 – Minimizing Formulas

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Outline

Propositional Logic Proofs

Disjunctive Normal Form

Minimization of Propositional Logic Formulas

KV-Diagrams – Motivation

Quine-McCluskey Algorithm

Minimizing

- ▶ Given: a propositional formula P
- ▶ Wanted: a propositional formula Q
- ▶ where $P \equiv Q$
- ▶ Q should be minimal according to some criterion (e.g., by number of variables, length of formula, ...)
- ▶ We “go” from P to Q by tiny steps which are already proven
- ▶ These small steps are the well known properties of the compositions

Proofs

- ▶ Given: two propositional formulas P and Q
- ▶ Wanted: a “way” from P to Q
- ▶ in order to show that $P \equiv Q$
- ▶ The “way” is again gone by tiny steps which are already proven

Summary and Motivation for Training

- ▶ Minimizing and proofing require the same skills
- ▶ Manipulate propositional formulas by using our well-known properties
- ▶ Two birds with one stone!
- ▶ So lets start doing some training

A First Example

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$$C \vee \neg(B \wedge C) \equiv \quad \text{[DeMorgan]} \quad (1)$$

$$C \vee (\neg B \vee \neg C) \equiv \quad \text{[Commut.]} \quad (2)$$

$$C \vee (\neg C \vee \neg B) \equiv \quad \text{[Assoc.]} \quad (3)$$

$$(C \vee \neg C) \vee \neg B \equiv \quad \text{[Taut.]} \quad (4)$$

$$\mathbf{T} \vee \neg B \equiv \quad \text{[Taut.]} \quad (5)$$

$$\mathbf{T} \quad \square \quad (6)$$

A Second Example

Your Turn

Proof that $(P \vee \neg Q) \Rightarrow Q \equiv (\neg P \vee Q) \wedge Q$

Bring the lines below in the right order and write down the properties used.

Hint: Here a property $A \Rightarrow B \equiv \neg A \vee B$ is used. We call it the Implication Elimination Rule (IER).

$$(P \vee \neg Q) \Rightarrow Q \equiv \quad (1)$$

$$(\neg P \vee Q) \wedge Q \quad \square \quad (2)$$

$$(\neg P \vee Q) \wedge (Q \vee Q) \equiv \quad (3)$$

$$(\neg P \wedge Q) \vee Q \equiv \quad (4)$$

$$\neg(P \vee \neg Q) \vee Q \equiv \quad (5)$$

A Third Example

Again Your Turn

Proof that $A \Rightarrow B \equiv A \wedge \neg B \Rightarrow \mathbf{F}$

$A \Rightarrow B \equiv$	$[IER]$	(1)
$\neg A \quad _ \quad _ \equiv$	$[DeMorgan]$	(2)
$_ (\neg \neg _ \quad _ \quad _ \quad _) \equiv$	$[DoubleNegation]$	(3)
$_ (_ \quad _ \quad _ \quad _) \equiv$	$[Tautology]$	(4)
$_ (_ \quad _ \quad _ \quad _) _ \mathbf{F} \equiv$	$[IER]$	(5)
$_ \quad _ \quad _ \quad _ \Rightarrow _ \quad \square$		(6)

Disjunctive Normal Form – Motivation

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- ▶ Find a logically equivalent formula P'

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- ▶ Given the truth table of a propositional formula P
- ▶ Find a logically equivalent formula P'
- ▶ No unambiguous solution
- ▶ Disjunctive normal form is one (and easy to find) solution
- ▶ Is a basis for several methods of several algorithms for automated formula simplification

Disjunctive Normal Form

Definition

A propositional logic formula of the form

$$\begin{aligned} & (A_{11} \wedge A_{12} \wedge \dots \wedge A_{1n}) \vee \\ & (A_{21} \wedge A_{22} \wedge \dots \wedge A_{2n}) \vee \\ & \dots \\ & (A_{m1} \wedge A_{m2} \wedge \dots \wedge A_{mn}) \end{aligned}$$

is called a *formula in disjunctive normal form* (DNF).

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$$Q_j = \left(\begin{pmatrix} A_1 \\ \text{or} \\ \neg A_1 \end{pmatrix} \wedge \begin{pmatrix} A_2 \\ \text{or} \\ \neg A_2 \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} A_n \\ \text{or} \\ \neg A_n \end{pmatrix} \right).$$

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- ▶ Choose A_i for the i^{th} variable if the truth table holds a t and $\neg A_i$ otherwise.
- ▶ The complete formula P' is realized by assembling all the sub-formulas Q_1, Q_2, \dots, Q_m as follows:

$$P' = Q_1 \vee Q_2 \vee \dots \vee Q_m$$

Example

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Consider the formula $P(A, B) = A \Rightarrow B$. The truth table would be as follows:

A	B	$P(A, B)$	$Q_j(A, B)$
t	t		

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t	f		

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f	f	t	$\neg A \wedge \neg B$

Therefore, P could be rewritten in disjunctive normal form as

$$P'(A, B) = (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B).$$

Minimizing Formulas – Motivation

- ▶ Easier to understand
- ▶ Save money when constructing integrated circuits

KV-Diagrams

Consider the following formula in DNF

$$P'(A, B) = (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B).$$

In the two dimensional case we construct a diagram in the following way:

	$\neg A$	A
$\neg B$		
B		

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B	1	1

Now we try to find boxes of size 2 or 4 or 8, etc. These boxes should be as large as possible. The number of boxes should be as small as possible. The boxes may overlap.

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$\neg A$

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In the two dimensional case we construct a diagram in the following way:

	$\neg A$	A
$\neg B$	1	
B	1	1

$$\neg A \vee B$$

KV-Diagrams – Another Example

Example

$$\begin{aligned} &(\neg A \wedge B \wedge C \wedge \neg D) \vee (A \wedge \neg B \wedge \neg C \wedge \neg D) \quad \vee (A \wedge \neg B \wedge \neg C \wedge D) \vee \\ &(A \wedge \neg B \wedge C \wedge D) \vee (A \wedge \neg B \wedge C \wedge \neg D) \quad \vee (A \wedge B \wedge \neg C \wedge \neg D) \vee \\ &(A \wedge B \wedge \neg C \wedge D) \vee (A \wedge B \wedge C \wedge \neg D) \end{aligned}$$

Another Example – Sample Solution

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	A	A	$\neg A$	$\neg A$	
B		1			D
B	1	1		1	$\neg D$
$\neg B$	1	1			$\neg D$
$\neg B$	1	1			D
	C	$\neg C$	$\neg C$	C	

$$(A \wedge \neg C) \vee (A \wedge \neg B) \vee (B \wedge C \wedge \neg D)$$

KV-Table for Three Variables

	A	A	$\neg A$	$\neg A$	
B					
$\neg B$					
	C	$\neg C$	$\neg C$	C	

Minterm and Literal

Definition

Given a propositional logic formula $P = Q_1 \vee Q_2 \vee \dots \vee Q_m$, where all $Q_j (1 \leq j \leq m)$ are of the form

$$Q_j = \left(\left(\begin{array}{c} A_1 \\ \text{or} \\ \neg A_1 \end{array} \right) \wedge \left(\begin{array}{c} A_2 \\ \text{or} \\ \neg A_2 \end{array} \right) \wedge \dots \wedge \left(\begin{array}{c} A_n \\ \text{or} \\ \neg A_n \end{array} \right) \right)$$

we call Q_j a *minterm*. The A_i or $\neg A_i$ ($1 \leq i \leq n$) are called *literals*.

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Remark

Sometimes we also call a row of a truth table which result is true a minterm.

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Remark

In case of a multiple appearance of a literal A_i in a Q_j , Q_j is called a product term.

Implicant

Definition

Given a propositional logic formula $P = Q_1 \vee \dots \vee Q_m$ with each minterm Q_i having n literals A_1, A_2, \dots, A_n . Furthermore we have a product term

$$I = \left(\begin{pmatrix} A_1 \\ \text{or} \\ \neg A_1 \end{pmatrix} \wedge \begin{pmatrix} A_2 \\ \text{or} \\ \neg A_2 \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} A_n \\ \text{or} \\ \neg A_n \end{pmatrix} \right).$$

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I is called a *prime implicant* of P if $I \Rightarrow P$, i.e., that whenever the evaluation of I with a specific set of truth values for A_1, A_2, \dots, A_n yields true, the evaluation of P with the same set of truth values also yields true.

Example and Remark

Consider the following propositional logic formula P :

$$P = (A_1 \wedge A_2) \vee (A_2 \wedge A_3) \vee A_4$$

This formula is implied by the formulas $A_1 \wedge A_2$, $A_2 \wedge A_3$, A_4 , $A_1 \wedge A_2 \wedge A_3$, $A_1 \wedge A_2 \wedge A_4$ and many others. These are the implicants of P .

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Consider the following propositional logic formula P :

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Remark

Implicants are sometimes called “coverings” of one or more minterms.

Prime Implicant

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Given a propositional logic formula P . An implicant I is called a *prime implicant* if it can't be reduced any further, i.e., any removal of a literal would make it a “non-implicant”.

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Example

Let $P = (A_1 \wedge A_2) \vee (A_2 \wedge A_3) \vee A_4$ be a propositional logic formula. The prime implicants of P would be $A_1 \wedge A_2$, $A_2 \wedge A_3$, and A_4 would be the prime implicants of P .

Prime Implicant and Minimal Formulas

Definition

A disjunction of prime implicants covering a complete formula P is called a minimal formula of P .

Remark

There may exist more than one minimal formula.

Example

Given the following truth table M :

x_1	x_2	x_3	y
f	f	f	f
f	f	t	f
f	t	f	f
f	t	t	t
t	f	f	t
t	f	t	t
t	t	f	f
t	t	t	t

Example continued

▶ $\neg A_1 \wedge A_2 \wedge A_3$
 $A_1 \wedge \neg A_2 \wedge \neg A_3$
 $A_1 \wedge \neg A_2 \wedge A_3$
 $A_1 \wedge A_2 \wedge A_3$
are the minterms of M .

Example continued

▶

$$\begin{array}{ccccc} \neg A_1 & \wedge & A_2 & \wedge & A_3 \\ A_1 & \wedge & \neg A_2 & \wedge & \neg A_3 \\ A_1 & \wedge & \neg A_2 & \wedge & A_3 \\ A_1 & \wedge & A_2 & \wedge & A_3 \end{array}$$

are the minterms of M .

▶ $A_1 \wedge \neg A_2$ and $A_2 \wedge A_3$ are prime implicants of M .

Example continued

▶

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are the minterms of M .

- ▶ $A_1 \wedge \neg A_2$ and $A_2 \wedge A_3$ are prime implicants of M .
- ▶ $A_1 \wedge \neg A_2 \vee A_2 \wedge A_3$ is a minimal formula representing M .

Quine-McCluskey Algorithm

- ▶ An algorithm to find minimal formulas
- ▶ Finds all prime implicants of a given formula S
- ▶ Finds a combination of prime implicants which covers S
- ▶ Example as given in class