Final Project
Physics 304: Computational Physics
Two-Fluid Approximation for Calculating the CMB Anisotropy

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In this project, we will go through a simple approximation for calculating the cosmic microwave background anisotropy in adiabatic process. It consists of numerically solving for evolution of a two-fluid model until the epoch of recombination and then integrating over the sources to obtain the CMB anisotropy power spectrum.

1. INTRODUCTION

Our universe was created in a space-time symmetry called Big Bang. After Big Bang, the universe expanded exponentially. Quantum fluctuations fill empty spaces by creating particles and anti-particles. However, during the exponential expansion of the universe these fluctuations went from microscopic to macroscopic and created density fluctuations. The places in the universe with higher density created potential wells where it started pulling matter in. Oppositely, at spots with lower density, potential hills were created.

The early universe was also extremely hot and dense. The temperature was so hot that neutral atoms could not form. The mean free path for each electron was extremely short. However, when the universe cooled down significantly enough neutral atoms started forming and the photons that were trapped so far in this opaque plasma were now free to go and roam the universe. As the universe expanded the wavelength of these photons were stretched so now we receive these photons as microwave radiation also known as cosmic microwave background radiation.

Turns out these radiation is largely homogeneous and has a temperature of 2.73. The fluctuations are 1 part in 100000. CMB anisotropy power spectrum shows temperature fluctuations of this radiation over the entire visible sky. We decompose the monopole of the temperature field using spherical harmonics and two dimensional temperature function T(θ , ϕ). This can be written as:

$$T(\theta, \phi) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi). \quad (1)$$

The theory of temperature fluctuations define a_{lm} such that each coefficient average depends on l and not m. The average can then be written as:

$$C_\ell = \left\langle |a_{lm}|^2 \right\rangle. \quad (2)$$

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In this project I use Ref. [4]'s method for calculating this using two fluid approximation.

2. THEORY

In the early universe, gravitational force of cold dark matter tried to keep the baryon-photon plasma contained. But since the universe was also extremely hot and ionized there was outward pressure. This created sound waves. The CMB photons allow us to see the patterns that resulted from the oscillations of the sound waves in early universe.

The calculation method for power spectra in this project is based on the idea that photons and baryons are lightly coupled prior to recombination which allows a two fluid description of the perturbation. The analysis is restricted to adiabatic perturbation.

Adiabatic perturbation is the driving effect of forced oscillations in the photon-baryon fluid. It related to our early discussion on potential wells and potential hills that was generated during inflation. As the baryon-photon plasma begins to compress, photon pressure resists the increase in the density perturbation. This allows the gravitational potential to decay. Therefore in a highly compressed state, the fluid oscillates as a cosine wave with enhanced amplitude as seen in fig b. The other effect included in the two fluid approximation is the Doppler Effect. The motion of an observer with relativistic velocity relative to an isotropic radiation field produces a Doppler-shifted temperature pattern. Power spectra calculation for this project is based on the line-of-sight integration along the photon past light cone. The approximation starts with the idea that the two fluids we are concerned with are of baryons and photons. The temperature fluctuation in k-space in some direction $\vec{n}$ as a line of sight integral can be written as :

$$\Delta(\vec{n}) = \int_0^{\tau_0} \left[\dot{\mu} \left(\phi + \frac{\delta_\gamma}{4} + \vec{n} \cdot \overrightarrow{v_b} \right) + 2\dot{\phi} \right] e^{-\mu} d\tau, \quad (3)$$

where τ is the conformal time, *phi* is gravitational potential, δ_γ is density of photons and ν_b is electron density. Conformal time is the amount of time it would

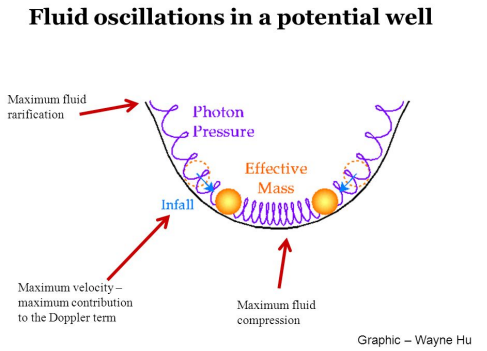


Figure 1: Adiabatic perturbation from density fluctuations created during inflation from Ref. [3].

take a photon to travel from where we are located to the furthest observable distance. It can be defined as a function of the scale factor, a which is the parameterized relative expansion of the universe as $\tau = \int_0^t \frac{dt'}{a(t')}$. The term $\mu(\tau)$ is Thomson opacity along the past light cone $\mu(\tau) = \int_\tau^{\tau_0} \dot{\mu}(\tau')\, d\tau'$ with $\dot{\mu} = a x_e n_e \sigma_T$ and n_e is electron number density. In this case the Freedman Equation is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G a^2}{3}\left(\bar{\rho}_\gamma + \bar{\rho}_\nu + \bar{\rho}_b + \bar{\rho}_c\right). \tag{4}$$

The solution to this equation is:

$$y = \frac{a}{a_{eq}} = (\alpha x)^2 + 2\alpha x \ , \ x = \frac{\tau}{\tau_r}. \tag{5}$$

In the tight coupling limit $\mu >> 1$, photons and baryons are coupled into a single fluid with $\nu_b = \nu_\gamma$ and density $\delta_b = \frac{3}{4}\delta_\gamma$. We also redefine the Fourier space as $\kappa = k\tau_r$. The equations of baryon-photon plasma evolution and cold dark matter perturbation can be written as:

$$\dot{\delta}_c = -\kappa v_c + 3\dot{\phi}, \tag{6}$$

$$\dot{\delta}_\gamma = -\frac{4}{3}\kappa v_\gamma + 4\dot{\phi}, \tag{7}$$

$$\phi = -\frac{3}{2}(\eta/\kappa)^2(\delta + 3\eta v/\kappa), \tag{8}$$

$$\delta = \frac{\delta_\gamma\left(1 + \frac{3}{4}(y - y_c)\right] + y_c\delta_c}{1+y}, \tag{9}$$

$$\dot{\nu}_c = -\eta\nu_c + \kappa\phi, \tag{10}$$

$$\dot{\nu}_\gamma = \left(\frac{4}{3} + y_b\right)^{-1}\left[-\eta y_b \nu_\gamma + \frac{\kappa\delta_\gamma}{3} + \kappa\phi\left(\frac{4}{3} + y_b\right)\right], \tag{11}$$

$$\dot{\phi} = -\eta\phi + \frac{3\eta^2\nu}{2\kappa}, \tag{12}$$

$$\nu = \frac{v_\gamma\left(\frac{4}{3} + y - y_c\right) + y_c\nu_c}{1+y}. \tag{13}$$

These are coupled first order equations. Here all the derivatives are with respect to x and the the Fourier space is over $\kappa = k\tau_r$ as opposed to simply k. And $y_b = 1.68\frac{\Omega_b}{\Omega_m}y$, $y_c = \left(1 - \frac{\Omega_b}{\Omega_m}/1.68\right)y$ and $\eta = 2\alpha(\alpha x + 1)/\left(\alpha^2x^2 + 2\alpha x\right)$. For the limit $\kappa\eta << 1$, the initial conditions are:

$$\phi = 1, \quad \delta_\gamma = -2\phi\left(1 + \frac{3y}{16}\right), \quad \delta_c = \frac{3}{4}\delta_\gamma,$$
$$v_\gamma = v_c = -\frac{\kappa}{\eta}\left[\frac{\delta_\gamma}{4} + \frac{2\kappa^2(1+y)\phi}{9\eta^2\left(\frac{4}{3}+y\right)}\right]. \tag{14}$$

To get the power spectra the equations for CDM density and velocity, photon density and velocity needs to be evolved till $x_rec = [(\alpha^2+1)^{1/2} - 1]/\alpha$. The power spectra Cl can be written as :

$$C_l = 4\pi A\int_0^\infty \kappa^n T(\kappa) d\ln\kappa \left[\left(\phi + \frac{\delta_\gamma}{4} + 2\Delta\phi\right)j_l(\kappa x_0) + v_\gamma j_l'(\kappa x_0)\right]^2. \tag{15}$$

Here $\Delta\phi$ is $\left[2 - 8/y\,(x_{rec}) + 16x_{rec}/y^3\,(x_{rec})\right]/10y\,(x_{rec})$. $T(\kappa)$ is the damping factor most of which comes from Silk damping and finite width of LSS. During recombination around $t_s \sim H^{-1}$, some scattering still occurs. This causes baryons to experience a drag from the photons. Perturbation analysis shows that the result is damping of baryon fluctuations on scales below the characteristic length the photons propagate. This phenomenon is called silk damping. $T(\kappa)$ is defined as:

$$T(\kappa) \approx e^{-\kappa^2\left(2x_s^2 + \sigma^2 x_{rec}^2\right)}. \tag{16}$$

Here $\sigma = 0.03$ and silk damping scale x_s is $x_s = 0.6\Omega_m^{1/4}\Omega_b^{-1/2}a_{rec}^{3/4}h^{-1/2}$. We now solve Eq. 16 numerically to get CMB anisotropy.

3. NUMERICAL METHODS

Eq. 6-13 are all related to each other with 4 coupled differential equations. To solve for C_ℓ we need to solve them but we can not do so analytically so we use numerical method for differential equation solving called the Runge-Kutta Method. We also need to do numerical integration for actually calculating the Cl over the k-space. I used trapezoidal rule for that. In this section we would go over briefly about what each of these numerical methods are.

3.1. Runge-Kutta Method

Runge-Kutta method is a simple differential solving method. The idea for it simple: if know the value of a function f at some point x_0, and we want to find its

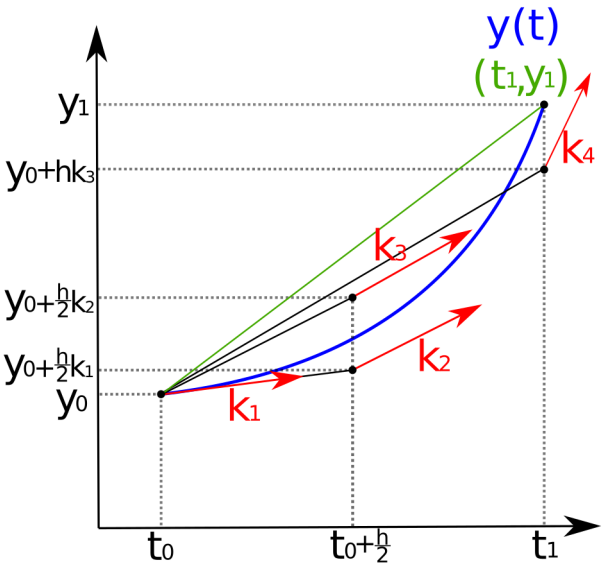


Figure 2: 4th Order Runge Kutta Method, Image Credit: Wikipedia

value at some point (x+h), we can multiply the derivative of f at x_0 with h to get f(x+h)-f(x) since f'_{x_0} accounts for the change in the function. For smaller and smaller values of h, we will have better and better approximation for f(x+h) as we set it equal to $x_0 + hf'(x_0)$. We do this process over and over for updating values for x_0. For Runge-Kutta depending on what order Runge-Kutta we are doing, for each iteration we take the slope at that number of points. This project uses a fourth order Runge-Kutta Method so we evaluate the slope at 4 different points per iteration as shown in Fig rk4. The process can be summarized as follows:

$$\begin{aligned} k_1 &= f\left(y\left(t_0\right), t_0\right), \\ k_2 &= f\left(y\left(t_0\right) + k_1 \tfrac{h}{2}, t_0 + \tfrac{h}{2}\right), \\ k_3 &= f\left(y\left(t_0\right) + k_2 \tfrac{h}{2}, t_0 + \tfrac{h}{2}\right), \\ k_4 &= f\left(y\left(t_0\right) + k_3 h, t_0 + h\right). \end{aligned} \tag{17}$$

k_1 here is the slope of at the beginning step. If we use k_1 for halfway through the time step then k_2 is the slope at midpoint to halfway to the end. We use k_2 to get a second estimate at the midpoint determined by k_3. We then take k_3 all the way to the end point. Finally, k_4 is the slope at the end point.

Then we do a weighted sum to get the slope by summing the initial value with the multiple of h as follows:

$$y_1 = y_0 + \frac{h(k_1 + 2k_2 + 2k_3 + k_4)}{6}. \tag{18}$$

It is extremely important to do enough number of iterations because otherwise it is easy to miss feature of the curve or run into non-physical solutions.

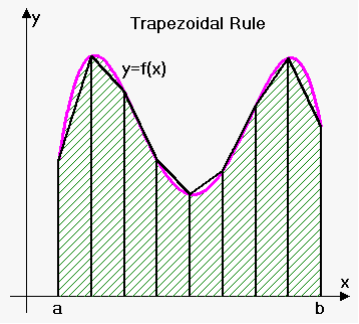


Figure 3: Trapezoidal rule for Integration, Image Credit: EmathHelp.

3.2. Numerical Integration: Trapezoid Method

For this integration method, we divide the function into small increments as seen in Fig. trapz.

Each small slice in there can be thought of as a small trapezoid with height f(a). The area of the trapezoid can be written as: $.5 * f(y_0) + f(y_1)(\Delta x)$. To get area under the curve, at each iteration, we can take area of the small trapezoid and keep adding them up. However, in this way each slice gets double counted so we take that into account and the actual integration is then:

$$\sum_{k=0}^{N-1} \left(x_{k+1} - x_k\right) \left(\frac{f\left(x_k\right) + f\left(x_{k+1}\right)}{2}\right). \tag{19}$$

For a fairly spaced number of points, this will give us a good estimate for area under the curve for the whole function.

3.3. Interpolation and Extrapolation

The basic idea behind interpolation is fitting unknown values of a function using given data points. In Fig. interp, we can see different methods of interpolating the same function. Although all these curves look reasonable, there are some issues with interpolation if the data is over-fitted or under-fitted. In both cases, it is possible for us to get artificial behavior behavior for the data points we are trying to predict. In which case, as we use the interpolated values to calculate some physical behavior, we will notice anomaly. For this project, I will be using linear interpolation as it is pretty standard and is likely to not over-fit the data. I also make sure to take enough data points before interpolation so that it is not under-fitted.

I also use extrapolation for this project. To extrapolate data points, I assume that the trends will continue even past the end points of the data set. In other words, this is a method of using predicting future data using previous data which is a common practice in statistics. Just interpolation this can also be done with our choice

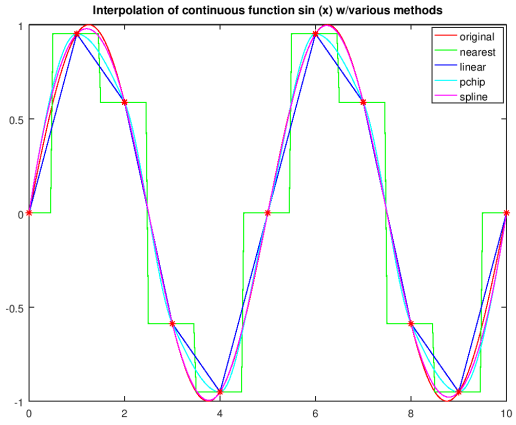


Figure 4: Different Methods of interpolation, Image Credit: Octave-Forge.

of curve fitting. I will continue to use linear interpolation in this case too.

4. IMPLEMENTING THE NUMERICAL METHODS, RESULTS

For this project, we use some built in libraries: numpy, scipy and matplolib. We first import these libraries to conveniently substitute well established numerical methods. After that, at first we use scipy's numerical integration to calculate conformal time at the time of recombination and matter-radiation equality. Matter-radiation equality happens when their density becomes equal and matter takes over radiation. Although in earlier discussion, τ was shown as an integral of 1/a over time, we can also calculate conformal time as the comoving distance multiplied with the speed of light. For calculating the comoving distance, I used values for Ω_m, Ω_l, Ω_r, H_0 which are matter density, radiation density, dark energy density and value of Hubble parameter respectively from Planck 2018 data [1].

To numerically solve comoving distance I use Romberg Integration . Romberg integration provides us an iterative way to minimize error in trapezoid rule and therefore has great accuracy. Once we have a function that can take care of conformal time, I calculate the necessary conformal times at the two interesting "time-stamps" for this project.

z	a	τ
1100(Rec)	0.000303	0.000909
3300(Matter-Radiation Eq)	278.435226	115.675811

I also calculate the angular distance to the LSS which we later use defined as $x_0 = \tau_(1)/\tau(a_{rec})$ then define all the coupled differential equations and necessary composite variables seen in Eq. (6-13) in the DEtoSolve function. The function takes dimensionless time parameter x, initial conditions and the newly defined Fourier space κ. Next I define my the Runge-Kutta fourth order differential equation solving method. Using Eq. 14, I get the necessary initial condition to solve for photon density, cold dark matter density and their respective velocity. For k = .2, the solution for the sources as a function of time for different values of cosmological constant in flat and curved universes are shown below desolve.

The solution for differential equations in Fig. 5 shows similar behavior with the evolution of these sources seen in Ref. [2]'s figure 6 upto recombination. After recombination their photon density damps as their intermediate calculation for these parameters include damping factors that are not included in Ref. [4] on which this project is based on.

Since for calculating anisotropy, for each ℓ, I have to calculate over the Fourier space, instead of doing the calculation for κ over and over, I instead calculate values for each parameter an array of κ. I then interpolate the functions and extrapolate for other points not in the array using the scipy 1d interpolation function.

Then I calculate the integrand in Eq. 15 for values of k that I will integrate over. The j_ℓ and j'_ℓ in Eq. 15 is harmonic Bessel function. These Bessel functions describe the geometry of how different ℓ functions are related to each other. This is defined as:

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z), \tag{20}$$

where J_n is regular Bessel function that we are familiar with. Bessel function of the first kind is defined as:

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+p+1)} \left(\frac{z}{2}\right)^{2n+p}. \tag{21}$$

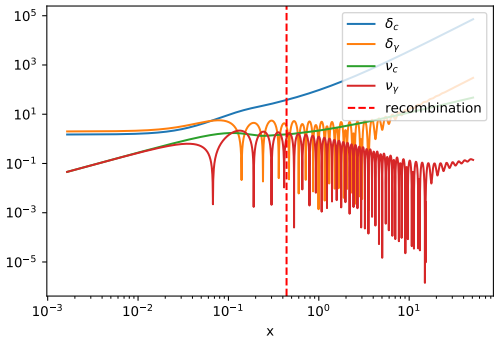
For integer or positive p, these Bessel functions are finite at the origin. Bessel functions of the first kind diverge as x approaches zero. Eq. 10 also uses the derivative of the spherical Bessel function for the Doppler effect. This is defined as:

$$\begin{aligned} j'_n(z) &= j_{n-1}(z) - \tfrac{n+1}{z} j_n(z) \\ j'_0(z) &= -j_1(z). \end{aligned} \tag{22}$$

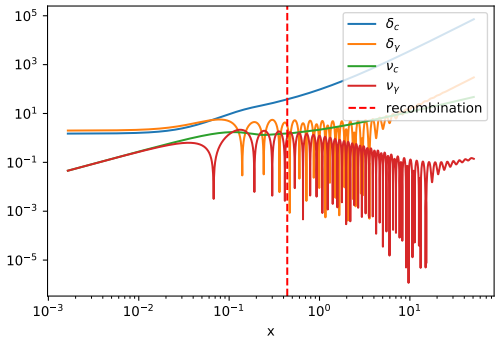
Another important thing to notice in Eq.10 is that the integral is in log-space. This makes a big difference as it forces to take more steps in the increments. That is why in the code I substitute e^u where u is $ln\kappa$ and perform the integral. Finally, I multiply C_ℓ with $\ell(\ell+1)$ since the spherical harmonics we talked about in introduction section, has l and m with ℓ+1 number of l's and ℓ number of m's.

5. COMPUTATIONAL CHALLENGE AND LIMITATIONS

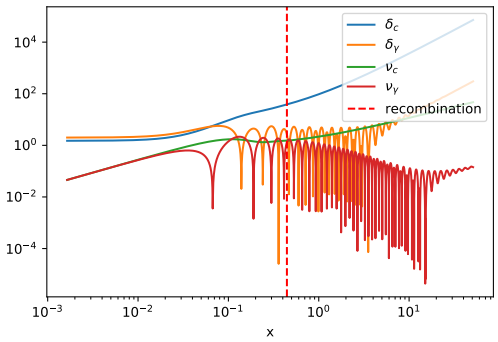
The whole code takes a very long time to run. The Runge-Kutta method or the integration does not take



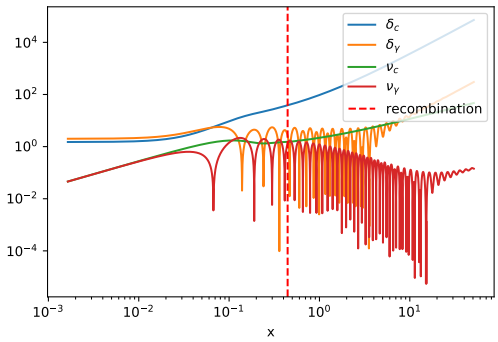
(a) Evolution of source terms using Planck bestfit (TT,TE,EE+lowE+lensing) assuming curvature



(b) Evolution of source terms using Planck bestfit (TT,TE,EE+lowE+lensing) assuming no curvature



(c) Evolution of source terms using Planck bestfit (TT,TE,EE+lowE+lensing+BAO) assuming curvature



(d) Evolution of source terms using Planck bestfit (TT,TE,EE+lowE+lensing+BAO) assuming no curvature

Figure 5: Evolution of δ_c, δ_γ, ν_c,ν_γ as a function of time for different cosmological parameters and curvature, the red line is the time of recombination in this new timescale.

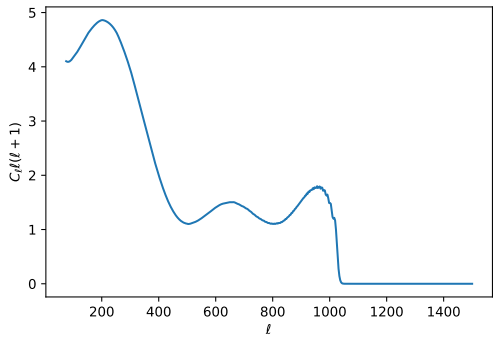


Figure 6: CMB Anisotropy power spectra due to Adiabatic Effect

too long to run but when I try to do the Runge-Kutta for 1000 values of k and append them at the time of recombination, it takes a long time. Seljak considers only adiabatic contribution and Doppler effect and neglects a lot of other physical effect, this anisotropy plot is not exactly what one would expect but it still has the peaks and almost right amplitude. Also the extrapolation here does not work very well on this as we see in Fig. desolve , there are a lot of oscillations in the terms associated with the photons. Also since the sources are dependent on our choice of cosmological parameters, deciding bestfit in this requires further assessment.

6. COLLABORATIONS AND ACKNOWLEDGEMENTS

I did not work with anyone else in the class. Prof. Daniel Grin helped me to debug my code during office hours. I used Numpy, Scipy and Matplotlib package for this project.

7. CONCLUSION

In this project, I used Seljak's theory for approximating CMB anisotropy for adiabatic process. The approx-

imation for calculating anisotropy used in the Seljak's paper a generalization of the Sachs-Wolfe approximation, which is only valid on scales larger than the Hubble sphere radius(radius of the observable universe) at recombination. The approximation is useful both for developing the physical understanding of processes that affect CMB fluctuations. So, although this is a good approximation, it can be made more accurate. If I had more time, I would evolve the source terms for more k values so that the behavior of the anisotropy past l =1000 was better. I also look into why my function is poorly behaved at lower l's.

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[2] Santanu Das and Anh Phan. Cosmic microwave background anisotropy numerical solution (CMBAns). part i. an introduction to ic/isubil/i/sub calculation. *Journal of Cosmology and Astroparticle Physics*, 2020(05):006–006, may 2020.

[3] Wayne Hu. Cmb introduction.

[4] Uros Seljak. A two-fluid approximation for calculating the cosmic microwave background anisotropies. *The Astrophysical Journal*, 435:L87, nov 1994.