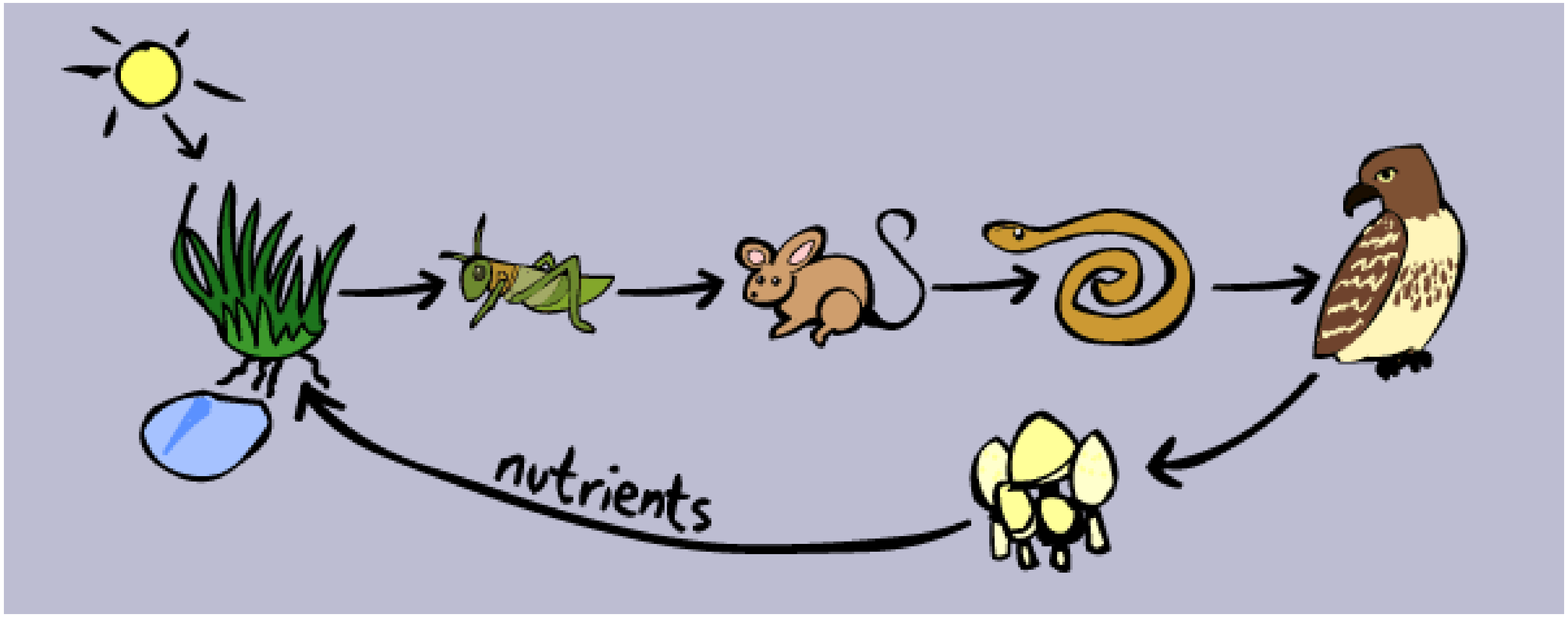
A Study on Population Changes in Food Chain and Its Affectedness by Contagions, via Mathematical Modelling

Yuan Ning, Huilin Tong and Yushi Liu
School of Engineering, Tufts University
May 10, 2018

Tufts
UNIVERSITY

Biological interaction



Biological interaction

- Analyzing population dynamics
    - Biological competition;
    - Predation.

- Analyzing change of population of species
    - with contagions.

- Birth rate is related to its population
    - internal competition;
    - Population of prey.
- Death rate is related to population of predator
    - Contagions

Model I: A→B

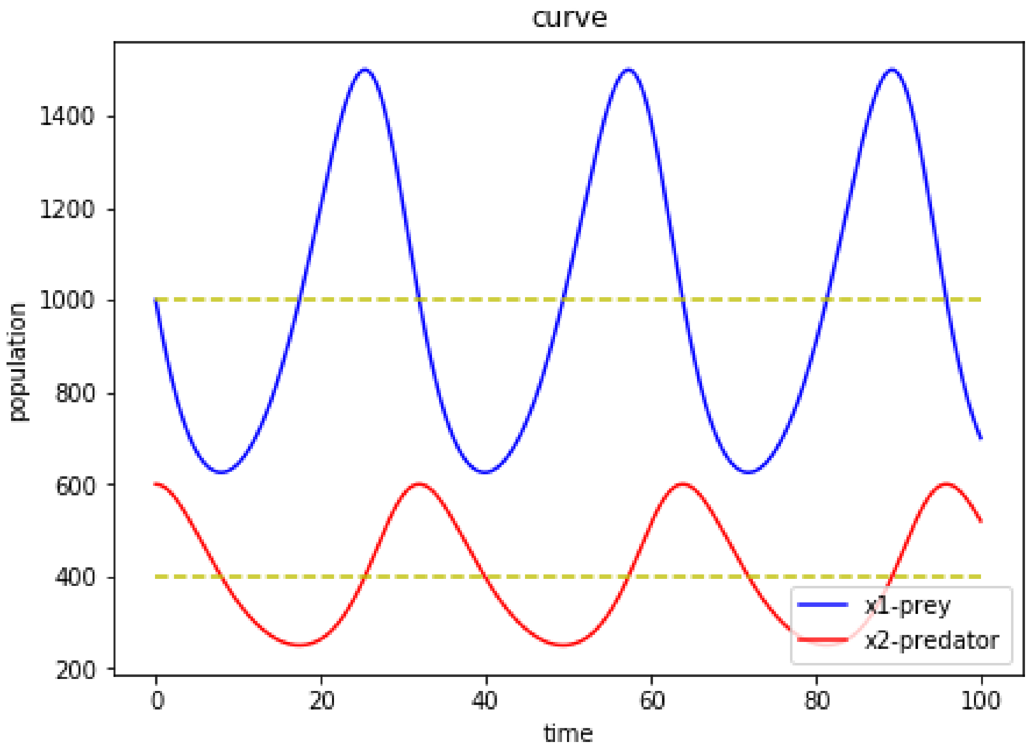
- Differential equations based on the assumption can be described as follows:

$$X_1'[t] = K_1 * X_1[t] - K_2 * X_1[t] * X_2[t]$$
$$X_2'[t] = K_3 * X_1[t] * X_2[t] - K_4 * X_2[t]$$

The birth rate of the predator is $K_1 * X_1[t]$, the mortality rate is $K_2 * X_1[t] * X_2[t]$, and the birth rate of the prey is $K_3 * X_1[t] * X_2[t]$, and the mortality rate is $K_4 * X_2[t]$.

Model I: A→B

- The predator's period is in delay.



K_1=0.2
K_2=0.0005
K_3=0.0002
K_4=0.2
K1/K2=400
K4/K3=1000

Initial :
X1(0)=1000
X2(0)=600

Model I: A$\rightarrow$B

- Differential equations based on the assumption can be described as follows:

$$X_1'[t] = K_1 * X_1[t] - K_2 * X_1[t] * X_2[t]$$
$$X_2'[t] = K_3 * X_1[t] * X_2[t] - K_4 * X_2[t]$$

The birth rate of the predator is $K_1 * X_1[t]$, the mortality rate is $K_2 * X_1[t] * X_2[t]$, and the birth rate of the prey is $K_3 * X_1[t] * X_2[t]$, and the mortality rate is $K_4 * X_2[t]$.

Internal competition?

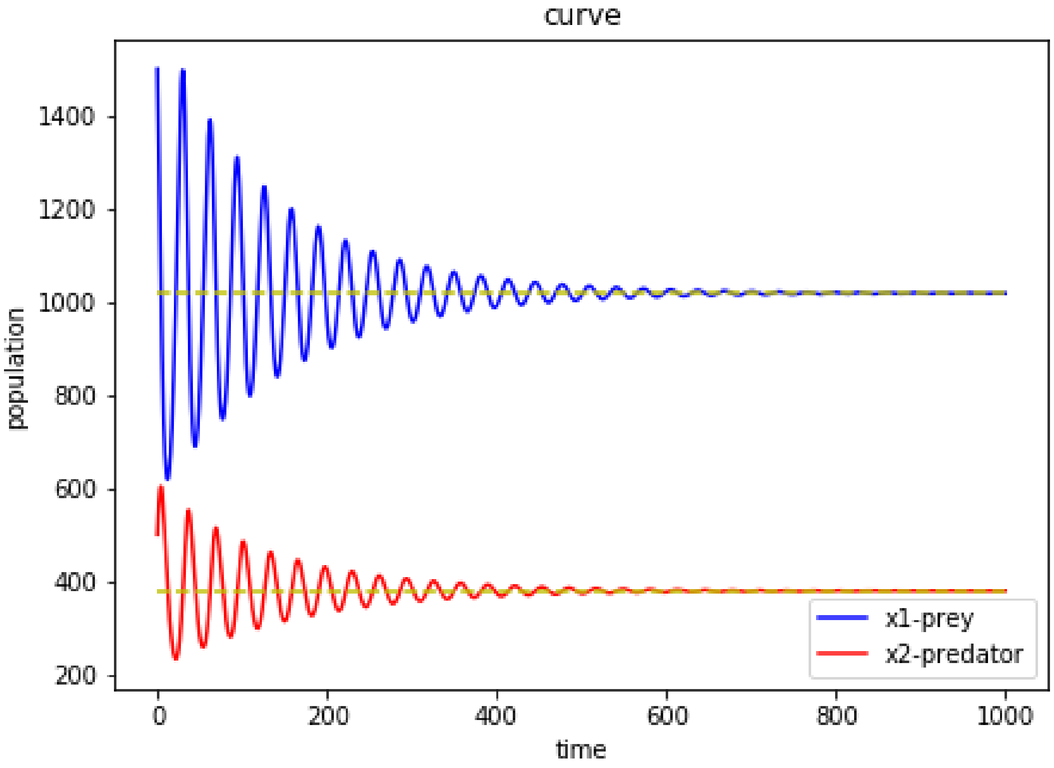
Model I: A →B

- Considering the constraint due to internal competition.

$$X_1'[t] = X_1[t]*(K_1 - K_2 * X_1[t] - K_3 * X_2[t])$$
$$X_2'[t] = X_2[t]*(K_4 * X_1[t] - K_5 * X_2[t] - K_6)$$

- Considering stability, when the increase or decrease rate is zero (derivatives are zero), system is stable.

Model I: A→B



K_1=0.2
K_3=0.0005
K_4=0.0002
K_6=0.2
K_2=0.00001
K_5=0.00001

Initial:
$X_1(0)$=1500
$X_2(0)$=500

- The population become stable after periods of oscillation.

Model II: A $\rightarrow$ B; A $\rightarrow$ C

X: prey
Y: predators

- Considering one more predator.
  - Differential equations
  - Similar assumption

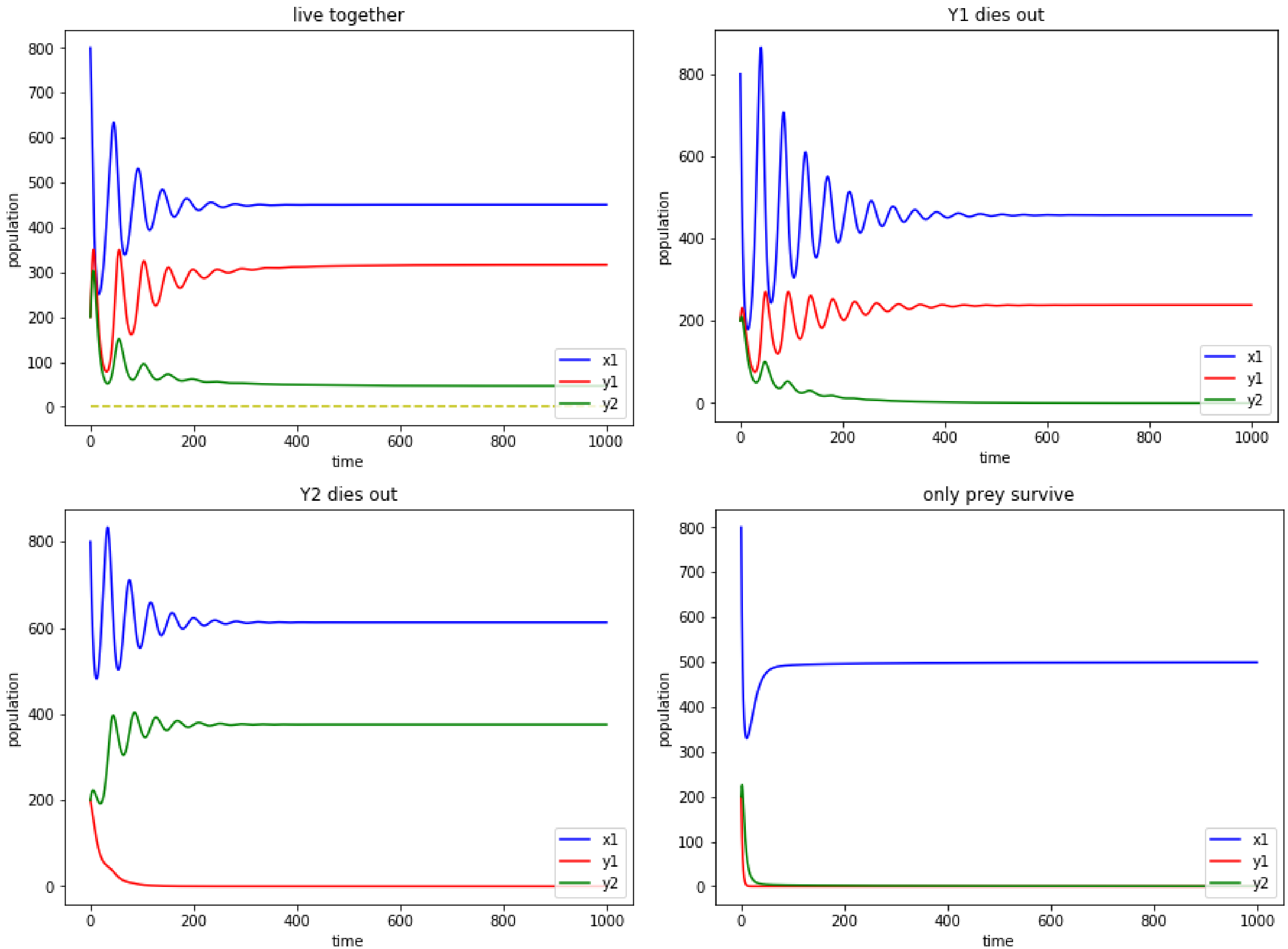
$$Y_1'[t] = Y_1[t](K_1 X[t] - K_2 Y_1[t] - K_3)$$

$$Y_2'[t] = Y_2[t](K_4 X[t] - K_5 Y_2[t] - K_6)$$

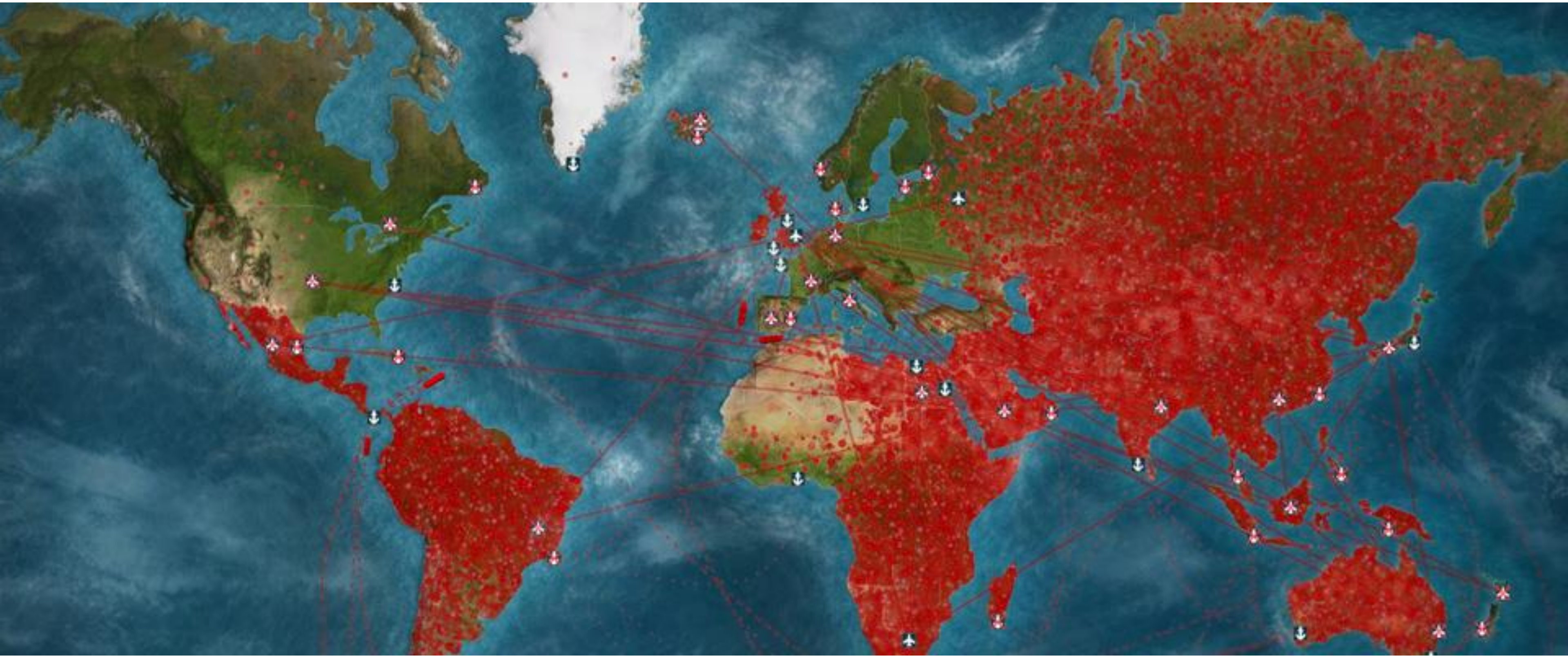
$$X'[t] = X[t](K_7 - K_8 X[t] - K_9 Y_1[t] - K_{10} Y_2[t])$$

Model II: A → B; A → C

X: prey
Y: predators



Biological interaction w/ Contagions



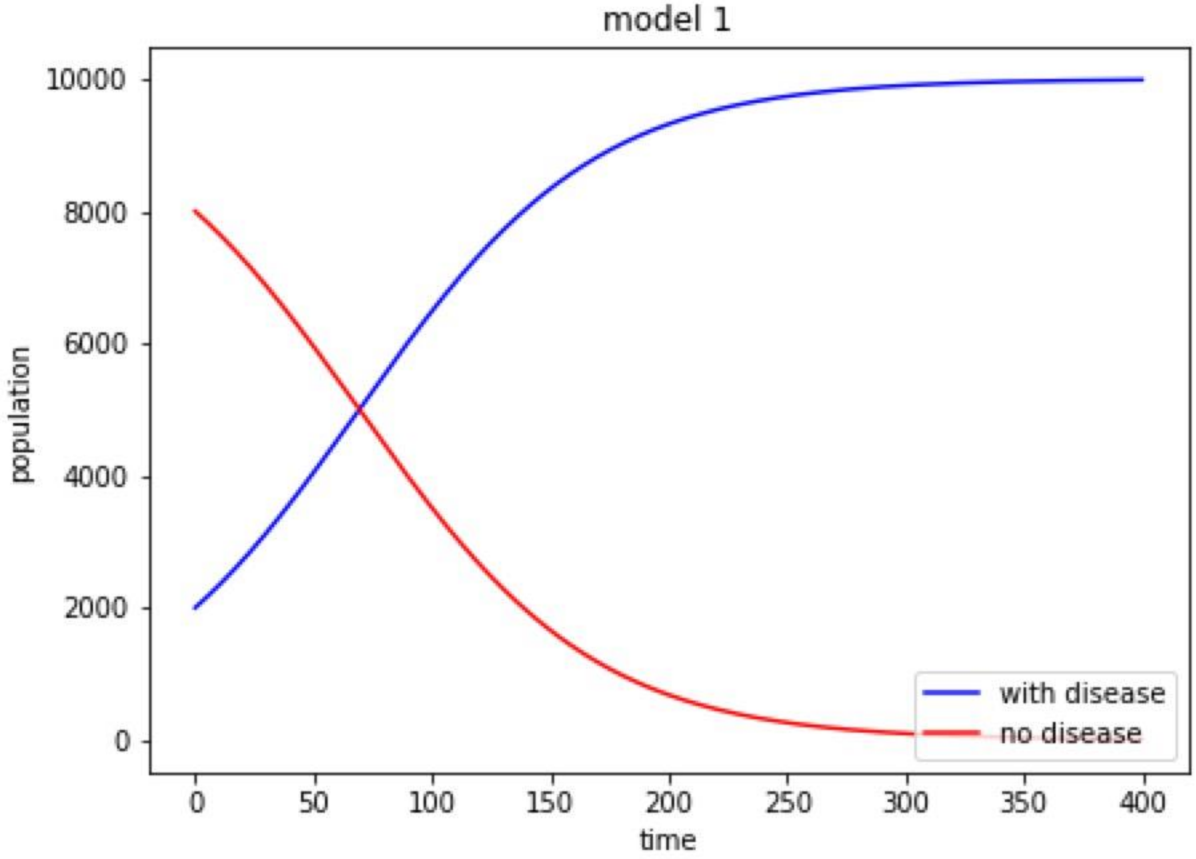
Model III: Contagions Constant population

- Contagions could constraint increase of population
- Simple model: no recovery.
- Assuming constant total population.
- Population: N(t) = X(t) + Y(t) (= constant)
    - Uninfected: X(t)
    - Infected: Y(t)

X'[t] = K*X[t]*Y[t]
N=X[t] +Y[t]
Y'[t] = K*(N-Y[t])*Y[T]

Model III: Contagions

Constant population



- analytic solution

$$Y[t] = \frac{e^{KNT} N Y_0}{N - Y_0 + e^{KNT} Y_0}$$

N=10000
Y0=2000
K=0.000002

Model III: Contagions

Constant population

- Contagion could constraint the speed of increase of population
- Simple model: no recovery.
- Assuming constant total population.
- Population: N(t) = X(t) + Y(t) (= constant)
  - Uninfected: X(t)
  - Infected: Y(t)

X’[t] = K*X[t]*Y[t]

N=X[t] +Y[t]

Y’[t] = K*(N-Y[t])*Y[T]

Model IV: Contagions

Decreased population

- Infected individuals recovers with immunity

- Population: $N(t) = X(t) + Y(t) + Z(t)$
  - Uninfected: $X(t)$
  - Infected: $Y(t)$
  - Recovery with immunity: $Z(t)$

Model IV: Contagions

Decreased population

- Not considering natural growth rate, the increase of population is from net migration.
- Individual death rate is a constant, ignore the restriction of environment recourse.

$$X'[t] = K_1 - K_2 X[t] - K_3 X[t]Y[t] + K_4 Z[t]$$

$$Y'[t] = K_3 X[t]Y[t] - (K_2 + K_5 + K_6)Y[t]$$

$$Z'[t] = K_6 Y[t] - (K_2 + K_4)Z[t]$$

$$N'[t] = K_1 - K_2 N[t] - K_5 Y[t]$$

K_1: #healthy individual coming from external environment in unit time
K_2: normal death rate
K_3: healthy individual infection rate(not include individuals with immunity)
K_4: rate of individuals lost their immunity
K_5: fatality rate
K_6: recovery rate

Model IV: Contagions

Decreased population

- Trivial Solution: X = $\frac{K_1}{K_2}$, Y = 0 , Z = 0;

- Stable solution:

$$X = \frac{K_2+K_5+K_6}{K_3}$$

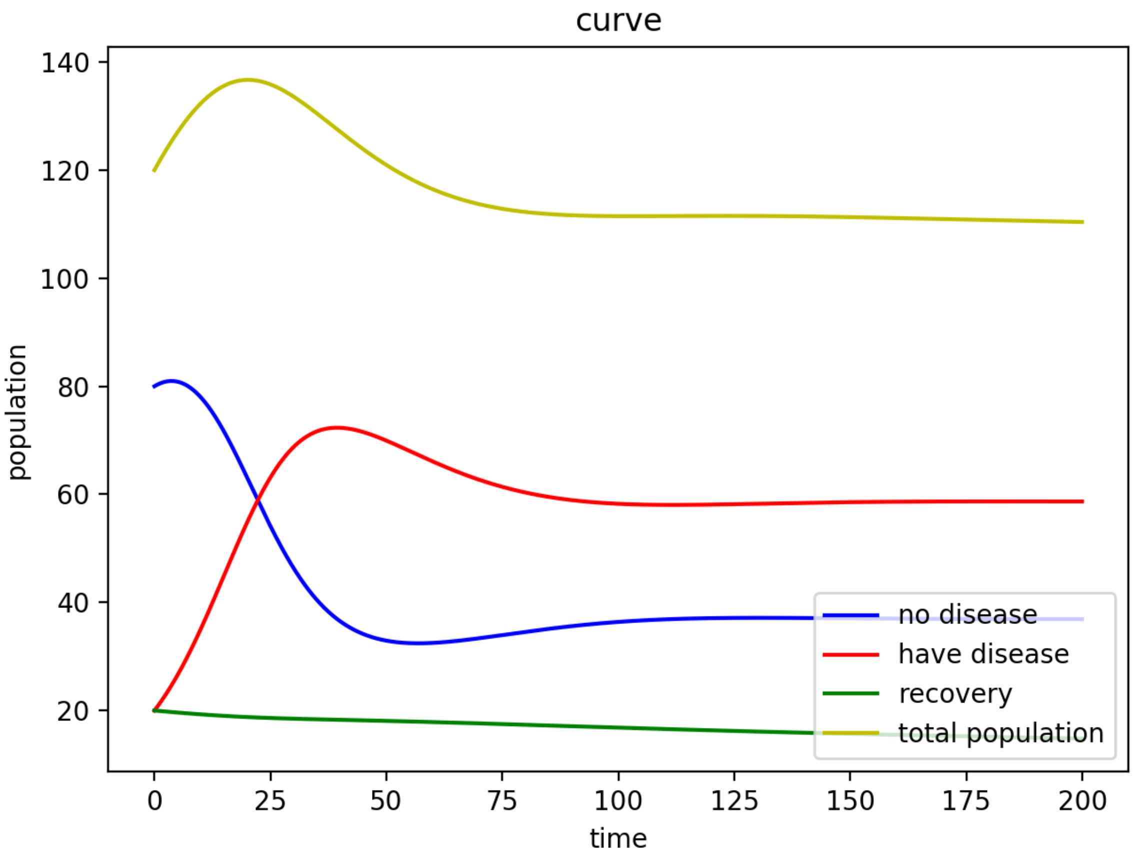
$$Y = \frac{-(K_2+K_4)(K_2^2-K_1K_3+K_2K_5+K_2K_6)}{K_3(K_2^2+K_2K_4+K_2K_5+K_4K_5+K_2K_6}$$

$$Z = \frac{-K_6\ (K_2^2-K_1K_3+K_2K_5+K_2K_6)}{K_3(K_2^2+K_2K_4+K_2K_5+K_4K_5+K_2K_6}$$

- Stable condition: $K_1/K_2 > (K_2 + K_5 + K_6)/K_3$

Model IV: Contagions

Decreased population



K1=3
K2=0.005
K3=0.0013
K4=0
K5=0.042
K6=0.001
X(0)=80
Y(0)=20
Z(0)=20
N(0) =120

Model V: Contagions

- Not considering natural growth rate, the increase of population is from net migration.
- Individual death rate is a constant, ignore the restriction of environment recourse.

$$X'[t] = K_1 - K_2 X[t] - K_3 X[t]Y[t] + K_4 Z[t]$$

$$Y'[t] = K_3 X[t]Y[t] - (K_2 + K_5 + K_6)Y[t]$$

$$Z'[t] = K_6 Y[t] - (K_2 + K_4)Z[t]$$

$$N'[t] = K_1 - K_2 N[t] - K_5 Y[t]$$

K_1: #healthy individual coming from external environment in unit time
K_2: normal death rate
K_3: healthy individual infection rate(not include individuals with immunity)
K_4: rate of individuals lost their immunity
K_5: fatality rate
K_6: recovery rate

Model V: Contagions

With reproduction

- Considering natural growth rate.
- Individual death rate is a constant, ignore the restriction of environment recourse.

$$X'[t] = K_1 N[t] - K_2 X[t] - K_3 X[t] Y[t] + K_4 Z[t]$$

$$Y'[t] = K_3 X[t] Y[t] - (K_2 + K_5 + K_6) Y[t]$$

$$Z'[t] = K_6 Y[t] - (K_2 + K_4) Z[t]$$

$$N'[t] = (K_1 - K_2) N[t] - K_5 Y[t]$$

K_1: birth rate
K_2: normal death rate
K_3: healthy individual infection rate(not include individuals with immunity)
K_4: rate of individuals lost their immunity
K_5: fatality rate
K_6: recovery rate

Model V: Contagions

With reproduction

- Trivial Solution: X = Y = Z =0

- Stable solution:

$$X = \frac{K_2+K_5+K_6}{K_3}$$

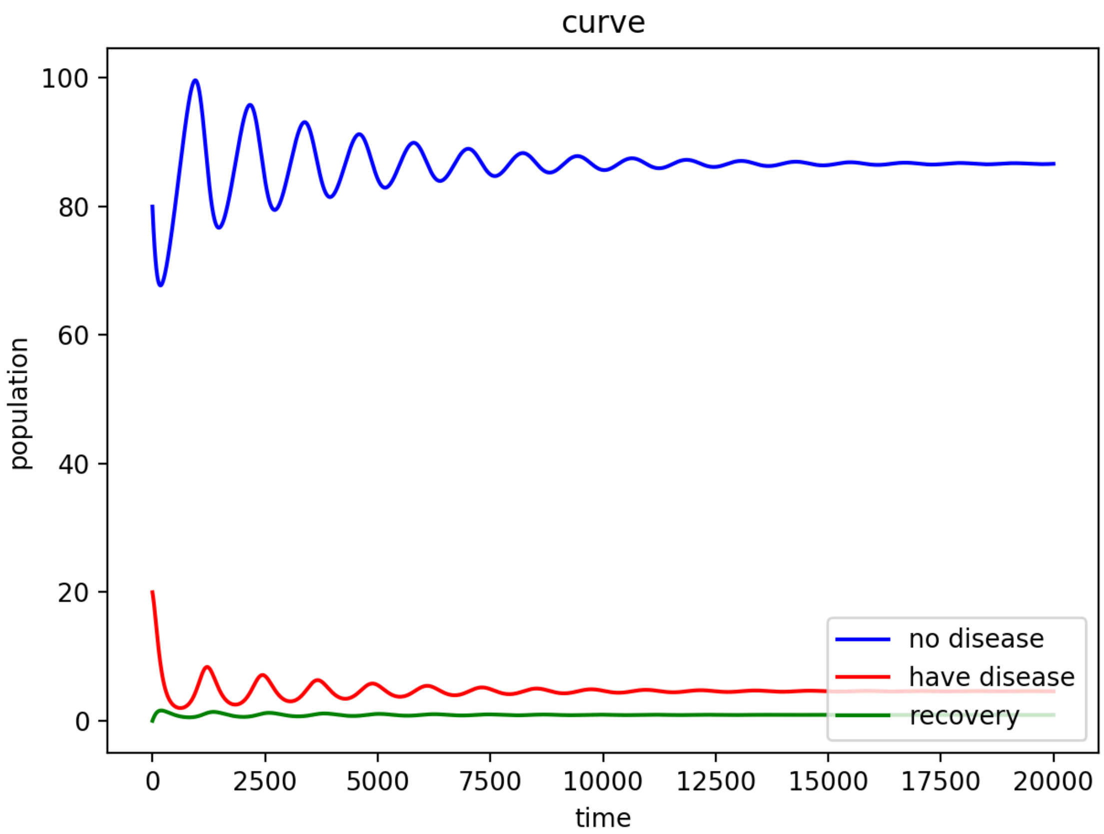
$$Y = \frac{(K_1-K_2)(K_2+K_4)(K_2+K_5+K_6)}{K_3(K_1K_2-K_2^2+K_1K_4-K_2K_4-K_2K_5-K_4K_5+K_1K_6-K_2K_6}$$

$$Z = \frac{(K_1-K_2)K_6\ (K_2+K_5+K_6)}{K_3(K_1K_2-K_2^2+K_1K_4-K_2K_4-K_2K_5-K_4K_5+K_1K_6-K_2K_6}$$

- Stable condition: $K_5 > (K_1 - K_2)(1 + \frac{K_6}{K_2+K_4})$

Model V: Contagions

With reproduction



K1=0.006
K2=0.005
K3=0.0003
K4=0
K5=0.02
K6=0.001
X(0)=80
Y(0)=20
Z(0)=0
N(0) = 120


Thank
You