EE194/BIO196: Modeling Biological Systems
HW 3: Stochastic tortoises

Extra credit

- We have been using l_x-m_x tables for our population models. Another common way to structure the vital rates is by using *Leslie matrices*. A Leslie matrix holds the same data as L_x-m_x tables, but uses a single sparse matrix. You can find more information on Leslie matrices from Wikipedia, or from Chapter 5 of *Plant and Animal Populations, Methods in Demography* by Thomas Ebert (on reserve at Tisch).
- According to the Perron-Frobenius theorem, only one of the eigenvalues of a Leslie matrix can be positive. The reason this is important is that any eigenvector of a Leslie matrix represents a stable population state, and the magnitude of the eigenvalue tells how robust the population growth is. Why might that be? (It is actually quite intuitive, given the basic definition of what an eigenvector is).

Solution:

Let $\boldsymbol{L}$ be the Leslie matrix for a particular set of vital rates. From the Perron-Frobenius theorem, it can have at most one positive eigenvalue. Let that eigenvalue be λ, and let the corresponding eigenvector be $\boldsymbol{v}$. Then, by the definition of an eigenvector and eigenvalue, $\boldsymbol{Lv}=\lambda\boldsymbol{v}$.

Multiplying a vector by a scalar will of course scale each component of the vector equally. Since every component of the vector represents the population of one age class, then this implies that every age class will scale identically – which exactly matches what we expect from a population that is experiencing exponential growth.

After two years, the population vector will be (by the definition of a Leslie matrix) $\boldsymbol{L}(\boldsymbol{Lv})$; i.e., the Leslie matrix $\boldsymbol{L}$ times the population vector at the end of the first year (i.e., $\boldsymbol{Lv}$). (Note that the parenthesis are used to represent simple grouping, not for function application). But $\boldsymbol{Lv}=\lambda\boldsymbol{v}$, so this is equal to $\boldsymbol{L}(\lambda\boldsymbol{v}) = \lambda(\boldsymbol{Lv}) = \lambda(\lambda\boldsymbol{v})= \lambda^2\boldsymbol{v}$. Thus, after two years, every age class in the population has been scaled identically by λ^2.

By a similar argument, after *n* years every age class in the population will have been scaled identically by λ^2. So we indeed have exponential growth, with a growth rate of λ. Thus, the eigenvalue of a Leslie matrix represents the stable growth rate of the corresponding population.