Author's problems on Armenian school physics olympiads

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Contents

*The number in the braces represents the difficulty.

Cylinder on the Corner

Problem

A cylinder of mass m is held on the corner of a table by a long homogenous plank of mass M in position described by α as shown on (Fig.1.1). What friction coefficients must there be between the cylinder and the table, between the plank and the table and between the cylinder and the plack for this situation to be possible? The plank is much longer than the radius of the cylinder.

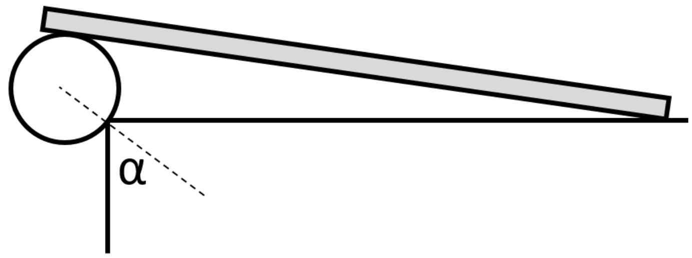
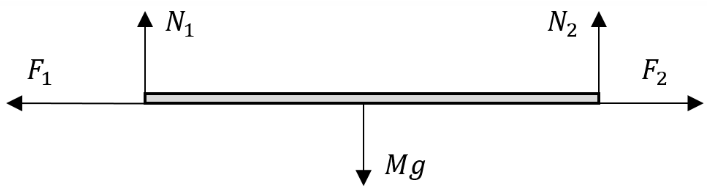
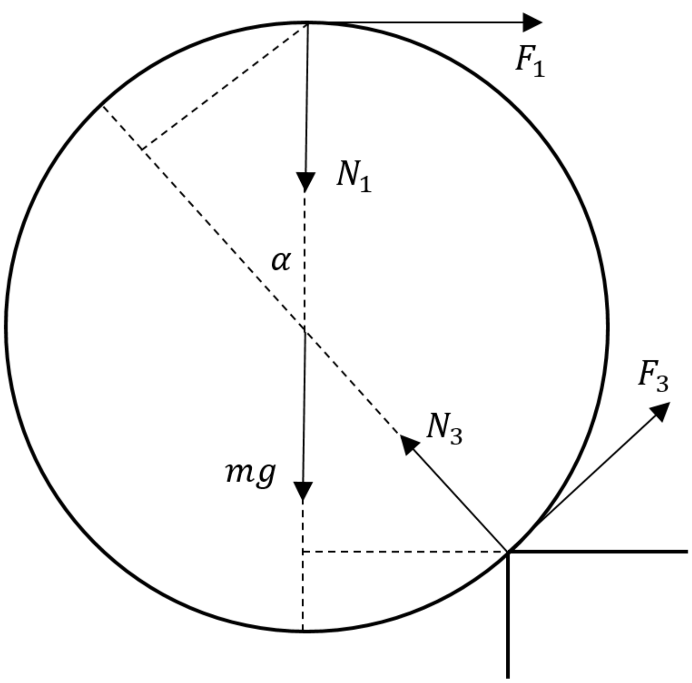


Figure 1.1

Solution



(a) Forces on the plank.



(b) Forces on the cylinder.

Figure 1.2: The forces in the system. F_i are the friction forces.

The forces acting on the cylinder and the plank are shown on (Fig.1.2)

From the equilibrium (both translational and rotational) on the plank we get

$$\begin{aligned} N_1 = N_2 &= \frac{Mg}{2} \\ F_1 &= F_2 \end{aligned} \tag{1.1}$$

and the equations for equilibrium on the cylinder, particularly torques against the center, torques against the cylinder's top point and torques against the cylinder-table touching point are respectively

$$\begin{aligned} F_1 &= F_3 \\ RN_3 \sin\alpha &= RF_3(1+\cos\alpha) \\ R(mg+N_1)\sin\alpha &= RF_1(1+\cos\alpha) \end{aligned} \tag{1.2}$$

where R is the radius of the cylinder. It is now easy to see, that

$$F_1 = \frac{(2m+M)g\sin\alpha}{2(1+\cos\alpha)} \tag{1.3}$$

For the required values of friction coefficients we simply get

$$\begin{aligned} \mu_{c,t} &> \frac{F_3}{N_3} = \frac{\sin\alpha}{1+\cos\alpha} \\ \mu_{c,p/p,t} &> \frac{F_2}{N_2} = \frac{F_1}{N_1} = \frac{\sin\alpha}{1+\cos\alpha}\left(1+\frac{2m}{M}\right) \end{aligned} \tag{1.4}$$

Pulley with Friction

Problem

The system supposed to lift weights consists of a stationary and a movable pulleys as shown in (Fig.2.1). The radii of a pulley and its axis are R and r correspondingly. Assume that the hole in the pulley is slightly bigger than the axis. There is a friction between the axis and the pulley with a given friction coefficient μ. There is no sliding between the ropes and the pulleys, as well as between the ropes and the axes. Determine the energy efficiency coefficient of the system

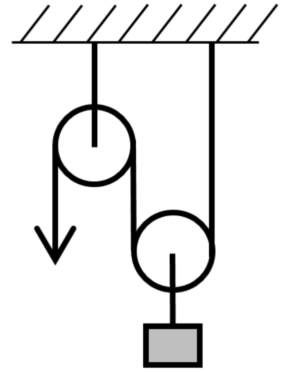


Figure 2.1

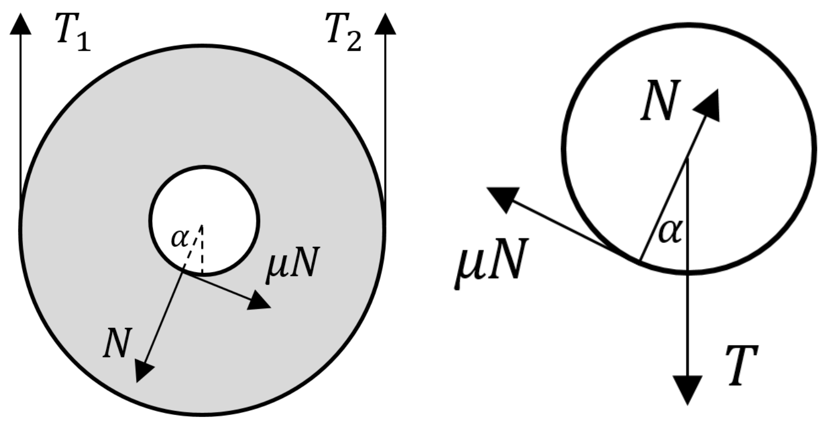
Solution

Consider the movable pulley. Let's assume, that in the process of lifting the touching point between the pulley and its axis is shifted left by some angle α. The forces acting on the pulley and the axis are shown in (Fig.2.2). From the equilibrium on the axis we have

$$\begin{aligned} N\sin\alpha &= \mu N\cos\alpha \\ N\cos\alpha + \mu N\sin\alpha &= T \end{aligned} \tag{2.1}$$

from where we get

$$\begin{aligned} \tan\alpha &= \mu \\ \mu N &= T\frac{\mu}{\sqrt{1+\mu^2}} \end{aligned} \tag{2.2}$$



(a) Forces on the pulley.

(b) Forces on the axis.

Figure 2.2: The forces on the movable pulley.

Notice that we are not allowed to write toque equilibrium as there is nothing known about the torque of interaction between the axis and the rope attached to it.

The equilibrium of the pulley itself is written as

$$\begin{aligned} T_1 + T_2 = N\cos\alpha + \mu N\sin\alpha &\implies T_1 + T_2 = T \\ RT_1 - RT_2 = r\mu N &\implies T_1 - T_2 = T\frac{r}{R}\frac{\mu}{\sqrt{1+\mu^2}} \end{aligned} \tag{2.3}$$

and the other equation is identical to that of the axis. From there we get

$$T_{1/2} = T\frac{1\pm\varepsilon}{2}, \qquad \varepsilon = \frac{r}{R}\frac{\mu}{\sqrt{1+\mu^2}} \tag{2.4}$$

The situation and the equations for the stationary pulley are absolutely the same as the ones written here. So the tension of the free end of the rope should be

$$T_0 = mg\cdot\frac{1+\varepsilon}{2}\cdot\frac{1+\varepsilon}{1-\varepsilon} \tag{2.5}$$

so for the efficiency coefficient we get

$$\eta = \frac{mgl}{T_0\cdot 2l} = \frac{1-\varepsilon}{(1+\varepsilon)^2} \tag{2.6}$$

Combined Pulley

Problem

How much will the weight m shown on (Fig.3.1) descend after hanging it on the rope? The coaxial pulleys are attached to each other and the ratio of their radii is $n > 1$. The stiffness of the spring is k. There is no sliding.

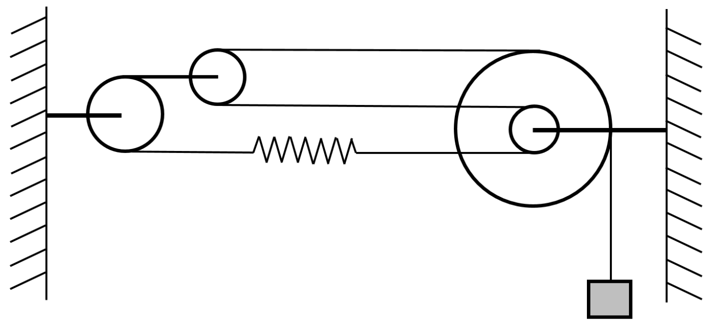


Figure 3.1

Solution

For the final state (Fig.3.2) we have the following equilibrium conditions

$$\begin{aligned} T &= T' \\ F &= T + T' \\ F + nmg &= T' + nT \end{aligned} \tag{3.1}$$

from where the value of F, and consequently the spring deformation x can be found.

$$x = \frac{F}{k} = \frac{2n}{n-1}\frac{mg}{k} \tag{3.2}$$

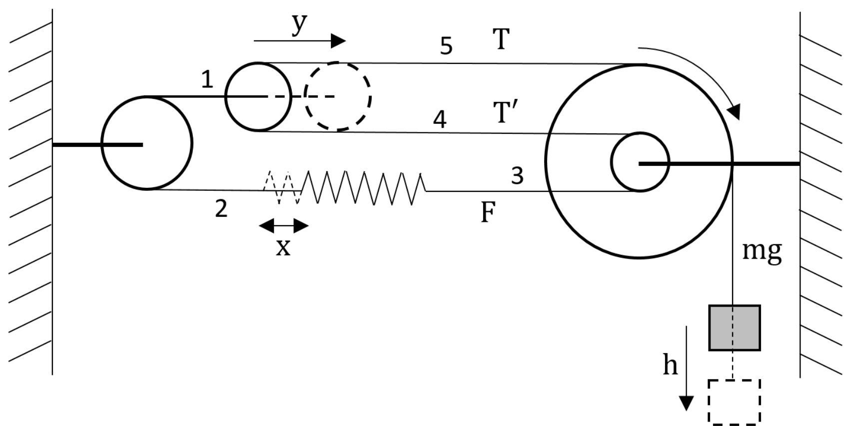


Figure 3.2: The initial (solid lines) and the final (dashed lines) states of the system.

Consider the amounts of movement of the points $1-5$ shown on (Fig.3.2). With choosing positive direction to be towards right, we have

$$\begin{aligned} \Delta_1 &= y \\ \Delta_2 &= -y \\ \Delta_3 &= -y + x \\ \Delta_4 &= y - x \\ \Delta_5 &= 2\Delta_1 - \Delta_4 = y + x \end{aligned} \tag{3.3}$$

Also we know that $n\Delta_4 = \Delta_5$, which provides a connection between y and x. Further solving gives

$$h = \Delta_5 = \frac{2n}{n-1}x = \left(\frac{2n}{n-1}\right)^2 \frac{mg}{k} \tag{3.4}$$

Inclined Rod

Problem

A rod is hanged form the ceiling by its ends with strings of length l_1 and l_2 in such a way, that both strings are vertical. Find the natural frequencies of rods oscillations.

Solution

There are three modes of oscillation. The first one is the oscillation within the strings' plane, and the other two include oscillations perpendicular to it alongside with rotational oscillations.

Consider the first mode. Let's denote the small inclinations of the strings with length l_1 and l_2 by α and β correspondingly. The heights of the ends of the rod don't change, so the constancy of rod length is written as $l_1\alpha = l_2\beta$. We will be doing substitutions $\sin\alpha \to \alpha$ and $\cos\alpha \to 1$ without mentioning it. There is also no rotation, so $T_1 = T_2$ (the string tension forces).

If the displacement of the rod is x, then $\alpha = x/l_1$ and $\beta = x/l_2$. The overall horizontal force will be

$$F = -T_1 \sin\alpha - T_2 \sin\beta = \frac{mg}{2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)x \tag{4.1}$$

As $F = m\ddot{x}$, the corresponding frequency will be

$$\omega_1 = \sqrt{\frac{g}{2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)} \tag{4.2}$$

For the other two modes let's use the same α and β, but here they will be showing the inclinations perpendicular ot strings' plane. The displacements of rod's edges will be $l_1\alpha$ and $l_2\beta$. The displacement x of rod's center and rod's rotation γ around the vertical axis are then give by

$$\begin{aligned} x &= \frac{l_1\alpha + l_2\beta}{2} \\ \gamma &= \frac{l_1\alpha - l_2\beta}{d} \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \alpha &= \frac{2x + \gamma d}{2l_1} \\ \beta &= \frac{2x - \gamma d}{2l_2} \end{aligned} \tag{4.3}$$

where d is the horizontal distance between the strings. The string tensions are also equal to $mg/2$ in this case. The motion equations are then written as

$$\begin{aligned} m\ddot{x} &= -T_1 \sin\alpha - T_2 \sin\beta = -\frac{mg}{2}(\alpha + \beta) = -\frac{mx}{2}\Sigma - \frac{md\gamma}{4}\Delta \\ I\ddot{\gamma} &= -T_1 \sin\alpha\frac{d}{2} + T_2 \sin\beta\frac{d}{2} = -\frac{mgd}{2}(\alpha - \beta) = -\frac{mdx}{4}\Delta - \frac{md^2\gamma}{8}\Sigma \end{aligned} \tag{4.4}$$

where $I = ml^2/12$ is the moment of inertia of the rod against the vertical axis through its center, $\Sigma = g(l_1^{-1} + l_2^{-1})$ and $\Delta = g(l_1^{-1} - l_2^{-1})$. Denoting $x = kd\gamma$ for a specific mode we get

$$\begin{aligned} k\ddot{\gamma} &= -\frac{k\Sigma}{2}\gamma - \frac{\Delta}{4}\gamma \\ \frac{\ddot{\gamma}}{12} &= -\frac{k\Delta}{4}\gamma - \frac{\Sigma}{8}\gamma \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \omega^2 &= -\frac{\Sigma}{2} - \frac{\Delta}{4k} \\ \omega^2 &= -3k\Delta - \frac{3\Sigma}{2} \end{aligned} \tag{4.5}$$

After some trivial calculations one can get

$$\omega_{2,3} = \sqrt{\frac{g}{l_1} + \frac{g}{l_2} \pm g\sqrt{\frac{1}{l_1^2} + \frac{1}{l_2^2} - \frac{1}{l_1 l_2}}} \tag{4.6}$$

Notice that two of the frequencies reduce to $\sqrt{g/l}$ in case $l_1 = l_2 = l$.

Audio Casette

Problem

Find the thickness h and density ρ of the tape of an audio casette. The coil cores have radius $r = 11.0$ mm and the mass of the whole casette is $M = 40$ g

Equipment given: audio casette, pencil, calipers.

Solution

In order to measure something, we first have to transport all the tape to one of the coils (namely first). Then we can do N rotations of the second coil and measure the radius ratio of the coils and the position of the center of mass.

The radius of the coils is measured as follows: you turn one of the coils by some angle (can be measured using the teeth of the coil), and follow the rotation of the other coil. The ratio of the angles is the ratio of radii. The process should be reversed before continuing. The mass center position can be found by just pushing the casette of something's edge and measure the position using the calipers. It's better to use the calipers itself as the "something" for better precision.

The measurements are as follows

N	$\alpha_1[2\pi]$	$\alpha_2[2\pi]$	r_1/r_2	x_c[m]
0	5	$2 + 1/6$	2.31	0.0935
50	3	$1 + 5/12$	2.12	0.0932
100	3	$1 + 13/24$	1.95	0.0928
150	3	$1 + 2/3$	1.80	0.0924
200	2	$1 + 1/4$	1.60	0.0920
250	4	$2 + 3/4$	1.45	0.0914
300	2	$1 + 1/2$	1.33	0.0909
350	5	$4 + 1/12$	1.22	0.0902
400	4	$3 + 7/12$	1.12	0.0894
450	2	$1 + 11/12$	1.04	0.0887
500	4	$4 + 1/4$	0.94	0.0880
550	4	$4 + 3/4$	0.84	0.0874
600	3	$3 + 11/12$	0.77	0.0867
650	3	$4 + 5/12$	0.68	0.0857
700	3	$5 + 1/12$	0.59	0.0848
750	1	2	0.50	0.0840
800	1	$2 + 1/2$	0.40	0.0831

we will later see, that the 0 point for x_c measurement doesn't matter.

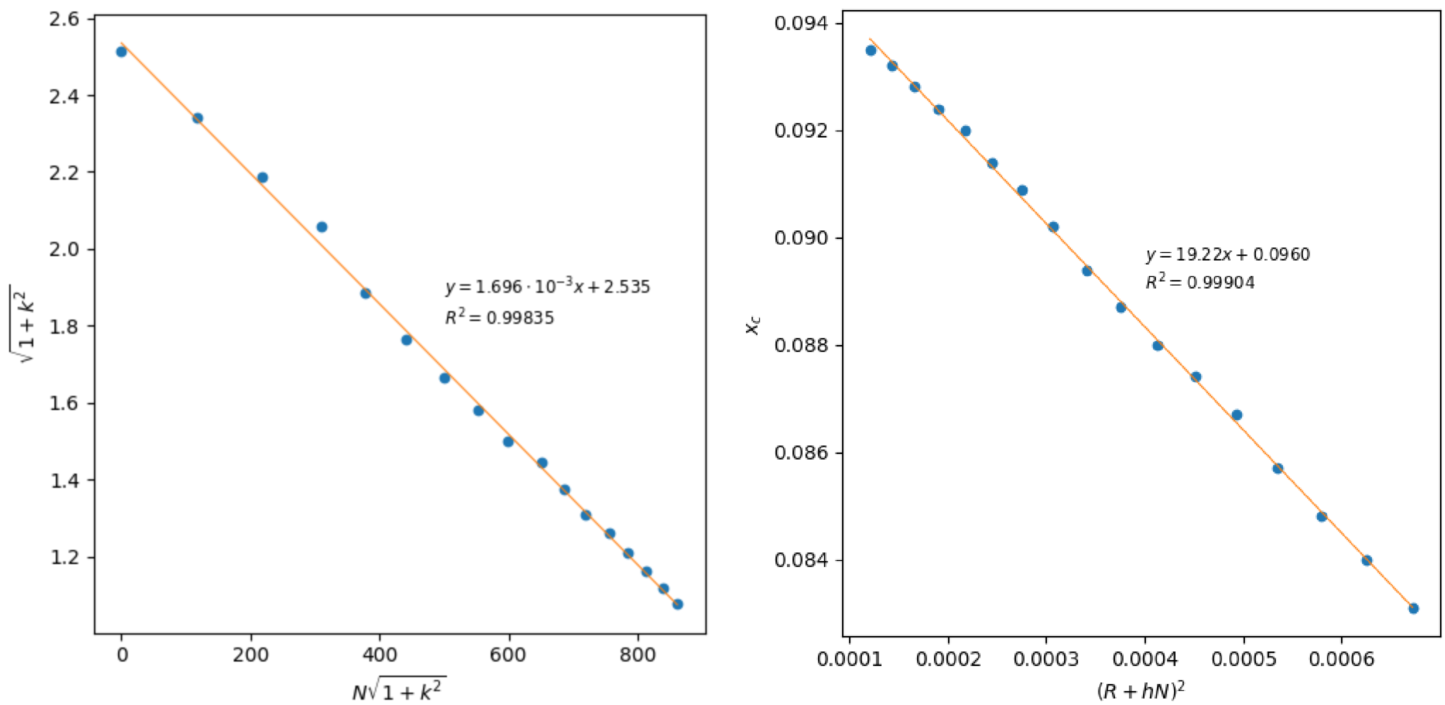
Theoretically we can write the radius ratio as

$$\frac{r_1}{r_2} = k = \frac{\sqrt{(r + hN_0)^2 + r^2 - (r + hN)^2}}{r + hN} \tag{5.1}$$

as the side surface of the tape $\propto r_1^2 + r_2^2$ conserves. Here N_0 is full number of rotations. This can be linearized to

$$\sqrt{1 + k^2} = -\frac{h}{r}\sqrt{1 + k^2}N + \sqrt{(1 + N_0 h/r)^2 + 1} \tag{5.2}$$

and by plotting $\sqrt{1 + k^2}$ on $N\sqrt{1 + k^2}$ (Fig.5.1a) one can get $h = r \cdot 1.7 \cdot 10^{-3} = 1.9 \cdot 10^{-5}$ m.



(a) (b)

Figure 5.1

The position of the center of mass is theoretically given as

$$x_c = \frac{m_* x_* - \rho \pi w h l (r + hN)^2}{M} \tag{5.3}$$

where w is the width of the tape and l is the distance between coil centers, which can be measured directly. $w = 3.8 \cdot 10^{-3}$ m, $l = 4.23 \cdot 10^{-2}$ m. m_* and x_* are some arbitrary units which include the casette box and some residual terms from the calculation.

So by plotting x_c on $(r + hN)^2$ (Fig.5.1b) we get $\rho = M/\pi w l \cdot 19.2$ m^{-1} $= 1.5 \cdot 10^3$ kg/m^3.

Leaky Container

Problem

There is a cup of radius R on the table. A container that is higher than the cup by H is fully filled with water and is placed next to the cup in such a way that the distance between cup's center and its nearest edge is D. Where on the container you should poke a small hole so there is maximum possible amount of water in the cup after the water flow stops?

Solution

We take the cup's top as 0 of height. Forget about the finiteness of the container for now. If the hole is made on height h, than the water should have velocity $v = d\sqrt{g/2h}$ upon exiting the container to hit distance d (basic kinematics). To gain velocity v the water level l should be $v^2/2g$ higher than the hole (Bernoulli equation). So if we poke the hole at h, the water levels that will hit the nearest and the farthest edges of the cup are

$$l_\pm = h + \frac{(D \pm R)^2}{4h} \tag{6.1}$$

and all the water between l_- and l_+ will end up in the cup. One may notice, that the difference $l_+ - l_-$ is decreasing on h, and hits infinity at $h = 0$. Here comes the finiteness of the container.

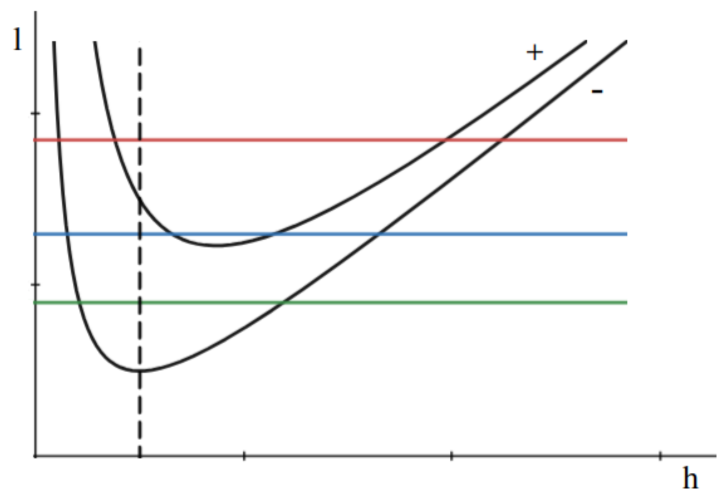


Figure 6.1: The water levels, at which water will hit the edges of the cup (black solid lines) for specific position of the hole (horizontal axis). Colorful lines demonstrate different possible values of H. Dashed black line indicates the minimum of l_-.

In order to better understand the situation let's qualitatively portray $l_\pm$ and H on h (Fig.6.1). The minima of $l_\pm$ are located at $h_\pm = (D \pm R)/2$, and thereby the minimum of l_- is left to that of l_+. We have to choose such h, that the section between l_+ and l_- clamped by H is maximal.

If we remember, that the distance without clamping decreases on h, it is easy to verify, that in case the first intersection of H and l_+ is left from the minimum of l_- (the red line case on (Fig.6.1)), that the intersection point itself is the optimal place for a hole. The intersection point is given by the smaller solution of $H = l_+$

$$h = \frac{H - \sqrt{H^2 - (D+R)^2}}{2} \tag{6.2}$$

If the intersection is right from h_- (the blue line case) or there is no intersection at all (the green line case) then h is just h_-.

So the optimal h for the hole is given by

$$h = \min\left(\frac{H - \sqrt{H^2 - (D+R)^2}}{2}, \frac{D-R}{2}\right) \tag{6.3}$$

and when the first argument is undefined we automatically take the second one.

There is also a possibility that no water can end up in the cup. In that case we are free to make a hole wherever we want.

1 + 1 (4)

Problem

The resistance of a thermistor depends on its absolute temperature as $R = \beta T^2$, where $\beta = 7.00 \cdot 10^{-4}$ Ohm/K. Surface area of the thermistor is $S = 1.00 \cdot 10^{-2}$ m^2 and the heat transfer coefficient with the medium is $\alpha = 200$ W/m^{2}K. The temperature of the medium is $T_0 = 293$ K.

1. Find the current in such a thermistor under voltage $U_0 = 1000$ V.

2. Find the current in a system with two consequently connected thermistors under voltage $U_0 = 1000$ V.

Useful information

- The solution of equation $x^3 + kx^2 + b = 0$ is given by

$$x = \frac{1}{3}\left[\left(\frac{r}{2}\right)^{1/3} + k^2\left(\frac{r}{2}\right)^{-1/3} - k\right], \qquad r = 3^{3/2}\sqrt{27b^2 + 4bk^3} - 27b - 2k^3 \tag{7.1}$$

- A solution of any equation can be found even if the equation can not be solved analytically.

Solution

The heat generated under voltage U in case of temperature T is given by $Q_+ = U^2/\beta T^2$. The heat loss is given by $Q_- = \alpha S(T - T_0)$. The equilibrium condition $Q_+ = Q_-$ gives

$$T^3 - T_0 T^2 - \frac{U^2}{\alpha\beta S} = 0 \tag{7.2}$$

For voltage $U = U_0$ we will get $T = 1000$ K and $I = U/\beta T^2 = 1.42$ A.

For the case of two consequently connected thermistors one might think that each of them will get voltage $U = U_0/2$ because of symmetry. Then we calculate the temperature $T = 680$ K and current $I = 1.55$ A. Immediately we see that there is some problem, because we got higher current for lower voltage. One may also check the current is locally decreasing at that point (this can simply done numerically).

This means that the solution is not stable. Indeed. Suppose the voltage slightly increases on the first thermistor and thereby decreases on the second one. This will result in current decrease on the first thermistor and increase on the second one, which in its turn will result in negative charge accumulation in the center thereby further increasing the voltage deviation.

So we need to find another solution, which obviously will be non-symmetric. The equation to be solved is

$$I(U) = I(U^*) \qquad U^* = U_0 - U \tag{7.3}$$

where $I(U)$ is $U/\beta T^2$ and T is the solution of (7.2).

This is to be done numerically. Remember that the current is decreasing on U at $U = U_0/2$, so for slightly less U-s $I(U) > I(U^*)$. Now we need to find the U, where $I(U)$ becomes bigger than $I(U^*)$. The calculations are

U [V]	U^* [V]	T_U [K]	T_{U^*} [K]	I_U [A]	I_{U^*} [A]
0	1000	293	1003	0	1.420
500	500	680	680	1.546	1.546
250	750	484	850	1.526	1.484
150	850	396	913	1.369	1.458
200	800	440	881	1.473	1.471
190	810	431	888	1.458	1.468
195	805	436	885	1.466	1.470

At this point we are sure that the equilibrium current is $I = 1.47$ A.

Thick Barrel (4)

Problem

What thickness should a cylindrical barrel with internal radius R have, in order to withstand internal pressure P? The barrel is made of material with strength σ. What happens in case $P \ll \sigma$?

Useful information

- The solutions to differential equation $y'' = y/x^2$ obviously have form $y = Cx^a$.
- When $k > 1$

$$\left(\frac{2}{5+\sqrt{5}} + \frac{2}{5-\sqrt{5}}k^{\sqrt{5}}\right) > k^{\frac{\sqrt{5}-1}{2}}$$

Solution

We will be describing the state of our barrel with the radial displacement of each point $x(r)$, $r \to r + x(r)$. The tangential component of relative deformation then will be

$$\varepsilon_t(r) = \frac{x(r)}{r} \tag{8.1}$$

as the ring with initial length $2\pi r$ now has length $2\pi(r + x(r))$. The radial deformation is given by

$$\varepsilon_r(r) = \frac{\mathrm{d}x(r)}{\mathrm{d}r} \tag{8.2}$$

as the point at initial positions r and $r + \Delta r$ are now at $r + x(r)$ and $r + \Delta r + x(r + \Delta r) = r + \Delta r + x(r) + x'(r)\Delta r$. So the point now have distance $(1 + x'(r))\Delta r$ instead of initial Δr.

The internal (inside the barrel walls) equilibrium is generated as a result of opposition of the radial forces to tangential forces. Consider a section of a barrel at radius r, with thickness $\mathrm{d}h$, length (along the axis) l and angular size $\mathrm{d}\alpha$. The equilibrium condition of it is written as

$$2E\varepsilon_t \cdot l\mathrm{d}h \cdot \frac{\mathrm{d}\alpha}{2} = E\frac{\mathrm{d}\varepsilon_r}{\mathrm{d}r}\mathrm{d}h \cdot lr\mathrm{d}\alpha \tag{8.3}$$

where the left-hand side of the equation are the tangential pressure forces projected on radial direction, and the right-hand side is the difference between the radial pressure forces from above and below (in terms of radii) of the section.

In terms of $x(r)$, this equation can be rewritten as simply $x'' = x/r^2$. The solutions are easily found in form Cx^a,, from where we get the general solution of the equation

$$x = C_+r^{\beta_+} + C_-r^{\beta_-} \qquad \beta_\pm = \frac{1 \pm \sqrt{5}}{2} \tag{8.4}$$

The constants are determined by edge conditions $E\varepsilon_r(R) = -P$ and $E\varepsilon_r(R_o) = 0$, where R_o is the outer radius of the barrel. To apply these conditions we need the expression for ε_r.

$$\varepsilon_r(r) = x'(r) = C_+\beta_+r^{\beta_+-1} + C_-\beta_-r^{\beta_--1} \tag{8.5}$$

The edge conditions result in

$$\beta_+C_+ = \frac{P}{ER^{\beta_+-1}\left(\left(\frac{R_o}{R}\right)^{\sqrt{5}} - 1\right)} \qquad \beta_-C_- = -\beta_+C_+R_o^{\sqrt{5}} \tag{8.6}$$

where $\sqrt{5}$ comes from $\beta_+ - \beta_-$. Note that $C_\pm > 0$.

Now we have to figure out where and in which direction the barrel will break. To this end we have to know where and in which directions the existing tension (which is proportional to ε-s) is the highest.

ε_r is a sum of increasing functions, as $C_+\beta_+ > 0$, $\beta_+ - 1 > 0$ and $C_-\beta_- < 0$, $\beta_- - 1 < 0$. Given that at $r = R_o$ it is 0, it has its highest absolute value at $r = R$.

The expression for ε_t is just $\varepsilon_t = C_+ r^{\beta_+ - 1} + C_- r^{\beta_- - 1}$. As a sum of increasing and decreasing positive monomials ($C_\pm > 0$, $\beta_+ - 1 > 0$, $\beta_- - 1 < 0$), ε_t is a function with a single minimum, which means that for any given interval $[a, b]$, the maximum value of the function is either at a or at b.

So the candidates of the highest tension are

$$
\begin{aligned}
|\varepsilon_r(R)| &= P/E \\
\varepsilon_t(R) &= \frac{P}{E(k-1)}\left(\frac{1}{\beta_+} + \frac{k}{\beta_-}\right) \qquad\qquad k = \left(\frac{R_o}{R}\right)^{\sqrt{5}} \\
\varepsilon_t(R_o) &= \frac{P\sqrt{5}}{E(k-1)}\left(\frac{R_o}{R}\right)^{\beta_-}
\end{aligned}
\tag{8.7}
$$

The last two expressions seem remotely similar to terms in inequation in the hint. After some transformations one can show the connection, with k in the hint being R_o/R. A a result we learn that $\varepsilon_t(R) > \varepsilon_t(R_o)$.

It is left to compare $|\varepsilon_r(R)|$ and $\varepsilon_t(R)$, that is

$$
1 * \frac{\beta_- + \beta_+ k}{(k-1)\beta_+\beta_-}
\tag{8.8}
$$

It is straightforward to show that the right-hand side is bigger.

Finally we have to demand $E\varepsilon_t(R) < \sigma$ for the barrel to remain integrate. This leads to a linear equation on k which results in

$$
k = \frac{1 + \frac{P}{\sigma\beta_+}}{1 - \frac{P}{\sigma\beta_-}}
\tag{8.9}
$$

The corresponding thickness is $h = R(k^{1/\sqrt{5}} - 1)$.

For very small values of pressure the expression can be decomposed

$$
h_{\ll} = R\left(1 + \frac{P}{\sqrt{5}\sigma\beta_+} + \frac{P}{\sqrt{5}\sigma\beta_-} - 1\right) = \frac{RP}{\sigma}
\tag{8.10}
$$

Field in Medium

Problem

1. **Dielectrics**

    1.1. What dipole moment will a conducting ball of radius a gain in electric field strength E?

    1.2. What is the strength of electric field inside a capacitor with dipole moment density p?

    1.3. What is the electric permittivity of a medium with concentration of atoms n, if we consider the atoms as conducting ball of radius a.

2. **Ferromagnetics**

    It is know that atoms have a magnetic moment, which can be explained by the rotation of the electron around the nucleus (but it's not). According to quantum theory, in case of existence of an external magnetic field that magnetic moment will either align with it or take the direction opposite to it. Furthermore, the probability of each of these states is given by Boltzmann distribution, i.e.

    $$\varphi \propto \exp\left(-\frac{E}{k_B T}\right) \tag{9.1}$$

    where φ is the probability of the state, E is its energy, k_B is Boltzmann constant and T is the absolute temperature.

    2.1. Find the magnetic moment expected value of a magnetic dipole with moment μ inside a uniform magnetic field $\vec{B}$.

    2.2. What is the strength of magnetic field inside a coil with magnetic moment density $\vec{\kappa}$?

    2.3. What is the magnetic permeability of a medium with concentration of atoms n which have magnetic moment μ, if we consider the field generated by the medium itself to be much smaller than the external field?

    It turns out that ferromagnetics have a domain structure. This means that there are regions (domains) if atoms with same magnetic moment direction rather then each atom being independent. As a result the magnetic field at any given point is considerably stronger than the mean magnetic field.

    To encapsulate all the effects of the domain structure, suppose that there is some effective magnetic field $B_{eff} = B + bI$ acting at each point in direction of $\vec{B}$, where B is the external field, I is the mean field generated by the medium and $b > 1$ is some parameter describing the matter.

    2.4. Write down an equation for I (an equation, where only I is unknown).

    2.5. At what temperatures it is possible for a matter with parameters n, b and μ to be permanent magnet? The critical temperature is called Curie temperature.

    2.6. Calculate the numerical value of b for iron, if known that $k_B = 1.4 \cdot 10^{-23}$ J/K, $\mu_0 = 1.3 \cdot 10^{-6}$ N/A^2, $\mu = 9.3 \cdot 10^{-24}$ J/T, $m_{\text{proton}} = 1.7 \cdot 10^{-27}$ kg, $M_{Fe} = 56$ g/mol, $\rho_{Fe} = 7.9 \cdot 10^3$ kg/m^3, $T^C_{Fe} = 1.0 \cdot 10^3$ K.

Useful information

- Energy of a magnetic dipole $\vec{\mu}$ in magnetic field $\vec{B}$ is given by $E = -\vec{\mu} \cdot \vec{B}$.
- Magnetic moment of a flat contour with current I and area S is $\vec{\mu} = I\vec{S}$.
- The solution of an equation is the intersection point of graphs of expressions on the two sides of the equation.

Solution

1. Dielectrics

1.1. Dipole moment of a ball in an external field

This is a famous problem. The only reason of including it here is that the result is needed later. If we consider two uniformly charged balls of radius a and charges $\pm q$, with displacement $\vec{d} \ll a$, the result will be a non-uniformly charged sphere of radius a.

To calculate the field inside this object we have to sum up the fields of the two balls. It is known, that at some distance r form the center only the charge "inside" that distance crates field. Suppose the ball with charge $-q$ is at $0-0$ and the other one is at $\vec{d}$. The dipole moment is $\vec{P} = q\vec{d}$. Their field is then given by

$$E(\vec{r}) = \frac{-kq}{r^2} \cdot \frac{r^3}{a^3} \cdot \frac{\vec{r}}{r} + \frac{kq}{r_d^2} \cdot \frac{r_d^3}{a^3} \cdot \frac{\vec{r}_d}{r_d} = -\frac{kq}{a^3}(\vec{r} - \vec{r}_d) = -\frac{k\vec{P}}{a^3} \tag{9.2}$$

where $\vec{r}_d = \vec{r} - \vec{d}$.

As the field inside is uniform, once inside an external electric field, the ball will charge in way we just discussed to compensate the external field inside of it (the field in a conductor should be 0). This will be achieved if the ball gains dipole moment $\vec{P} = \vec{E}a^3/k$.

1.2. Electric field in a capacitor

Consider a capacitor of area S and thickness d, that is charged with surface charge density $\pm\sigma$. The overall dipole moment in this case is $P = \sigma S \cdot d$ (in direction towards the plate charged with σ) and its density is thereby $\vec{p} = \sigma$.

The electric field inside can be given by $E = \sigma/\varepsilon_0$ towards the plate with $-\sigma$. So the field can be expressed in terms of $\vec{p}$ as

$$\vec{E} = -\vec{p}/\varepsilon_0 \tag{9.3}$$

1.3. Electric permittivity of a medium

Electric permittivity is the ratio of the initial external field to what we have in the medium as a result of superposition with the medium's field. The source of medium's field is the polarization of atoms under the influence of both external and medium-generated fields. If we denote the later by $\vec{I}$, the polarization density of the medium is then

$$\vec{p} = \frac{n\left(\vec{E} + \vec{I}\right)a^3}{k} \tag{9.4}$$

We already know the strength of the field that this will create. By plugging (9.4) into (9.3) we get

$$\vec{I} = -\frac{n\left(\vec{E} + \vec{I}\right)a^3}{k\varepsilon_0} = -4\pi n\left(\vec{E} + \vec{I}\right)a^3 \tag{9.5}$$

From here we can get $\vec{I}$ and the permittivity is given by $\varepsilon = \vec{E}/\left(\vec{E} + \vec{I}\right) = 1 + 4\pi n a^3$

2. Ferromagnetics

2.1. Magnetic moment expected value

Let's denote $x = \mu B/k_B T$. Also let's have axis z in direction of $\vec{B}$. The probability of $\vec{\mu}$ being along $\vec{B}$ is $\varphi_\uparrow = ae^x$, and for being opposite to $\vec{B}$ it is $\varphi_\downarrow = ae^{-x}$. The coefficient a can be found using the fact that these are the only possible states. For expected value of μ we then have

$$\langle \mu_z \rangle = \mu\varphi_\uparrow - \mu\varphi_\downarrow = \mu \frac{e^x - e^{-x}}{e^x + e^{-x}} = \mu \tanh x \tag{9.6}$$

2.2. Magnetic field in a coil

Consider a coil with section area S, length L and winding distance d, with current I in it. The current flows up and counterclockwise when looking from above. The magnetic field inside it is given by $B = \mu_0 I/d$, pointing up. The magnetic moment of a single winding is IS, also pointing up. The direction can be understood by supposing a square contour that is not perpendicular to some magnetic field, and observing the forces that are acting on it, which we know, want to alight the moment with the field, as that state has the lowest energy.

The magnetic moment density is then $\kappa = I/d$. Finally $\vec{B}$ in terms of $\vec{\kappa}$ is given as $\vec{B} = \mu_0 \vec{\kappa}$.

2.3. Magnetic permeability of a medium

Magnetic permeability is the ratio of the field inside the medium to the initial magnetic field. The source of medium's field is the polarization of atoms under the external fields (as we can neglect the medium's field here). Notice that unlike the dielectrics, the medium generates a field along the external filed and not opposite to it. If we denote the internally created field by I, then the magnetic permeability is

$$\mu^m = \frac{\vec{B} + \vec{I}}{\vec{B}} = 1 + \frac{\mu_0 n \mu \tanh x}{B} \approx 1 + \frac{\mu_0 n \mu x}{B} = 1 + \frac{n \mu_0 \mu^2}{k_B T} \tag{9.7}$$

2.4. Strong internal field

According to the model, we should consider the action of the effective field on each individual atom, which as a result will generate the internal field. The corresponding equation is

$$I = n\mu_0 \mu \tanh\left(\frac{\mu(B + bI)}{k_B T}\right) \tag{9.8}$$

2.5. Permanent magnet

The term "permanent magnet" implies it can sustain its own magnetic field without any external field. So the problem reduces to determining the temperatures, for which (9.8) has a solution with $B = 0$.

On the left-hand side of the equation is just I, and on the right-hind side it's an I in tanh(). Both these functions are 0 at $I = 0$. Moreover, $\tanh(x)$ has a decreasing derivative, which eventually reaches 0. This means, that the equation has a solution other than 0 if and only if the derivative of right-hand side at 0 is greater than 1. Formally it is written as

$$n\mu_0\mu \frac{\mathrm{d}}{\mathrm{d}I} \tanh\left(\frac{\mu b I}{k_B T}\right)\Bigg|_{I=0} = \frac{n\mu_0\mu^2 b}{k_B T} \cosh^{-2}\left(\frac{\mu b I}{k_B T}\right)\Bigg|_{I=0} = \frac{n\mu_0\mu^2 b}{k_B T} > 1 \tag{9.9}$$

So the temperature should be $T < T^C = n\mu_0\mu^2 b/k_B$.

2.6. b of iron

The only parameter not known to calculate b is the concentration, which can be easily found using the molar mass, density and the mass of proton as $n_{Fe} = \rho_{Fe}/m_p[M_{Fe}\text{ mol/g}]$, where the square braces emphasize that we take a dimensionless parameter, namely 56.

Then for b we get $b_{Fe} = k_B T^C_{Fe}/\mu_0\mu^2 n_{Fe} \approx 1500$.

Light in Medium

Problem

1. **Dispersion**

One can model the atom as a stationary object with a heavy nucleus and a light electron with mass m and charge $q < 0$. We assume that electron's charge is uniformly distributed in a ball of radius R with its center being near the nucleus.

1.1. Find the natural frequency ω_0 of the atom's oscillations.

The emission and absorption of photons by the atom is the likeliest near the natural frequency of the atom. As long as we want the medium to be transparent for the light we will be using frequencies far away form the natural frequency.

An electromagnetic wave is called linearly polarized if its electric field strength is given by $\vec{E} = \vec{E_0}\cos(\omega t - kz)$ with $\vec{E_0} \perp \hat{z}$.

1.2. Find the low of motion of the atom in a linearly polarized light with amplitude $\vec{E}$.

1.3. What is the strength of electric field inside a capacitor with dipole moment density p?

1.4. What is the refractive index of the medium with atom concentration ρ for light of frequency ω?

2. **Faraday effect**

An electromagnetic wave is called circularly polarized if its electric field strength is given by $E_x = E_0\cos(\omega t - kz)$, $E_y = \pm E_0\sin(\omega t - kz)$.

2.1. How will such waves be written in complex representation, if we want the x component to be the real part and the y component to be the imaginary part? How is k given in terms of refractive index of the medium and other known parameters?

2.2. Find the low of motion of the atom described in previous section in such field.

Suppose the medium is placed in a magnetic field of strength B along the direction of light propagation.

2.3. How will the magnetic field affect the motion of atoms, if we consider that effect to be very small?

2.4. Find the refractive index of the medium with concentration ρ inside magnetic field B for clockwise and counterclockwise polarized light.

The linearly polarized wave can be represented as a sum of two circularly polarized waves.

2.5. How will the linearity polarized light be different at distance l in the medium?

Useful information

- The light speed in medium is given by $c = 1/\sqrt{\varepsilon_0\mu_0\varepsilon\mu}$.

Solution

1. Dispersion

1.1. Atom's natural frequency

In the whole problem we will consider a static nucleus and a moving electron cloud, as the later's mass is much smaller

The interaction force between the nucleus and the electron is linearly proportional to the cloud's center's displacement $\vec{x}$, as only the part of the cloud within radius x matters. The force on the electron is given by

$$\vec{F} = -\frac{kq}{x^2} \cdot q\frac{x^3}{R^3} \cdot \frac{\vec{x}}{x} = -\frac{kq^2}{R^3} \cdot \vec{x} \tag{10.1}$$

So the natural frequency will be $\omega_0 = \sqrt{kq^2/mR^3}$

1.2. Stimulated motion of an atom

The motion equation inside an external electric field if given by $m\ddot{\vec{x}} = -m\omega_0^2 x + Eq$. If $E = E_0 \cos(\omega t)$ at some given point, we should look for x in form $x = x_0 \cos(\omega t)$. Simple calculations lead to result

$$x_0 = \frac{E_0 q}{m(\omega_0^2 - \omega^2)} \tag{10.2}$$

It might seem unnatural that x_0 changes the sign when ω passes ω_0, but one should remember, that everything we discuss is correct only for ω-s far form ω_0.

1.3. Electric field in a capacitor

Consider a capacitor of area S and thickness d, that is charged with surface charge density $\pm\sigma$. The overall dipole moment in this case is $P = \sigma S \cdot d$ (in direction towards the plate charged with σ) and its density is thereby $\vec{p} = \sigma$.

The electric field inside can be given by $E = \sigma/\varepsilon_0$ towards the plate with $-\sigma$. So the field can be expressed in terms of $\vec{p}$ as

$$\vec{E} = -\vec{p}/\varepsilon_0 \tag{10.3}$$

1.4. Refractive index

The existence of a non-identity refractive index is a result of the medium-created electric field, which is a source of non-identity permittivity.

Electric permittivity is the ratio of the initial external field to what we have in the medium as a result of superposition with the medium's field. The source of medium's field is the polarization of atoms under the influence of both external and medium-generated fields. We will assume medium's field to have the same ω as the external field. If we denote the later by $\vec{I} = \vec{I}_0 \cos(\omega t)$, using (10.2) will produce the polarization of a single atom

$$\vec{P} = ex = \frac{\left(\vec{E} + \vec{I}\right) q^2}{m(\omega_0^2 - \omega^2)} \tag{10.4}$$

We already know the strength of the field that this will create. Plugging (10.4) into (10.3) gives

$$\vec{I} = -\frac{\rho\left(\vec{E} + \vec{I}\right) q^2}{m(\omega_0^2 - \omega^2)\varepsilon_0} \implies \varepsilon = \frac{\vec{E}}{\vec{E} + \vec{I}} = 1 + \frac{\rho q^2}{m(\omega_0^2 - \omega^2)\varepsilon_0} \tag{10.5}$$

The corresponding refractive index is then just $n = \sqrt{\varepsilon}$.

2. Faraday effect

2.1. Complex representation

In the complex representation $\vec{E}$ is just written as $E = E_x + iE_y = e^{\pm i(\omega t - kz)}$. k is found from condition $k\lambda = 2\pi$ and the fact that light travels distance λ in time $n\lambda/c$. So $k = n\omega/c$.

2.2. Stimulated motion of an atom

The motion equation is the same: $m\ddot{x} = -m\omega_0^2 + qE$. The difference is, that we look for the solution in form $x = x_0 e^{\pm\omega t}$. It is straightforward to see, that x_0 is that same as (10.2).

2.3. Motion with magnetic field

The magnetic field acts on the moving particles with force $\vec{F} = q\vec{V} \times \vec{B}$. If V in in complex representation and B is along z, it is possible to see that the force in complex representation is given by $F = -iqVB = -iq\dot{x}B$. This term should be added to the motion equation, which then becomes $m\ddot{x} = -m\omega_0^2 + qE - iq\dot{x}B$. The solution is then easily fount in the same form.

$$x = x_0' e^{\pm i(\omega t - kz)} \qquad x_0' = \frac{E_0 q}{m(\omega_0^2 - \omega^2) \mp qB\omega} \approx \frac{E_0 q}{m(\omega_0^2 - \omega^2)}\left(1 \pm \frac{qB\omega}{m(\omega_0^2 - \omega^2)}\right) \tag{10.6}$$

2.4. Refractive indices

The refractive indices are found just like in 1.4. The permittivities for the two cases are

$$\varepsilon_\pm = 1 + \frac{\rho q^2}{m(\omega_0^2 - \omega^2)\varepsilon_0}\left(1 \pm \frac{qB\omega}{m(\omega_0^2 - \omega^2)}\right) \tag{10.7}$$

And the refractive indices are

$$n_\pm = \sqrt{\varepsilon_\pm} \approx n_0 \pm \Delta n \qquad n_0 = \sqrt{1 + \frac{\rho q^2}{m(\omega_0^2 - \omega^2)\varepsilon_0}} \qquad \Delta n = \frac{1}{n_0}\frac{\rho q^3 B\omega}{2m^2(\omega_0^2 - \omega^2)^2\varepsilon_0} \tag{10.8}$$

The approximation here is a simple Taylor expansion.

2.4. Linearly polarized light

The decomposition into two circularly polarized waves is simply written as

$$E = E_0 e^{i(\omega t - k_+ z)} + E_0 e^{-i(\omega t - k_- z)} \tag{10.9}$$

Outside of the medium, where $k_+ = k_-$, this is indeed a linearly polarized light with $E = 2E_0\cos(\omega t - kz)$. Inside the medium $k_\pm = (n_0 \pm \Delta n)\omega/c = k_0 \pm \Delta k$. The expression for the field then can be rewritten as

$$E = E_0 e^{i(\omega t - k_0 z)} e^{-i\Delta kz} + E_0 e^{-i(\omega t - k_0 z)} e^{-i\Delta kz} = E_{B=0} e^{-i\Delta kz} \tag{10.10}$$

Here we have an additional rotation of plane of E compared to the case without magnetic field. As $q < 0$, $\Delta k < 0$. This means that the rotation is clockwise. After distance l the plane will have been rotated by angle $\beta = l\Delta n\omega/c$ counterclockwise (β itself is negative).