Author's problems on Armenian school physics olympiads

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February 13, 2023

Contents

*The number in the braces represents the difficulty.

Cylinder on the Corner

Problem

A cylinder of mass m is held on the corner of a table by a long homogenous plank of mass M in position described by α as shown on (Fig.1.1). What friction coefficients must there be between the cylinder and the table, between the plank and the table and between the cylinder and the plack for this situation to be possible? The plank is much longer than the radius of the cylinder.

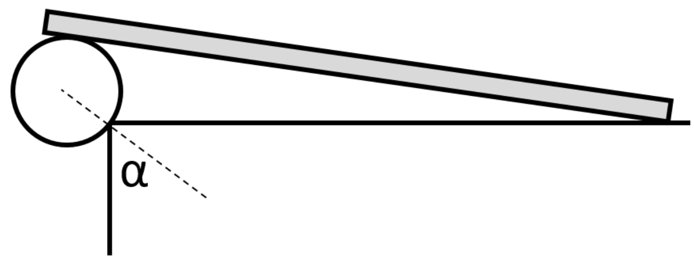
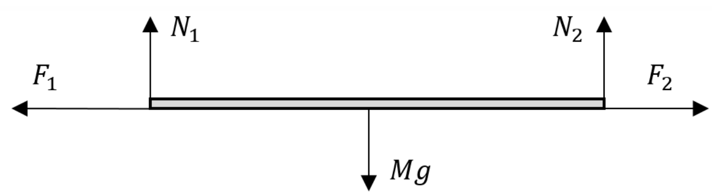
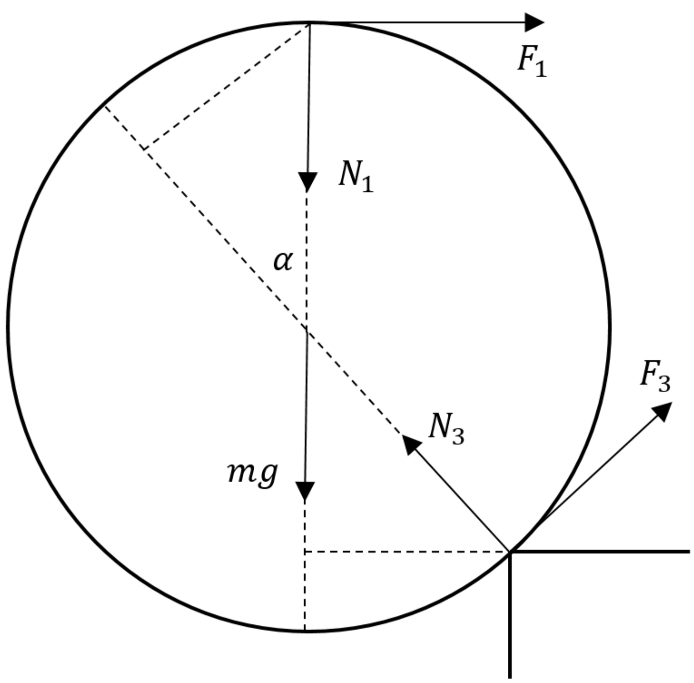


Figure 1.1

Solution



(a) Forces on the plank.



(b) Forces on the cylinder.

Figure 1.2: The forces in the system. F_i are the friction forces.

The forces acting on the cylinder and the plank are shown on (Fig.1.2)

From the equilibrium (both translational and rotational) on the plank we get

$$\begin{aligned} N_1 = N_2 &= \frac{Mg}{2} \\ F_1 &= F_2 \end{aligned} \tag{1.1}$$

and the equations for equilibrium on the cylinder, particularly torques against the center, torques against the cylinder's top point and torques against the cylinder-table touching point are respectively

$$\begin{aligned} F_1 &= F_3 \\ RN_3 \sin\alpha &= RF_3(1+\cos\alpha) \\ R(mg+N_1)\sin\alpha &= RF_1(1+\cos\alpha) \end{aligned} \tag{1.2}$$

where R is the radius of the cylinder. It is now easy to see, that

$$F_1 = \frac{(2m+M)g\sin\alpha}{2(1+\cos\alpha)} \tag{1.3}$$

For the required values of friction coefficients we simply get

$$\begin{aligned} \mu_{c,t} &> \frac{F_3}{N_3} = \frac{\sin\alpha}{1+\cos\alpha} \\ \mu_{c,p/p,t} &> \frac{F_2}{N_2} = \frac{F_1}{N_1} = \frac{\sin\alpha}{1+\cos\alpha}\left(1+\frac{2m}{M}\right) \end{aligned} \tag{1.4}$$

Pulley with Friction

Problem

The system supposed to lift weights consists of a stationary and a movable pulleys as shown in (Fig.2.1). The radii of a pulley and its axis are R and r correspondingly. Assume that the hole in the pulley is slightly bigger than the axis. There is a friction between the axis and the pulley with a given friction coefficient μ. There is no sliding between the ropes and the pulleys, as well as between the ropes and the axes. Determine the energy efficiency coefficient of the system

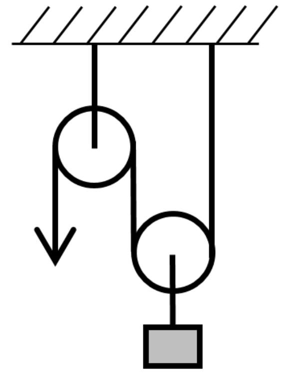


Figure 2.1

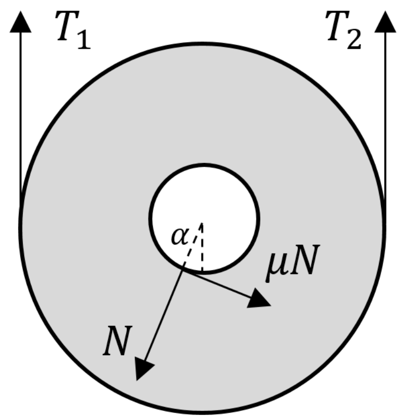
Solution

Consider the movable pulley. Let's assume, that in the process of lifting the touching point between the pulley and its axis is shifted left by some angle α. The forces acting on the pulley and the axis are shown in (Fig.2.2). From the equilibrium on the axis we have

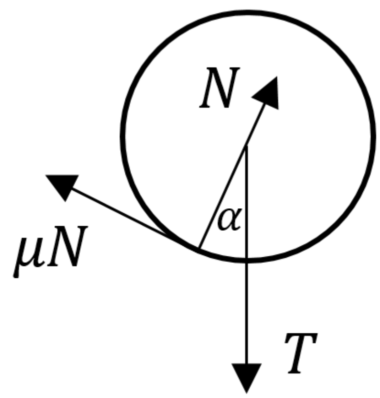
$$\begin{aligned} N\sin\alpha &= \mu N\cos\alpha \\ N\cos\alpha + \mu N\sin\alpha &= T \end{aligned} \tag{2.1}$$

from where we get

$$\begin{aligned} \tan\alpha &= \mu \\ \mu N &= T\frac{\mu}{\sqrt{1+\mu^2}} \end{aligned} \tag{2.2}$$



(a) Forces on the pulley.



(b) Forces on the axis.

Figure 2.2: The forces on the movable pulley.

Notice that we are not allowed to write toque equilibrium as there is nothing known about the torque of interaction between the axis and the rope attached to it.

The equilibrium of the pulley itself is written as

$$\begin{aligned} T_1 + T_2 = N\cos\alpha + \mu N\sin\alpha &\implies T_1 + T_2 = T \\ RT_1 - RT_2 = r\mu N &\implies T_1 - T_2 = T\frac{r}{R}\frac{\mu}{\sqrt{1+\mu^2}} \end{aligned} \tag{2.3}$$

and the other equation is identical to that of the axis. From there we get

$$T_{1/2} = T\frac{1\pm\varepsilon}{2}, \qquad \varepsilon = \frac{r}{R}\frac{\mu}{\sqrt{1+\mu^2}} \tag{2.4}$$

The situation and the equations for the stationary pulley are absolutely the same as the ones written here. So the tension of the free end of the rope should be

$$T_0 = mg\cdot\frac{1+\varepsilon}{2}\cdot\frac{1+\varepsilon}{1-\varepsilon} \tag{2.5}$$

so for the efficiency coefficient we get

$$\eta = \frac{mgl}{T_0\cdot 2l} = \frac{1-\varepsilon}{(1+\varepsilon)^2} \tag{2.6}$$

Combined Pulley

Problem

How much will the weight m shown on (Fig.3.1) descend after hanging it on the rope? The coaxial pulleys are attached to each other and the ratio of their radii is $n > 1$. The stiffness of the spring is k. There is no sliding.

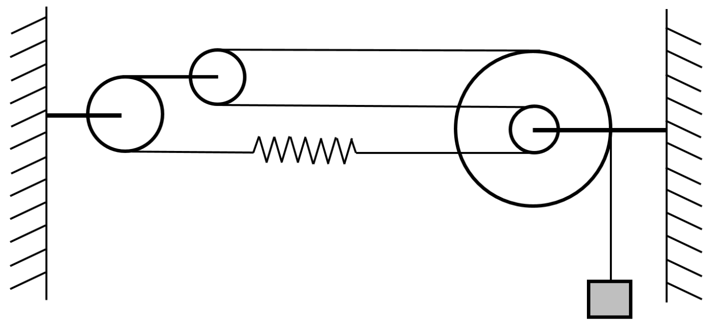


Figure 3.1

Solution

For the final state (Fig.3.2) we have the following equilibrium conditions

$$\begin{aligned} T &= T' \\ F &= T + T' \\ F + nmg &= T' + nT \end{aligned} \tag{3.1}$$

from where the value of F, and consequently the spring deformation x can be found.

$$x = \frac{F}{k} = \frac{2n}{n-1}\frac{mg}{k} \tag{3.2}$$

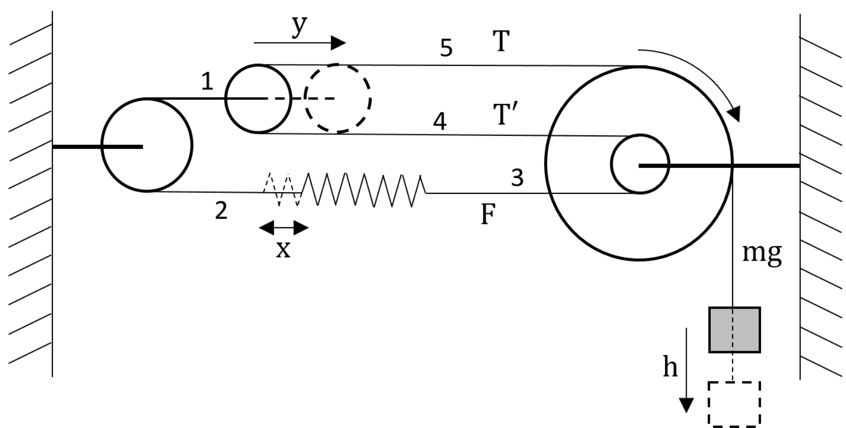


Figure 3.2: The initial (solid lines) and the final (dashed lines) states of the system.

Consider the amounts of movement of the points $1-5$ shown on (Fig.3.2). With choosing positive direction to be towards right, we have

$$\begin{aligned} \Delta_1 &= y \\ \Delta_2 &= -y \\ \Delta_3 &= -y + x \\ \Delta_4 &= y - x \\ \Delta_5 &= 2\Delta_1 - \Delta_4 = y + x \end{aligned} \tag{3.3}$$

Also we know that $n\Delta_4 = \Delta_5$, which provides a connection between y and x. Further solving gives

$$h = \Delta_5 = \frac{2n}{n-1}x = \left(\frac{2n}{n-1}\right)^2 \frac{mg}{k} \tag{3.4}$$

Inclined Rod

Problem

A rod is hanged form the ceiling by its ends with strings of length l_1 and l_2 in such a way, that both strings are vertical. Find the natural frequencies of rods oscillations.

Solution

There are three modes of oscillation. The first one is the oscillation within the strings' plane, and the other two include oscillations perpendicular to it alongside with rotational oscillations.

Consider the first mode. Let's denote the small inclinations of the strings with length l_1 and l_2 by α and β correspondingly. The heights of the ends of the rod don't change, so the constancy of rod length is written as $l_1\alpha = l_2\beta$. We will be doing substitutions $\sin\alpha \to \alpha$ and $\cos\alpha \to 1$ without mentioning it. There is also no rotation, so $T_1 = T_2$ (the string tension forces).

If the displacement of the rod is x, then $\alpha = x/l_1$ and $\beta = x/l_2$. The overall horizontal force will be

$$F = -T_1 \sin\alpha - T_2 \sin\beta = \frac{mg}{2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)x \tag{4.1}$$

As $F = m\ddot{x}$, the corresponding frequency will be

$$\omega_1 = \sqrt{\frac{g}{2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)} \tag{4.2}$$

For the other two modes let's use the same α and β, but here they will be showing the inclinations perpendicular ot strings' plane. The displacements of rod's edges will be $l_1\alpha$ and $l_2\beta$. The displacement x of rod's center and rod's rotation γ around the vertical axis are then give by

$$\begin{aligned} x &= \frac{l_1\alpha + l_2\beta}{2} \\ \gamma &= \frac{l_1\alpha - l_2\beta}{d} \end{aligned} \quad\Longrightarrow\quad \begin{aligned} \alpha &= \frac{2x + \gamma d}{2l_1} \\ \beta &= \frac{2x - \gamma d}{2l_2} \end{aligned} \tag{4.3}$$

where d is the horizontal distance between the strings. The string tensions are also equal to $mg/2$ in this case. The motion equations are then written as

$$\begin{aligned} m\ddot{x} &= -T_1 \sin\alpha - T_2 \sin\beta = -\frac{mg}{2}(\alpha + \beta) = -\frac{mx}{2}\Sigma - \frac{md\gamma}{4}\Delta \\ I\ddot{\gamma} &= -T_1 \sin\alpha \frac{d}{2} + T_2 \sin\beta \frac{d}{2} = -\frac{mgd}{2}(\alpha - \beta) = -\frac{mdx}{4}\Delta - \frac{md^2\gamma}{8}\Sigma \end{aligned} \tag{4.4}$$

where $I = ml^2/12$ is the moment of inertia of the rod against the vertical axis through its center, $\Sigma = g(l_1^{-1} + l_2^{-1})$ and $\Delta = g(l_1^{-1} - l_2^{-1})$. Denoting $x = kd\gamma$ for a specific mode we get

$$\begin{aligned} k\ddot{\gamma} &= -\frac{k\Sigma}{2}\gamma - \frac{\Delta}{4}\gamma \\ \frac{\ddot{\gamma}}{12} &= -\frac{k\Delta}{4}\gamma - \frac{\Sigma}{8}\gamma \end{aligned} \quad\Longrightarrow\quad \begin{aligned} \omega^2 &= -\frac{\Sigma}{2} - \frac{\Delta}{4k} \\ \omega^2 &= -3k\Delta - \frac{3\Sigma}{2} \end{aligned} \tag{4.5}$$

After some trivial calculations one can get

$$\omega_{2,3} = \sqrt{\frac{g}{l_1} + \frac{g}{l_2} \pm g\sqrt{\frac{1}{l_1^2} + \frac{1}{l_2^2} - \frac{1}{l_1 l_2}}} \tag{4.6}$$

Notice that two of the frequencies reduce to $\sqrt{g/l}$ in case $l_1 = l_2 = l$.

Audio Casette

Problem

Find the thickness h and density ρ of the tape of an audio casette. The coil cores have radius $r = 11.0$ mm and the mass of the whole casette is $M = 40$ g

Equipment given: audio casette, pencil, calipers.

Solution

In order to measure something, we first have to transport all the tape to one of the coils (namely first). Then we can do N rotations of the second coil and measure the radius ratio of the coils and the position of the center of mass.

The radius of the coils is measured as follows: you turn one of the coils by some angle (can be measured using the teeth of the coil), and follow the rotation of the other coil. The ratio of the angles is the ratio of radii. The process should be reversed before continuing. The mass center position can be found by just pushing the casette of something's edge and measure the position using the calipers. It's better to use the calipers itself as the "something" for better precision.

The measurements are as follows

N	$\alpha_1[2\pi]$	$\alpha_2[2\pi]$	r_1/r_2	x_c[m]
0	5	$2+1/6$	2.31	0.0935
50	3	$1+5/12$	2.12	0.0932
100	3	$1+13/24$	1.95	0.0928
150	3	$1+2/3$	1.80	0.0924
200	2	$1+1/4$	1.60	0.0920
250	4	$2+3/4$	1.45	0.0914
300	2	$1+1/2$	1.33	0.0909
350	5	$4+1/12$	1.22	0.0902
400	4	$3+7/12$	1.12	0.0894
450	2	$1+11/12$	1.04	0.0887
500	4	$4+1/4$	0.94	0.0880
550	4	$4+3/4$	0.84	0.0874
600	3	$3+11/12$	0.77	0.0867
650	3	$4+5/12$	0.68	0.0857
700	3	$5+1/12$	0.59	0.0848
750	1	2	0.50	0.0840
800	1	$2+1/2$	0.40	0.0831

we will later see, that the 0 point for x_c measurement doesn't matter.

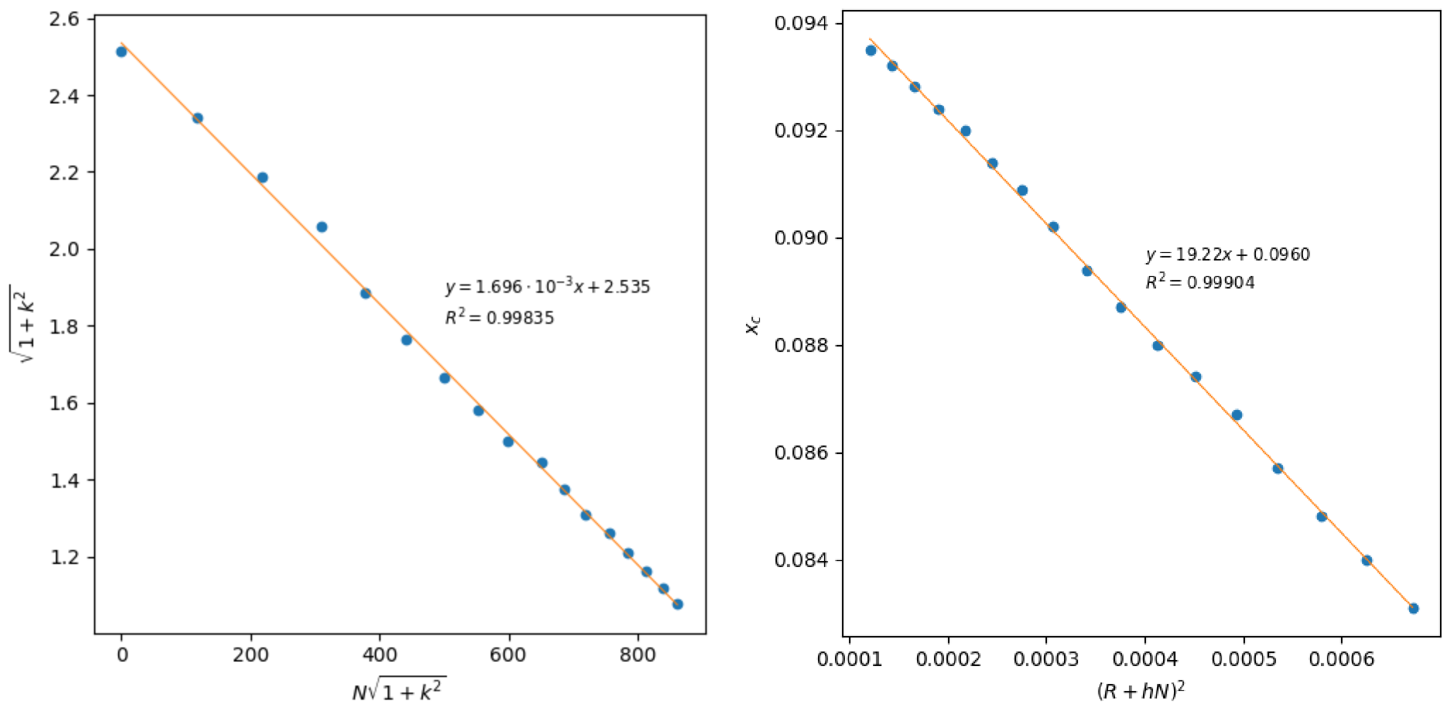
Theoretically we can write the radius ratio as

$$\frac{r_1}{r_2} = k = \frac{\sqrt{(r + hN_0)^2 + r^2 - (r + hN)^2}}{r + hN} \tag{5.1}$$

as the side surface of the tape $\propto r_1^2 + r_2^2$ conserves. Here N_0 is full number of rotations. This can be linearized to

$$\sqrt{1 + k^2} = -\frac{h}{r}\sqrt{1 + k^2}N + \sqrt{(1 + N_0 h/r)^2 + 1} \tag{5.2}$$

and by plotting $\sqrt{1 + k^2}$ on $N\sqrt{1 + k^2}$ (Fig.5.1a) one can get $h = r \cdot 1.7 \cdot 10^{-3} = 1.9 \cdot 10^{-5}$ m.



(a) (b)

Figure 5.1

The position of the center of mass is theoretically given as

$$x_c = \frac{m_* x_* - \rho \pi w h l (r + hN)^2}{M} \tag{5.3}$$

where w is the width of the tape and l is the distance between coil centers, which can be measured directly. $w = 3.8 \cdot 10^{-3}$ m, $l = 4.23 \cdot 10^{-2}$ m. m_* and x_* are some arbitrary units which include the casette box and some residual terms from the calculation.

So by plotting x_c on $(r + hN)^2$ (Fig.5.1b) we get $\rho = M/\pi w l \cdot 19.2\ \mathrm{m}^{-1} = 1.5 \cdot 10^3\ \mathrm{kg/m}^3$.

Leaky Container

Problem

There is a cup of radius R on the table. A container that is higher than the cup by H is fully filled with water and is placed next to the cup in such a way that the distance between cup's center and its nearest edge is D. Where on the container you should poke a small hole so there is maximum possible amount of water in the cup after the water flow stops?

Solution

We take the cup's top as 0 of height. Forget about the finiteness of the container for now. If the hole is made on height h, than the water should have velocity $v = d\sqrt{g/2h}$ upon exiting the container to hit distance d (basic kinematics). To gain velocity v the water level l should be $v^2/2g$ higher than the hole (Bernoulli equation). So if we poke the hole at h, the water levels that will hit the nearest and the farthest edges of the cup are

$$l_\pm = h + \frac{(D \pm R)^2}{4h} \tag{6.1}$$

and all the water between l_- and l_+ will end up in the cup. One may notice, that the difference $l_+ - l_-$ is decreasing on h, and hits infinity at $h = 0$. Here comes the finiteness of the container.

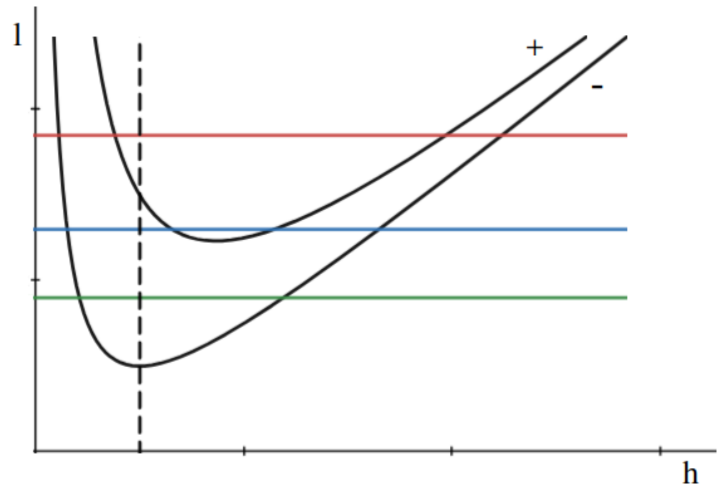


Figure 6.1: The water levels, at which water will hit the edges of the cup (black solid lines) for specific position of the hole (horizontal axis). Colorful lines demonstrate different possible values of H. Dashed black line indicates the minimum of l_-.

In order to better understand the situation let's qualitatively portray $l_\pm$ and H on h (Fig.6.1). The minima of $l_\pm$ are located at $h_\pm = (D \pm R)/2$, and thereby the minimum of l_- is left to that of l_+. We have to choose such h, that the section between l_+ and l_- clamped by H is maximal.

If we remember, that the distance without clamping decreases on h, it is easy to verify, that in case the first intersection of H and l_+ is left from the minimum of l_- (the red line case on (Fig.6.1)), that the intersection point itself is the optimal place for a hole. The intersection point is given by the smaller solution of $H = l_+$

$$h = \frac{H - \sqrt{H^2 - (D+R)^2}}{2} \tag{6.2}$$

If the intersection is right from h_- (the blue line case) or there is no intersection at all (the green line case) then h is just h_-.

So the optimal h for the hole is given by

$$h = \min\left(\frac{H - \sqrt{H^2 - (D+R)^2}}{2}, \frac{D-R}{2}\right) \tag{6.3}$$

and when the first argument is undefined we automatically take the second one.

There is also a possibility that no water can end up in the cup. In that case we are free to make a hole wherever we want.

1 + 1 (4)

Problem

The resistance of a thermistor depends on its absolute temperature as $R = \beta T^2$, where $\beta = 7.00 \cdot 10^{-4}$ Ohm/K. Surface area of the thermistor is $S = 1.00 \cdot 10^{-2}$ m^2 and the heat transfer coefficient with the medium is $\alpha = 200$ W/m^{2}K. The temperature of the medium is $T_0 = 293$ K.

1. Find the current in such a thermistor under voltage $U_0 = 1000$ V.

2. Find the current in a system with two consequently connected thermistors under voltage $U_0 = 1000$ V.

   You might need the following information while solving the problem.

   - The solution of equation $x^3 + kx^2 + b = 0$ is given by

   $$x = \frac{1}{3}\left[\left(\frac{r}{2}\right)^{1/3} + k^2\left(\frac{r}{2}\right)^{-1/3} - k\right], \qquad r = 3^{3/2}\sqrt{27b^2 + 4bk^3} - 27b - 2k^3 \tag{7.1}$$

   - A solution of any equation can be found even if the equation can not be solved analytically.

Solution

The heat generated under voltage U in case of temperature T is given by $Q_+ = U^2/\beta T^2$. The heat loss is given by $Q_- = \alpha S(T - T_0)$. The equilibrium condition $Q_+ = Q_-$ gives

$$T^3 - T_0 T^2 - \frac{U^2}{\alpha\beta S} = 0 \tag{7.2}$$

For voltage $U = U_0$ we will get $T = 1000$ K and $I = U/\beta T^2 = 1.42$ A.

For the case of two consequently connected thermistors one might think that each of them will get voltage $U = U_0/2$ because of symmetry. Then we calculate the temperature $T = 680$ K and current $I = 1.55$ A. Immediately we see that there is some problem, because we got higher current for lower voltage. One may also check the current is locally decreasing at that point (this can simply done numerically).

This means that the solution is not stable. Indeed. Suppose the voltage slightly increases on the first thermistor and thereby decreases on the second one. This will result in current decrease on the first thermistor and increase on the second one, which in its turn will result in negative charge accumulation in the center thereby further increasing the voltage deviation.

So we need to find another solution, which obviously will be non-symmetric. The equation to be solved is

$$I(U) = I(U^*) \qquad U^* = U_0 - U \tag{7.3}$$

where $I(U)$ is $U/\beta T^2$ and T is the solution of (7.2).

This is to be done numerically. Remember that the current is decreasing on U at $U = U_0/2$, so for slightly less U-s $I(U) > I(U^*)$. Now we need to find the U, where $I(U)$ becomes bigger than $I(U^*)$. The calculations are

U [V]	U^* [V]	T_U [K]	T_{U^*} [K]	I_U [A]	I_{U^*} [A]
0	1000	293	1003	0	1.420
500	500	680	680	1.546	1.546
250	750	484	850	1.526	1.484
150	850	396	913	1.369	1.458
200	800	440	881	1.473	1.471
190	810	431	888	1.458	1.468
195	805	436	885	1.466	1.470

At this point we are sure that the equilibrium current is $I = 1.47$ A.

Thick Barrel (4)

Problem

What thickness should a cylindrical barrel with internal radius R have, in order to withstand internal pressure P? The barrel is made of material with strength σ. What happens in case $P \ll \sigma$?

You might need the following information while solving the problem.

- The solutions to differential equation $y'' = y/x^2$ obviously have form $y = Cx^a$.
- When $k > 1$
$$\left(\frac{2}{5+\sqrt{5}} + \frac{2}{5-\sqrt{5}}k^{\sqrt{5}}\right) > k^{\frac{\sqrt{5}-1}{2}}$$

Solution

We will be describing the state of our barrel with the radial displacement of each point $x(r)$, $r \to r + x(r)$. The tangential component of relative deformation then will be

$$\varepsilon_t(r) = \frac{x(r)}{r} \tag{8.1}$$

as the ring with initial length $2\pi r$ now has length $2\pi(r + x(r))$. The radial deformation is given by

$$\varepsilon_r(r) = \frac{\mathrm{d}x(r)}{\mathrm{d}r} \tag{8.2}$$

as the point at initial positions r and $r + \Delta r$ are now at $r + x(r)$ and $r + \Delta r + x(r + \Delta r) = r + \Delta r + x(r) + x'(r)\Delta r$. So the point now have distance $(1 + x'(r))\Delta r$ instead of initial Δr.

The internal (inside the barrel walls) equilibrium is generated as a result of opposition of the radial forces to tangential forces. Consider a section of a barrel at radius r, with thickness $\mathrm{d}h$, length (along the axis) l and angular size $\mathrm{d}\alpha$. The equilibrium condition of it is written as

$$2E\varepsilon_t \cdot l\mathrm{d}h \cdot \frac{\mathrm{d}\alpha}{2} = E\frac{\mathrm{d}\varepsilon_r}{\mathrm{d}r}\mathrm{d}h \cdot lr\mathrm{d}\alpha \tag{8.3}$$

where the left-hand side of the equation are the tangential pressure forces projected on radial direction, and the right-hand side is the difference between the radial pressure forces from above and below (in terms of radii) of the section.

In terms of $x(r)$, this equation can be rewritten as simply $x'' = x/r^2$. The solutions are easily found in form Cx^a,, from where we get the general solution of the equation

$$x = C_+ r^{\beta_+} + C_- r^{\beta_-} \qquad \beta_\pm = \frac{1 \pm \sqrt{5}}{2} \tag{8.4}$$

The constants are determined by edge conditions $E\varepsilon_r(R) = -P$ and $E\varepsilon_r(R_o) = 0$, where R_o is the outer radius of the barrel. To apply these conditions we need the expression for ε_r.

$$\varepsilon_r(r) = x'(r) = C_+\beta_+ r^{\beta_+-1} + C_-\beta_- r^{\beta_--1} \tag{8.5}$$

The edge conditions result in

$$\beta_+ C_+ = \frac{P}{ER^{\beta_+-1}\left(\left(\frac{R_o}{R}\right)^{\sqrt{5}} - 1\right)} \qquad \beta_- C_- = -\beta_+ C_+ R_o^{\sqrt{5}} \tag{8.6}$$

where $\sqrt{5}$ comes from $\beta_+ - \beta_-$. Note that $C_\pm > 0$.

Now we have to figure out where and in which direction the barrel will break. To this end we have to know where and in which directions the existing tension (which is proportional to ε-s) is the highest.

ε_r is a sum of increasing functions, as $C_+\beta_+ > 0$, $\beta_+ - 1 > 0$ and $C_-\beta_- < 0$, $\beta_- - 1 < 0$. Given that at $r = R_o$ it is 0, it has its highest absolute value at $r = R$.

The expression for ε_t is just $\varepsilon_t = C_+ r^{\beta_+ - 1} + C_- r^{\beta_- - 1}$. As a sum of increasing and decreasing positive monomials ($C_\pm > 0$, $\beta_+ - 1 > 0$, $\beta_- - 1 < 0$), ε_t is a function with a single minimum, which means that for any given interval $[a, b]$, the maximum value of the function is either at a or at b.

So the candidates of the highest tension are

$$\begin{aligned} |\varepsilon_r(R)| &= P/E \\ \varepsilon_t(R) &= \frac{P}{E(k-1)}\left(\frac{1}{\beta_+} + \frac{k}{\beta_-}\right) \qquad\qquad k = \left(\frac{R_o}{R}\right)^{\sqrt{5}} \\ \varepsilon_t(R_o) &= \frac{P\sqrt{5}}{E(k-1)}\left(\frac{R_o}{R}\right)^{\beta_-} \end{aligned} \tag{8.7}$$

The last two expressions seem remotely similar to terms in inequation in the hint. After some transformations one can show the connection, with k in the hint being R_o/R. A a result we learn that $\varepsilon_t(R) > \varepsilon_t(R_o)$.

It is left to compare $|\varepsilon_r(R)|$ and $\varepsilon_t(R)$, that is

$$1 * \frac{\beta_- + \beta_+ k}{(k-1)\beta_+\beta_-} \tag{8.8}$$

It is straightforward to show that the right-hand side is bigger.

Finally we have to demand $E\varepsilon_t(R) < \sigma$ for the barrel to remain integrate. This leads to a linear equation on k which results in

$$k = \frac{1 + \frac{P}{\sigma\beta_+}}{1 - \frac{P}{\sigma\beta_-}} \tag{8.9}$$

The corresponding thickness is $h = R(k^{1/\sqrt{5}} - 1)$.

For very small values of pressure the expression can be decomposed

$$h_{\ll} = R\left(1 + \frac{P}{\sqrt{5}\sigma\beta_+} + \frac{P}{\sqrt{5}\sigma\beta_-} - 1\right) = \frac{RP}{\sigma} \tag{8.10}$$