Modulation-Transfer Function and Phase-Structure Function of an Optical Wave in a Turbulent Medium

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Expressions are derived for the dependence of the modulation transfer function (MTF) and the phase structure function (D_φ) on the Kolmogorov turbulence parameters. In particular, it is demonstrated that certain measurements of the MTF and D_φ can be interpreted in terms of an outer scale of turbulence of the order 10–50 cm, values which typically can be expected under inversion conditions.

With regard to our interpretation of the measurements of D_φ and the MTF we note the following formulas.

First, the complex field at the point $\mathbf{r}_i$ can be written, correct through terms of second order in the fluctuations of refractive index,[1] n_1

$$U(\mathbf{r}_i)=U_0\exp[\psi_1(\mathbf{r}_i)+\varphi_2(\mathbf{r}_i)-\tfrac{1}{2}\psi_1^2(\mathbf{r}_i)], \tag{1}$$

where U_0 is the field in the absence of turbulence, ψ_1 is the (first order) Rytov approximation ($=U_1/U_0$ where U_1 is the first-order Born approximation), and $\varphi_2=U_2/U_0$ where U_2 is the second-order Born approximation. Noting that conservation of the average energy through second order in n_1 implies $\langle\mathrm{Re}\varphi_2\rangle+\frac{1}{2}\langle|\psi_1|^2\rangle=0$, we obtain for the MTF, to second order in n_1,

$$M=\langle U(\mathbf{r}_1)U(\mathbf{r}_2)^*\rangle=\exp\{-\tfrac{1}{2}\langle|\psi_1(\mathbf{r}_1)-\psi_1(\mathbf{r}_2)|^2\rangle\}, \tag{2}$$

where, for convenience, we have taken $|U_0|^2=1$, and the asterisk denotes the complex conjugate of the appropriate quantity.

Tatarski[2] calculates the phase and log-amplitude structure functions in a plane perpendicular to the direction of propagation of a plane wave with wave number k which has propagated a distance L to be

$$D_{\varphi l}(\rho)=4\pi^2k^2L\int_0^\infty[1-J_0(K\rho)]\times\left[1\pm\frac{k}{K^2L}\sin\left(\frac{K^2L}{k}\right)\right]\Phi_n(K)KdK, \tag{3}$$

where J_0 is the Bessel function of zero order, $\rho=|\mathbf{r}_1-\mathbf{r}_2|$, and $\Phi_n(K)$ is the three-dimensional spectral density of refractive-index fluctuations. The MTF for plane waves is given by

$$M(\rho)=\exp[-\tfrac{1}{2}\{D_l(\rho)+D_\varphi(\rho)\}]=\exp\left\{-4\pi^2k^2L\int_0^\infty[1-J_0(K\rho)]\Phi_n(K)KdK\right\},\quad L<L_c. \tag{4}$$

Equation (4) has been shown to be correct to all orders in n_1 when the fluctuation of refractive index is a gaussian-random process and has been demonstrated to be correct through second order in n_1 for arbitrary $\Phi_n(K)$. For spherical waves the MTF given by Eq. (4) is modified by replacing $J_0(K\rho)$ by $z^{-1}\int_0^z J_0(K\rho s/z)ds$.

The Kolmogorov theory yields for the refractive-index structure function

$$D_n(r)=\begin{cases}C_n^2r^{2/3} & \text{for } l_0\ll r\ll L_0\\ C_n^2l_0^{2/3}(r/l_0)^2 & \text{for } r\ll l_0,\end{cases} \tag{5}$$

where l_0 and L_0 are the inner and outer scales of turbulence, respectively, and the multiplicative factors were chosen to make

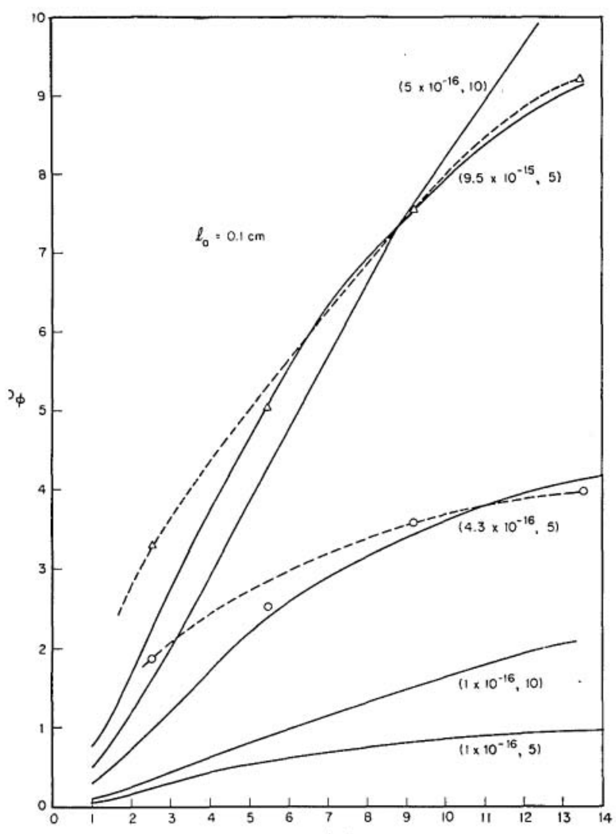

FIG. 1. Comparison of computed phase structure function with data of Bertolotti *et al.* at a range of 0.5 km. Labels indicate values of C_n^2 in cm$^{-2/3}$ and $(L_0/2\pi)$ in cm, respectively.

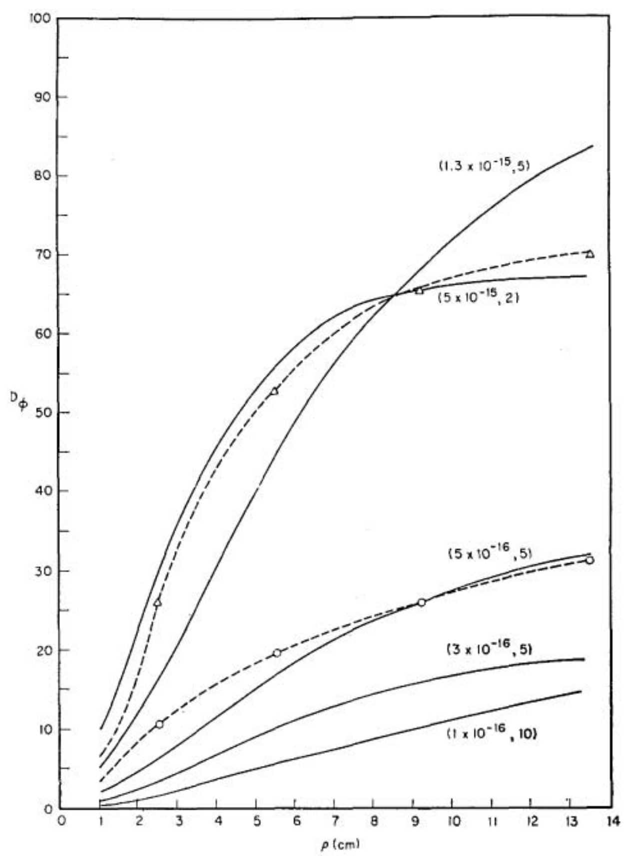

FIG. 2. Comparison of computed phase structure function with data of Bertolotti *et al.* at a range of 3.5 km. Labels indicate values of C_n^2 in cm$^{-2/3}$ and $(L_0/2\pi)$ in cm, respectively.

$D_n(r)$ continuous at $r=l_0$. Tatarski demonstrates the insensitivity of D_l and D_φ to the behavior of $\Phi_n(K)$ for $K \lesssim 1/L_0$ when $(\lambda L)^{1/2} \ll L_0$. Hence the refractive-index spectral density (assuming isotropic turbulence)

$$\Phi_n(K)=\frac{0.033C_n^2\exp[-(Kl_0/5.92)^2]}{(K^2+\bar{L}_0^{-2})^{11/6}} \tag{6}$$

extrapolated from Tatarski's, which implies a flat spectrum for $K<1/\bar{L}_0$ ($\bar{L}_0=L_0/2\pi$), should give an adequate quantitative dependence of D_φ and M on the turbulence parameters.

In Figs. 1 and 2, we have reproduced some experimental curves[3] of D_φ for propagation paths of 0.5 and 3.5 km, respectively. On the same axis we have plotted D_φ obtained by numerically integrating Eq. (3) for plane waves (which corresponds to the experimental situation in Ref. 3). The results are very insensitive to l_0; a nominal value of 0.1 cm was assumed. The phase structure function will scale as C_n^2, and L_0 will be roughly determined by the transverse distance corresponding to the inflection point in D_φ. Comparison shows that reasonable agreement can be obtained with $C_n^2 \sim (5\times10^{-16}-5\times10^{-15})\text{cm}^{-2/3}$ and $L_0 \sim (12\text{–}30)$ cm. Larger values of L_0 would not yield a leveling off of D_φ within $\sim$10 cm, to the extent obtained in the experiment. If the measurements had been made in the daytime rather than after sunset, larger L_0's and a correspondingly greater range of validity for the "$\frac{5}{3}$ law" would be expected.

In Fig. 3, Djurle and Bäck's[4] experimental curves of the MTF for a path length of 11 km have been reproduced. With $l_0=0.1$ cm, the curves can be fairly well represented by using spherical waves and choosing $C_n^2 \sim 10^{-16}$ cm$^{-2/3}$ and $L_0 \sim (25\text{–}50)$ cm. The interpretation of the shape of the MTF's implying outer scales of the magnitude suggested depends on the experimental values being correct to within $\sim$10%. It is noteworthy that no estimates of the experimental errors are given for either D_φ or the MTF. This seems to be characteristic of most optical experiments, which makes theoretical interpretation difficult.

Measurements made under temperature-lapse conditions[5] are usually inferred to yield an outer scale of turbulence of the order of a meter (when the path is $\sim$1 m above the ground). However, most optical experiments, including the ones discussed here, are performed after dark, when the probability of a temperature inversion is high. During the semi-stable conditions that occur under inversion conditions, a reduction of turbulent energy generation takes place. The usual concept is of energy cascading from larger to smaller turbulent eddies; with a decrease of energy generation the outer scale size would decrease. This phenomena is also supported by the experiments of Deitz[5] and Tsvang.[6] Nighttime measurements of the temperature-structure function between two sensors led Deitz to infer an outer scale of 30 cm or less. Tsvang's measurements display a shifting of the center of gravity of Φ_n to larger K as the vertical temperature gradient changed from lapse to inversion conditions.

For physical reasons, a lower limit to the MTF must be obtained when the transverse distance ρ is sufficiently large that the optical wave arriving at the points $\mathbf{r}_1$ and $\mathbf{r}_2$ has been scattered through (statistically) independent media. In this case, as $\rho \to \infty$,

$$M=\langle U(\mathbf{r}_1)U(\mathbf{r}_2)^*\rangle \to \langle U(\mathbf{r}_1)\rangle\langle U(\mathbf{r}_2)^*\rangle \sim \exp[-2L/L_c].^1$$

Hence if the optical path is sufficiently short, the medium does not limit the resolution that can be obtained.

A decrease of the inertial subrange could thus explain the saturation in the phase structure function and the MTF. If the limiting values of the MTF were sufficiently high, optical resolution could be improved by employing larger receiver optics.

A similar consideration is involved in evaluating the performance of an optical heterodyne detector, which involves a double integral of the MTF over the receiving aperture. Based on an MTF that rapidly tends to zero, Fried[7] has predicted a limit of the achievable average signal-to-noise ratio, no matter how large the detector collection aperture is. A high saturation value of the MTF under inversion conditions suggests that further improvement of the performance with larger apertures might still be reasonably expected for sufficiently short paths.

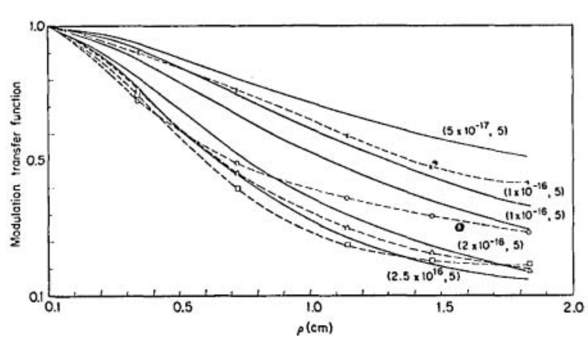

FIG. 3. Comparison of computed modulation transfer function with data of Djurle and Bäck at a range of 11 km. Labels indicate values of C_n^2 in cm$^{-2/3}$ and ($L_0/2\pi$) in cm, respectively.

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1 H. T. Yura, J. Opt. Soc. Am. **59**, 111 (1969).
2 V. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw–Hill Book Co., New York, 1961).
3 M. Bertolotti, M. Carnevale, L. Muzii, and D. Sette, Appl. Opt. **7**, 11 (1968).
4 E. Djurle and A. Bäck, J. Opt. Soc. Am. **51**, 9 (1961).
5 P. H. Deitz and N. J. Wright, J. Opt. Soc. Am. **59**, 527 (1969).
6 L. R. Tsvang, Izvestia ANSSSR, Geophys. Ser. No. 8 (1960).
7 D. L. Fried, Proc. IEEE **55**, 1, (1967).

Relationship between the Real and Imaginary Parts of the Refractive Index

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This letter summarizes an experimental investigation of the relationship between the real and imaginary parts of the complex refractive index $\tilde{n}$ of a liquid medium ($\tilde{n}=n-ik$). It is generally assumed that the Kramers–Kronig relations hold,[1] but this assumption has rarely been experimentally verified[2–4] owing to instrumental difficulties arising when the measurement of $n(\nu)$ is attempted through the whole range of an absorption band. However, by means of an apparatus recently developed in our laboratory,[5,6] easier and reliable measurements were possible. It was concluded that the Kramers–Kronig relationship holds well in the case studied.

Other apparatus employed to check this relationship include:

(1) ellipsometric[7] or energetic refractometers,[8] which are suitable for the investigation of strongly absorbing media. However, this method suffers the disadvantage that the absorption curve $k(\nu)$ cannot be recorded under the same experimental conditions (i.e., same sample and same concentration). This may lead to significant errors in comparisons of the $n(\nu)$ and $k(\nu)$ functions.[8]