Moment-based estimation for the shape parameters of the Gamma-Gamma atmospheric turbulence model.

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Abstract: We study the parameter estimation problem for the Gamma-Gamma turbulence model for free-space optical communication. An estimation scheme for the shape parameters of the Gamma-Gamma distribution is proposed based on the concept of fractional moments and convex optimization. To improve the estimation performance, we further propose a modified scheme which exploits the relationship between the Gamma-Gamma shape parameters in free-space optical communication. Simulation results reveal that the modified estimation scheme can achieve satisfactory performance for a wide range of turbulence conditions.

OCIS codes: (010.1330) Atmospheric turbulence; (010.1300) Scintillation.

References and links

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1. Introduction

Being capable of establishing full-duplex high-speed wireless communication links over a distance of several kilometers using license-free spectrums, free-space optical (FSO) communication has attracted much attention in the past decade. Because of ease and low cost of implementation, FSO system is considered as an alternative to optical fiber for the 'last mile' problem when fiber optic links are unavailable or too expensive to implement.

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As a typical line-of-sight optical communication technology, FSO communication differs from most radio frequency (RF) systems which suffer from fading due to multipath propagation. In FSO communication, the main impairment is caused by atmospheric turbulence-induced fluctuations. Therefore, multipath fading models are no longer applicable to system design and performance analysis of FSO systems. Instead we focus on the study of atmospheric turbulence models.

For weak turbulence conditions, Parry [1] and Phillips and Andrews [2] independently proposed to use a log-normal probability density function (PDF) to model the irradiance, which is the power density of the optical beam. When turbulence becomes stronger, the negative exponential distribution was introduced as a limit distribution for the irradiance. This limit distribution can only provide sufficient accuracy when the system goes far into the saturation regime [3]. The *K*-distribution, which is based on an assumed modulation process, was later introduced to model the irradiance in strong turbulence scenarios [4]. Being a widely accepted turbulence model for FSO communications under strong turbulence conditions, the *K*-distribution is, however, incapable of modeling the irradiance when turbulence is weak. Because the scintillation index given by the *K* distributed irradiance is always greater than unity, which is not valid for weak turbulence scenarios.

Another modulation-based model, the Gamma-Gamma distribution, was later proposed by Al-Habash *et al.* [5] to model the irradiance in FSO systems. The PDF of the Gamma-Gamma distribution is given by

$$f_G(I) = \frac{2(\alpha\beta)^{(\alpha+\beta)/2}}{\lambda\Gamma(\alpha)\Gamma(\beta)}\left(\frac{I}{\lambda}\right)^{\frac{\alpha+\beta}{2}-1} K_{\alpha-\beta}\left(2\sqrt{\alpha\beta I/\lambda}\right),\ \alpha>0, \beta>0, \lambda>0 \qquad (1)$$

where λ is a scale parameter, α and β are the shape parameters, and $K\nu(\cdot)$ is the νth order modified Bessel function of the second kind. The Gamma-Gamma turbulence model is desirable because for both weak and strong turbulence scenarios, this model can provide a good fit to the experimental measurements of irradiance [5]. Thus, the key advantage of using the Gamma-Gamma turbulence model is that it covers a wide-range of turbulence conditions.

To apply turbulence models to the analyses of practical FSO systems, we are often required to estimate the corresponding unknown parameters. Parameter estimation methods for the log-normal distribution and the *K*-distribution have been well studied in [6–9]. However, to the authors' best knowledge, estimator for the parameters of the Gamma-Gamma PDF has not been reported in literature. The parameter estimation problem for the Gamma-Gamma distribution is challenging because a maximum-likelihood approach will involve derivatives of $K\nu(\cdot)$, with respect to both its argument and the order index. For the same reason, the Cramér-Rao lower bound of the estimators cannot be easily derived. Current method for determining the shape parameters of the Gamma-Gamma turbulence model has focused on calculating the Rytov variance, which requires the knowledge of link distance L and refractive-index structure parameter C_n^2 [10]. However, this requirement is not always desirable for practical FSO systems, especially when terminals have some degrees of portability which can change the link parameters frequently. For FSO systems with slant propagation path, the refractive-index structure parameter cannot even be measured accurately because it is a function of altitude, which will change along the slant path.

The remainder of this paper is organized as follows. Section 2 reviews some important statistical properties of the Gamma-Gamma distribution which are useful for our estimation problem. In Section 3 we first propose an estimation scheme for the Gamma-Gamma turbulence model based on the concept of fractional moments and convex optimization. Then a modified estimator which makes use of the relationship between the Gamma-Gamma shape parameters in FSO applications is also proposed. Simulation results show that significant performance improvement in terms of mean square error (MSE) can be achieved by the modified estimation scheme. Finally, Section 4 makes some concluding remarks.

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2. Statistical properties of the Gamma-Gamma turbulence model

Similar to the *K*-distribution, the Gamma-Gamma turbulence model is developed based on a modulation process, in which small scale irradiance fluctuation is modulated by large scale irradiance fluctuation. In the Gamma-Gamma PDF specified in (1), the parameter α represents the effective number of large-scale cells of the scattering process, and the parameter β represents the effective number of small-scale cells [10]. We also emphasize that parameters α and β cannot be arbitrarily chosen in FSO applications, they are related through a parameter called Rytov variance, which is a measure of optical turbulence strength. Under an assumption of plane wave and negligible inner scale, which corresponds to long propagation distance and small detector area, the shape parameters of the Gamma-Gamma model satisfy the following relationships [10]

$$\alpha = g(\sigma_R) = \left[\exp\left(\frac{0.49\sigma_R^2}{\left(1+1.11\sigma_R^{12/5}\right)^{7/6}} \right) - 1 \right]^{-1} \tag{2a}$$

$$\beta = h(\sigma_R) = \left[\exp\left(\frac{0.51\sigma_R^2}{\left(1+0.69\sigma_R^{12/5}\right)^{5/6}} \right) - 1 \right]^{-1} \tag{2b}$$

where σ_R^2 is the Rytov variance. Though the relationships described in (2a) and (2b) can change when spherical wave and a finite inner scale are taken into account [5], our estimation approach can be similarly applied to the other scenarios considered in [5].

It can be shown that $\alpha = g(\sigma_R)$ in (2a) is a convex function of σ_R on $(0, \infty)$, and $\beta = h(\sigma_R)$ in (2b) is a monotonically decreasing function on $(0, \infty)$. In addition, the relationship $\alpha > \beta$ always holds, and the smaller shape parameter β is lower bounded above 1 as σ_R approaches ∞. Figure 1 plots α and β as functions of σ_R.

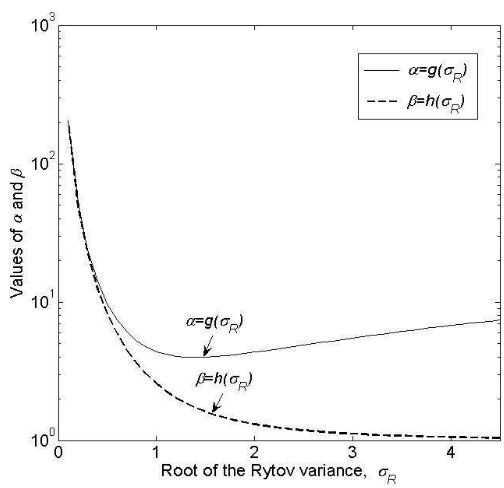

Fig. 1. Gamma-Gamma shape parameters α and β as functions of σ_R.

As a measure of optical turbulence strength, the Rytov variance can also be used to characterize different turbulence levels [11]: weak-turbulence regime refers to $\sigma_R^2 \le 0.3$; moderate-turbulence regime has $0.3 < \sigma_R^2 \le 5$; and strong-turbulence regime corresponds to $\sigma_R^2 > 5$. However, above definition for fluctuation regimes by the Rytov variance is not

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unique, and other classifications have also been used in literature. For example, in [12], Voelz and Xiao used Rytov variance values in [1, 10) to define the moderate turbulence regime for plane wave scenario. Gamma-Gamma PDFs for weak, moderate, and strong turbulence scenarios are plotted in Fig. 2, where the corresponding Rytov variance values are σ_R^2 = 0.25, 2, and 11, and the scale parameter λ is set to unity. We observe that when the strength of turbulence becomes strong, the Gamma-Gamma distribution approaches a negative exponential distribution.

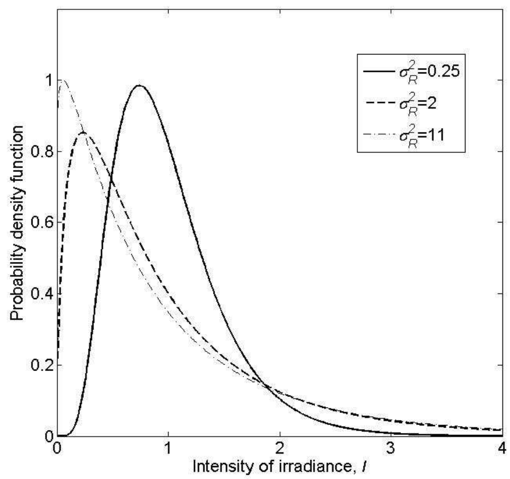

Fig. 2. Gamma-Gamma PDF's with Rytov variance σ_R^2 = 0.25, 2, and 11.

The closed-form expression for the *kth* order moment of the Gamma-Gamma distribution is given by [13]

$$\mu_k = E\left[I^k\right] = \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{\lambda}{\alpha\beta}\right)^k. \tag{3}$$

In this work we normalize the first moment by setting $\lambda = 1$. It can be shown that the moment expression in (3) is valid for $k > -1$, where k can take both integer and non-integer values.

3. Moment-based estimation for the Gamma-Gamma turbulence model

3.1 The MoM/CVX estimation approach

Taking the ratio of the $(k + 1)th$ and the *kth* order moments of the Gamma-Gamma distribution, we obtain

$$\frac{\mu_{k+1}}{\mu_k} = 1 + \frac{k}{\alpha} + \frac{k}{\beta} + \frac{k^2}{\alpha\beta}. \tag{4}$$

From (3), we also find that the second-order moment of the Gamma-Gamma distribution is

$$\mu_2 = 1 + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha\beta}. \tag{5}$$

Using (4) and (5), a nonlinear equation set involving variables α and β is formulated as

#126383 - $15.00 USD Received 1 Apr 2010; revised 15 May 2010; accepted 21 May 2010; published 1 Jun 2010

$$\begin{cases} \frac{1}{\alpha} + \frac{1}{\beta} = c \\ \frac{1}{\alpha} \cdot \frac{1}{\beta} = d \end{cases} \tag{6}$$

where

$$c = \frac{k^2 \mu_2 - \frac{\mu_{k+1}}{\mu_k} - (k^2 - 1)}{k^2 - k}, \tag{7a}$$

$$d = \frac{k \mu_2 - \frac{\mu_{k+1}}{\mu_k} - (k - 1)}{k - k^2}. \tag{7b}$$

After some algebraic manipulations to (6), α and β values can be found as the roots of the following quadratic equation

$$x^2 - \frac{c}{d} x + \frac{1}{d} = 0. \tag{8}$$

For FSO applications, since the shape parameter α is always greater than the shape parameter β, we designate the larger root of (8) to be α, and the smaller one to be β. A moment-based shape parameter estimator for the Gamma-Gamma turbulence model can thus be expressed as

$$\hat{\alpha} = \frac{\hat{c}}{2\hat{d}} + \frac{1}{2} \sqrt{\frac{\hat{c}^2}{\hat{d}^2} - \frac{4}{\hat{d}}}, \tag{9a}$$

$$\hat{\beta} = \frac{\hat{c}}{2\hat{d}} - \frac{1}{2} \sqrt{\frac{\hat{c}^2}{\hat{d}^2} - \frac{4}{\hat{d}}} \tag{9b}$$

where $\hat{c}$ and $\hat{d}$ are c and d values in (7) calculated using sample moments.

It is known that moment-based estimators with higher order moments may suffer from outlier samples. The outlier problem can be alleviated by choosing smaller k values. To achieve better performance, we propose to use fractional moments ($0< k <1$) instead of positive integer moments in our moment-based shape parameter estimators. The application of fractional moments in the study of atmospheric laser scintillation has been discussed by Consortini and Rigal [14]. It has been shown that using fractional moments of orders less than two can significantly reduce the fitting error of moments. Even with the presence of noise and background which cannot be removed directly from fractional moments, the fitting accuracy can be guaranteed as along as we have small enough width of the noise of the experimental setup.

Although the denominators of expressions in (7a) and (7b) become zero when $k = 0$, it can be shown that the equalities hold for $k = 0$ by applying the L'Hôpital's rule as

$$\lim_{k \to 0} c = \lim_{k \to 0} \frac{k^2 \mu_2 - \frac{\mu_{k+1}}{\mu_k} - (k^2 - 1)}{k^2 - k} = \lim_{k \to 0} \frac{2k\mu_2 - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2k}{\alpha\beta} \right) - 2k}{2k - 1} = \frac{1}{\alpha} + \frac{1}{\beta}, \tag{10a}$$

$$\lim_{k \to 0} d = \lim_{k \to 0} \frac{k \mu_2 - \frac{\mu_{k+1}}{\mu_k} - (k - 1)}{k - k^2} = \lim_{k \to 0} \frac{\mu_2 - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2k}{\alpha\beta} \right) - 1}{1 - 2k} = \frac{1}{\alpha} \cdot \frac{1}{\beta}. \tag{10b}$$

In order to obtain real-valued roots, Eq. (8) must have a positive discriminant

$$\Delta = \left(\frac{\hat{c}}{\hat{d}}\right)^2 - \frac{4}{\hat{d}} > 0. \quad (11)$$

However, one may experience negative-valued discriminant especially when the Rytov variance becomes small ($\sigma_R < 1$), which corresponds to weak turbulence scenarios. In that case, the moment-based estimator in (9) will not give meaningful real-valued estimates for α and β.

To address this problem, we observe that the left-hand side of (8) is a convex function. Let $f(x) = x^2 - \hat{c}x/\hat{d} + 1/\hat{d}$, a suboptimal solution to the estimation problem can be formulated as a convex optimization problem

$$\begin{aligned} &\underset{\alpha,\beta}{\text{minimize}} \quad [f(\alpha) - 0]^2 + [f(\beta) - 0]^2 \\ &\text{subject to} \quad \alpha > 0, \beta > 0, \alpha \geq \beta. \end{aligned} \quad (12)$$

The minimizer for the convex optimization problem described by (12) can be found as

$$\hat{\alpha} = \hat{\beta} = \frac{\hat{c}}{2\hat{d}}. \quad (13)$$

From Fig. 1, it can be seen that when $\sigma_R < 1$, α and β values are close to each other. Hence it is intuitively correct to have suboptimal estimates with $\hat{\alpha} = \hat{\beta}$.

By combining the fractional moment-based estimator (9) and the convex optimization estimator (13), we arrive at a robust estimation scheme for the shape parameters α and β. We name this scheme as the method-of-moments/convex-optimization (*MoM/CVX*) approach.

We use MSE as the metric for assessing the estimation performance. Monte Carlo simulations were carried out for the *MoM/CVX* estimator with $k = 0.5$ and σ_R value from 0.5 to 4.5, the data sample size was chosen to be $N = 100{,}000$.

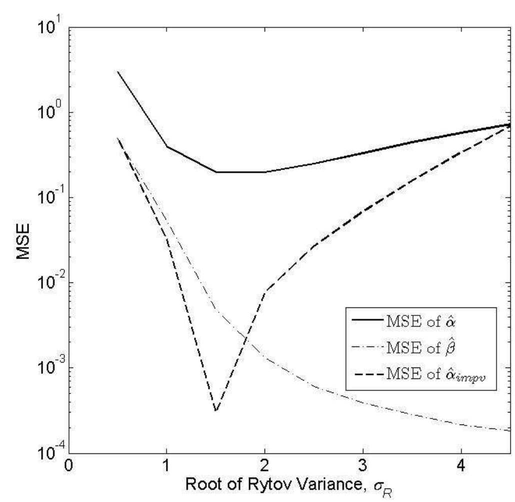

Fig. 3. MSE performance of the *MoM/CVX* estimator, the modified *MoM/CVX* estimator, as well as the estimator for the root of Rytov variance with $k = 0.5$ and sample size $N = 100{,}000$.

From the simulation results shown in Fig. 3, we observe that the *MoM/CVX* estimator for β can provide good estimates over a wide range of σ_R values. However, the estimation performance of the *MoM/CVX* estimator for α is poorer. For $\sigma_R = 0.5$, the MSE of $\hat{\alpha}$ can be as

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large as 2.97, which corresponds to an average relative error of 17.8%. Therefore, we are motivated to further improve the estimation performance for parameter α.

3.2 A modified estimation scheme for shape parameter α

An alternative method is to use $\hat{\beta}$ to estimate σ_R via

$$\hat{\sigma}_R = h^{-1}\left(\hat{\beta}\right) \tag{14}$$

where $h^{-1}(\cdot)$ denotes the inverse function of $h(\cdot)$ in (2b). Replacing σ_R in (2a) with its estimates in (14), a new estimator for α can be obtained as

$$\hat{\alpha}_{impv} = g\left(h^{-1}(\hat{\beta})\right). \tag{15}$$

The analytical expression of $h^{-1}(\cdot)$ is cumbersome; however, the built-in function **solve** in MATLAB can be used to find numerical results for $h^{-1}(\cdot)$.

We can observe in Fig. 3 that the MSE performance of the estimates of α is significantly improved by the modified method (dashed line). In our sample points, the largest improvement is achieved at $\sigma_R = 1.5$, the MSE is reduced by 99.85%.

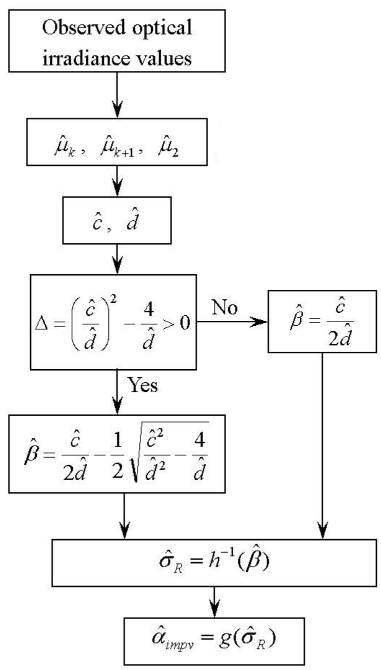

Fig. 4. Flow chart of the modified *MoM/CVX* Gamma-Gamma shape parameter estimator.

Figure 4 summarizes the estimation process of the modified *MoM/CVX* estimator. In the modified estimation scheme, we first calculate sample moments $\hat{\mu}_k$, $\hat{\mu}_{k+1}$ and $\hat{\mu}_2$ of the Gamma-Gamma turbulence model from the observed optical irradiance sample values. Parameters $\hat{c}$ and $\hat{d}$ in (7) can then be determined by using the sample moments. If the discriminant Δ of the quadratic equation in (8) is greater than zero, we use the quadratic solution in (9) to obtain the estimate of parameter β; otherwise, an estimate of β will be given

#126383 - $15.00 USD Received 1 Apr 2010; revised 15 May 2010; accepted 21 May 2010; published 1 Jun 2010

by the convex optimization solution (13). With an estimate of β, we can finally find an improved estimator $\hat{\alpha}_{impv}$ via (15).

As a side product, one can estimate the refractive-index structure parameter C_n^2 from the estimates of σ_R obtained in the modified estimation process by using the definition of the Rytov variance $\sigma_R^2 = 1.23 C_n^2 k_w^{7/6} L^{11/6}$ [10], provided the system has a horizontal propagation path and known link distance L and wavenumber k_w.

4. Conclusions

We have proposed a composite estimation scheme for the shape parameters of the Gamma-Gamma atmospheric turbulence model based on concepts of fractional moments and convex optimization. With the proposed method, estimates of the Gamma-Gamma shape parameters are directly obtained from signal intensity measurements. Our approach has advantage over the exiting method because the proposed shape parameter estimators do not require the knowledge of physical quantities such as the link distance and the refractive-index structure parameter. Our estimation technique can be used to characterize the atmospheric turbulence model over a wide range of turbulence conditions and can facilitate the performance analysis and system design of FSO systems.

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