Statistics of a Geometric Representation of Wavefront Distortion

D. L. FRIED
Electro-Optical Laboratory, North American Aviation, Inc., Space and Information Systems Division, Torrance, California

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A precise statistical definition is established for the geometric "shape" of a randomly distorted wavefront. Relationships between the phase-structure function and the statistics governing the shape are derived. The most significant portion of wavefront distortion caused by atmospheric turbulence is a random tilting of the plane-wave front. A procedure is outlined for calculating the influence of wavefront distortion on optical systems. Estimates are formed of the effect of wave-front distortion on photographic resolution and optical heterodyne efficiency.

I. INTRODUCTION

OPTICAL wave propagation through a randomly inhomogeneous medium produces wavefront deformation. The spatial (as distinguished from the temporal) statistics of this deformation are generally described by a quantity called the phase-structure function.[1] If, at points $\mathbf{x}$ and $\mathbf{x}'$,the phase variations associacted with the deformation are denoted by $\phi(\mathbf{x})$ and $\phi(\mathbf{x}')$, respectively, then the phase-structure function is defined as

$$\mathfrak{D}(r)=\langle[\phi(\mathbf{x})-\phi(\mathbf{x}')]^2\rangle, \tag{1.1}$$

where

$$r=|\mathbf{x}-\mathbf{x}'|, \tag{1.2}$$

and the brackets $\langle\ \rangle$ denote an ensemble average. (Isotropy and homogeneity of the deformation statistics are implicit in this definition.) Theoretical and experimental procedures for the evaluation of $\mathfrak{D}(r)$ have been developed.[2]

Inasmuch as $\mathfrak{D}(r)$ represents all the statistics of phase fluctuation,[3] it should be possible to obtain information concerning the "shape" of the deformed wavefront from $\mathfrak{D}(r)$. Knowledge of the shape would provide insight into the nature of the effects of wavefront deformation on an optical system. For example, if wave deformation can be approximated by a random tilting of a plane wavefront, then shorter exposures will clearly result in improved photographic resolution. The resolution can be computed directly from the phase-structure function,[4] but a quick and physically meaningful glimpse into the problem, providing quantitative results, can be obtained if we have data on the statistical "shape" of wavefront deformation. Similarly, the performance of an optical heterodyne receiver detecting a signal with a deformed wavefront can be estimated from wavefront "shape."[4]

The key to the whole concept is in formulating a precise statistical definition of the deformed wavefront's "shape." In this paper, this has been accomplished by examining the wavefront over the circular aperture of an unspecified optical system. The deformation in that region is represented by a series of orthonormal polynomials. These polynomials are chosen so that each term or group of terms in the series can be associated with a specific geometric shape. The shape of the wavefront can then be related to the mean square value of the coefficients of the polynomials in the series. The mean square error for a truncated series provides a measure of the shape of the deformed wavefront.

The polynomials used are the first six terms of an infinite sequence. Their sum can be cast in the form of a general quadratic; i.e.,

$$a+bx+cy+dx^2+exy+fy^2.$$

This can represent any wavefront distortion which can be built up from an average phase change, a tilt, a spherical deformation, and a hyperbolic deformation. The extension of the results obtained in this paper to include higher order types of distortion is a straightforward process.

II. NOTATION

Consider a plane perpendicular to the nominal direction of propagation of a distorted plane wave. Two position vectors, $\mathbf{x}$ and $\mathbf{x}'$ are defined in terms of a coordinate system in this plane. The vectors $\mathbf{x}$ and $\mathbf{x}'$ have components (x,y) and (x',y'), and magnitudes x and x'. As indicated previously, the phase at $\mathbf{x}$ and $\mathbf{x}'$ is denoted by $\phi(\mathbf{x})$ or $\phi(\mathbf{x}')$, for a particular realization of the wave-front distortion. Brackets $\langle\ \rangle$ are used to denote an ensemble average. The function $W(\mathbf{x},D)$ defines a circular region of diameter D, radius R. It equals unity

[1] V. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw-Hill Book Company, Inc., New York, 1961).

[2] See Ref. 1. R. E. Hufnagel and N. R. Stanley, J. Opt. Soc. Am. 54, 52 (1964). D. L. Fried and J. D. Cloud, J. Opt. Soc. Am. **54**, 574A (1964).

[3] It can be shown theoretically that phase fluctuation, because of the central limit theorem, has a Gaussian distribution. Since a Gaussian distribution is completely described (except for a mean value) by its second moment, and since $\mathfrak{D}(r)$ is the second moment for differential-phase fluctuation (which difference has zero mean), we conclude that $\mathfrak{D}(r)$ completely specifies the statistics of phase fluctuation.

[4] D. L. Fried, "The Effect of Wave-Front Distortion on the Performance of an Ideal Optical Heterodyne Receiver and an Ideal Camera," presented at the Conference on Atmospheric Limitations to Optical Propagation at the U. S. National Bureau of Standards CRPL, 18–19 March 1965.

inside this circle and zero outside; i.e.,

$$W(\mathrm{x},D)\equiv\begin{cases}1, & \text{if } |x|<R\\ 0, & \text{if } |x|>R.\end{cases} \quad (2.1)$$

The six orthonormal polynomials worked with in this paper are defined in Eqs. (2.2a–f). The first polynomial

$$F_1(\mathrm{x})\equiv(\pi R^2)^{-\frac{1}{2}} \quad (2.2a)$$

can be used to represent a change in the average phase over the circular region. The second pair of polynomials

$$F_2(\mathrm{x})\equiv(\pi R^4/4)^{-\frac{1}{2}}(x), \quad (2.2b)$$

$$F_3(\mathrm{x})\equiv(\pi R^4/4)^{-\frac{1}{2}}(y), \quad (2.2c)$$

can be used to represent an average tilt. The next

$$F_4(\mathrm{x})\equiv(\pi R^6/12)^{-\frac{1}{2}}(x^2+y^2-R^2/2) \quad (2.2d)$$

can be used to represent a spherical deformation. Finally,

$$F_5(\mathrm{x})\equiv(\pi R^6/6)^{-\frac{1}{2}}(x^2-y^2), \quad (2.2e)$$

$$F_6(\mathrm{x})\equiv(\pi R^6/24)^{-\frac{1}{2}}(xy), \quad (2.2f)$$

can be used to represent a hyperbolic deformation.

These polynomials are closely related to Zernike's polynomials. It is easy to generate higher-order terms; i.e., $F_7(\mathrm{x})$ and up (from Table XXI of Ref. 5). It is easy to verify the fact that these polynomials satisfy the orthonormality condition

$$\int d\mathrm{x}W(\mathrm{x},D)F_\mu(\mathrm{x})F_\nu(\mathrm{x})=\delta_{\mu\nu}, \quad (2.3)$$

where $\delta_{\mu\nu}$ is the Kronecker delta

$$\delta_{\mu\nu}\equiv\begin{cases}1, & \text{if } \mu=\nu\\ 0, & \text{if } \mu\neq\nu.\end{cases} \quad (2.4)$$

The integration in Eq. (2.3) and all other integrations throughout this paper are understood to cover the infinite plane, unless explicitly indicated otherwise. Finite bounds for the integraion are provided, in effect, by the aperture function.

III. WAVEFRONT APPROXIMATION

The distorted wavefront's phase $\phi(\mathrm{x})$ is approximated by the infinite series

$$\Phi(\mathrm{x})=\sum_{\mu=1}^{\infty}a_\mu F_\mu(\mathrm{x}), \quad (3.1)$$

where the coefficients a_μ are chosen to optimize the approximation. The error in the approximation is Δ, the aperture-averaged-square difference between $\Phi(\mathrm{x})$ and $\phi(\mathrm{x})$; i.e.,

$$\Delta\equiv(1/\pi R^2)\int d\mathrm{x}W(\mathrm{x},D)[\phi(\mathrm{x})-\Phi(\mathrm{x})]^2. \quad (3.2)$$

To minimize Δ, a_μ is chosen so that

$$(\partial/\partial a_\mu)\Delta=0. \quad (3.3)$$

It is shown in Sec. IV that extrapolation to two-dimensional form from the usual one-dimensional result for the coefficients of an orthonormal series is applicable; i.e., that

$$a_\mu=\int d\mathrm{x}W(\mathrm{x},D)\phi(\mathrm{x})F_\mu(\mathrm{x}). \quad (4.3)$$

New coefficients a_C, a_L, a_S, and a_Q, are defined by

$$(a_C)^2\equiv(a_1)^2, \quad (3.4a)$$

$$(a_L)^2\equiv(a_2)^2+(a_3)^2, \quad (3.4b)$$

$$(a_S)^2\equiv(a_4)^2, \quad (3.4c)$$

$$(a_Q)^2\equiv(a_4)^2+(a_5)^2+(a_6)^2. \quad (3.4d)$$

Examining the polynomials $F_1(\mathrm{x})$ to $F_6(\mathrm{x})$ and visualizing the surface each represent, we see that the quantities $\langle(a_C)^2\rangle$, $\langle(a_L)^2\rangle$, $\langle(a_S)^2\rangle$, and $\langle(a_Q)^2\rangle$ can be considered measures of the mean square amount of average phase fluctuation over the circular region of interest, of the average tilt of the wavefront, of the extent of spherical deformation of the wavefront, and of the amount of quadratic (i.e., spherical plus astigmatic) deformation, respectively.

In addition to evaluating these four quantities, it would be desirable to know how good an approximation to the deformed wavefront is provided by a finite number of terms in the series $\Phi(\mathrm{x})$ of Eq. (3.1). For this purpose, define the finite series $\Phi_C(\mathrm{x})$, $\Phi_L(\mathrm{x})$, $\Phi_S(\mathrm{x})$ and $\Phi_Q(\mathrm{x})$, as

$$\Phi_j(\mathrm{x})=\sum_{\mu=1}^{n_j}a_\mu F_\mu(\mathrm{x}), \quad (3.5)$$

where j takes the values C, L, S, and Q and n_j has the values 1, 3, 4, and 6, respectively. The geometric significance of the four Φ_j's defined above is obvious. Now, let us define Δ_j as

$$\Delta_j\equiv(1/\pi R^2)\int d\mathrm{x}W(\mathrm{x},D)[\phi(\mathrm{x})-\Phi_j(\mathrm{x})]^2. \quad (3.6)$$

If the ensemble average $\langle\Delta_j\rangle$ can be computed, we obtain a measure of the goodness of various possible finite series approximations to the distorted wave front.

IV. DERIVATION OF UNAVERAGED RELATIONSHIPS

Substituting for $\Phi_j(\mathrm{x})$ in Eq. (3.3) by using Eq. (3.1), expanding the square, and interchanging the order of

[5] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1959).

summation and integration, we find

$$\Delta=\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)\phi^2(\mathbf{x})-\frac{2}{\pi R^2}\sum_{\mu=1}^{\infty}a_\mu\int d\mathbf{x}W(\mathbf{x},D)\phi(\mathbf{x})F_\mu(\mathbf{x})+\sum_{\mu,\nu=1}^{\infty}a_\mu a_\nu\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)F_\mu(\mathbf{x})F_\nu(\mathbf{x}). \quad (4.1)$$

Invoking the orthonormality condition, Eq. (2.3), we can perform the last integration and then one of the two summations in the double sum, we obtain

$$\Delta=\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)\phi^2(\mathbf{x})-\frac{2}{\pi R^2}\sum_{\mu=1}^{\infty}a_\mu\int d\mathbf{x}W(\mathbf{x},D)\phi(\mathbf{x})F_\mu(\mathbf{x})+\frac{1}{\pi R^2}\sum_{\mu=1}^{\infty}(a_\mu)^2. \quad (4.2)$$

Substituting Eq. (4.2) into Eq. (3.2), differentiating, and solving for a_μ, we get

$$a_\mu=\int d\mathbf{x}W(\mathbf{x},D)\phi(\mathbf{x})F_\mu(\mathbf{x}), \quad (4.3)$$

as indicated previously. Substituting Eq. (4.3) into (4.2) to eliminate the second integration, we get

$$\Delta=\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)\phi^2(\mathbf{x})-\frac{1}{\pi R^2}\sum_{\mu=1}^{\infty}(a_\mu)^2. \quad (4.4)$$

If we substitute Eq. (4.3) into (4.2) to eliminate a_μ, make double integrals out of the product of two integrals, and interchange the order of integration and summation, we obtain

$$\Delta=\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)\phi^2(\mathbf{x})-\frac{1}{\pi R^2}\int\int d\mathbf{x}d\mathbf{x}'W(x,D)\times W(\mathbf{x}',D)\sum_{\mu=1}^{\infty}F_\mu(\mathbf{x})F_\mu(\mathbf{x}')\phi(\mathbf{x})\phi(\mathbf{x}'). \quad (4.5)$$

Working with the finite series approximation, we can easily show the results corresponding to Eq. (4.4) and (4.5) to be

$$\Delta_j=\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)\phi^2(\mathbf{x})-\frac{1}{\pi R^2}\sum_{\mu=1}^{n_j}(a_\mu)^2, \quad (4.4')$$

$$\Delta_j=\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)\phi^2(\mathbf{x})-\frac{1}{\pi R^2}\int\int d\mathbf{x}d\mathbf{x}'W(x,D)\times W(\mathbf{x}',D)\sum_{i=1}^{n_j}F_\mu(\mathbf{x})F_\mu(\mathbf{x}')\phi(\mathbf{x})\phi(\mathbf{x}'). \quad (4.5')$$

From Eqs. (3.4b, c, and d) and Eq. (4.4′) we can see that

$$(a_{\mathrm{L}})^2=\Delta_{\mathrm{C}}-\Delta_{\mathrm{L}}, \quad (4.6a)$$

$$(a_{\mathrm{S}})^2=\Delta_{\mathrm{L}}-\Delta_{\mathrm{S}}, \quad (4.6b)$$

$$(a_{\mathrm{Q}})^2=\Delta_{\mathrm{L}}-\Delta_{\mathrm{Q}}. \quad (4.6c)$$

The problem at this point is reduced to computation of the ensemble average of Δ_j as given by Eq. (4.5′) and relating the result to the phase-structure function.

V. DERIVATION OF AVERAGED RELATIONSHIPS

Examining Eqs. (2.2a–f), we note that $F_1(\mathbf{x})$ is independent of $\mathbf{x}$ so that $\int d\mathbf{x}'W(\mathbf{x}',D)F_1(\mathbf{x})F_1(\mathbf{x}')$ equals unity. Also for μ not equal to one, $\int d\mathbf{x}W(\mathbf{x}',D)F_\mu(\mathbf{x})\times F_\mu(\mathbf{x}')$ vanishes. Consequently,

$$\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)\phi^2(\mathbf{x})=\frac{1}{\pi R^2}\int\int d\mathbf{x}d\mathbf{x}'W(\mathbf{x},D)\times W(\mathbf{x}',D)\sum_{\mu=1}^{n_j}F_\mu(\mathbf{x})F_\mu(\mathbf{x}')\phi^2(\mathbf{x}). \quad (5.1)$$

Symmetrizing between $\mathbf{x}$ and $\mathbf{x}'$ in the right-hand side of Eq. (5.1) gives

$$\frac{1}{\pi R^2}\int d\mathbf{x}W(\mathbf{x},D)\phi^2(\mathbf{x})=\frac{1}{2\pi R^2}\int\int d\mathbf{x}d\mathbf{x}'W(\mathbf{x},D)\times W(\mathbf{x}',D)\sum_{\mu=1}^{n_j}F_\mu(\mathbf{x})F_\mu(\mathbf{x}')[\phi^2(\mathbf{x})+\phi^2(\mathbf{x}')]. \quad (5.2)$$

Substituting Eq. (5.2) into (4.5′), taking ensemble averages of both sides, converting the variables of integration from $\mathbf{x}$, $\mathbf{x}'$ to $\mathbf{r}$, $\mathbf{r}'$, where $\mathbf{r}$, $\mathbf{r}'$ are defined as

$$\mathbf{r}'=\tfrac{1}{2}(\mathbf{x}+\mathbf{x}'), \quad (5.3a)$$

$$\mathbf{r}=\mathbf{x}-\mathbf{x}', \quad (5.3b)$$

and recognizing the presence of the phase-structure function in the integrand, we find

$$\langle\Delta_j\rangle=\frac{1}{2\pi R^2}\int\int d\mathbf{r}d\mathbf{r}'W(|\mathbf{r}'+\tfrac{1}{2}\mathbf{r}|,D)W(|\mathbf{r}'-\tfrac{1}{2}\mathbf{r}|,D)\times\sum_{\mu=1}^{n_j}F_\mu(\mathbf{r}'+\tfrac{1}{2}\mathbf{r})F_\mu(\mathbf{r}'-\tfrac{1}{2}\mathbf{r})\mathfrak{D}(r). \quad (5.4)$$

The $\mathbf{r}'$ integration in Eq. (5.4) can be performed explicitly. For this purpose, define the function $\mathfrak{F}_j(r,D)$ as

$$\mathfrak{F}_j(r,D)=\int d\mathbf{r}'W(|\mathbf{r}'+\tfrac{1}{2}\mathbf{r}|,D)W(|\mathbf{r}'-\tfrac{1}{2}\mathbf{r}|,D)\times\sum_{\mu=1}^{n_j}F_\mu(\mathbf{r}'+\tfrac{1}{2}\mathbf{r})F_\mu(\mathbf{r}'-\tfrac{1}{2}\mathbf{r}). \quad (5.5)$$

This function is evaluated in Appendix A for $j=C$, L,

S, and Q. The results are

$$\mathfrak{F}_C(r,D)=(1/\pi)\{2\cos^{-1}(r/D)-2(r/D)[1-(r/D)^2]^{\frac{1}{2}}\}W(r,2D),\quad (5.6a)$$

$$\mathfrak{F}_L(r,D)=(1/\pi)\{6\cos^{-1}(r/D)-[14(r/D)-8(r/D)^3]\times[1-(r/D)^2]^{\frac{1}{2}}\}W(r,2D),\quad (5.6b)$$

$$\mathfrak{F}_S(r,D)=(1/\pi)\{8\cos^{-1}(r/D)-[24(r/D)-(80/3)(r/D)^3+(32/3)(r/D)^5]\times[1-(r/D)^2]^{\frac{1}{2}}\}W(r,2D),\quad (5.6c)$$

$$\mathfrak{F}_Q(r,D)=(1/\pi)\{12\cos^{-1}(r/D)-[44(r/D)-64(r/D)^3+32(r/D)^5]\times[1-(r/D)^2]^{\frac{1}{2}}\}W(r,2D).\quad (5.6d)$$

The evaluation of $\mathfrak{F}_j(r,D)$ given in Eqs. (5.6a–d), obtained in the Appendix through a rather extensive and quite uninteresting calculation, is fundamental to the program of this paper. With this in hand, the rest of the work is straightforward. Substituting from Eq. (5.5) into Eq. (5.4) and noting that the integrand is isotropic in $\mathbf{r}$, so that the angular integration can be performed, we find

$$\langle\Delta_j\rangle=\frac{1}{R^2}\int_0^D rdr\mathfrak{F}_j(r,D)\mathfrak{D}(r),\quad (5.7)$$

$$\langle(a_L)^2\rangle=\frac{1}{R^2}\int_0^D rdr[\mathfrak{F}_C(r,D)-\mathfrak{F}_L(r,D)]\mathfrak{D}(r),\quad (5.8a)$$

$$\langle(a_S)^2\rangle=\frac{1}{R^2}\int_0^D rdr[\mathfrak{F}_L(r,D)-\mathfrak{F}_S(r,D)]\mathfrak{D}(r),\quad (5.8b)$$

$$\langle(a_Q)^2\rangle=\frac{1}{R^2}\int_0^D rdr[\mathfrak{F}_L(r,D)-\mathfrak{F}_Q(r,D)]\mathfrak{D}(r).\quad (5.8c)$$

VI. THE PHASE-STRUCTURE FUNCTION

Several theoretical studies of the phase-structure function have been performed utilizing various approaches to the propagation problem.[2] Based on the Kolomogoroff similarity theory of turbulence,[6] which predicts a spatial correlation of turbulence which decreases proportionally to the two-thirds power of the spatial separation, it is possible to show that the phase structure function may be written as

$$\mathfrak{D}(r)=\mathcal{A}r^{5/3},\quad (6.1)$$

where $\mathcal{A}$ is a constant determined by the path of propagation, the wavelength, and the particular environmental conditions. Equation (6.1) is exact only in the near field; i.e., for short propagation paths. For longer distances of propagation, there is an additional r-dependent factor which varies from one half to one, which is suppressed in (6.1). For the body of this paper, we ignore this factor. (A discussion of this matter is given in Appendix D.)

In this paper, particular values of $\mathcal{A}$ are not of concern. (The necessary data for computing $\mathcal{A}$ are provided in Appendix C.) In fact, it is convenient to replace $\mathcal{A}$ with a new quantity r_0 which has the dimensions of length and is defined by

$$r_0\simeq(6.88/\mathcal{A})^{3/5}.\quad (6.2)$$

Correspondingly, the phase structure function, expressed in terms of r_0, is

$$\mathfrak{D}(r)=6.88(r/r_0)^{5/3}.\quad (6.3)$$

The apparently arbitrary constant 6.88 was chosen on the basis of the analysis of the performance of an optical heterodyne detection system.[4] In Ref. 4, r_0 is shown to be that diameter of a heterodyne collector for which distortion effects begin to seriously limit performance. It is seen later that it is also that diameter for which $\langle\Delta_C\rangle$ is essentially unity. Typical values of r_0 for visible and near infrared wavelengths and for approximately vertical propagation paths down through the atmosphere are of the order of several centimeters.

VII. EVALUATIONS

Substituting Eq. (6.6) into (5.7), replacing the variable r by u, where

$$u=r/D,\quad (7.1)$$

and noting that

$$\mathfrak{F}_j(r,D)=\mathfrak{F}_j(u,1),\quad (7.2)$$

we get

$$\langle\Delta_j\rangle=27.5(D/r_0)^{5/3}\int_0^1 udu\mathfrak{F}_j(u,1)u^{5/3}.\quad (7.3)$$

Utilizing the known relations[7]

$$\int_0^1 z^{2\alpha+1}(1-z^2)^\beta dz=\tfrac{1}{2}B(\alpha+1,\beta+1),\quad (7.4a)$$

$$\int_0^1 z^\alpha\cos^{-1}(z)dz=\frac{1}{2(\alpha+1)}B\left(\frac{\alpha+2}{2},\frac{1}{2}\right),\quad (7.4b)$$

where $B(\alpha,\beta)$ is the well-known beta function, and defining I_j as

$$I_j=\int_0^1 u^{8/3}\mathfrak{F}_j(u,1)du,\quad (7.5)$$

we find that

$$I_C\simeq3.68\times10^{-2},\quad (7.6a)$$

$$I_L\simeq4.73\times10^{-3},\quad (7.6b)$$

$$I_S\simeq3.96\times10^{-3},\quad (7.6c)$$

$$I_Q\simeq2.29\times10^{-3}.\quad (7.6d)$$

[6] A. Kolomogoroff in *Turbulence, Classic Papers on Statistical Theory*, edited by S. K. Friedlander and L. Topper (Interscience Publishers, Inc., New York, 1961), p. 151.

[7] W. Grobner and N. Hofreiter, *Integraltafel* (Springer-Verlag, Berlin/Vienna, 1961), Vol II, Eqs. (121.1) and (341.5a).

Thus,

$$\langle\Delta_C\rangle \simeq 1.013(D/r_0)^{5/3}, \tag{7.7a}$$

$$\langle\Delta_L\rangle \simeq 0.1301(D/r_0)^{5/3}, \tag{7.7b}$$

$$\langle\Delta_S\rangle \simeq 0.1090(D/r_0)^{5/3}, \tag{7.7c}$$

$$\langle\Delta_Q\rangle \simeq 0.0630(D/r_0)^{5/3}. \tag{7.7d}$$

From these results we can see that

$$\langle(a_L)^2\rangle \simeq 0.883(D/r_0)^{5/3}, \tag{7.8a}$$

$$\langle(a_S)^2\rangle \simeq 0.0211(D/r_0)^{5/3}, \tag{7.8b}$$

$$\langle(a_Q)^2\rangle \simeq 0.0671(D/r_0)^{5/3}. \tag{7.8c}$$

It is principally through $\langle(a_j)^2\rangle$ and $\langle\Delta_j\rangle$ that a precise definition of the deformed wave front's "shape" is developed. A quantitative physical insight into the nature of the deformation can be obtained, however, by computing the quantity D_j^*. The quantity D_j^* is the diameter of the circular region for which $\langle\Delta_j\rangle$ takes on some critical value Δ^*. Thus, it is the aperture diameter over which the average deformation may be considered to consist of no significant amount of deformations of higher order than j. What constitutes a significant amount is defined by Δ^*. The aperture diameter D_j^* is defined by Eq. (7.9).

$$\Delta^* = \left\langle \frac{4}{\pi(D_j^*)^2} \int d\mathbf{x} W(x,D_j^*)[\phi(\mathbf{x})-\Phi_j(\mathbf{x})]^2 \right\rangle. \tag{7.9}$$

The corresponding aperture area A_j^* is simply

$$A_j^* \equiv (\pi/4)(D_j^*)^2. \tag{7.10}$$

The quantity A_j^*/n_j is of interest in that it provides a measure of the "average utility" of each degree of freedom represented by one of the n_j adjustable coefficients in Φ_j. Utilizing Eqs. (7.7a–d), we can evaluate D_j^* and A_j^*/n_j.

$$D_C^* \simeq 0.992 r_0(\Delta^*)^{3/5}, \tag{7.11a}$$

$$D_L^* \simeq 3.40 r_0(\Delta^*)^{3/5}, \tag{7.11b}$$

$$D_S^* \simeq 3.79 r_0(\Delta^*)^{3/5}, \tag{7.11c}$$

$$D_Q^* \simeq 5.26 r_0(\Delta^*)^{3/5}. \tag{7.11d}$$

$$A_C^*/n_C \simeq 0.77 r_0^2(\Delta^*)^{6/5}, \tag{7.12a}$$

$$A_L^*/n_L \simeq 3.03 r_0^2(\Delta^*)^{6/5}, \tag{7.12b}$$

$$A_S^*/n_S \simeq 2.82 r_0^2(\Delta^*)^{6/5}, \tag{7.12c}$$

$$A_Q^*/n_Q \simeq 3.63 r_0^2(\Delta^*)^{6/5}. \tag{7.12d}$$

These are all the quantities that were to be computed. In the next section, their physical significance is discussed.

VIII. APPLICATION AND DISCUSSION OF RESULTS

To get a quick picture of the shape of the deformed wave front, we consider the coefficients in Eqs. (7.8a–c). Since the coefficient of the linear term is so much larger than that of the spherical or other quadratic terms, we can conclude that a large part of the deformation consists of wavefront tilting. We might have expected this result from the fact that, as proven in Appendix B,

$$\frac{1}{R^2}\int_0^D r dr \mathfrak{F}_j(r,D) r^{6/3} = \begin{cases} R^2/2 & \text{if } j=C \\ 0 & \text{if } j\neq C \end{cases} \tag{8.1}$$

Considering this in conjunction with Eqs. (5.7) and (5.8a–c), we see that a phase structure function which has a six-thirds dependence on r [instead of the five-thirds power of Eq. (6.1)], corresponds exactly to a randomly tilted, but otherwise undistorted, plane wave.

Noting that the spherical coefficient $\langle(a_S)^2\rangle$ corresponds to a single degree of freedom in the series approximation, that the quadratic coefficient $\langle(a_Q)^2\rangle$ corresponds to three degrees of freedom, and noting further that $\langle(a_S)^2\rangle/\langle(a_Q)^2\rangle$ as determined from Eqs. (7.8b c) is almost exactly one-third, we can conclude that spherical deformation of the wavefront is no more and no less significant than the other two quadratic forms of deformation.

Considering Eqs. (7.12a–d), we see that the area per degree of freedom over which a series approximation can provide a given quality match (as specified by Δ^*) to the wavefront jumps abruptly between the one-term $j=C$ approximation, and the three-term $j=L$ approximation. If this jump had not occurred, the conclusion would have to be drawn that the wavefront is distorted in a way which is not subject to a geometric interpretation. In fact, quite the contrary is the case.

To get some insight into the photographic resolution to be expected when the wavefront being collected is atmospherically deformed, note that the Strehl definition of a diffraction-limited optical system collecting a deformed wave front is set by the mean square phase variation over the aperture.[8] For a mean square phase variation $\Delta^*=1$ rad^2, the Strehl definition is about 30%, i.e., $\exp[-(\Delta^*)^2]$. For long exposure photography, the applicable mean square phase deviation is $\langle\Delta_C\rangle$ and should be computed from Eq. (7.11a). A 30% Strehl definition is thus achieved with a lens diameter about equal to r_0 and an angular resolution of the order of λ/r_0. It can be shown[4] that increasing the lens diameter beyond r_0 increases the phase deviation so rapidly that the achievable angular resolution is not improved beyond λ/r_0. For a very short exposure, wavefront tilt, as distinguished from higher-order-type distortion, does not reduce the resolution of the system. Wavefront tilt displaces the image of a point but does not blur the image. Uncorrelated displacement of different points in an image does, however, distort the picture. Consequently, for a very short-exposure photograph, the

[8] E. L. O'Neill, *Introduction to Statistical Optics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1963), pp. 87, 106.

mean square phase deviation used to compute the Strehl definition should be taken as $\langle\Delta_L\rangle$, not $\langle\Delta_C\rangle$. From Eq. (7.11b) it is seen that 30% Strehl definition for very short exposures is achieved when the lens diameter is about $3.40r_0$. The corresponding angular resolution is of the order of $\lambda/3.4r_0$, which is 3.4 times as much as the long-exposure resolution. (In a more precise calculation considering the effect of wavefront distortion on the integrated modulation-transfer function, we have found that the resolution for high-speed exposure is increased by a factor of 2, not 3.4, over the best that can be achieved with a long exposure. The peak resolution is achieved at a lens diameter of approximately $3.8r_0$.)

For an optical heterodyne receiver, as indicated in Sec. VI, the performance starts to deteriorate significantly when the collector diameter becomes larger than r_0. This can be understood from Eq. (7.7a) which indicates that the rms phase deviation for this size aperture is about one radian and from consideration of the fact that if the rms difference is this large then there are many pairs of points in the aperture where the phase difference is π. The signal generated by these pairs of points is very small, contributing very little to the total performance of the detector.

For a heterodyne receiver which can track at a sufficiently high rate any tilting of the signal wavefront, the mean square phase deviation as far as the system performance is concerned, should be computed from Eq. (7.7b) for $\langle\Delta_L\rangle$ rather than from $\langle\Delta_C\rangle$. The rms phase deviation, according to Eq. (7.11b) does not become one radian until the collector diameter is $3.40r_0$. Thus a tracking system could utilize a collector diameter 3.4 times as large as a nontracking system. This would yield a shot-noise-limited signal-to-noise-ratio improvement over the nontracking system of 20 log3.4$\simeq$10 dB. (A more precise calculation would probably yield a somewhat smaller improvement.)

By use of a rapidly adjustable variable focal-length lens system, spherical deformation of the collected wavefront as well as tilt could be tracked out. However, comparison of Eqs. (7.11b, c) shows that the resulting gain over a system which only tracked tilt would be trivial.

Similar semiquantitative analyses of the effect of wavefront deformation on other types of optical systems can be computed in the same manner as the cases treated in this section.

In the discussion of the imaging and the heterodyne detection systems, it has been assumed that intensity variations across the aperture were not present, or could be ignored. This consideration is discussed in Appendix D.

IX. CONCLUSIONS

It has been shown that a precise definition of the shape of a deformed wavefront can be generated and that the statistics of the deformation shape can be computed from the phase-structure function. Formulas have been provided for these computations for the first four types of deformation. It has been shown that according to current theory of optical propagation in the atmosphere a large part of the wavefront deformation can be understood as wavefront tilting. Sample techniques for interpreting the deformation results in terms of performance of certain types of optical systems have been provided, the implication being that quantitative estimate can be generated of the performance of any type of optical system whose behavior is limited by wavefront distortion.

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APPENDIX A. EVALUATION OF $\mathfrak{F}_j(r)$

In this appendix we outline the procedure used in evaluating the function $\mathfrak{F}_j(r,D)$ for $j=C$, L, S and Q. As intermediate steps in the process, we have to evaluate the functions $K_i(r,D)$ for $i=0$, 1, 2, and 3; $J(m,n;r,D)$ for $(m,n)=(0,0)$, (2,0), (0,2), (4,0), (2,2), and (0,4); and $L(m,n;r,D)$ for $(m,n)=(4,0)$, (3,0), (2,0), (1,0), (0,0) (2,2), (1,2), (0,2), and (0,4). These functions are defined as

$$K_0(r,D)=\int d\mathbf{r}'W(|\mathbf{r}'+\tfrac{1}{2}\mathbf{r}|,D)W(|\mathbf{r}'-\tfrac{1}{2}\mathbf{r}|,D),\quad \text{(A1a)}$$

$$K_0(r,D)=\int d\mathbf{r}'W(|\mathbf{r}'+\tfrac{1}{2}\mathbf{r}|,D)W(|\mathbf{r}'-\tfrac{1}{2}\mathbf{r}|,D)r'^2,\quad \text{(A1b)}$$

$$K_2(r,D)=\int d\mathbf{r}'W(|\mathbf{r}'+\tfrac{1}{2}\mathbf{r}|,D)W(|\mathbf{r}'-\tfrac{1}{2}\mathbf{r}|,D)r'^4,\quad \text{(A1c)}$$

$$K_3(r,D)=\int d\mathbf{r}'W(|\mathbf{r}+\tfrac{1}{2}\mathbf{r}|,D)\times W(|\mathbf{r}'-\tfrac{1}{2}\mathbf{r}|,D)(\mathbf{r}\cdot\mathbf{r}')^2,\quad \text{(A1d)}$$

$$J(m,n;r,D)=2\int_0^{R-\frac{1}{2}r}dp\int_{-[R^2-(R+\frac{1}{2}r)^2]^{\frac{1}{2}}}^{+[R^2-(R+\frac{1}{2}r)^2]^{\frac{1}{2}}}dq\,p^mq^n,\quad \text{(A2)}$$

$$L(m,n;r,D)=\int_{r/D}^{1}dv\,v^m(1-v^2)^{(n+1)/2}.\quad \text{(A3)}$$

Starting from Eqs. (5.3a, b), we can show that

$$\sum_{i=1}^{n_C}F_i(\mathbf{x})F_i(\mathbf{x}')=\frac{1}{\pi R^2},\quad \text{(A4a)}$$

$$\sum_{i=1}^{n_L}F_i(\mathbf{x})F_i(\mathbf{x}')=\frac{1}{\pi R^4}(4r'^2-r^2+R^2),\quad \text{(A4b)}$$

$$\sum_{i=1}^{n_S} F_i(\mathbf{x})F_i(\mathbf{x}')=\frac{1}{\pi R^6}[12r'^4+(6r^2-8R^2)r'^2+(\tfrac{3}{4}r^4-4R^2r^2+4R^4)-12(\mathbf{r}\cdot\mathbf{r}')^2], \quad \text{(A4c)}$$

$$\sum_{i=1}^{n_Q} F_i(\mathbf{x})F_i(\mathbf{x}')=\frac{1}{\pi R^6}\{18r'^4+(-3r^2-8R^2)r'^2+[(9/8)r^4-4R^2r^2+4R^4]-6(\mathbf{r}\cdot\mathbf{r}')^2\}. \quad \text{(A4d)}$$

From this we see that

$$\mathfrak{F}_C(r,D)=(1/\pi R^2)K_0(r,D), \quad \text{(A5a)}$$

$$\mathfrak{F}_L(r,D)=(1/\pi R^4)[4K_1(r,D)+(R^2-r^2)K_0(r,D)], \quad \text{(A5b)}$$

$$\mathfrak{F}_S(r,D)=(1/\pi R^6)[12K_2(r,D)+(6r^2-8R^2)K_1(r,D)+(\tfrac{3}{4}r^4-4R^2r^2+4R^4)K_0(r,D)-12K_3(r,D)], \quad \text{(A5c)}$$

$$\mathfrak{F}_Q(r,D)=(1/\pi R^6)\{18K_2(r,D)+(-3r^2-8R^2)K_1(r,D)+[(9/8)r^4-4R^2r^2+4R^4]K_0(r,D)-6K_3(r,D)\}. \quad \text{(A5d)}$$

We define a rectangular coordinate system with axes parallel and perpendicular to the vector $\mathbf{r}$. Let p and q be the components of $\mathbf{r}'$ in this system, with p denoting the coefficient parallel to $\mathbf{r}$. Considering the range of p and q in which $W(|\mathbf{r}'+\frac{1}{2}\mathbf{r}|,D)\ W(|\mathbf{r}'-\frac{1}{2}\mathbf{r}|,D)$ is non-zero, i.e., the region of overlap of two circles of diameter D whose centers are at $(p=\frac{1}{2}r,\ q=0)$ and $(p=-\frac{1}{2}r,\ q=0)$ we see that

$$K_0(r,D)=J(0,0;r,D)W(r,2D), \quad \text{(A6a)}$$

$$K_1(r,D)=[J(2,0;r,D)+J(0,2;r,D)]W(r,2D), \quad \text{(A6b)}$$

$$K_2(r,D)=[J(4,0;r,D)+2J(2,2;r,D)+J(0,4;r,D)]W(r,2D), \quad \text{(A6c)}$$

$$K_3(r,D)=r^2J(2,0;r,D)W(r,2D), \quad \text{(A6d)}$$

where we have made use of the fact that

$$r'^2=p^2+q^2, \quad \text{(A7a)}$$

$$r'^4=p^4+2p^2q^2+q^4, \quad \text{(A7b)}$$

$$(\mathbf{r}\cdot\mathbf{r}')^2=r^2p^2. \quad \text{(A7c)}$$

Examining Eq. (A2), performing the q integration and making the substitution

$$v=(2p+r)/D, \quad \text{(A8)}$$

for p, we get

$$J(m,n;r,D)=\frac{4}{n+1}R^{m+n+2}\int_{r/D}^{1}dv\left(v-\frac{r}{D}\right)^m(1-v^2)^{(n+1)/2}. \quad \text{(A9)}$$

Expanding $[v-(r/D)]^m$ and considering Eq. (A3) we see that

$$J(0,0;r,D)=4R^2[L(0,0;r,D)], \quad \text{(A10a)}$$

$$J(2,0;r,D)=4R^4[L(2,0;r,D)-2(r/D)L(1,0;r,D)+(r/D)^2L(0,0;r,D)], \quad \text{(A10b)}$$

$$J(0,2;r,D)=\tfrac{4}{3}R^4[L(0,2;r,D)], \quad \text{(A10c)}$$

$$J(4,0;r,D)=4R^6[L(4,0;r,D)-4(r/D)L(3,0;r,D)+6(r/D)^2L(2,0;r,D)-4(r/D)^3L(1,0;r,D)+(r/D)^4L(0,0;r,D)], \quad \text{(A10d)}$$

$$J(2,2;r,D)=\tfrac{4}{3}R^6[(2,2;r,D)-2(r/D)L(1,2;r,D)+(r/D)^2L(0,2;r,D)], \quad \text{(A10e)}$$

$$J(0,4;r,D)=\tfrac{4}{5}R^6[L(0,4;r,D)]. \quad \text{(A10f)}$$

The $L(m,n;r,D)$ functions can be evaluated using integration formulas from Dwight's "Table of Integrals". The results are

$$L(4,0;r,D)=\tfrac{1}{16}\cos^{-1}(r/D)+[1-(r/D)^2]^{\frac{1}{2}}[-\tfrac{1}{6}(r/D)^5+(1/24)(r/D)^3+\tfrac{1}{16}(r/D)], \quad \text{(A11a)}$$

$$L(3,0;r,D)=[1-(r/D)^2]^{\frac{1}{2}}[-\tfrac{1}{5}(r/D)^4+(1/15)(r/D)^2+(2/15)], \quad \text{(A11b)}$$

$$L(2,0;r,D)=\tfrac{1}{8}\cos^{-1}(r/D)+[(1-(r/D)^2]^{1/2}\times[-\tfrac{1}{4}(r/D)^3+\tfrac{1}{8}(r/D)], \quad \text{(A11c)}$$

$$L(1,0;r,D)=[1-(r/D)^2]^{\frac{1}{2}}[-\tfrac{1}{3}(r/D)^2+\tfrac{1}{3}], \quad \text{(A11d)}$$

$$L(0,0;r,D)=\tfrac{1}{2}\cos^{-1}(r/D)+[1-(r/D)^2]^{\frac{1}{2}}[-\tfrac{1}{2}(r/D)], \quad \text{(A11e)}$$

$$L(2,2;r,D)=\tfrac{1}{16}\cos^{-1}(r/D)+[1-(r/D)^2]^{\frac{1}{2}}[\tfrac{1}{6}(r/D)^5-(7/24)(r/D)^3+\tfrac{1}{16}(r/D)], \quad \text{(A11f)}$$

$$L(1,2;r,D)=[1-(r/D)^2]^{\frac{1}{2}}\times[\tfrac{1}{5}(r/D)^4-\tfrac{2}{5}(r/D)^2+\tfrac{1}{5}], \quad \text{(A11g)}$$

$$L(0,2;r,D)=\tfrac{3}{8}\cos^{-1}(r/D)+[1-(r/D)^2]^{\frac{1}{2}}\times[\tfrac{1}{4}(r/D)^3-\tfrac{5}{8}(r/D)], \quad \text{(A11h)}$$

$$L(0,4;r,D)=\tfrac{5}{16}\cos^{-1}(r/D)+[1-(r/D)^2]^{\frac{1}{2}}[-\tfrac{1}{6}(r/D)^5+(13/24)(r/D)^3-\tfrac{11}{16}(r/D)]. \quad \text{(A11i)}$$

Substituting these equations into Eqs. (A10a–f) and the results into Eqs. (A6a–d), and that set of results into Eqs. (A5a–d), we obtain Eqs. (5.6a–d).

APPENDIX B. A THEOREM

In this Appendix we prove the theorem that

$$\frac{1}{R^2}\int_0^D rdr\mathfrak{F}_j(r,D)r^2=\begin{cases} R^2/2 & \text{if } j=C \\ 0 & \text{if } j\neq C. \end{cases} \quad \text{(B1)}$$

We start by noting that

$$r^2=(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')=x^2+y^2+x'^2+y'^2-2xx'-2yy'. \quad \text{(B1)}$$

This can be expressed in terms of the functions $F_\mu(\mathrm{x})$ and $F_\mu(\mathrm{x}')$ giving

$$r^2=[\pi^2R^8/12]^{\frac{1}{2}}[F_1(\mathrm{x})F_4(\mathrm{x}')+F_1(\mathrm{x}')F_4(\mathrm{x})]+\pi R^4F_1(\mathrm{x})F_1(\mathrm{x}')-(\pi R^4/2)\times[F_2(\mathrm{x})F_2(\mathrm{x}')+F_3(\mathrm{x})F_3(\mathrm{x}')]. \quad \text{(B2)}$$

We note that because of the orthonormality conditions

$$\frac{1}{2\pi R^2}\int\int d\mathrm{x}d\mathrm{x}'W(\mathrm{x},D)W(\mathrm{x}',D)r^2\sum_{\mu=1}^{n_j}F_\mu(\mathrm{x})F_\mu(\mathrm{x}')=\begin{cases}R^2/2 & \text{if } j=C\\ 0 & \text{if } j\neq C,\end{cases} \quad \text{(B3)}$$

since the F_1F_4 term cannot contribute, the F_1F_1 term always contributes $R^2/2$ to the result, and the F_2F_2 and F_3F_3 terms each contribute $-R^2/4$ whenever n_j is three or greater; i.e., whenever $j\neq C$. In this latter case, the net contribution is zero. However, transforming to $\mathbf{r}$ and $\mathbf{r}'$ coordinates and considering the definition of $\mathfrak{F}_j(r,D)$ as given in Eq. (5.5), we see that

$$\frac{1}{2\pi R^2}\int\int d\mathrm{x}d\mathrm{x}'W(\mathrm{x},D)W(\mathrm{x}',D)r^2\sum_{i=1}^{n_j}F_i(\mathrm{x})F_i(\mathrm{x}')=\frac{1}{2\pi R^2}\int d\mathbf{r}r^2\mathfrak{F}_j(r,D)=\frac{1}{R^2}\int_0^D rdrr^2\mathfrak{F}_j(r,D). \quad \text{(B4)}$$

From Eqs. (B3) and (B4) we see that

$$\frac{1}{R^2}\int_0^D rdr\mathfrak{F}_j(r,D)r^2=\begin{cases}R^2/2 & \text{if } j=C\\ 0 & \text{if } j\neq C,\end{cases} \quad \text{(B5)}$$

thus proving the theorem.

Similar relationships, such as that

$$\frac{1}{R^2}\int_0^D rdr\mathfrak{F}_j(r,D)=\tfrac{1}{2}, \quad \text{(for all } j) \quad \text{(B6)}$$

can be proven just as easily.

APPENDIX C. EVALUATION OF $\mathcal{A}$

It may be shown[2] that $\mathcal{A}$ can be written as

$$\mathcal{A}=2.91\left(\frac{2\pi}{\lambda}\right)^2\int_{\substack{\text{path of}\\ \text{propagation}}} d\Lambda C_N^2 \quad \text{(C1)}$$

where λ is the optical wavelength, Λ is a variable defining length along the path of propagation. The quantity C_N is called the atmospheric refractive-index structure constant. It is a function of altitude and consequently depends on Λ because altitude may vary along the path of propagation. $C_N{}^2$ may be written in terms of the atmospheric refractive-index variance A and the outer scale of turbulence L_0 defined here as the distance in which turbulence correlation falls to one-half of the maximum value. This expression is

$$C_N{}^2=AL_0^{-\frac{2}{3}}. \quad \text{(C2)}$$

A reasonable fit to available data for A and L_0 is obtained with expressions

$$A=6.7\times10^{-14}\exp(-h/3200), \quad \text{(C3a)}$$

$$L_0=2h^{\frac{1}{2}}, \quad \text{(C3b)}$$

where h is altitude, and all units of measure are understood to be in mks units. Substituting (C3a, b) into Eq. (C2), and the result into Eq. (C1), we obtain a closed-form result for $\mathcal{A}$. This result for the most general type of propagation path (ignoring curvature of the earth) is expressible in terms of the incomplete gamma function of order two-thirds.

APPENDIX D. SCINTILLATION EFFECTS

This entire article thus far has considered wavefront distortion with the implicit assumption that intensity is uniform over the area of interest. For some problems this is indeed the case and the results derived and conclusions drawn need no qualifications. In this Appendix, we sketch out a proof of the fact that the results also apply when scintillation is present, but with a slightly different interpretation.

Instead of working with only the phase deviation $\phi(\mathrm{x})$, let us also consider the log amplitude $l(\mathrm{x})$, defined as the natural logarithm of the ratio of the instantaneous amplitude at x, to what the amplitude would be if there were no scintillation. Thus, $\phi(\mathrm{x})-il(\mathrm{x})$ is the complex phase which completely specifies the instantaneous wave fluctuation at x. Now rather than consider the simple phase-structure function as defined in (1.1), we consider what we may call the wave-structure function for which we also use the symbol $\mathfrak{D}(r)$.

$$\mathfrak{D}(\mathrm{r})=\langle|[\phi(\mathrm{x})-il(\mathrm{x})]-[\phi(\mathrm{x}')-il(\mathrm{x}')]|^2\rangle=\langle[\phi(\mathrm{x})-\phi(\mathrm{x}')]^2\rangle+\langle[l(\mathrm{x})-l(\mathrm{x}')]^2\rangle, \quad \text{(D1)}$$

where the vertical bars denote taking an absolute value. We now go back through the paper asking that the infinite series $\Phi(\mathrm{x})$ and the finite polynomials $\Phi_j(\mathrm{x})$, define in (3.1) and (3.5), provide the best fit in the sense of Eqs. (3.2) and (3.3), not simply to $\phi(\mathrm{x})$ but to $\phi(\mathrm{x})-il(\mathrm{x})$. It is to be understood in (3.2) and appropriate places thereafter that absolute values are to be taken in squaring. Of course, the coefficients a_μ are to be allowed to take on complex values and Eq. (3.3) must be understood as two equations for each value of μ, one for the real and one for the imaginary part of a_μ. The coefficients $a_C{}^2$, $a_L{}^2$, $a_S{}^2$ and $a_Q{}^2$, Eqs. (3.4a–d), are to be understood as the sum of magnitudes.

Equation (4.1) now takes the form

$$\Delta=\frac{1}{\pi R^2}\int d\mathbf{x}W(x,D)[\phi^2(\mathbf{x})+l^2(\mathbf{x})]$$
$$-\frac{1}{\pi R^2}\sum_{\mu=1}^{\infty}\int d\mathbf{x}W(x,D)$$
$$\times\{[\phi(\mathbf{x})-il(\mathbf{x})]\bar{a}_\mu+[\phi(\mathbf{x})+il(\mathbf{x})]a_\mu\}F_\mu(\mathbf{x})$$
$$+\frac{1}{\pi R^2}\sum_{\mu,\nu=1}^{\infty}\bar{a}_\mu a_\nu\int d\mathbf{x}W(x,D)F_\mu(\mathbf{x})F_\nu(\mathbf{x}),\quad \text{(D2)}$$

where the overbar denotes a complex conjugate. Equation (4.2) takes a corresponding form. Now, applying the modified form of (3.2) to determine the real and imaginary parts of a_μ separately, we find that Eq. (4.3) takes the form

$$a_\mu=\int d\mathbf{x}W(x,D)[\phi(\mathbf{x})-il(\mathbf{x})]F_\mu(\mathbf{x}),$$

as might have been expected. All the rest of the equations of Sec. 4 follow immediately with $\phi(\mathbf{x})$ replaced by $[\phi(\mathbf{x})-il(\mathbf{x})]$, and squares understood to involve absolute values.

Likewise, all the equations of Sec. 5 follow with the phase $\phi(\mathbf{x})$ replaced by the complex phase $[\phi(\mathbf{x})-il(\mathbf{x})]$, and with $\mathfrak{D}(r)$, previously the phase-structure function as defined in (1.1), now understood to be the wave-structure function as defined in (D1). *Thus Eqs. (5.7) and (5.8a–c) are seen to be correct as stated when scintillation is present except that $\mathfrak{D}(r)$ has a different interpretation than given in Eq. (1.1).*

The balance of the paper is better with this interpretation of $\mathfrak{D}(r)$ than it was initially. Equation (6.1) and (C1) are precise when $\mathfrak{D}(r)$ denotes the wave-structure function. If $\mathfrak{D}(r)$ is simply the phase-structure function then these two equations are accurate only when the propagation path is short. For a long path, the coefficient a contains an r-dependent factor which varies slowly and monotonically from a value of about one-half at $r=0$ to unity for large r. This factor was arbitrarily set equal to unity in (6.1) and (C1).

Offhand, while pleased with the increased precision of the structure function for propagation in the atmosphere, obtained by considering scintillation and working with the complex phase we have to ask if we can now assign any meaning to the complex polynomial used to match the complex phase. For a problem in which there is no scintillation, or when the propagation path is short, the results derived in the main body of the paper are correct with $\mathfrak{D}(r)$ interpreted as the phase-structure function and we can assign exact physical meaning to the quantities we have been evaluating. For long-path propagation, where scintillation may be significant, we make no more than a factor-of-two error by still interpreting $\mathfrak{D}(r)$ as the phase-structure function, and we retain some of the physical insight into the quantities evaluated.

To the extent that only phase fluctuation is of importance in a process, we overestimate the extent of the distortion by a factor of two. In those processes in which intensity fluctuations also are significant we may not be overestimating the effect of distortion. In the case of the image-forming process of a lens, for instance, intensity variations across the lens result in additional diffraction which limits resolution. If the intensity variations are random, the log amplitude appears in image statistics in a form analogous to the phase so that the wave-structure function is the quantity which appears. Thus, use of $D_C{}^*$, as computed from (6.1) and (C1), to estimate photographic resolution should be more precise than the derivation, based on phase fluctuation alone, would justify. On the other hand, the use of $D_L{}^*$, when intensity fluctuations are significant, to estimate high-speed resolution may be more optimistic than warranted since the high-speed exposure, freezing the distortion, can compensate only for wavefront tilt, but not for a linear variation of log amplitude. When intensity fluctuations are not significant, i.e., when $\langle[l(\mathbf{x})-l(\mathbf{x}')]^2\rangle\ll\langle[\phi(\mathbf{x})-\phi(\mathbf{x}')]^2\rangle$, the use of $D_L{}^*$ to estimate high-speed photographic resolution should be quite accurate.

How to use the geometric statistics of wavefront distortion, when to ignore, and how to consider the effect of intensity variations requires careful judgment by the user.