0.1 Strehl Ratio

The Strehl ratio is a metric for determining the performance of an optical system by comparing the power delivered to the theoretical diffraction limit.

If looking at a target in the far field, the electric field can be determined using the Fraunhofer approximation, which says that the field is proportional to the Fourier transform of the emitted field, scaled relative to the propagation distance and wavelength.

Since we are looking at the ratio between the unit device and an ideal device, those scaling terms will cancel out and can be ignored. Our field ratio is then given by

$$\frac{\mathscr{F}\{E_{Device}\}}{\mathscr{F}\{E_{Ideal}\}}$$

and our power ratio is given by

$$\frac{|\mathscr{F}\{E_{Device}\}|^2}{|\mathscr{F}\{E_{Ideal}\}|^2}$$

For any point in this far field distribution, the distance from the origin indicates some angle of the Poyntnig vector. As the propagation distance approaches infinity, any such angle will eventually result in enough of a shift off axis to miss whatever target is being aimed at. For arbitrary distance then, only the power values at the origin are relevant for calculating our Strehl ratio. So the Strehl ratio can be determined as:

$$S = \frac{|\mathscr{F}\{E_{Device}\}(0,0)|^2}{|\mathscr{F}\{E_{Ideal}\}(0,0)|^2}$$

The Fraunhofer equation includes additional scaling and phase factors, but these either cancel out in the ratio or are irrelevant at the origin, so we can just use the Fourier transform for the time being.

Looking at the example of a Gaussian illuminated lens aperture, we should be able to calculate the theoretical Strehl ratio.

We will define an aperture of radius R and define our electric field such that the intensity of light entering the aperture follows a Gaussian distribution with a unit peak amplitude.

$$I_0 = exp(-0.5(r/\sigma)^2)$$

We will also define this beam to have no spatial phase variation at the aperture. In this case, the electric field entering the aperture can be expressed simply as the square root of this intensity.

$$E_0 = \sqrt{I_{Device}} = exp(-0.25(r/\sigma)^2)$$

Our ideal device will be defined as having the same total power as our initial field, except uniformly distributed within an equivalent circular aperture to our test device. We can find the total power by integrating over intensity of our full initial plane.

$$P_{Ideal} = \int\int I_0 = \int_0^{2\pi}\int_0^{\infty} exp(-0.5(r/\sigma)^2)rdrd\theta$$

Using the substitutions:

$$u = -0.5(r/\sigma)^2$$

$$du = r/\sigma^2$$

This integral becomes solvable, giving

$$P_{Ideal} = 2\pi\sigma^2$$

Which means our ideal aperture is defined as

$$I_{Ideal} = circ(r/R)\frac{\int\int I_{Ideal}}{\int\int circ(r/R)} = \frac{2\sigma^2}{R^2}circ(r/R)$$

Again, since we have no spatial variation our electric field is just the square root of the intensity

$$E_{Ideal} = \sqrt{I_{Ideal}} = \frac{\sqrt{2}\sigma}{R}circ(r/R)$$

With this expression, we can calculate the denominator of our Strehl ratio

$$\mathscr{F}\{E_{Ideal}\}(0,0) = \int_0^{2\pi}\int_0^{R}\frac{\sqrt{2}\sigma}{R}exp(0)rdrd\theta = \sqrt{2}\sigma\pi R$$

$$P_{IFarField} = |\mathscr{F}\{E_{Ideal}\}(0,0)|^2 = 2\sigma^2\pi^2R^2$$

We can now take the same approach to finding the numerator. Our device field is given by our original Gaussian electric field E0 bounded by a circular aperture.

$$E_{Device} = circ(r/R)E_0 = circ(r/R)exp(-0.25(r/\sigma)^2)$$

$$\mathscr{F}\{E_{Device}\}(0,0) = \int_0^{2\pi}\int_0^R exp(-0.25(r/\sigma)^2)exp(0)rdrd\theta$$

$$= -4\pi\sigma^2 exp(-0.25(r/\sigma)^2)|_0^R = 4\pi\sigma^2[1 - exp(-0.25(R/\sigma)^2)]$$

$$P_{DFarField} = |\mathscr{F}\{E_{Device}\}(0,0)|^2 = 16\pi^2\sigma^4[1 - exp(-0.25(R/\sigma)^2)]^2$$

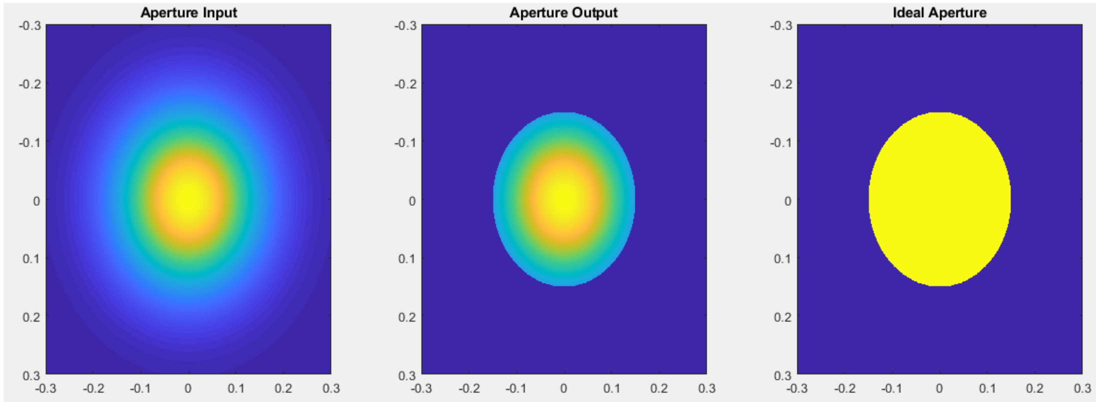
With these terms we can now calculate our theoretical Strehl ratio for a Gaussian illuminated lens.

$$S = \frac{|\mathscr{F}\{E_{Device}\}(0,0)|^2}{|\mathscr{F}\{E_{Ideal}\}(0,0)|^2} = \frac{16\pi^2\sigma^4[1 - exp(-0.25(R/\sigma)^2)]^2}{2\sigma^2\pi^2R^2}$$

$$S = 8(\sigma/R)^2(1 - 2exp(-0.25(R/\sigma)^2) + exp(-0.5(R/\sigma)^2))$$

Based on this equation, it can be seen that the Strehl ratio of a Gaussian illuminated lens is determined by the ratio of the intensity distribution standard deviation and the radius of the aperture. If we sweep this ratio from 0 to 1, there is a maximum S value of 0.8145 at $\sigma = 0.446$.

This theoretical value can be tested by simulating a Gaussian illuminated aperture and numerically calculating the Strehl ratio. We can use the original formula for this purpose, by creating an ideal aperture as well as a test aperture and taking the FFT of both, then taking the ratio squared of the center value of each. The example shown below is for a Gaussian illuminated aperture of diameter 30 cm.

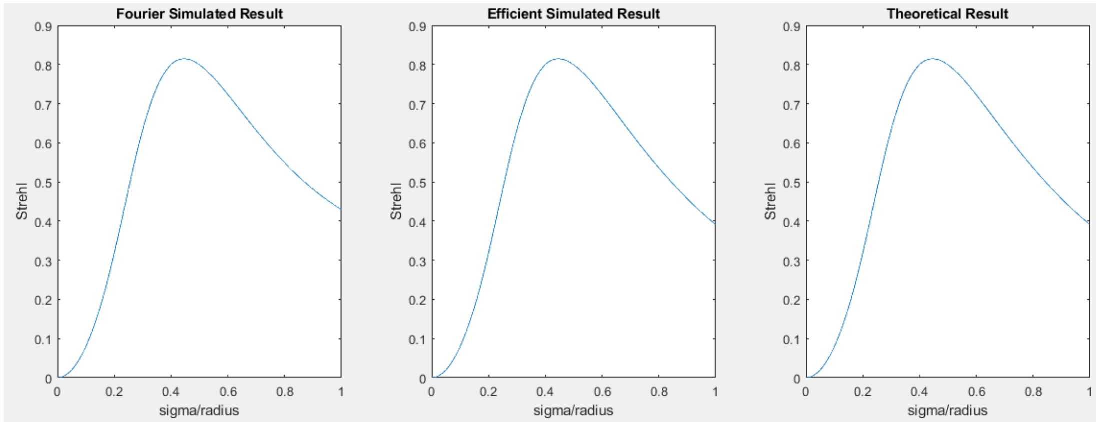


This, however, might be unnecessarily complex. Since we are only calculating the central value of each field's Fourier transform, it is a faster operation to just integrate over the whole field, since the exponential term in the Fourier transformation becomes 1 at the origin. Additionally, since the Gaussian distribution has a known integral which was already solved for, we can directly use the Ideal Power term to produce our ideal aperture. Thus our Strehl function can be more quickly computed with the following

$$S = \left|\frac{\int\int E_{Device}}{\int\int\sqrt{2\pi\sigma^2/\pi R^2}}\right|^2$$

If both simulation methods are applied and compared to the derived theoretical equation, the plots shown below are the result. There is close agreement between all three calculations. The Fourier simulated result shows some deviation from the theoretical model as the σ-radius ratio approaches 1. This illustrates a potential source of error which needs to be accounted for in numerical simulations. At high σ values, the Gaussian intensity distribution is wide enough to extend past the bounds defined for the simulation. As a result, the calculated ideal far-field power is decreased, and the Strehl ratio increases. This can easily be rectified by increasing the bounds of the calculation plane, but doing so incurs a longer calculation time.

Alternately, using the second calculation method avoids this problem entirely. The total power used to generate the ideal aperture is determined directly from the σ value, so any part of the input field which is clipped is irrelevant.



The results shown here relate directly to the field of optics because a Gaussian intensity distribution is characteristic of the Gaussian beam, which is given by the field equation:

$$E(r,z) = E_0 \frac{w_0}{w(z)} exp(\frac{-r^2}{w(z)^2}) exp(-i(kz + k\frac{r^2}{2R(z)} - \phi(z)))$$

If we are looking at the beam waist the end phase term disappears and w(z) = w0, simplifying this a bit.

$$E(r,z) = E_0 exp(\frac{-r^2}{w_0^2})$$

Which means that intensity becomes the square of this expression

$$I(r,z) = E_0^2 exp(-2(r/w_0)^2)$$

In this simplified form it becomes apparent that this matches the Gaussian profile we used to define our intensity, and therefore

$$w_0 = 2\sigma$$

We can use our previous analysis then to conclude that the optimal w0 for an aperture to maximize Strehl ratio is 2 x 0.446 = 0.892 R.