Huong Phan
MATH 320 - Lab 4 Report

1. Problem statement and objectives
Problem statement: To compare two different strategies for managing a renewable resource, modeled by adding a source/sink term to the logistic equation.

The objective of this lab is to develop two differential equation models for the change in fish population over time, which are Constant Effort Harvesting and Constant Yield Harvesting.

2. Methodology and Results and interpretation/conclusion(s)

(a) Constant Effort Harvesting
Constant Effort Harvesting model is based on the assumption that the more fish there are, the easier it is to catch them. For this model, the rate of change in fish population of fish can be modeled by the following differential equation:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey$$

where y is the population of fish over time t, r is the intrinsic growth rate of fish population, K is the maximum sustainable population of fish, and E is the positive constant with units 1/time that measures the total effort made to harvest the species of fish.

We find the equilibrium points by equating $\frac{dy}{dt}$ to 0. Solving for y, we have two solutions y_1 = 0 and y_2 = $\left(1 - \frac{E}{r}\right)K$. Thus, when E < r, $\frac{E}{r}$< 1 and y_2 > 0. We have the following phase line:

From the phase line above, y = y_1 is unstable and y = y_2 is asymptotically stable. Thus, as long as E < r, there is an infinitely stable population of fish of y_2 $= \left(1 - \frac{E}{r}\right)K$.

The sustainable yield Y of the fishery is $Ey_2 \Rightarrow$ Y = E$\left(1 - \frac{E}{r}\right)K$

To determine the maximum sustainable yield Y_m, we equate $\frac{dY}{dE}$ to 0. Solving for $\frac{dY}{dE}$, we have

$$\frac{dY}{dE} = K\left(1 - \frac{2E}{r}\right)$$

Equating $\frac{dY}{dE}$ to 0, $K\left(1 - \frac{2E}{r}\right) = 0 \Rightarrow E = \frac{r}{2}$

Therefore, at $E = \frac{r}{2}$, Y is maximum.

The maximum sustainable yield Y_m is $\frac{r}{2}\left(1 - \frac{1}{2}\right)K = \frac{rK}{4}$

(b) Constant Yield Harvesting

Constant Effort Harvesting model is based on the assumption that fish are caught at a constant rate h independent of the size of the fish population, which is the harvesting rate $H(y, t) = h$. For this model, the rate of change in fish population of fish can be modeled by the following differential equation:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - h = f(y)$$

where y is the population of fish over time t, r is the intrinsic growth rate of fish population, K is the maximum sustainable population of fish.

We find the equilibrium points by equating $\frac{dy}{dt}$ to 0. Solving for y, we have two solutions $y_1 = \frac{K}{2} - \sqrt{\frac{K^2}{4} - \frac{hK}{r}}$ and $y_2 = \frac{K}{2} + \sqrt{\frac{K^2}{4} - \frac{hK}{r}}$, when $h < \frac{rK}{4}$.

From the phase line above, $y = y_1$ is unstable and $y = y_2$ is asymptotically stable. Thus, as long as $h < \frac{rK}{4}$, there is an infinitely stable population of fish of $y_2 = \sqrt{\frac{K^2}{4} - \frac{hK}{r}} + \frac{K}{2}$.

We can also plot the graph of f(y) versus y as followed:

Thus, if initial population $y_0 > y_1$, when t increases towards infinity, y increases towards y_2 since $dy/dt > 0$ because y_2 is the infinitely stable population of fish in the presence of exploitation.
If initial population $y_0 < y_1$, when t increases, y decreases to zero since $dy/dt < 0$. Since y_0 is non-zero, the population will go extinct at a finite time.

If $h > rK/4$, $\dfrac{dy}{dt} = r\left(1 - \dfrac{y}{K}\right)y - h = -\left(y - \dfrac{K}{2}\right)^2 + \dfrac{K^2}{4} - \dfrac{hK}{r} < 0$ for all values of y $\Rightarrow$ y decreases to 0 as t increases regardless of the value of y_0.

If $h = rK/4$, $\dfrac{dy}{dt} = r\left(1 - \dfrac{y}{K}\right)y - h = r\left(1 - \dfrac{y}{K}\right)y - \dfrac{rK}{4} = -\left(y - \dfrac{K}{2}\right)^2$

We find the equilibrium points by equating $\dfrac{dy}{dt}$ to 0. Solving for y, $y = \dfrac{K}{2}$

From the phase line above, $y = \dfrac{K}{2}$ is semistable since $\left(y - \dfrac{K}{2}\right)^2 > 0$ for all values of y.

Thus, the maximum sustainable yield h_m is rK/4 which coincides with maximum sustainable yield Y_m in part (a) and if y falls below K/2, the population is considered to be overexploited.