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MATH 320 - Lab 2 Report

1. Problem statement and objectives

Problem statement: To predict the current population of U.S. by developing two different differential equation models using data from U.S. Census Bureau, and justify whether each of the model really applies to the current situation.

The objective of this lab report is to develop two different differential equation models for the total population of U.S., using census data for the period 1900 through 1999. The two differential equation models explored in this lab report will be exponential model and logistic model. We will construct the equations and compare the predicted current population figures from each model to the real-time population figure using the data from U.S. Census Bureau, then discuss about the possible limitations of these models.

2. Methodology

(a) Exponential model

The exponential model involves modeling the rate of change of total U.S. population over time as followed

$$\frac{dy}{dt} = ay$$

in which y is the total U.S. population (in people), t is the number of years from 1900 (in years), with 1900 is set to be year zero, $\frac{dy}{dt}$ is the rate of change of total U.S. population over time (in people per year), a is the growth rate of U.S. population.

Solving the above differential equation, we have the following equation

$$y = Ce^{at}$$

in which C is a constant. At t = 0, y = the total population of U.S. at year zero (which is 1900) = C, thus we have y(0) = C = 76094000. We now have

$$y = 76094000e^{at}$$

Since we are using the data of total U.S. population from 1900 to 1999, it may be better to use t = 99 to find the growth rate as it may be more representative of the data. Also 1999 is also closer to 2017 than other years so it may be more reflective of the situation.
Substitute t = 99 (for year 1999) and y = 272690813 to solve for a, we have 272690813 = $76094000e^{99a}$ ⇒ a = 0.012893

Therefore, by exponential model, $y = 7609400e^{0.012893t}$.

(b) Logistic model
The logistic model involves modeling the rate of change of total U.S. population over time as followed

$$\frac{dy}{dt} = ay(1 - \frac{y}{k})$$

in which y is the total U.S. population (in people), t is the number of years from 1900 (in years), with 1900 is set to be year zero, $\frac{dy}{dt}$ is the rate of change of total U.S. population over time (in people per year), a is the intrinsic growth rate of U.S. population, and k is the carrying capacity, or maximum sustainable total population of U.S.

Solving the above differential equation, we have the following equation

$$y = \frac{k}{1 + Ce^{-at}}$$

in which C is a constant.
We choose k to be 400000000. At t = 0, y = the total population of U.S. at year zero (which is 1900) = $\frac{400000000}{1+C}$. Thus, $76094000 = \frac{400000000}{1+C}$. Solve for C, we have C = 4.2567.
For the same reason as above, substitute t = 99 (for year 1999) and y = 272690813 to solve for a, we have 272690813 = $\frac{400000000}{1+4.2567e^{-99a}} \Rightarrow$ a = 0.022325.

Thus, by logistic model, $y = \frac{400000000}{1+4.2567e^{-0.022325t}}$.

We can now plot the graphs of total U.S. population against time.

3. Results and interpretation/conclusion(s)

<u>Results:</u>
(a) Exponential model
Substitute t = 117 (for year 2017), we have y = $76094000e^{0.012893*117}$ = 343918849. Thus by exponential model, we predict the total population of U.S to be 343918849 in 2017
(b) Logistic model
Substitute t = 117 (for year 2017), we have y = $\frac{400000000}{1+4.2567e^{-0.022325*117}}$ = . Thus by exponential model, we predict the total population of U.S to be 304789474 in 2017.

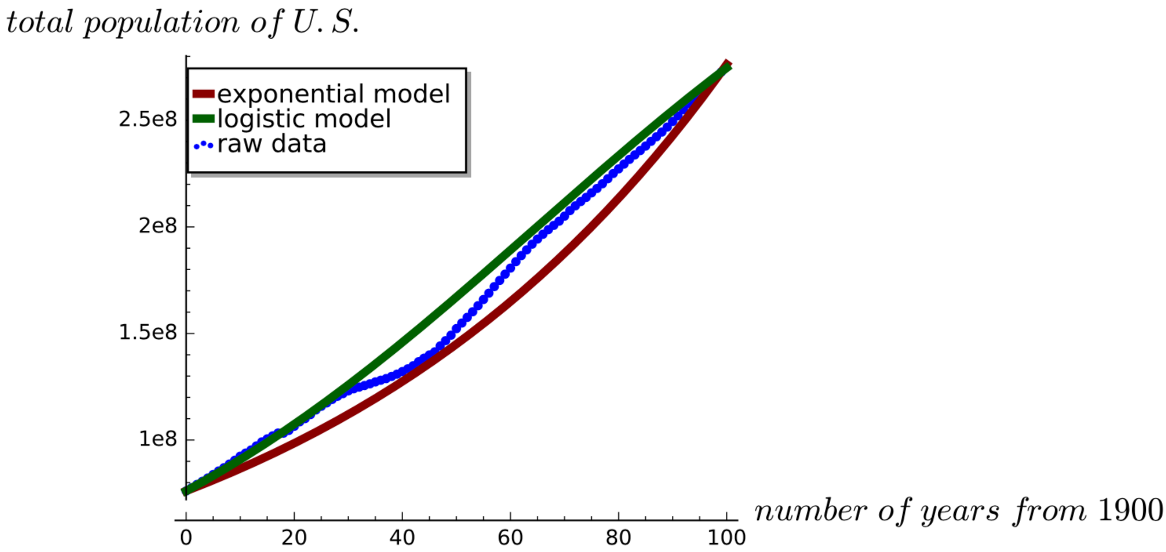

Figure 1. The total U.S. population as a function of number of years from 1900, with the census data, exponential model and logistic model of population

Interpretation/conclusion

Total population of U.S. until September 12, 2017 is 325,858,624. (Source: US Census Bureau. Available at https://www.census.gov/popclock/). Thus, the result for the above exponential model is closer to the real result than the result of the above logistic model.

There are certain limitations for both of these two models. Exponential model seems to be not very realistic as the Earth has only a limited amount of resources so a carrying capacity should be considered as a factor in determining the size of a population. Meanwhile, though logistic model does take in account of the carrying capacity of the Earth, it is very difficult to estimate the maximum sustainable population as it involves a lot of factors and may change in the future, for example, due to the change in the quantity and quality of limited resources. Moreover, the data may be outdated and not reflect the possible changes in recent population landscape since the last year recorded was 1999, which was almost two decades ago. If there is a greater amount of data from earlier than 1900 to 2016, logistic model may be better modeled since we may see where the growth rate slows down, if any, then estimate the carrying capacity to be double of the population at the point with the greatest gradient on the graph.