Huong Phan
MATH 320 - Lab 5 Report

1. Problem statement and objectives

Problem statement: To measure respective annual military expenditures in some common units of currency of two countries in competition with each other as a function of time by developing two different differential equation models, and justify whether each of the model really applies to the current situation.

The objective of this lab is to develop two differential equation models for the rate of change in their military expenditures, with respect to time, which are the mutual fear model and The Richardson model.

2. Methodology and Results and interpretation/conclusion(s)

Let G(t) be the military expenditure of country Green at time t, in years, and P(t) = military expenditure of country Purple at time t, in years. We are going to use two different models to measure the rate of change in military expenditure spent by each nation over time as follows:

(a) The mutual fear model

The mutual fear model is based on the assumption that each nation increases its spending in direct proportion to the other nation's current level of spending. For this model, the rate of change in their military expenditures, with respect to time, can be modeled by the following differential equations:

$$\frac{dG}{dt} = aP$$
$$\frac{dP}{dt} = bG$$

where a, b are the positive constants for mutual fear of the other between Green nation and Purple nation; $\frac{dG}{dt}$ is the rate of change in military expenditure of Green nation over time in years; $\frac{dP}{dt}$ is the rate of change in military expenditure of Purple nation over time in years.

$$\frac{dG}{dP} = \frac{dG}{dt} \times \frac{dt}{dP} = \frac{dG}{dt} \div \frac{dP}{dt} = \frac{aP}{bG}$$

Solving the above differential equation, we have the following equation

$$\frac{b}{2}G^2 = \frac{a}{2}P^2 + c$$

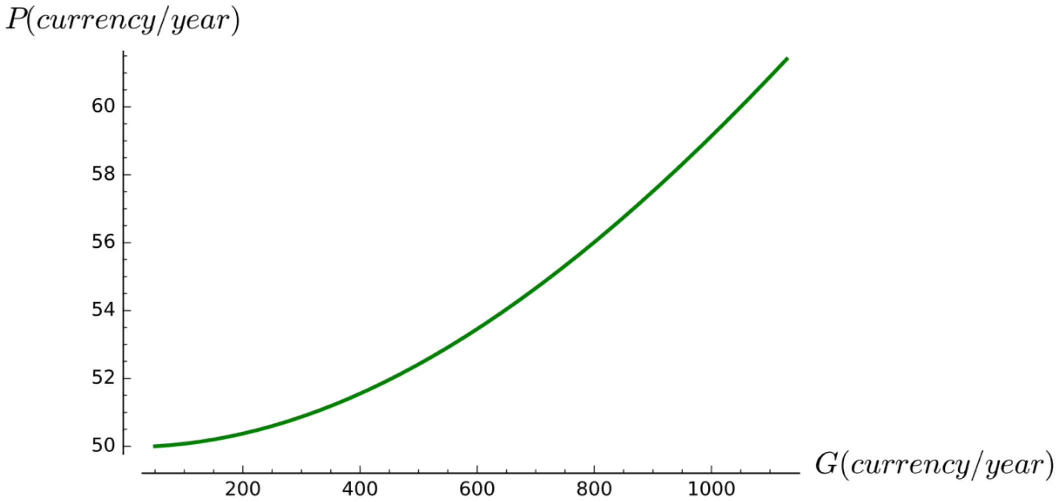
At t = 0, G = g_0, P = p_0, c = $\frac{b}{2}G^2 - \frac{a}{2}P^2 = \frac{b}{2}{g_0}^2 - \frac{a}{2}{p_0}^2$. Thus,
$\frac{b}{2}G^2 = \frac{a}{2}P^2 + \frac{b}{2}{g_0}^2 - \frac{a}{2}{p_0}^2$. Writing this in hyperbola form, we have

$$\frac{G^2}{\left(\pm\sqrt{\frac{b{g_0}^2 - a{p_0}^2}{b}}\right)^2} - \frac{P^2}{\left(\pm\sqrt{\frac{b{g_0}^2 - a{p_0}^2}{a}}\right)^2} = 1.$$

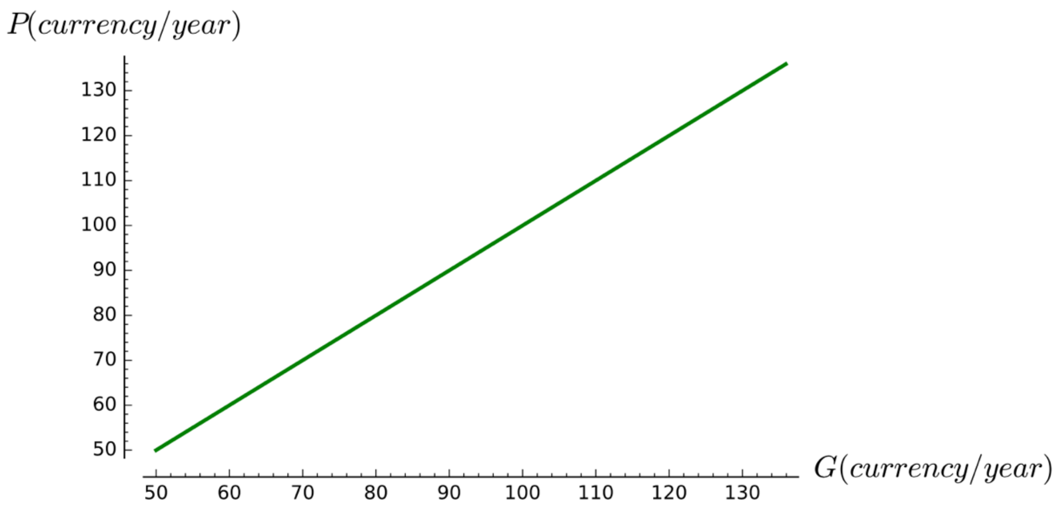
$\frac{b{g_0}^2 - a{p_0}^2}{b}$ and $\frac{b{g_0}^2 - a{p_0}^2}{a}$ must not zero for the equation to be valid. Since a, b > 0, $b{g_0}^2 \neq a{p_0}^2$.

The long term relationship between P and G depends on the values of a and b. If a > b, as G increases, P increases at an increasing rate. If a < b, as G increases, P increases at a decreasing rate. If a = b, as G increases, P increases at a constant rate. For all values and b > 0, eigenvalues for equilibrium solution are all real, distinct and of opposite values. Thus, all equilibrium solutions are semistable. The phase plane of typical trajectories are as follows:

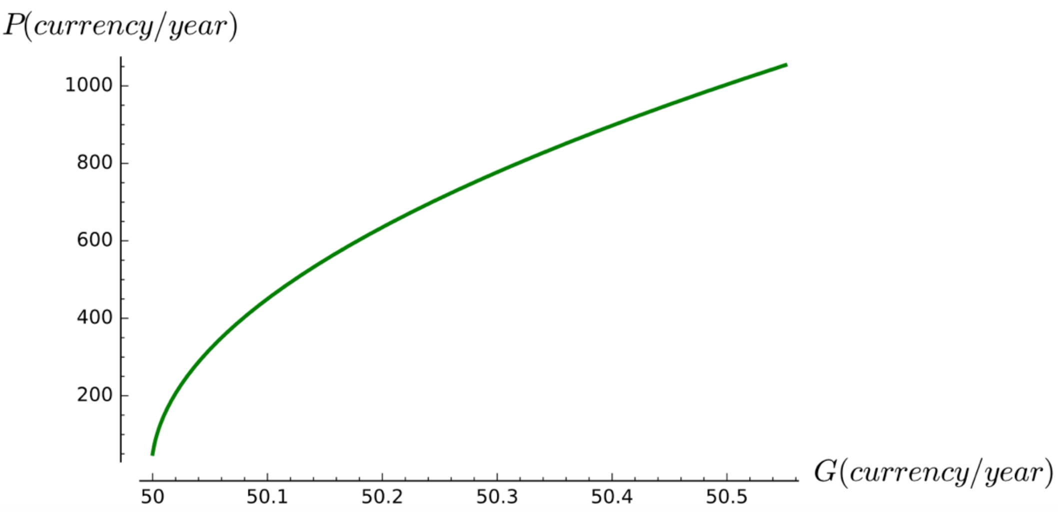
a = 0.20; b = 0.0002 (a > b)



a = 0.01; d = 0.01 (a = b)



a = 0.00001; b = 0.2 (a < b)



(b) The Richardson model

The Richardson model is based on the following assumptions:

(i) Each nation increases its spending in direct proportion to the other nation's current level of spending (mutual fear).

(ii) As each nation increases its spending in military arms race, it has to decrease its spending in other sectors leading to a slower growth/negative growth of its economy, which eventually leads to a decrease its military expenditure in direct proportion to its current military spending.

(iii) Each nation may have hostile or friendly relationship with the other nation, which leads to a positive or negative change, respectively, in its military expenditure.

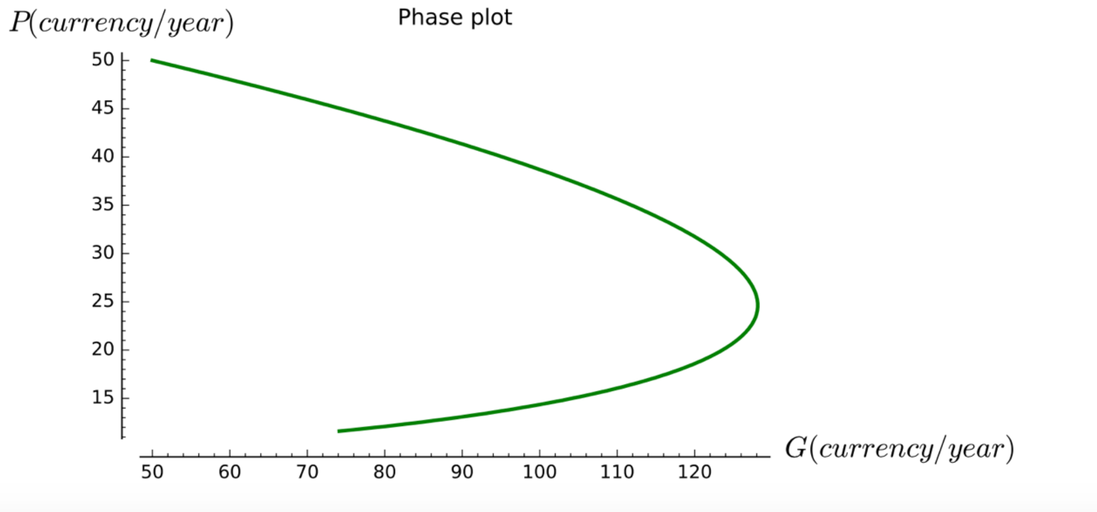
For this model, the rate of change in their military expenditures, with respect to time, can be modeled by the following differential equations:

$$\frac{dG}{dt} = aP - cG + m$$
$$\frac{dP}{dt} = bG - dP + n$$

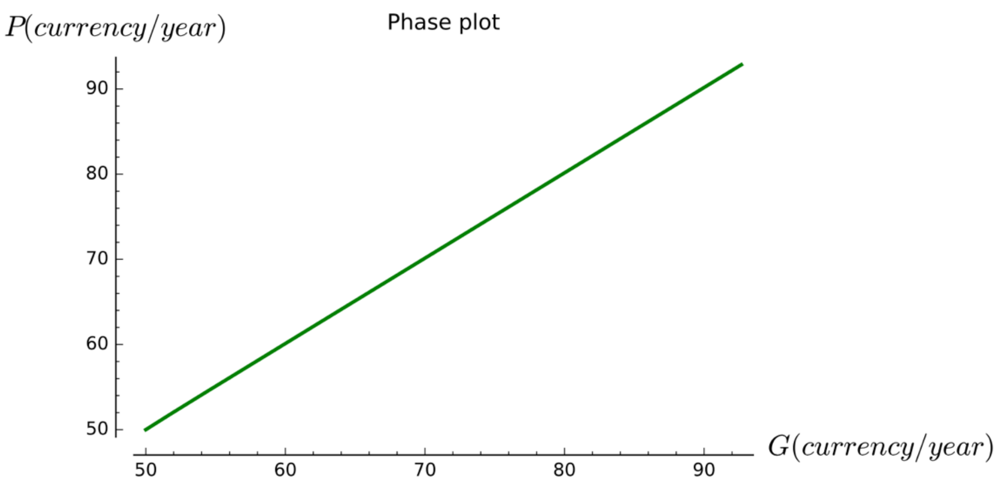
where a, b are the positive constants for mutual fear of the other between Green nation and Purple nation (assumption (i)); c, d are positive constants which represents that the current increasing military expenditure can have a debilitating effect on the economy and thus decrease military expenditure in the future (assumption (ii)); m, n are positive or negative constants depending on the nature of relationships between the two nations (assumption (iii)). $\frac{dG}{dt}$ is the rate of change in military expenditure of Green nation over time in years; $\frac{dP}{dt}$ is the rate of change in military expenditure of Purple nation over time in years.

Solving for $\frac{dG}{dt} = 0$ and $\frac{dP}{dt} = 0$ using Sage, we have equilibrium solution $G = \frac{dm + an}{cd - ab}$, $P = \frac{bm + cn}{cd - ab}$. Thus, for these equations to be valid, cd –ab $\neq 0 \Rightarrow$ cd $\neq$ ab.
Assuming that m, n are positive, i.e., there is a history of mutual grievance between the nations, a, b, c, d, m, n are now all positive. Thus, $dm + an > 0$ and $bm + cn > 0$. If cd – ab > 0, both G and P are positive and the equilibrium solution lies in the 1st quadrant. If cd – ab < 0, both G and P are negative and the equilibrium solution lies in the 3rd quadrant. When cd – ab > 0, the real parts eigenvalues of equilibrium solutions are both negative ⇒ the equilibrium solutions are stable. When cd – ab < 0, the real parts of eigenvalues of equilibrium solutions are of opposite signs ⇒ the equilibrium solutions are semistable. The phase plane of typical trajectories are as follows:

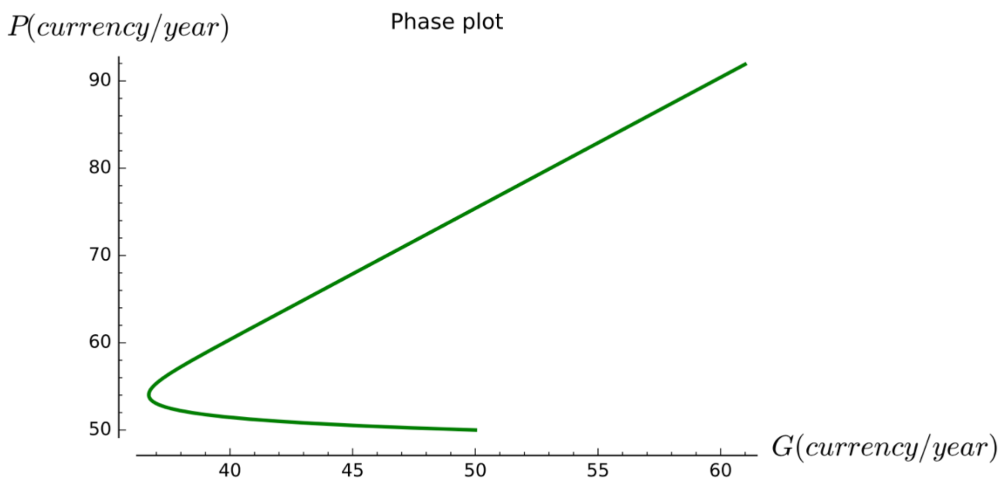
a = 0.10; b = 0.0002; c = 0.02; d = 0.02; m = 0.1; n = 0.2 (cd – ab > 0)



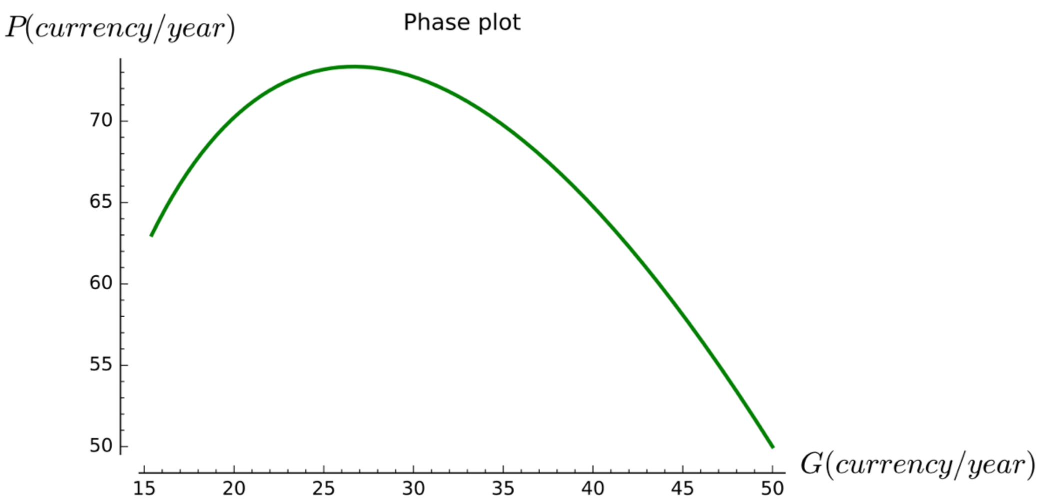
a = 0.10; b = 0.0002; c = 0.1; d = 0; m = 0.2; n = 0.2 (cd – ab < 0)



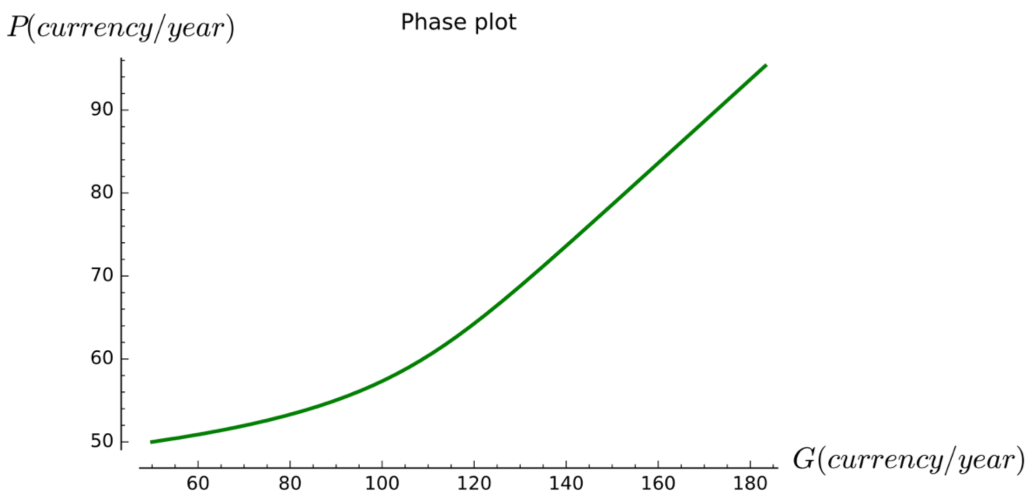
a = 0.10; b = 0.0002; c = 0.15; d = 0;m = 0.1; n = 0.2 (cd – ab < 0)



a = 0; b = 0.02; c = 0.01; d = 0.01;m = 0.1; n = 0.2 (cd – ab > 0)



a = 0.1; b = 0.0002; c = 0.05; d = 0; m = 0.1; n = 0.2 (cd – ab < 0)



(c) More extensions of the model

From the Richardson model, we replace the first term in each differential equation, which is essentially an exponential growth term, with logistic growth. Thus, the rate of change in their military expenditures, with respect to time, can be modeled by the following differential equations:

$$\frac{dG}{dt} = aP\left(1 - \frac{G}{K_G}\right) - cG + m$$
$$\frac{dP}{dt} = bG\left(1 - \frac{P}{K_P}\right) - dP + n$$

where a, b, c, d, m, n, $\frac{dP}{dt}, \frac{dG}{dt}$ are defined as above in the Richardson model; K_G, K_P are the maximum sustainable spending of Green nation and Purple nation on military.