Carnegie Mellon
THE ROBOTICS INSTITUTE

Probability Review

Robot Localization and Mapping 16-833

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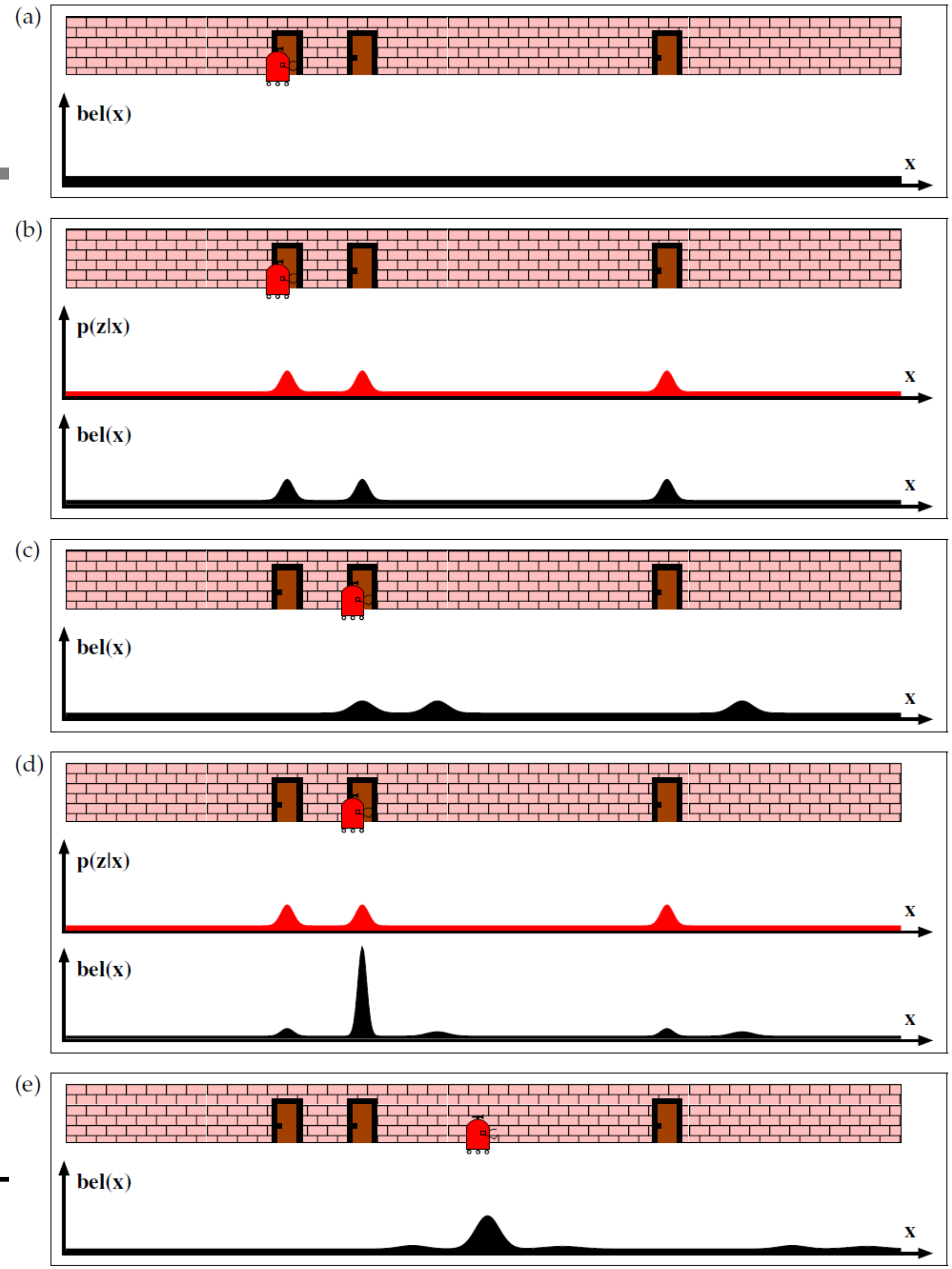
Slides based on probabilistic-robotics.org

Why Probabilities?

Global localization example:

- 1D world
- Map is known
- Sensors:
  - Door detector
  - Wheel odometry



Thrun, Burgard, Fox, 2005

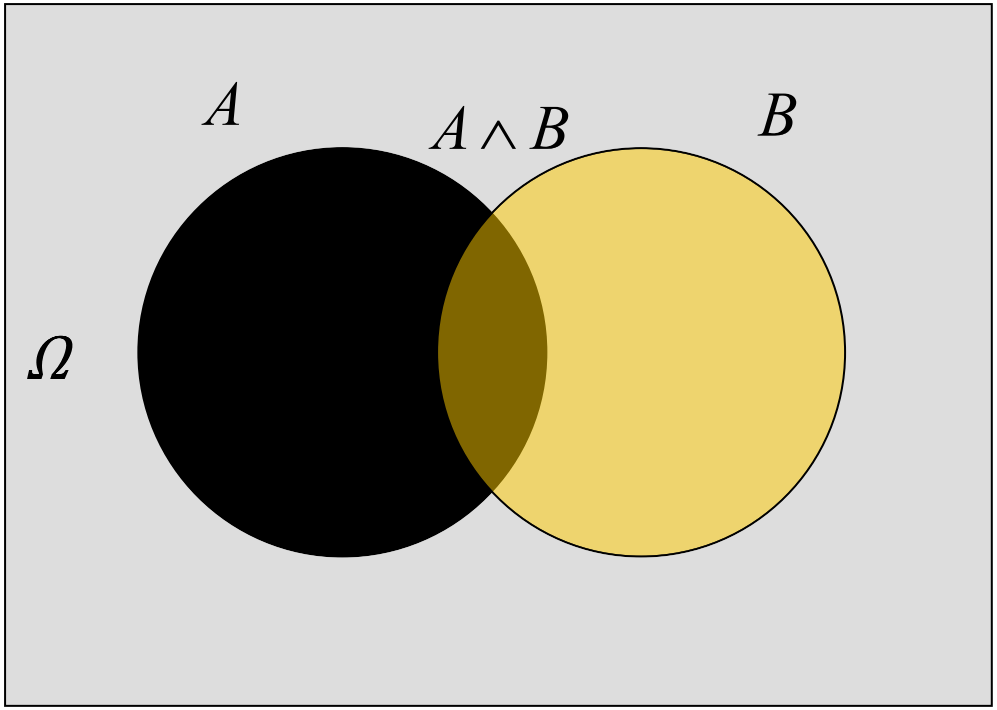
Axioms of Probability Theory

$\Pr(A)$ denotes probability that proposition A is true.

- $0 \leq \Pr(A) \leq 1 \qquad \forall$ valid $A \in \Omega$
- $\Pr(True) = 1 \qquad \Pr(False) = 0$
- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

A Closer Look at Axiom 3

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$



Venn diagram

Using the Axioms

$$
\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}
$$

Discrete Random Variables

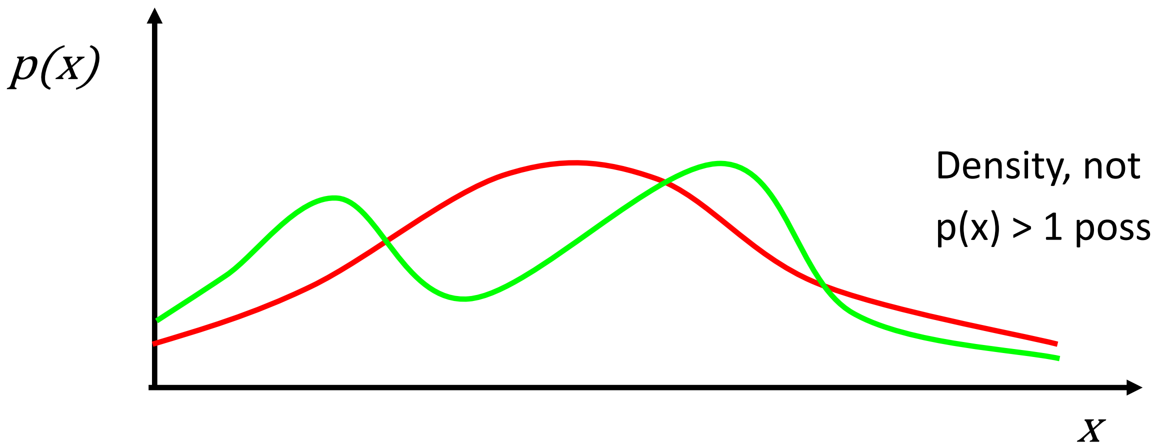
- X denotes a **random variable**.
- X can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the **probability** that the random variable X takes on value x_i.
- $P(\cdot)$ is called **probability mass function**.
- e.g. $$P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$$

Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$, or $p(x)$, is a **probability density function**.

$$\Pr(x \in (a,b)) = \int_a^b p(x)dx$$

- e.g.



Density, not probability!
p(x) > 1 possible

- **Cumulative distribution function** $P(X \leq x)$

Joint and Conditional Probability

- $P(x,y) = P(X=x \text{ and } Y=y)$
- If X and Y are **independent** then

$$P(x,y) = P(x)\,P(y)$$

- $P(x|y)$ is the probability of ***x* given *y***

$$P(x|y) = P(x,y) \,/\, P(y)$$
$$= P(y|x)\,P(x) \,/\, P(y)$$

- If X and Y are **independent** then

$$P(x|y) = P(x)$$

Law of Total Probability, Marginals

Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$\int p(x)dx = 1$$

$$p(x) = \int p(x, y)dy$$

$$p(x) = \int p(x \mid y) p(y)dy$$

Bayes Formula

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$P(x|y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

$$P(x|y) = \frac{P(y \mid x)P(x)}{\sum_{x'} P(y \mid x')P(x')}$$

Via Law of Total Probability

Normalization

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \eta\ P(y \mid x)\, P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)}$$

Algorithm:

$$\forall x : \mathrm{aux}_{x|y} = P(y \mid x)\ P(x)$$

$$\eta = \frac{1}{\sum_x \mathrm{aux}_{x|y}}$$

$$\forall x : P(x \mid y) = \eta\ \mathrm{aux}_{x|y}$$

Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

Conditioning

- Law of total probability:

$$p(x) = \int p(x, z) dz$$

$$p(x) = \int p(x | z) p(z) dz$$

$$p(x | y) = \int p(x | y, z) p(z | y) dz$$

Conditional Independence

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

equivalent to $$P(x \mid z) = P(x \mid z, y)$$

and $$P(y \mid z) = P(y \mid z, x)$$

Checkpoint

#Legs	Species	*P(L=#Legs, S=Species)*
2	Dog	0.001
2	Cat	0.001
2	Bird	0.2
3	Dog	0.057
3	Cat	0.04
3	Bird	0.001
4	Dog	0.4
4	Cat	0.3
4	Bird	0

- $P(\#legs = 2 \vee \#legs = 3 \vee \#legs = 4)$
- $P(Dog \vee Cat \vee Bird)$
- $P(Bird)$
- $P(Bird, \#legs = 2)$
- $P(Bird \mid \#legs = 2)$
- $P(\#legs = 2|Bird)$