Kalman Filter

state at time t $\quad x_t \quad$ n-vector $\qquad\qquad$ $z \quad \tilde{z}$

control $\quad u_t \quad$ m-vector

measurement $\quad z_t \quad$ k-vector

State update:
$$\underset{n\times 1}{x_t} = \underset{n\times n}{A_t}\,\underset{n\times 1}{x_{t-1}} + \underset{n\times m}{B_t}\,\underset{m\times 1}{u_t} + \underset{n\times 1}{\varepsilon_t}$$

└ process noise, covariance R_t

└ how control changes the state from $t-1$ to t

└ how state evolves from $t-1$ to t without control + noise

Measurement update:
$$\underset{k\times 1}{z_t} = \underset{k\times n}{C_t}\,\underset{n\times 1}{x_t} + \underset{k\times 1}{\delta_t}$$

└ measurement noise, covariance Q_t

└ projection of state to an observation

independent
normally distributed
0-mean

Initial State: $Bel(x_0) = p(x_0) = \mathcal{N}(x_0; \mu_0, \Sigma_0)$

Bayes Filter: $Bel(x_t) = p(x_t \mid u_{1:t}, z_{1:t})$

$$= \eta\, p(z_t \mid x_t) \overbrace{\int \underbrace{p(x_t \mid u_t, x_{t-1}) \cdot Bel(x_{t-1})}_{p(x_t\, x_{t-1} \mid u_{1:t},\, z_{1:t-1})}\, dx_{t-1}}^{\overline{Bel}(x_t)}$$

State Update

$P(x_t, x_{t-1} \mid u_{1:t}, z_{1:t-1})$ followed by marginalization

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} A_t \\ I \end{bmatrix} x_{t-1} + \begin{bmatrix} B_t \\ 0 \end{bmatrix} u_t + \begin{bmatrix} I \\ 0 \end{bmatrix} \varepsilon_t$$

$$\xi = F_t x_{t-1} + G_t u_t + H_t \varepsilon_t$$

$$E[\varepsilon_t \varepsilon_t^T] = E[(\varepsilon_t - \overset{0}{\mu_{\varepsilon_t}})(\varepsilon_t - \mu_{\varepsilon_t})^T] = R_t$$

$$\mu_\xi = E[\xi] = F_t \mu_{t-1} + G_t u_t + \cancel{H_t 0} = \boxed{\begin{bmatrix} A_t \\ I \end{bmatrix} \mu_{t-1} + \begin{bmatrix} B_t \\ 0 \end{bmatrix} u_t}$$

$$\Sigma_\xi = E[(\xi - \mu_\xi)(\xi - \mu_\xi)^T] = E[((F_t(x_{t-1} - \mu_{t-1}) + H_t \varepsilon_t)(\ldots)^T]$$

$$= F_t \Sigma_{t-1} F_t^T + F_t \cancel{E[(x_{t-1} - \mu_{t-1}) \varepsilon_t^T]} H_t^T + H_t \cancel{E[\varepsilon_t (x_{t-1} - \mu_{t-1})^T]} \cdot F_t^T + H_t R_t H_t^T$$

0 under the assumption that ε is independent of x

$$= \begin{bmatrix} A_t \\ I \end{bmatrix} \Sigma_{t-1} [A_t^T \; I] + \begin{bmatrix} I \\ 0 \end{bmatrix} R_t [I \; 0]$$

$$= \begin{bmatrix} \boxed{A_t \Sigma_{t-1} A_t^T + R_t} & A_t \Sigma_{t-1} \\ \Sigma_{t-1} A_t^T & \Sigma_{t-1} \end{bmatrix}$$

marginalization:

$$\left.\begin{aligned} \bar{\mu}_t &= A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t &= A_t \Sigma_{t-1} A_t^T + R_t \end{aligned}\right\} \text{Kalman prediction}$$

$$\overline{Bel}(x_t) = p(x_t \mid u_{1:t}, z_{1:t-1})$$

$$= \int p(x_t, x_{t-1} \mid u_{1:t}, z_{1:t-1}) \, dx_{t-1} \sim \mathcal{N}(x_t; \bar{\mu}_t, \bar{\Sigma}_t)$$

Measurement Update

$p(x_t, z_t \mid z_{1:t-1}, u_{1:t})$ followed by conditioning $p(x_t \mid z_t, z_{1:t-1}, u_{1:t}) = Bel(x_t)$

$p(z_t \mid x_t)\,\overline{Bel}(x_t)$

$z_t = C_t x_t + \delta_t$

$$\begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} I \\ C_t \end{bmatrix} x_t + \begin{bmatrix} 0 \\ I \end{bmatrix} \delta_t$$

$$\xi_t = D_t x_t + E_t \delta_t$$

$D_t(x_t - \bar{\mu}_t) + E_t \delta_t$

$$E[\delta_t \delta_t^T] = E[(\delta_t - \overset{0}{\mu_{\delta_t}})(\delta_t - \mu_{\delta_t})^T] = Q_t$$

$$\mu_\xi = E[\xi_t] = D_t \bar{\mu}_t = \begin{bmatrix} \bar{\mu}_t \\ C_t \bar{\mu}_t \end{bmatrix}$$

$$\Sigma_\xi = E\{(\xi_t - \mu_\xi)(\xi_t - \mu_\xi)^T\} = D_t \bar{\Sigma}_t D_t^T + E_t Q_t E_t^T$$

$$= \begin{array}{cc} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \alpha \\ \beta \end{array} & \begin{bmatrix} \bar{\Sigma}_t & \bar{\Sigma}_t C_t^T \\ C_t \bar{\Sigma}_t & C_t \bar{\Sigma}_t C_t^T + Q_t \end{bmatrix} \end{array}$$

$\begin{array}{|c|c|} \hline \alpha\alpha & \alpha\beta \\ \hline \beta\alpha & \beta\beta \\ \hline \end{array}$

$$\mu = \mu_\alpha + \Sigma_{\alpha\beta} \Sigma_{\beta\beta}^{-1} (\beta - \mu_\beta)$$

$$\Sigma = \Sigma_{\alpha\alpha} - \Sigma_{\alpha\beta} \Sigma_{\beta\beta}^{-1} \Sigma_{\beta\alpha}$$

$$Bel(x_t) = p(x_t \mid z_t, z_{1:t-1}, u_{1:t}) \sim N(x_t; \mu_t, \Sigma_t)$$

conditioning:

$$\mu_t = \bar{\mu}_t + \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1} (z_t - C_t \bar{\mu}_t)$$

$$\Sigma_t = \bar{\Sigma}_t - \underbrace{\bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}}_{K_t \text{ Kalman gain}} C_t \bar{\Sigma}_t$$

Kalman update:

$$\left.\begin{aligned} \mu_t &= \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t) \\ \Sigma_t &= \bar{\Sigma}_t - K_t C_t \bar{\Sigma}_t \\ &= (I - K_t C_t) \bar{\Sigma}_t \end{aligned}\right\}$$

$$
\begin{tikzcd}
x_{t-1} & & x_t & & x_t \\
\mu_{t-1}, \Sigma_{t-1} \arrow[rr, "\text{control } u_t"] & & \bar{\mu}_t, \bar{\Sigma}_t \arrow[rr, "\text{measurement}"] & & \mu_t, \Sigma_t \arrow[llll, bend left, "\text{recursively update}"]
\end{tikzcd}
$$