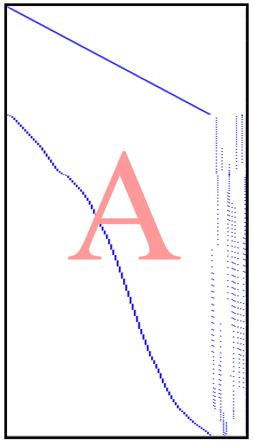
SLAM and Graphical Models

Robot Localization and Mapping
16-833

Michael Kaess

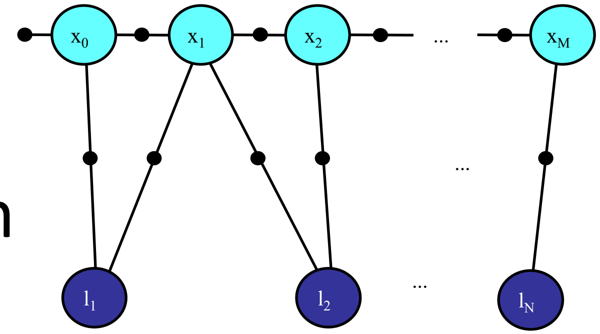
April 12+14, 2021

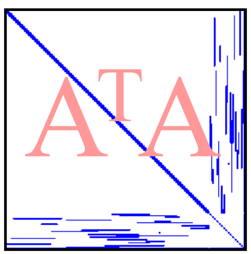
Matrix vs. Graph



Measurement Jacobian

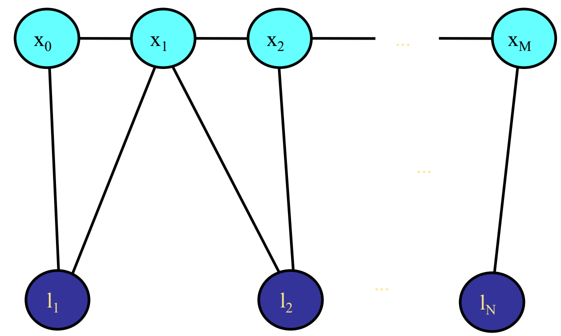
Factor Graph

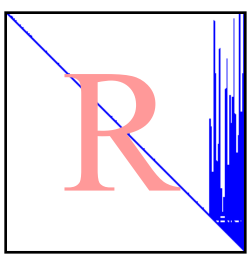




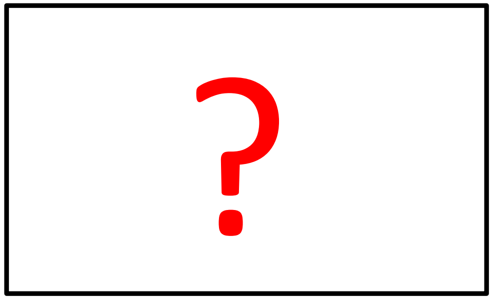
Information Matrix

Markov Random Field





Square Root Inf. Matrix



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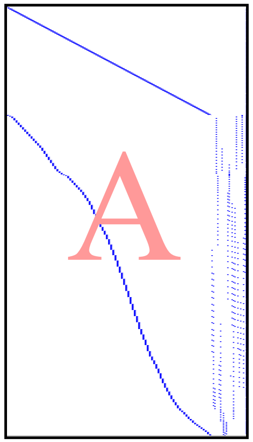

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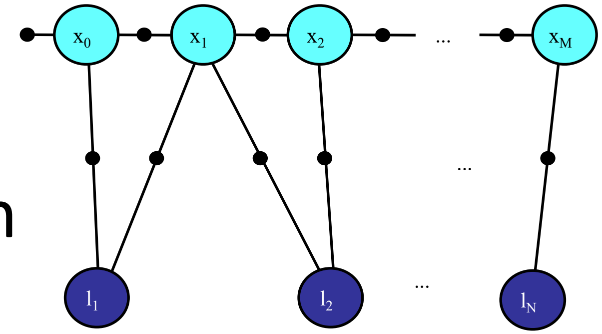

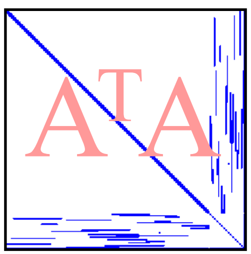

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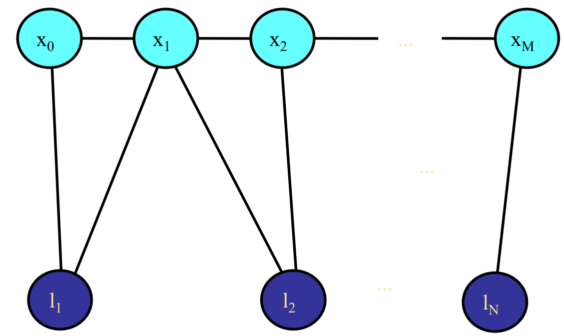
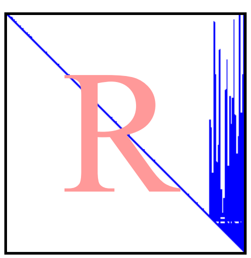
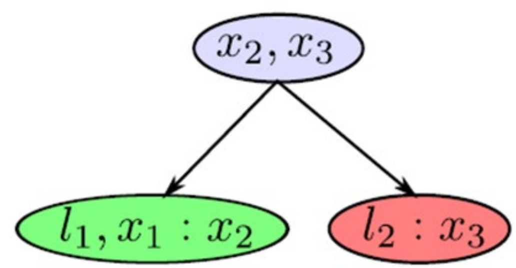


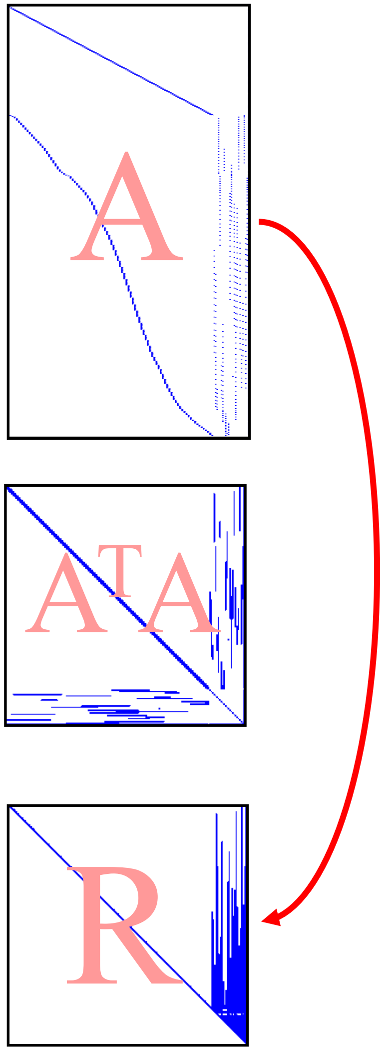


Square Root Inf. Matrix

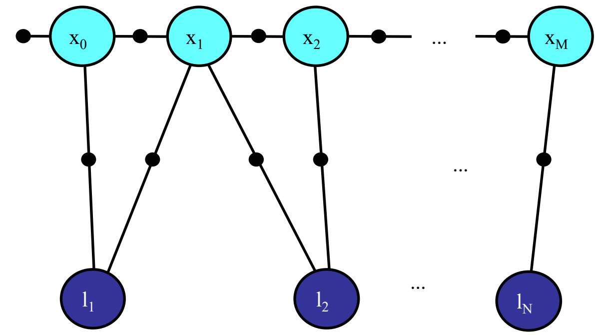
Bayes Tree

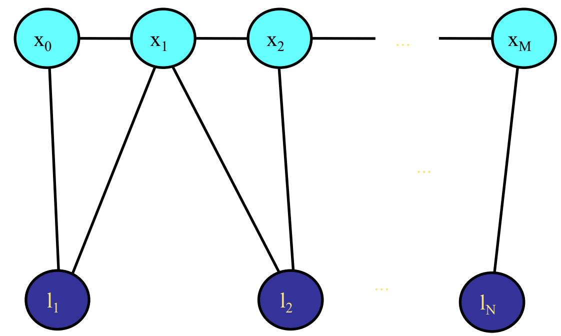


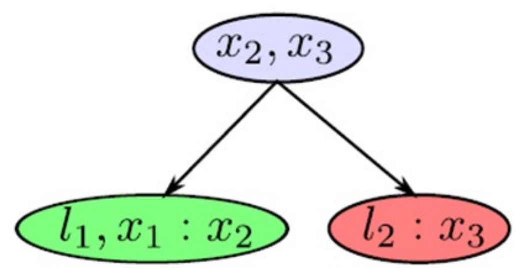
Matrix vs. Graph



Matrix factorization







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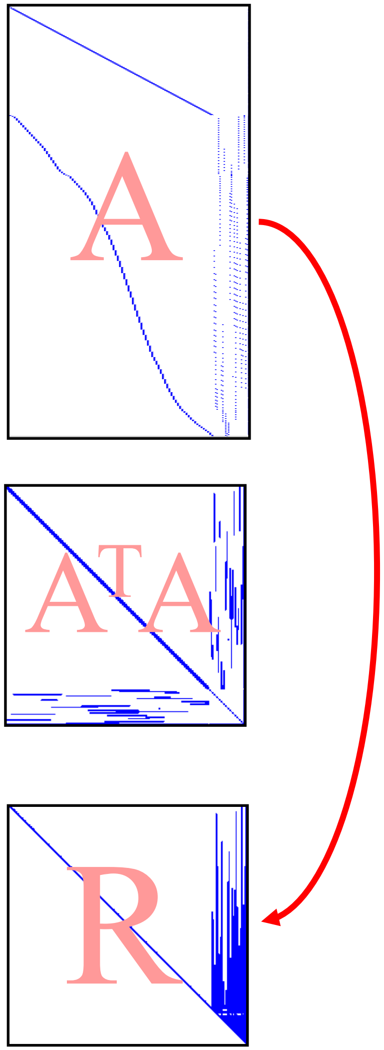
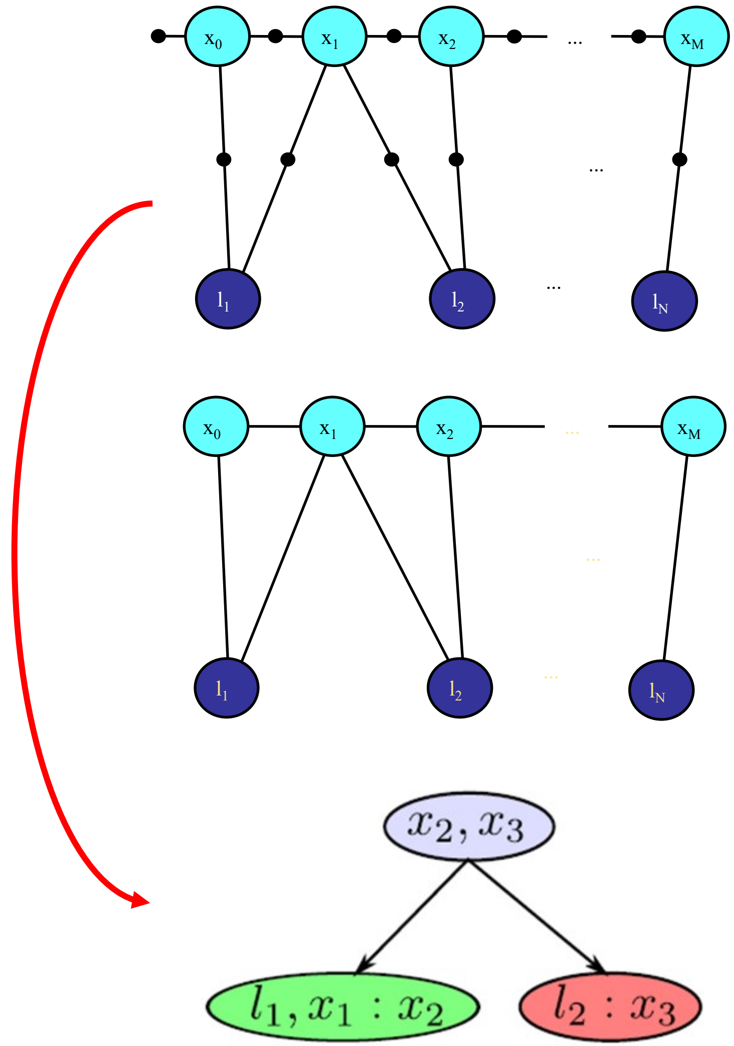
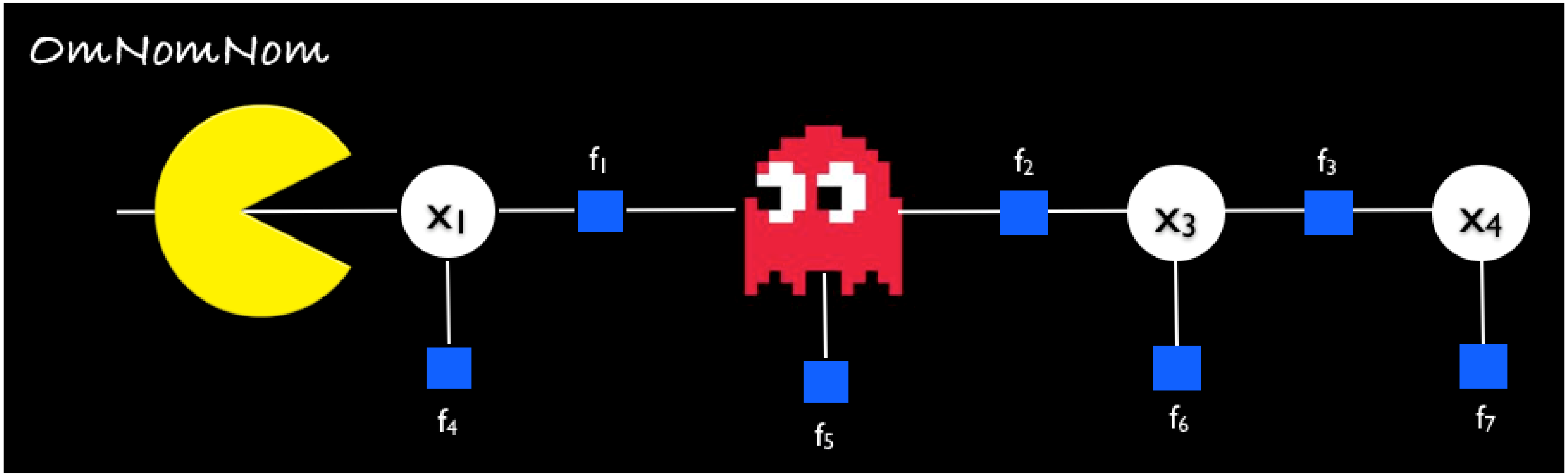


Matrix factorization

Variable elimination



Variable Elimination



Courtesy of Daniel Kohlsdorf (Georgia Tech)

iSAM2: Bayes Tree

Inference in tree structure is easy

Idea: Convert factor graph to tree structure

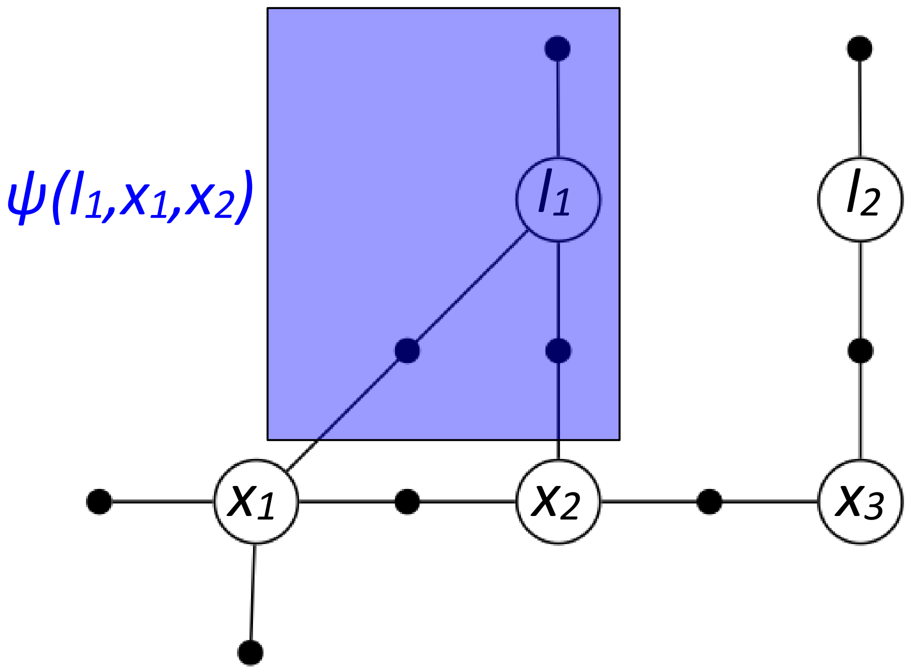
Two stage process:

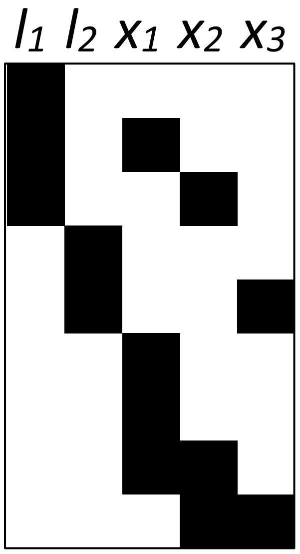
- Variable elimination converts factor graph to Bayes net
- Discovering cliques provides Bayes tree

"iSAM2: Incremental Smoothing and Mapping Using the Bayes Tree"
M. Kaess, H. Johannsson, R. Roberts, V. Ila, J.J. Leonard, and F. Dellaert.
IJRR 2012

Variable Elimination – Example

- Choose ordering: l_1, l_2, x_1, x_2, x_3
- Eliminate one node at a time



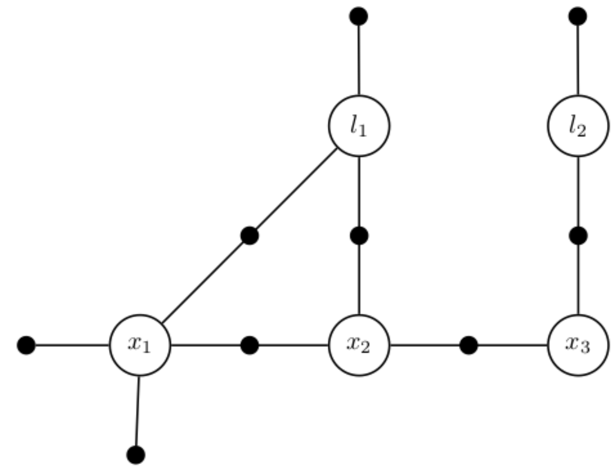


$$\psi(l_1,x_1,x_2) = \psi(l_1|x_1,x_2)\ \psi(x_1,x_2)$$

Variable Elimination

Factorization of the factors connected to l_1

(on the board)



$$[A|b] = \begin{array}{c} \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{array} \begin{array}{c} \begin{array}{cccccc} \delta l_1 & \delta l_2 & \delta x_1 & \delta x_2 & \delta x_3 & rhs \end{array} \\ \left[\begin{array}{ccccc|c} & & A_{13} & & & b_1 \\ & & A_{23} & A_{24} & & b_2 \\ & & & A_{34} & A_{35} & b_3 \\ A_{41} & & & & & b_4 \\ & A_{52} & & & & b_5 \\ & & A_{63} & & & b_6 \\ A_{71} & & A_{73} & & & b_7 \\ A_{81} & & & A_{84} & & b_8 \\ & A_{92} & & & A_{95} & b_9 \end{array}\right] \end{array}$$

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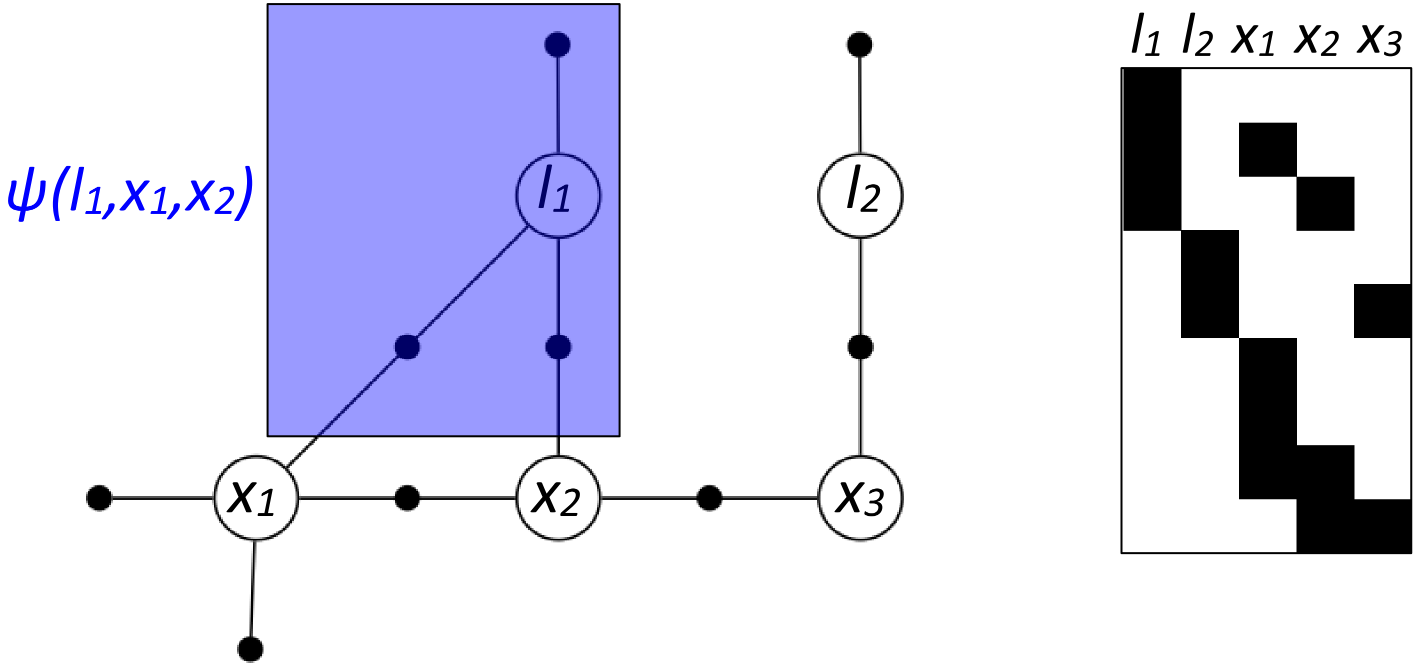

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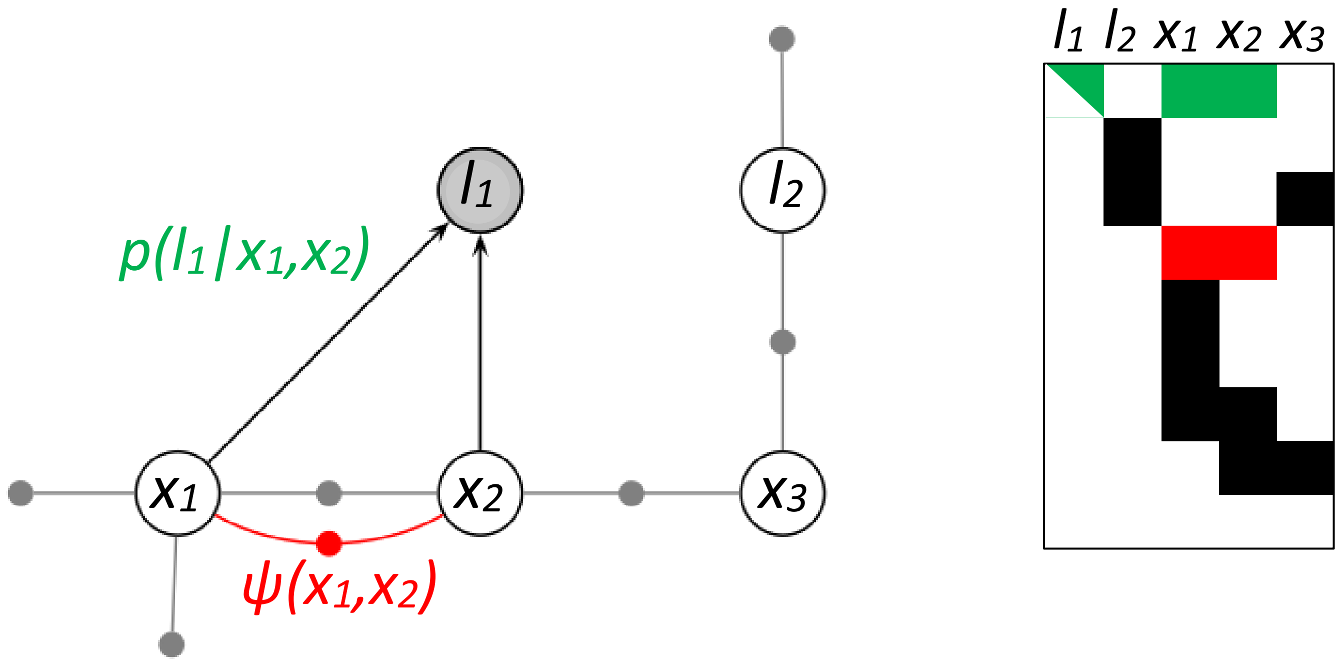

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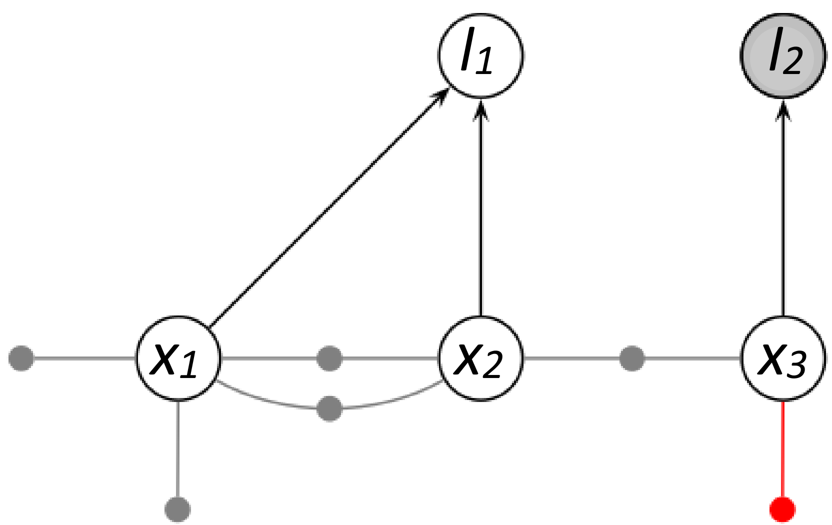

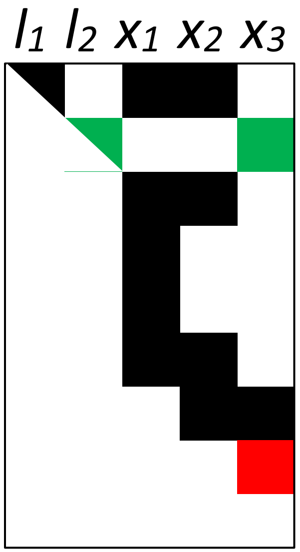

$\psi(l_2,x_3) = \psi(l_2|x_3)\ \psi(x_3)$

Variable Elimination – Example

- Choose ordering: l_1, l_2, x_1, x_2, x_3
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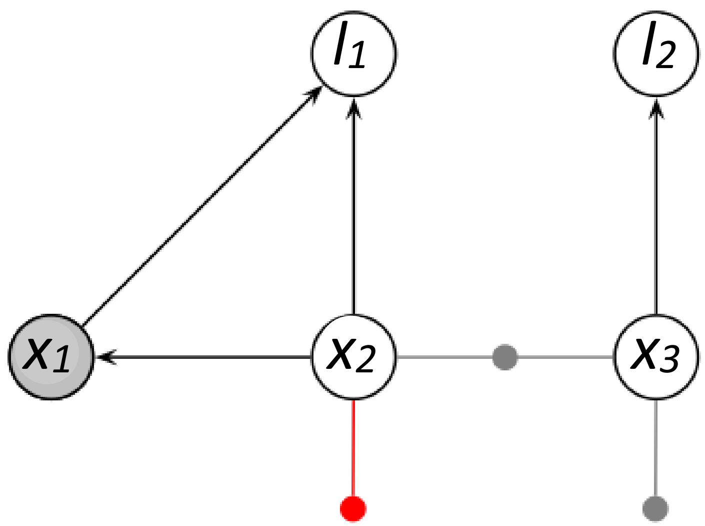

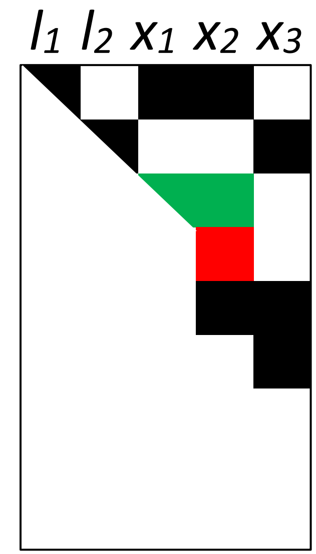

$$\psi(x_1,x_2) = \psi(x_1|x_2)\,\psi(x_2)$$

Variable Elimination – Example

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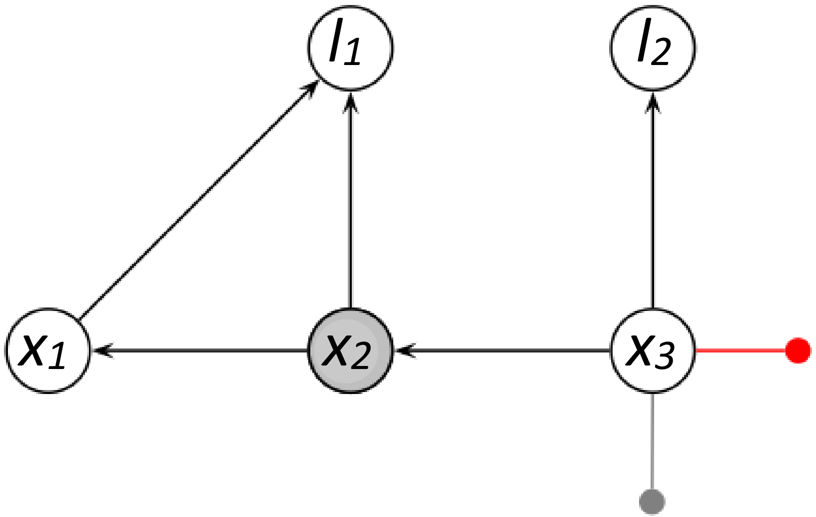

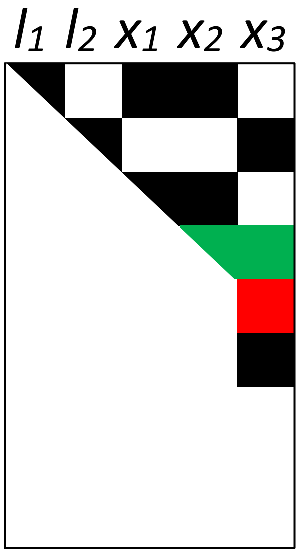

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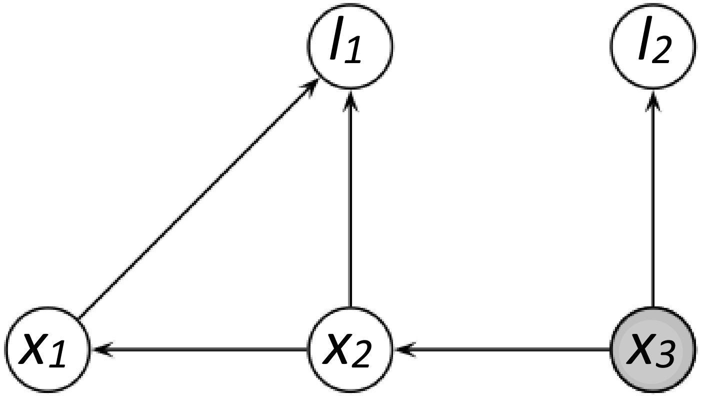
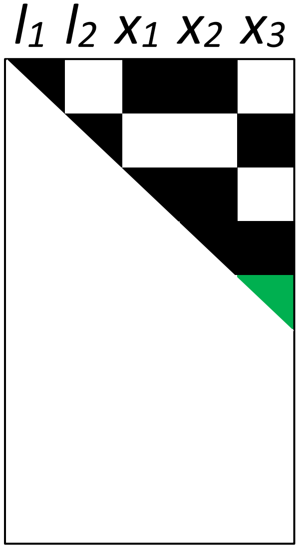




$\psi(x_3)$

Variable Elimination – Algorithm

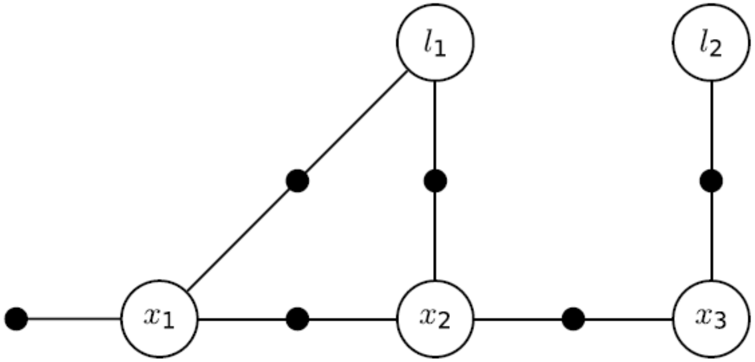
Algorithm 3.1 The Variable Elimination Algorithm

1: **function** ELIMINATE($\Phi_{1:n}$) ▷ given a factor graph on n variables
2: **for** $j = 1...n$ **do** ▷ for all variables
3: $p(x_j|S_j), \Phi_{j+1:n} \leftarrow \text{EliminateOne}(\Phi_{j:n}, x_j)$ ▷ eliminate x_j
4: **return** $p(x_1|S_1)p(x_2|S_2)\ldots p(x_n)$ ▷ return Bayes net

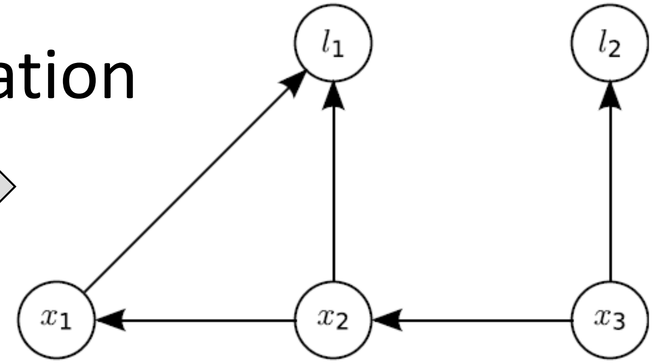
Algorithm 3.2 Eliminate variable x_j from a factor graph $\Phi_{j:n}$.

1: **function** ELIMINATEONE($\Phi_{j:n}, x_j$) ▷ given reduced graph $\Phi_{j:n}$
2: Remove all factors $\phi_i(X_i)$ that are adjacent to x_j
3: $\mathcal{S}(x_j) \leftarrow$ all variables involved excluding x_j ▷ the separator
4: $\psi(x_j, S_j) \leftarrow \prod_i \phi_i(X_i)$ ▷ create the product factor ψ
5: $p(x_j|S_j)\tau(S_j) \leftarrow \psi(x_j, S_j)$ ▷ factorize the product ψ
6: Add the new factor $\tau(S_j)$ back into the graph
7: **return** $p(x_j|S_j), \Phi_{j+1:n}$ ▷ Conditional and reduced graph

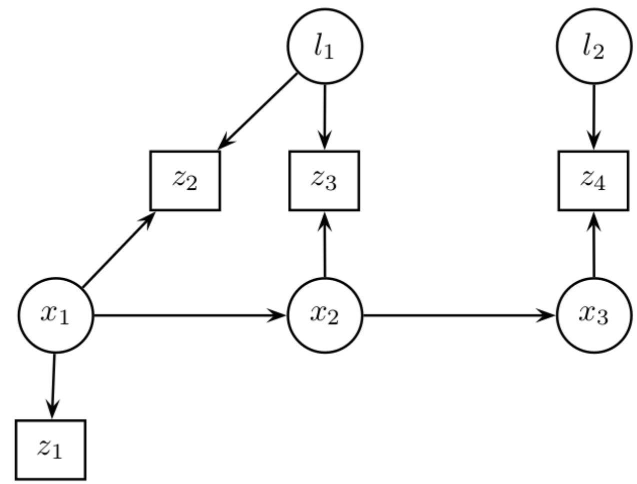
Bayes Tree Data Structure



Elimination



Not the same as the original Bayes net!



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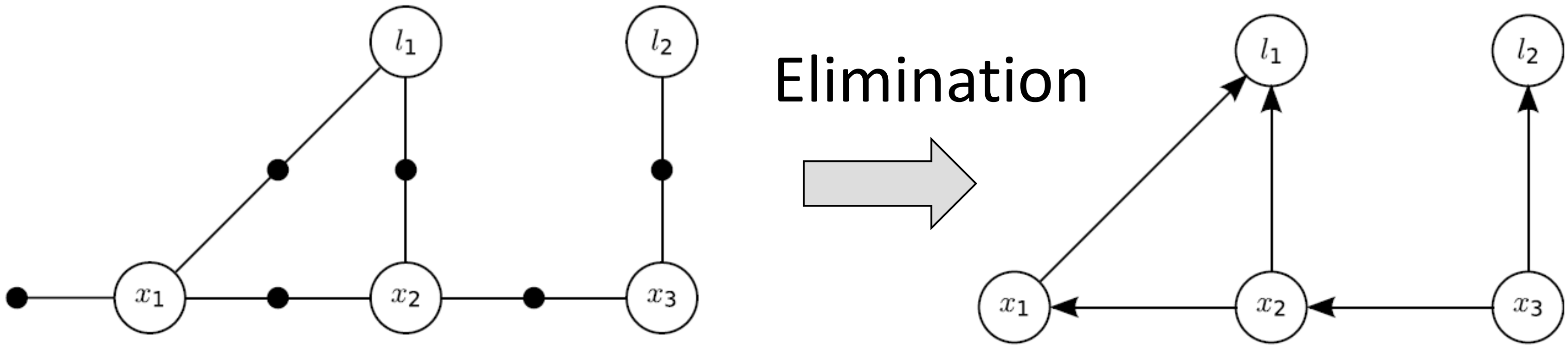
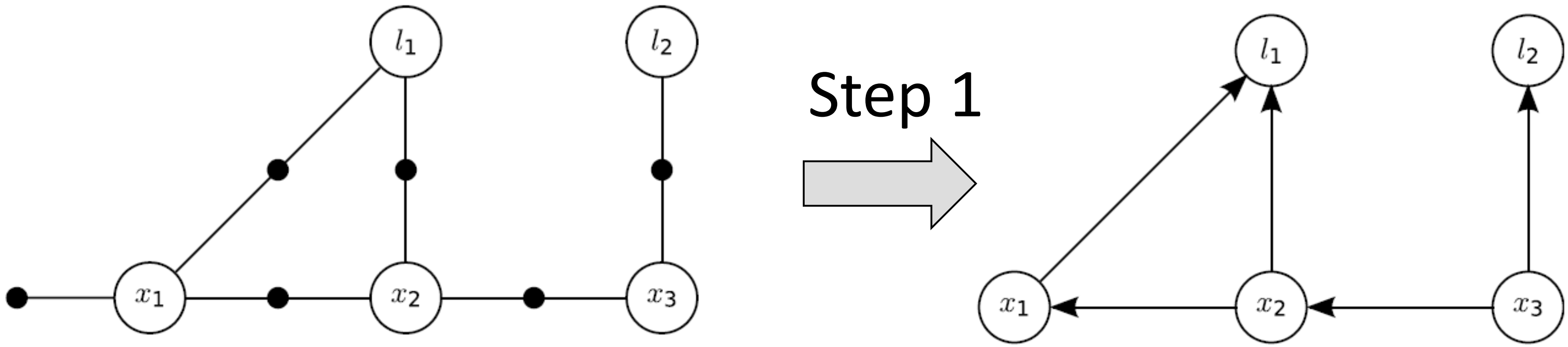


The Bayes net has a special property: its undirected equivalent is chordal by construction

Chordal: There is no cycle greater than 3 that has no shortcut

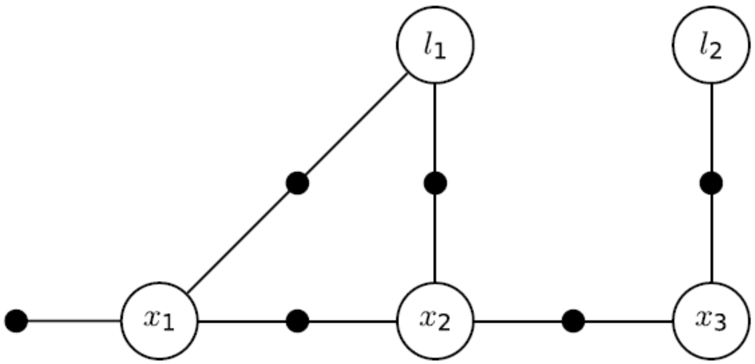
[Kaess et al., IJRR 12]

Bayes Tree Data Structure



Step 2: Find cliques in reverse elimination order:

Bayes Tree Data Structure



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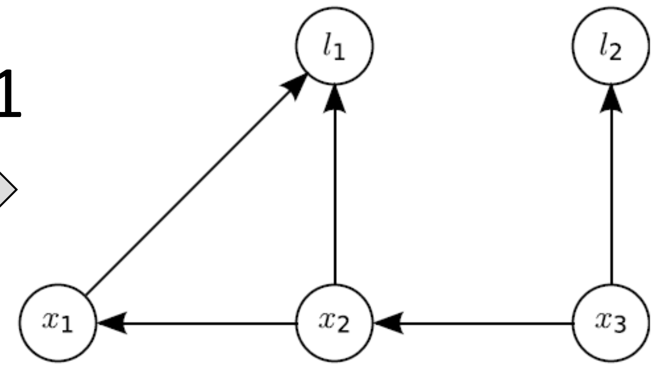

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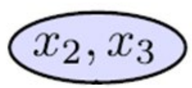
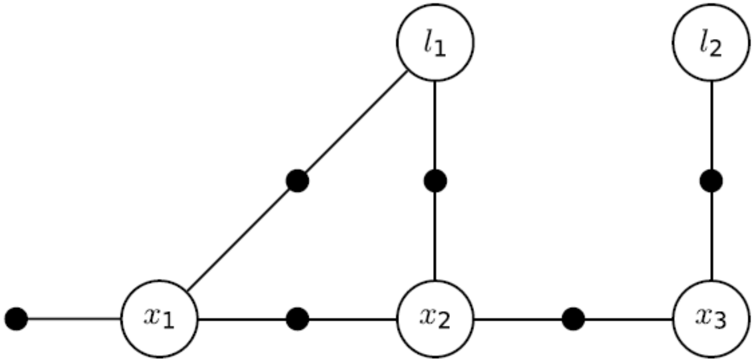


[Kaess et al., IJRR 12]

Bayes Tree Data Structure



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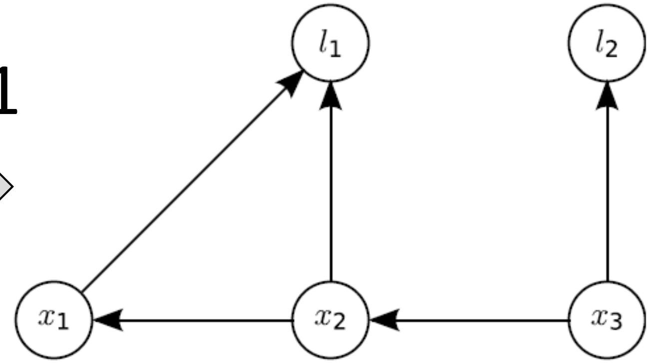

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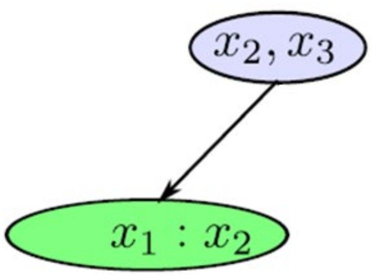

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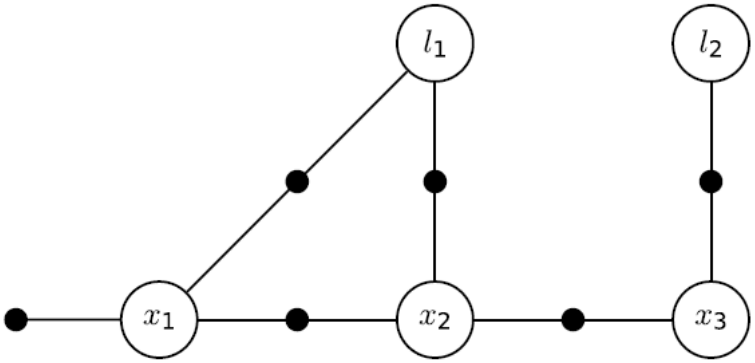

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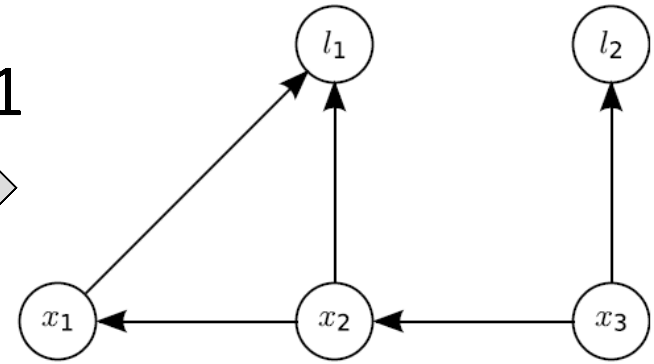
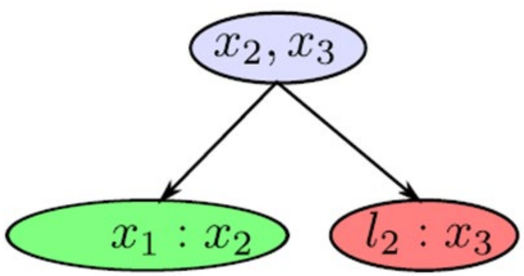


Step 2: Find cliques in reverse elimination order:



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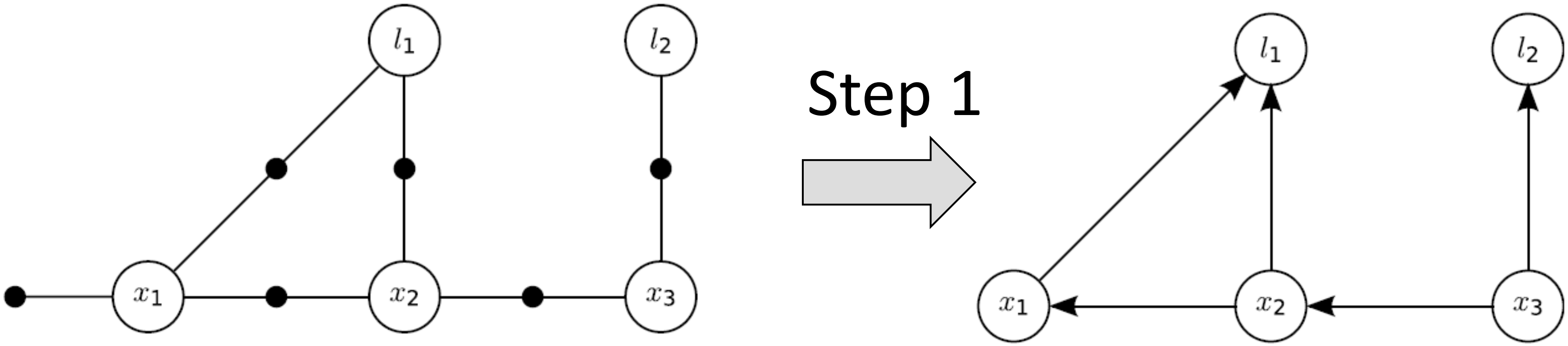

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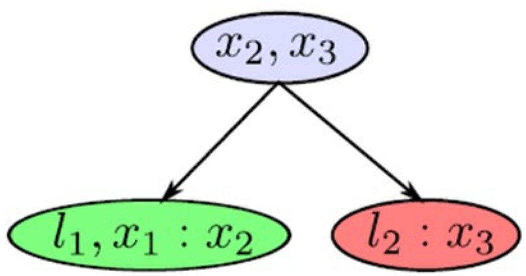

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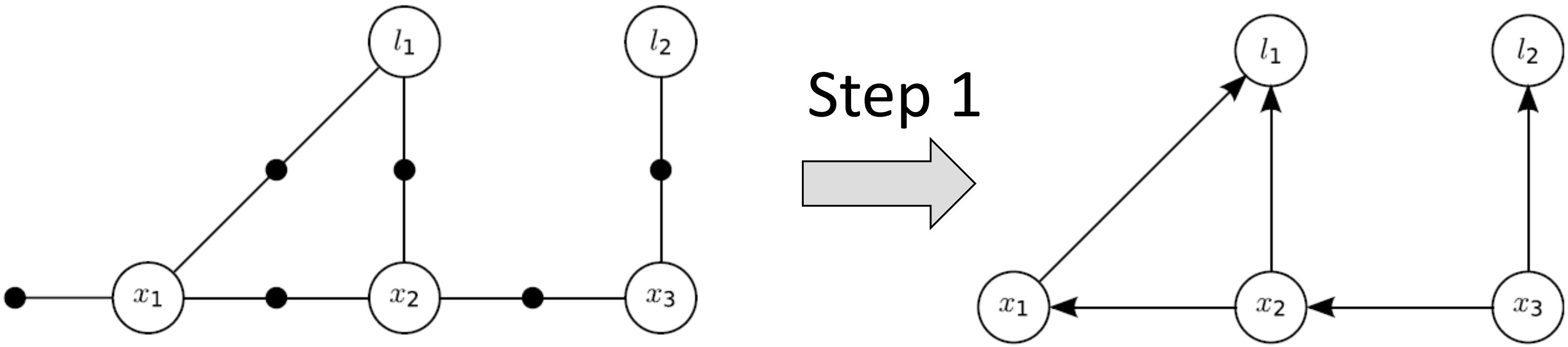

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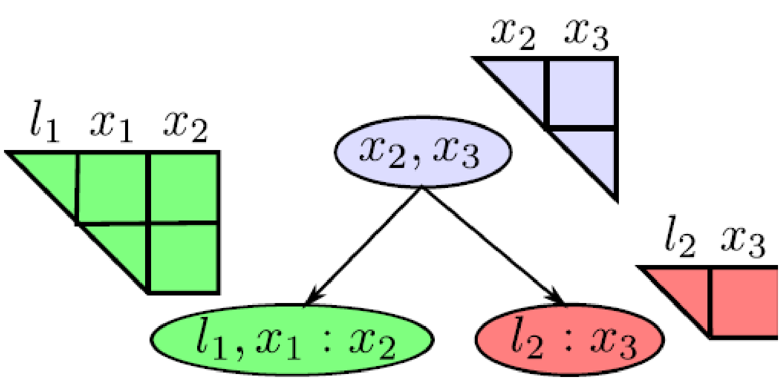
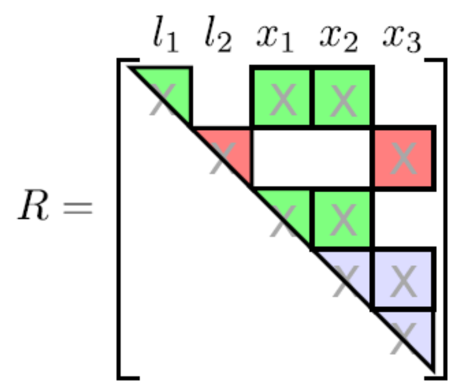




$$P(x_j|S_j) \propto \exp\left\{-\frac{1}{2\sigma^2}\left(x_j + rS_j - d\right)^2\right\}$$

Bayes Tree – Algorithm

Alg. 3 Creating a Bayes tree from the chordal Bayes net resulting from elimination (Alg. 2).

For each conditional density $P(\theta_j|S_j)$ of the Bayes net, in *reverse* elimination order:

If no parent ($S_j = \{\}$)

  start a new root clique F_r containing θ_j

else

  identify parent clique C_p that contains the first eliminated variable of S_j as a frontal variable

  if nodes $F_p \cup S_p$ of parent clique C_p are equal to separator nodes S_j of conditional

    insert conditional into clique C_p

  else

    start new clique C' as child of C_p containing θ_j

Bayes Tree Example

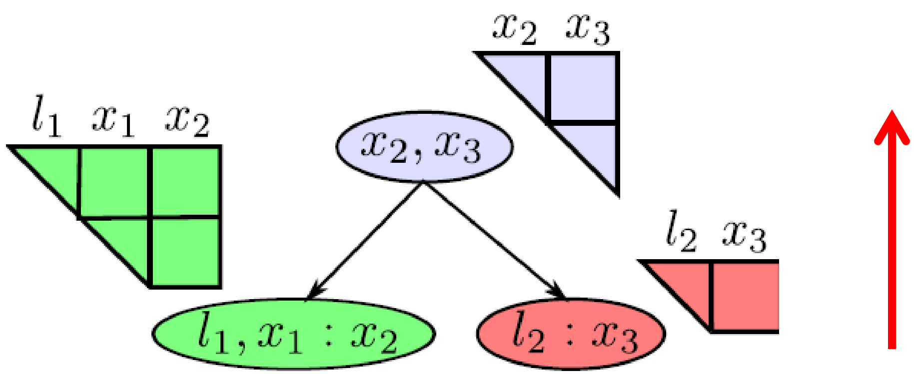
- On the board:
  - Example with 4 nodes, 3 different orderings

Question

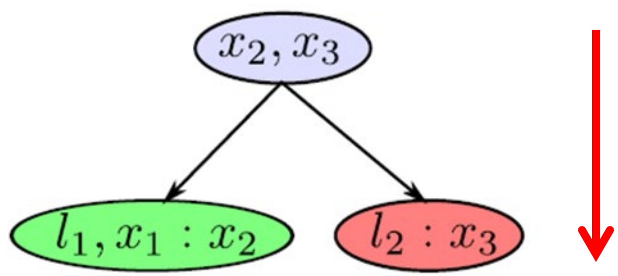
- How to do backsubstitution in the graph?

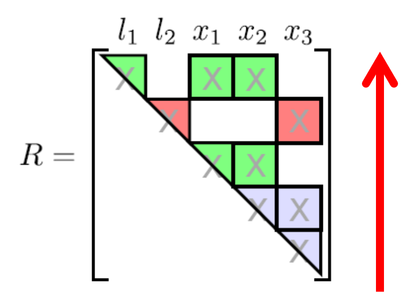
Backsubstitution in the Graph

- Inference is a two-step process:
  - Elimination starts at leaves and proceeds to the root



  - Solving starts at root and proceeds to the leaves





Question

- In the matrix factorization, which entries correspond to the root in the Bayes tree?

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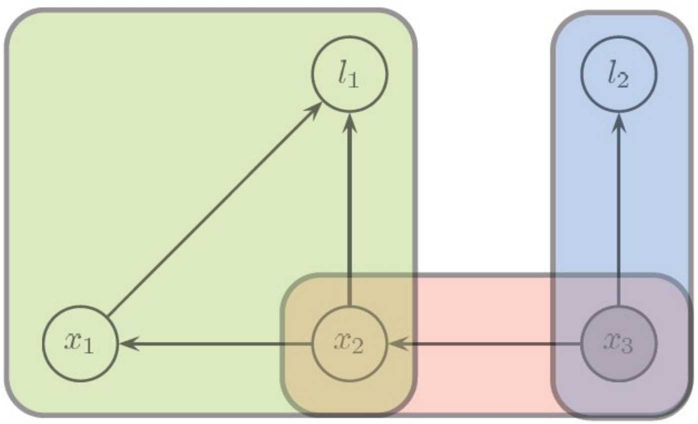
- In the matrix factorization, which entries correspond to the root in the Bayes tree?

  The bottom-/right-most entries correspond to the root – they are not conditioned on any other variables

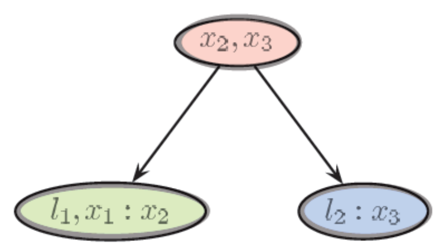
Bayes Tree vs. Junction Tree/Clique Tree

BT = direct(ed) result from elimination

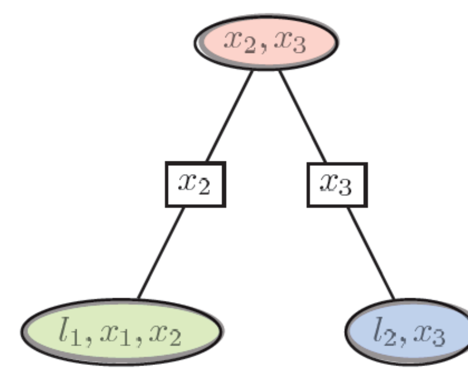
More intuitive, directly encodes square root inform. factor, but also less general: reflects an ordering



Chordal Bayes Net and cliques

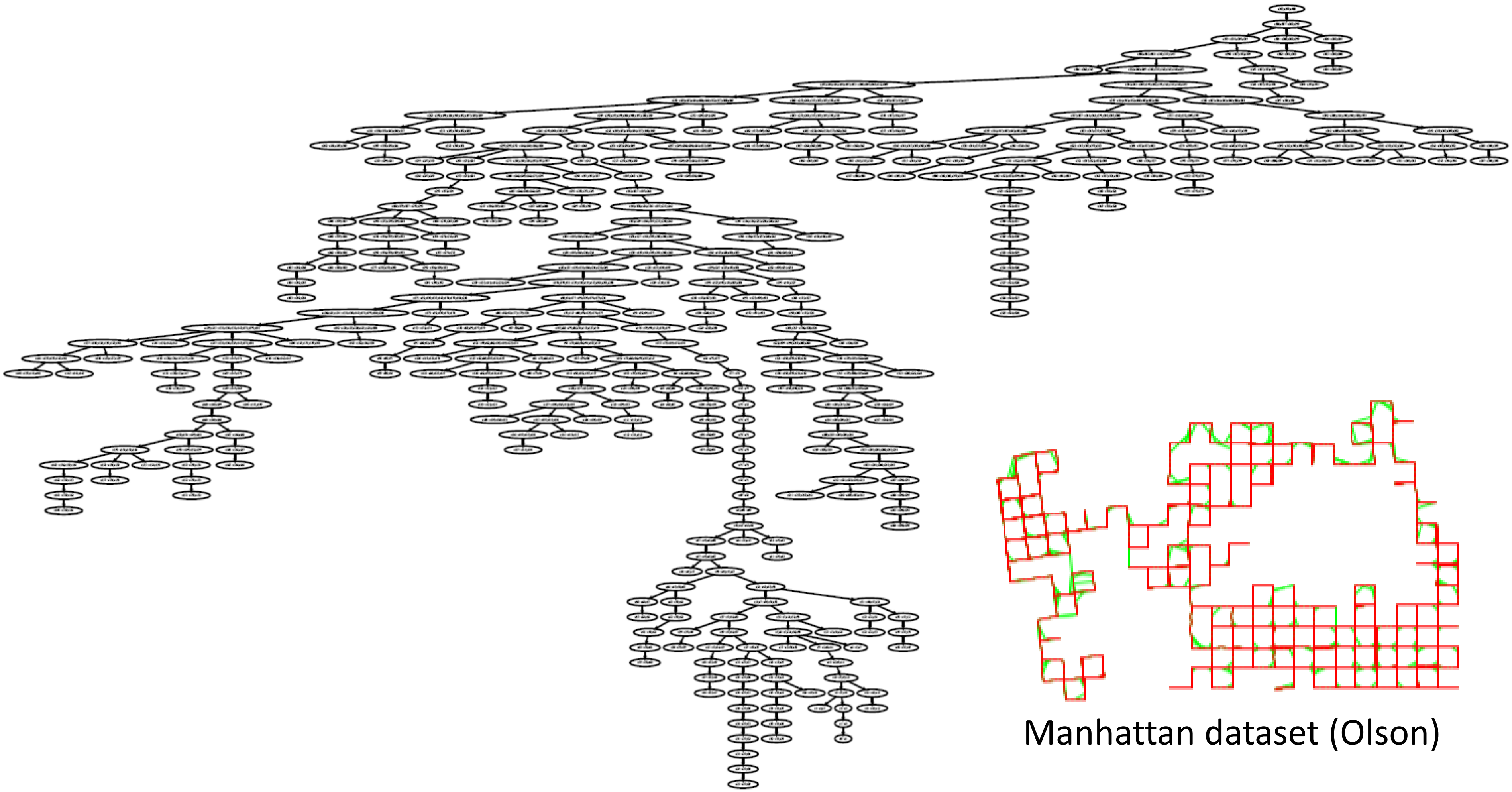


Bayes Tree



Junction Tree

iSAM2: Bayes Tree Example



Manhattan dataset (Olson)

Complexity depends on the size of the largest clique