Solution of INTEGRATION LECTURE 3

1) $\int x \ln x \, dx$

$= \ln x \int x \, dx - \int \left\{ \frac{d}{dx}(\ln x) \int x \, dx \right\} dx$

$= \ln x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx$

$= \ln x . \frac{x^2}{2} - \frac{1}{2} \int x \, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} . \frac{x^2}{2} + c$

$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$

2) $\int \left(e^{\sqrt{x}} - e^{-\sqrt{x}} \right) dx$

$= \int (e^z - e^{-z}) \, 2z \, dz$ — Let, $\sqrt{x} = z$

$= 2 \left[z \int (e^z - e^{-z}) \, dz - \int \left\{ \frac{d}{dz}(z) \int (e^z - e^{-z}) \, dz \right\} dz \right]$ — $\Rightarrow \frac{1}{2\sqrt{x}} dx = dz$

$= 2 \left[z(e^z + e^{-z}) - \int (e^z + e^{-z}) \, dz \right]$ — $\Rightarrow dx = 2\sqrt{x} dz$

$= 2[z(e^z + e^{-z}) - (e^z - e^{-z})] + c$ — $\Rightarrow dx = 2z dz$

$= 2\left[\sqrt{x}\left(e^{\sqrt{x}} + e^{-\sqrt{x}}\right) - \left(e^{\sqrt{x}} - e^{-\sqrt{x}}\right) \right] + c$

3) $\int (\sin^{-1} x)^2 \, dx$

$= (\sin^{-1} x)^2 \int 1 \, dx - \int \left\{ \frac{d}{dx} (\sin^{-1} x)^2 \int 1 \, dx \right\} dx$ — Let, $\sin^{-1} x = z$

$= (\sin^{-1} x)^2 . x - \int 2 \sin^{-1} x \frac{1}{\sqrt{1 - x^2}} \; x \; dx$ — $\Rightarrow \frac{1}{\sqrt{1 - x^2}} dx = dz$

$= (\sin^{-1} x)^2 . x - 2 \int z \sin z \; dz$ — and, $x = \sin z$

$$= x\,(\sin^{-1}x)^2 - 2\left[z\int \sin z\ dz - \int \left\{\frac{d}{dz}(z)\int \sin z\ dz\right\} dz\right]$$

$$= x\,(\sin^{-1}x)^2 - 2\left[-z\cos z + \int \cos z\, dz\right]$$

$$= x\,(\sin^{-1}x)^2 - 2[-z\cos z + \sin z] + c$$

$$= x\,(\sin^{-1}x)^2 - 2[-(\sin^{-1}x)\cos(\sin^{-1}x) + \sin(\sin^{-1}x)] + c$$

4) $\int x^2 \sin^2 x\ dx$

$$= \frac{1}{2}\int x^2\,(1-\cos 2x)\ dx$$

$$= \frac{1}{2}\int x^2\ dx - \frac{1}{2}\int x^2 \cos 2x\ dx$$

$$= \frac{1}{2}.\frac{x^3}{3} - \frac{1}{2}\left[x^2\int \cos 2x\ dx - \int \left\{\frac{d}{dx}x^2\int \cos 2x\, dx\right\} dx\right]$$

$$= \frac{x^3}{6} - \frac{1}{2}\left[x^2\frac{\sin 2x}{2} - \int 2x\frac{\sin 2x}{2}dx\right]$$

$$= \frac{x^3}{6} - \frac{x^2\sin 2x}{4} + \frac{1}{2}\int x\sin 2x\, dx$$

$$= \frac{x^3}{6} - \frac{x^2\sin 2x}{4} + \frac{1}{2}\left[x\int \sin 2x\ dx - \int \left\{\frac{d}{dx}x\int \sin 2x\, dx\right\} dx\right]$$

$$= \frac{x^3}{6} - \frac{x^2\sin 2x}{4} + \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) + \int \left(\frac{\cos 2x}{2}\right) dx\right]$$

$$= \frac{x^3}{6} - \frac{x^2\sin 2x}{4} - \frac{x\cos 2x}{4} + \frac{\sin 2x}{8} + c$$

5) $\int \frac{x}{1+\cos x}\ dx$

$$= \int \frac{x}{2\cos^2\frac{x}{2}}\ dx$$

$$= \frac{1}{2}\int\ x\sec^2\frac{x}{2}\ dx$$

$$= \frac{1}{2}\left[x\int \sec^2\frac{x}{2}\ dx - \int \left\{\frac{d}{dx}x\int \sec^2\frac{x}{2}\ dx\right\} dx\right]$$

$$= \frac{1}{2}\left[x\frac{\tan\frac{x}{2}}{\frac{1}{2}} - \int \frac{\tan\frac{x}{2}}{\frac{1}{2}}\,dx\right]$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} - \int 2\,\frac{\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}}{\sec\frac{x}{2}}\,dx$$

$$= x\tan\frac{x}{2} - 2\ln\left|\sec\frac{x}{2}\right| + c$$

6) $\displaystyle\int \frac{x+\sin x}{1+\cos x}\,dx$

$$= \int \frac{x + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} + c$$

7) $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$

$$= \int \ln z\,dz \qquad\qquad \text{Let, } \ln x = z$$

$$= \ln z\int 1\,dz - \int\left\{\frac{d}{dz}(\ln z)\int 1\,dz\right\}dz \qquad\qquad \Rightarrow \frac{1}{x}\,dx = dz$$

$$= \ln z\,.\,z - \int \frac{1}{z}.\,z\,dz$$

$$= z\ln z - \int 1\,dz$$

$$= z\ln z - z + c$$

$$= \ln x\,(\ln(\ln x)) - \ln x + c$$

8) $\int \frac{\ln(x+1)}{\sqrt{x+1}}\ dx$

$= \ln(x+1)\int \frac{1}{\sqrt{x+1}}\ dx - \int\left\{\frac{d}{dx}(\ln(x+1))\int \frac{1}{\sqrt{x+1}}\ dx\right\}dx$

$= \ln(x+1)\,2\sqrt{x+1} - \int \frac{1}{x+1}.2\sqrt{x+1}\,dx$

$= 2\sqrt{x+1}\ln(x+1) - 2\int \frac{1}{\sqrt{x+1}}\ dx$

$= 2\sqrt{x+1}\ln(x+1) - 4\sqrt{x+1} + c$

9) $\int \frac{x}{\sec x + 1}\ dx$

$= \int \frac{x}{\frac{1}{\cos x}+1}\ dx$

$= \int \frac{x\cos x}{1+\cos x}\ dx$

$= \int \frac{x\left(2\cos^2\frac{x}{2}-1\right)}{2\cos^2\frac{x}{2}}\ dx$

$= \int x\ dx - \frac{1}{2}\int\ x\sec^2\frac{x}{2}\ dx$

$= \frac{x^2}{2} - \frac{1}{2}\left[x\int \sec^2\frac{x}{2}\ dx - \int\left\{\frac{d}{dx}x\int \sec^2\frac{x}{2}\ dx\right\}dx\right]$

$= \frac{x^2}{2} - \frac{1}{2}\left[x\frac{\tan\frac{x}{2}}{\frac{1}{2}} - \int \frac{\tan\frac{x}{2}}{\frac{1}{2}}dx\right]$

$= \frac{x^2}{2} - x\tan\frac{x}{2} + \int \tan\frac{x}{2}dx$

$= \frac{x^2}{2} - x\tan\frac{x}{2} + \int 2\ \frac{\frac{1}{2}\ \sec\frac{x}{2}\tan\frac{x}{2}}{\sec\frac{x}{2}}dx$

$= \frac{x^2}{2} - x\tan\frac{x}{2} + 2\ln\sec\frac{x}{2} + c$

10) $\int (\ln x)^2\,dx$

$= (\ln x)^2 \int 1\,dx - \int \left\{\frac{d}{dx}(\ln x)^2 \int 1\,dx\right\} dx$

$= (\ln x)^2 . x - \int 2\ln x . \frac{1}{x} . x\,dx$

$= x(\ln x)^2 - \int 2\ln x\,dx$

$= x(\ln x)^2 - 2\left[\ln x \int 1\,dx - \int \left\{\frac{d}{dx}\ln x \int 1\,dx\right\} dx\right]$

$= x(\ln x)^2 - 2\left[x\ \ln x - \int \frac{1}{x} . x\,dx\right]$

$= x(\ln x)^2 - 2\left[x\ \ln x - \int 1\,dx\right]$

$= x(\ln x)^2 - 2[x\ \ln x - x] + c$

<u>Solution of INTEGRATION LECTURE 4</u>

<u>H.W:</u>

1) $\int \frac{x+1}{x^2-5x+6}dx$ Ans: $\int \frac{x+1}{(x-2)(x-3)}dx = -3\ln(x-2) + 4\ln(x-3) + c$

2) $\int \frac{2x+3}{x^3+x^2-2x}dx$ Ans: $\int \frac{2x+3}{x(x+2)(x-1)}dx = \frac{-3}{2}\ln x - \frac{1}{6}\ln(x+2) + \frac{5}{3}\ln(x-1) + c$

3) $\int \frac{1}{x^2(x-1)}dx$ Ans: $-\ln x + \frac{1}{x} + \ln(x-1) + c$

4) $\int \frac{x+1}{x^2-7x+10}dx$ Ans: $\int \frac{x+1}{(x-2)(x-5)}dx = 2\ln(x-5) - \ln(x-2) + c$

5) $\int \frac{1}{x\,(x+1)^2}dx$ Ans: $\ln x - \ln(x+1) + \frac{1}{x+1} + c$