Line Integral

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Problem 1:

If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$. Evaluate $\int_c \vec{F}\cdot d\vec{r}$ where, c is curve in xy plane $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution:

In xy plane, $z = 0$

$$\vec{r} = x\hat{i} + y\hat{j}$$
$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_c (3xy\hat{i} - y^2\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$
$$= \int_c (3xy\,dx - y^2\,dy)$$

Since, $y = 2x^2 \quad \therefore dy = 4x\,dx$

x varies from 0 to 1.

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_0^1 3x\cdot 2x^2\,dx - (2x^2)^2\,4x\,dx$$
$$= \int_0^1 (6x^3 - 16x^5)\,dx$$
$$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1$$
$$= \frac{6}{4} - \frac{16}{6} = \frac{36-64}{24} = \frac{-28}{24} = -\frac{7}{6} \text{ (Ans)}$$

Problem 2:

If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\,\hat{j} + 20xz^2\hat{k}$.

Evaluate $\int \vec{F} \cdot d\vec{r}$ along following path c

a) $x = t,\ y = t^2,\ z = t^3$ from $(0,0,0)$ to $(1,1,1)$

b) Straight line from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$.

c) Straight line joining $(0,0,0)$ to $(1,1,1)$.

Solution:

Here, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = \int_c \left((3x^2 + 6y)\hat{i} - 14yz\,\hat{j} + 20xz^2\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \int_c (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz.$$

(a) Here, $x = t,\ y = t^2,\ z = t^3$

$$dx = dt,$$

$$dy = 2t\,dt$$

$$dz = 3t^2\,dt$$

$\therefore t$ varies from limit 0 to 1.

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_0^1 (3t^2 + 6t^2)\,dt - 28t^6\,dt + 20t^7 \cdot 3t^2\,dt)$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= 9\left[\frac{t^3}{3}\right]_0^1 - 28\left[\frac{t^7}{7}\right]_0^1 + 60\left[\frac{t^{10}}{10}\right]_0^1$$

$$= 3 - 4 + 6$$

$$= 5 \quad \text{(Ans)}$$

Solution b:

Straight line from $(0,0,0)$ to $(1,0,0)$

$\therefore y = 0, \quad z = 0$

$dy = 0, \quad dz = 0$

x varies from 0 to 1.

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_c (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

$$= \int_0^1 3x^2\,dx$$

$$= 1$$

→ Straight line from $(1,0,0)$ to $(1,1,0)$

Here, $x = 1$, $z = 0$,

$dx = 0$, $dz = 0$,

y varies from 0 to 1.

$$\therefore \int_c \vec{F} \cdot d\vec{r} = \int_0^1 0 - 14y(0) \cdot dy + 0$$

$$= 0$$

→ Straight line from $(1,1,0)$ to $(1,1,1)$

Here, $x = 1$, $y = 1$

$dx = 0$, $dy = 0$

z varies from 0 to 1.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 20xz^2 \, dz$$

$$= 20 \left[\frac{z^3}{3} \right]_0^1$$

$$= \frac{20}{3}$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = 1 + 0 + \frac{20}{3}$$

$$= \frac{23}{3}$$

(Ans)

Solution c

Straight line joining (0,0,0) to (1,1,1) is given in parametric form.

$x = t, \quad y = t, \quad z = t.$

$dx = dt, \quad dy = dt, \quad dz = dt.$

$\therefore t$ varies from 0 to 1.

$$\therefore \int_c \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + 6t)\,dt - 14t^2\,dt + 20t^3\,dt$$

$$= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3)\,dt$$

$$= \int_0^1 (20t^3 - 11t^2 + 6t)\,dt$$

$$= 20\left[\frac{t^4}{4}\right]_0^1 - 11\left[\frac{t^3}{3}\right]_0^1 + 6\left[\frac{t^2}{2}\right]_0^1$$

$$= \frac{20}{4} - \frac{11}{3} + \frac{6}{2}$$

Formula: $W = \int \vec{F} \cdot d\vec{r}$

Problem 1:

Find work done moving a particle once around a circle c in xy plane, if the circle has centre at origin and radius 3 and if force field is given by

$\vec{F} = (2x - y + z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y + 4z)\hat{k}$

Solution: in xy plane, $z = 0$

so, $\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}$

$\therefore$ work $= \int_c \vec{F} \cdot d\vec{r} = \int_c \{(2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}\} \cdot (dx\hat{i} + dy\hat{j})$

$= \int_c (2x - y)\,dx + (x+y)\,dy$

Now, $x = r\cos t = 3\cos t$, $dx = -3\sin t\,dt$

$y = r\sin t = 3\sin t$, $dy = 3\cos t\,dt$.

$\therefore$ Work $= \int_0^{2\pi} (2x-y)\,dx + (x+y)\,dy$ $\rightarrow$ full circle at $0 \rightarrow 2\pi$

$= \int_0^{2\pi} (6\cos t - 3\sin t)(-3\sin t\,dt) + (3\cos t + 3\sin t)(3\cos t\,dt)$

$= \int_0^{2\pi} (-18\cos t\sin t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cos t)\,dt$

$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt$

$= 9\int_0^{2\pi} \left(1 - \frac{1}{2}\sin 2t\right)dt$

$= 9\left[t - \frac{1}{4}\cos 2t\right]_0^{2\pi}$

$= 9\left[2\pi - 0 - \frac{1}{4}(\cos\pi - \cos 0)\right]$

$= 18\pi$

(Ans)

$2\sin$

$= \sin$

$\sin$

$= \frac{1}{}$

Problem 2

Around a curve c where $x = t^2+1, y = 2t^2, z = t^3$ from $t=1$, to $t=2$ and if force field is given by $F^{\to} = 3xy\,\hat{i} - 5z\,\hat{J} + 10x\,\hat{k}$.

Solution:

$$\text{work} = \int_c \vec{F}\cdot d\vec{r}$$

$$= \int_c (3xy\,\hat{i} - 5z\,\hat{J} + 10x\,\hat{k})(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_c 3xy\,dx - 5z\,dy + 10x\,dz$$

now, $x = t^2+1$ $\quad y = 2t^2$ $\quad z = t^3$

$dx = 2t\,dt \quad \Rightarrow dy = 4t\,dt \quad \Rightarrow dz = 3t^2\,dt$.

$$\therefore \text{work} = \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt.$$

$$= 12\left[\frac{t^6}{6}\right]_1^2 + 12\left[\frac{t^4}{4}\right]_1^2 - 20\left[\frac{t^5}{5}\right]_1^2 + 30\left[\frac{t^5}{5}\right]_1^2 + 3$$

Gradient, Divergence, Curl.

Gradient

Formula → $\text{grad } \vec{F} = \vec{\nabla} F$

$$= \left(\frac{\delta}{\delta x} \hat{i} + \frac{\delta}{\delta y} \hat{j} + \frac{\delta}{\delta z} \hat{k} \right) F$$

$$= \frac{\delta F}{\delta x} \hat{i} + \frac{\delta F}{\delta y} \hat{j} + \frac{\delta F}{\delta z} \hat{k}$$

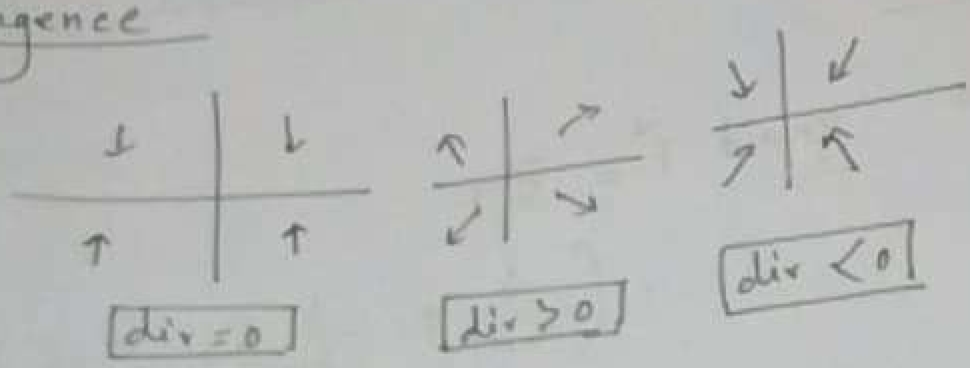
Problem 1: if $F = 3x^2y^2 - y^3z^2$, find grad F.

$$\therefore \text{grad } F = \frac{\delta}{\delta x}\left(3x^2y^2 - y^3z^2\right)\hat{i} + \frac{\delta}{\delta y}\left(3x^2y^2 - y^3z^2\right)\hat{j} + \frac{\delta}{\delta z}\left(3x^2y^2 - y^3z^2\right)\hat{k}$$

$$= 6xy^2\,\hat{i} + \left(6x^2y - 3y^2z^2\right)\hat{j} - 2y^3z\,\hat{k}$$

(Ans)

Divengence



considen field $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\text{div } \vec{F} = \vec{\nabla}\cdot\vec{F}$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(P\hat{i} + Q\hat{j} + R\hat{k}\right)$$
$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\Rightarrow$ if div $\vec{F} = 0$ then vecton field is called solenoidal.

Problem 1: if $\vec{F} = x^2 z\,\hat{i} - 2y^3 z^2\,\hat{j} + xy^2 z\,\hat{k}$, Find div $\vec{F}$

$$\therefore \text{div } \vec{F} = \frac{\partial}{\partial x}(x^2 z) - \frac{\partial}{\partial y}(2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z)$$
$$= 2xz - 6y^2 z^2 + xy^2$$

(2) $\rightarrow$ now, at $(1, -1, 1)$ check the vecton field is solenoidal on

Solution div, $\vec{F} = 2 - 6 + 1 = -3$

...ce, div $\vec{F} \neq 0$, the vecton field is not solenoidal.

Curl

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \times \left(P\hat{i} + Q\hat{j} + R\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ P & Q & R \end{vmatrix}$$

→ if curl $\vec{F} = 0$ then vector field is called conservative or irrotational.

Problem 1:

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 z & -2y^3 z^2 & xy^2 z \end{vmatrix}$$

$$= i\left\{\frac{\delta}{\delta y}(xy^2 z) - \frac{\delta}{\delta z}(-2y^3 z^2)\right\} - \hat{j}\left\{\frac{\delta}{\delta x}(xy^2 z) - \frac{\delta}{\delta z}(x^2 z)\right\} + \hat{k}\left\{\frac{\delta}{\delta x}(-2y^3 z^2) - \frac{\delta}{\delta y}(x^2 z)\right\}$$

$$= (2xyz + 4y^3 z)\hat{i} - (y^2 z - x^2)\hat{j}$$

(Ans)

Problem

$\vec{F} = (x^2y^3 - z^4)\,\hat{i} + 4x^5y^2z\,\hat{j} + -y^4z^6\,\vec{k}$

a) is vector field $\vec{F}$ is solenoidal at $(0,0,0)$?

b) is vector field $\vec{F}$ is conservative at $(0,0,0)$?

c) Find div (curl $\vec{F}$)

Solution (a)

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F}$$

$$= \frac{\delta}{\delta x}(x^2y^3 - z^4) + \frac{\delta}{\delta y}(4x^5y^2z) - \frac{\delta}{\delta z}(y^4z^6)$$

$$= 2xy^3 + 8x^5yz - 6y^4z^5$$

now, at $(0,0,0)$ point $= 0 + 0 - 0 = 0$

since, div $\vec{F} = 0$, the vector field is solenoidal.

Solution (b)

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ (x^2y^3 - z^4) & (4x^5y^2z) & (-y^4z^6) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(-y^4z^6) - \frac{\partial}{\partial z}(4x^5y^2z)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-y^4z^6) - \frac{\partial}{\partial z}(x^2y^3 - z^4)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(4x^5y^2z) - \frac{\partial}{\partial y}(x^2y^3 - z^4)\right\}$$

$$= \hat{i}(-4y^3z^6 - 4x^5y^2) - \hat{j}(4z^3) + \hat{k}(20x^4y^2z - 3x^2y^2)$$

At point $(0,0,0)$ $= 0$

$\therefore$ curl $\vec{F} = 0$, the vector field is conservative.

(c) div (curl $\vec{F}$)

$$= \frac{\partial}{\partial x}(-4y^3z^6 - 4x^5y^2) - \frac{\partial}{\partial y}(4z^3) + \frac{\partial}{\partial z}(20x^4y^2z - 3x^2y^2)$$

$$= -20x^4y^2 + 20x^4y^2$$

$$= 0$$

$\Rightarrow$ curl এর পর div করলে result always 0 হবে.

Green's theorem

only for 2 dimensional

<u>statement:</u> let, c is a piecewise smooth simple closed curve bounded by a simply connected region R. if $P, Q, \frac{\delta P}{\delta y}, \frac{\delta Q}{\delta x}$ are continious in R. then,

$$\oint_c P\,dx + Q\,dy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dA$$

<u>Problem 1:</u> Apply green's theorem Evaluate

$\oint_c (10x^4 + 2xy^3)\,dx - 3x^2y^2\,dy$ along path c

$x^4 - 6xy^3 - 4y^2$ from $(0,0)$ to $(2,1)$

<u>Solution:</u> let, $P = 10x^4 + 2xy^3$

$Q = -3x^2y^2$

$$\frac{\delta P}{\delta y} = 6xy^2, \quad \frac{\delta Q}{\delta x} = -6xy^2$$

By green's theorem.

$$\oint_C (10x^4 + 2xy^3)\,dx - 3x^2y^2\,dy$$

$$= \iint_R (-6xy^2 - 6xy^2)\,dA$$

$$= \int_{x=0}^{2} \int_{y=0}^{1} -12xy^2 \,dy\,dx$$

$$= -12 \int_0^2 x \left[\frac{y^3}{3}\right]_0^1 dx$$

$$= -4 \int_0^2 x\,(1^3 - 0)\,dx$$

$$= -4 \left[\frac{x^2}{2}\right]_0^2$$

$$= -2\,(2^2 - 0)$$

$$= -8$$

Ans)

Prob 2

Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$ along path C is the circle $x^2+y^2=9$.

Solution

$P = 3y - e^{\sin x}$, $\frac{\partial P}{\partial y} = 3$

$Q = 7x + \sqrt{y^4+1}$, $\frac{\partial Q}{\partial x} = 7$

By green's theorem,

$$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy = \iint_R (7-3)\,dA$$
$$= 4\iint dA$$

Now, double integral $\iint_R dA$ gives area of region R bounded by circle of radius 3. So, area of circle is $= \pi r^2$

$= \pi 3^2$

$= 9\pi$

$$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy = 4\cdot 9\pi$$
$$= 36\pi$$

(Ans)

Problem 3: Evaluate $\oint_c (y-\sin x)\,dx + \cos x\,dy$, where c is triangular path with vertices $(0,0)$, $(\frac{\pi}{2},0)$, $(\frac{\pi}{2},1)$ with positive orientation.

Solution:

$P = y - \sin x$, $\dfrac{\partial P}{\partial y} = 1$

$Q = \cos x$, $\dfrac{\partial Q}{\partial x} = -\sin x$

Now, x varies from 0 to $\pi/2$

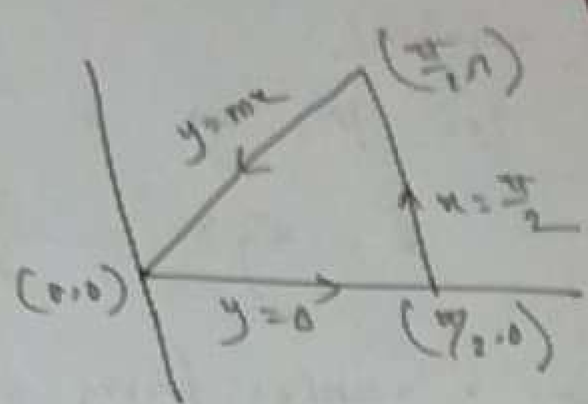
y varies from 0 to $\dfrac{2x}{\pi}$.

By green's theorem,

$$\oint_c (y-\sin x)\,dx + \cos x\,dy$$

$$= \iint_R (-\sin x - 1)\,dA$$

$$= \int_0^{\pi/2}\int_0^{\frac{2x}{\pi}} (-\sin x - 1)\,dy\,dx$$



$$y = mx$$
$$= \left(\frac{\Delta y}{\Delta x}\right)x$$
$$= \left(\frac{1-0}{\pi/2-0}\right)x$$
$$= \frac{2x}{\pi}$$

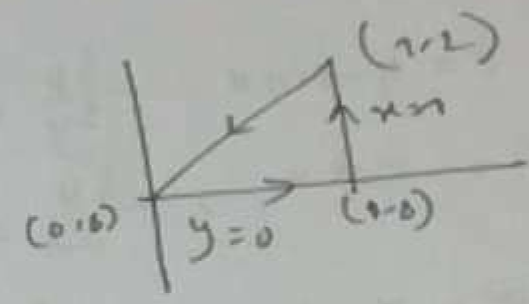
Ans: $\dfrac{-2}{\pi} - \dfrac{\pi}{4}$

Problem 4: Apply green's theorem to evaluate $\oint_c xy\,dx + x^2y^3\,dy$ where c is triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Solution:

Let, $P = xy$, $\frac{\delta P}{\delta y} = x$

$Q = x^2y^3$, $\frac{\delta Q}{\delta x} = 2xy^3$



$y = mx$

$= \left(\frac{2-0}{1-0}\right)x$

$= 2x$

Now, x varies from 0 to 1. y varies from 0 to $2x$,

By green's theorem,

$$\oint_c xy\,dx + x^2y^3\,dy$$

$$= \iint_R (2xy^3 - x)\,dA$$

$$= \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$$

$$= \int_0^1 \left[\frac{xy^4}{2} - xy\right]_0^{2x}$$

$$= \int_0^1 (8x^5 - 2x^2)\,dx$$

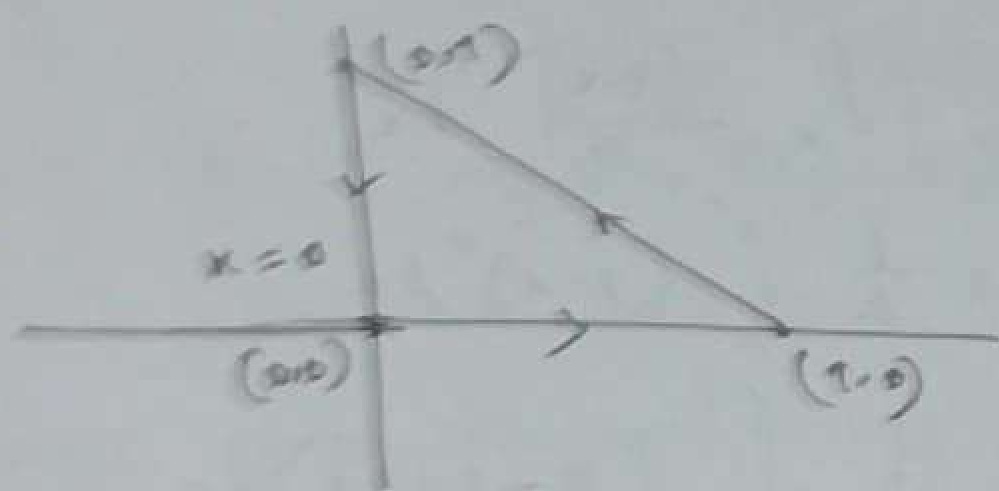
$$= \left[\frac{8x^6}{6} - \frac{2x^3}{3}\right]_0^1 = \frac{2}{3} \quad \text{(Ans)}$$

Problem 5: Apply Green's theorem to evaluate $\oint_C x^4 dx + xy\,dy$ where C is triangle with vertices $(0,0), (1,0), (0,1)$ with positive orientation.

Solution:

let, $P = x^4$, $\frac{\partial P}{\partial y} = 0$

$Q = xy$, $\frac{\partial Q}{\partial x} = y$



$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\Rightarrow \frac{y - 0}{0 - 1} = \frac{x - 1}{1 - 0}$$

$$\Rightarrow y = -x + 1$$

Here, x varies from 0 to 1, y varies from 0 to $(-x+1)$

$$\therefore \oint_C x^4\,dx + xy\,dy = \iint_R (y-0)\,dA$$

$$= \int_0^1 \int_0^{-x+1} y\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^{-x+1} dx$$

$$= \int_0^1 \frac{(1-x)^2}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 (x^2-2x+1)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - x^2 + x\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3} - 1 + 1\right)$$

$$= \frac{1}{6}$$

(Ans)

Surface Integral / Flux

If $\vec{F}$ is continious vector field defined on an oriented surface S, with unit normal vector $\hat{n}$ then surface integral or flux of $\vec{F}$, over S is

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n}\, ds$$

Divergence Theorem:

Let G be a solid whose surface S is oriented outward. if $\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$, where, f, g, h have continious first partial derivatives on some open set containing G. and if $\hat{n}$ is the outward unit normal vector on S, then,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_G \operatorname{div}(\vec{F})\, dv.$$

Problem 1

Apply divergence theorem to find outward flux of $\vec{F} = 2x\hat{i} + 3y\hat{j} + z^2\hat{k}$ across the cube of length 1.

Solution:

The divergence ~~theorem~~ of vector field is

$$\text{div } \vec{F} = \frac{\delta}{\delta x}(2x) + \frac{\delta}{\delta y}(3y) + \frac{\delta}{\delta z}(z^2)$$

$$= 2 + 3 + 2z$$

$$= 5 + 2z$$

By divergence theorem, the flux is,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (5+2z)\, dv$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (5+2z)\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 [5+z^2]_0^1\, dy\, dx$$

$$= \int_0^1\int_0^1 6\, dy\, dx$$

$$= \int_0^1 [6]_0^1\, dx$$

$$= 6$$

(Ans)

Problem:
Apply divergence theorem to find outward flux of $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$ across surface of region bounded by sphere $x^2+y^2+z^2 = 16$.

Solution:

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(5z)$$

$$= 3+4+5 = 12$$

By divergence theorem outward flux is,

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \operatorname{div} \vec{F}\, dV$$

$$= \iiint_V 12\, dV$$

$$= 12 \iiint_V dV$$

Here, $\iiint_V dV$ is the volume of sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\pi 4^3 = \frac{256}{3}\pi$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, dS = 12 \cdot \frac{256}{3}\pi$$

$$= 1024\pi$$

(Ans)

Problem 4

Apply Divergence theorem to find outward flux of $\vec{F} = y^2z^2\,\hat{i} + z^2x^2\,\hat{j} + z^2y^2\,\hat{k}$, where surface is part of sphere $x^2+y^2+z^2=1$ above xy plane and bounded by this plane.

Solution:

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)$$
$$= 2zy^2$$

By divergence theorem, the flux is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_G 2zy^2\,dv$$

Let, $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$

$\phi \to z$ [illegible] [illegible] [illegible] limit $0 \to 2\pi$

$$dx\,dy\,dz = R^2\sin\theta\,dR\,d\theta\,d\phi$$

$$= \iiint_{G} 2z\, y^2\, dv$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{R=0}^{1} 2R\cos\theta\, R^2\sin^2\theta \sin^2\phi\, R^2\sin\theta\, dR\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 R^5 \sin^3\theta \cos\theta \sin^2\phi\, dR\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi} \int_0^{\pi/2} \left[\frac{R^6}{6}\right]_0^1 \underline{\sin^3\theta \cos\theta} \sin^2\phi\, dR\, d\theta\, d\phi$$

$$= \frac{2}{6}\int_0^{2\pi} \int_{u=0}^{1} \underline{u^3\, du} \sin^2\phi\, d\phi$$

$$= \frac{1}{3}\int_0^{2\pi} \left[\frac{u^4}{4}\right]_0^1 \sin^2\phi\, d\phi$$

$$= \frac{1}{12} \int_0^{2\pi} \frac{1}{2}(1-\cos 2\phi)\, d\phi$$

$$= \frac{1}{24}\left[\phi - \frac{\sin 2\phi}{2}\right]_0^{2\pi}$$

$$= \frac{1}{24}\left[(2\pi - 0) - \frac{1}{2}(\sin 4\pi - \sin 0)\right]$$

$$= \frac{2\pi}{24} = \frac{\pi}{12}$$

Ans

let,

$\sin\theta = u$

$\Rightarrow \frac{du}{d\theta} = \cos\theta$

$\Rightarrow du = \cos\theta\, d\theta$

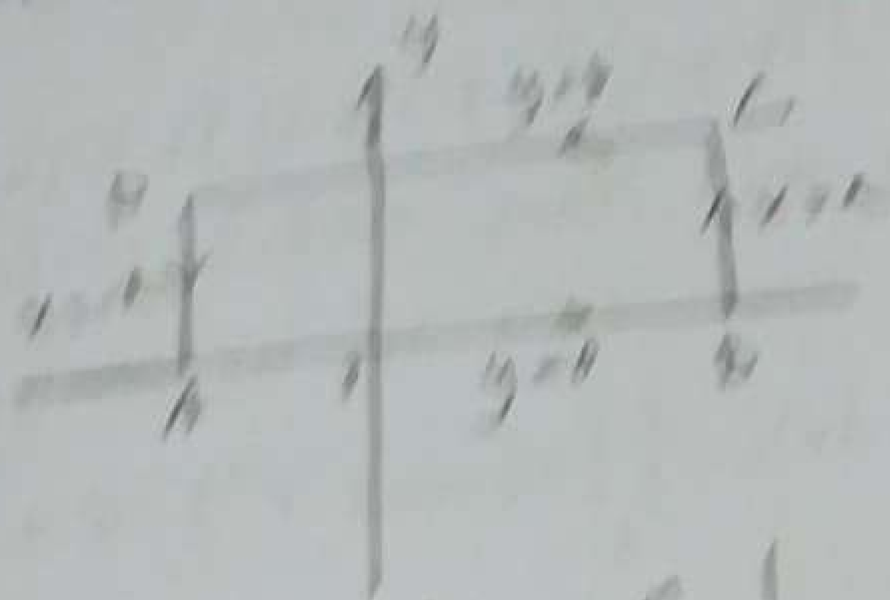
θ	0	$\pi/2$
u	0	1

$2\sin^2\phi = 1 - \cos 2\phi$

Problem 4:

Verify Stoke's theorem for $F = (x+y)\,i - 4xy\,j$ in the plane $z=0$, integral around the rectangle whose sides are along the lines $x=\pm a$, $y=0$, $y=b$, with positive orientation (anti clockwise)

Solution:



$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & -4xy & 0 \end{vmatrix}$$

$$= -4y\,k$$

In the plane $z=0$, so, $\hat{n}=\hat{k}$ and $ds = dx\,dy$

R.H.S

$$\iint_S (\text{curl } \vec{F}) \cdot \hat{n}\, ds = \iint_S -4y\,\hat{k}\cdot\hat{k}\, dx\,dy$$

$$= -4 \int_{y=0}^{b} \int_{x=-a}^{a} y\, dx\, dy$$

$$= -4 \int_0^b y\,[x]_{-a}^{a}\, dy$$

$$= -8a \left[\frac{y^2}{2}\right]_0^b$$

$$= -4ab^2$$

Now, $\oint_C \vec{F}\cdot d\vec{r} = \oint_C (x^2+y^2)\,dx - 2xy\,dy$

Along line AB, $y=0$, $dy=0$, x varies from $-a$ to a

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along line BC, $x=a$, $dx=0$, y varies from 0 to b

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^b -2ay\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$$

Along line CD, $y = b$, $dy = 0$, x varie from a

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_{a}^{-a} (x^2+b^2)\,dx = \left[\frac{x^3}{3}+b^2x\right]_{a}^{-a}$$

$$= \frac{1}{3}\left(-a^3-a^3\right)+b^2(-a-a)$$

$$= -\frac{2a^3}{3} - 2ab^2$$

Along line DA, $x=-a$, $dx = 0$, y varies from b to 0.

$$\int_{DA} \vec{F}\cdot d\vec{r} = \int_{b}^{0} 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_{b}^{0} = -ab^2$$

$$\therefore \oint_{C} \vec{F}\cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

$$= -4ab^2$$

$$\therefore \oint \vec{F}\cdot d\vec{r} = \iint_{S} (\text{curl } \vec{F})\cdot \hat{n}\, ds.$$

(Proved)

Md. Bagher DA

11-6-19

① Line Integral

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

1) If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$ Evalute $\int_c \vec{F} \cdot d\vec{r}$. where line integreal c is curve in xy plane $y = 2x^2$ from $(0,0)$ to $(1,2)$.

$\boxed{\begin{aligned} y &= 2x \\ x &= \frac{y}{2} \end{aligned}}$

$\Rightarrow$ In xy plane, $z = 0$

$\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}$

$\therefore \int_c \vec{F} \cdot d\vec{r} = \int_c (3xy\hat{i} - y^2\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$

$= \int_c (3xy\,dx - y^2 dy)$

since, $y = 2x^2$

$dy = 4x \cdot dx$

x varies from 0 to 1.

(3)

2) If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$

Evaluate $\int_c \vec{F} \cdot d\vec{r}$ along following path c

a) $x = t$, $y = t^2$, $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$.

(b) straight line from $(0, 0, 0)$ to $(1, 0, 0)$ then to $(1, 1, 0)$ and then to $(1, 1, 1)$

(c) straight line joining $(0, 0, 0)$ to $(1, 1, 1)$

$\Rightarrow$ Here, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = \int_c \left((3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_c (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(4)

(a) Here, $x = t$, $y = t^2$, $z = t^3$

$dx = dt$, $dy = 2t\,dt$, $dz = 3t^2 dt$

$\therefore$ t varies from 0 to 1

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3t^2 + 6t^2) dt - 28t^6 dt + 20t^7 \cdot 3t^2 \cdot dt)$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9) dt$$

$$= 9\left[\frac{t^3}{3}\right]_0^1 - 28\left[\frac{t^7}{7}\right]_0^1 + 60\left[\frac{t^{10}}{10}\right]_0^1$$

$$= 9\left(\frac{1}{3} - \frac{0}{3}\right) - 28\left(\frac{1}{7} - \frac{0}{7}\right) + 60\left(\frac{1}{10}\right)$$

$$= \frac{9}{3} - \frac{28}{7} + \frac{60}{10}$$

Ans: 5

$$= 3 - \frac{28}{7} + 6$$

$$= \frac{21 - 28 + 42}{7} = 9 - 4$$

$f(x) = \begin{cases} x, & 0<x<2 \\ 3, & 2<x<5 \end{cases}$ (b)

$\int_0^5 f(x)\,dx$

$= \int_0^2 x\,dx + \int_2^5 3\,dx$.

(b) ⇒ straight line from (0,0,0) to (1,0,0)

$y=0,\ z=0$, $dy=0,\ dz=0$

∴ x varies from 0 to 1

$\int_c \vec{F}\cdot d\vec{r} = \int_c (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$

$= \int_c (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$

$= \int_0^1 3x^2\,dx = 1$.

2nd ∴ straight line from (1,0,0) to (1,1,0)

Here, $x=1,\ z=0,\ dx=0, dz=0$.

∴ y varies from 0 to 1.

(c) straight line joining (0,0,0) to (1,1,1).
is given in parametric from.

$x=t$, $y=t$, $z=t$

$dx=dt$, $dy=dt$, $dz=dt$

t varies from 0 to 1.

$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_c (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$

$= \int_0^1 (3t^2+6t)dt - 14t.t.\,dt + 20t.t^2 dt$

$= \int_0^1 3\frac{t^3}{3} + 6\times\frac{t^2}{2} - 14\frac{t^3}{3} + 20\frac{t^4}{4}$

$= 3\times\left[\frac{t^3}{3}\right]_0^1 + 6\times\left[\frac{t^2}{2}\right]_0^1 - 14\times\left[\frac{t^3}{3}\right]_0^1 + 20\times\left(\frac{t^4}{4}\right)_0^1$

$= \cancel{3}\times\frac{1}{\cancel{3}} + \overset{3}{\cancel{6}}\times\frac{1}{\cancel{2}} - 14\times\frac{1}{3} + \overset{5}{\cancel{20}}\times\frac{1}{\cancel{4}}$

$= 4 - \frac{14}{3} + 5$

$= \frac{12-14+5}{3} = \frac{12+1}{3} = \frac{13}{3}$ Ans)

(9)

$$= \int_0^{2\pi} (9 - 9\sin t \cos t)\, dt$$

$$= \int_0^{2\pi} \left(1 - \frac{1}{2}\sin 2t\right) dt$$

$$= 9\left[t - \frac{1}{4}\cos 2t\right]_0^{2\pi}$$

$$= 9\left(2\pi - 0 - \frac{1}{4}(\cos 4\pi - \cos 0)\right)$$

$$= 18\pi \text{ (Ans:)}$$

$2\sin t \cos t = \sin 2t$

$\Rightarrow \sin t \cos t = \frac{1}{2}\sin 2t$

2) Find work done moving a particle once around a curve C where $x = t^2 + 1$, $y = 2t^2$, $z = t^3$. From $t = 1$ to $t = 2$ and if force field is given by $\vec{F} = 3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$

$$\Rightarrow \text{work} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C (3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_C \{3xy\,dx - 5z\,dy + 10x\,dz$$

জরুরি

Gradient:

Formula

$$\text{grad } F = \vec{\nabla} F$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)F$$

$$= \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

1) If $F = 3x^2y^2 - y^3z^2$. Find grad F.

স্কেলার field

$$\Rightarrow \text{grad } F = \frac{\partial}{\partial x}(3x^2y^2 - y^3z^2)\hat{i} + \frac{\partial}{\partial y}(3x^2y^2 - y^3z^2)\hat{j} + \frac{\partial}{\partial z}(3x^2y^2 - y^3z^2)\hat{k}$$

$$= 6xy^2\hat{i} + (6x^2y - 3y^2z^2)\hat{j} - 2y^3z\hat{k}$$

(Ans:)

$= 2xz - 6y^2z^2 + xy^2$

At $(1, -1, 1)$

$\text{div} \vec{F} = 2 - 6 + 1$

$= -3$

Since, div $\vec{F} \neq 0$, the vector field is not solenoidal. (Ans)

$\boxed{1}$ curl ~~cost~~ consider field $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

curl $\vec{F} = \vec{\nabla} \times \vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(P\hat{i} + Q\hat{j} + R\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

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(15)

1) $\vec{F} = (x^2y^3 - z^4)\hat{i} + 4x^5y^2z\hat{j} - y^4z^6\hat{k}$

a) Is vector field $\vec{F}$ is solenoidal at (0,0,0)?

b) Is vector field $\vec{F}$ is conservative at (0,0,0)?

c) Find div $(\text{curl } \vec{F})$

a) $\text{div } \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x^2y^3 - z^4)\hat{i} + 4x^5y^2z\hat{j} - y^4z^6\hat{k}$

$= \frac{\partial}{\partial x}(x^2y^3 - z^4) + \frac{\partial}{\partial y}(4x^5y^2z) + \frac{\partial}{\partial z}(-y^4z^6)$

$= 2xy^3 + \cancel{4x^5 8y}\ 8x^5yz - y^4 6z^5$

at (0,0,0)

$\text{div } \vec{F} = 0$

$\therefore$ field $\vec{F}$ is solenoidal (Ans:)

(17)

at $(0,0,0)$

$\text{curl } \vec{F} = 0$

since, $\text{curl } \vec{F} = 0$ the vector field is conservative.

(c) $\text{curl } \vec{F} = -4x^5y^2\hat{i} + 4z^3\hat{j} + (20x^4y^2z - 3x^2y^2)\hat{k}$

$\therefore \text{div}(\text{curl } \vec{F}) = \vec{\nabla} \cdot \text{curl } \vec{F}$

$= -20x^4y^2 + 20x^4y^2$

$= 0$ (Ans.)

(19)

$= \int_{x=0}^{2}\int_{y=0}^{1} -12xy^2\,dy\,dx \quad // \int_{y=0}^{1}\int_{x=0}^{2} -12xy^2\,dy\,dx.$

$= -12\int_{x=0}^{2} x\left[\frac{y^3}{3}\right]_0^1 dx$

$= -4\int_{x=0}^{2} x\,(1^3-0)\,dx$

$= -4\left[\frac{x^2}{2}\right]_0^2$

$= -2\,(2^2-0)$

$= 8$ (Ans:)

$\frac{2^2}{2} - \frac{0}{2}$

$2-0$

$-4(2-0)$

মূলবিন্দুগামী না হলে

straight line $\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$

$= (\pi/2, 1), (0,0)$

$y-1$

২

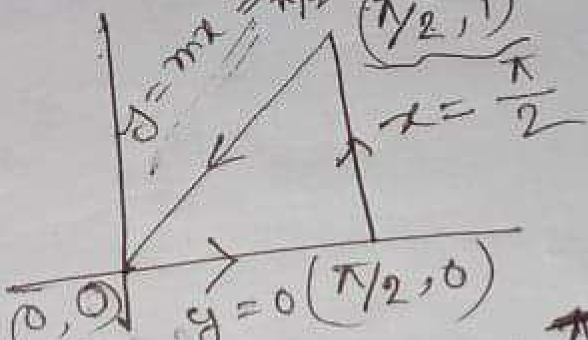
$= 36\pi$ (Ans:-)

* Apply Green's Theorem Evaluate $\oint_C (y-\sin x)\,dx + \cos x\,dy$, where C is triangular path with vertices $(0,0), (\frac{\pi}{2}, 0), (\frac{\pi}{2}, 1)$ with positive orientation,

$\Rightarrow$ Let $P = y - \sin x \quad , \frac{\partial P}{\partial y} = 1$

$Q = \cos x \quad , \frac{\partial Q}{\partial x} = -\sin x$.

$y = mx = \frac{1-0}{\pi/2-0}x = \frac{2x}{\pi} \Rightarrow$ মূলবিন্দুগামী হলে



Now, x varies from 0 to $\pi/2$, y varies from 0 to $2x/\pi$.

By Green's Theorem,

2-6-19

23

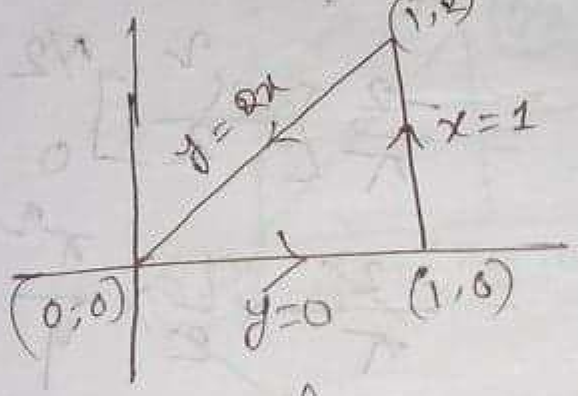
1) Apply Green's Theorem to evaluate $\oint_C xy\,dx + x^2y^2\,dy$ where C is triangle with vertices $(0,0),(1,0),(1,2)$ with positive orientation.

$\Rightarrow$ Let, $P = xy$; $\frac{\partial P}{\partial y} = x$

$Q = x^2y^3$, $\frac{\partial Q}{\partial x} = 2xy^3$

By, Green's theorem,

$$\oint_C xy\,dx + x^2y^2\,dx = \iint_R \cancel{(x - 2xy^3)\,dA}\ (2xy^3 - x)\,dA$$



$y = mx$
$= \frac{2-0}{1-0}x$
$= 2x$

Now, x varies from 0 to 1, y varies from 0 to $2x$.

$$\therefore \oint_C xy\,dx + x^2y^2\,dx = \int_0^1\int_0^{2x} (2xy^3 - x)\,dy\,dx$$

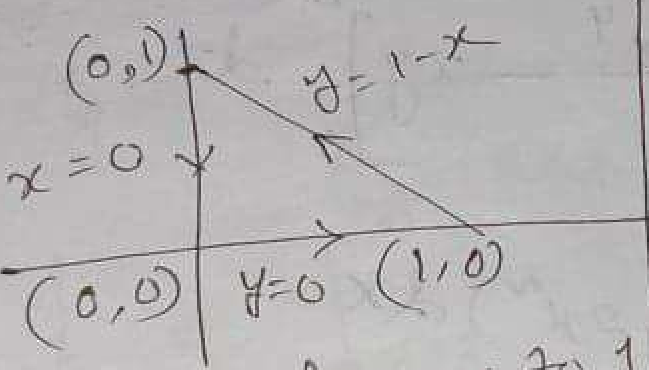
line integral + Green's the

Next [9.7.19] CT (25)

2) Apply Green's theorem to evaluate $\oint_C x^4 dx + xy\,dy$ where C is triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ with positive orientation.

$\Rightarrow$ Let, $P = x^4$ $\qquad \frac{\partial P}{\partial y} = 0$

$Q = xy$ $\qquad \frac{\partial Q}{\partial x} = y$



$\overset{x_1\,y_1}{(1,0)}$, $\overset{x_2\,y_2}{(0,1)}$

when,

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

$$\frac{y-0}{0-1} = \frac{x-1}{1-0}$$

$$\Rightarrow \frac{y}{-1} = \frac{x-1}{1}$$

$$\Rightarrow y = -x+1.$$

Here, x varies from 0 to 1,

y varies from 0 to $-x+1$.

$$\therefore \oint_C x^4 dx + xy\,dy = \iint (y-0)\, dA$$

$$= \int_0^1 \int_0^{1-x} y\, dy\, dx$$

(III) Surface Integral / Flux:- If $\vec{F}$ is continous vectore field defined on an oriented surface S, with unit normal vector $\hat{n}$, then surface integral on flux of $\vec{F}$ over S is,

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n}\, ds$$

Divergence Theorem:

Let, G be a solid whose surface S is oriented outward. If, $\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$, where f, g, h have continuous first partial derivatives on some open set containing G, and if $\hat{n}$ is the outward unit normal vectore on S, then,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_G \operatorname{div}(\vec{F})\, dv.$$

(29)

$= \int_{x=0}^{1}\int_{y=0}^{1} \left[5z + 2\cdot\frac{z^2}{2}\right]_0^1 dy\,dx$

$= \int_0^1\int_0^1 (5+1)\,dy\,dx \quad = \int_0^1\int_0^1 6\,dy\,dx$

$= 6\int_0^1\int_0^1 dy\,dx$

$= 6\int_0^1 [y]_0^1\,dx$

$= 6\int_0^1 1\,dx \quad = 6\,[x]_0^1 \quad = 6$ Ans:)

cartesian → cylindrical

(r, θ, z)

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

cartesian → spherical (R, θ, ϕ)

$x = R\sin\theta\cos\phi$

$y = R\sin\theta\sin\phi$

$z = R\cos\theta$

Here, $x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dv = dx\,dy\,dz$

$= r\,dr\,d\theta\,dz$

$$= \int_{\theta=0}^{2\pi} \left[6r^2 - 4r^3 \sin\theta + \frac{9r^2}{2} \right]_0^2 d\theta$$

$$= \int_{\theta=0}^{2\pi} (24 - 32\sin\theta + 18)\, d\theta$$

$$= \int_{\theta=0}^{2\pi} (42 - 32\sin\theta)\, d\theta$$

$$= \left[42\theta + 32\cos\theta \right]_0^{2\pi}$$

$$= 42 \cdot 2\pi + 32(\cos 2\pi - \cos 0)$$

$$= 84\pi \quad (\text{Ans:})$$

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(33)

3) Apply Divergence Theorem to find outward flux of $\vec{F} = y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}$, which in enclosed by sphere $x^2+y^2+z^2 = 1^2$ above xy plane and bounded by this plane.

$\Rightarrow \text{div}\,\vec{F} = \frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2) = 2zy^2$

By Divergence Theorem, the flux is,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_G 2zy^2\,dv.$$

let,
$x = R\sin\theta\cos\phi$
$y = R\sin\theta\sin\phi$
$z = R\cos\theta$
$dx\,dy\,dz = R^2\sin\theta\,dR\,d\theta\,d\phi$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{R=0}^{1} 2R\cos\theta\,R^2\sin^2\theta\sin^2\phi$$

$$= 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{R=0}^{1} R^5\sin^3\theta\cos\theta\sin^2\phi\,dR\,d\theta\,d\phi$$

$$= 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left[\frac{R^6}{6}\right]_0^1 \sin^3\theta\cos\theta\sin^2\phi\,d\theta\,d\phi$$

(3)

curl $\vec{F}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = -4y\hat{k}$

In the plane, $z=0$, so, $\hat{n} = \hat{k}$ and $ds = dxdy$.

$$\iint_S (\text{curl } \vec{F}).\hat{n}.ds = \iint_S -4y\hat{k}.\hat{k}.dxdy$$

$$= -4\int_{y=0}^{b}\int_{x=-a}^{a} y\, dx\, dy$$

$$= \cancel{-4\int_0^b \left[\frac{y^2}{2}\right]^a} \qquad = -4\int_a^b y\,[x]_{-a}^{a}\,dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b$$

$$= -8a.\frac{b^2}{2}$$

$$= -4ab^2$$

Along, DA, $x=-a$, $dx=0$, y varies form b to 0.

$$\int_{DA} \vec{F}\cdot d\vec{r} = \int_b^0 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_b^0 = -ab^2$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - 2a^3/3 - 2ab^2 - ab^2.$$

$$= -4ab^2$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } \vec{F})\cdot\hat{n}\,ds$$

Thus, stoke's Theorem is varified

2) verify stoke's theorem for $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the rectangle in the plane $z=0$ whose sides are along lines $x=0$, $x=a$, $y=0$, $y=a$, with positive orientation.

Ans:- $a^3/2$.

(40)

$= \frac{1}{2} \cdot a^2 \cdot 2\pi$

$= \pi a^2$.

$\therefore \oint_c \vec{F} \cdot d\vec{r}$

$= \oint_c (z\hat{i} + x\hat{j} + y\hat{k}) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$= \oint_c z\,dx + x\,dy + y\,dz$

$= \oint_c x\,dy$

$= \int_0^{2\pi} a\cos\theta \cdot a\cos\theta\, d\theta$.

Let,
$x = a\cos\theta$
$y = a\sin\theta$
$dy/d\theta = a\cos\theta$
$dy = a\cos\theta\, d\theta$

$= \int_0^{2\pi} a^2\cos^2\theta\, d\theta$

$= \frac{a^2}{2} \int_0^{2\pi} (1 + \cos 2\theta)\, d\theta$

$= \frac{a^2}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$