Integration- Method of Substitution

Ex. 1) $\displaystyle \int \frac{\sin x}{\sqrt{1+\cos x}}\,dx$

$= -\displaystyle\int \frac{1}{\sqrt{z}}\,dz$ Let, $1+\cos x = z$

$= -2\sqrt{z} + c$ $\Rightarrow -\sin x\,dx = dz$

$= -2\sqrt{1+\cos x} + c$

Ex. 2) $\displaystyle \int e^{\tan^{-1}x}.\frac{1}{1+x^2}\,dx$

$= \displaystyle\int e^z\,dz$ Let, $\tan^{-1}x = z$

$= e^z + c$ $\Rightarrow \dfrac{1}{1+x^2}\,dx = dz$

$= e^{\tan^{-1}x} + c$

Ex. 3) $\displaystyle \int \frac{1}{\sqrt{x}}\cos\sqrt{x}\,dx$

$= \displaystyle\int \cos z\;2\,dz$ Let, $\sqrt{x} = z$

$= 2\sin z + c$ $\Rightarrow \dfrac{1}{2\sqrt{x}}\,dx = dz$

$= 2\sin\sqrt{x} + c$ $\Rightarrow \dfrac{1}{\sqrt{x}}\,dx = 2\,dz$

Ex. 4) $\displaystyle \int \frac{a\cos x - b\sin x}{a\sin x + b\cos x + d}\,dx$

$= \displaystyle\int \frac{dz}{z}$ Let, $a\sin x + b\cos x + d = z$

$= \ln z + c$ $\Rightarrow (a\cos x - b\sin x)\,dx = dz$

$= \ln(a\sin x + b\cos x + d) + c$

Ex. 5) $\displaystyle\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,dx$

$= \displaystyle\int z^2\,dz$ Let, $\sin^{-1} x = z$

$= \dfrac{z^3}{3} + c$ $\Rightarrow \dfrac{1}{\sqrt{1-x^2}}\,dx = dz$

$= \dfrac{(\sin^{-1} x)^3}{3} + c$

Ex. 6) $\displaystyle\int \cos^3 x\,dx$

$= \displaystyle\int \cos^2 x.\cos x\,dx$ Let, $\sin x = z$

$= \displaystyle\int (1-\sin^2 x).\cos x\,dx$ $\Rightarrow \cos x\,dx = dz$

$= \displaystyle\int (1-z^2)\,dz$

$= z - \dfrac{z^3}{3} + c$

$= \sin x - \dfrac{(\sin x)^3}{3} + c$

Ex. 7) $\displaystyle\int \sin^4 x\cos^3 x\,dx$

$= \displaystyle\int \sin^4 x\,.\cos^2 x.\cos x\,dx$ Let, $\sin x = z$

$= \displaystyle\int \sin^4 x\,(1-\sin^2 x).\cos x\,dx$ $\Rightarrow \cos x\,dx = dz$

$= \displaystyle\int z^4\,(1-z^2)\,dz$

$= \displaystyle\int (z^4 - z^6)\,dz$

$= \dfrac{z^5}{5} - \dfrac{z^7}{7} + c = \dfrac{(\sin x)^5}{5} - \dfrac{(\sin x)^7}{7} + c$

Ex. 8) $\displaystyle\int \tan^3 2x \sec 2x\, dx$

$= \displaystyle\int \tan^2 2x \ \tan 2x \ \sec 2x\, dx$ Let, $\sec 2x = z$

$= \displaystyle\int (\sec^2 2x - 1)\tan 2x \sec 2x\, dx$ $\Rightarrow \sec 2x \tan 2x\, dx\, .2 = dz$

$= \displaystyle\int (z^2 - 1)\frac{1}{2} dz$ $\Rightarrow \sec 2x \tan 2x\, dx = \dfrac{1}{2} dz$

$= \dfrac{1}{2}\left(\dfrac{z^3}{3} - z\right) + c$

$= \dfrac{1}{2}\left(\dfrac{\sec^3 2x}{3} - \sec 2x\right) + c$

Ex. 9) $\displaystyle\int \frac{\tan x}{\ln \cos x} dx$

$= \displaystyle\int \frac{-dz}{z}$ Let, $\ln \cos x = z$

$= -\ln z + c$ $\Rightarrow -\dfrac{\sin x}{\cos x} dx = dz$

$= -\ln(\ln \cos x) + c$ $\Rightarrow \tan x\, dx = -dz$

Ex. 10) $\displaystyle\int \frac{\sin 2x}{(a + b\cos x)^2} dx$

$= \displaystyle\int \frac{2\sin x \cos x}{(a + b\cos x)^2}\ dx$ Let, $a + b\cos x = z$

$= 2\displaystyle\int \frac{\frac{z-a}{b}}{z^2}\left(-\frac{1}{b} dz\right)$ $\Rightarrow -b \sin x\, dx = dz$

$= \dfrac{-2}{b^2}\displaystyle\int \frac{z-a}{z^2}\ dz$ $\Rightarrow \sin x\, dx = -\dfrac{1}{b} dz$

$= \dfrac{-2}{b^2}\displaystyle\int \left(\frac{1}{z} - \frac{a}{z^2}\right)\ dz$ and, $\cos x = \dfrac{z-a}{b}$

$= \dfrac{-2}{b^2}\left(\ln z + \dfrac{a}{z}\right) + c = \dfrac{-2}{b^2}\left(\ln(a + b\cos x) + \dfrac{a}{a + b\cos x}\right) + c$

Ex. 11) $\displaystyle\int \frac{\sin^4 x}{\cos^8 x}\,dx$

$= \displaystyle\int \tan^4 x\ \sec^4 x\,dx$ Let, $\tan x = z$

$= \displaystyle\int \tan^4 x\ \sec^2 x \sec^2 x\,dx$ $\Rightarrow \sec^2 x\,dx = dz$

$= \displaystyle\int \tan^4 x\ (1 + \tan^2 x)\sec^2 x\,dx$

$= \displaystyle\int z^4\ (1 + z^2)\,dz$

$= \displaystyle\int (z^4 + z^6)$

$= \dfrac{z^5}{5} + \dfrac{z^7}{7} + c$

$= \dfrac{\tan^5 x}{5} + \dfrac{\tan^7 x}{7} + c$

H.W::

1) $\displaystyle\int \frac{\sqrt{1+\ln x}}{x}\,dx$	6) $\displaystyle\int \frac{\tan x \sec^2 x}{(a^2 + b^2\tan^2 x)^2}\,dx$
2) $\displaystyle\int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx$	7) $\displaystyle\int \frac{\sin x}{\sin(x-a)}\,dx$
3) $\displaystyle\int \frac{1-\sin x}{x+\cos x}\,dx$	8) $\displaystyle\int \frac{dx}{x\ln x\ [\ln(\ln x)]}$
4) $\displaystyle\int \frac{e^{\sec^{-1}x}}{x\sqrt{x^2-1}}\,dx$	9) $\displaystyle\int \frac{\cos 2x}{(\sin x + \cos x)^2}\,dx$
5) $\displaystyle\int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x}\,dx$	10) $\displaystyle\int \sqrt{\sin x}\cos^3 x\,dx$