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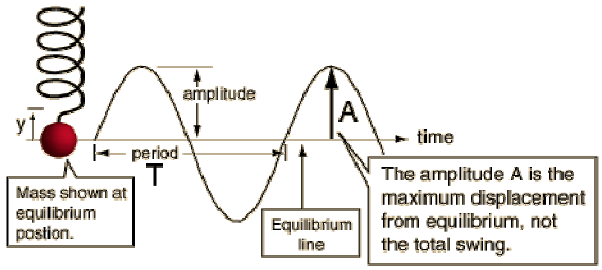
Simple harmonic motion



Definition: ***Simple harmonic motion is "motion where the force acting on a body and thereby acceleration of the body is proportional to, and opposite in direction to the displacement from its equilibrium position" (i.e. $F = -kx$).***

Period (T)

The period of wave motion is the time in which a particle of the medium completes one oscillation.

Frequency (f)

It is the number of waves produced per second or the number of oscillations made by a particle of the medium per second.

Wavelength (λ)

It is the distance travelled by the wave in the time in which a particle of the medium completes one oscillation. In the case of a transverse wave, the distance between the centres of two nearest crests or between the centres two nearest troughs gives the wavelength. In the case of a longitudinal wave, the distance between the centres of two nearest compressions or between the centres of two nearest rarefactions gives the wavelength.

Velocity (v)

The distance travelled by the wave in one second gives its velocity. In a time equal to the period, the wave covers a distance equal to wavelength (λ).

Amplitude (A)

It is the maximum displacement of a particle of the medium from its mean position.

Phase

When a wave passes through a medium, the particles of the medium vibrate about their respective mean positions in the same manner, but reach the corresponding positions in their paths at different instants of time. These relative positions represent the phase of the motion. It is measured either in terms of the angle that the particle has described (denoted as a fraction of 2π)

or the time that has elapsed (measured as a fraction of the time period T), since the particle last passed through its mean position in the positive direction.

The phase difference between any two particles indicates the extent by which they are out of step with each other. For example, a particle on the crest and a particle on the adjacent trough of a wave differ in phase by 180^o or π radians.

Wavefront

A surface on which all the particles of the medium are in identical state of motion at a given instant, is called a wavefront. It is the locus of all the points which are in the same phase. In a homogenous and isotropic medium, the wave front is always perpendicular to the direction of propagation of the wave.

Angular frequency (ω)

It is defined as the rate of change of phase with time. One full wave (i.e., a crest and a trough) is associated with a phase change of 360^o or 2π radians. The time required for this change is the period T. Thus, the angular frequency $= \frac{2\pi}{T}$.

Propagation constant (K)

It is defined as the space rate of change of phase.

Harmonic oscillator (Definition and types)

A harmonic oscillator is a system which, when displaced from its equilibrium position, experiences a restoring force F proportional to the displacement x according to Hooke's law:

$F = -kx$, where k is a positive constant.

If F is the only force acting on the system, the system is called a **simple harmonic oscillator**,

If a frictional force (damping) proportional to the velocity is also present, the harmonic oscillator is described as a damped oscillator. In such situation, the amplitude of the oscillations decreases with time.

If an external time-dependent force is present, the harmonic oscillator is described as a driven oscillator.

Simple harmonic oscillator (Finding out its differential equation and total energy)

The simple harmonic oscillator has no driving force, and no friction (damping), so the net force is just:

$$F = -kx$$

Using Newton's Second Law of motion,

$$F = ma = -kx$$

The acceleration, ***a*** is equal to the second derivative of ***x.***

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

If we define ${\omega_0}^2 = k/m$, then the equation can be written as follows,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + {\omega_0}^2 x = 0$$

, ***this is the differential equation of simple harmonic oscillator***.

Now let us define**,**

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$$

We observe that:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\dot{x}$$

and substituting

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\dot{x} + {\omega_0}^2 x = 0$$

$$\mathrm{d}\dot{x}\cdot\dot{x} + {\omega_0}^2 x\cdot\mathrm{d}x = 0$$

Integrating, $\dot{x}^2 + {\omega_0}^2 x^2 = K$ where, K is the integration constant, set $K = (A\,\omega_0)^2$

$$\dot{x}^2 = A^2{\omega_0}^2 - {\omega_0}^2 x^2$$

$$\dot{x} = \pm\omega_0\sqrt{A^2 - x^2}$$

$$\frac{\mathrm{d}x}{\pm\sqrt{A^2 - x^2}} = \omega_0 \mathrm{d}t$$

integrating, the results (including integration constant φ) are

$$\begin{cases} \arcsin\frac{x}{A} = \omega_0 t + \phi \\ \arccos\frac{x}{A} = \omega_0 t + \phi \end{cases}$$

and has the general solution

$$x = A\cos(\omega_0 t + \phi)$$

where the amplitude A and the phase ϕ are determined by the initial conditions.

Alternatively, the general solution can be written as

$$x = A\sin(\omega_0 t + \phi)$$

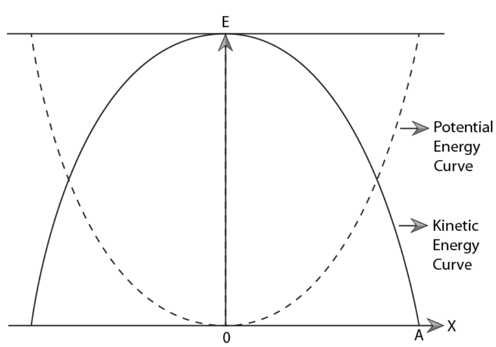
where the value of ϕ is shifted by $\pi/2$ relative to the previous form;

The kinetic energy is $T = \frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{1}{2}kA^2\sin^2(\omega_0 t + \phi)$.

and the potential energy is, $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

so the total energy of the system has the constant value,

$$E = \frac{1}{2}kA^2.$$



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Composition of two simple harmonic vibrations of equal time periods acting at right angles (Formation of Lissajous' figures)

Definition: ***When a particle is influenced simultaneously by two simple harmonic motions at right angles to each other, the resultant motion of the particle traces a curve. These curves are called Lissajous figures.***

let $x = a\sin(\omega t + \delta)$ …………. (1)

and $y = b\sin\omega t$ …………... (2)

Represent the displacements of a particle along the *X* and *Y* axes due to influence of two simple harmonic vibrations acting simultaneously on a particle in perpendicular directions. Here, the two vibrations are of the same time period but are of different amplitudes and phase angles.

From eq (2), $\sin\omega t = \dfrac{y}{b}$ and $\cos\omega t = \sqrt{1 - \dfrac{y^2}{b^2}}$

From eq (1), $\dfrac{x}{a} = [\sin\omega t\cos\delta + \cos\omega t\sin\delta]$ ……… (3)

Substituting the values of *sinωt* and *cosωt* in eq (3)

$$\left[\frac{x}{a} = \frac{y}{b}\cos\delta + \sqrt{1 - \frac{y^2}{b^2}}\sin\delta\right]$$

or
$$\frac{x}{a} - \frac{y}{b}\cos\delta = \sqrt{1 - \frac{y^2}{b^2}}\sin\delta$$

Squaring,
$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\delta - \frac{2xy}{ab}\cos\delta = \left(1 - \frac{y^2}{b^2}\right)\sin^2\delta$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\left[\sin^2\delta + \cos^2\delta\right] - \frac{2xy}{ab}\cos\delta = \sin^2\delta$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\delta=\sin^2\delta \qquad \text{.............} \ (4)$$

This represents the general equation of an ellipse. Thus due to the superimposition of two simple harmonic vibrations, the displacement of the particle will be along a curve given by eqation (4). The resultant vibration of the particle will depend upon the value of δ.

Special Cases. (1) When $\delta = 0 \sin\delta = 0$ and $\cos\delta = 1$

From equation (*iii*)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}=0$$

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2=0$$

or
$$y=\frac{bx}{a}$$

This is the equation of a straight line. Therefore, the emergent light will be plane polarized (Fig. 10.32).

(2) When $\delta = \frac{\pi}{2}$, $\cos\delta = 0, \sin\delta = 1$

From equation (*iii*)

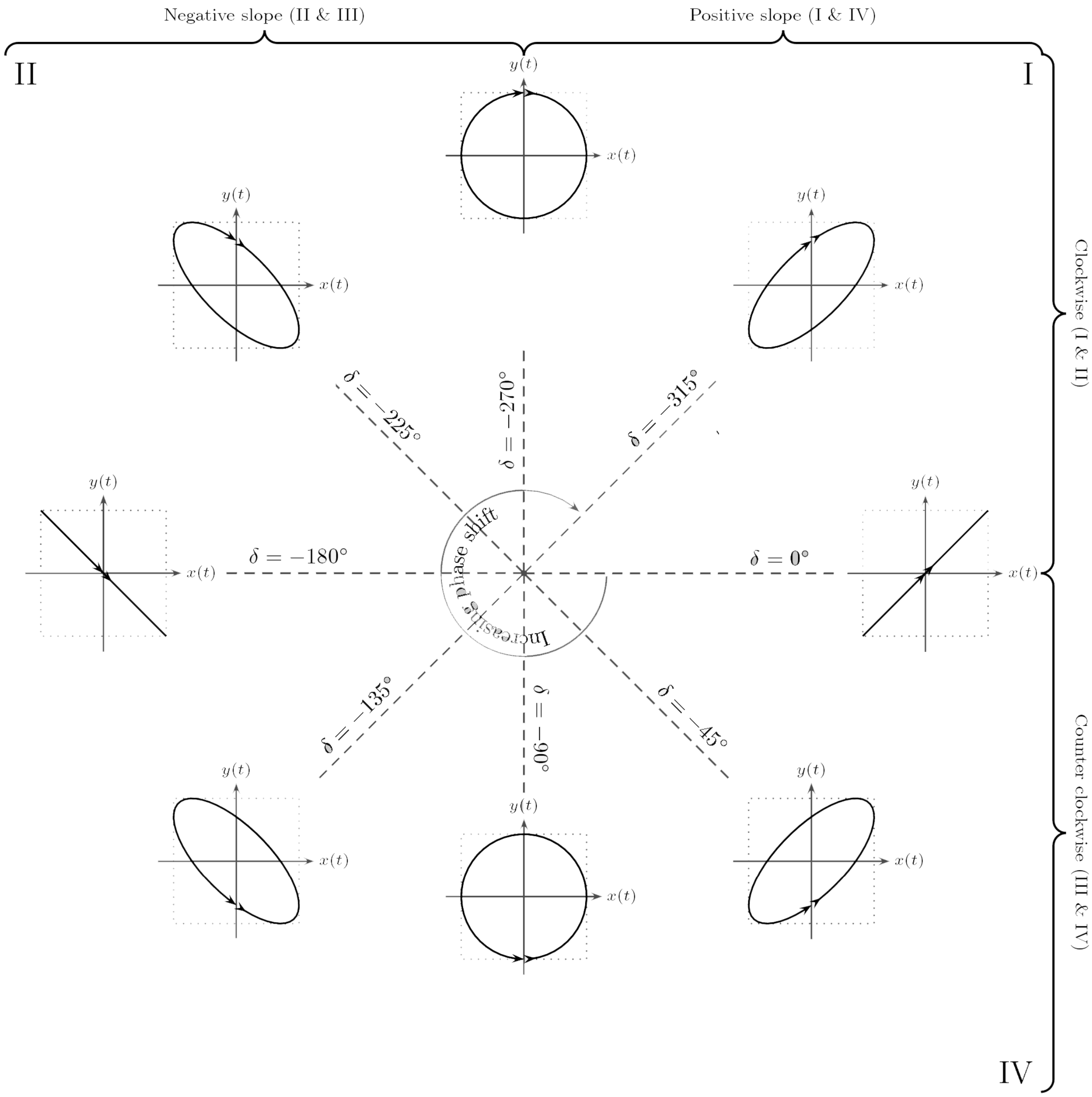
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This represents the equation of a symmetrical ellipse. The emergent light in this case will be **elliptically polarized** provided $a \neq b$.

(3) When $\delta = \frac{\pi}{2}$ and $a = b$

From equation (*iii*),

$$x^2+y^2=a^2$$



Negative slope (II & III)
Positive slope (I & IV)
II
I
IV
Clockwise (I & II)
Counter clockwise (III & IV)
$y(t)$
$x(t)$
$\delta = 0°$
$\delta = -45°$
$\delta = -90°$
$\delta = -135°$
$\delta = -180°$
$\delta = -225°$
$\delta = -270°$
$\delta = -315°$
Increasing phase shift

Transverse and Longitudinal wave

In this type of wave motion, the particles of the medium vibrate at right angles to the direction of propagation of the wave.

Longitudinal waves are waves that have vibrations along or parallel to their direction of travel.

Equation for a Progressive Wave

The simplest type of wave is the one in which the particles of the medium are set into simple harmonic vibrations as the wave passes through it. Consider a particle *O* in the medium. The displacement at any instant of time is given by

$$y = A\sin\omega t \text{ - - - - - } (1.3)$$

Where *A* is the amplitude, ω is the angular frequency of the wave. Consider a particle *P* at a distance *x* from the particle *O* on its right. Let the wave travel with a velocity *v* from left to right. Since it takes some time for the disturbance to reach *P*, its displacement can be written as

$$y = A\sin(\omega t - \phi) \text{ - - - - - } (1.4)$$

Where ϕ is the phase difference between the particles *O* and *P*. We know that a path difference of λ corresponds to a phase difference of 2π radians. Hence a path difference of *x* corresponds to a phase difference of $\frac{2\pi x}{\lambda}$.

$$\therefore \phi = \frac{2\pi x}{\lambda} \text{ - - - - - } (1.5)$$

Substituting equation (1.5) in equation (1.4), we get,

$$y = A\sin\left(\omega t - \frac{2\pi x}{\lambda}\right)$$

But $\omega = \frac{2\pi}{T}$

$$\therefore y = A\sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$$

$$\text{or } y = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \text{-----} (1.6)$$

$$\text{But } v = \frac{\lambda}{T} \text{ or } T = \frac{\lambda}{v} \qquad \therefore y = A\sin 2\pi\left(\frac{vt}{\lambda} - \frac{x}{\lambda}\right)$$

$$\text{or } y = A\sin\frac{2\pi}{\lambda}(vt - x) \text{-----} (1.7)$$

Similarly, for a particle at a distance *x* to the left of *0*, the equation for the displacement is given by

$$y = A\sin\frac{2\pi}{\lambda}(vt + x)$$

Differentiating equation (1.7) with respect to *x*, we get,

$$\frac{dy}{dx} = -A\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x) \text{-----} (1.8)$$

$\frac{dy}{dx}$ represents the strain or the compression. When $\frac{dy}{dx}$ is positive, a rarefaction takes place and when $\frac{dy}{dx}$ is negative, a compression takes place.

The velocity of the particle, whose displacement *y* is represented by equation (1.7), is obtained by differentiating it with respect to ***t***, since velocity is the rate of change of displacement with respect to time.

$$\therefore \frac{dy}{dt} = A\frac{2\pi v}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x) \text{-----} (1.9)$$

Comparing equations (1.8) and (1.9) we get,

$$\frac{dy}{dt} = -v\frac{dy}{dx} \text{-----} (1.10)$$

∴ Particle velocity = wave length x slope of the displacement curve or strain

Differentiating equation (1.8),

$$\frac{d^2y}{dx^2} = -A\frac{4\pi^2}{\lambda^2}\sin\frac{2\pi}{\lambda}(vt-x) \text{ - - - - - } (1.11)$$

Differentiating equation (1.9)

$$\frac{d^2y}{dt^2} = -A\frac{4\pi^2}{\lambda^2}v^2\sin\frac{2\pi}{\lambda}(vt-x) \text{ - - - - - } (1.12)$$

Comparing equation (1.11) and (1.12) we get,

$$\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2} \text{ - - - - - } (1.13)$$

This is the differential equation of wave motion.

Phase velocity

The phase velocity (or phase speed) of a wave is the rate at which the phase of the wave propagates in space. For such a component, any given phase of the wave (for example, the crest) will appear to travel at the phase velocity. The phase speed is given by

$$v_p = \frac{\lambda}{T}.$$

Or, equivalently, in terms of the wave's angular frequency ω and wavenumber k by

$$v_p = \frac{\omega}{k}.$$

In a dispersive medium, the phase velocity varies with frequency and is not necessarily the same as ***the group velocity of the wave***, which ***is the rate that changes in amplitude (known as the envelope of the wave) will propagate.***

The phase velocity of electromagnetic radiation may under certain circumstances (e.g. in the case of anomalous dispersion) exceed the speed of light in a vacuum.

Group velocity

The group velocity of a wave is the velocity with which the variations in the shape of the wave's amplitude (known as the modulation or envelope of the wave) propagate through space. [*For example, imagine what happens if you throw a stone into the middle of a very still pond. When the stone hits the surface of the water, a circular pattern of waves appears. It soon turns into a circular ring of waves with a quiescent center. The ever expanding ring of waves is the* ***group****, within which one can discern individual wavelets of differing wavelengths traveling at different speeds. The longer waves travel faster than the group as a whole, but they die out as they approach the leading edge. The shorter waves travel slower and they die out as they emerge from the trailing boundary of the group.*]

The group velocity v_g is defined by the equation

$$v_g \equiv \frac{\partial \omega}{\partial k}$$

where: ω is the wave's angular frequency; k is the wave number.

Reverberation

Reverberation is the persistence of sound after a sound is produced. Reverberation is created when a sound or signal is reflected causing a large number of reflections to build up and then decay as the sound is absorbed by the surfaces of objects in the space – which could include furniture, people, and air. This is most noticeable when the sound source stops but the reflections continue, decreasing in amplitude, until they reach zero amplitude.

Sabine's Formula

Prof. Wallace C. Sabine (1868 - 1919) of Harvard University investigated architectural acoustics scientifically. He deduced experimentally that the reverberation time is:

- directly proportional to the volume of the hall
- inversely proportional to the effective absorbing surface area of the walls and the materials inside the hall

$$\therefore T \alpha \frac{V}{\sum aA}$$

where, V is the volume of the hall, **a** is the absorption coefficient of an area A. If the volume is measured in cubic feet and area in square feet, then the experimentally obtained value of the constant of proportionality, according to Sabine is 0.05. Then,

$$T = \frac{0.05V}{\sum aA}$$

If there are different absorbing surfaces of area A_1, A_2, A_3, A_4, etc., having absorption coefficients a_1, a_2, a_3, a_4 etc., then,

$$T = \frac{0.05V}{a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 + \ldots}$$

If the area is measured in square meters and the volume in cubic meters, then Sabine's formula can be written as:

$$T = \frac{0.16V}{\sum aA}$$

Increasing the effective area of complete absorption like, changing the wall materials or adding more furniture may decrease an excessive reverberation time for a hall. But this also decreases the intensity of a steady tone. Also, too much absorption will make the reverberation time too short and cause the room to sound acoustically 'dead'. Hence, the optimum reverberation time is a compromise between clarity of sound and its intensity.