Integration by Method of Partial Fraction

Ex. 1) $\int \frac{\mathbf{1}}{(\mathbf{x}+\mathbf{1})(\mathbf{x}-\mathbf{5})}\mathbf{dx}$

Solution: Let,

$$\frac{1}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(i)}$$

$$\Rightarrow 1 = \frac{(x+1)(x-5)A}{x+1} + \frac{(x+1)(x-5)B}{x-5}$$

$$\Rightarrow 1 = A(x-5) + B(x+1) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}$$

Putting $x = 5$ in (ii) we get,

$$\Rightarrow 1 = A.0 + B.6$$

$$\Rightarrow 6B = 1 \Rightarrow B = \frac{1}{6}$$

Putting $x = -1$ in (ii) we get,

$$\Rightarrow 1 = A.-6 + B.0$$

$$\Rightarrow -6A = 1 \Rightarrow A = \frac{-1}{6}$$

From (i),

$$\frac{1}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5}$$

$$\Rightarrow \frac{1}{(x+1)(x-5)} = \frac{\frac{-1}{6}}{x+1} + \frac{\frac{1}{6}}{x-5}$$

$$\Rightarrow \int \frac{dx}{(x+1)(x-5)} = \int \frac{\frac{-1}{6}}{x+1}dx + \int \frac{\frac{1}{6}}{x-5}dx$$

$$\Rightarrow \int \frac{dx}{(x+1)(x-5)} = \frac{-1}{6}\int \frac{1}{x+1}dx + \frac{1}{6}\int \frac{1}{x-5}dx$$

$$\Rightarrow \int \frac{dx}{(x+1)(x-5)} = \frac{-1}{6}\ln(x+1) + \frac{1}{6}\ln(x-5) + c$$

Ex. 2) $\int \frac{\mathbf{1}}{\mathbf{x^2 + x}} \mathbf{dx}$

Solution: Let,

$$\frac{1}{x^2 + x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(i)}$$

$$\Rightarrow 1 = A(x+1) + Bx \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}$$

Putting $x = 0$ in (ii) we get,

$$\Rightarrow 1 = A.1 + B.0$$

$$\Rightarrow A = 1$$

Putting $x = -1$ in (ii) we get,

$$\Rightarrow 1 = A.0 + B.-1$$

$$\Rightarrow -B = 1 \Rightarrow B = -1$$

From (i),

$$\frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$$

$$\Rightarrow \int \frac{1}{x(x+1)} dx = \int \frac{1}{x} dx - \int \frac{1}{x+1} dx$$

$$\Rightarrow \int \frac{1}{x(x+1)} dx = \ln x - \ln(x+1) + c$$

Ex. 3) $\int \frac{\mathbf{x - 1}}{\mathbf{(x-2)(x-3)}} \mathbf{dx}$

Solution: Let,

$$\frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(i)}$$

$$\Rightarrow x - 1 = A(x-3) + B(x-2) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}$$

Putting $x = 2$ in (ii) we get,

$\Rightarrow 1 = A.-1 + B.0$

$\Rightarrow A = -1$

Putting $x = 3$ in (ii) we get,

$\Rightarrow 2 = A.0 + B.1$

$\Rightarrow B = 2$

From (i),

$$\frac{x-1}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{2}{x-3}$$

$$\Rightarrow \int \frac{x-1}{(x-2)(x-3)}\,dx = \int \frac{-1}{x-2}\,dx + \int \frac{2}{x-3}\,dx$$

$$\Rightarrow \int \frac{x-1}{(x-2)(x-3)}\,dx = -\ln(x-2) + 2\ln(x-3) + c$$

Ex. 4) $\int \frac{\mathbf{2x-1}}{\mathbf{x(x-1)(x-2)}}\,\mathbf{dx}$

Solution: Let,

$$\frac{2x-1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\Rightarrow 2x-1 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Putting $x = 0$ in (ii) we get,

$\Rightarrow -1 = A.(-1)(-2) + B.0 + C.0$

$\Rightarrow 2A = -1 \Rightarrow A = \frac{-1}{2}$

Putting $x = 1$ in (ii) we get,

$\Rightarrow 1 = A.0 + B.1.(-1) + C.0$

$\Rightarrow B = -1$

Putting $x = 2$ in (ii) we get,

$\Rightarrow 3 = A.0 + B.0 + C.2.1$

$\Rightarrow 2C = 3 \Rightarrow C = \frac{3}{2}$

From (i),

$$\frac{2x-1}{x(x-1)(x-2)} = \frac{\frac{-1}{2}}{x} + \frac{-1}{x-1} + \frac{\frac{3}{2}}{x-2}$$

$$\Rightarrow \int \frac{2x-1}{x(x-1)(x-2)}dx = \frac{-1}{2}\int \frac{1}{x}dx - \int \frac{1}{x-1}dx + \frac{3}{2}\int \frac{1}{x-2}dx$$

$$\Rightarrow \int \frac{2x-1}{x(x-1)(x-2)}dx = -\frac{1}{2}\ln x - \ln(x-1) + \frac{3}{2}\ln(x-2) + c$$

5) $\int \frac{\mathbf{x}}{\mathbf{(x-1)^2(x+2)}}\mathbf{dx}$

Solution: Let,

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(i)}$$

$$\Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}$$

Putting $x = 1$ in (ii) we get,

$\Rightarrow 1 = A.0 + 3B + C.0$

$\Rightarrow B = \frac{1}{3}$

Putting $x = -2$ in (ii) we get,

$\Rightarrow -2 = A.0 + B.0 + C.(-2-1)^2$

$\Rightarrow -2 = 9C \Rightarrow C = \frac{-2}{9}$

Calculating the co − efficients of x^2 from both sides,

$$\Rightarrow 0 = A + C \Rightarrow 0 = A - \frac{2}{9} \Rightarrow A = \frac{2}{9}$$

From (i),

$$\frac{x}{(x-1)^2(x+2)} = \frac{\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} + \frac{-\frac{2}{9}}{x+2}$$

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)}\,dx = \frac{2}{9}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{(x-1)^2}\,dx - \frac{2}{9}\int \frac{1}{x+2}\,dx$$

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)}\,dx = \frac{2}{9}\ln(x-1) - \frac{1}{3}\left(\frac{1}{x-1}\right) - \frac{2}{9}\ln(x+2) + c$$

H.W:

1) $\int \frac{x+1}{x^2-5x+6}\,dx$

2) $\int \frac{2x+3}{x^3+x^2-2x}\,dx$

3) $\int \frac{1}{x^2(x-1)}\,dx$

4) $\int \frac{x+1}{x^2-7x+10}\,dx$

5) $\int \frac{1}{x\,(x+1)^2}\,dx$