LIMIT & CONTINUITY

Definition of Limit: If the values of **f (x)** become arbitrarily close to a single number **l** as the values of a variable **x** approaches to **a** from both sides of **a** (right and left), then **l** is called the limit of the function $\mathbf{f(x)}$. It is denoted by $\lim\limits_{x\to a} \mathbf{f(x)} = \mathbf{l}$.

$$\textbf{Ex. 1)}\ \lim_{x\to 1}(x^2 - 2x + 1)$$

$$= \lim_{x\to 1}\ (x^2 - 2x + 1)$$

$$= 1 - 2 + 1 = 0$$

$$\textbf{Ex. 2)} \lim_{x\to 3} \frac{x^2 - 2x}{x + 1}$$

$$= \lim_{x\to 3} \frac{x^2 - 2x}{x + 1}$$

$$= \frac{9 - 6}{3 + 1} = \frac{3}{4}$$

$$\textbf{Ex. 3)}\ \lim_{x\to -1} \frac{x + 1}{x^2 - 1}$$

$$= \lim_{x\to -1} \frac{x + 1}{x^2 - 1}$$

$$= \lim_{x\to -1} \frac{(x + 1)}{(x + 1)(x - 1)}$$

$$= \lim_{x\to -1} \frac{1}{x - 1}$$

$$= -\frac{1}{2}$$

$$\textbf{Ex. 4)}\ \lim_{x\to \frac{\pi}{2}} \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\left(x-\frac{\pi}{2}\right)\left(x^2+x.\frac{\pi}{2}+\frac{\pi^2}{4}\right)}{\left(x-\frac{\pi}{2}\right)}$$

$$= \lim_{x\to\frac{\pi}{2}} \left(x^2+x.\frac{\pi}{2}+\frac{\pi^2}{4}\right)$$

$$= \frac{\pi^2}{4}+\frac{\pi^2}{4}+\frac{\pi^2}{4} = \frac{3\pi^2}{4}$$

Ex. 5) $\lim_{x\to-\infty} \frac{\sqrt{3x^4+x}}{x^2-8}$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^4\left(3+\frac{x}{x^4}\right)}}{x^2\left(1-\frac{8}{x^2}\right)}$$

$$= \lim_{x\to-\infty} \frac{x^2\sqrt{\left(3+\frac{1}{x^3}\right)}}{x^2\left(1-\frac{8}{x^2}\right)}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{3+\frac{1}{x^3}}}{1-\frac{8}{x^2}}$$

$$= \frac{\sqrt{3+0}}{1-0}$$

$$= \sqrt{3}$$

H.W

1) $\lim_{x\to1} \frac{x^4-1}{x-1}$

2) $\lim_{x\to\infty} \frac{5x^2+7}{3x^2-x}$

3) $\lim_{x\to\infty} \frac{2^x-2^{-x}}{2^x+2^{-x}}$

4) $\lim_{x\to0} \frac{\sqrt{1+x}-1}{x}$

<u>Continuity:</u> If $\mathbf{f(a)}$is defined, $\lim\limits_{\mathbf{x\to a}} \mathbf{f(x)}$ exists and $\lim\limits_{\mathbf{x\to a}} \mathbf{f(x)} = \mathbf{f(a)}$ then $\mathbf{f(x)}$is called continuous at $\mathbf{x = a}$. Here, $\lim\limits_{\mathbf{x\to a}} \mathbf{f(x)}$ means the value of limit of $\mathbf{f\ (x)}$ at $\mathbf{x = a}$ and $\mathbf{f(a)}$means the value of $\mathbf{f(x)}$ at $\mathbf{x = a.}$

If $\lim\limits_{\mathbf{x\to a_-}} \mathbf{f(x)} = \mathbf{f\ (a)}$, then $\mathbf{f(x)}$ is called left continuous at $\mathbf{x = a}$.

If $\lim\limits_{\mathbf{x\to a_+}} \mathbf{f(x)} = \mathbf{f\ (a)}$, then $\mathbf{f(x)}$ is called right continuous at $\mathbf{x = a}$.

If $\mathbf{f(x)}$ is not continuous at $\mathbf{x = a}$ then it is said to be discontinuous.

Ex. 1) At $x = 1$ and $x = 0$ discuss the continuity of the function $f: R \to R$,

$$\text{where, } f(x) = \begin{cases} x^2 + 1 \text{ when } x < 0 \\ x \quad \text{when } 0 \le x \le 1 \\ \dfrac{1}{x} \text{ when } x > 1 \end{cases}$$

Solution:

At $x = 1$,

$$\lim_{x\to 1_+} f(x) = \lim_{x\to 1_+} \frac{1}{x} = \frac{1}{1} = 1$$

$$\lim_{x\to 1_-} f(x) = \lim_{x\to 1_-} (x) = 1$$

and,

when, $x = 1$ then, $f(x) = f(1) = x = 1$

$$\therefore \lim_{x\to 1_+} f(x) = \lim_{x\to 1_-} f(x) = f(1)$$

So, the function $f(x)$ is continuous at $x = 1$.

At $x = 0$,

$$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_+} (x) = 0$$

$$\lim_{x\to 0_-} f(x) = \lim_{x\to 0_-} (x^2 + 1) = 1$$

Since, $\lim\limits_{x\to 0_+} f(x) \neq \lim\limits_{x\to 0_-} f(x)$

So, the function $f(x)$ is not continuous at $x = 0$.

Ex. 2) Investigate the continuity of the function at $x = 0$ and $x = \frac{3}{2}$,

$$\text{where, } f(x) = \begin{cases} 3 + 2x \text{ when } -\frac{3}{2} \leq x < 0 \\ 3 - 2x \text{ when } 0 \leq x \leq \frac{3}{2} \\ -3 - 2x \quad \text{when } x \geq \frac{3}{2} \end{cases}$$

Solution:

At $x = 0$,

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} (3 - 2x) = 3$$

$$\lim_{x \to 0_-} f(x) = \lim_{x \to 0_-} (3 + 2x) = 3$$

and,

when, $x = 0$ then, $f(x) = f(0) = 3 - 0 = 3$

$$\therefore \lim_{x \to 0_+} f(x) = \lim_{x \to 0_-} f(x) = f(0)$$

So, the function $f(x)$ is continuous at $x = 0$.

At $x = \frac{3}{2}$,

$$\lim_{x \to \frac{3}{2}_+} f(x) = \lim_{x \to \frac{3}{2}_+} (-3 - 2x) = -3 - 2.\frac{3}{2} = -3 - 3 = -6$$

$$\lim_{x \to \frac{3}{2}_-} f(x) = \lim_{x \to \frac{3}{2}_-} (3 - 2x) = 3 - 2.\frac{3}{2} = 3 - 3 = 0$$

Since, $\lim_{x \to \frac{3}{2}_+} f(x) \neq \lim_{x \to \frac{3}{2}_-} f(x)$

So, the function $f(x)$ is not continuous at $x = \frac{3}{2}$.

Ex. 3) Show that, the function f (x) is discontinuous at $x = 2$.

Also Define the function **f** in such a way so that it is continuous at $x = 2$.

$$\text{where, } f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{when } x \neq 2 \\ 3 & \text{when } x = 2 \end{cases}$$

Solution:

At $x = 2$,

$$\lim_{x \to 2_+} f(x) = \lim_{x \to 2_+} \left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \to 2_+} (x + 2) = 4$$

$$\lim_{x \to 2_-} f(x) = \lim_{x \to 2_-} \left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \to 2_-} (x + 2) = 4$$

when, $x = 2$ then, $f(x) = f(2) = 3$

Since, $\lim_{x \to 2_+} f(x) = \lim_{x \to 2_-} f(x) \neq f(2)$. So, the function f(x) is not continuous at $x = 2$.

For continuity f(x) is defined at $x = 2$ in the following way,

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$$

<u>H.W:</u>

1) Show that, the function f (x) is discontinuous at $x = 2$.

$$\text{where, } f(x) = \begin{cases} x + 2 & \text{when } x < 2 \\ x^2 - 1 & \text{when } x \geq 2 \end{cases}$$

2) Show that, the function f (x) is continuous at $x = 0$ and discontinuous at $x = 1$,

$$\text{where, } f(x) = \begin{cases} -x & \text{when } x \leq 0 \\ x & \text{when } 0 < x < 1 \\ 1 - x & \text{when } x \geq 1 \end{cases}$$

3) Investigate the continuity of the function at $x = 1$,

$$\text{where, } f(x) = \begin{cases} x & \text{when } x \leq 1 \\ 2x - 1 & \text{when } x > 1 \end{cases}$$

4) Show that, the function $f(x)$ is continuous at $x = 0$ and discontinuous at $x = -1$.

$$\text{where, } f(x) = \begin{cases} 1 + x \quad \text{when } -4 \le x < -1 \\ 4 \text{ when } -1 \le x \le 0 \\ 4 + x^2 \text{ when } 0 < x \le 4 \end{cases}$$

5) Investigate the continuity of the function at $x = 0,1,2$

$$\text{where, } f(x) = \begin{cases} -x^2 \text{ when } x \le 0 \\ 5x - 4 \text{ when } 0 < x \le 1 \\ 4x^2 - 3x \text{ when } 1 < x < 2 \\ 3x + 4 \text{ when } x \ge 2 \end{cases}$$

6) Investigate the continuity of the function at $x = 1$ and $x = 2$,

$$\text{where, } f(x) = \begin{cases} \ln x \quad \text{when } 0 < x \le 1 \\ 0 \text{ when } 1 < x \le 2 \\ 1 + x^2 \text{ when } x > 2 \end{cases}$$