Partial Differentiation

Ex. 1) If $\mathbf{u = x^2 + 5xy - 6y^2 + 8}$**, then Find** $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}$

Solution: Given,

$$u = x^2 + 5xy - 6y^2 + 8$$

$$\frac{\partial u}{\partial x} = 2x + 5y$$

$$\frac{\partial u}{\partial y} = 5x - 12y$$

$$\frac{\partial^2 u}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(5x - 12y) = 5$$

$$\frac{\partial^2 u}{\partial y \, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(2x + 5y) = 5$$

Ex. 2) If $\mathbf{u = ax^2 + 2hxy + by^2}$**, then Find** $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial y^3}$

Solution: Given,

$$u = ax^2 + 2hxy + by^2$$

$$\frac{\partial u}{\partial x} = 2ax + 2hy$$

$$\frac{\partial u}{\partial y} = 2hx + 2by$$

$$\frac{\partial^2 u}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(2hx + 2by) = 2h$$

$$\frac{\partial^2 u}{\partial y \, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(2ax + 2hy) = 2h$$

Now,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 2a$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial}{\partial x}(2a) = 0$$

And,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 2b$$

$$\frac{\partial^3 u}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial y}(2b) = 0$$

Ex. 3) If $\mathbf{u = \sin xy}$, then Find $\boldsymbol{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}}$

Solution: Let, $z = xy$

Then, $u = \sin z$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial x} = \frac{\partial}{\partial z}(\sin z).\frac{\partial}{\partial x}(z) = \cos z\,.\frac{\partial}{\partial x}(xy) = \cos z\,.\,y = y\cos xy$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial y} = \frac{\partial}{\partial z}(\sin z).\frac{\partial}{\partial y}(z) = \cos z\,.\frac{\partial}{\partial y}(xy) = \cos z\,.\,x = x\cos xy$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x\cos xy) = x(-\sin xy)\,.\,y + \cos xy = \cos xy - xy\sin xy$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(x\cos xy) = x(-\sin xy).\,x + \cos xy\,.\,0 = -x^2\sin xy$$

Ex. 4) If $\mathbf{u = x\ \sin y + y \sin x}$, then Find $\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}$

Solution: Given,

$$u = x\ \sin y + y \sin x$$

$$\frac{\partial u}{\partial x} = \sin y + y \cos x$$

$$\frac{\partial u}{\partial y} = x \cos y + \sin x$$

Therefore,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \sin y + y \cos x + x \cos y + \sin x$$

Ex. 5) If $\mathbf{u = 3x^2 + 7xy + 4y^2}$, then Find $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \frac{\partial \mathbf{u}}{\partial \mathbf{y}}, \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x} \partial \mathbf{y}}, \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y} \partial \mathbf{x}}$ at the point $(\mathbf{1}, -\mathbf{1})$.

Solution: Given,

$$u = 3x^2 + 7xy + 4y^2$$

$$\frac{\partial u}{\partial x} = 6x + 7y$$

$$\frac{\partial u}{\partial y} = 7x + 8y$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(7x + 8y) = 7$$

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(6x + 7y) = 7$$

Therefore, at the point (1,-1)

$$\frac{\partial u}{\partial x} = 6 - 7 = -1$$

$$\frac{\partial u}{\partial y} = 7 - 8 = -1$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = 7$$

$$\frac{\partial^2 u}{\partial y\, \partial x} = 7$$

Ex. 6) If $u = \sin^{-1}\frac{y}{x} + \tan^{-1}\frac{x}{y}$, then prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Solution: Given,

$$u = \sin^{-1}\frac{y}{x} + \tan^{-1}\frac{x}{y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \frac{-y}{x^2} + \frac{\frac{1}{y}}{1+\left(\frac{x}{y}\right)^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = x\left(\frac{\frac{-y}{x^2}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot + \frac{\frac{1}{y}}{1+\left(\frac{x}{y}\right)^2}\right) = \frac{\frac{-y}{x}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot + \frac{\frac{x}{y}}{1+\left(\frac{x}{y}\right)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \frac{1}{x} + \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = y\left(\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \frac{1}{x} + \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2}\right) = \frac{\frac{y}{x}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} + \frac{\frac{-x}{y}}{1+\left(\frac{x}{y}\right)^2}$$

L. H. S: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$= \frac{\frac{-y}{x}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} . + \frac{\frac{x}{y}}{1+\left(\frac{x}{y}\right)^2} + \frac{\frac{y}{x}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} + \frac{\frac{-x}{y}}{1+\left(\frac{x}{y}\right)^2} = 0 = \text{R.H.S (proved)}$$

Ex. 7) If $\mathbf{u = \sqrt{x^2 + y^2}}$**, then prove that,** $\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \dfrac{\mathbf{1}}{\mathbf{u}}$

Solution: Given,

$$u = \sqrt{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}} . 2x = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\sqrt{x^2 + y^2} . \frac{\partial}{\partial x}(x) - x \frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2}\right)}{\left(\sqrt{x^2 + y^2}\right)^2}$$

$$= \frac{\sqrt{x^2 + y^2} - x\frac{x}{\sqrt{x^2 + y^2}}}{x^2 + y^2}$$

$$= \frac{\frac{x^2 + y^2 - x^2}{\sqrt{x^2 + y^2}}}{x^2 + y^2}$$

$$= \frac{y^2}{(x^2 + y^2)\sqrt{x^2 + y^2}}$$

Now,

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2}} . 2y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\sqrt{x^2 + y^2} . \frac{\partial}{\partial y}(y) - y \frac{\partial}{\partial y}\left(\sqrt{x^2 + y^2}\right)}{\left(\sqrt{x^2 + y^2}\right)^2}$$

$$= \frac{\sqrt{x^2+y^2} - y\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{\frac{x^2+y^2-y^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\text{L.H.S: } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{u} = \text{R.H.S} \ \text{(proved)}$$

H.W:

1) If $u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, then prove that, $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

2) If $u = x^2 + y^2 + 1$, then prove that, $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

3) If $u = \ln(x^2 + y^2)$, then prove that, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$