Fuzzy Control

Lect 3 Membership Function and Approximate Reasoning

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  - Features of Membership Function
  - Fuzzy Membership Functions
  - Types of Membership Function
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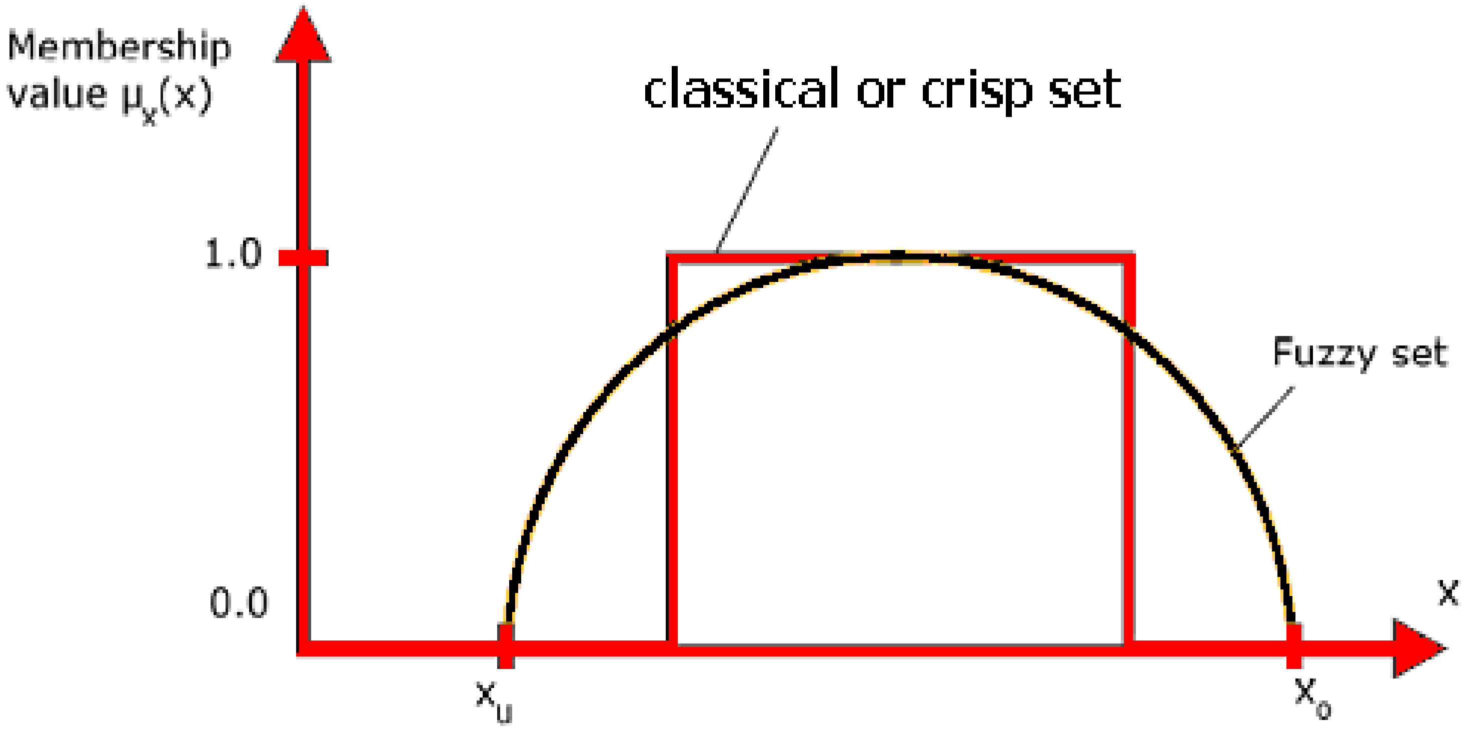
Membership Function

Membership Functions characterize the fuzziness of fuzzy sets. There are an infinite # of ways to characterize fuzzy $\rightarrow$ infinite ways to define fuzzy membership functions.

Membership function essentially embodies all fuzziness for a particular fuzzy set, its description is essential to fuzzy property or operation.

Membership Function



Membership Function

Core: comprises of elements x of the universe, such that

$$\mu_A(x) = 1$$

Support: comprises of elements x of universe, such that

$$\mu_A(x) > 0$$

Boundaries: comprise the elements x of the universe

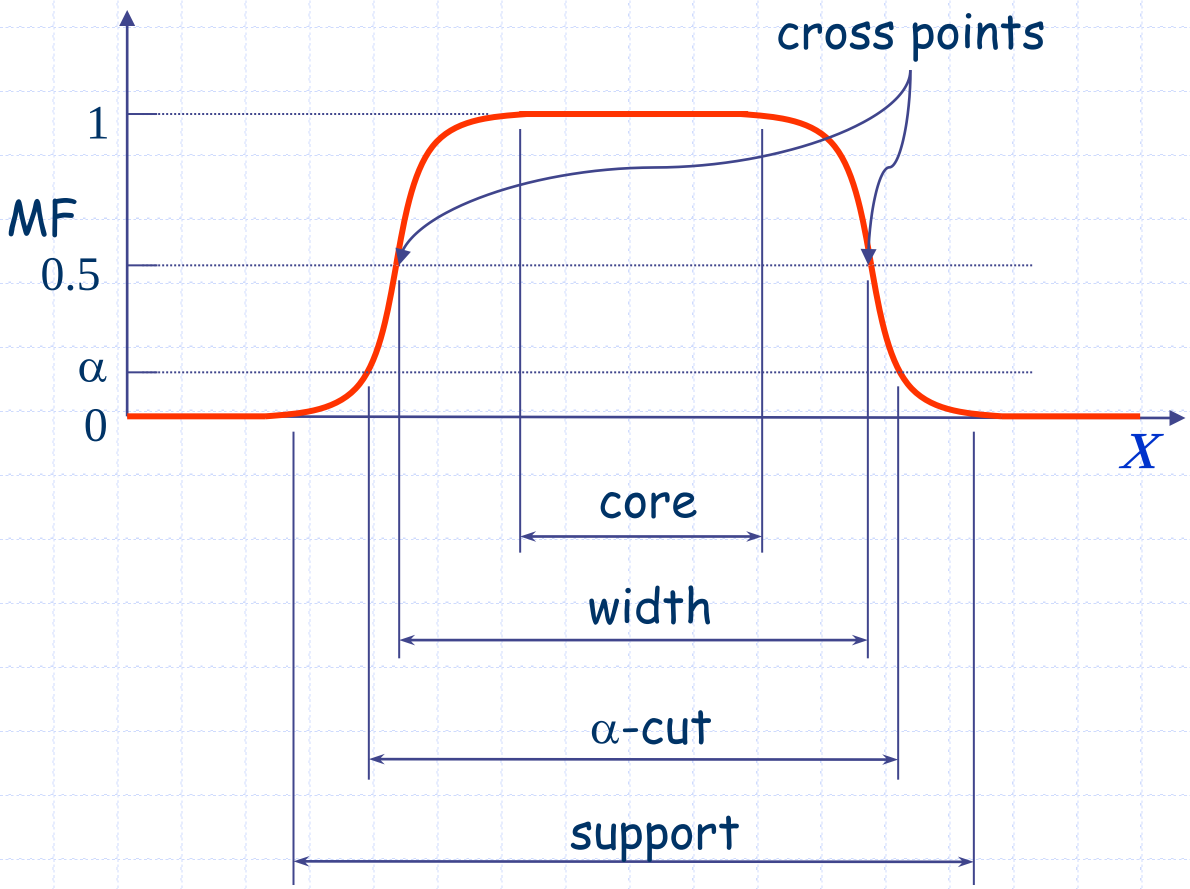
$$0 < \mu_A(x) < 1$$

A normal fuzzy set has at least one element with membership 1

A subnormal fuzzy set has no element with membership=1.

MF Terminology

(1) Core $\mu_A(x) = 1$
(2) Support $\mu_A(x) > 0$
(3) Boundary $\mu_A(x) < 1$
(4) Height $\max\left(\mu_A(x)\right)$
(5) Crossover point $\mu_A(x) = 0.5$



Alpha-Cut:

An α cut or α level set of a fuzzy set A $\subseteq$ X is a set $A_\alpha \subseteq X$,such that:

$$A_\alpha = \{\mu_A(x) \geq \alpha,\ \forall x \in X\}$$

Ex: Consider X = {1, 2, 3} and set A

A = 0.3/1 + 0.5/2 + 1/3

Then $A_{0.5}$ = {2, 3}, $A_{0.1}$ = {1, 2, 3}, A_1 = {3}

Fuzzy Set Math Operations

$$aA = \{a\mu_A(x), \forall x \in X\}$$

Let a =0.5, and A = {0.5/a, 0.3/b, 0.2/c, 1/d}

then $a\ A$ = {0.25/a, 0.15/b, 0.1/c, 0.5/d}

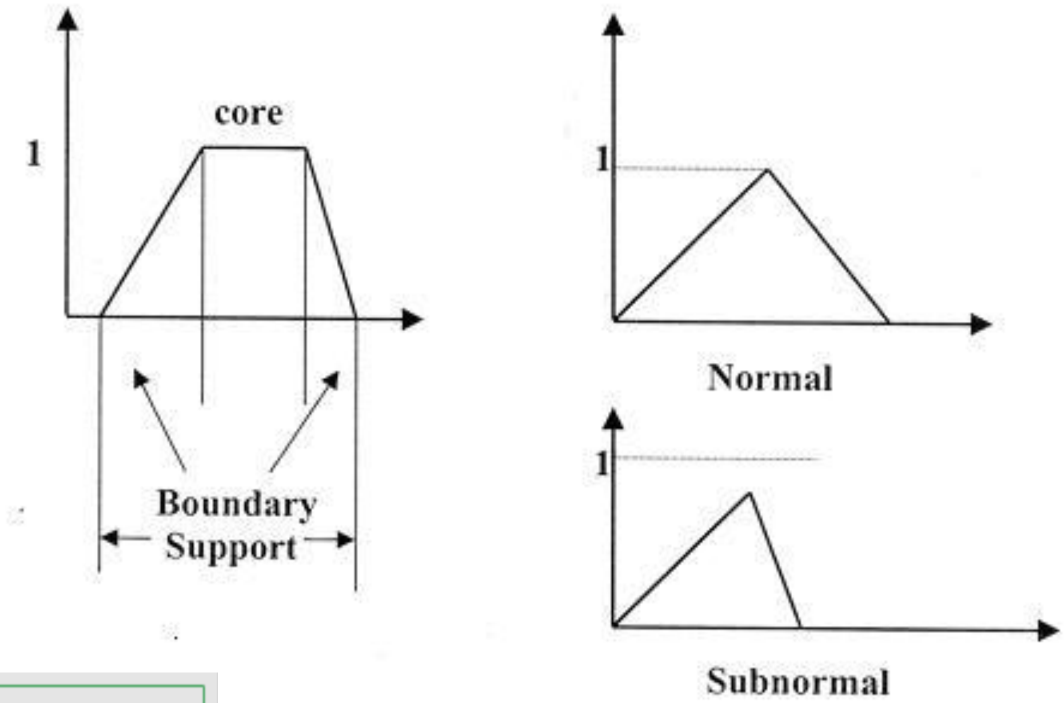
$$A^a = \{\mu_A(x)^a, \forall x \in X\}$$

Let a =2, and A = {0.5/a, 0.3/b, 0.2/c, 1/d}

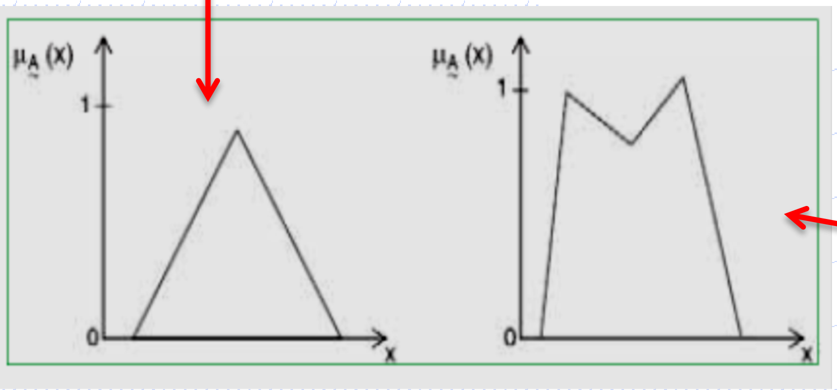
then A^a = {0.25/a, 0.09/b, 0.04/c, 1/d}

Features of Membership Function

Graphically,



Convex fuzzy set



Nonconvex fuzzy set

Features of Membership Function

Example Consider two fuzzy subsets of the set X,

$X = \{a, b, c, d, e\}$

referred to as A and B

$A = \{1/a, 0.3/b, 0.2/c\ 0.8/d, 0/e\}$

And $B = \{0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e\}$

Core:

$core(A) = \{a\}$

$core(B) = \{o\}$

Support:

$supp(A) = \{a, b, c, d\}$

$supp(B) = \{a, b, c, d, e\}$

Cardinality:

$card(A) = 1+0.3+0.2+0.8+0 = 2.3$

$card(B) = 0.6+0.9+0.1+0.3+0.2 = 2.1$

α -cut:

$A_{0.2} = \{a, b, c, d\}$,

$A_{0.3} = \{a, b, d\}$

$A_{0.8} = \{a, d\}$, $A_1 = \{a\}$

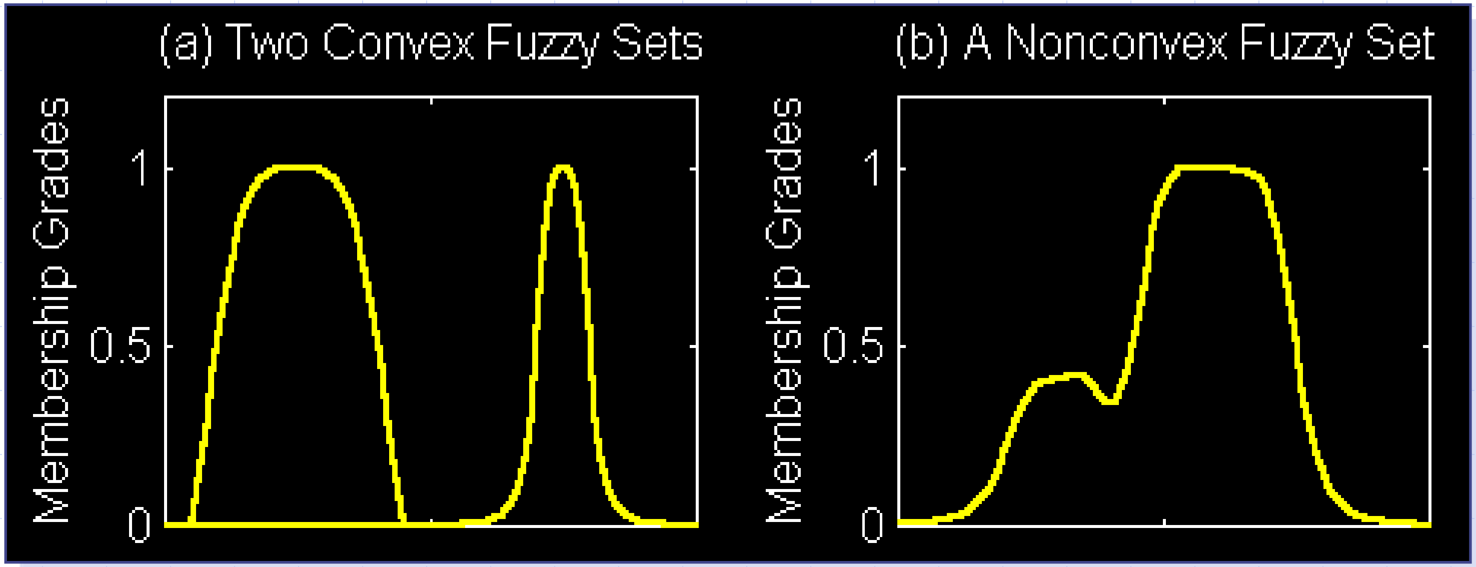
Membership Functions (MF's)

- The most common forms of membership functions are those that are normal and convex.
- However, many operations on fuzzy sets, hence operations on membership functions, result in fuzzy sets that are subnormal and nonconvex.
- Membership functions can be symmetrical or asymmetrical.

Convexity of Fuzzy Sets

- A fuzzy set A is convex if for any λ in [0, 1].

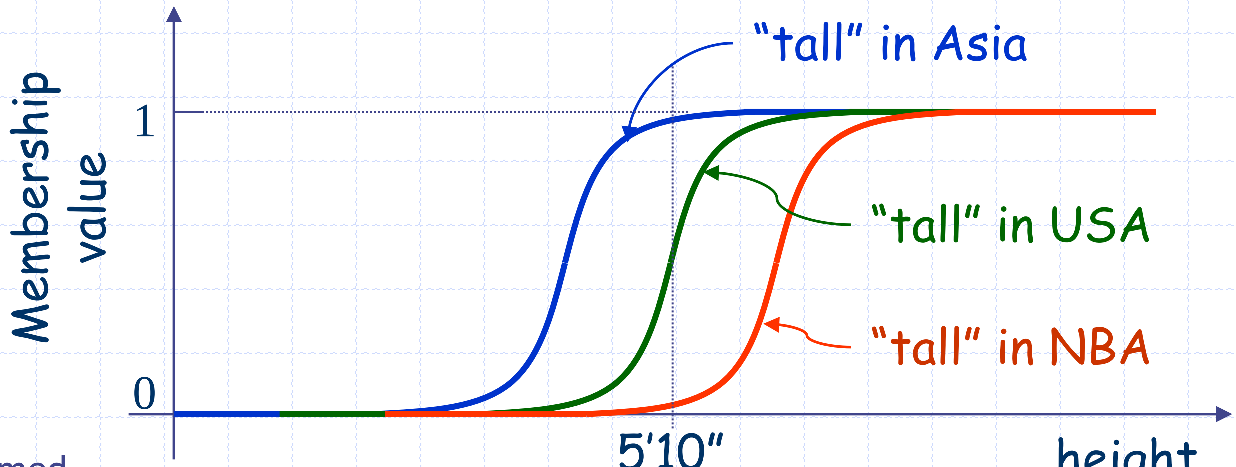
$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$



Membership Functions (MF's)

A fuzzy set is completely characterized by a membership function.

- a subjective measure.
- *not* a probability measure.



Membership Functions (MF’s)

The degree of the fuzzy membership function $\mu\ (x)$ can be define as possibility function not probability function.

What is The Different Between Probability Function and Possibility Function

Example:

When you and your best friend go to visit another friend, in the car your best friend asks you, “Are you sure our friend is at the home?"

you answer,

- "yes, I am sure but I do not know if he is in the bed room or on the roof “
- "I think 90% he is there"

What is The Different Between Probability Function and Possibility Function

Look to the answer. In the first answer, you are sure he is in the house but you do not know where he is in the house exactly.

However, in the second answer you are not sure he may be there and may be not there. That is the different between possibility and the probability

What is The Different Between Probability Function and Possibility Function

In the possibility function the element is in the set by certain degree,

Meanwhile the probability function means that the element may be in the set or not .

So if the probability of (x) = 0.7 that means (x) may be in the set by 70%. But in possibility if the possibility of (x) is 0.7 that means (x) is in the set and has degree 0.7.

Fuzzy Membership Functions

- One of the key issues in all fuzzy sets is how to determine fuzzy membership functions
- The membership function fully defines the fuzzy set
- A membership function provides a measure of the degree of similarity of an element to a fuzzy set
- Membership functions can take any form, but there are some common examples that appear in real applications

Fuzzy Membership Functions

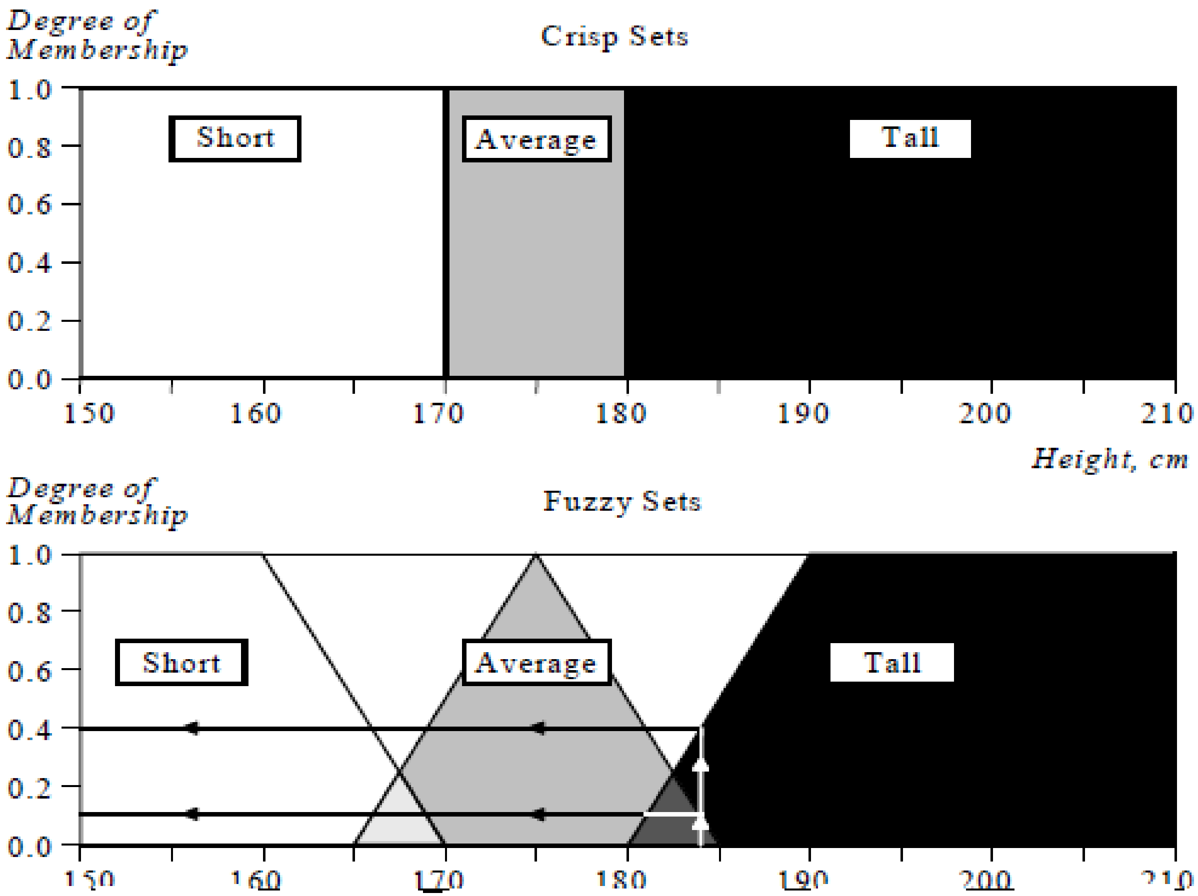
- Membership functions can
  - either be chosen by the user arbitrarily, based on the user's experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)
  - Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)
- There are different shapes of membership functions; triangular, trapezoidal, piecewise-linear, Gaussian, bell-shaped, etc.

Types of MF

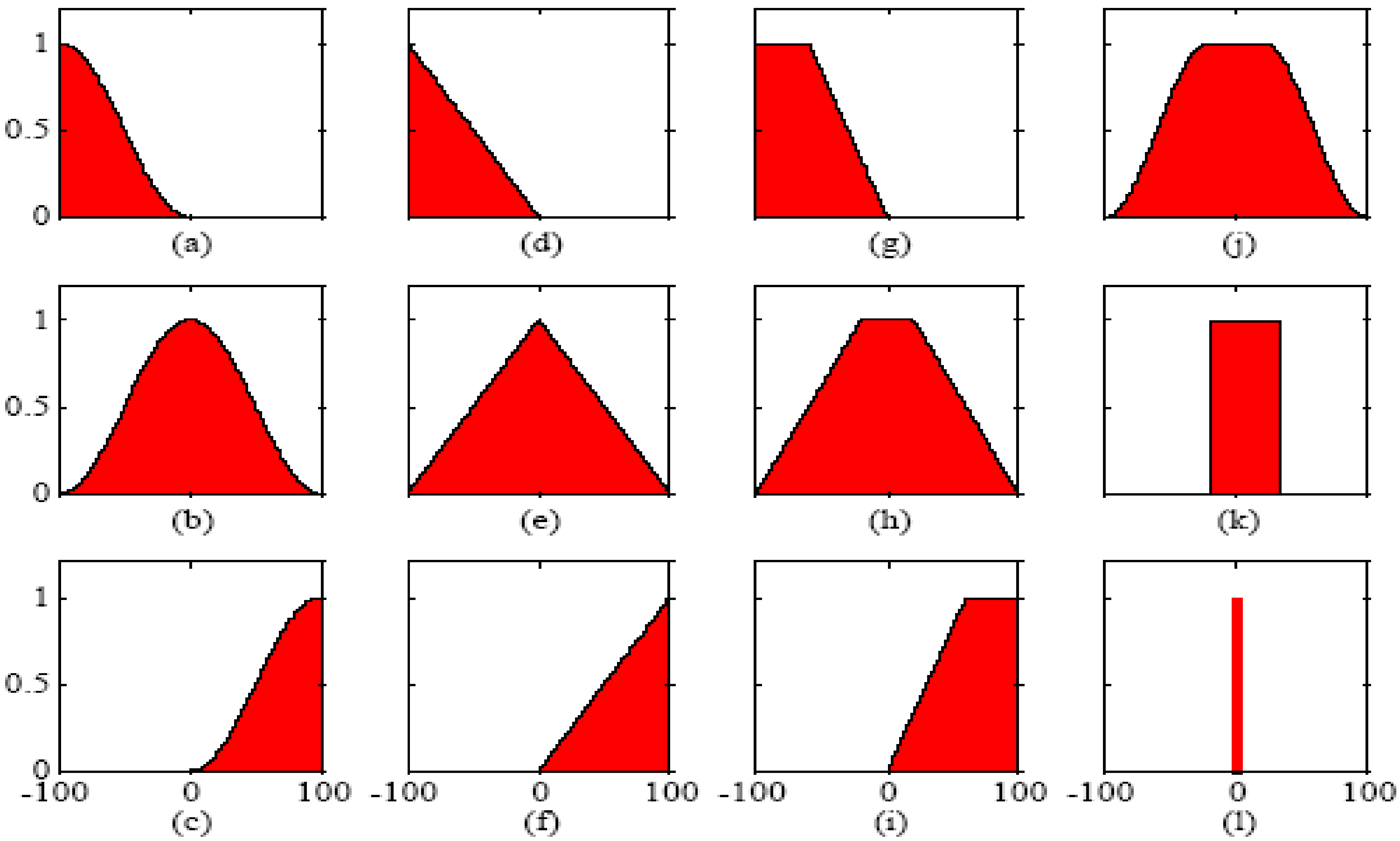
In the classical sets there is one type of membership function but in fuzzy sets there are many different types of membership function, now will show some of these types.

Membership Function

- **Tall men set:** consists of three sets: short, average and tall men.



Types of MF



Different shapes of membership functions (a) s_function, (b) п_function, (c) z_function, (d-f) triangular (g-i) trapezoidal (J) flat п_function, (k) rectangle, (L) singleton

MF Formulation

- Triangular MF

$$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

- Trapezoidal MF

$$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

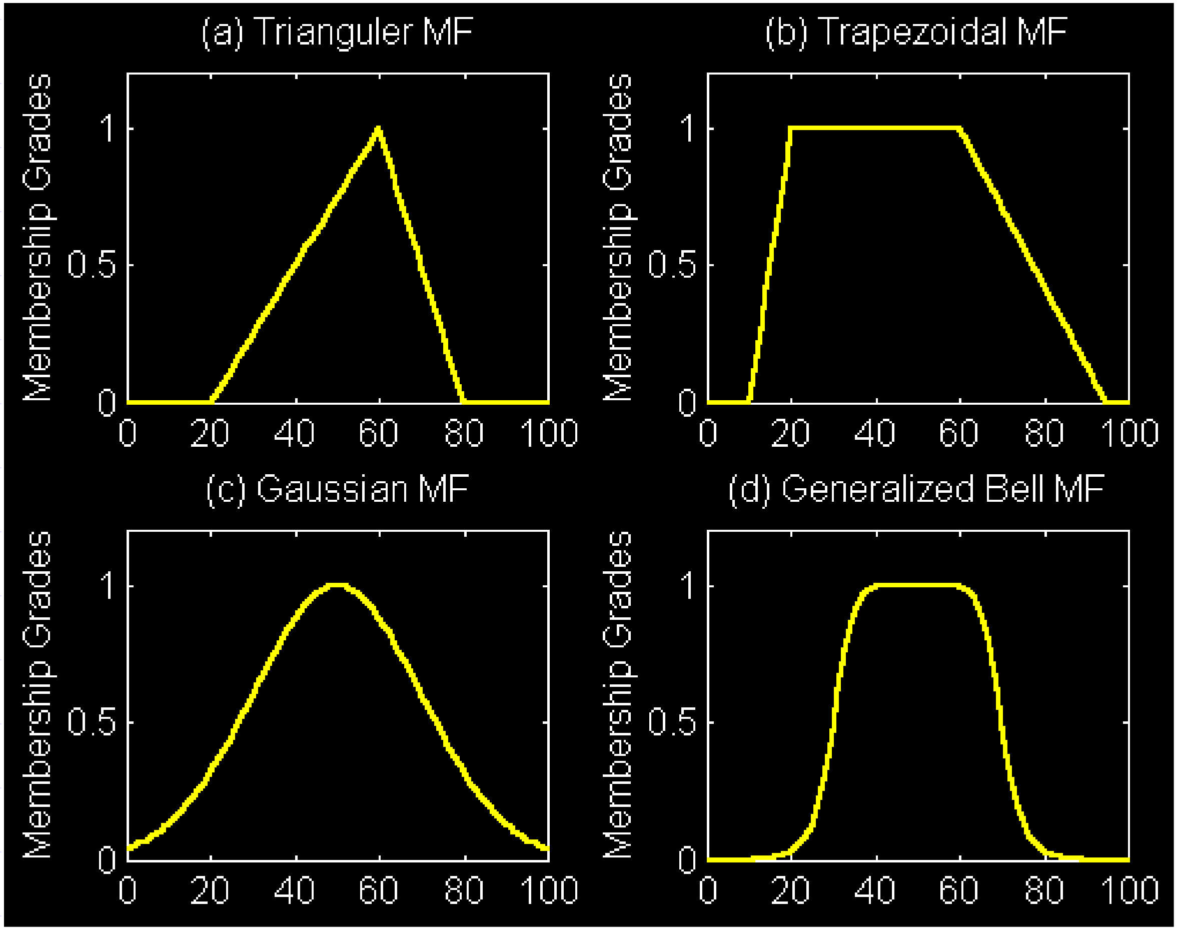
- Gaussian MF

$$gaussmf(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

- Generalized bell MF

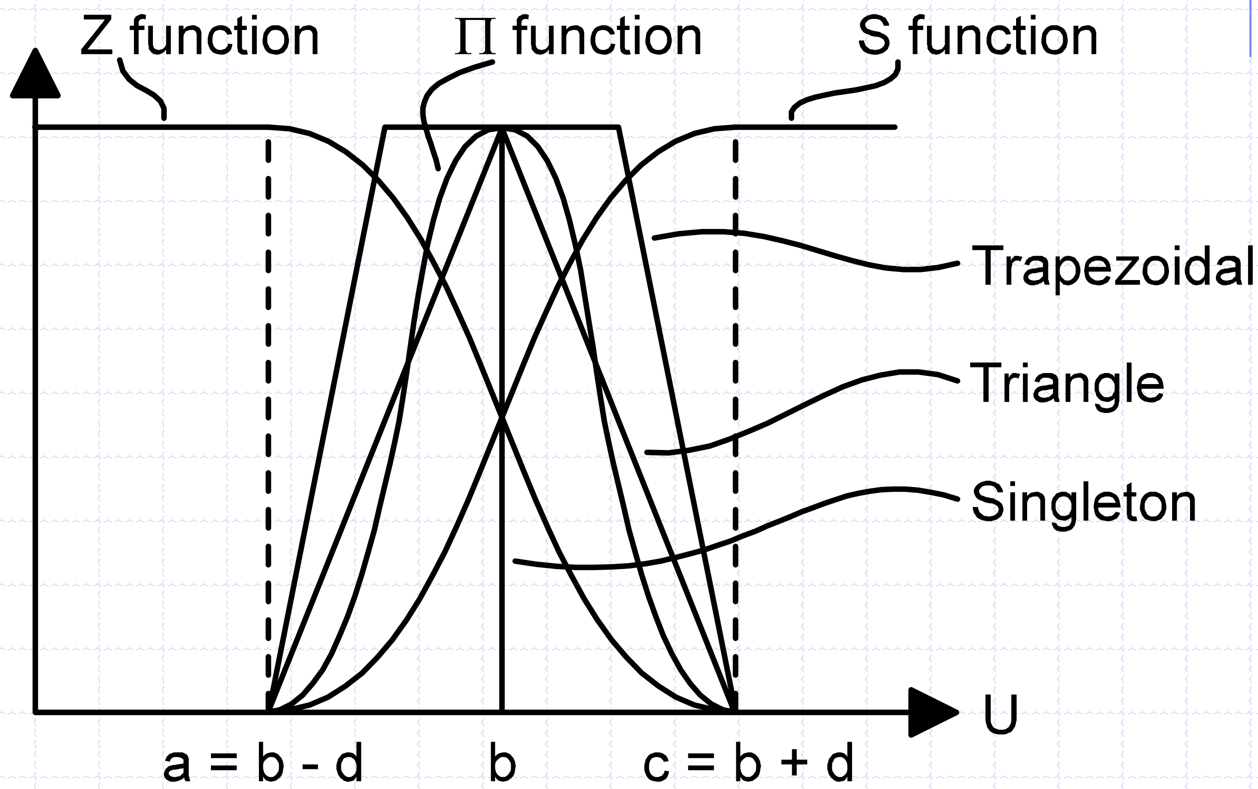
$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

MF Formulation



Types of MF

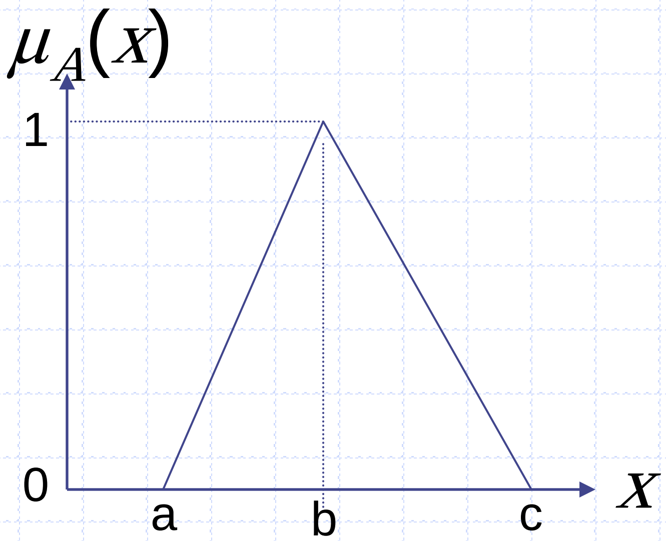
Standard types of membership functions: Z function; Π function; S function; trapezoidal function; triangular function; singleton.



Triangular membership function

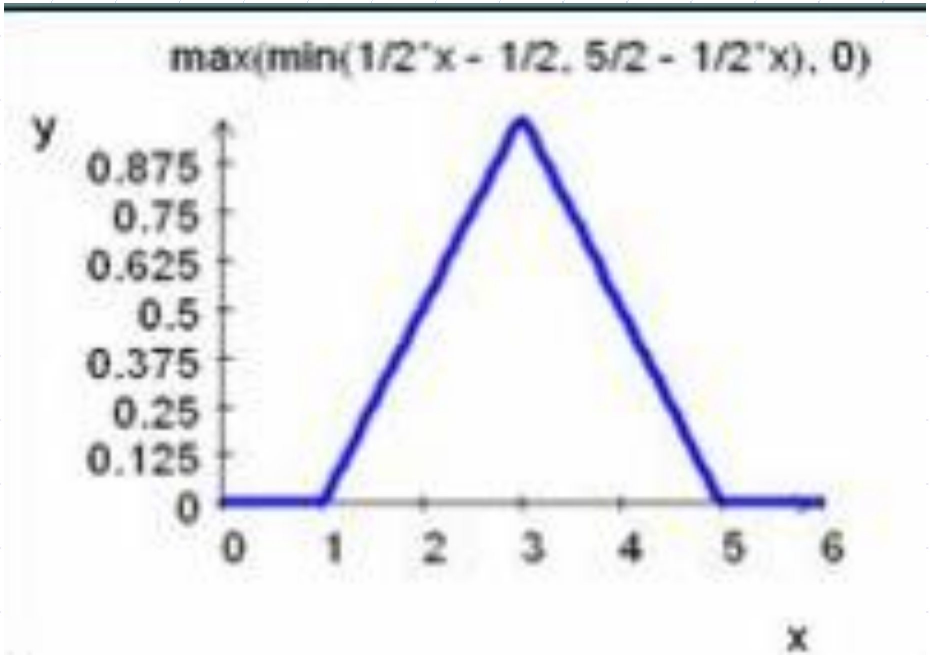
- a, b and c represent the *x* coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1)

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ \dfrac{c-x}{c-b} & \text{if } b \le x \le c \\ 0 & \text{if } x \ge c \end{cases}$$



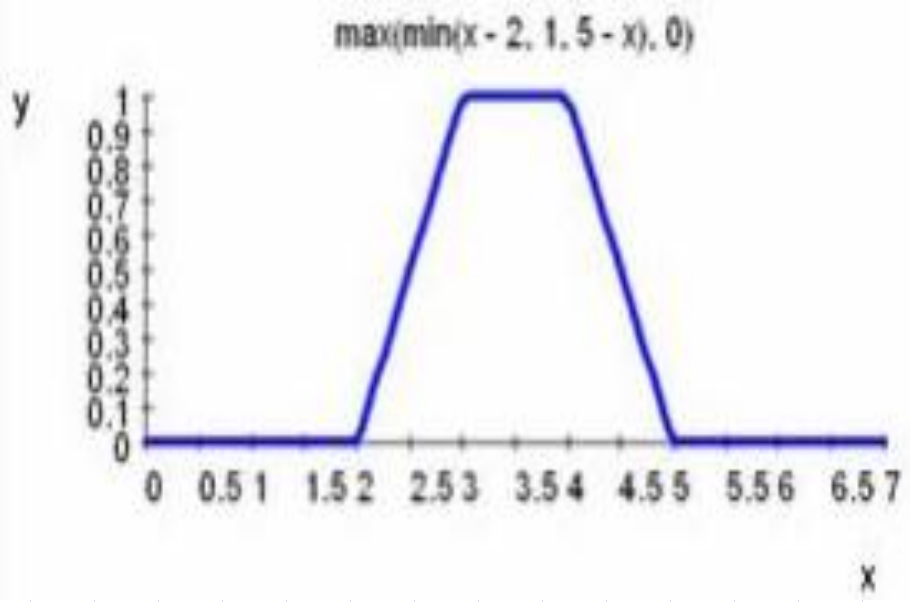
Triangular MF:

$$\mathrm{triangularmf}(x,\ c,\ h) = \max\left(\min\left(\frac{h-c+x}{h},\ \frac{c+h-x}{h}\right),\ 0\right)$$



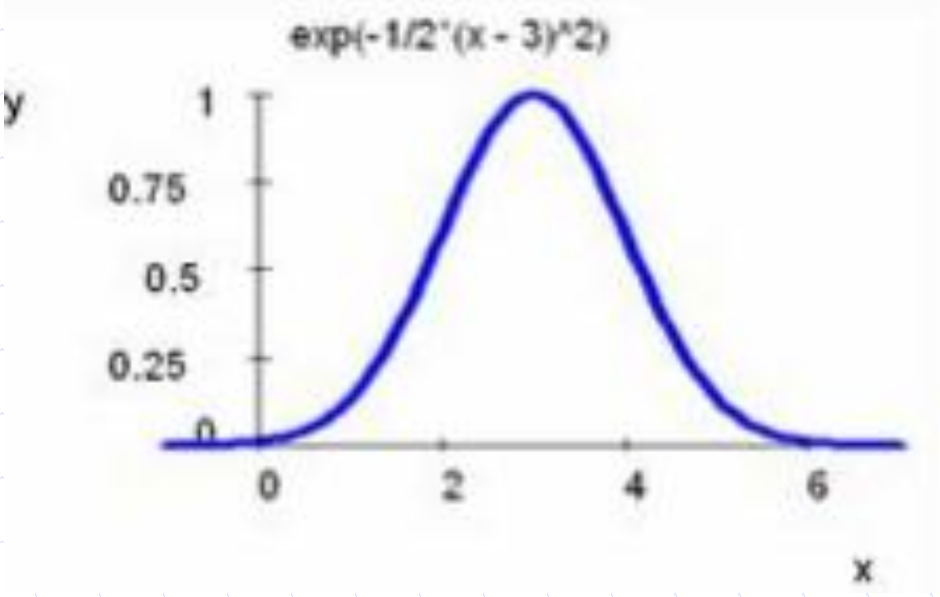
Trapezoidal MF:

$$\mathrm{trapmf}(x, a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$



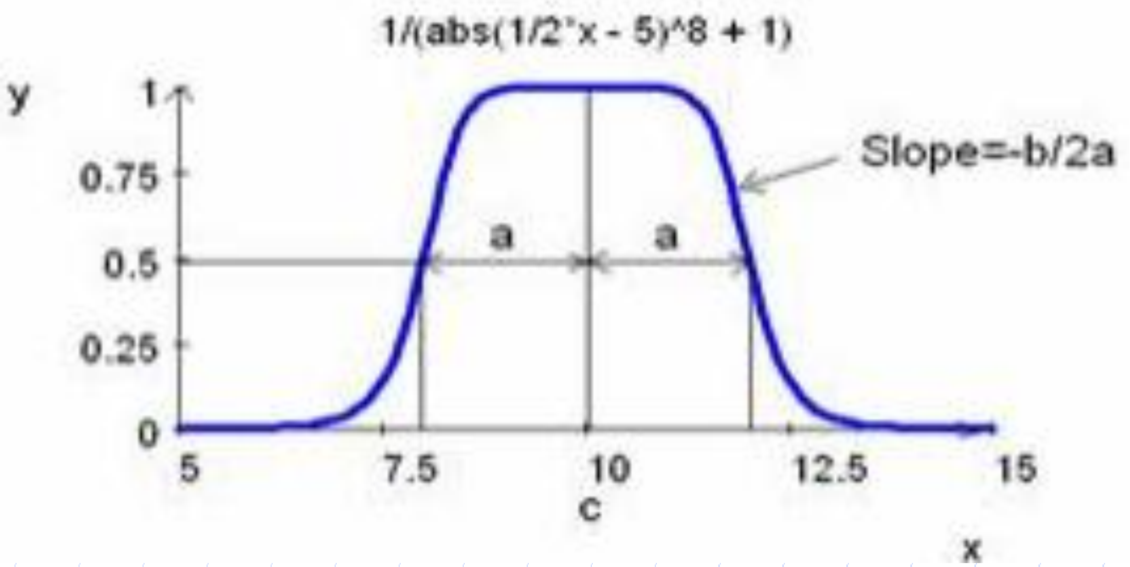
Gaussian MF:

$$\text{gaussmf}(x, c, s) = e^{-\frac{(x-c)^2}{2 \cdot s^2}}$$



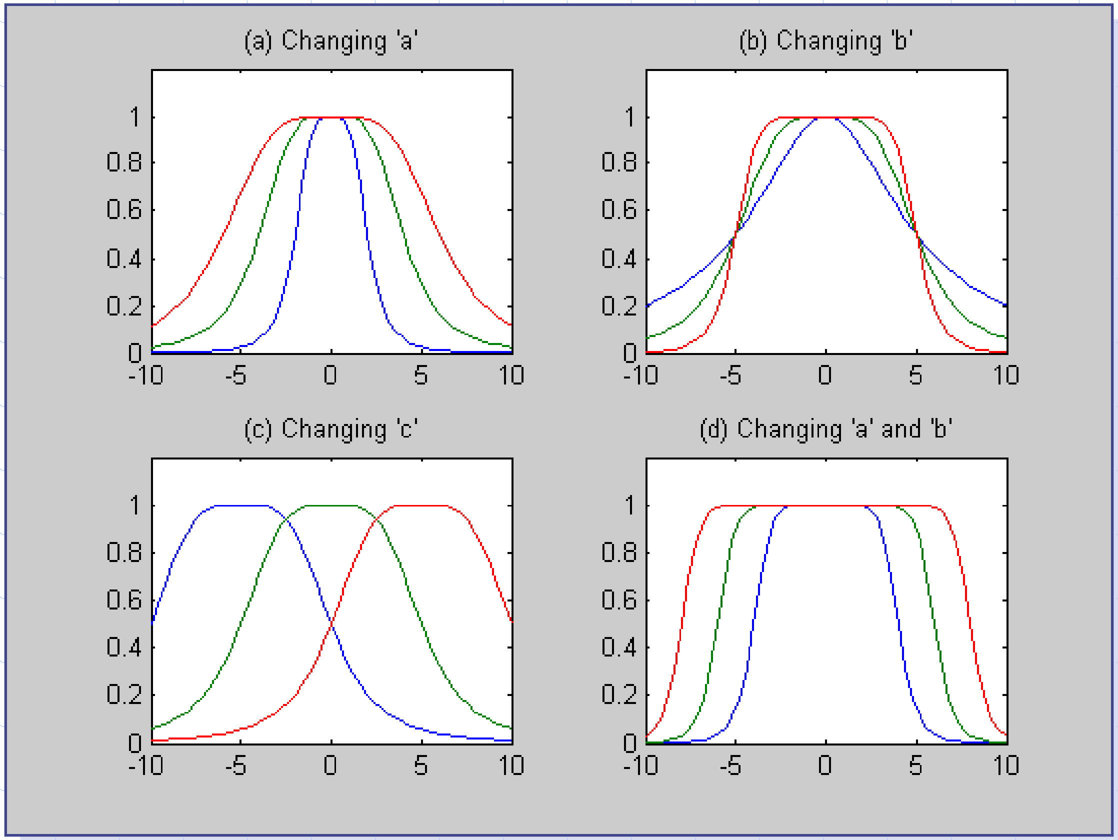
Generalized bell MF:

$$\text{gbellmf}(x, a, b, c) = \frac{1}{\left|\frac{x}{a} - \frac{c}{a}\right|^{2 \cdot b} + 1}$$



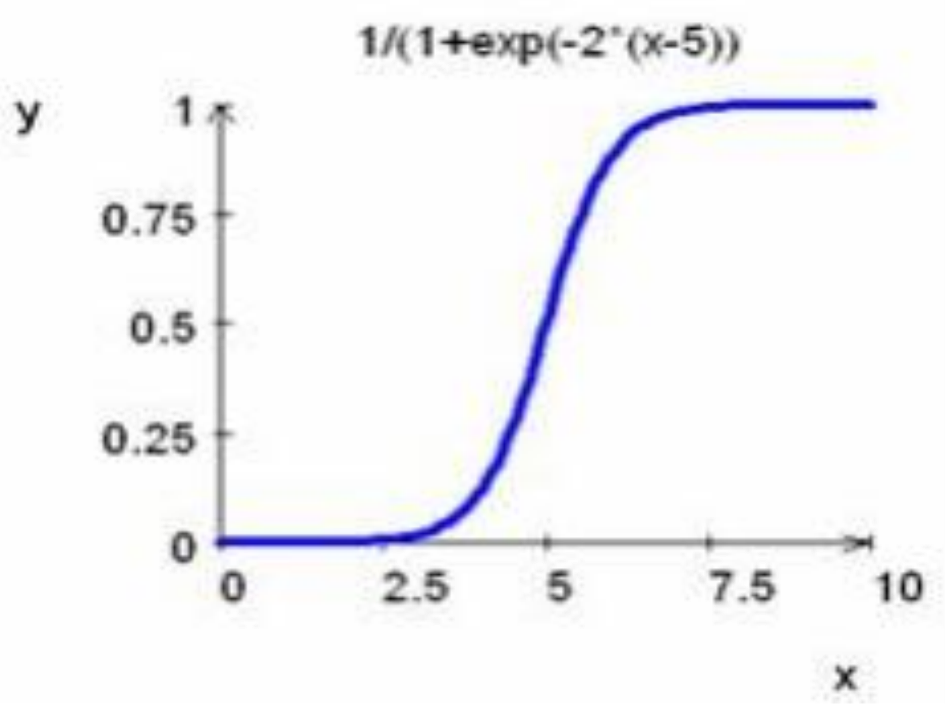
Manipulating Parameter of the Generalized Bell Function

$$gbellmf(x;a,b,c)=\frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$



Sigmoidal MF:

$$\mathrm{sigmf}(x, a, c) = \frac{1}{e^{-a(x-c)} + 1}$$



Common Membership Functions

⑤

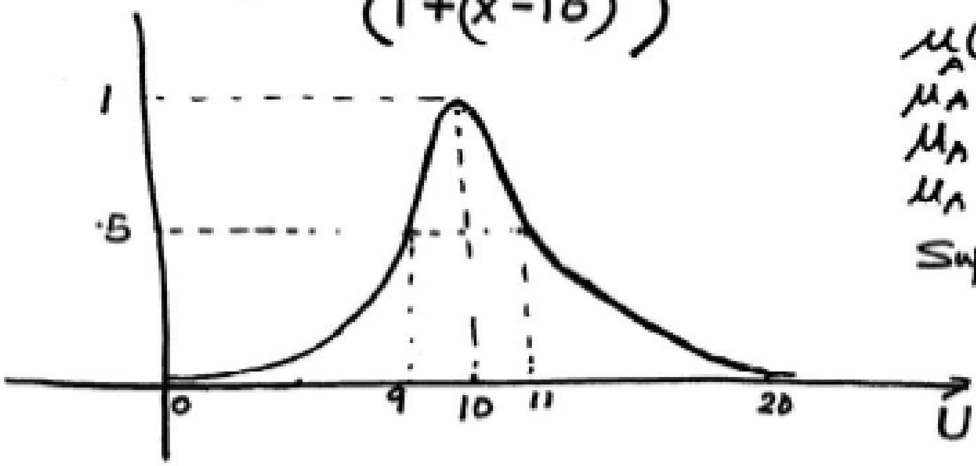
- bell-shaped functions
- trapezoidal
- triangular
- S-functions
- Z-functions

In the examples, we will use the universe $U = [0, \infty]$

Bell shaped Membership Function

- A = "number closed to 10"

$$\mu_A(x) = \frac{1}{(1+(x-10)^2)}$$



$\mu_A(10) = 1$
$\mu_A(9) = \mu_A(11) = 0.5$
$\mu_A(8) = \mu_A(12) = 0.2$
$\mu_A(0) = \mu_A(20) = 0.01$
$Supp(A) = [0, \infty)$

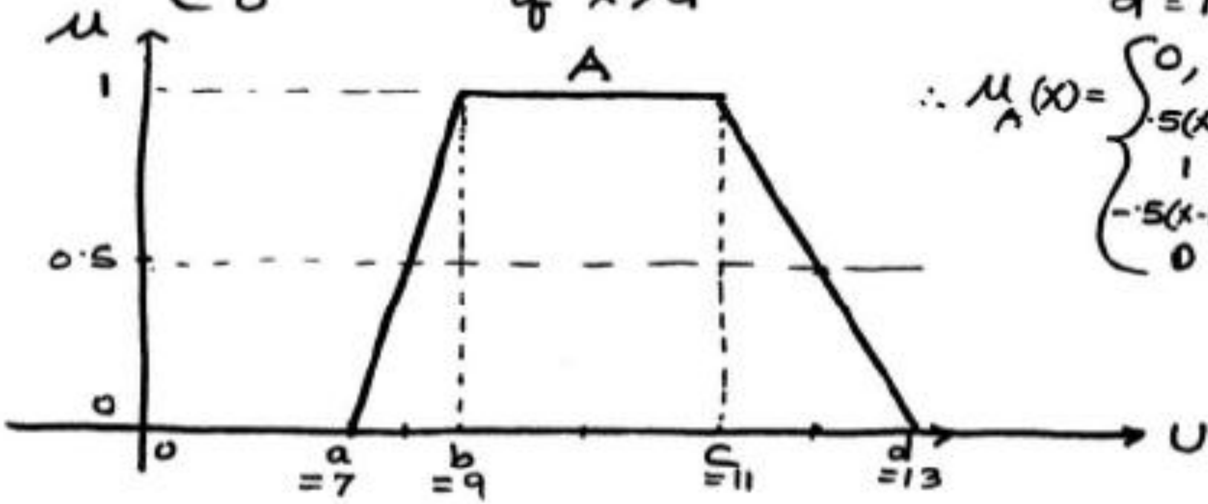
Trapezoidal Membership Function

⑥

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le a \\ (x-a)/(b-a) & \text{if } a < x < b \\ 1 & \text{if } b < x \le c \\ (d-x)/b-a & \text{if } c < x < d \\ 0 & \text{if } x > d \end{cases}$$

Let $a = 7$, $b = 9$, $c = 11$, $d = 13$

$$\therefore \mu_A(x) = \begin{cases} 0, & x \le 7 \\ .5(x-7), & 7 < x \le 9 \\ 1, & 9 \le x \le 11 \\ -.5(x-13), & 11 \le x \le 13 \\ 0, & x \ge 13 \end{cases}$$



$\text{Core}(A) = [9, 11]$

$\text{Supp}(A) = [7, 13]$

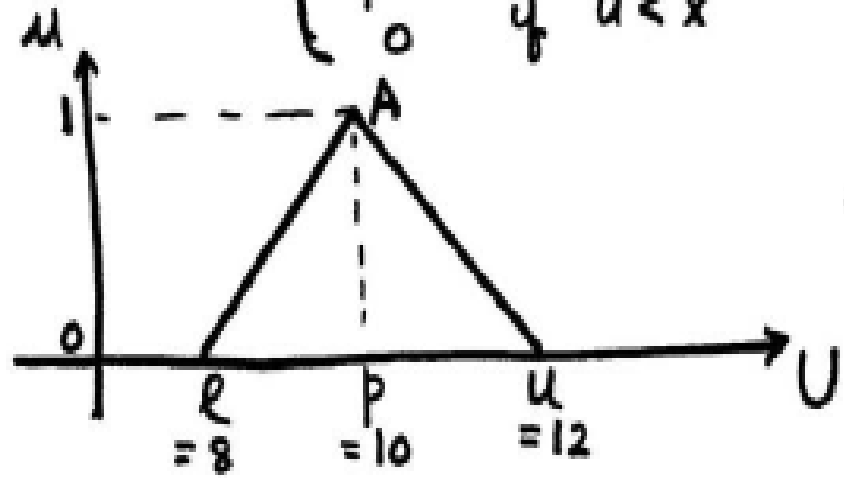
$|A| = 4 \quad = 0.5 + 1 + 1 + 1 + 0.5 = 4$

Cardinality : when U is a finite set, the cardinality of fuzzy set A is defined by $|A| = \sum_{x \in U} \mu_A(x)$

Triangular Membership Function

- Center point = p, l – lower bound of support
  u – upper bound of support

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le l \\ \dfrac{x-l}{p-l} & \text{if } l < x \le p \\ \dfrac{x-u}{p-u} & \text{if } p < x \le u \\ 0 & \text{if } u < x \end{cases}$$



Example:

$l = 8$

$p = 10$

$u = 12$

$$\therefore \mu_A(x) = \begin{cases} 0, & x \le 8 \\ .5(x-8), & 8 < x \le 10 \\ -.5(x-12), & 10 \le x < 12 \\ 0, & x \ge 12 \end{cases}$$

Core (A) = 10

Supp (A) = (8, 12)

|A| = 2

S Membership Function

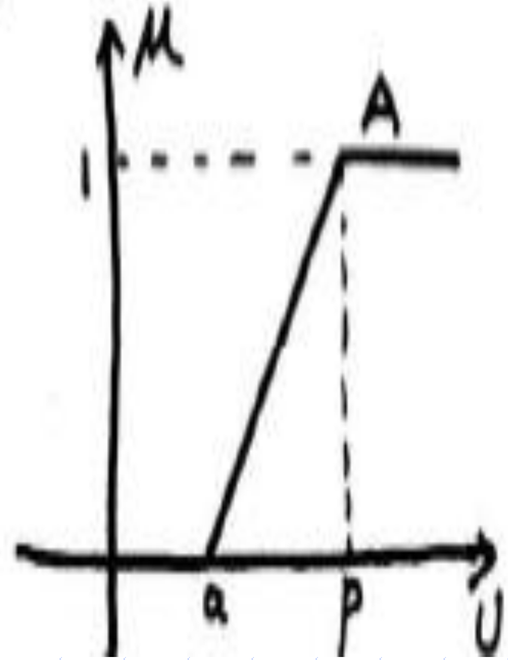
- $U = [0,1]$ and A defines *large* numbers in U
- peak p: point at which μ_A becomes 1
- bandwidth b: length of interval from $\mu_A(x) = 0$ to $\mu_A(x) = 1$

$$
\mu_A(x) = \begin{cases} 0 & \text{if } x \le p-b \\ 2(x-p+b)^2/b^2 & \text{if } p-b < x \le p - b/2 \\ 1 - 2(u-p)^2/b^2 & \text{if } p - \frac{b}{2} < x \le p \\ 1 & \text{if } p \le x \end{cases}
$$

'Linear' S Membership Function

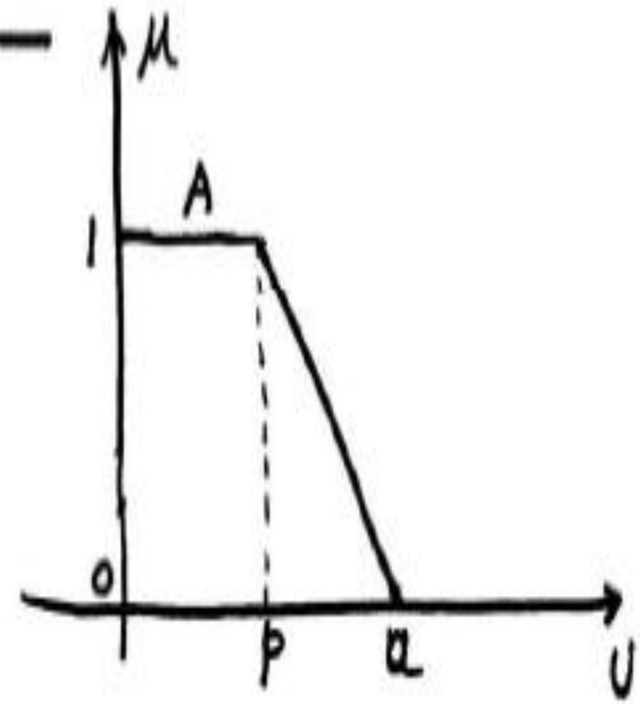
- peak p: point at which μ_A becomes 1.
- a: rightmost zero element

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le a \\ (x-a)/(p-a) & \text{if } p-a < x \le p \\ 1 & \text{if } p \le x \end{cases}$$



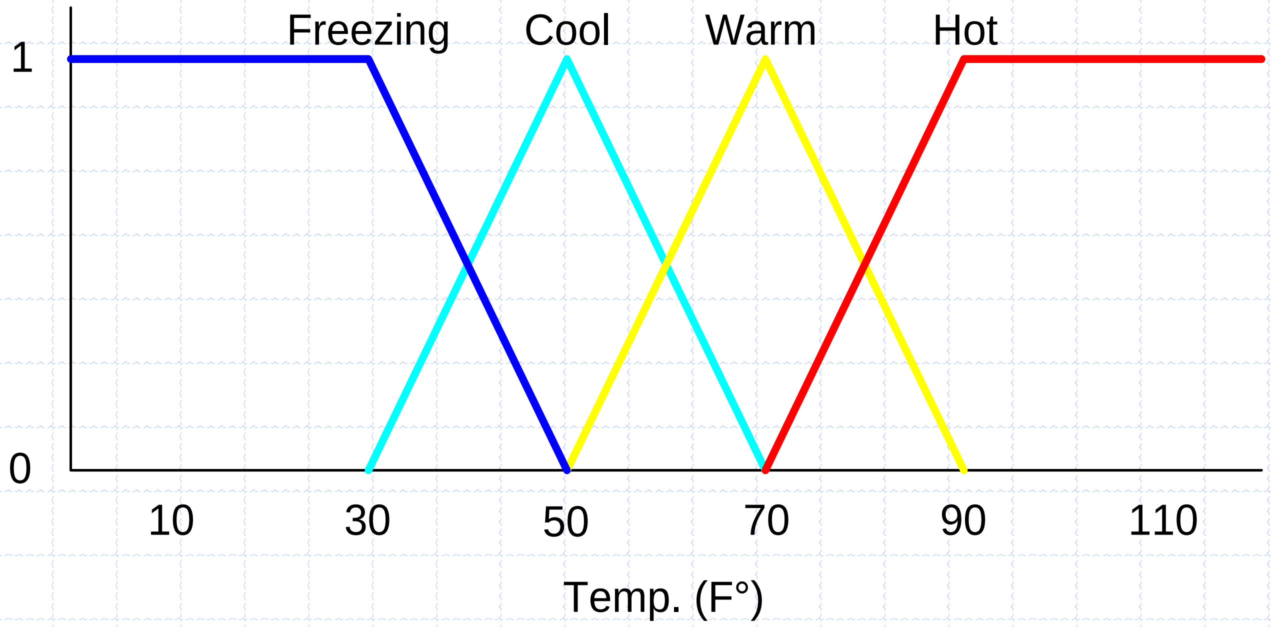
'Linear' Z Membership Function

$$\mu_A(x) = \begin{cases} 1 & \text{if } p \geq x \\ -(x-a)/(p-a) & \text{if } p < x \leq p\text{-}a \\ 0 & \text{if } x \geq a \end{cases}$$



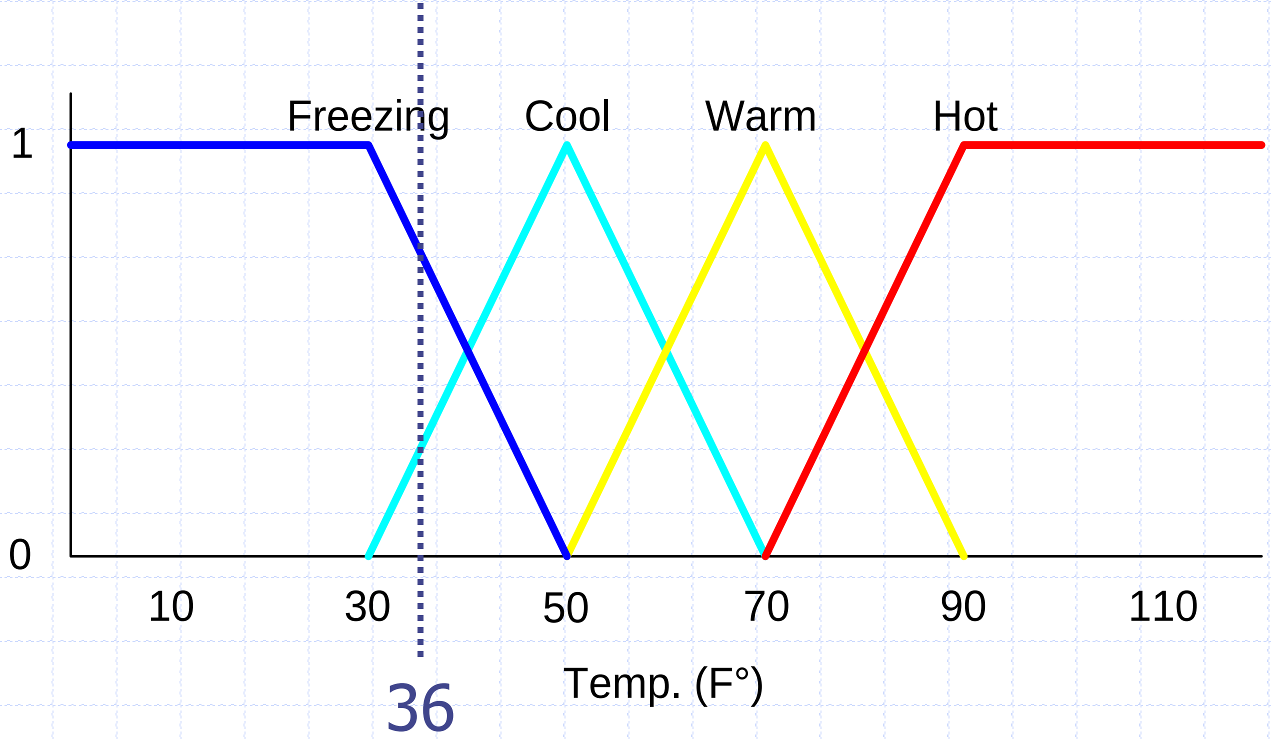
Membership Functions

- Degree of Truth or "Membership"
- Temp: {Freezing, Cool, Warm, Hot}



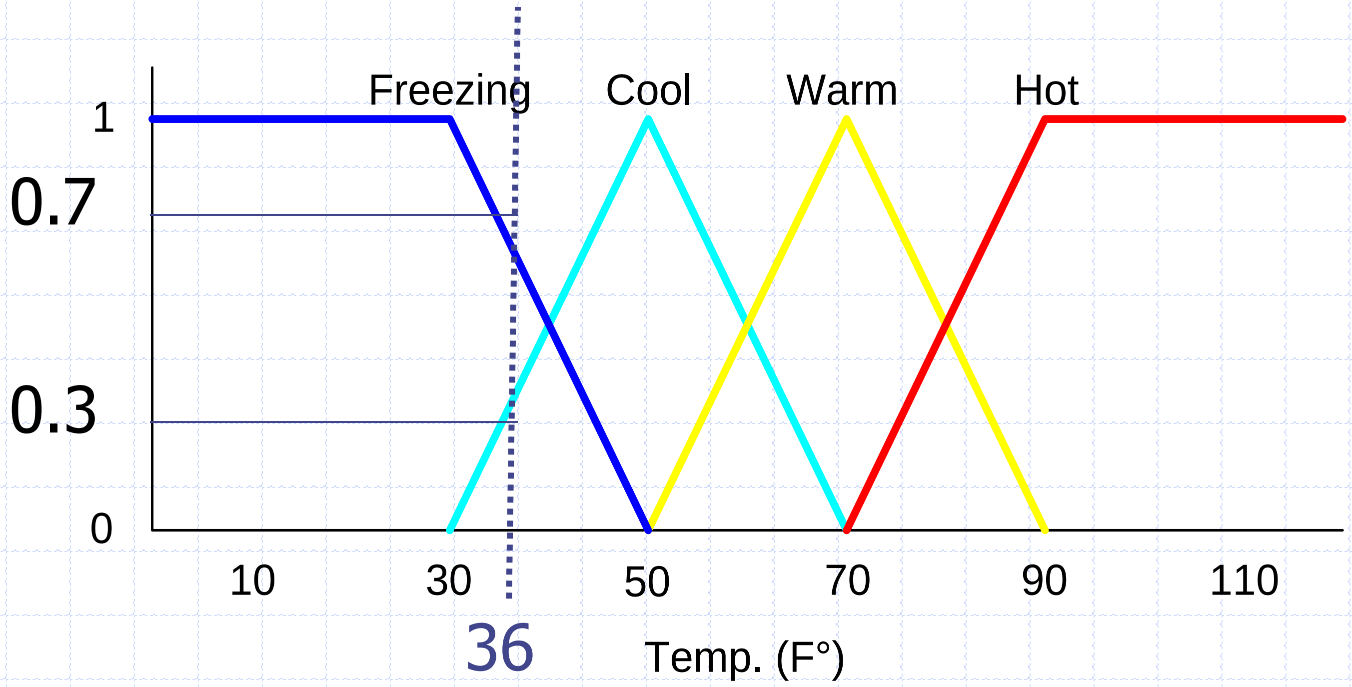
Membership Functions

How cool is 36 F° ?



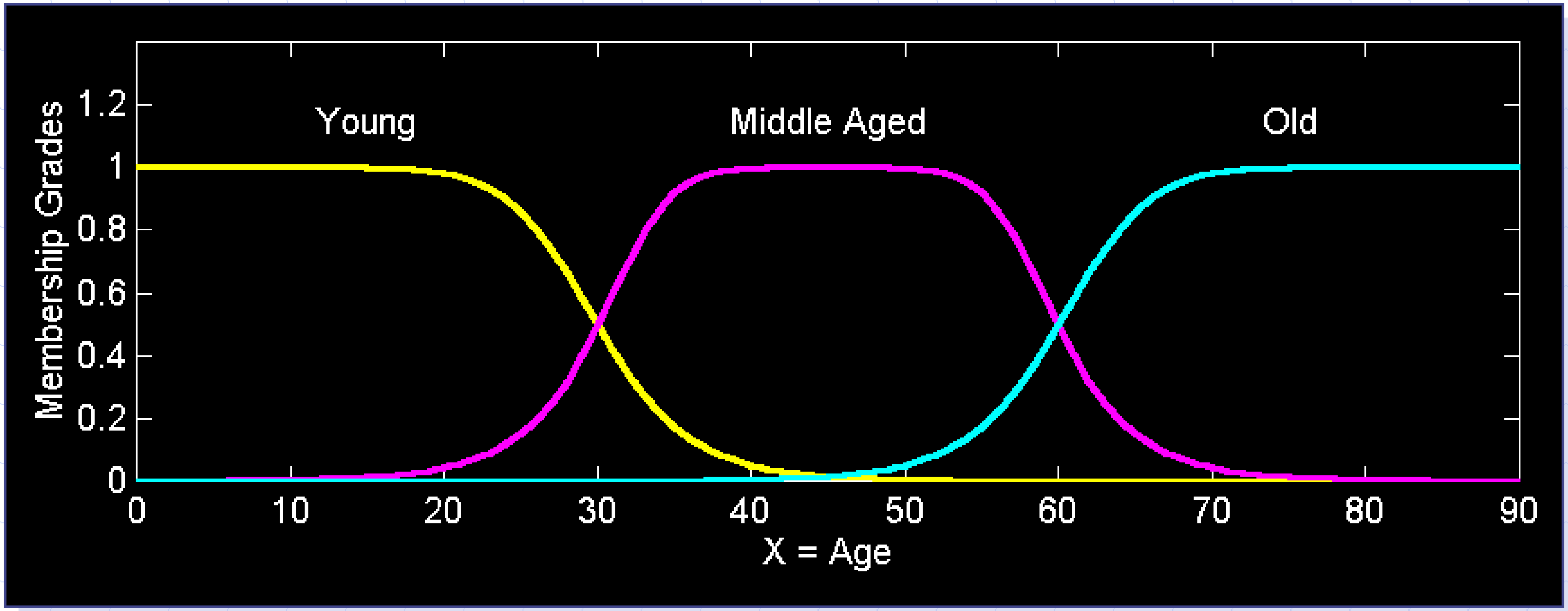
Membership Functions

- How cool is 36 F° ?
- It is 30% Cool and 70% Freezing



Fuzzy Partition

- Fuzzy partitions formed by the linguistic values "young", "middle aged", and "old":



Approximate Reasoning

Approximate Reasoning

The idea of linguistic fuzzy models imitating the human way of thinking was proposed by Zadeh in his pioneering work

"The term Approximate reasoning refers to methods and methodologies that enable reasoning with imprecise inputs to obtain meaningful outputs".

Approximate Reasoning

- Fuzzy logic with approximate reasoning describes relations by if-then rules, such as, "if heating valve is close then temperature is low". The uncertainty in the linguistic terms (e.g. low temperature) is represented by fuzzy sets. Each linguistic fuzzy set in this case shows the temperature range from its one extreme to the other
- The smooth outcome of reasoning with fuzzy sets is a kind of interpolation

Linguistic Variables

When fuzzy sets are used to solve the problem without analyzing the system; but the expression of the concepts and the knowledge of it in human communication are needed. Human usually do not use mathematical expression but use the linguistic expression.

For example, if you see heavy box and you want to move it, you will say, "I want strong motor to move this box" we see that, we use *strong expression* to describe the *force* that we need to move the box. In fuzzy sets we do the same thing we use linguistic variables to describe the fuzzy sets.

Linguistic Variables

- Linguistic variable is “a variable whose values are words or sentences in a natural or artificial language“.

- Each linguistic variable may be assigned one or more linguistic values, which are in turn connected to a numeric value through the mechanism of membership functions.

Natural Language

- Consider:
  - Joe is tall -- what is tall?
  - Joe is very tall -- what does this differ from tall?
- Natural language (like most other activities in life and indeed the universe) is not easily translated into the absolute terms of 0 and 1.

"false" "true"

Linguistic Variables

In 1973, Professor Zadeh proposed the concept of linguistic or "fuzzy" variables. Think of them as linguistic objects or words, rather than numbers.

The sensor input is a noun, e.g. "temperature", "displacement", "velocity", "flow", "pressure", etc. Since error is just the difference, it can be thought of the same way.

The fuzzy variables themselves are adjectives that modify the variable (e.g. "large positive" error, "small positive" error ,"zero" error, "small negative" error, and "large negative" error). As a minimum, one could simply have "positive", "zero", and "negative" variables for each of the parameters. Additional ranges such as "very large" and "very small" could also be added

Fuzzy Linguistic Variables

- Fuzzy Linguistic Variables are used to represent qualities spanning a particular spectrum
- Temp: **{Freezing, Cool, Warm, Hot}**
- Membership Function
- Question: What is the temperature?
- Answer: It is warm.
- Question: How warm is it?

Example

if temperature is cold and oil is cheap

then heating is high

Example

$$\text{if } \underbrace{\overbrace{\text{temperature}}^{\text{Linguistic Variable}} \text{ is } \overbrace{\text{cold}}^{\text{Linguistic Value}}}_{\mu_{\text{cold}}} \text{ and } \underbrace{\overbrace{\text{oil}}^{\text{Linguistic Variable}} \text{ is } \overbrace{\text{cheap}}^{\text{Linguistic Value}}}_{\mu_{\text{cheap}}}$$

$$\text{then } \overbrace{\underbrace{\text{heating}}_{\text{Linguistic Variable}} \text{ is } \underbrace{\text{high}}_{\text{Linguistic Value}}}^{\mu_{\text{high}}}$$

Definition [Zadeh 1973]

A linguistic variable is characterized by a quintuple

$$\left(x, T(x), U, G, M\right)$$

Name

Term Set

Universe

Syntactic Rule

Semantic Rule

Example

A linguistic variable is characterized by a quintuple

$$\big(x, T(x), U, G, M\big)$$

age

$$G(\text{age}) = \left\{ \begin{array}{l} \text{old, very old, not so old,} \\ \text{more or less young,} \\ \text{quite young, very young} \end{array} \right\}$$

[0, 100]

Example semantic rule:

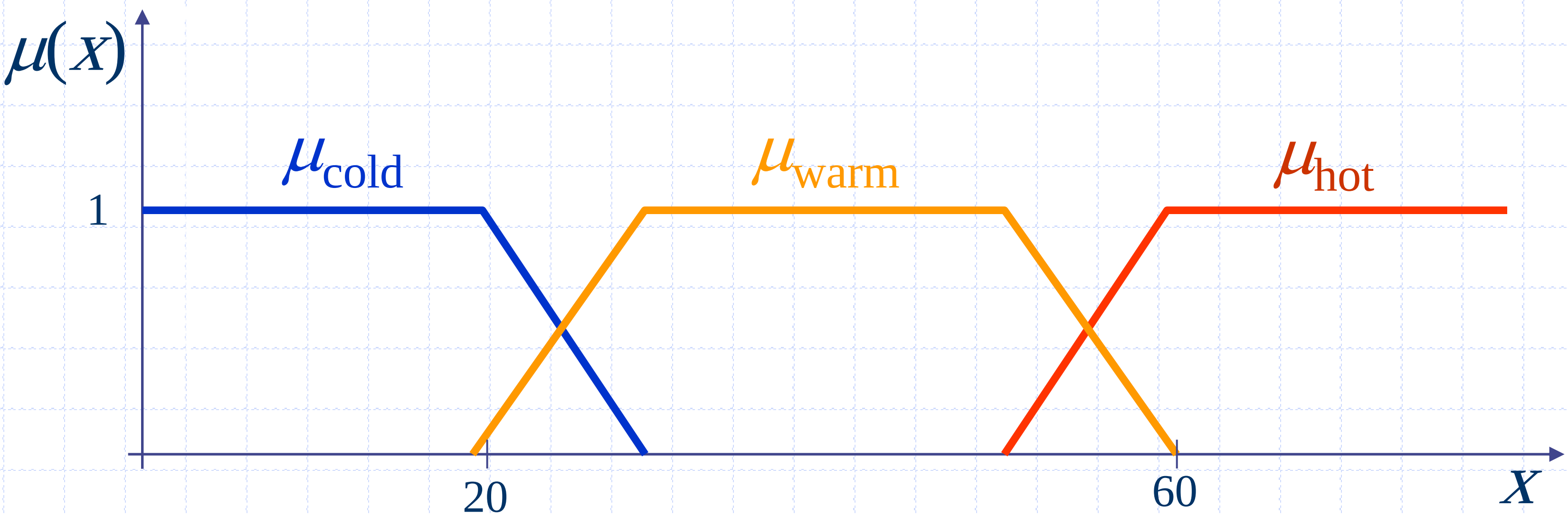
$$M(\text{old}) = \left\{ \big(u, \mu_{\text{old}}(u)\big) \,\middle|\, u \in [0,100] \right\}$$

$$\mu_{\text{old}}(u) = \begin{cases} 0 & u \in [0,50] \\ \left[1 + \left(\dfrac{u-50}{5} \right)^{-2} \right]^{-1} & u \in [50,100] \end{cases}$$

Example

Linguistic Variable : temperature

Linguistics Terms (Fuzzy Sets) : {cold, warm, hot}



Fuzzy If-Than Rules

IF (input1 is MFa) AND (input2 is MFb) AND…AND (input n is MFn) THEN (output is MFc)

Where MFa, MFb, MFn, and MFc are the linguistic values of the fuzzy membership functions that are in input 1,input 2…input n, and output.

Example

In a system where the inputs of the system are Serves and Food and the output is the Tip.

Food may be (good, ok, bad), and serves can be (good, ok, bad), the output tip can be (generous, average, cheap), where good, ok, bad, generous, average, and cheap are the linguistic variables of fuzzy membership function of the inputs (food, and serves) and the output (tip).

Example

We can write the rules such as.

IF (**Food** is **bad**) and (**Serves** is **bad**) **THEN** (**Tip** is **cheap**)

IF (**Food** is **good**) and (**Serves** is **good**) **THEN** (**Tip** is **generous**)

: : :

antecedent
or
premise

consequence
or
conclusion

Fuzzy If-Than Rules

$$A \Rightarrow B \equiv \text{If } \underbrace{x \text{ is } A}_{\substack{\text{antecedent}\\ \text{or}\\ \text{premise}}} \text{ then } \underbrace{y \text{ is } B}_{\substack{\text{consequence}\\ \text{or}\\ \text{conclusion}}}.$$

Fuzzy If-Than Rules

- Premise 1 (fact): *x* is *A*,
- Premise 2 (rule): IF *x* is *A* THEN *y* is *B*,
- Consequence (conclusion): *y* is *B*.

Examples

$$A \rightarrow B \equiv \text{If } x \text{ is } A \text{ then } y \text{ is } B.$$

- If *pressure* is *high*, then *volume* is *small*.
- If the *road* is *slippery*, then *driving* is *dangerous*.
- If a *tomato* is *red*, then *it* is *ripe*.
- If the *speed* is *high*, then *apply the brake* a *little*.

Fuzzy Rules as Relations

$$\underbrace{A \rightarrow B}_{R} \equiv \text{If } x \text{ is } A \text{ then } y \text{ is } B.$$

A fuzzy rule can be defined as a binary relation with MF

$$\underbrace{\mu_R(x, y) = \mu_{A \rightarrow B}(x, y)}_{\text{Depends on how to interpret } A \rightarrow B}$$

Fuzzy Rules as Relations

- Generally, there are two ways to interpret the fuzzy rule $A \to B$. One way to interpret the implication $A \to B$ is that A is coupled with B, and in this case,

$$R = A \to B = \int_{X \times Y} \mu_A(x) * \mu_B(y) / (x, y),$$

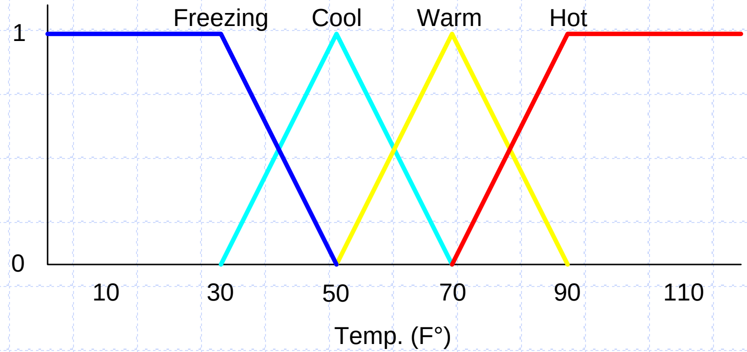
- The other interpretation of implication $A \to B$ is that A entails B, and in this case it can be written as

$$R = A \to B = \bar{A} \cup B.$$

EXAMPLE

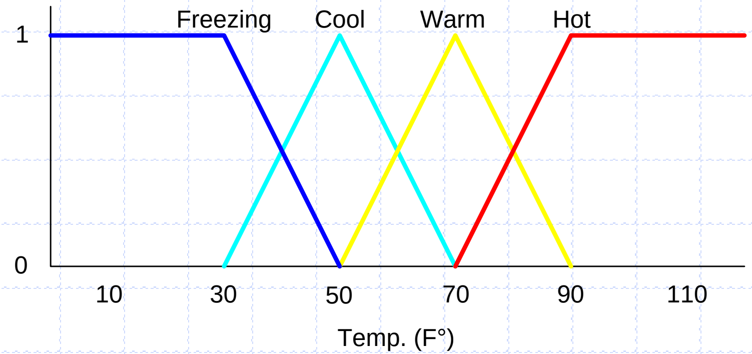
Inputs: Temperature

- Temp: {Freezing, Cool, Warm, Hot}

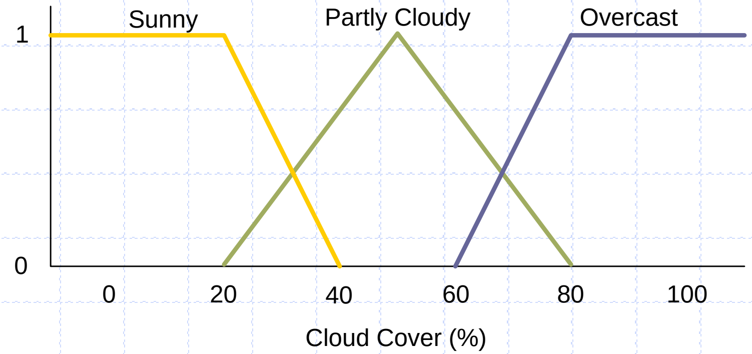


Inputs: Temperature, Cloud Cover

- Temp: {Freezing, Cool, Warm, Hot}

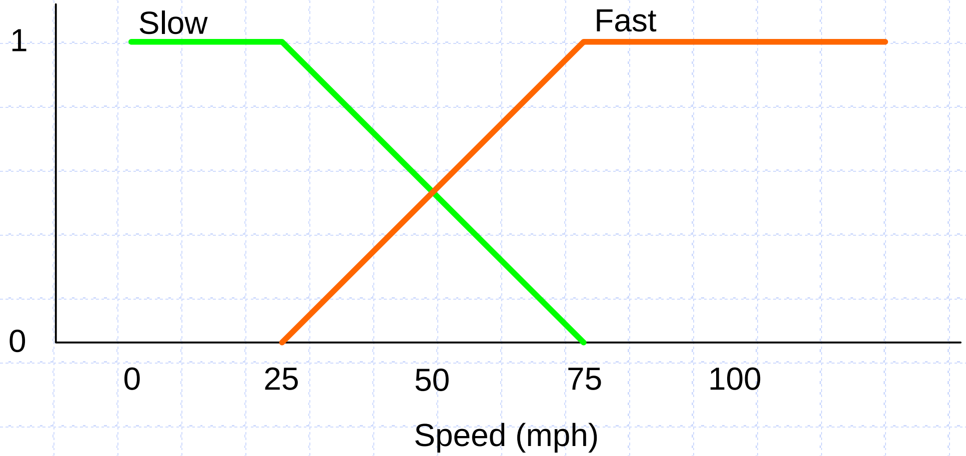


- Cover: {Sunny, Partly, Overcast}



Output: Speed

- Speed: {Slow, Fast}



Rules

- If it's Sunny and Warm, drive Fast

  $\text{Sunny}(\text{Cover}) \wedge \text{Warm}(\text{Temp}) \Rightarrow \text{Fast}(\text{Speed})$

- If it's Cloudy and Cool, drive Slow

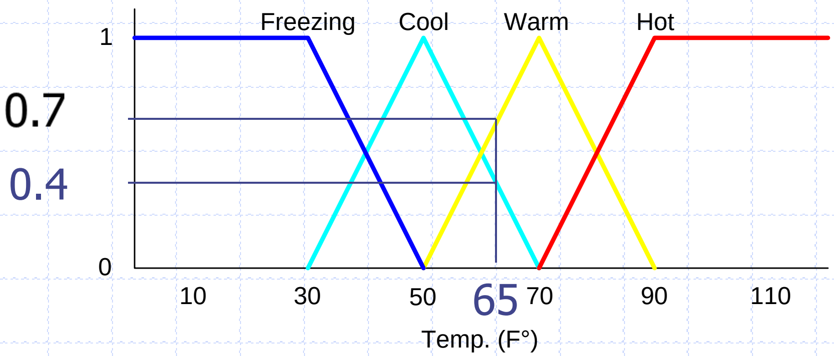
  $\text{Cloudy}(\text{Cover}) \wedge \text{Cool}(\text{Temp}) \Rightarrow \text{Slow}(\text{Speed})$

- Driving Speed is the combination of output of these rules...

Example Speed Calculation

- How fast will I go if it is
  - 65 F°
  - 25 % Cloud Cover ?

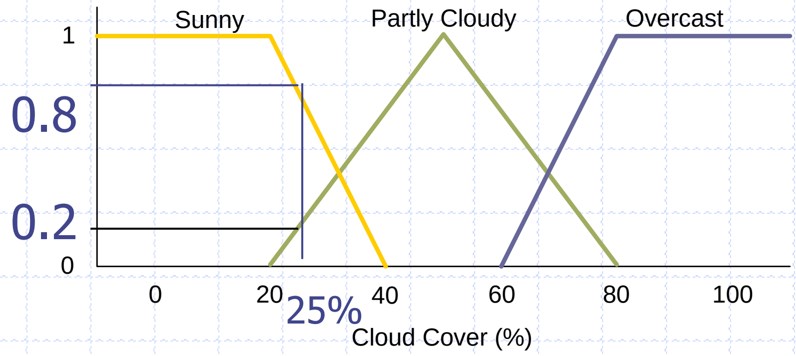
Calculate Input Membership Levels



- 65 F° $\Rightarrow$ Cool = 0.4, Warm= 0.7

Calculate Input Membership Levels

- 25% Cover $\Rightarrow$ Sunny = 0.8, Cloudy = 0.2



...Calculating...

- If it's Sunny and Warm, drive Fast

$Sunny(Cover) \wedge Warm(Temp) \Rightarrow Fast(Speed)$

$$0.8 \wedge 0.7 = 0.7$$

$\Rightarrow$ **Fast = 0.7**

- If it's Cloudy and Cool, drive Slow

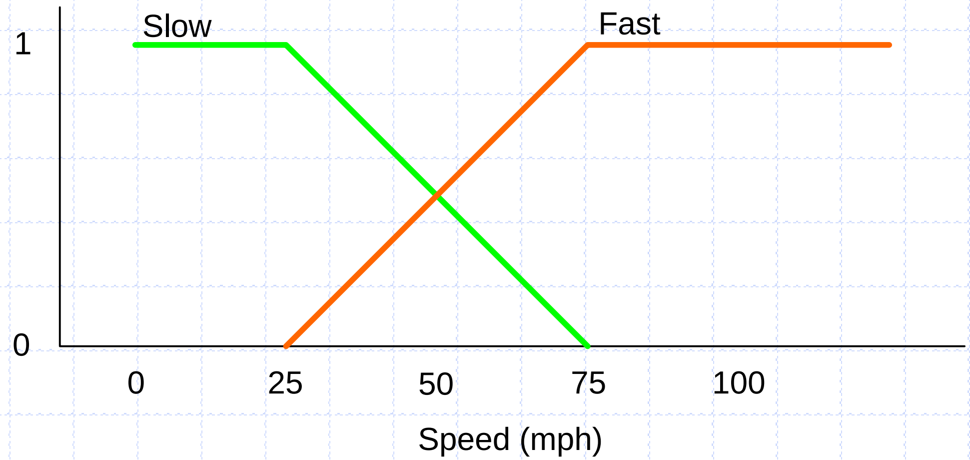
$Cloudy(Cover) \wedge Cool(Temp) \Rightarrow Slow(Speed)$

$$0.2 \wedge 0.4 = 0.2$$

$\Rightarrow$ **Slow = 0.2**

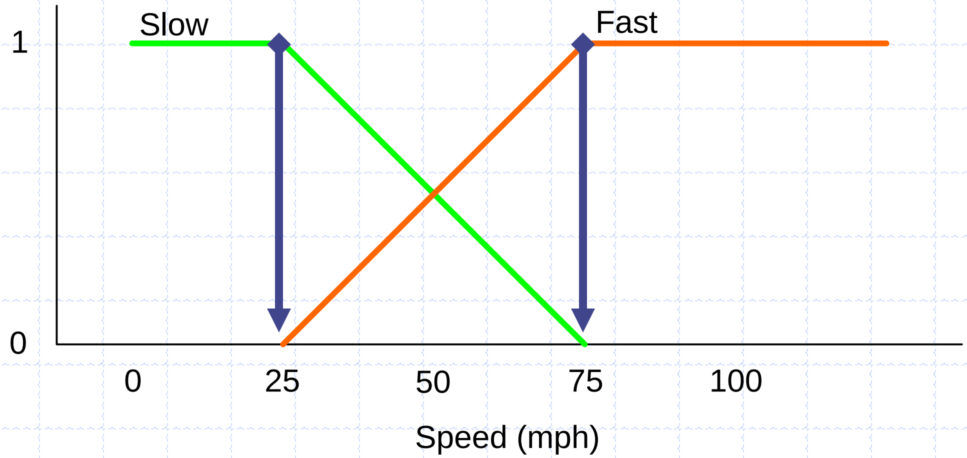
Constructing the Output

- Speed is 20% Slow and 70% Fast



Constructing the Output

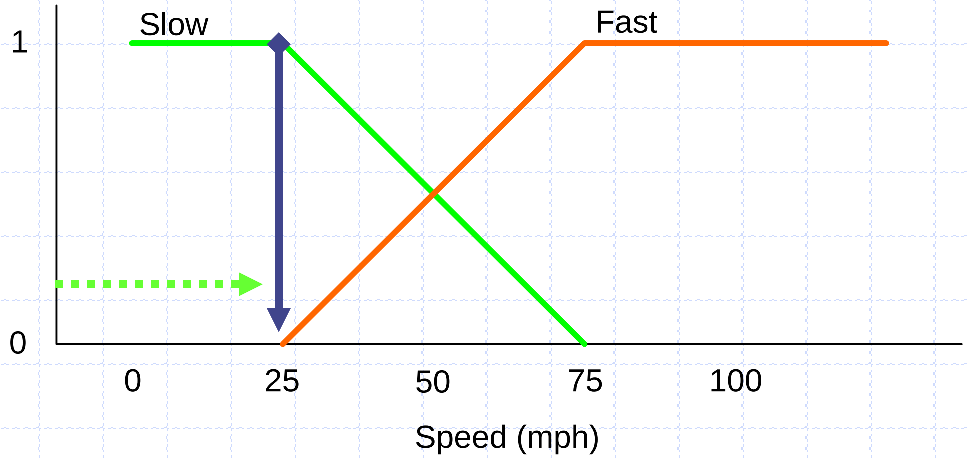
- Speed is 20% Slow and 70% Fast



- Find centroids: Location where membership is 100%

Constructing the Output

- Speed is 20% Slow and 70% Fast

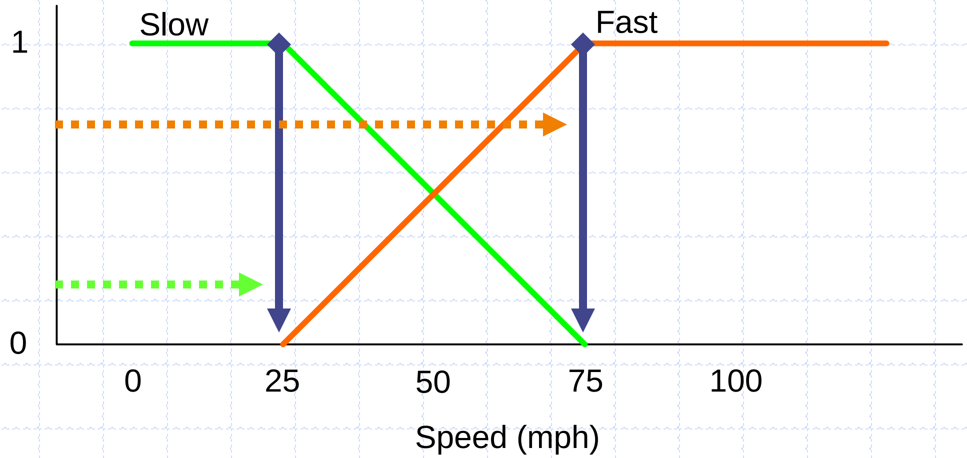


- Speed = weighted mean

  = (2*25+...

Constructing the Output

- Speed is 20% Slow and 70% Fast



- Speed = weighted mean

  = (2*25+7*75)/(9)

  = 63.8 mph