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Individual Final Project

Life Insurance: Annuity Fair Payment Model

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Abstract

Purchasing a life insurance plan is crucial for individuals who assume many responsibilities and whose absence can bare devastating financial impact to those who depend on them. Life insurance offers a sense of financial security and peace of mind, knowing that one's family will be taken care of in the face of adversity. Therefore, it is important to understand how insurance premiums are priced and how much one should be willing to pay given their circumstances. In this project, we consider life expectancy as a random variable function of age, and develop a mathematical model to calculate the correct annuity payment. This model will incorporate factors such as mortality probabilities, interest rates, and benefit amounts, allowing us to calculate the actuarial present value (APV) of the life insurance policy. By equating the APV to the present value of annuity payments, we can determine the fair annuity amount that a policyholder should pay to secure the desired financial protection for their family.

1 Introduction

Life insurance is a vital financial instrument that provides protection and financial security to families in the event of an unexpected death. By paying a predetermined annual premium or annuity, the policyholder ensures that their beneficiaries receive a lump sum or a series of payments upon their passing. This financial support can help cover various expenses including mortgage payments, educational expenses, and daily living needs, ensuring the well-being of loved ones during challenging times. In this project, we will explore the concept of life insurance and develop a mathematical model to determine the fair annuity payment based on the policyholder's age and life expectancy.

To achieve our objectives, we will first delve into the fundamentals of life insurance and actuarial science, familiarizing ourselves with the concepts and techniques employed in the industry. We review relevant data sources, such as life tables, which provide key information on mortality probabilities and life expectancy. Using this information, we will construct a mathematical model that takes into account the policyholder's age, and evaluate its effectiveness in determining the fair annuity payment.

Ultimately, this project aims to provide valuable insights into the complex world of life insurance and the role of mathematical models in determining fair annuity payments. By enhancing our understanding of these concepts, we hope to empower individuals to make informed decisions about securing financial protection for their families and achieving peace of mind.

2 Data Collection

2.1 Types of Life Insurance

There are several types of life insurance policies designed to cater to different needs and preferences. In this paper we introduce the four most popular types of life insurance plans:

1. Term Life Insurance: Term life insurance provides coverage for a specific period, typically ranging from 10 to 30 years. If the policyholder dies within the term, the death benefit is paid to the beneficiaries. Term life insurance policies do not have a cash value component and are generally more affordable than permanent life insurance. If the policyholder outlives the term,

the coverage ends, and no benefit is paid.

2. Whole Life Insurance: Whole life insurance is a type of permanent life insurance that provides lifetime coverage. It combines a death benefit with a cash value component that grows over time. Part of the premium paid goes towards the death benefit, while the remaining portion is invested, allowing the cash value to accumulate. Policyholders can borrow against the cash value or even withdraw a portion of it during their lifetime, subject to certain conditions.

3. Universal Life Insurance: Universal life insurance is another form of permanent life insurance that offers more flexibility than whole life insurance. It also combines a death benefit with a cash value component. However, universal life policies allow policyholders to adjust their premiums and death benefit, within certain limits, to suit their changing needs. The cash value grows based on a minimum guaranteed interest rate, and the policyholder can also borrow against or withdraw from the cash value.

4. Variable Life Insurance: Variable life insurance is a permanent life insurance policy with an investment component. Policyholders can allocate a portion of their premium payments to various investment options, such as stocks, bonds, and mutual funds. The death benefit and cash value fluctuate based on the performance of the chosen investments. This type of policy carries more risk, as the policy's value is subject to market fluctuations.

For this paper, we will derive a simple yet effective model for a Term Life Insurance plan. Other types of life insurance plan can be derived from this model with added considerations for the cash value component or investment option.

2.2 Actuarial Table

Since life expectancy - a random variable function of age - is crucial for determining the annuity fair an individual should pay for their life insurance, we need to first understand how insurance companies calculate life expectancy and death probabilities. Actuaries and statisticians in the insurance industry use statistical models and historical data to make actuarial tables (or life tables) to showcase death probabilities and life expectancy as a function of age. Actuarial tables often feature a hypothetical population of 100000 people divided by gender. At any given age, three data points are included:

Death probability: the chance of death within the next year given age and gender

Number of lives: expected total number of survivors out of the initial 100000 people

Life expectancy: expected age for an individual given age and gender

Data in actuarial tables utilize the concept of mortality force (hazard function) which is the chances of death among humans by age. The underlying principle follows Gompertz law which describes how the force of mortality increases exponentially with age. Makeham law adds an age-independent component to account for external causes of death.

The Gompertz-Makeham distribution is described by the following equation for the force of mortality:

$$u(x) = A + B * e^{(C*x)} \tag{1}$$

Here, A, B, and C are positive constants, and x represents age.

Details of life tables calculations can be found here, data given by the Social Security Administration (SSA):

https://www.ssa.gov/OACT/HistEst/CohLifeTables/LifeTableDefinitions.pdf

Period Life Table, 2020, as used in the 2023 Trustees Report

Exact age	Male			Female		
	Death probability [a]	Number of lives [b]	Life expectancy	Death probability [a]	Number of lives [b]	Life expectancy
0	0.005837	100,000	74.12	0.004907	100,000	79.78
1	0.000410	99,416	73.55	0.000316	99,509	79.17
2	0.000254	99,376	72.58	0.000196	99,478	78.19
3	0.000207	99,350	71.60	0.000160	99,458	77.21
4	0.000167	99,330	70.62	0.000129	99,442	76.22
5	0.000141	99,313	69.63	0.000109	99,430	75.23
6	0.000123	99,299	68.64	0.000100	99,419	74.24
7	0.000113	99,287	67.65	0.000096	99,409	73.25
8	0.000108	99,276	66.65	0.000092	99,399	72.25
9	0.000114	99,265	65.66	0.000089	99,390	71.26

Figure 1: SSA actuarial table

3 Model

3.1 Assumptions

Below are our assumptions of the model:

1. Homogeneous population: the insured population is homogeneous, meaning that all policyholders have the same mortality rates as represented in the life table. In reality, individuals can have different mortality rates due to health conditions, occupation, or lifestyle choices.

2. Stable interest rate: we assume that the annual interest rate (i) remains constant over the policy term. In reality, interest rates can fluctuate over time with high volatility, especially in times of financial panic, which may affect the calculation of actuarial present values and premium amounts.

3. Static mortality rates: the probability of death at each age (P(a)), remain constant over the policy term. In reality, mortality rates can change due to improvements in healthcare, changes in lifestyle, or other factors. This assumption makes the model rely on actuarial tables.

4. Independent events: finally, we assume that the death events are independent and that the probabilities of death at different ages are not correlated. This assumption may not hold true in certain situations, such as the occurrence of a pandemic or other large-scale events that impact mortality rates.

3.2 Variables and Formula

We adapt the actuarial present value (APV) concept to create a mathematical model that takes into account the probability of death within the term, the interest rate, and the benefit amount. Here's a proposed model:

Let P(a) be the probability of death at age a, derived from a life table.

Let T be the term of the insurance policy in years.

Let A be the policyholder's current age.

Let B be the benefit amount paid to the beneficiary upon the insured's death.

Let i be the annual interest rate, which is used to discount future cash flows.

Let APV be the actuarial present value of the term life insurance policy, which is the expected present value of future cash flows (i.e., the benefit paid upon death) within the term.

The APV equation is:

$$\text{APV}_{\text{term}} = B \cdot \sum_{a} [P(a) \cdot v^{(a+1)}] \tag{2}$$

, where v = 1 / (1 + i) and a ranges from the current age (A) to the maximum age within the term (A+T-1).

Let Annuity be the annual premium the insured pays to maintain the term life insurance policy for the specified term. The fair Annuity can be calculated by solving the equation

$$\text{Annuity}_{\text{term}} \cdot \sum_{a} [p(a) \cdot v^{(a)}] \tag{3}$$

, where p(a) is the probability of surviving to age a, and the sum ranges from the current age (A) to the maximum age within the term (A+T-1). Using the life table data provided by the Social Security Administration and the current annual interest rate, we can calculate the fair Annuity by plugging the values into the equations above.

3.3 Limitations

While the greatest strengths of the model are the simplicity yet effective nature in calculating the premiums amount based on known approved principles, it certainly has limitations and weaknesses to work on.

1. Reliance on life tables: The accuracy of the model depends on the quality of the life tables used, which may not accurately represent the mortality experience of the insured population.

2. Constant interest rate: The model assumes that the interest rate (i) remains constant throughout the term of the policy, which may not be the case in real-world situations.

3. Consideration of policy expenses: The model does not take into account the expenses associated with issuing and maintaining the insurance policy, such as administrative costs, commissions, and taxes.

4. Simplistic risk assessment: The model does not consider individual risk factors, such as the insured's health, occupation, or lifestyle, which may affect the probability of death and the

cost of insurance.

4 Results

We use Python to calculate the recommended premium amount for both genders at various ages. The full code is provided in the appendix for the adaptation of the project. Below are outputs for the annuity fair with different parameters (age range, T, B, i):

```
# Constants
start_age = 20          # Starting age
end_age = 75            # Ending age
T = 20                  # Term of the insurance policy in years
B = 100000              # Benefit amount paid to the beneficiary upon
i = 0.02                # Annual interest rate

# Calculate annuity payments for all males and females from 20 to 60 y
results = annuity_payment_results(start_age, end_age, T, B, i)
headers = ["Age", "Annuity Amount (Male)", "Annuity Amount (Female)"]
print(tabulate(results, headers=headers, floatfmt=".2f"))
```

```
  Age    Annuity Amount (Male)    Annuity Amount (Female)
-----  -----------------------  -------------------------
   20                   209.54                      90.30
   25                   257.10                     122.38
   30                   320.15                     166.85
   35                   421.33                     233.43
   40                   592.64                     338.91
   45                   866.50                     502.37
   50                  1257.43                     738.20
   55                  1805.56                    1098.69
   60                  2685.00                    1716.34
   65                  4173.25                    2828.15
   70                  6890.31                    4954.22
```

```
# Constants
start_age = 20          # Starting age
end_age = 75            # Ending age
T = 30                  # Term of the insurance policy in years
B = 500000              # Benefit amount paid to the beneficiary upo
i = 0.02                # Annual interest rate

# Calculate annuity payments for all males and females from 20 to 60
results = annuity_payment_results(start_age, end_age, T, B, i)
headers = ["Age", "Annuity Amount (Male)", "Annuity Amount (Female)"
print(tabulate(results, headers=headers, floatfmt=".2f"))
```

```
  Age    Annuity Amount (Male)    Annuity Amount (Female)
-----  -----------------------  -------------------------
   20                  1300.56                     628.09
   25                  1683.41                     877.95
   30                  2282.52                    1255.40
   35                  3203.61                    1821.09
   40                  4540.10                    2644.56
   45                  6496.61                    3920.14
   50                  9659.31                    6075.59
   55                 14854.85                    9824.26
   60                 23761.86                   16653.92
   65                 39482.55                   29144.61
   70                 65911.87                   50529.67
```

In both cases, we found that on average, male annuity fair is a lot higher than those for females at any given age. Moreover, the rate of increase is very rapid and skewed towards the large age numbers, reflecting the Gompertz-Makeham distribution. We also notice that the premium gets very high for people age 60 and above in both scenarios (above 20000 dollars for T=30, B=500000). This is because these individuals have very high chance of passing before the end of term. Hence, the company must charge a high price to offset the verylikely chance it must cover the benefit amount. Most yield data shows consistent results with major insurance companies prices.

5 Alternative Scenarios

In the alternative scenarios, we suggest running stochastic models with many variables being random variables in place of the current deterministic model. Below are some considerations:

1. Dynamic interest rate: Instead of using a constant interest rate (i) throughout the term, the model could incorporate a dynamic interest rate that varies over time to reflect changes in market conditions and the insurer's investment returns.

2. Incorporating policy expenses: The model could be adjusted to account for policy-related expenses, such as administrative costs, commissions, and taxes. This would provide a more accurate representation of the costs associated with issuing and maintaining the insurance policy.

3. Individual risk factors: The model could be modified to consider individual risk factors, such as the insured's health, occupation, or lifestyle, which may affect the probability of death and the cost of insurance. This could lead to more accurate and personalized premium calculations.

6 Conclusion

In this project, by utilizing the probabilities of death and survival derived from life tables, along with other factors such as the term of the insurance policy, the policyholder's age, and the interest rate, we have been able to determine the actuarial present value of a term life insurance policy and a fair annual premium amount.

While our model has provided a solid foundation for understanding and calculating life insurance premiums, we recognize that it is not without its limitations. Some of the key areas for potential improvement include incorporating dynamic interest rates, policy expenses, individual risk factors, stochastic modeling, as well as considering alternative scenarios and policyholder behavior.

As we move forward, we recommend further exploration of these alternative scenarios and enhancements to our model, as well as the integration of more sophisticated modeling techniques to better capture the complexities of the life insurance market. Ultimately, our goal is to develop a more accurate and comprehensive model that can help insurers better assess and manage risks, while offering policyholders fair and personalized premium calculations.

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8 Appendix

```
import numpy as np
!pip install tabulate


#Set up life table using data from SSA actuarial table
def probability_of_death(age, gender):
    life_table_data = {
        "male": {
        i: value for i, value in enumerate([
                0.005837, 0.000410, 0.000254, 0.000207, 0.000167,
                0.000141, 0.000123, 0.000113, 0.000108, 0.000114,
                0.000127, 0.000146, 0.000174, 0.000228, 0.000312,
                0.000435, 0.000604, 0.000814, 0.001051, 0.001250,
                0.001398, 0.001524, 0.001612, 0.001682, 0.001747,
                0.001812, 0.001884, 0.001974, 0.002070, 0.002172,
                0.002275, 0.002368, 0.002441, 0.002517, 0.002590,
                0.002673, 0.002791, 0.002923, 0.003054, 0.003207,
                0.003333, 0.003464, 0.003587, 0.003735, 0.003911,
                0.004137, 0.004452, 0.004823, 0.005214, 0.005594,
                0.005998, 0.006500, 0.007081, 0.007711, 0.008394,
                0.009109, 0.009881, 0.010687, 0.011566, 0.012497,
                0.013485, 0.014595, 0.015702, 0.016836, 0.017908,
                0.018943, 0.020103, 0.021345, 0.022750, 0.024325,
```

```
        0.026137, 0.028125, 0.030438, 0.033249, 0.036975,
        0.040633, 0.044710, 0.049152, 0.054265, 0.059658,
        0.065568, 0.072130, 0.079691, 0.088578, 0.098388,
        0.109139, 0.120765, 0.133763, 0.148370, 0.164535,
        0.182632, 0.202773, 0.223707, 0.245124, 0.266933,
        0.288602, 0.309781, 0.330099, 0.349177, 0.366635,
        0.384967, 0.404215, 0.424426, 0.445648, 0.467930,
        0.491326, 0.515893, 0.541687, 0.568772, 0.597210,
        0.627071, 0.658424, 0.691346, 0.725913, 0.762209,
        0.800319, 0.840335, 0.882352])},
"female": {
    i: value for i, value in enumerate([
        0.004907, 0.000316, 0.000196, 0.000160, 0.000129,
        0.000109, 0.000100, 0.000096, 0.000092, 0.000089,
        0.000092, 0.000104, 0.000123, 0.000145, 0.000173,
        0.000210, 0.000257, 0.000314, 0.000384, 0.000440,
        0.000485, 0.000533, 0.000574, 0.000617, 0.000655,
        0.000700, 0.000743, 0.000796, 0.000851, 0.000914,
        0.000976, 0.001041, 0.001118, 0.001186, 0.001241,
        0.001306, 0.001386, 0.001472, 0.001549, 0.001637,
        0.001735, 0.001850, 0.001950, 0.002072, 0.002217,
        0.002383, 0.002573, 0.002777, 0.002984, 0.003210,
        0.003476, 0.003793, 0.004136, 0.004495, 0.004870,
        0.005261, 0.005714, 0.006227, 0.006752, 0.007327,
        0.007926, 0.008544, 0.009173, 0.009841, 0.010529,
        0.011265, 0.012069, 0.012988, 0.014032, 0.015217,
        0.016634, 0.018294, 0.020175, 0.022321, 0.025030,
        0.027715, 0.030631, 0.033900, 0.037831, 0.042249,
        0.047148, 0.052545, 0.058685, 0.065807, 0.074052,
        0.083403, 0.093798, 0.104958, 0.117435, 0.131540,
```

```
                0.146985, 0.163592, 0.181562, 0.200724, 0.219958,
                0.239460, 0.258975, 0.278225, 0.296912, 0.314727,
                0.333610, 0.353627, 0.374844, 0.397335, 0.421175,
                0.446446, 0.473232, 0.501626, 0.531724, 0.563627,
                0.597445, 0.633292, 0.671289, 0.711567, 0.754261,
                0.799516, 0.840335, 0.882352])}
        }
    return life_table_data[gender][age]



#APV formula
def actuarial_present_value(A, T, B, i, gender):
    v = 1 / (1 + i)
    sum_death_probs = sum(probability_of_death(a, gender) * v**(a+1)
    for a in range(A, A + T-1))
    return B * sum_death_probs

#Annuity payment formula
def annuity_payment(A, T, B, i, gender):
    APV_term = actuarial_present_value(A, T, B, i, gender)
    v = 1 / (1 + i)
    sum_survival_probs = sum((1 - probability_of_death(a, gender)) * v**a
    for a in range(A, A + T-1))
    return APV_term / sum_survival_probs

#Set up a table to display annuity fair amount for both gender at
various ages
from tabulate import tabulate
def annuity_payment_results(start_age, end_age, T, B, i):
    results = []
```

```
    for age in range(start_age, end_age, 5):
        annuity_term_male = annuity_payment(age, T, B, i, 'male')
        annuity_term_female = annuity_payment(age, T, B, i, 'female')
        results.append((age, annuity_term_male, annuity_term_female))
    return results

# Constants
start_age = 20    # Starting age
end_age = 75      # Ending age
T = 20            # Term of the insurance policy in years
B = 300000        # Benefit amount paid to the beneficiary upon the insured's death
i = 0.05          # Annual interest rate

# Calculate annuity payments for all males and females from 20 to 70 years old
results = annuity_payment_results(start_age, end_age, T, B, i)
headers = ["Age", "Annuity Amount (Male)", "Annuity Amount (Female)"]
print(tabulate(results, headers=headers, floatfmt=".2f"))
```