Decision Making and Reinforcement Learning

Module 4: Dynamic Programming

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Topics

- Dynamic programming for solving MDPs
- Value iteration
- Policy iteration
- Algorithm implementation and properties

Learning Objectives

- **Use** the idea of dynamic programming to solve the nonlinear Bellman equations
- **Implement** the value iteration and policy iteration algorithms
- **Understand** properties of algorithm complexity and convergence
- **Compare & contrast** dynamic programming approaches

MDPs and Bellman Equations

- **MDPs**: Stochastic, sequential decision problems
- **Policy** $\pi: S \rightarrow A$, assignment of action to each state
- **Value** $V^{\pi}: S \rightarrow \mathbb{R}$, expected state utilities from following π

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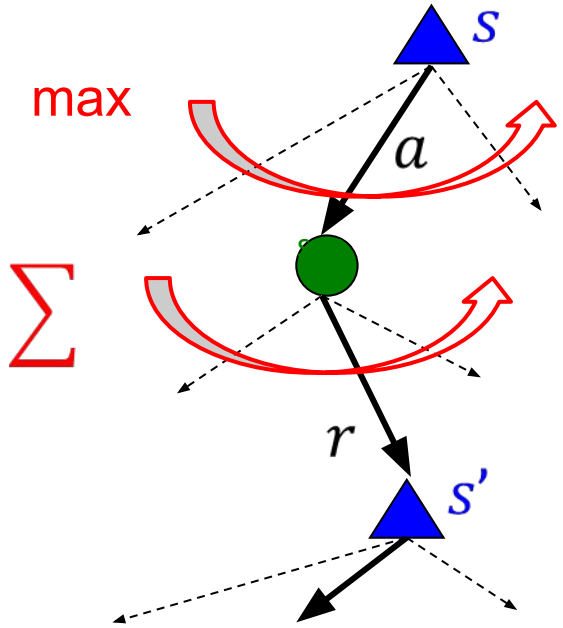
- **Bellman optimality equations**:

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

$$\pi^*(s) = \operatorname{argmax}_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$



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- Idea #2: The utility of a sequence with $i + 1$ rewards can be computed if we have the utility of its *subsequence* with i rewards
- To solve for V^*, we will compute the **time-limited values** of successively larger *finite-horizon* MDPs

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- $i = 2$: $V_2(s) = \max_a \sum_{s'} T(s, a, s')[R(s, a, s') + \gamma V_1(s')]$

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- Given time-limited values V_i, we can compute V_{i+1} using the **Bellman update** using *discounted* V_i:

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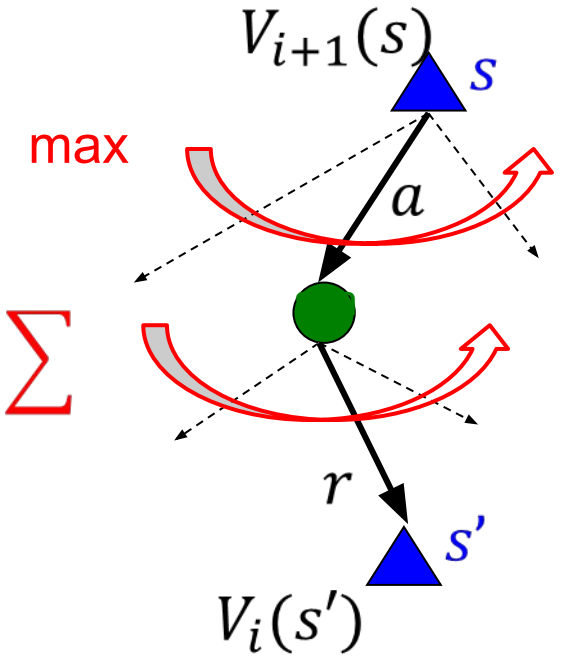

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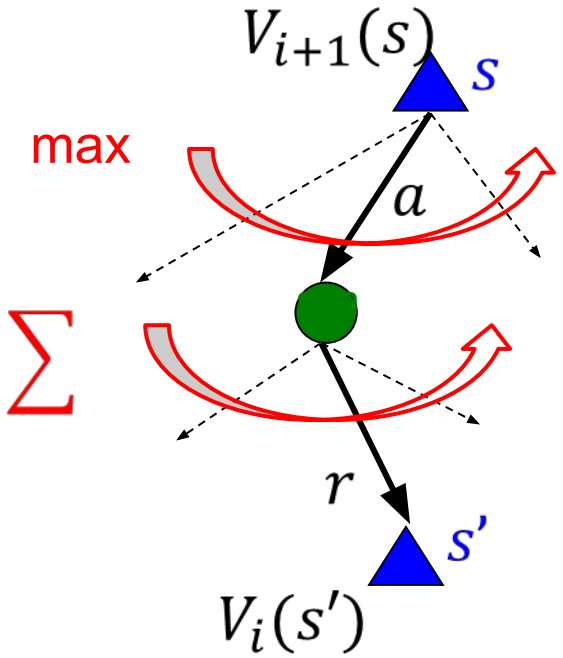
- The function V_i approaches V^* as $i \to \infty$

- Practically, we only need to find V_i for "sufficiently large" i assuming $0 < \gamma < 1$



Value Iteration

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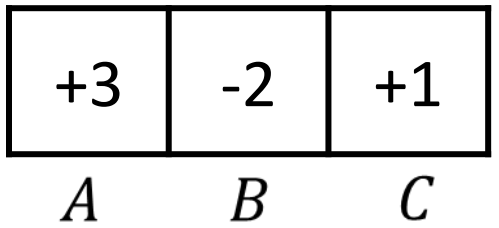

- States, actions, rewards as shown; $\gamma = 0.5$
- Transitions: $\Pr(\text{intended direction}) = 0.8$, $\Pr(\text{opposite direction}) = 0.2$
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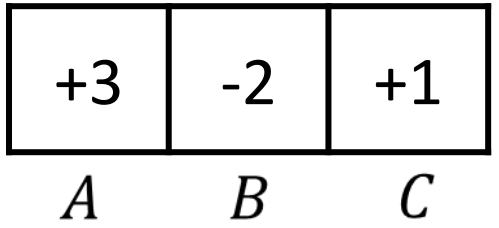

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L action

$$V_1(A) = \max\big[0.8\big(3 + 0.5(0)\big) + 0.2\big(-2 + 0.5(0)\big), \quad\big]$$

$s' = A$ $\qquad$ $s' = B$

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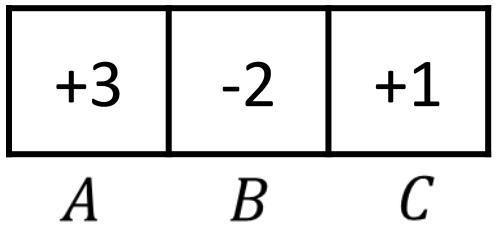

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$$V_1(A) = \max\Big[\underbrace{0.8\big(3 + 0.5(0)\big) + 0.2\big(-2 + 0.5(0)\big)}_{L \text{ action}}, \underbrace{0.8\big(-2 + 0.5(0)\big) + 0.2\big(3 + 0.5(0)\big)}_{R \text{ action}}\Big]$$

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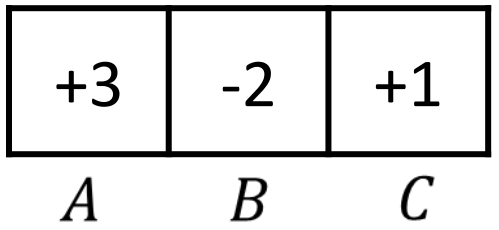

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L action $\qquad\qquad$ R action

$$V_1(B) = \max\big[0.8\big(3 + 0.5(0)\big) + 0.2\big(1 + 0.5(0)\big), 0.8\big(1 + 0.5(0)\big) + 0.2\big(3 + 0.5(0)\big)\big]$$

$s' = A$ $\qquad$ $s' = C$ $\qquad$ $s' = C$ $\qquad$ $s' = A$

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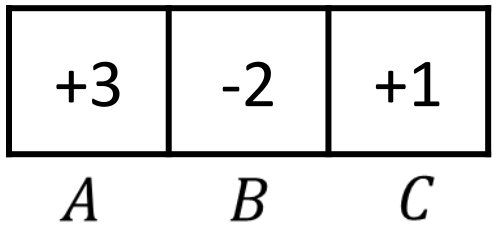

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$$V_1(C) = \max\Big[\underbrace{0.8\big(-2 + 0.5(0)\big)}_{s'=B} + \underbrace{0.2\big(1 + 0.5(0)\big)}_{s'=C}, \underbrace{0.8\big(1 + 0.5(0)\big)}_{s'=C} + \underbrace{0.2\big(-2 + 0.5(0)\big)}_{s'=B}\Big]$$

(L action: first two terms; R action: last two terms)

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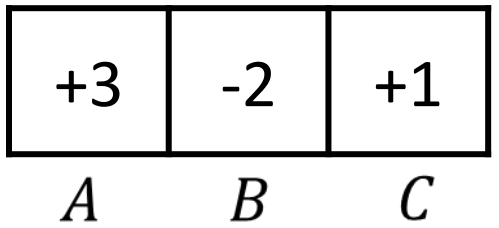

- Time-limited values V_1: $\begin{pmatrix} V_1(A) \\ V_1(B) \\ V_1(C) \end{pmatrix} = \begin{pmatrix} 2 \\ 2.6 \\ 0.4 \end{pmatrix}$

- From each state, we choose the *best action* that produces the *maximum expected value* due to *one transition*

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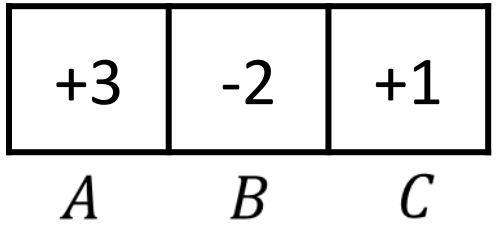

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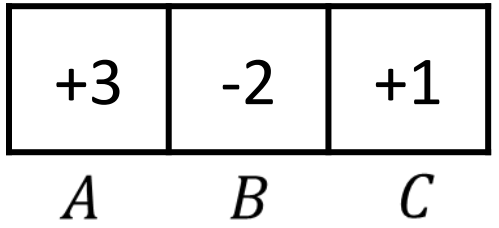

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- From each state, we choose the *best action* that produces the *maximum expected value* due to *one transition*
- Try solving for V_2! $\begin{pmatrix} V_2(A) \\ V_2(B) \\ V_2(C) \end{pmatrix} = \begin{pmatrix} 3.06 \\ 3.44 \\ 0.82 \end{pmatrix}$

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- Stop when values do not change by more than ϵ, a predefined threshold (e.g., a specified fraction of the values themselves)

Checkpoint: Value Iteration Implementation

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- What happen if we set $\epsilon = \infty$?
- What happens if we set $\epsilon = 0$?

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- Fact 1: Bellman update does not change optimal values V^* (*fixed point*)

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

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- For example, suppose largest true value, and max initial error, is $\frac{|r_{\max}|}{1-\gamma}$
- After k passes of value iteration, max error is bounded by $\gamma^k \frac{|r_{\max}|}{1-\gamma}$

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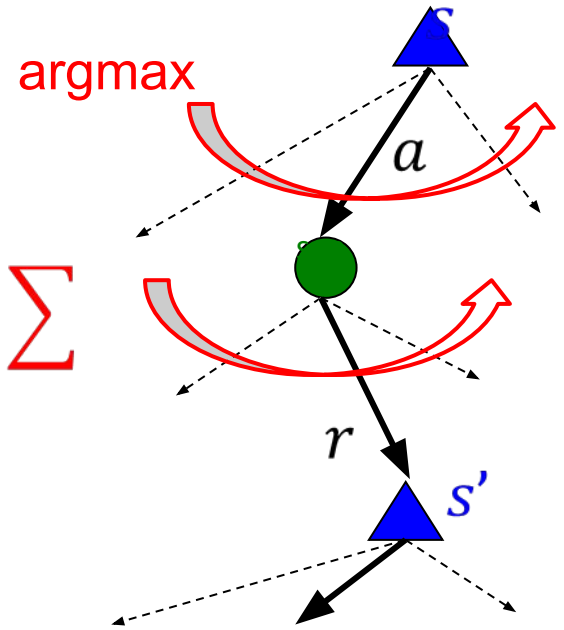
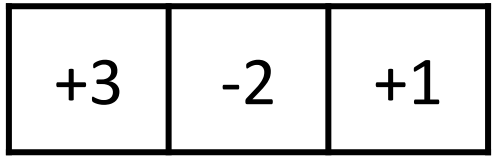
- Given V^*, we can solve for π^* in $O(|S|^2|A|)$ time



Example: Mini-Gridworld

- Given $V^*(A), V^*(B), V^*(C), \gamma = 0.5$
- Find $\pi^*(B)$:



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- $\pi^*(B) = \text{argmax}\big(V^{left}, V^{right}\big)$

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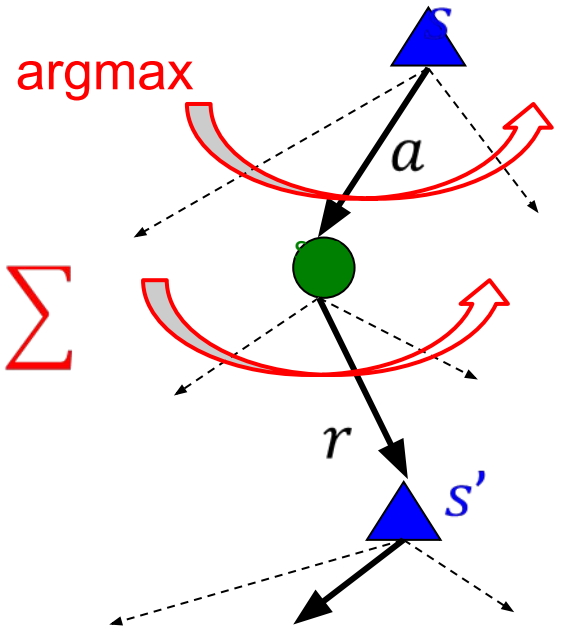
- Observation: Only *relative*, not absolute, V^* required
- We may not need to wait for V^* to converge



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- **Until** $\pi_{i+1} = \pi_i$

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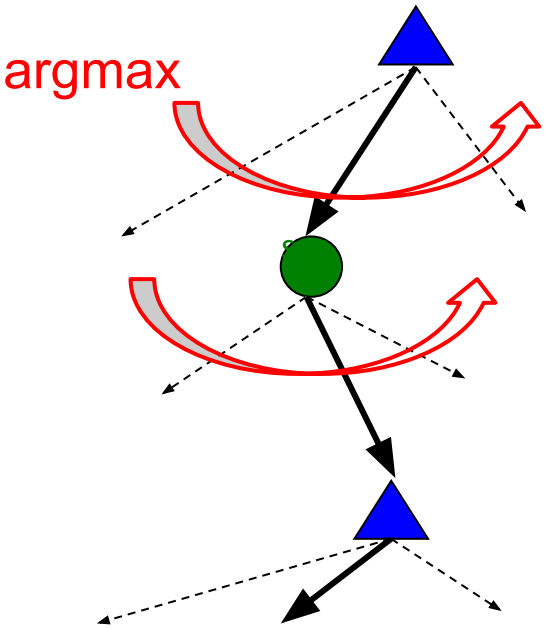
$$V_{i+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^\pi(s')]$$

- **Until** $\max_s |V_{i+1}^\pi(s) - V_i^\pi(s)| < \epsilon$ (small threshold)

Policy Improvement

- Given values for a policy π_i, how can we improve it?
- Consider taking "greediest" action at each state:



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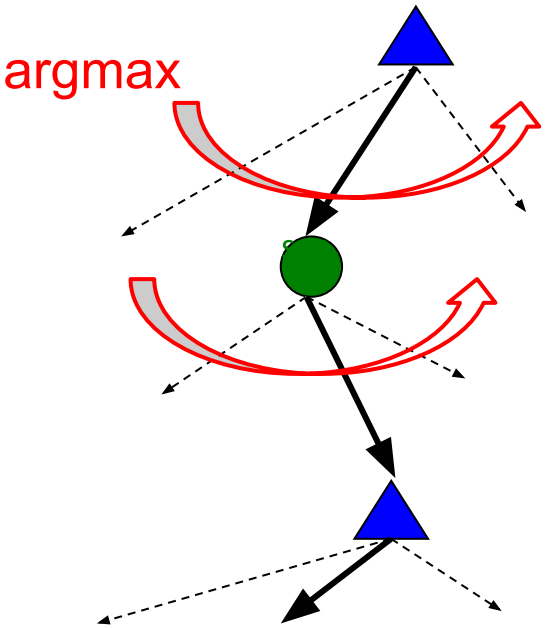

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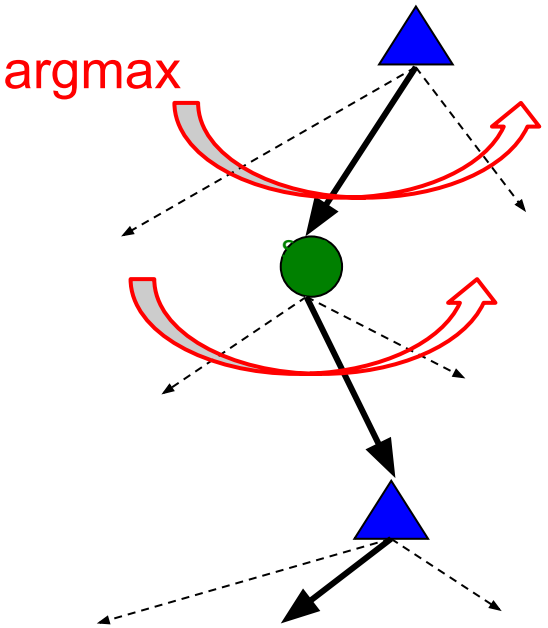

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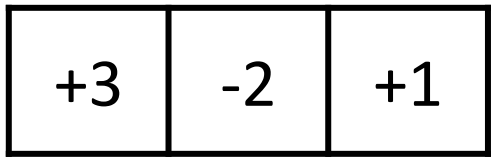
- Otherwise, V^{π_i} can be moved closer to V^* by changing some actions
- Full proof of convergence shown by *policy improvement theorem*

Example: Mini-Gridworld

- Suppose we initialize $\left(\pi_0(A), \pi_0(B), \pi_0(C)\right) = (R, R, R)$



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- **Evaluate** policy using iterative policy evaluation:

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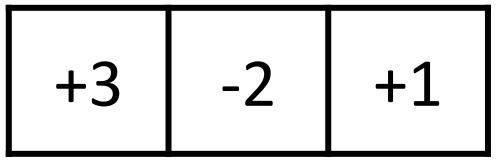

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- **Repeat** the previous steps for π_1

Algorithm Complexity

- Value iteration: Each sweep consists of one full round of iterative policy evaluation (sum) followed by policy improvement (max)

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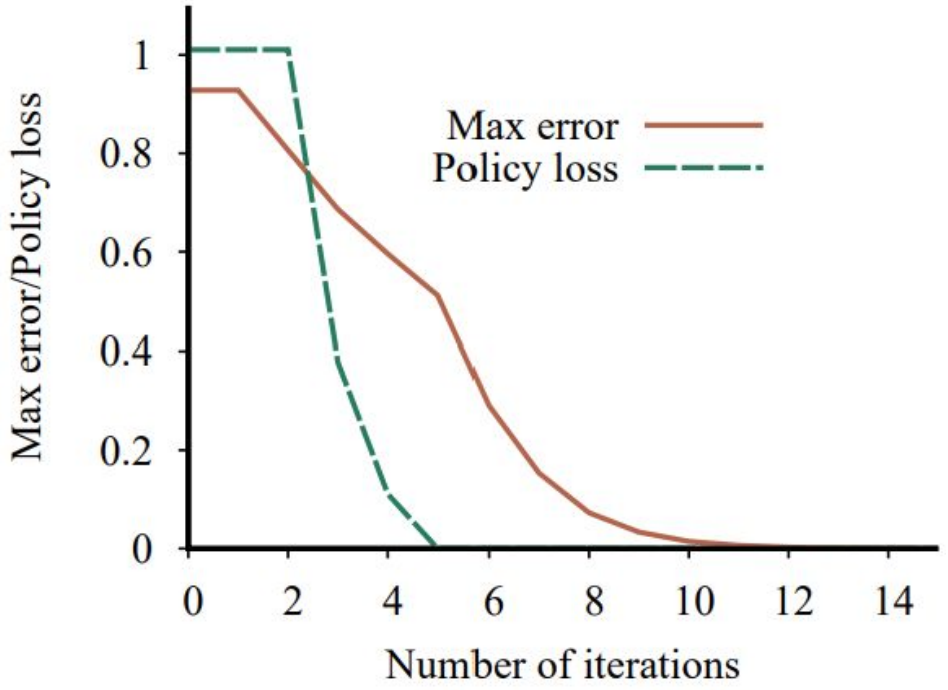

Example comparison: Max error (VI) requires more iterations to converge than does policy loss (PI)

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- We do the same in iterative policy evaluation (new values computed using old values)
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- In practice, it is also correct to make all updates *in-place* in the same array
- Can lead to faster convergence since we use more accurate information
- This is called **asynchronous dynamic programming**

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- Convergence in values in some or all states drives convergence in policy
- Convergence in policy in some or all states drives convergence in values
- Both processes stabilize at joint optimal solution

Summary

- Dynamic programming solves MDPs by exploiting recursive relationships among the state values in the Bellman equations
- Bellman updates push both values and policies toward optimal solution
- Value iteration: Compute and converge toward optimal values for all states, then extract policy
- Policy iteration: Alternative between evaluating and improving a current policy