Decision Making and Reinforcement Learning

Module 2: Bandit Problems

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Topics

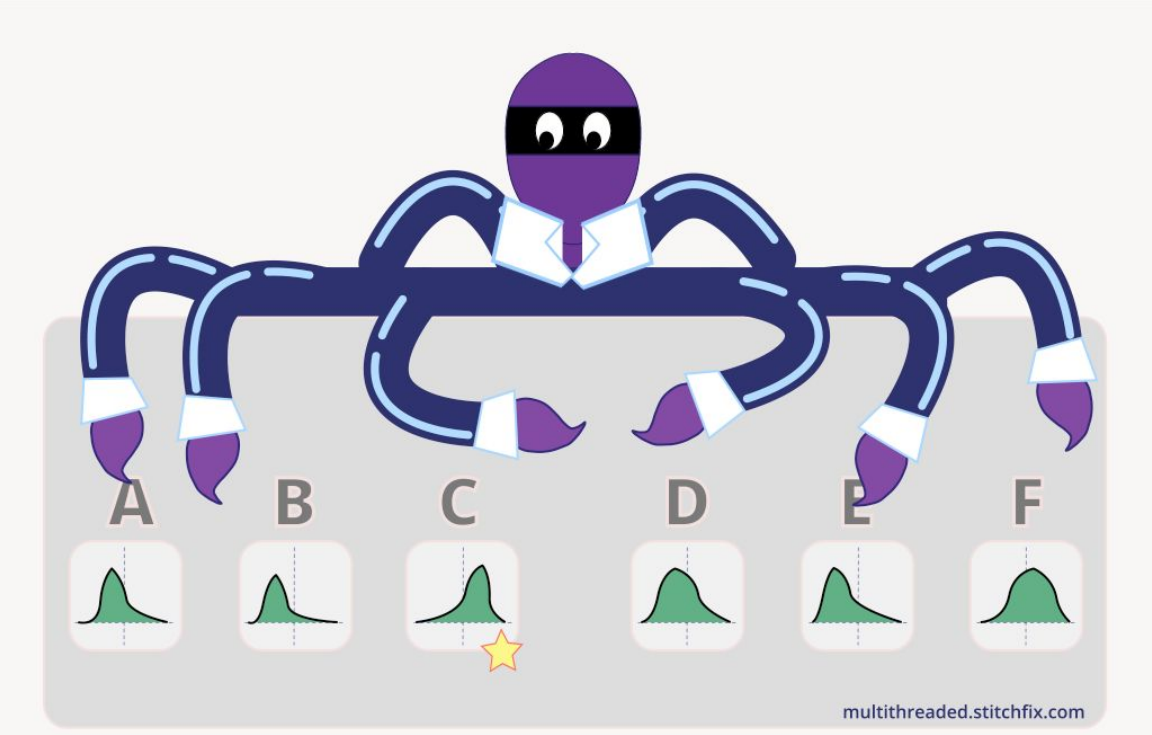
- Multi-armed bandit problems
- Action values and sample averaging
- ε-greedy action selection
- Upper confidence bound

Learning Objectives

- **Define** the multi-armed bandit problem and associated action values
- **Compute** action value updates using uniform and weighted averages
- **Understand** the exploration and exploitation tradeoff and **implement** ε-greedy selection policies
- **Understand and implement** upper confidence bound selection policies

Multi-Armed Bandits

- Suppose we have several slot machines with different rewards and odds
- We can only learn about the machine by trying them (taking actions)
- We want to maximize the overall rewards received



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- Applications: Resource allocation for maximizing productivity, clinical trials to explore different treatments, financial portfolio design

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- A_t is the action chosen and R_t is the resultant reward at time t
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- First initialize all $Q_0(a)$, e.g. by trying each action once and recording reward

- *Sample-averaging*: $$Q_t(a) = \frac{\text{sum of rewards from taking } a \text{ prior to } t}{\text{number of times taking } a \text{ prior to } t}$$

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- Suppose R_t is the nth reward from a; we have that

$$Q_{t+1}(a) = \frac{1}{n}\big((n-1)Q_t(a) + R_t\big) = Q_t(a) + \frac{1}{n}\big(R_t - Q_t(a)\big)$$

- Update form: "new estimate" = "old estimate" + "step size" × "error"

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- Moving average: $Q_1 = 5$ $\qquad Q_2 = Q_1 + \frac{1}{2}(-1 - Q_1) = 5 + \frac{1}{2}(-1 - 5) = 2$

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Nonstationary Problems

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- For **nonstationary** problems in which reward distributions change over time, we can use a *constant* step size α between 0 and 1:

$$Q_{t+1}(a) = Q_t(a) + \alpha\big(R_t - Q_t(a)\big)$$

Exponential Recency-Weighted Average

- For constant α, the action value update rule ends up weighting all rewards, with weights on past rewards *decaying exponentially*

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$$= \cdots = (1-\alpha)^{t-1} R_1 + \alpha \sum_{i=2}^{t} (1-\alpha)^{t-i} R_i$$

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- Equivalent calculation using a *weighted average*:

$$Q_4 = 0.8\left(0.2^0(4) + 0.2^1(0) + 0.2^2(-1)\right) + 0.2^3(5) = 3.208$$

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- *Optimistic* initial values can be used to encourage exploration
- Set all Q_0 much higher than 0, perhaps even higher than actual rewards

Initial Values and Exploration

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- **ε-greedy** action selection: Select greedy action *most* of the time, but with small probability ε, pick a random action to *explore* instead

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- ε-greedy action selection: a_3 is chosen with probability $1 - \varepsilon$, while a_1 and a_2 may be chosen each with probability 0.5ε

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- **ε-first**: Set $\varepsilon = 1$ for a fixed number of trials, then set $\varepsilon = 0$ afterward
- **ε-decreasing**: Set ε to be a high initial value and decrease it over time

Optimism Under Uncertainty

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- When exploring, ε-greedy picks actions completely randomly
- Better idea: Targeted exploration, to focus on more "promising" actions

Upper Confidence Bound

- **Upper confidence bound** combines both exploitation and exploration
  - t = current time step
  - $N_t(a)$ = number of times a taken prior to t
  - c = exploration parameter ≥ 0

$$UCB(a) = Q_t(a) + c\sqrt{\frac{\ln t}{N_t(a)}}$$

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- For proper initialization, we can take each possible action once so that each $N(a)$ starts with value equal to 1

- Subsequent action selection then follows $A_t = \text{argmax}_a UCB(a)$

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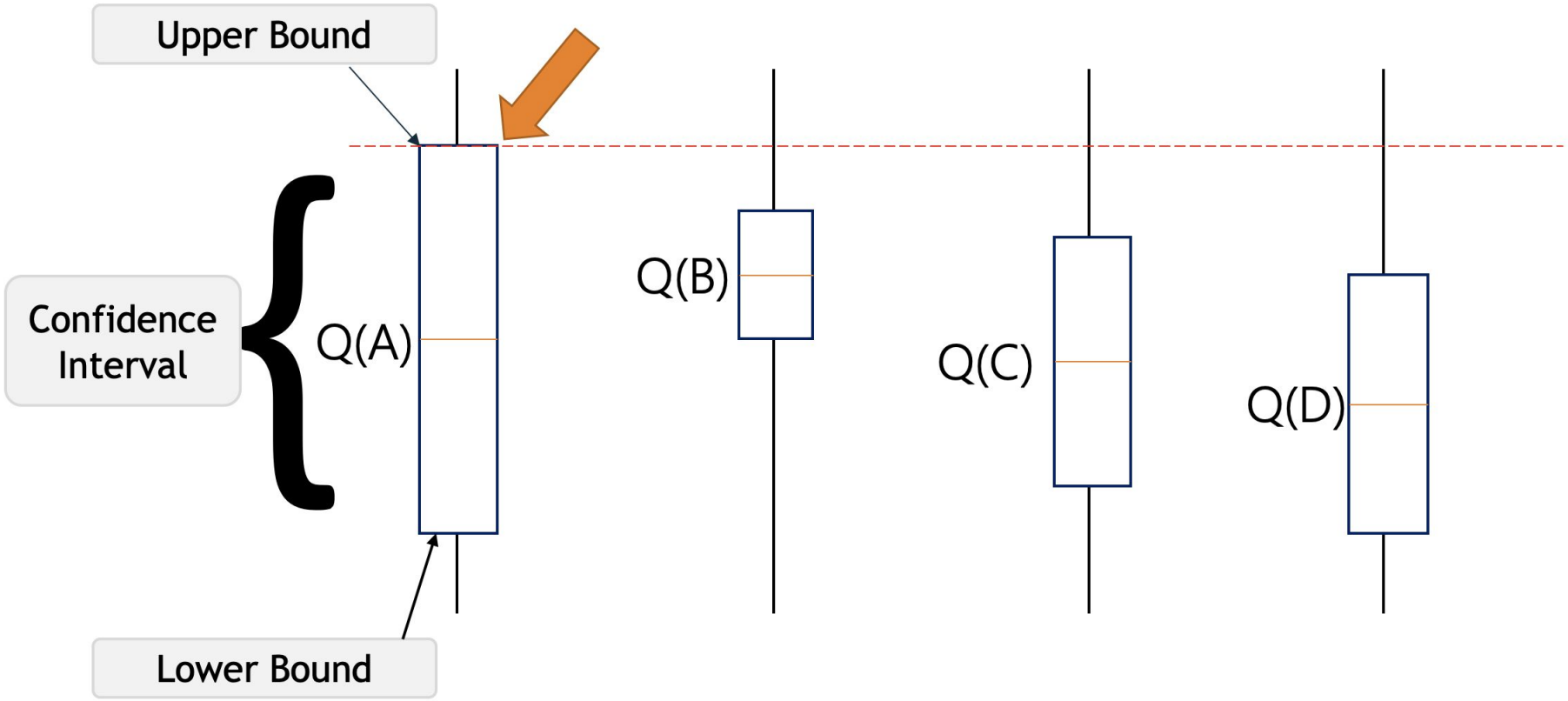
- Exploration is more frequent when N_t is small and tapers out as N_t increases, leading to more exploitation
- $\ln(t)$ term *slowly* increases over time so that exploration is never ruled out

Visualization of UCB



https://www.geeksforgeeks.org/upper-confidence-bound-algorithm-in-reinforcement-learning/

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- **Loop:**
  - Take action $a = \text{argmax}_{a'} UCB(a')$ and observe reward r
  - Update $Q(a) \leftarrow Q(a) + \alpha\big(r - Q(a)\big)$ and $N(a) \leftarrow N(a) + 1$
  - Update α and/or c as desired

ε-Greedy Implementation

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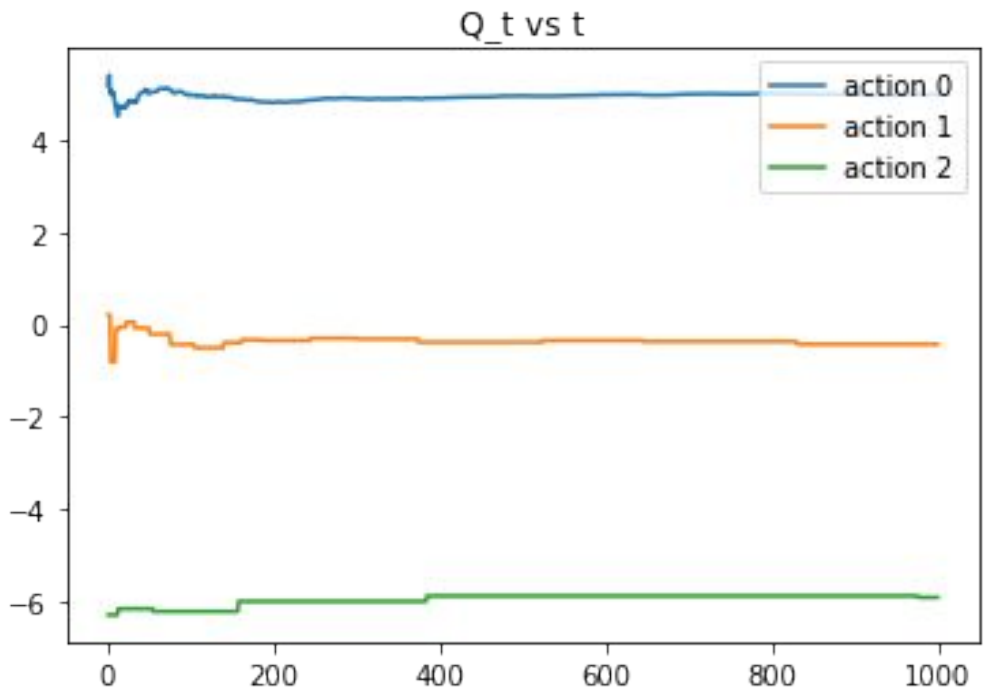
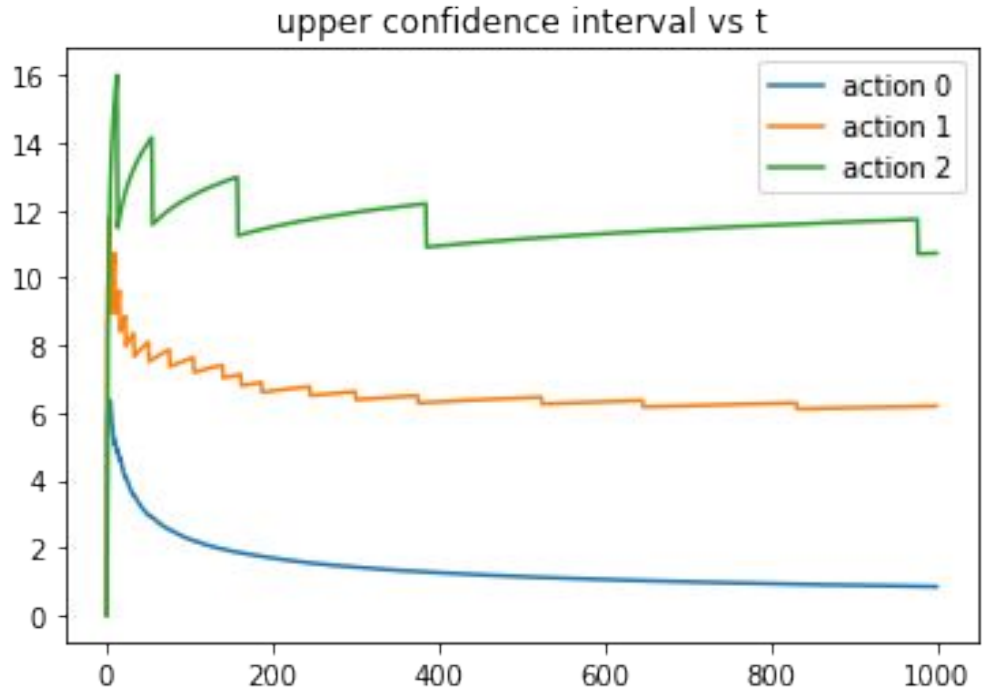
Example: UCB Simulation





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- Baseline decreases as N increases and increases (very slowly) as t increases
- *Likelihood* that error is larger than baseline decreases as t increases

Summary

- Multi-armed bandits model decision making with unknown consequences
- Fundamental tradeoff between exploration and exploitation

- We can keep track and summarize what we know using action values
- Action values can be efficiently updated online while decision making

- Exploration can be encouraged probabilistically, e.g. ε-greedy strategies
- Exploration can also take into account uncertainty in action value estimates, e.g. UCB strategies