COMS W4732 Homework 2: Machine Learning Basics

Following the machine learning content covered in class, in this assignment we will explore some crucial concepts in gradient descent and backpropgation.

Specifically, in Section 1, we will work with a **feedforward neural network** (also known as a **multi-layer perceptron**) implemented solely in numpy, reflect on the associated details of the **forward** pass and implement the **backpropagation** parts of our layers to train our MLP. In Section 2, we will independently look at some gradient-based optimization techniques that are popular with convex functions and have been shown to be useful in finding sufficiently satisfactory optima on the loss manifolds on parametrized models.

Your job is to implement the sections marked with `TODO` to complete the tasks. Your tasks on this homework will be:

Section 1 (40 points)

Review the details on the chain rule and the backproprapagation step from lecture. You should take a look at the guide we provide to get familiar with the forward pass and backward pass equations for a Multi-Layer Perceptron. You are required to implement all parts marked with a **TO DO:**. The goal of this assignment is to leave you with a very through understanding of forward pass as well as the backpropgation mechanics of a Multi-layer Peceptron. Namely, you will be working with:

- a **linear** layer with **Leaky ReLU**
- a **linear** layer with a custom activation function that has a learnable parameter
- a **linear** output layer
- a **softmax cross-entropy** Loss layer

Section 2 (60 points).

We will introduce different gradient-based iterative optimization techniques which came from the domain of convex optimization and which have since been adapted for loss functions of modern neural networks that have millions of parameters. Try out a few of these to appreciate the improvements they make on each other. Specifically, we will look at:

- **Full Gradient Descent**
- **Stochastic Gradient Descent**
- **Stochastic Gradient Descent + Momentum**
- **AdaGrad**
- **Adaptive Moment Estimation (ADAM)**

About Submission

- Please submit the notebook (ipynb and pdf) including the output of all cells. Please note that you should export your completed notebook as a PDF (CV2_HW2_UNI.pdf) and upload it to GradeScope.
- Then, please submit the executable completed notebook (CV2_HW2_UNI.ipynb) to Cousework.
- For both files, 1) remember to replace with your own uni; 2) the completed notebook should show your code, as well as the final image you created.

Before your implementation

- Please check the packages listed in the **requirements.txt**. You can also use `pip install -r requirements.txt` to install the packages directly.

Section 1: Backpropagation

This assignment is aimed at leaving you with a very solid understanding of how backpropogation works in the context of a Neural Network. We have provided most of the code for a MLP (Multi-layer Perceptron) written completely in Numpy with some functions left for you to implement to get the network up and running.

On correct completion, you will be able to successfully train your MLP for any classification problem where the input feature vector is relatively low-dimensional. We've provided code that pre-processes and trains your MLP on the **Red Wine Classification** dataset for binary-classification. This is to enable you to quickly run and test your MLP.

Lets get to it!

As the building blocks of our MLP, we define three layer classes: Hidden, Output and Loss. Each of these inherit from our Base class and thus implement self.forward_pass(), self.backward_pass(), and contain self.update_weighhts(), which is already implemented.

Before we dive in, we would like to highlight the distinction between the Hidden Layer and the Hidden_Vondrick layer as you will see defined below in the code.

Hidden Layer

This is a standard Hidden Layer that uses Leaky ReLU as its activation function. Remember that Leaky ReLU is defined as a piecewise function:

$$g(x)=\{x: \quad x>0, \quad \text{¿}0.01x: \quad x<\text{¿}0$$

Hidden_Vondrick Layer (utilizes cuztom activation function with learnable parameter)

This layer is similar to the standard Hidden layer, but with one notable exception. We will use a custom activation function g'(x). Furthermore, we will make exponent parameter 'n' learnable, and update it also leveraging the chain rule and backpropgation.

$g'(x) = \{x^n: \quad x > 0, \quad ¿ 0.01x: \quad x < ¿ 0$

Instructions

Go over the code for the MLP throughly and understand each update equation implemented as code. The forward_pass() method is implemented for you for every layer and thus you may find printing out variables and their shapes useful, before you begin implementing the backpropogation methods. Having a clear understanding of the forward and backward pass formulae is crucial for this Section. Your job is to implement only sections marked as **TO DO:** (7 in total) (The PDF in the zip file is for your reference)

```
import numpy as np
import random
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline
from scipy.special import softmax
```

```
class Base:
    def __init__(self, input_dims:int, output_dims:int):
        self.input_dims = input_dims
        self.output_dims = output_dims

    def forward_pass(self):
        pass

    def backward_pass(self):
        pass

    def update_weights(self, W, b, del_W, del_b, learning_rate):
        W-=(learning_rate*del_W)
        b-=(learning_rate*del_b)
        return W, b

class Hidden(Base):
    def __init__(self, input_dims:int, output_dims:int):
        super().__init__(input_dims, output_dims)

        self.W = np.random.random((input_dims, output_dims)) - 0.5
        self.c = np.random.random((1, output_dims)) - 0.5

    def forward_pass(self, X):
        U =X@ self.W + self.c
        activations= self.leaky_relu(U)
        return activations

    def backward_pass(self, X, h, dLdh, alpha, learning_rate):
        """
        TO DO:  Finish this backward_pass method by completing the 
lines marked by the #s.
```

```
        Remember to use the helper functions update_weights(),
leaky_relu, and leaky_relu_derivative wherever needed.
        """

        relu_derivative = self.leaky_relu_derivative(h)
        dLdU = np.multiply(relu_derivative, dLdh) #
        dLdW = np.matmul(np.transpose(X), dLdU) #
        dLdc = np.sum(dLdU,axis=0) #
        for_prev = np.matmul(dLdU, np.transpose(self.W))#


        self.W, self.c = self.update_weights(self.W, self.c, dLdW,
dLdc, learning_rate)
        return for_prev
    """Note: for_prev is the gradient dL/dh' we pass onto the previous
layer h' """

    def leaky_relu(self, inp):
        activation_mask = 1.0 * (inp >0) + 0.01*(inp<0)
        activations= np.multiply(inp, activation_mask)
        return activations

    def leaky_relu_derivative(self, h):
        """
        TO DO: Implement the leaky_relu_derivative method.
        This should return a numpy ndarray of shape (batch_size x
self.output_dims)
        """
        if np.where(h>0):
            leaky_relu_derivative = 1
        else:
            leaky_relu_derivative = 0.01
        return leaky_relu_derivative

class Hidden_Vondrick(Base):
    def __init__(self, input_dims:int, output_dims:int):
        super().__init__(input_dims, output_dims)

        self.W = np.random.random((input_dims, output_dims))
        self.c = np.random.random((1, output_dims))
        self.U= None
        self.vondrick_exponent= np.random.uniform(1.4,2) #The
learnable exponent, called Vondrick Exponent, for our custom
activation function is initilized from a Unifom(1.4,2) distribution.
You may change this if you really want to, but keep it close to this
range to ensure training stability.

        print("Intital Value of Vondrick Exponent: "+
str(self.vondrick_exponent) )
```

```
    def forward_pass(self, X):
        self.U =X@ self.W + self.c
        #Applys the custom activation elementwise
        activations= self.vondrick_activation(self.U)
        return activations

    def backward_pass(self, X, h, dLdh, alpha, learning_rate=0.0005):

        """
        # TO DO: Fill in this  backward pass method.
        """
        derivative_wrt_U, derivative_wrt_exponent =
self.vondrick_activation_derivative()
        dL_dexponent_scalar =
np.mean(np.matmul(np.transpose(derivative_wrt_exponent), dLdh))
        dLdU = np.multiply(derivative_wrt_U, dLdh)
        dLdW = np.matmul(np.transpose(X), dLdU) #
        dLdc = np.sum(dLdU, axis=0) #
        for_prev = np.matmul(dLdU, np.transpose(self.W))

        #Note, that for the purposes of training stablity, we have
hard-coded the learning rate here to be 0.0005
        self.W, self.c = self.update_weights(self.W, self.c, dLdW,
dLdc, 0.0005)

        #Gradient Descent on our learnable activation function
parameter: Updating exponent, but clipping it's lower range to 1.01
        self.vondrick_exponent= max(1.01,self.vondrick_exponent-
0.001*dL_dexponent_scalar )

        return for_prev

    def vondrick_activation(self, U):
        """
        TO DO:
        Implement this helper function that the forward pass uses
compute to compute the activation map for the given input U.

        return activations: (batch_size x output_dims)
        """
        vondrick_activation_mask = np.power(U, self.vondrick_exponent-
1)*(U>0) + 0.01*(U<0)
        activations = np.multiply(U, vondrick_activation_mask)
        return activations

    def vondrick_activation_derivative(self):
        """
        # TO DO:
```

```
        Implement this helper function that uses the stored self.U to
do a backward pass and return both dh/dU and dh/dexponent. Both should
be numpy matrices dimensions batch_size x self.output_dims

        return activations: (batch_size x self.output_dims)
        """
        # Derivative wrt U
        if np.where(self.U>0):
            derivative_wrt_U = self.vondrick_exponent*np.power(self.U,
self.vondrick_exponent-1)
        else:
            derivative_wrt_U = 0.01

        # Derivative wrt Exponent
        if np.where(self.U>0):
            derivative_wrt_exponent = np.log(self.U)*np.power(self.U,
self.vondrick_exponent) # derivative of a^x = ln(a)*a^x
        else:
            derivative_wrt_exponent = 0

        return derivative_wrt_U, derivative_wrt_exponent


class Output(Base):

    def __init__(self, input_dims, output_dims):
        super().__init__(input_dims, output_dims)
        self.w = np.random.random((input_dims, output_dims)) -0.5
        self.b = np.random.random((1, output_dims)) -0.5

    def forward_pass(self, h):
        z = h @self.w + self.b
        z = z - np.max(z, axis = 1).reshape(z.shape[0], 1) # trick:
subtracting maz z as softmax is not effected: prevents overflow when
we do exponentation
        return z

    def backward_pass(self, h, dLdz, alpha, learning_rate):
        """
        # TO DO: Implement the backward pass for the output layer.
        Finally, update the Weight matrix and bias vector
appropriately, and then return dLdh, which will be passed backed to
previous layers during backpropgation
        """
        dLdw = np.matmul(np.transpose(h), dLdz)
        dLdb = np.sum(dLdz, axis=0)
        dLdh = np.matmul(dLdz, np.transpose(self.w))
        self.w, self.b = self.update_weights(self.w, self.b, dLdw,
dLdb, learning_rate)
        return dLdh
```

```
class Loss(Base):

    def __init__(self, input_dims, output_dims):
        super().__init__(input_dims, output_dims)

    def forward_pass(self, z, y):

        temp = -z + np.log(np.sum(np.exp(z), axis =
1)).reshape(z.shape[0], 1) #Computing Softmax Cross Entropy Loss terms
for each z_i. Note dimensions of temp: batch_size x output layer
output_dims
        L = temp[np.arange(z.shape[0]), y.flatten().astype(int)]
#Extracts Loss term corresponding only to ground truth class from each
row (sample).
        L = np.mean(L) #Mean Loss over the batch
        return L

    def backward_pass(self, z, y):
        #Recall the simplified expression we get for dL_i/dz_k= p_k-
I(y_i=k) (Details in the guide)
        temp1 = np.zeros(z.shape)
        for i in range(z.shape[0]):
            true_class = int(y[i].item())
            temp1[i][true_class] = -1     #-1 is added to the loss
term corresponding to the true class

        temp2 = np.exp(z)/ np.sum(np.exp(z), axis
=1 ).reshape(z.shape[0], 1) #Matrik of p_k terms, aka, elements
replaced by softmaxed probabilities
        for_previous = temp1 + temp2
        return for_previous

class NN:
    def __init__(self):
        self.output_layer= self.loss_layer =  None
        self.hidden_layers = []

    def add_layer(self, name, input_dims, output_dims):
        if name.lower() == 'hidden':
            self.hidden_layers.append(Hidden(input_dims, output_dims))
        elif name.lower() == 'hidden_vondrick':
            self.hidden_layers.append(Hidden_Vondrick(input_dims,
output_dims))
        elif name.lower() =='output':
            self.output_layer = Output(input_dims, output_dims)
        elif name.lower() =='loss':
            self.loss_layer = Loss(input_dims, output_dims)

    def forward_prop(self, X, y, alpha):
```

```
        hidden_outputs = []
        z = L = h= None
        for layer in self.hidden_layers:
            h = layer.forward_pass(X)
            hidden_outputs.append(h)
            X = h
        z = self.output_layer.forward_pass(h)
        L = self.loss_layer.forward_pass(z, y)
        for layer in self.hidden_layers:
            L += 0.5*alpha*np.linalg.norm(layer.W)**2
        L+= 0.5*alpha* np.linalg.norm(self.output_layer.w)**2
        return hidden_outputs, z, L

    def backward_prop(self, X, hidden_outputs, z, y, alpha =0.01,
learning_rate =0.01):
        dLdz = self.loss_layer.backward_pass(z, y)
        for_previous =
self.output_layer.backward_pass(hidden_outputs[-1], dLdz, alpha,
learning_rate)
        for i in range(len(self.hidden_layers)-1,0,-1):
            temp =
self.hidden_layers[i].backward_pass(hidden_outputs[i-1],
hidden_outputs[i], for_previous, alpha, learning_rate)
            for_previous = temp
        self.hidden_layers[0].backward_pass(X, hidden_outputs[0],
for_previous, alpha, learning_rate)

    def train(self, X, y, epochs, batch_size, learning_rate, alpha,
show_training_accuracy=True):

        loss = []
        for epoch in range(epochs):
            predicted = self.predict(X)
            correct = 0
            for i in range(len(predicted)):
                if predicted[i] == y[i]:
                    correct+=1
            if show_training_accuracy:
                print(f'the accuracy on the training data after epoch
{epoch + 1} is {correct/X.shape[0]}')
            temp = total = 0
            for k in range(0, X.shape[0], batch_size):
                inp = X[k:k+batch_size]
                out = y[k:k+batch_size]

                hidden_outputs, z, L = self.forward_prop(inp, out,
alpha)
                temp+=L
                total+=1
                self.backward_prop(inp, hidden_outputs, z, out, alpha,
```

```
learning_rate)

            loss.append(temp/total)

        return loss

    def predict(self, X):
        """
        TO DO:
        Implement the predict() method that takes in a batch input X
(number_of_samples x feauture_vector_dims) and returns an nparray y of
predictions (number_of_samples x 1)
        """
        hidden_outputs, z, L = self.forward_prop(X,
np.zeros(X.shape[0]), 0)
        softmax_predictions = softmax(z)
        #print(softmax_predictions)
        predictions = np.argmax(softmax_predictions,
axis=1).astype(int) #Since Softmax outputs probabilities for each
class
        return predictions

    def compute_accuracy(self, X, Y):
        predicted_Y= self.predict(X)
        correct=0
        for i in range(len(predicted_Y)):
            if predicted_Y[i] == Y[i]:
                correct+=1
        return correct/len(Y)

def plot_loss(loss_li):
    #Given a list of losses over the epochs, plots the loss curve.
    plt.xlabel("epoch")
    plt.ylabel("loss")
    plt.title("loss of the neural network per epoch")
    plt.plot(loss_li)
    plt.show()
```

Testing your MLP: Red Wine Quality Classification Dataset

More about the dataset:

https://archive.ics.uci.edu/ml/datasets/wine+quality

```
from sklearn.neural_network import MLPClassifier
from sklearn.preprocessing import MinMaxScaler
from sklearn.model_selection import train_test_split
```

```
# If you are using colab, pls refer to commands below for file
uploading.
# Otherwise, just ignore it.
'''
from google.colab import files
uploaded = files.upload()
'''
wine_dataset = pd.read_csv('./winequality-red.csv')

#Converting Labels to a Binary Classification Problem
def Convert_Labels(data):
    data.loc[:,'quality'] = np.where(data.loc[:,'quality']>=6, 1, 0)
    return data

#Scales features to constrain them to lie within the default range
(0,1)
def DataScaler(data):
    scaler = MinMaxScaler()
    data = scaler.fit_transform(data)
    return data

all_columns = list(wine_dataset)
target = ['quality']
print(all_columns)
features = list(set(all_columns)-set(target))
print(features)
wine_dataset.loc[:,features] =
DataScaler(wine_dataset.loc[:,features])

wine_dataset.head()
```

```
['fixed acidity', 'volatile acidity', 'citric acid', 'residual sugar',
'chlorides', 'free sulfur dioxide', 'total sulfur dioxide', 'density',
'pH', 'sulphates', 'alcohol', 'quality']
['free sulfur dioxide', 'chlorides', 'alcohol', 'citric acid',
'residual sugar', 'volatile acidity', 'sulphates', 'density', 'fixed
acidity', 'pH', 'total sulfur dioxide']
```

```
   fixed acidity  volatile acidity  citric acid  residual sugar
chlorides  \
0       0.247788          0.397260         0.00        0.068493
0.106845
1       0.283186          0.520548         0.00        0.116438
0.143573
2       0.283186          0.438356         0.04        0.095890
0.133556
3       0.584071          0.109589         0.56        0.068493
0.105175
4       0.247788          0.397260         0.00        0.068493
0.106845
```

```
   free sulfur dioxide  total sulfur dioxide   density        pH  
sulphates  \
0             0.140845              0.098940  0.567548  0.606299   
0.137725   
1             0.338028              0.215548  0.494126  0.362205   
0.209581   
2             0.197183              0.169611  0.508811  0.409449   
0.191617   
3             0.225352              0.190813  0.582232  0.330709   
0.149701   
4             0.140845              0.098940  0.567548  0.606299   
0.137725   

    alcohol  quality  
0  0.153846        5  
1  0.215385        5  
2  0.215385        5  
3  0.215385        6  
4  0.153846        5  

label_converted_dataset = Convert_Labels(wine_dataset)
print(label_converted_dataset)
#As you can see, the quality column (our labels) now has either 0 (for 
quality<6) and 1 (for quality>=6)

      fixed acidity  volatile acidity  citric acid  residual sugar  
chlorides  \
0          0.247788          0.397260         0.00        0.068493   
0.106845   
1          0.283186          0.520548         0.00        0.116438   
0.143573   
2          0.283186          0.438356         0.04        0.095890   
0.133556   
3          0.584071          0.109589         0.56        0.068493   
0.105175   
4          0.247788          0.397260         0.00        0.068493   
0.106845   
...             ...               ...          ...             ...   
...   
1594       0.141593          0.328767         0.08        0.075342   
0.130217   
1595       0.115044          0.294521         0.10        0.089041   
0.083472   
1596       0.150442          0.267123         0.13        0.095890   
0.106845   
1597       0.115044          0.359589         0.12        0.075342   
0.105175   
1598       0.123894          0.130137         0.47        0.184932   
0.091820   
```

```
      free sulfur dioxide  total sulfur dioxide   density        pH  \
0                0.140845              0.098940  0.567548  0.606299
1                0.338028              0.215548  0.494126  0.362205
2                0.197183              0.169611  0.508811  0.409449
3                0.225352              0.190813  0.582232  0.330709
4                0.140845              0.098940  0.567548  0.606299
...                   ...                   ...       ...       ...
1594             0.436620              0.134276  0.354626  0.559055
1595             0.535211              0.159011  0.370778  0.614173
1596             0.394366              0.120141  0.416300  0.535433
1597             0.436620              0.134276  0.396476  0.653543
1598             0.239437              0.127208  0.397944  0.511811

      sulphates   alcohol  quality
0      0.137725  0.153846        0
1      0.209581  0.215385        0
2      0.191617  0.215385        0
3      0.149701  0.215385        1
4      0.137725  0.153846        0
...         ...       ...      ...
1594   0.149701  0.323077        0
1595   0.257485  0.430769        1
1596   0.251497  0.400000        1
1597   0.227545  0.276923        0
1598   0.197605  0.400000        1

[1599 rows x 12 columns]
```

```
#Quick Sanity check that our dataset it indeed relatively balanced
label_converted_dataset['quality'].mean()
```

```
0.5347091932457786
```

```
y_wine = label_converted_dataset.loc[:,'quality']
X_wine = label_converted_dataset.drop(target,axis=1)

X_wine_np= np.asarray(X_wine)
y_wine_np= np.asarray(y_wine)
X_wine_train, X_wine_test, y_wine_train, y_wine_test =
train_test_split(X_wine_np, y_wine_np, test_size=0.25, random_state=1)
```

Here, you compare your Neural Network's performance (Training and Test Accuracy) with that of Scikit-learn's built-in model MLPClassifier. We, of course, do not expect you to exceed their performance, but your accuracies should be reasonably close to MLPClassifier's.

For the Red Wine Dataset, you should be getting over 70% for both your training as well as test set accuracies. If your MLP's training accuracy hovers around 50%, that is a sign that your model is not learning and you need to go back and fix a bug in your implementation.

```
wine_quality_classifier = MLPClassifier(solver='sgd',
                                        alpha=0.01,
                                        learning_rate_init=0.001,
                                        batch_size=16,

hidden_layer_sizes=(12,8,2),
                                        random_state=1,
                                        max_iter=200
                                        )

wine_quality_classifier.fit(X_wine_train, y_wine_train)
```

```
MLPClassifier(alpha=0.01, batch_size=16, hidden_layer_sizes=(12, 8,
2),
              random_state=1, solver='sgd')
```

```
wine_quality_classifier.score(X_wine_test, y_wine_test)
```

```
0.75
```

Evaluating your MLP on the Red Wine Dataset

First, we instantiate and train a standard MLP (that uses the RELU activation function).

```
my_wine_NN_1 = NN()
num_epochs= 100
lambda_reg= 0.01
learning_rate= 0.001
batch_size= 32
random.seed(2)

my_wine_NN_1.add_layer('Hidden', 11, 16) #Note that the first layer's
weight matrix must be 11 x k , as the input feature vector is 11-
dimensional.
my_wine_NN_1.add_layer('Hidden', 16, 12)
my_wine_NN_1.add_layer('Hidden', 12, 8)
my_wine_NN_1.add_layer('Output', 8, 2)
my_wine_NN_1.add_layer('Loss', 0, 0)

loss_wine_li_1= my_wine_NN_1.train(X_wine_train, y_wine_train,
num_epochs, batch_size, learning_rate, lambda_reg)
plot_loss(loss_wine_li_1)
```

```
the accuracy on the training data after epoch 1 is 0.53628023352794
the accuracy on the training data after epoch 2 is 0.53628023352794
the accuracy on the training data after epoch 3 is 0.53628023352794
the accuracy on the training data after epoch 4 is 0.5446205170975813
the accuracy on the training data after epoch 5 is 0.5871559633027523
the accuracy on the training data after epoch 6 is 0.6246872393661385
the accuracy on the training data after epoch 7 is 0.6522101751459549
the accuracy on the training data after epoch 8 is 0.6638865721434529
the accuracy on the training data after epoch 9 is 0.6722268557130943
```

```
the accuracy on the training data after epoch 10 is 0.6763969974979149
the accuracy on the training data after epoch 11 is 0.6889074228523769
the accuracy on the training data after epoch 12 is 0.6897414512093412
the accuracy on the training data after epoch 13 is 0.6980817347789825
the accuracy on the training data after epoch 14 is 0.7047539616346956
the accuracy on the training data after epoch 15 is 0.713094245204337
the accuracy on the training data after epoch 16 is 0.7164303586321935
the accuracy on the training data after epoch 17 is 0.7264386989157632
the accuracy on the training data after epoch 18 is 0.7272727272727273
the accuracy on the training data after epoch 19 is 0.7289407839866555
the accuracy on the training data after epoch 20 is 0.731442869057548
the accuracy on the training data after epoch 21 is 0.7364470391993327
the accuracy on the training data after epoch 22 is 0.7389491242702252
the accuracy on the training data after epoch 23 is 0.7406171809841534
the accuracy on the training data after epoch 24 is 0.7414512093411176
the accuracy on the training data after epoch 25 is 0.7397831526271893
the accuracy on the training data after epoch 26 is 0.7381150959132611
the accuracy on the training data after epoch 27 is 0.7406171809841534
the accuracy on the training data after epoch 28 is 0.7447873227689742
the accuracy on the training data after epoch 29 is 0.7447873227689742
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the accuracy on the training data after epoch 59 is 0.7406171809841534
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the accuracy on the training data after epoch 60 is 0.7406171809841534
the accuracy on the training data after epoch 61 is 0.7406171809841534
the accuracy on the training data after epoch 62 is 0.7414512093411176
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the accuracy on the training data after epoch 70 is 0.7406171809841534
the accuracy on the training data after epoch 71 is 0.7406171809841534
the accuracy on the training data after epoch 72 is 0.7406171809841534
the accuracy on the training data after epoch 73 is 0.7406171809841534
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the accuracy on the training data after epoch 75 is 0.7406171809841534
the accuracy on the training data after epoch 76 is 0.7414512093411176
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the accuracy on the training data after epoch 88 is 0.7422852376980817
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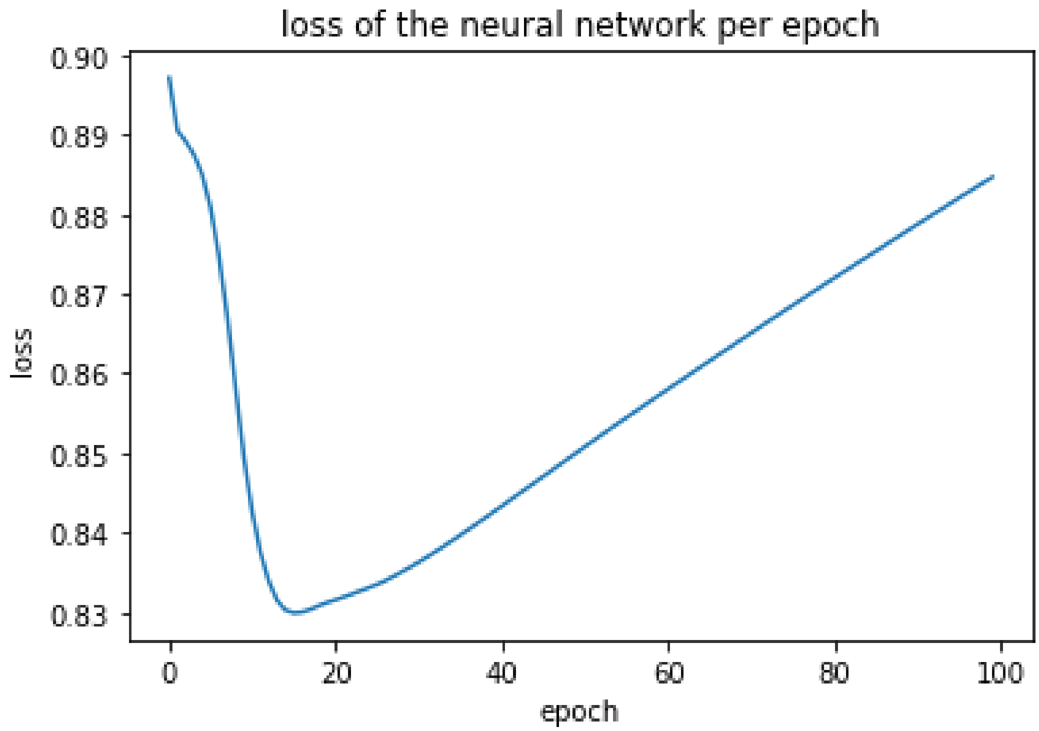
the accuracy on the training data after epoch 90 is 0.7422852376980817
the accuracy on the training data after epoch 91 is 0.7422852376980817
the accuracy on the training data after epoch 92 is 0.7422852376980817
the accuracy on the training data after epoch 93 is 0.7422852376980817
the accuracy on the training data after epoch 94 is 0.7422852376980817
the accuracy on the training data after epoch 95 is 0.7422852376980817
the accuracy on the training data after epoch 96 is 0.7431192660550459
the accuracy on the training data after epoch 97 is 0.7431192660550459
the accuracy on the training data after epoch 98 is 0.7431192660550459
the accuracy on the training data after epoch 99 is 0.7431192660550459
the accuracy on the training data after epoch 100 is
0.7431192660550459
```




```
print(my_wine_NN_1.compute_accuracy(X_wine_test, y_wine_test))
```

```
0.75
```

```
my_wine_NN_2 = NN()
num_epochs= 100
lambda_reg= 0.01
learning_rate= 0.001
batch_size= 32
random.seed(0)

my_wine_NN_2.add_layer('Hidden', 11, 16) #Note that the first layer's
weight matrix must be 11 x k , as the input feature vector is 11-
dimensional.
my_wine_NN_2.add_layer('Hidden', 16, 12)
my_wine_NN_2.add_layer('Hidden_Vondrick', 12, 8)
my_wine_NN_2.add_layer('Output', 8, 2)
my_wine_NN_2.add_layer('Loss', 2, 2)
```

```
Intital Value of Vondrick Exponent: 1.6664604968631305
```

```
loss_wine_li_2= my_wine_NN_2.train(X_wine_train, y_wine_train,
num_epochs, batch_size, learning_rate, lambda_reg)
plot_loss(loss_wine_li_2)
```

```
the accuracy on the training data after epoch 1 is 0.46371976647206004
the accuracy on the training data after epoch 2 is 0.5437864887406172
the accuracy on the training data after epoch 3 is 0.5554628857381151
the accuracy on the training data after epoch 4 is 0.5871559633027523
```

```
the accuracy on the training data after epoch 5 is 0.6238532110091743
the accuracy on the training data after epoch 6 is 0.6780650542118432
the accuracy on the training data after epoch 7 is 0.7055879899916597
the accuracy on the training data after epoch 8 is 0.7197664720600501
the accuracy on the training data after epoch 9 is 0.7197664720600501
the accuracy on the training data after epoch 10 is 0.718932443703086
the accuracy on the training data after epoch 11 is 0.7222685571309424
the accuracy on the training data after epoch 12 is 0.725604670558799
the accuracy on the training data after epoch 13 is 0.7272727272727273
the accuracy on the training data after epoch 14 is 0.7264386989157632
the accuracy on the training data after epoch 15 is 0.7306088407005839
the accuracy on the training data after epoch 16 is 0.7331109257714762
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the accuracy on the training data after epoch 29 is 0.7464553794829024
the accuracy on the training data after epoch 30 is 0.749791492910759
the accuracy on the training data after epoch 31 is 0.7481234361968306
the accuracy on the training data after epoch 32 is 0.749791492910759
the accuracy on the training data after epoch 33 is 0.749791492910759
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the accuracy on the training data after epoch 36 is 0.7481234361968306
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the accuracy on the training data after epoch 47 is 0.7464553794829024
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the accuracy on the training data after epoch 52 is 0.7456213511259383
the accuracy on the training data after epoch 53 is 0.7464553794829024
the accuracy on the training data after epoch 54 is 0.7456213511259383
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the accuracy on the training data after epoch 81 is 0.7431192660550459
the accuracy on the training data after epoch 82 is 0.7431192660550459
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the accuracy on the training data after epoch 84 is 0.7422852376980817
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the accuracy on the training data after epoch 86 is 0.7414512093411176
the accuracy on the training data after epoch 87 is 0.7414512093411176
the accuracy on the training data after epoch 88 is 0.7406171809841534
the accuracy on the training data after epoch 89 is 0.7397831526271893
the accuracy on the training data after epoch 90 is 0.7397831526271893
the accuracy on the training data after epoch 91 is 0.7414512093411176
the accuracy on the training data after epoch 92 is 0.7422852376980817
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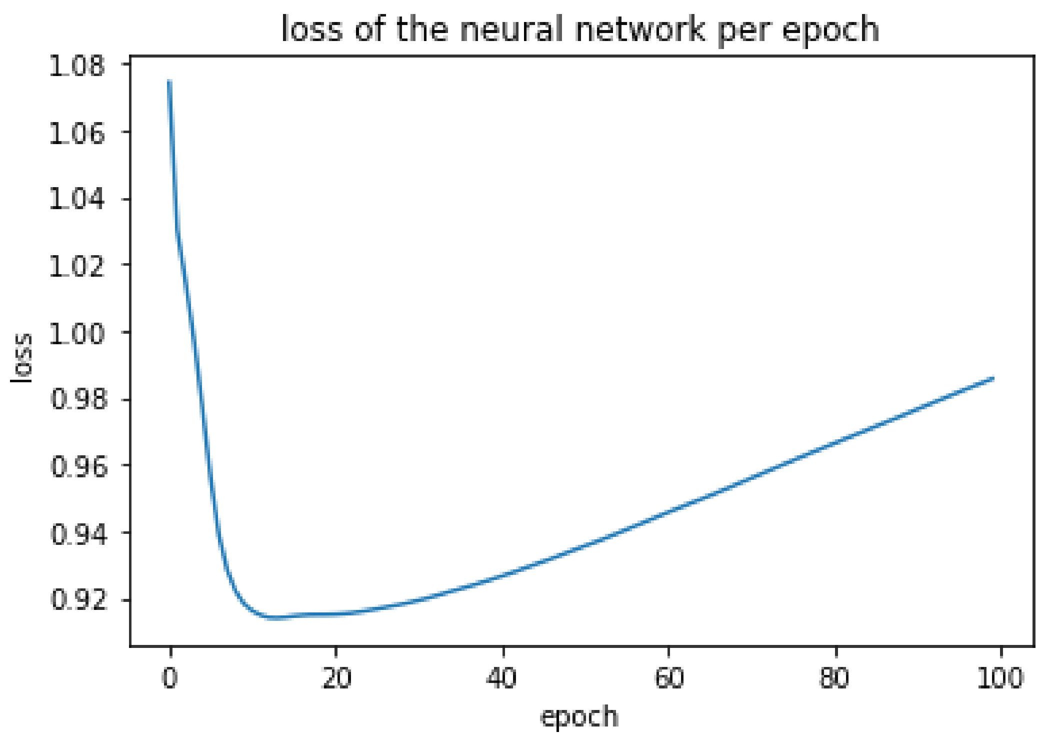
the accuracy on the training data after epoch 94 is 0.7422852376980817
the accuracy on the training data after epoch 95 is 0.7431192660550459
the accuracy on the training data after epoch 96 is 0.7431192660550459
the accuracy on the training data after epoch 97 is 0.7431192660550459
the accuracy on the training data after epoch 98 is 0.7431192660550459
the accuracy on the training data after epoch 99 is 0.74395329441201
the accuracy on the training data after epoch 100 is
0.7447873227689742




```
print(my_wine_NN_2.compute_accuracy(X_wine_test, y_wine_test))
```

```
0.7525
```

Section 2: Optimization

You now have intuition for how the backpropagation procedure updates every single node or layer in the neural network with the gradient of the loss function with respect to the specific parameters. Luckily, you will not have to repeat this tedious enumeration in the future as autograd packages can help you track and organize the gradient tracking process. Even better, most modern neural network libraries like PyTorch and TensorFlow have their own autograd versions which abstract gradient calculations into a single function call on your loss function, instantly tracking along a computational neural network graph to quickly update gradients.

Equally important to the machine learning pipeline is the process of optimization: actually using the calculated **gradient** at the current values and moving to the next values, which are closer to the optimal arguments to our function. A version of **stochastic gradient descent** was already used in section 1 to train the multilayer perceptron.

In a perfect world, once we have an analytical formulation for the gradient of a function, we can go back to the classic technique from Calc 2/Calc 3 of setting the gradient to 0 and calculating the values of the variables at the optimal location. Indeed take

$$f(x,y)=x^2+y^2$$

We have that:

$$\nabla f(x,y)=\begin{bmatrix}2x\\2y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

yielding the location of the optima at

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

However, this technique does not work well with most functions: it is the exception rather than norm to set values to 0 to get and classify the optima. Moreover as the order of the partial derivatives increases, the resulting polynomials become harder and harder to solve (even if a solution is possible) and the problem quickly becomes computationally intractable. Second order methods (Hessian matrices) are required to further classify if these points are in any way useful (saddle points for example would be severely detrimental stopping points for our optimization problem) and are notoriously difficult computationally.

We will now proceed to look at some well-known gradient-based iterative algorithms that have successfully been deployed in training deep learning models. In a typical machine learning pipeline, these optimizers will only be useful after the backpropagation stage is complete. At this stage we have that the network parameters θ have an associated gradient with respect to the loss $\frac{\delta L}{\delta \theta}$. This step involves using the gradient to nudge θ towards an optimum value, i.e, the θ that would yield the lowest possible loss.

In reality, loss functions that are encountered in neural networks are parametrized by hundreds, thousands and millions of parameters, and hence it is not always easy to visualize or study the exact properties of optimization algorithms on them. Typically such optimization functions are designed for **convex** functions, which are characterized, amidst others by **Jensen's inequality**, meaning that the function always lies below any surface connecting two points on this surface, or precisely for a function $f: X \mapsto Y$:

$$f(\theta\vec{x}+(1-\theta)\vec{y}) \leq \theta\vec{x}+(1-\theta)\vec{y}$$

for $x, y \in X$ and $0 \leq \theta \leq 1$. In R^3 the **bowl** or **sphere** function is the archetypical convex function.

$$f(x,y)=x^2+y^2$$

While convex theory gives convenient bounds on gradient based optimization, it is not enough to stop here as we do not expect the loss function for our machine learning models to be convex. Optimization research then focuses on studying the behavior of algorithms on test functions that accentuate some of the possible problematic optimization scenarios we might run into on a loss manifold in higher dimensions. For example, one potential issue we have studied in class is that in an iterative optimization process, we might get stuck in a local optima. A potential test function that could especially be indicative of if an optimization algorithm handles this issue, is the following function, which we will call the **mult** function:

$$f(x, y) = \sin¿$$

Another issue could be a point which has different signs to its curvature in different directions, but locally has no gradient, aka a saddle point. A "test" function to effectively evaluate an algorithm's performance on saddle points could be

$$f(x, y) = x^3 + 3x y^2$$

or otherwise known as the **monkey saddle**.

As a final example, it might be concerning if a point has a very high gradient in one direction but extremely low gradient aka a high condition number is associated with its eigendecomposition of the Hessian (if you didn't understand this last line, that's fine- the Hessian only comes into play while giving a proof of convergence for convex functions for gradient descent and such theory is beyond the scope of this class. An alternate way of thinking why this is an issue is because it will cause a zig-zag convergence to the optima when we can potentially save a lot of iterations by just taking a step along one axis). A function to test convergence performance emperically for this issue could be one shaped like a taco shell. A well known function of this kind is the **Matyas** function.

$$f(x, y) = 0.26(x^2 + y^2) - 0.48 x y$$

Bonus point for figuring out who the **Matyas** function is named after, because I looked forever in the hopes of adding a half-clever note on who Matyas was to improve the readability of this homework with a casual fun fact that has nothing to do with machine learning, but I ended up getting lost online and achieving nothing for 30 mins.

Implement the said functions below as bowl, mult, monkey and matyas and use the plot function below to visualize what they look like. Add a comment on each to explain what there utility might be as test functions.

```
from mpl_toolkits.mplot3d import Axes3D
from matplotlib.colors import LogNorm
import matplotlib.colors
from matplotlib import animation, rc
from IPython.display import HTML
import matplotlib.pyplot as plt
import autograd.numpy as np
from autograd import grad
from autograd import elementwise_grad as egrad
from scipy.optimize import minimize
from collections import defaultdict
from itertools import zip_longest
from functools import partial


def bowl():
    '''
    TODO
```

```
    Implement the bowl function as defined above.

    Add comment here explaining why it is a reasonable test function.
    '''
    # Reasoning
    '''
    The bowl() function is one of the simplest convex function for
which even
    Gradient Descent will find the global optimum. Thus it is a
reasonable
    function to get check the working of any optimizer.
    '''

    def f_bowl(x, y):

        out = np.square(x) + np.square(y)
        # CODE HERE

        return out

    def opt_bowl():

        return np.array([0., 0.])
    return f_bowl, opt_bowl



def mult():
    '''
    TODO

    Implement the mult function as defined above.

    Add comment here explaining why it is a reasonable test function.
    '''
    ## Reasoning
    '''
    Given that the mult() function is a  sinusoidal function which
implies
    presence of multiple local optimum. Hence, to check how an
optimizer would
    behave in such scenario mult() function is a reasonable test
function.
    '''
    def f_mult(x, y):

        out = np.sin(np.sqrt(np.square(x) + np.square(y)))
        # CODE HERE

        return out
```

```
    def opt_mult():

        return np.array([0., 0.])

    return f_mult, opt_mult

def monkey():
    '''
    TODO

    Implement the monkey saddle function as defined above.

    Add comment here explaining why it is a reasonable test function.
    '''
    # Reasoning
    '''
    In real world scenario while training neural networks we might
have
    instances where the gradient becomes zero in the vicinity of a
partricular
    point, also known as "saddle point" i.e. there exist a global
optimum but to
    the optimizer the saddle point seems to the global optimum. This
probelm can be
    difficult to address in higher dimensions. Thus, monkey() function
can be a resonable
    test function to check the performance of an optimizer.
    '''

    def f_monkey(x, y):

        out = np.power(x, 3) + 3*x*np.square(y)
        # CODE HERE

        return out

    def opt_monkey():

        return np.array([0., 0.])

    return f_monkey, opt_monkey


def matyas():
    '''
    TODO

    Implement the mult function as defined above.
```

```
    Add comment here explaining why it is a reasonable test function.
    '''
    # Reasoning
    '''
    There might be scenaios of real world data where the gradient
changes changes
    rapidly in one direction whereas there's neglible change in
gradient in other
    direction. In such cases the gradient descent will get stuck, and
adding momentum
    help in building inertia in the direction of higher gradient and
overcome noisy gradient
    oscillations. Thus matyas() function is a resaonable test function
because with
    with it taco shell shape.
    '''
    def f_matyas(x, y):

        out = 0.26*(np.square(x) + np.square(y)) - 0.48*x*y
        # CODE HERE

        return out

    def opt_matyas():

        return np.array([0., 0.])


    return f_matyas, opt_matyas


def plot_func(f):

    func, opt = f()

    #Set grid parameters
    xmin = -4.5
    xmax= 4.5
    ymin = -4.5
    ymax = 4.5
    step = 0.2

    x, y = np.meshgrid(np.arange(xmin, xmax + step, step),
np.arange(ymin, ymax + step, step))
    z = func(x, y)
    cp = opt()
    optima = cp.reshape(-1, 1)

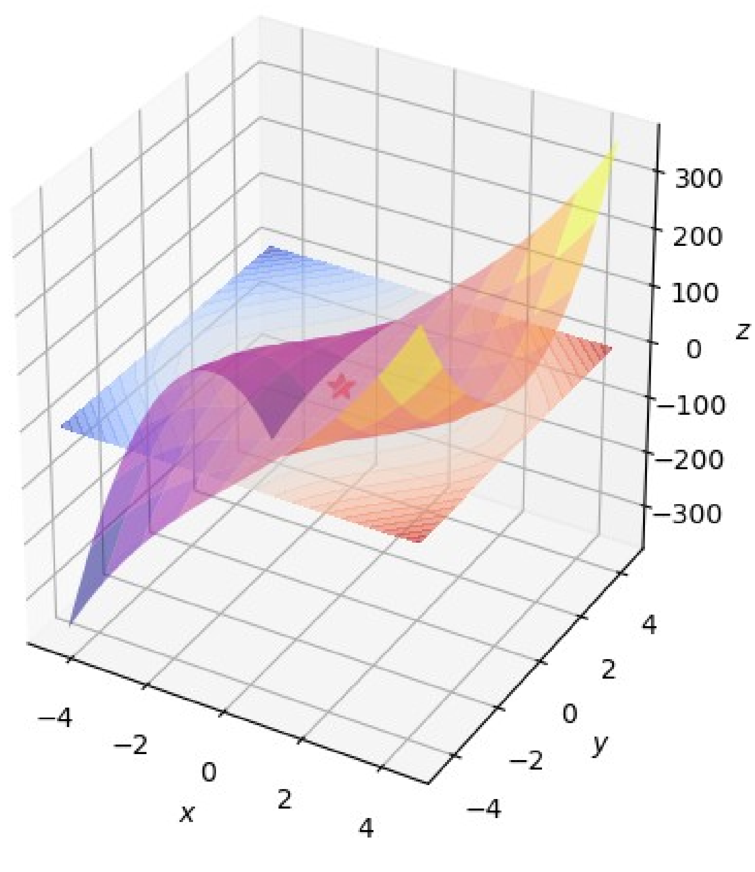
    fig = plt.figure(figsize=(12,6), dpi = 100)
```

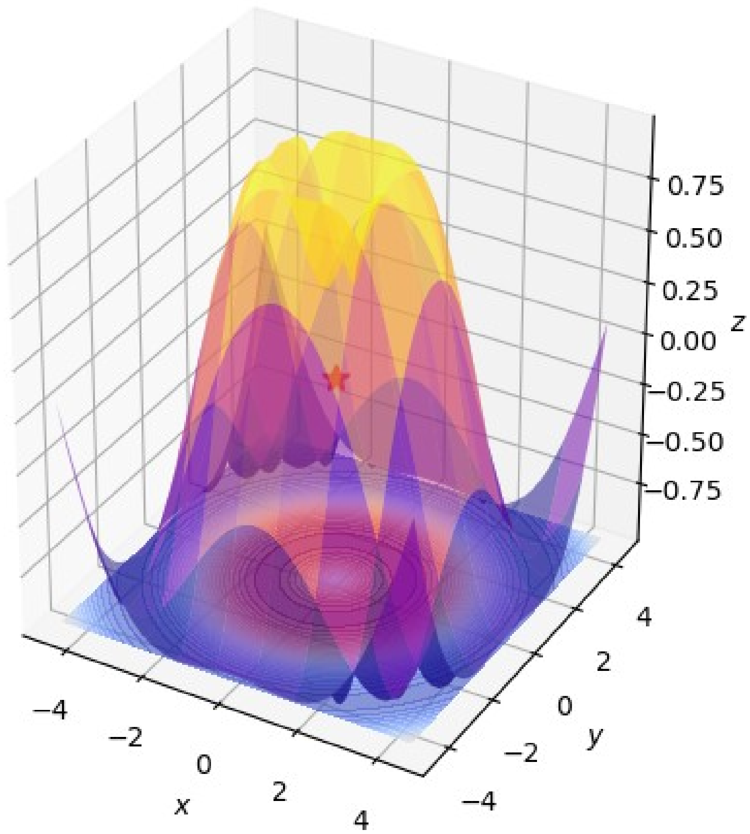
```
    ax = fig.add_subplot(1,2,1,projection='3d')
    ax.plot_surface(x, y, z,  rstride=5, cstride=5, alpha = 0.5,
cmap=plt.cm.plasma)
    cset = ax.contourf(x, y, z, 25, zdir='z', offset=-1, alpha=0.6,
cmap=plt.cm.coolwarm)
    out1 = func(*optima)

    ax.plot(*optima, out1 , 'r*', markersize=10)

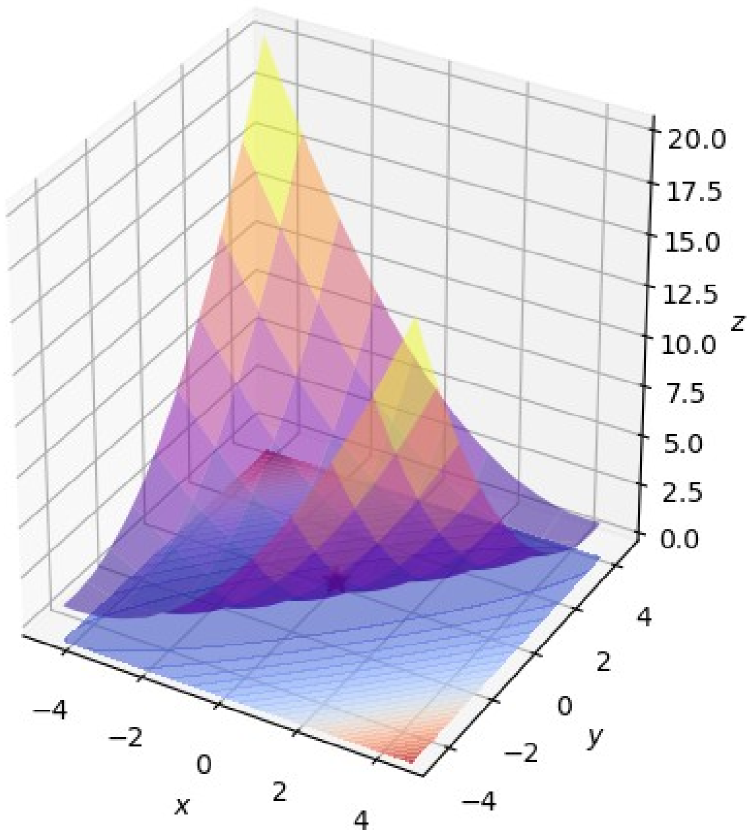
    ax.set_xlabel('$x$')
    ax.set_ylabel('$y$')
    ax.set_zlabel('$z$')

plot_func(monkey)
plot_func(mult)
plot_func(matyas)
plot_func(bowl)
```





0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
z
−4
−2
0
2
4
x
4
2
0
−2
−4
y



20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
z
−4
−2
0
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−2
−4
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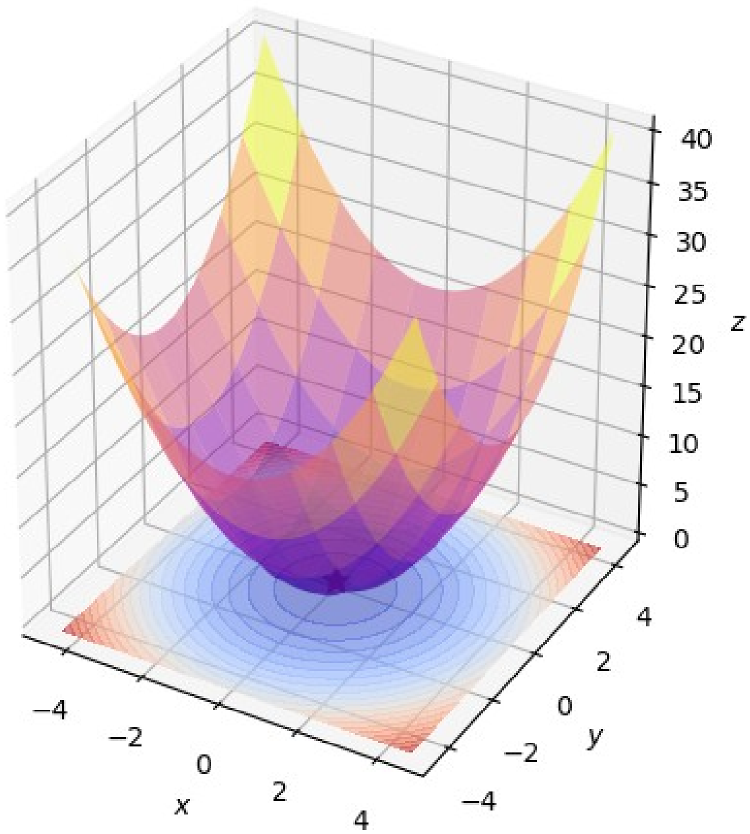


Gradient Descent

$$\theta = \theta - \alpha \nabla_{\theta} L(\theta)$$

We computed the gradient with respect to each of the parameters and make an update in the opposite direction of the local gradient. α which is the learning rate, is a hyperparam that controls how quickly gradient descent converges. If it is too low, too many updates may be required, especially as gradients are small, and if it is too large you may overshoot the optima. In the context of batch gradient descent- at each epoch where the above update is ran- batch gradient descent requires that the entire loss function be computed at each stage through a pass through the entire training data so that the calculation of the $L(\theta)$ and its subsequent gradient be accurate. As a result, this method is usually very slow. Additionally, it is only guaranteed to converge to the global minimum for convex functions such as the bowl functions and this is often not the case in machine learning, in which case it is guaranteed to only converge to local minima.

Implement a version of gradient descent below.

```
def gradient_descent(x, y, dx, dy, hparams):
    '''
    x: value of x before update
    y: value of y before update
```

```
    dx: derivative wrt x
    dy: derivative wrt y
    hparams must contain alpha

    TODO

    Implement the update rule and return the new value of x, y after
the update
    '''

    alpha = hparams['alpha']

    #CODE HERE
    x -= alpha*dx
    y -= alpha*dy
    return x, y
```

Stochastic Gradient Descent

$$\theta = \theta - \alpha \nabla_\theta L(\theta; x^i, y^i)$$

This method is very similar to full gradient descent, except that instead of calculating the loss function over all training examples, the loss function is only calculated over a single training data point at a time. As such the estimate of $L(\theta)$ and its gradient is not precise, but in return, SGD is much faster, and additionally reduces redundant time for recomputing gradients for similar examples.

Gradients received can be highly erratic because they are not calculated over the full dataset, and as a result the optimization path will often zig zag and occasionally spiral out of control. It has been shown that with enough control over the learning rate, sgd and batch gradient descent often achieve the same results, but SGD does it much faster in the context of machine learning. A commonly used variant of gradient descent that sits between full and stochastic gradient descent is mini-batch gradient descent where the gradient and loss function are calculated over a randomly chosen fixed size batch of training examples.

Implement a version of stochastic gradient descent below. Since we are not actually using a dataset or even a data distribution to generate our "loss" manifold that we are trying to optimize over, you should simulate the effect of approximation using gaussian noise on the gradients. Feel free to use the gauss function imported below for the same.

```
from random import gauss

def stochastic_gradient_descent(x, y, dx, dy, hparams):
    '''
    x: value of x before update
    y: value of y before update
    dx: derivative wrt x
    dy: derivative wrt y
    hparams must contain alpha
```

```
    # TODO

    Implement the update rule and return the new value of x, y after
the update
    '''

    alpha = hparams['alpha']

    #CODE HERE
    x -= alpha*(dx+gauss(0,2))
    y -= alpha*(dy+gauss(0,2))

    return x, y
```

Stochastic Gradient Descent and Momentum

$$v_t = \gamma v_{t-1} + \alpha \nabla_\theta L\left(\theta ; x^i, y^i\right)$$

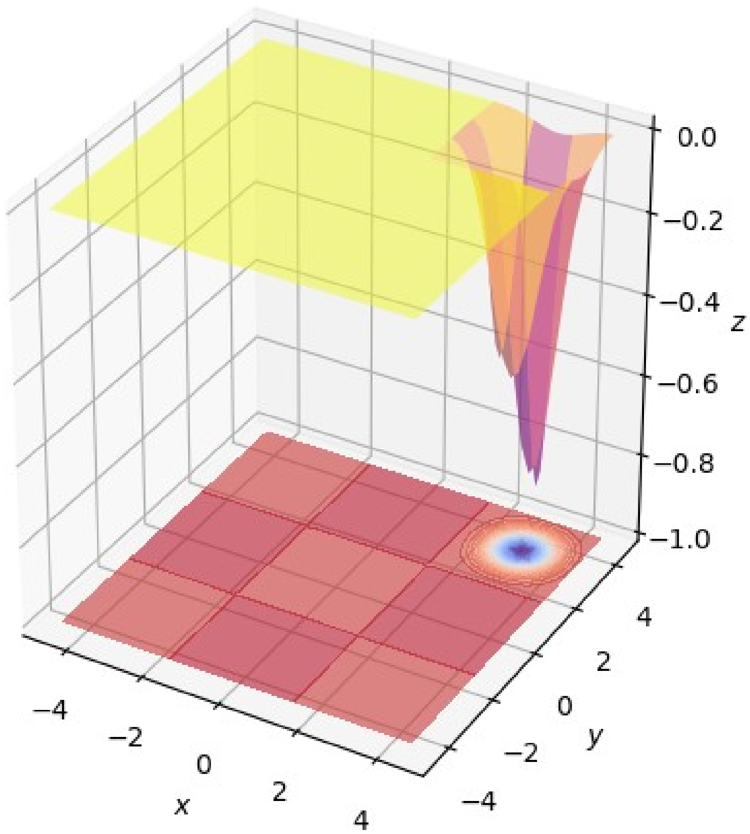
$$\theta = \theta - v_t$$

Stochastic Gradient Descent suffers very heavily if the loss function changes very quickly in one direction and slowly in another direction (eg. taco shell function) In this case the direction of the gradient does not align with the direction toward the minimum and a zig zag motion is direction with slower gradients. This is the case where the Hessian of the loss function wrt parameters has a high condition number, aka the eigenvalues of the representative matrix have a high ratio between the highest and the lowest eigenvalue.

The use of the word momentum is a metaphor in this context, as such a method allows the navigation of shallow local optima or the navigation of ravines using the build-up of gradients from the navigation. The idea is that we maintain what is a 'velocity' term v at each timestep that keeps track of how much gradient has so far been encountered. Hence this value builds up in each direction and even in cases when gradients received in training are poor (eg. around saddle points, or local minima), the algorithm is able to escape such points (much like a ball rolling down a hill). The term for gamma, which is a hyperparameter, can be thought of as friction for the build-up of this velocity, as it decides how much of the previous velocity to count at a certain timestep. Even in the case of a ravine, the zig-zag motion of stochastic gradient descent would reduce as the buildup term in one direction would carry us smoothly through the low-gradient sensitive dimension. The momentum vectors also help cancel/smooth some of the noise that results from approximating gradients using a single data point, aka, the stochastic way.

This method has its own flaws: it might settle in extremely deep minima. Such minima are not desirable even as global minima, as on a data manifold they may be "too good" to be true; representative of a kind of overfit. An example test function for this issue is the **Easom function**. The good news is that with some tuning of **gamma** more often that never SGD + momentum will settle in shallow, wide minima.

```python
def easom():
    '''
    TODO

    Implement the easom function as defined above.

    Add comment here explaining why it is a reasonable test function.
    '''
    # Reasoning
    '''
    In a the easom() function the area of the global minima is really
small compared to
    the search space thus this can be a good test function to check
how fast out optimizer
    can converge to the miminma. This function can be particularly
also used to check
    the performace of an evolutionary algorithm

    '''
    def f_easom(x, y):
        #CODE HERE
        outåå = -np.cos(x)*np.cos(y)*np.exp(-(np.square(x-np.pi)
+np.square(y-np.pi)))
        return outåå

    def opt_easom():

        return np.array([np.pi, np.pi])


    return f_easom, opt_easom

plot_func(easom)
```



Implement your version of stochastic gradient descent with momentum below.

```
def momentum(x, y, dx, dy, v_x, v_y, hparams):
    '''
    x: value of x before update
    y: value of y before update
    dx: derivative wrt x
    dy: derivative wrt y
    v_x: velocity parameter wrt x
    v_y: velocity parameter wrt y
    hparams must contain alpha and gamma

    # TODO

    Implement the update rule and return the new value of x, y, v_x,
v_y after
    the update. Don't forget to add the gaussian noise for the
stochasticity
    '''

    alpha = hparams['alpha']
    gamma = hparams['gamma']
```

```python
    #CODE HERE
    v_x = gamma*v_x + alpha*(dx+gauss(0,2))
    v_y = gamma*v_y + alpha*(dy+gauss(0,2))

    x -= v_x
    y -= v_y

    return x, y, v_x, v_y
```

Nesterov Momentum

$$v_t = \gamma v_{t-1} + \alpha \nabla_\theta L(\theta - \gamma v_{t-1})$$

$$\theta = \theta - v_t$$

A problem with momentum is that there is no way to control the slow-down of the optimizer even as we approach a minima- the optimizer isn't smart enough to decide whether or not it needs to continue up a slope once it has reached a (local or global) minima. A version of momentum, called Nesterov momentum helps deal with this problem by first changing parameters in the direction of the accumulated gradient, estimating the destination gradient and then making an update in that direction.

You do not have to implement nesterov updates.

AdaGrad

Adagrad is another approach toward solving some of the same problems that momentum attempts to solve. However, it does more by giving the optimizer the ability to adapt updates for individual parameters depending on their importance. It adapts its learning rate to make higher updates (higher learning rate) for dimensions/features with higher values and lower updates (low learning rate) for dimensions/features with lower values.

$$G_{i,i}^{t+1} = G_{i,i}^{t} + \left(\frac{\delta L(\theta)}{\delta \theta_i} \right)^2$$

$$\theta_{t,i} = \theta_{t-1,i} - \frac{\alpha}{\sqrt{G_{i,i} + \epsilon}} \nabla_\theta L(\theta_{t,i})$$

Here G starts off as a 0 matrix and builds up squared gradients for each feature/dimension which is later used to scale the value for the parameter updates. ϵ is generally a very small number to ensure that division by zero does not occur in case the initialization point provides no gradient. The biggest advantage of adagrad is that it takes away the need to manually tune the learning rate as is required with SGD, and provides feature-tuned learning rates. However, because of the way adagrad adjusts learning rates it sometimes tends to work well only with convex problems, as with non-convex problems in runs into issues with how to overshooting local minima/saddle points as it slows down rapidly due to accumulating squared gradients. It also provides really slow initial updates if the gradients at the initialization point are really high.

Implement your variant of Adagrad below.

```
def adagrad(x, y, dx, dy, v_x, v_y, hparams):
    '''
    x: value of x before update
    y: value of y before update
    dx: derivative wrt x
    dy: derivative wrt y
    v_x: velocity parameter wrt x
    v_y: velocity parameter wrt y
    hparams must contain eps and alpha

    # TODO

    Implement the update rule and return the new value of x, y, v_x,
v_y after
    the update. Don't forget to add the gaussian noise for the
stochasticity.
    '''

    eps = hparams['eps']
    alpha = hparams['alpha']

    #CODE HERE
    v_x += np.square(dx)
    v_y += np.square(dy)

    x -= (alpha/np.sqrt(v_x+eps))*(dx+gauss(0,2))
    y -= (alpha/np.sqrt(v_y+eps))*(dy+gauss(0,2))

    return x, y, v_x, v_y
```

RMSProp and **AdaDelta** are two different algorithms that combat the aggressive learning rate reduction that comes with AdaGrad.

Adaptive Momentum Estimation

This is the **best of all worlds** update algorithm. The second line (f) accumulated gradients much like momentum, while the third line (s) accumulates gradient squared for adjusting the learning rates like AdaGrad. These have to be adjusted in lines 3 and 4, because they are initialized to 0 at the beginning of optimization and since f is in the numerator of the final update we cannot afford to multiply by a really small number, neither can we afford to divide by a really small number. β_1 and β_2 must be less than 1 and are typically initialized high values such as 0.9 or 0.99. The unbiasing operations on line 4 and 5 above help bring up the values of f and s early in training.

$$g_{t,i} = \nabla_\theta L(\theta_{t,i})$$

$$f_{t,i} = \beta_1 f_{t-1,i} + (1-\beta_1) g_{t,i}$$

$$s_{t,i} = \beta_2 s_{t-1,i} + \left(1 - \beta_2\right) g_{t,i}^2$$

$$\widehat{f_{t,i}} = \frac{f_{t,i}}{1 - \beta_1^t}$$

$$\widehat{s_{t,i}} = \frac{s_{t,i}}{1 - \beta_2^t}$$

$$\theta_{t+1,i} = \theta_{t,i} - \frac{\alpha \widehat{f_{t,i}}}{\sqrt{\widehat{s_{t,i}} + \epsilon}}$$

Implement your function for adam below. Follow the comments closely for interpretation of inputs and outputs.

```
def adam(x, y, dx, dy, f_x, f_y, s_x, s_y, i, hparams):
    '''
    x: value of x before update
    y: value of y before update
    dx: derivative wrt x
    dy: derivative wrt y
    f_x, f_y: first order gradient accumulators
    s_x, s_y: second order gradient accumulators
    i: number of iteration
    hparams must contain alpha, eps, beta_1 and beta_2

    # TODO

    Implement the update rule and return the new value of x, y, f_x,
f_y, s_x,
    s_y, after the update. Don't forget to add the gaussian noise for
    the stochasticity.
    '''

    eps = hparams['eps']
    alpha = hparams['alpha']
    beta_1 = hparams['beta_1']
    beta_2 = hparams['beta_2']

    #CODE HERE
    f_x = (beta_1*f_x)+((1-beta_1)*(dx+gauss(0,2)))
    f_y = (beta_1*f_y)+((1-beta_1)*(dy+gauss(0,2)))

    s_x = (beta_2*s_x) + ((1-beta_2)*np.square(dx+gauss(0,2)))
    s_y = (beta_2*s_y) + ((1-beta_2)*np.square(dy+gauss(0,2)))

    f_x_hat = f_x/(1-np.power(beta_1, i+1))
    f_y_hat = f_y/(1-np.power(beta_1, i+1))

    s_x_hat = s_x/(1-np.power(beta_2, i+1))
```

```
    s_y_hat = s_y/(1-np.power(beta_2, i+1))

    x -= (alpha*f_x_hat)/np.sqrt(s_x_hat+eps)
    y -= (alpha*f_y_hat)/np.sqrt(s_y_hat+eps)

    return x, y, f_x, f_y, s_x, s_y
```

Study in detail the class Optimizer that has been implemented below, because you will be using it to study the effect of key hyperparameters on the optimization process, and the difference that the bells and whistles on Gradient Descent can make. The **fit** function calls the methods you have implemented above depending on how you initialize the class.

```
from random import seed
import math

seed(1)

class Optimizer:

    def __init__(self, x_init, y_init, method, func, hparams):

        '''
        x_init: Initialization x point
        y_init: Initialization y point
        method: adam, adagrad, sgrad, ... check the fit function below
        func: function to optimize [mult, easom, monkey, bowl, matyas,
...]
        hparams: alpha, gamma, eps, beta, beta_1, beta_2
        '''

        f, optima = func()

        self.x = x_init
        self.y = y_init
        self.hparams = hparams
        self.first = True
        self.iter = 0
        self.x_list = []
        self.y_list = []

        cp = optima()
        self.cp_x = np.asscalar(np.array([0]))
        self.cp_y = np.asscalar(np.array([1]))

        self.method = method
        self.f = f
        self.f_gradx = grad(f, 0)
        self.f_grady = grad(f, 1)
        self.count = 0
```

```python
    def distance(self, x, y):
        '''
        L2 Norm
        '''

        return math.sqrt((x-self.cp_x)**2 + (y-self.cp_y)**2)

    def cgrad(self, x, y):
        '''
        Uses autograd
        '''

        return self.f_gradx(x, y), self.f_grady(x, y)

    def fit(self, epochs):

        '''
        Epochs: max number of updates to be made
        '''



        self.x_list = []
        self.y_list = []
        self.z_list = []

        if self.method == 'grad':

            '''
            USES
            alpha: learning rate'
            '''

            for i in range(epochs):

                self.x_list.append(self.x)
                self.y_list.append(self.y)
                self.z_list.append(self.f(self.x, self.y))

                dx, dy = self.cgrad(self.x, self.y)

                self.x, self.y = gradient_descent(self.x, self.y, dx,
dy, self.hparams)

                if (self.distance(self.x, self.y) < 1 and self.first):
                    self.iter = i
                    self.first = False


        if self.method == 'sgrad':
```

```
            '''
            USES
            alpha: learning rate
            '''

            for i in range(epochs):

                self.x_list.append(self.x)
                self.y_list.append(self.y)
                self.z_list.append(self.f(self.x, self.y))

                dx, dy = self.cgrad(self.x, self.y)

                self.x, self.y = stochastic_gradient_descent(self.x,
self.y, dx, dy, self.hparams)

                if (self.distance(self.x, self.y) < 1 and self.first):
                    self.iter = i
                    self.first = False



        if self.method == 'grad_momentum':

            '''
            USES
            alpha: learning rate
            gamma: momentum factor
            '''

            v_x, v_y = 0., 0.
            for i in range(epochs):

                self.x_list.append(self.x)
                self.y_list.append(self.y)
                self.z_list.append(self.f(self.x, self.y))

                dx, dy = self.cgrad(self.x, self.y)

                self.x, self.y, v_x, v_y = momentum(self.x, self.y,
dx, dy, v_x, v_y, self.hparams)

                if (self.distance(self.x, self.y) < 1 and self.first):
                    self.iter = i
                    self.first = False

        if self.method == 'adagrad':

            '''
            USES
```

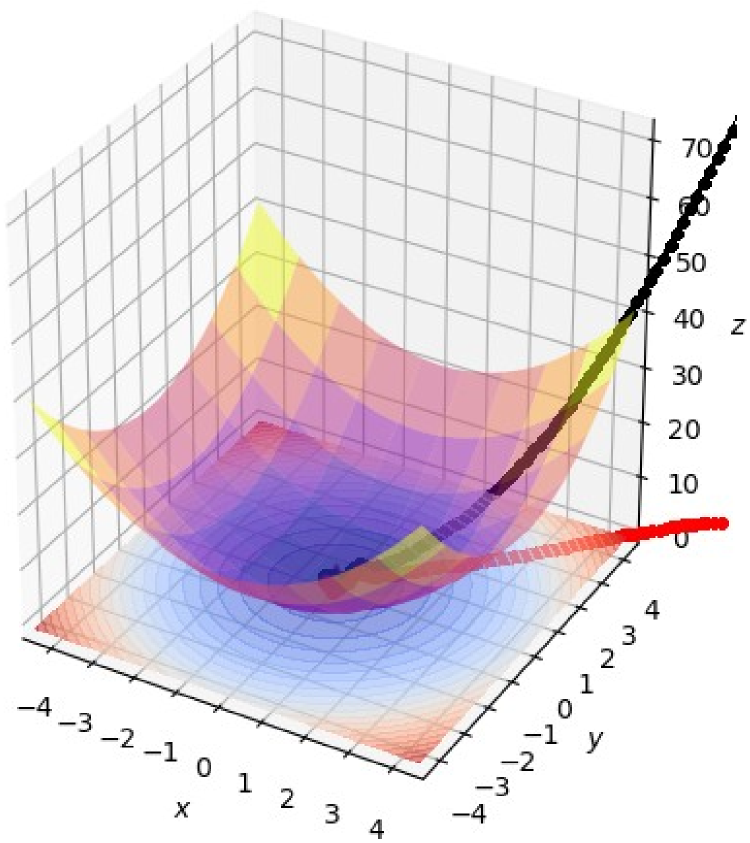
```
            alpha: learning rate
            eps:
            '''

            v_x, v_y = 0., 0.
            for i in range(epochs):

                self.x_list.append(self.x)
                self.y_list.append(self.y)
                self.z_list.append(self.f(self.x, self.y))

                dx, dy = self.cgrad(self.x, self.y)

                self.x, self.y, v_x, v_y = adagrad(self.x, self.y, dx,
dy, v_x, v_y, self.hparams)

                if (self.distance(self.x, self.y) < 1 and self.first):
                    self.iter = i
                    self.first = False


        if self.method == 'adam':

            '''
            USES
            alpha: learning rate
            beta_1:
            beta_2:
            eps:
            '''

            f_x, f_y = 0., 0.
            s_x, s_y = 0., 0.
            for i in range(epochs):

                self.x_list.append(self.x)
                self.y_list.append(self.y)
                self.z_list.append(self.f(self.x, self.y))

                dx, dy = self.cgrad(self.x, self.y)

                self.x, self.y, f_x, f_y, s_x, s_y = adam(self.x,
self.y, dx, dy, f_x, f_y, s_x, s_y, i, self.hparams)

                if (self.distance(self.x, self.y) < 1 and self.first):
                    self.iter = i
                    self.first = False
```

The code at the bottom provides helper functions to help you visualize your results and play around with your optimizer implementations.

```
def animate(i):
    '''Plotting helper'''

    i = int(i*(epochs/frames))
    line1.set_data(optim.x_list[:i+1], optim.y_list[:i+1])
    line1.set_3d_properties(optim.z_list[:i+1])
    line2.set_data(optim.x_list[:i+1], optim.y_list[:i+1])
    line2.set_3d_properties(np.zeros(i+1) -1)
    title.set_text('Epoch: {: d}, Error: {:.4f}'.format(i,
optim.z_list[i]))

    return line1, line2, title


def plot_function(epochs, frames, func, xmin = -4.5, xmax= 4.5, ymin =
-4.5, ymax = 4.5, step = 0.2, option = '3d'):

    '''
    Plot the optimization
    epochs: number of optimization improvements
    frames: number of displayed frames
    func: function to plot. eg. mult, monkey, ....
    xmin: Min value of x to be displayed
    xmax: Max value of x to be displayed
    ymin: Min value of y to be displayed
    ymax: Max value of y to be displayed
    step: Split between xmin and xmax, ymin and ymax
    option: keep this option '3d'

    '''


    x, y = np.meshgrid(np.arange(xmin, xmax + step, step),
np.arange(ymin, ymax + step, step))
    f, opt = func()
    z = f(x, y)
    cp = opt()

    optima = cp.reshape(-1, 1)

    if (option == '3d'):

        fig = plt.figure(figsize=(12,6), dpi = 100)
        ax = fig.add_subplot(1,2,1,projection='3d')

        ax.plot_surface(x, y, z,  rstride=5, cstride=5, alpha = 0.5,
cmap=plt.cm.plasma)
        cset = ax.contourf(x, y, z, 25, zdir='z', offset=-1,
```

```
alpha=0.6, cmap=plt.cm.coolwarm)
        out1 = f(*optima)
        ax.plot(*optima, out1 , 'r*', markersize=10)

        ax.set_xlabel('$x$')
        ax.set_ylabel('$y$')
        ax.set_zlabel('$z$')


        i = 0

        for i in range(epochs):
            line1, = ax.plot(optim.x_list[:i+1], optim.y_list[:i+1],
optim.z_list[:i+1], color ='black', marker = '.')

            line2, = ax.plot(optim.x_list[:i+1], optim.y_list[:i+1],
np.zeros(i+1)-1, color ='red', marker='.')


    if (option == '2d'):

        fig = plt.figure(dpi = 100)
        ax = plt.subplot(111)

        cset = ax.contourf(x, y, z, 25, zdir='z', offset=-1,
alpha=0.6, cmap=plt.cm.bwr)

        dz_dx = egrad(f, argnum=0)(x, y)
        dz_dy = egrad(f, argnum=1)(x, y)
        ax.quiver(x, y, x- dz_dx, y-dz_dy, alpha = 0.5)
        ax.plot(*optima, 'r*', markersize = 18)

        ax.set_xlabel('$x$')
        ax.set_ylabel('$y$')


    ax.set_xlim((xmin, xmax))
    ax.set_ylim((ymin, ymax))

    plt.show

'''
Test plot: Here, we plot optimization with adam on bowl function for
200 updates,
initialized at (6., 6.).
'''

frames = 20
epochs = 200
```

```
func = bowl
init_x, init_y = 6., 6.

hparams = {'alpha': 0.1, 'eps': 1e-7, 'beta_1': 0.8, 'beta_2': 0.99}
optim = Optimizer(init_x, init_y, 'adam', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```



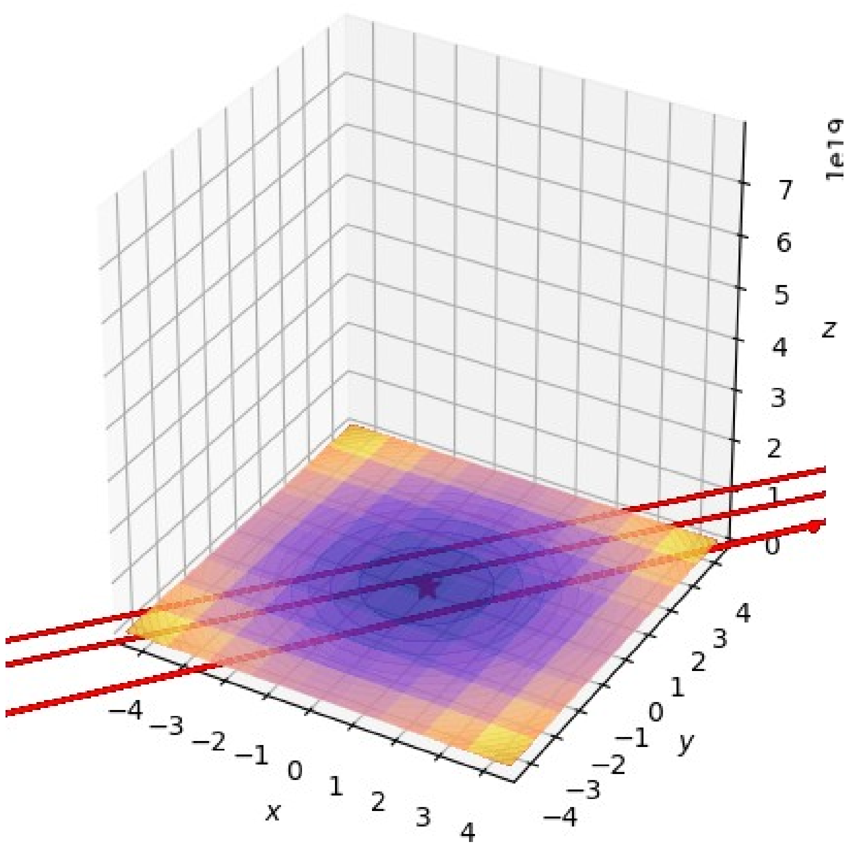
Importance of learning rate

Call the plotting function and Optimizer as shown above and make two plots to show how too high or too low of a learning rate could be a problem for stochastic gradient descent on the bowl function. Keep the number of iterations (epochs) constant across the two to demonstrate the effect. You are recommended to use the bowl function.

```
frames = 20
epochs = 20

'''
```

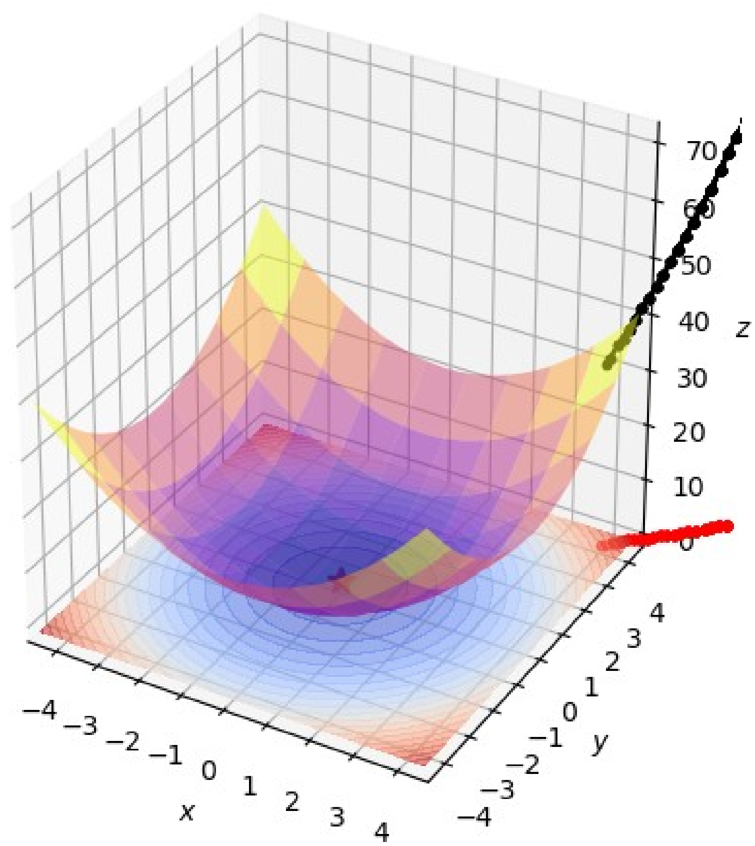
TODO

```
Make a plot to show what happens if the learning rate is too high with
sgrad.

Recommended hparams:

initialization: (6., 6.)
alpha: 2
'''

func = bowl
#CODE HERE
init_x, init_y = 6., 6.
hparams = {'alpha': 2}
optim = Optimizer(init_x, init_y, 'sgrad', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```

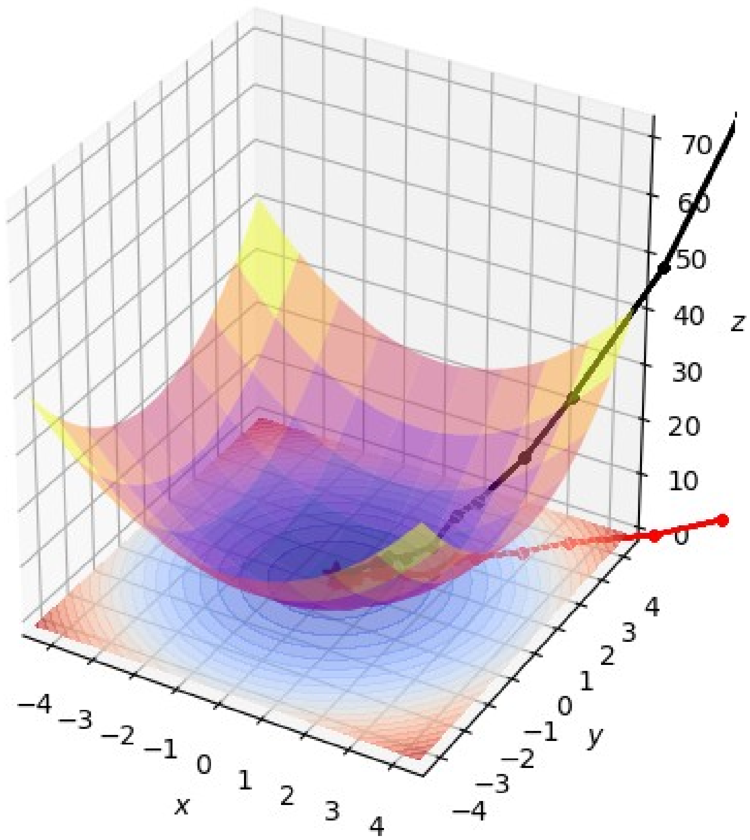



```
frames = 20
epochs = 20
```

```
'''
TODO

Make a plot to show what happens if the learning rate is too low with
sgrad.

Recommended hparams:

initialization: (6., 6.)
alpha: 0.01
'''

func = bowl
#CODE HERE
init_x, init_y = 6., 6.
hparams = {'alpha': 0.01}
optim = Optimizer(init_x, init_y, 'sgrad', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```



```python
frames = 20
epochs = 20

'''
TODO

Make a plot to show what happens if the learning rate is just right
with sgrad.

Recommended hparams:

initialization: (6., 6.)
alpha: 0.1
'''

func = bowl
#CODE HERE
init_x, init_y = 6., 6.
hparams = {'alpha': 0.1}
optim = Optimizer(init_x, init_y, 'sgrad', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```

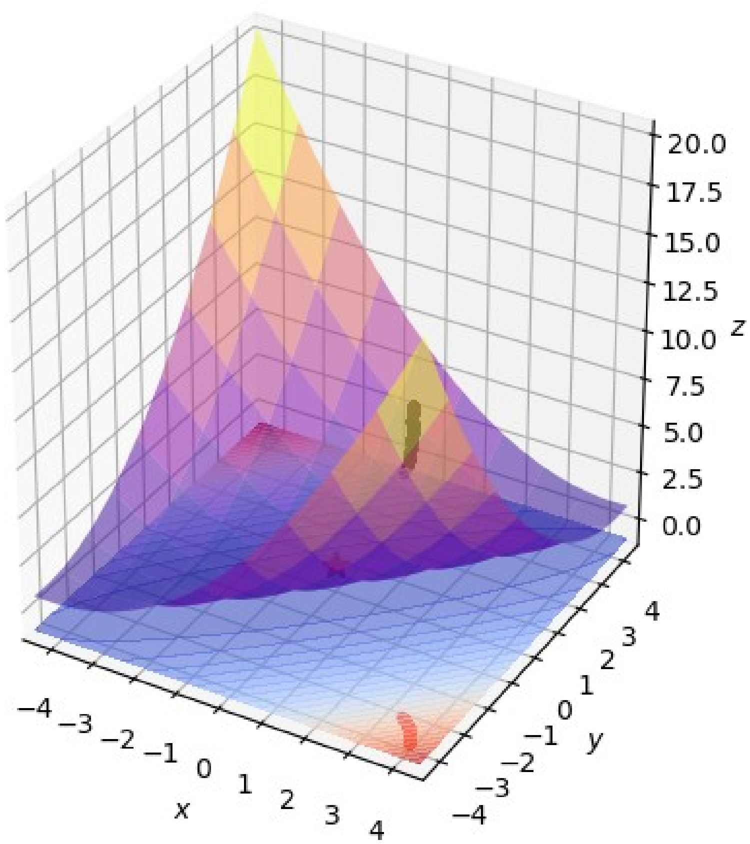


SGD and Matyas: momentum helps

Make two plots to show how momentum possibly helps with navigating a taco-shell function like Matyas, where gradient is low in one direction and high in another direction. Keep all parameters except the type of optimizer constant.

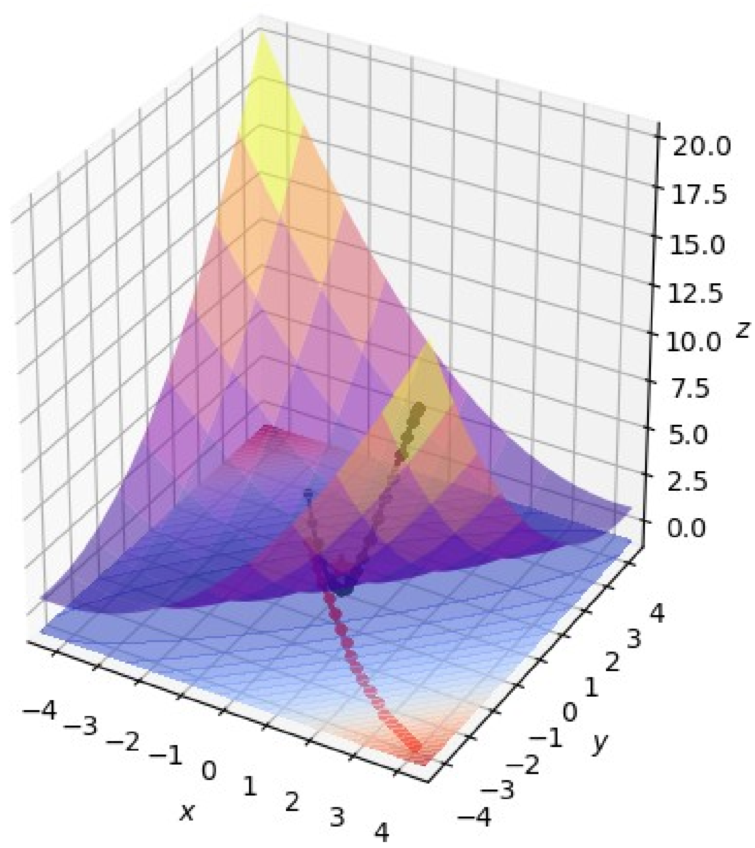
```
frames = 20
epochs = 20

'''
TODO

Make a plot to show what happens if stochastic gradient descent is
used on a
function like this.

Recommended hparams:

initialization: (4., -4.)
alpha: 0.01
'''
```

```
func = matyas
#CODE HERE
init_x, init_y = 4.,-4.
hparams = {'alpha': 0.01}
optim = Optimizer(init_x, init_y, 'sgrad', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```




```
frames = 20
epochs = 20

'''
TODO

Make a plot to show what happens if stochastic gradient descent with
momentum
is used on a function like this.

Recommended hparams:
```

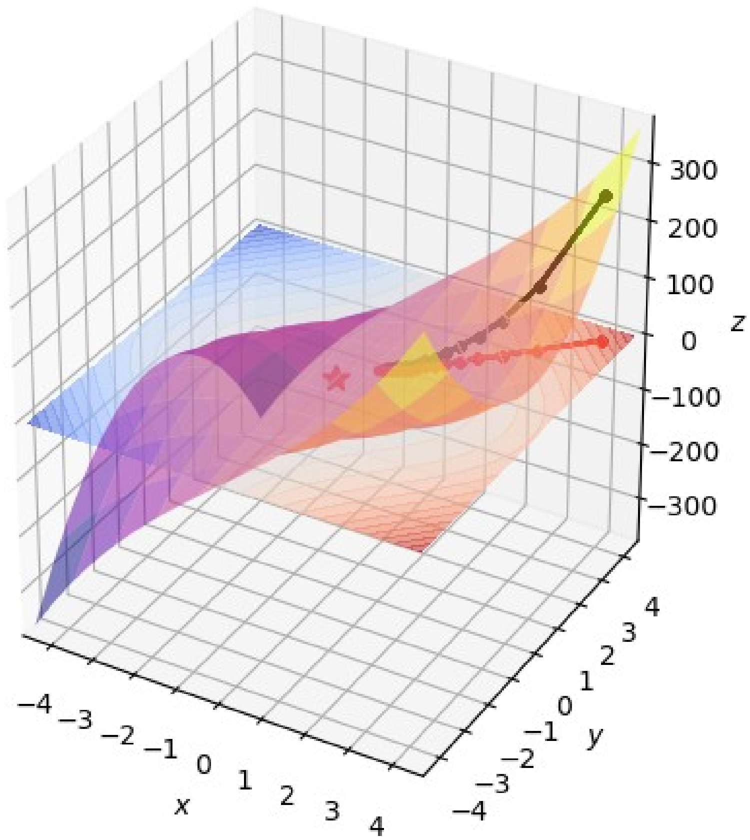
```
initialization: (4., -4.)
alpha: 0.01
gamma: 0.99
'''

func = matyas
#CODE HERE
init_x, init_y = 4., -4.
hparams = {'alpha': 0.01, 'gamma': 0.99}
optim = Optimizer(init_x, init_y, 'grad_momentum', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```



Exploring Saddle Points

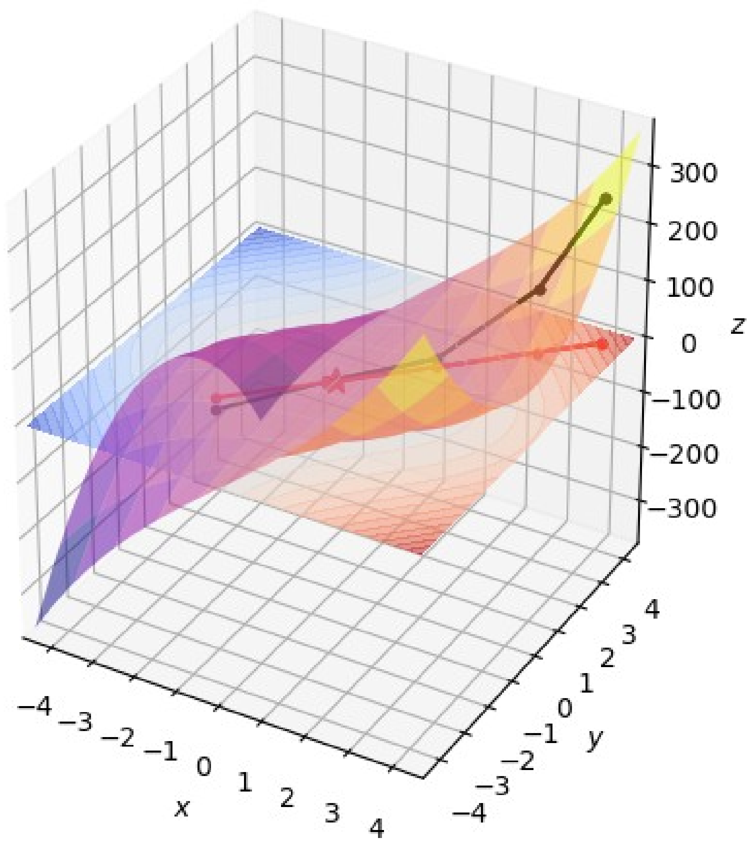
Make three plots to show how SGD, momentum, and adagrad behave on the monkey function. A good optimization algorithm will move past the flat region and continue its trajectory downward. Do not change the learning rate as this has been artificially reduced to make the distinction clear on this problem. Do you notice the observations made in the text above?

```python
frames = 20
epochs = 20

'''
TODO

Make a plot to show what happens if stochastic gradient descent is
used on a
function like this.

Recommended hparams:

initialization: (4., 4.)
alpha: 0.01
'''

func = monkey
#CODE HERE
init_x, init_y = 4., 4.
hparams = {'alpha': 0.01}
optim = Optimizer(init_x, init_y, 'sgrad', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```



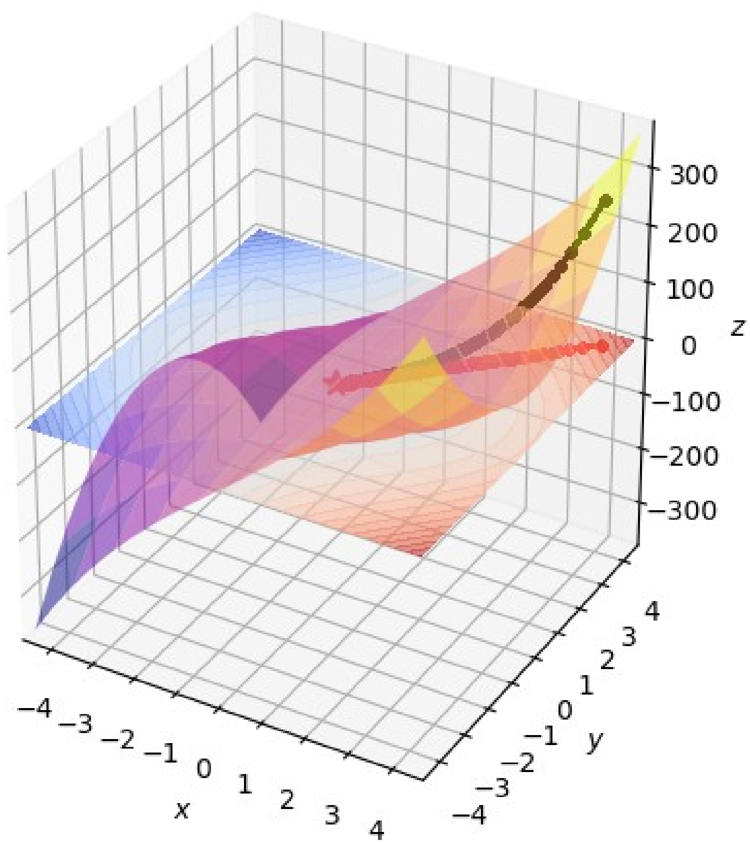

```
frames = 20
epochs = 5

'''
TODO

Make a plot to show what happens if stochastic gradient descent with
momentum
is used on a function like this.

Recommended hparams:

initialization: (4., 4.)
alpha: 0.01
gamma: 0.99
'''

func = monkey
#CODE HERE
init_x, init_y = 4., 4.
hparams = {'alpha': 0.01, 'gamma':0.99}
optim = Optimizer(init_x, init_y, 'grad_momentum', func, hparams)
```

```
optim.fit(epochs)

plot_function(epochs, frames, func)
```




```
frames = 20
epochs = 1000

'''
TODO

Make a plot to show what happens if adagrad
is used on a function like this.

Recommended hparams:

initialization: (4., 4.)
alpha: 0.3
eps: 1e-7
'''

func = monkey
#CODE HERE
```

```
init_x, init_y = 4., 4.
hparams = {'alpha': 0.3, 'eps': 1e-7}
optim = Optimizer(init_x, init_y, 'adagrad', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```



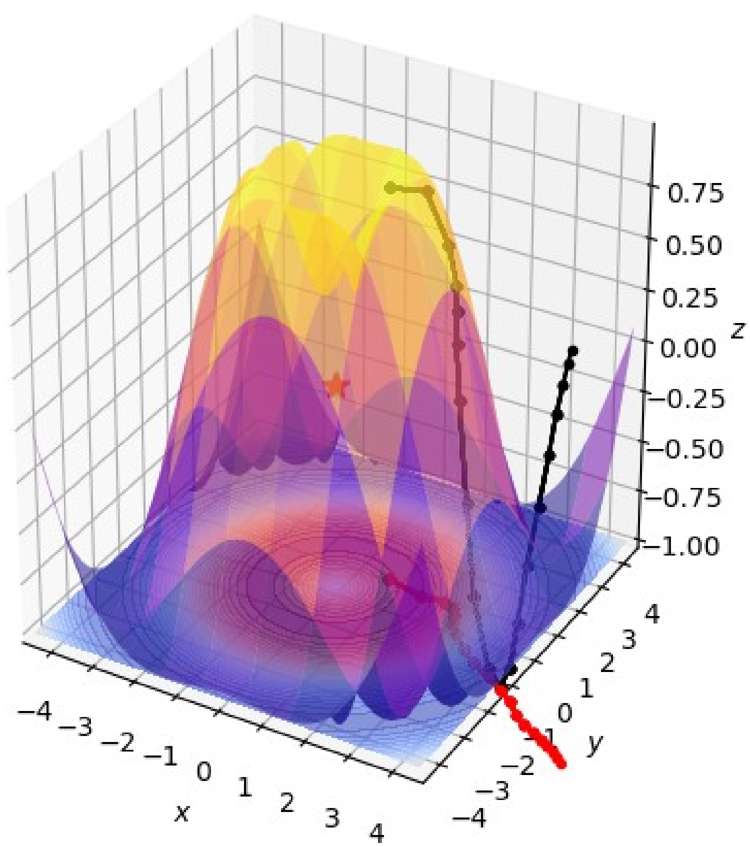
Escaping Local Minima

Make 3 or more plots (with atleast one plot using ADAM) to show how the different functions escape the local minima on the **mult** function. We will initialize at (0.5, 0.5).

```
frames = 20
epochs = 20

'''
TODO

Make a plot to show what happens if ADAM
is used on a function like this.
```

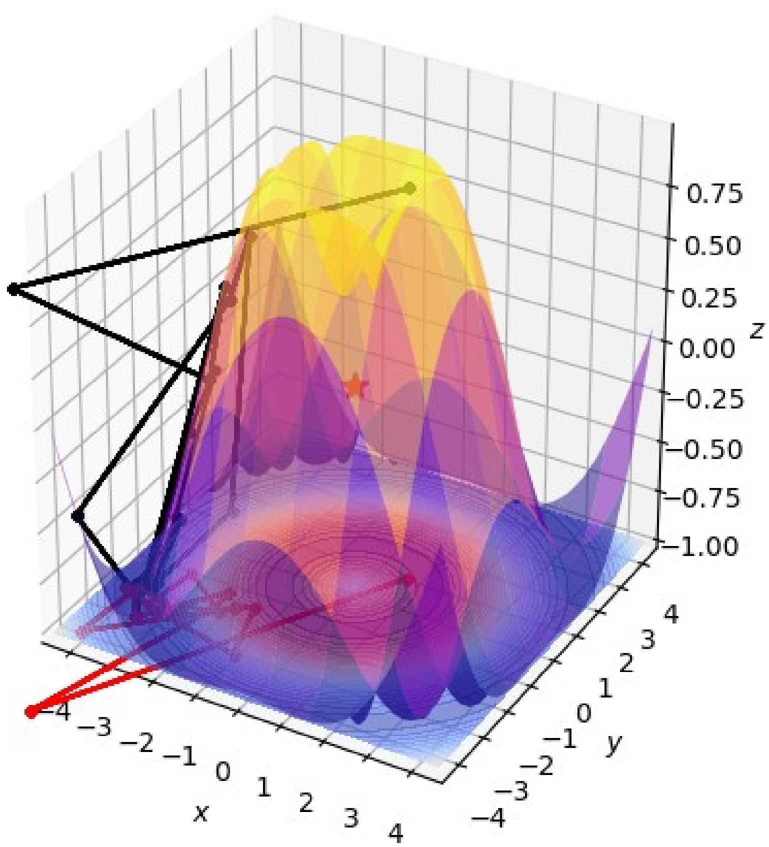
```
Recommended hparams:

initialization: (.8, .8)
alpha: 0.7
eps: 1e-7
beta_1: 0.99
beta_2: 0.9
'''
```

```
func = mult
#CODE HERE
init_x, init_y = .8, .8
hparams = {'alpha': 0.7, 'eps': 1e-7, 'beta_1': 0.99, 'beta_2': 0.9}
optim = Optimizer(init_x, init_y, 'adam', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```

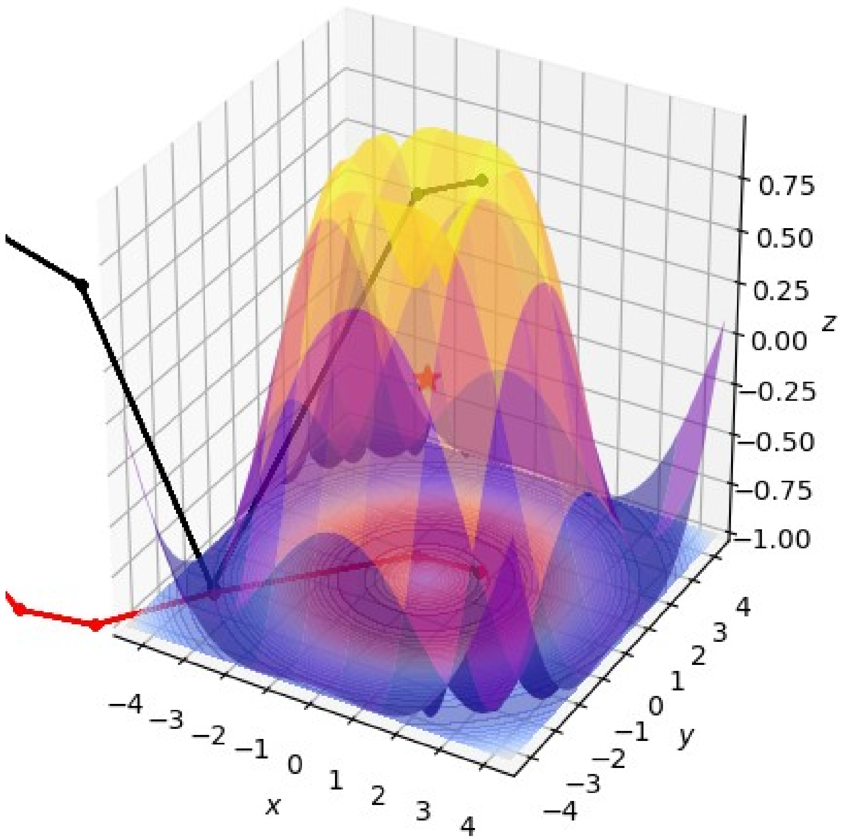



```
frames = 20
epochs = 20
```

```
'''
TODO

Make a plot to show what happens if adagrad
is used on a function like this.

Recommended hparams:

initialization: (.8, .8)
alpha: 0.7
eps: 1e-7
'''

func = mult
#CODE HERE
init_x, init_y = .8, .8
hparams = {'alpha': 0.7, 'eps': 1e-7}
optim = Optimizer(init_x, init_y, 'adagrad', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```

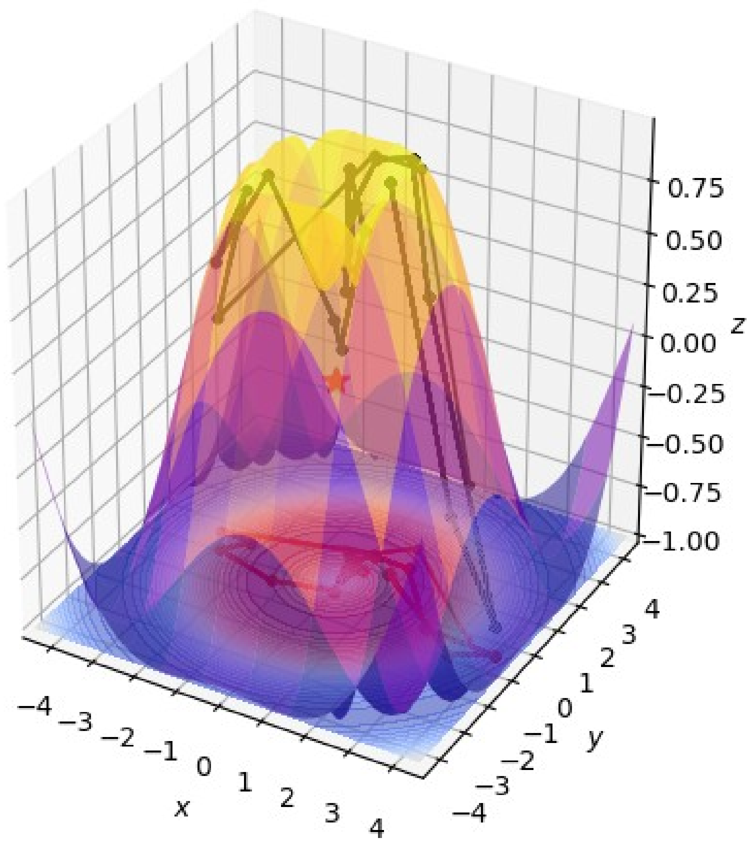


```python
frames = 20
epochs = 20


'''
TODO

Make a plot to show what happens if stochastic gradient descent with
momentum
is used on a function like this.

Recommended hparams:

initialization: (.8, .8)
alpha: 0.7
gamma: 0.99
'''



func = mult
#CODE HERE
init_x, init_y = .8, .8
hparams = {'alpha': 0.7, 'gamma': 0.99}
optim = Optimizer(init_x, init_y, 'grad_momentum', func, hparams)
optim.fit(epochs)

plot_function(epochs, frames, func)
```




```
frames = 20
epochs = 20


'''
TODO

Make a plot to show what happens if stochastic gradient descent
is used on a function like this.

Recommended hparams:

initialization: (.8, .8)
alpha: 0.7
'''

func = mult
#CODE HERE
init_x, init_y = .8, .8
hparams = {'alpha': 0.7}
optim = Optimizer(init_x, init_y, 'sgrad', func, hparams)
optim.fit(epochs)
```

```
plot_function(epochs, frames, func)
```



Collaboration

Discussed concepts and approaches with Pragyendra Bagediya (pb2861)