Revision Sheet: Pfaffian Equations & Compatible Systems

Mathematics & Computing

1 Pfaffian Differential Equations

1.1 Definition & Geometry

A **Pfaffian Equation** is a differential 1-form set to zero:

$$\mathbf{F} \cdot d\mathbf{r} = 0 \implies P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$$

Geometrically, it defines a field of hyperplanes. Solving it means finding a surface (integral manifold) tangent to these planes at every point.

1.2 Integrability (Frobenius Theorem)

An equation is integrable if there exists a family of surfaces $\phi(x, y, z) = C$.

- **Condition:** $\mathbf{F} \cdot (\nabla \times \mathbf{F}) = 0$.
- **Exact:** If $\nabla \times \mathbf{F} = \mathbf{0}$, the field is a gradient field ($\mathbf{F} = \nabla\phi$).
- **Non-Integrable:** If $\mathbf{F} \cdot (\nabla \times \mathbf{F}) \neq 0$ (Non-holonomic system).

1.3 Solution Methods

1. **Inspection:** Recognizing differentials like $d(xy) = xdy + ydx$ or $d(x/y) = \frac{ydx - xdy}{y^2}$.
2. **Variable Separation:** $f(x)dx + g(y)dy + h(z)dz = 0$.
3. **One Variable Constant:** Assume $dz = 0$, solve $Pdx + Qdy = 0$ to get $u(x, y, z) = f(z)$, then differentiate to find $f(z)$.
4. **Homogeneous:** If P, Q, R are degree n, use $x = uz, y = vz$.

2 Compatible Systems of first-order PDEs

2.1 Definition

Two equations $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ are **compatible** if they share a common solution.

2.2 Compatibility Condition (Poisson Bracket)

The system is compatible if the bracket $[f, g]$ vanishes:

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

Simplified Case: If z is absent ($f_z = g_z = 0$), then $f_x g_p - f_p g_x + f_y g_q - f_q g_y = 0$.

2.3 Solving Procedure

1. **Check** $[f, g] = 0$**.**
2. **Solve algebraically** for p and q in terms of x, y, z.
3. **Integrate the Pfaffian Form:** $dz = p\,dx + q\,dy$.

3 Example: Combined Application

Given System: $f = p - (x + y) = 0, \quad g = q - (x - y) = 0$.
1. Compatibility: $[f, g] = (1) + 0 + (-1) + 0 = 0$. (Compatible)
2. Pfaffian Form: $dz = (x + y)dx + (x - y)dy$.
3. Test Pfaffian Integrability: $\mathbf{F} = \langle x + y, x - y, -1\rangle$. curl $\mathbf{F} = \langle 0, 0, 1 - 1\rangle = \mathbf{0}$. (Exact)
4. Integrate: $z = \int(x + y)dx$ (keeping y const) $= \frac{x^2}{2} + xy + \psi(y)$.
Differentiating w.r.t y: $x + \psi'(y) = q = x - y \implies \psi'(y) = -y$.
Final Result: $z = \frac{x^2}{2} + xy - \frac{y^2}{2} + C$.

4 Charpit's Method

4.1 Purpose

The most general method for finding the **complete integral** of a non-linear first-order PDE: $f(x, y, z, p, q) = 0$. It works by finding a compatible equation $g(x, y, z, p, q, a) = 0$.

4.2 Charpit's Auxiliary Equations

To find p and q, we solve the following ratios:

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

4.3 Procedure

1. Compute partial derivatives: f_x, f_y, f_z, f_p, f_q.
2. Substitute into the auxiliary equations.
3. Pick the **simplest pair** to find a relation between p, q, x, y, or z (introducing a constant a).
4. Use this relation and the original PDE to solve for p and q explicitly.
5. Substitute into $dz = p\,dx + q\,dy$ and integrate.

4.4 Standard Forms (Derived from Charpit)

- **Type 1:** $f(p, q) = 0 \implies p = a, q = \text{const}$.
- **Type 2 (Clairaut):** $z = px + qy + f(p, q) \implies z = ax + by + f(a, b)$.
- **Type 3:** $f(z, p, q) = 0 \implies q = ap$ (then solve for p).
- **Type 4:** $f_1(x, p) = f_2(y, q) \implies f_1 = a, f_2 = a$.