Calculus 2 for CCS Students

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Term 3 2023-2024

Outline

Course Description

This second course in analysis covers differentiation and integration of exponential, logarithm and trigonometric functions; the concepts of the definite integral, the indefinite integral, and some applications of the definite integral.

Differentials

Differentials are used to approximate changes in function values near points where the function is differentiable.

Let $y = f(x)$ be a function.

If $f'(x)$ exists, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where $\Delta y = f(x + \Delta x) - f(x)$.

For a sufficiently small $|\Delta x|$, $f'(x)\,\Delta x$ is a good approximation to the value of Δy. That is,

$$\Delta y \approx f'(x)\,\Delta x$$

if $|\Delta x|$ sufficiently small.

Differentials of the Dependent Variable

Definition 1.1

If the function f is defined by the equation $y = f(x)$, then the differential of y, denoted by dy, is given by

$$dy = f'(x)\,\Delta x,$$

where x is in the domain of f' and Δx is an arbitrary increment of x.

Differential of the Independent Variable

Definition 1.2

If the function f is defined by the equation $y = f(x)$, then the differential of x, denoted by dx, is given by

$$dx = \Delta x,$$

where x is in the domain of f' and Δx is an arbitrary increment of x.

From Definitions 1.1 and 1.2,

$$dy = f'(x)\, dx.$$

Exercises

Find (a) Δy; (b) dy; and (c) $\Delta y - dy$.

1. $y = x^2 - 3x$; $x = -1$; $\Delta x = 0.02$
2. $y = \dfrac{1}{x}$; $x = 3$; $\Delta x = -0.2$
3. $y = x^3 + 1$; $x = -1$; $\Delta x = 0.1$

Find dy.

1. $y = \dfrac{3x}{x^2 + 2}$
2. $y = \sqrt{4 - x^2}$
3. $y = x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$
4. $y = \cot 2x \csc 2x$

1. The metal box in the form of a cube is to have an interior volume of 1000 cm^3. The six sides are to be made of metal $\frac{1}{2}$ cm thick. If the cost of the metal to be used is \$0.20 per cubic centimeter, use differentials to find the approximate cost of the metal to be used in the manufacture of the box.
2. The stem of a particular mushroom is cylindrical in shape, and a stem height of 2 cm and radius r centimeters had a volume of V cubic centimeters, where $V = 2\pi r^2$. Use differentials to find the approximate increase in the volume of the stem when the radius increases from 0.4 cm to 0.5 cm.
3. A certain bacterial cell is spherical in shape such that r micrometers is its radius and V cubic micrometers is its volume, then $V = \frac{4}{3}\pi r^3$. Use differentials to find the approximate increase in the volume of the cell when the radius increases from 2.2 μm to 2.3 μm.

Antidifferentiation

The inverse operation of differentiation is called *antidifferentiation*, which involves the computation of an *antiderivative*.

Definition 2.1 (Antiderivative)

A function F is called an antiderivative of the function f on an interval I if $F'(x) = f(x)$ for every value of x in I.

Theorem 2.1

If f and g are two functions defined on an interval I, such that

$$f'(x) = g'(x) \qquad \textit{for all } x \textit{ in } I$$

then there is a constant K such that

$$f(x) = g(x) + K \qquad \textit{for all } x \textit{ in } I.$$

Theorem 2.2

If F is a particular antiderivative of f on an interval I, then every antiderivative of f on I is given by

$$F(x) + C, \tag{1}$$

where C is an arbitrary constant, and all antiderivatives of f on I can be obtained from (1) by assigning particular values to C.

Antidifferentition is the process of finding the set of all antiderivatives of a given function. The symbol $\int$ denotes the operation of antidifferentiation, and we write

$$\int f(x)dx = F(x) + C, \tag{2}$$

where

$$F'(x) = f(x).$$

The expression $F(x) + C$ in (2) is the general antiderivative of f.

Theorem 2.3

$$\int dx = x + C$$

Theorem 2.4

$$\int af(x)\ dx = a\int f(x)\ dx$$

Theorem 2.5

$$\int [f(x) + g(x)]\ dx = \int f(x)\ dx + \int g(x)\ dx$$

Theorem 2.6

If $f_1, f_2, \ldots, f_n$ are defined on the same interval,

$$\int [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]\ dx = c_1 \int f_1(x)\ dx + c_2 \int f_2(x)\ dx + \cdots + c_n \int f_n(x)\ dx,$$

where $c_1, c_2, \ldots, c_n$ are constants.

Theorem 2.7

If n is a rational number,

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

Example

Find the most general antiderivative of the function $g(t) = \dfrac{1+t+t^2}{\sqrt{t}}$.

Solution

We rewrite the given function as follows:

$$g(t) \;=\; \frac{1+t+t^2}{\sqrt{t}} \;=\; \frac{1}{\sqrt{t}} + \frac{t}{\sqrt{t}} + \frac{t^2}{\sqrt{t}} \;=\; t^{-1/2} + \sqrt{t} + \sqrt{t^3}.$$

Thus we want to find

$$\int \left(t^{-1/2} + t^{1/2} + t^{3/2}\right)\, dt.$$

Example

Using Theorems 2.6 and 2.7, we obtain

$$\begin{aligned}\int \left(t^{-1/2} + t^{1/2} + t^{3/2}\right)\, dt &= \int t^{-1/2}\, dt + \int t^{1/2}\, dt + \int t^{3/2}\, dt \\ &= \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C \\ &= 2\sqrt{t} + \frac{2\sqrt{t^3}}{3} + \frac{2\sqrt{t^5}}{5} + C.\end{aligned}$$

We can check the answer by taking the derivative. We have

$$\begin{aligned}\frac{d}{dx}\left(2\sqrt{t} + \frac{2\sqrt{t^3}}{3} + \frac{2\sqrt{t^5}}{5} + C\right) &= 2\cdot\frac{1}{2}t^{\frac{1}{2}-1} + \frac{2}{3}\cdot\frac{3}{2}t^{\frac{3}{2}-1} + \frac{2}{5}\cdot\frac{5}{2}t^{\frac{5}{2}-1} \\ &= t^{-1/2} + t^{1/2} + t^{3/2} = \frac{1}{\sqrt{t}} + \sqrt{t} + \sqrt{t^3} \\ &= \frac{1+t+t^2}{\sqrt{t}}.\end{aligned}$$

Exercises

Perform the antidifferentiation.

1. $\displaystyle\int \frac{3}{t^5}\, dt$
2. $\displaystyle\int \frac{3}{\sqrt{y}}\, dy$
3. $\displaystyle\int x^4(5-x^2)\, dx$
4. $\displaystyle\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)\, dx$
5. $\displaystyle\int \frac{y^4+2y^2-1}{\sqrt{y}}\, dy$

Theorem 2.8

$$\int \sin x \; dx = -\cos x + C$$

Theorem 2.9

$$\int \cos x \; dx = \sin x + C$$

Theorem 2.10

$$\int \sec^2 x \; dx = \tan x + C$$

Theorem 2.11

$$\int \csc^2 x \ dx = -\cot x + C$$

Theorem 2.12

$$\int \sec x \tan x \ dx = \sec x + C$$

Theorem 2.13

$$\int \csc x \cot x \ dx = -\csc x + C$$

Fundamental Identities

Trigonometric identities are often used when computing antiderivatives involving trigonometric functions.

$$\sin x \csc x = 1 \qquad \cos x \sec x = 1 \qquad \tan x \cot x = 1$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Example

Find the most general antiderivative of the function $g(\theta) = 2\sin\theta - \sec^2\theta$.

Solution

Using Theorems 2.6 and 2.4, we have

$$\begin{aligned}\int \left(2\sin\theta - \sec^2\theta\right)\, d\theta &= \int 2\sin\theta\, d\theta - \int \sec^2\theta\, d\theta \\ &= 2\int \sin\theta\, d\theta - \int \sec^2\theta\, d\theta.\end{aligned}$$

Using Theorems 2.8 and 2.10, we obtain

$$2\int \sin\theta\, d\theta - \int \sec^2\theta\, d\theta = 2(-\cos\theta) - \tan\theta + C = -2\cos\theta - \tan\theta + C.$$

Exercises

Perform the antidifferentiation.

1. $\displaystyle\int (5\cos x - 4\sin x)\ dx$
2. $\displaystyle\int \frac{\cos x}{\sin^2 x}\ dx$
3. $\displaystyle\int (3\csc^2 t - 5\sec t \tan t)\ dt$
4. $\displaystyle\int \frac{3\tan\theta - 4\cos^2\theta}{\cos\theta}\ d\theta$

Some Techniques of Antidifferentiation

Theorem 2.14 (Chain Rule for Antidifferentiation)

Let g be a differentiable function, and let the range of g be an interval I. Suppose that f is a function defined on I and that F is an antiderivative of f on I. Then

$$\int f(g(x))\,[g'(x)\;dx] = F(g(x)) + C$$

Theorem 2.15

If g is a differentiable function, and n is a rational number,

$$\int [g(x)]^n \, [g'(x) \; dx] = \frac{[g(x)]^{n+1}}{n+1} + C \qquad\qquad n \neq -1$$

Example

Evaluate the indefinite integral $\int \sqrt{\cot x} \csc^2 x \; dx$.

Solution

Let $u = \cot x$. Then $du = -\csc^2 x \; dx$, so $\csc^2 x \; dx = -du$ and

$$\begin{aligned} \int \sqrt{\cot x} \csc^2 x \; dx &= -\int \sqrt{u} \; du = -\int u^{1/2} \; du \\ &= -\frac{2u^{3/2}}{3} + C = -\frac{2\sqrt{\cot^3 x}}{3} + C. \end{aligned}$$

Exercises

Perform the antidifferentiation.

1. $\displaystyle\int x(2x^2+1)^6\ dx$
2. $\displaystyle\int \frac{s}{\sqrt{3s^2+1}}\ ds$
3. $\displaystyle\int \frac{t}{\sqrt{t+3}}\ dt$
4. $\displaystyle\int \left(x^3+3\right)^{1/4} x^5\ dx$
5. $\displaystyle\int \frac{1}{2} t\cos 4t^2\ dt$
6. $\displaystyle\int r^2 \sec^2 r^3\ dr$

Exercises

Perform the antidifferentiation.

7. $\displaystyle\int \sqrt{\frac{1}{t} - 1}\ \frac{dt}{t^2}$
8. $\displaystyle\int \sin^3\theta\cos\theta\ d\theta$
9. $\displaystyle\int x\left(x^2+1\right)\sqrt{4-2x^2-x^4}\ dx$
10. $\displaystyle\int \left(t+\frac{1}{t}\right)^{3/2}\left(\frac{t^2-1}{t^2}\right)\ dt$
11. $\displaystyle\int \sec x\tan x\cos(\sec x)\ dx$

Differential Equations

Definition 2.2 (Differential Equation)

An equation containing a function and its derivatives, or just its derivatives, is called a differential equation.

The order of a differential is the order of the derivative of highest order that appears in the equation.

A function is a solution of a differential equation if its derivatives satisfy the equation.

Separable Differential Equations

We are interested in finding the solution of the differential equations of the form

$$\frac{dy}{dx} = f(x) \tag{3}$$

and

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}. \tag{4}$$

These are called separable differential equations.

Separable Differential Equations

Using differentials, we can write Equation (3) as

$$dy = f(x)\ dx. \tag{5}$$

The variables in Equation (4) can be separated by multiplying both sides of the equation by $h(y)\ dx$. We have

$$h(y)\ dy = g(x)\ dx. \tag{6}$$

We can see in Equations (5) and (6) that all the y's are one side of the equation and all x's are on the other side. Taking the antiderivative of both sides of the equation would give the solution.

Example

Find the particular solution of the differential equation $\dfrac{dy}{dx} = 5x^4 - 2x^5$ determined by the initial condition $y = 4$ when $x = 0$.

Solution

The given differential equation is separable so can rewrite it as follows

$$dy = (5x^4 - 2x^5)\ dx.$$

Antidifferentiating bothe sides, we have

$$\int dy = \int (5x^4 - 2x^5)\ dx$$

which leads to

$$y = x^5 - \frac{x^6}{3} + C.$$

Example

Solution

To find the particular solution, we substitute 0 for x and 4 for y, we obtain

$$4 = (0)^5 - \frac{(0)^6}{3} + C.$$

This gives $C = 4$. The particular solution is

$$y = x^5 - \frac{x^6}{3} + 4.$$

Exercises

Find the complete solution of the differential equation.

1. $\dfrac{dy}{dx} = 6 - 3x^2$
2. $\dfrac{ds}{dt} = 5\sqrt{s}$
3. $\dfrac{du}{dv} = \dfrac{\cos 2v}{\sin 3u}$
4. $\dfrac{dy}{dx} = \dfrac{x^2\sqrt{x^3 - 3}}{y^2}$
5. $\dfrac{d^2y}{dx^2} = \sqrt{2x - 3}$

Exercises

Find the particular solution of the differential equation determined by the initial condition.

1. $\dfrac{dy}{dx} = (x+1)(x+2)$; $y = -\frac{3}{2}$ when $x = -3$
2. $\dfrac{ds}{dt} = \cos \frac{1}{2}t$; $s = 3$ when $t = \frac{1}{3}\pi$
3. $\dfrac{d^2y}{dx^2} = -\dfrac{3}{x^4}$; $y = \frac{1}{2}$ and $\dfrac{dy}{dx} = -1$ when $x = 1$

The Definite Integral

Definition 2.3 (Definite Integral)

If f is a function defined for $a \leq x \leq b$, divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. Let $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

provided this limit exists. If it does exist, we say that f is integrable on $[a, b]$.

(See Figures 1 and 2.)

The Definite Integral

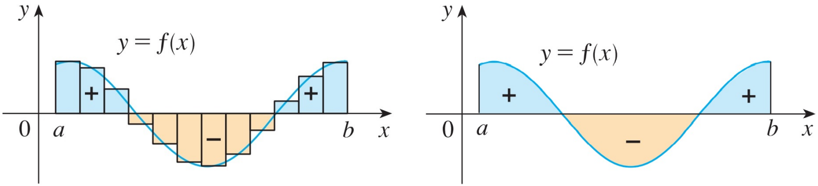
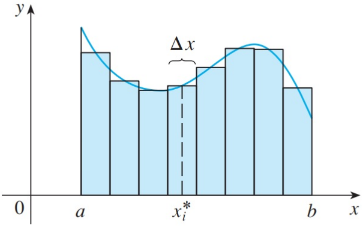


Figure 1: The expression $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is an approximation of the net area while $\int_a^b f(x)\,dx$ is the net area.

The Definite Integral



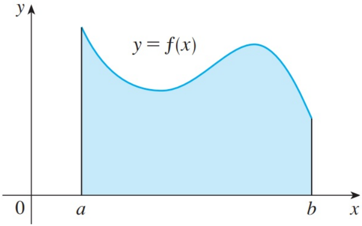


Figure 2: If $f(x) \geq 0$, then $\sum_{i=1}^{n} f(x_i^*)\, \Delta x$ is the sum of areas of rectangles while $\int_a^b f(x)\, dx$ is the area under the curve $y = f(x)$ from a to b.

Note

The symbol $\int$ was introduced by Leibniz and is called an integral sign. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit. The symbol dx indicates that the independent variable is x. The procedure of calculating an integral is called integration.

The sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ in Definition 2.3 is called a Riemann sum after the German mathematician Bernhard Riemann (1826–1866).

Theorem 2.16

If f is a continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

where F is any antiderivatives of f, that is, $F' = f$.

The notation $\int f(x)\, dx$ is traditionally used for an antiderivative of f and is called the indefinite integral.

Example

Find the exact value of the definite integral $\int_{-1}^{2} \left(4x^3 - 3x^2\right)\, dx$.

Solution

Using the properties of integrals, we have

$$\begin{aligned}
\int_{-1}^{2} \left(4x^3 - 3x^2\right)\, dx &= \int_{-1}^{2} 4x^3\, dx - \int_{-1}^{2} 3x^2\, dx \\
&= x^4\Big|_{-1}^{2} - x^3\Big|_{-1}^{2} \\
&= \left[(2)^4 - (-1)^4\right] - \left[(2)^3 - (-1)^3\right] \\
&= [16 - 1] - [8 - (-1)] \\
&= 15 - 9 = 6.
\end{aligned}$$

Properties

1. If $a > b$ and $\int_a^b f(x)\ dx$ exists, then
$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$$
2. If $f(a)$ exists, then
$$\int_a^a f(x)\ dx = 0.$$
3. If function f is integrable on the closed interval $[a, b]$, and if k is any constant, then
$$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx.$$

4. If the functions $f_1, f_2, \ldots, f_n$ are all integrable on $[a, b]$, then $(f_1 \pm f_2 \pm \cdots \pm f_n)$ is integrable on $[a, b]$ and

$$\int_a^b [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)] \, dx = \int_a^b f_1(x) \, dx \pm \int_a^b f_2(x) \, dx \pm \cdots \pm \int_a^b f_n(x) \, dx$$

5. If the function f is integrable on the closed intervals $[a, b]$, $[a, c]$ and $[c, b]$, then

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx,$$

where $a < c < b$.

Example

If $\int_0^9 f(x)\ dx = 37$ and $\int_0^9 g(x)\ dx = 16$, find $\int_0^9 [2f(x) + 3g(x)]\ dx$.

Solution

Using the Properties 4 and 3 of definite integrals, we have

$$\begin{aligned}
\int_0^9 [2f(x) + 3g(x)]\ dx &= \int_0^9 2f(x)\ dx + \int_0^9 3g(x)\ dx \\
&= 2\int_0^9 f(x)\ dx + 3\int_0^9 g(x)\ dx \\
&= 2(37) + 3(16) \ = \ 122.
\end{aligned}$$

Exercises

1. $\displaystyle\int_{-3}^{5}(y^3-4y)\ dy$
2. $\displaystyle\int_{0}^{\sqrt{5}} t\sqrt{t^2+1}\ dt$
3. $\displaystyle\int_{-1}^{3}\frac{1}{(y+2)^3}\ dy$
4. $\displaystyle\int_{1}^{3}\frac{x}{(3x^2-1)^3}\ dx$
5. $\displaystyle\int_{2}^{4}\frac{w^4-w}{w^3}\ dw$
6. $\displaystyle\int_{-4}^{4}|x-2|\ dx$
7. $\displaystyle\int_{-3}^{3}\sqrt{3+|x|}\ dx$
8. $\displaystyle\int_{0}^{1}\frac{x^3+1}{x+1}\ dx$
9. $\displaystyle\int_{1}^{4}\frac{x^5-x}{3x^3}\ dx$
10. $\displaystyle\int_{0}^{1}\sin\pi x\cos\pi x\ dx$
11. $\displaystyle\int_{\pi/8}^{\pi/4}3\csc^2 2x\ dx$
12. $\displaystyle\int_{0}^{\pi/3}\frac{\tan^3 x}{\sec x}\ dx$

Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

The Chain Rule

If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= f'(u)g'(x) \\ &= f'(g(x))g'(x). \end{aligned}$$

Example

Find $\dfrac{d}{dx}\displaystyle\int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz$.

Solution

We have to use the Chain Rule together with the FTC1. Let $u = \sqrt{x}$. We have

$$\begin{aligned}
\frac{d}{dx}\int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz &= \frac{d}{dx}\int_1^{u} \frac{z^2}{z^4+1}\,dz \\
&= \frac{d}{du}\left[\int_1^{u} \frac{z^2}{z^4+1}\,dz\right]\frac{du}{dx} \\
&= \frac{u^2}{u^4+1}\,\frac{du}{dx} \\
&= \frac{x}{x^2+1}\,\frac{1}{2\sqrt{x}}
\end{aligned}$$

Exercises

Compute the derivative.

1. $\dfrac{d}{dx}\displaystyle\int_x^3 \sqrt{1+t^4}\,dt$
2. $\dfrac{d}{dx}\displaystyle\int_2^x \frac{1}{t^4+4}\,dt$
3. $\dfrac{d}{dx}\displaystyle\int_{-x}^x \cos(t^2+1)\,dt$
4. $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \frac{1}{\sqrt{t^2+1}}\,dt$
5. $\dfrac{d}{dx}\displaystyle\int_3^{\sin x} \frac{1}{1-t^2}\,dt$

References

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3. J. Stewart. *Calculus: Early Transcendentals (7th ed)*. Brooks/Cole, Cengage Learning, 2010.