TWISTED DERIVED EQUIVALENCES AND ISOGENIES FOR ABELIAN SURFACES

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Abstract. In this paper, we study the twisted Fourier-Mukai partners of abelian surfaces. Following the work of Huybrechts [23], we introduce the twisted derived equivalences (or derived isogenies) between abelian surfaces. We show that there is a twisted derived Torelli theorem for abelian surfaces over fields with characteristic $\neq 2$. Over complex numbers, the derived isogenies correspond to rational Hodge isometries between the second cohomology groups, which is in analogy to the work of Huybrechts and Fu-Vial on K3 surfaces. Their proof relies on the global Torelli theorem over $\mathbb{C}$, which is missing in positive characteristics. To overcome this issue, we firstly extend Shioda's trick [44] on integral Hodge structures, to rational Hodge structures, ℓ-adic Tate modules and F-crystals. Then we make use of Tate's isogeny theorem to give a characterization of the twisted derived equivalences between abelian surfaces via isogenies. As a consequence, we show the two abelian surfaces are principally isogenous if and only if they are derived isogenous.

1. Introduction

Let X and Y be abelian varieties of dimension g over a field k of characteristic p. We say X and Y are isogenous over k if there exists a surjective map $\phi : X \to Y \in \mathrm{Hom}_k(X, Y)$ with finite kernel. When k is finitely generated over its prime field, there is a canonical bijection for each $\ell \neq p$:

$$\mathrm{Hom}_k(X, Y) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(k^s/k)}(T_\ell(X), T_\ell(Y)),$$

where $\mathrm{Hom}_{\mathrm{Gal}(k^s/k)}(T_\ell(X), T_\ell(Y))$ denotes the Galois-equivariant maps between the Tate modules of X and Y. Any invertible element in the left-hand side will be called a $\mathbb{Z}_\ell$-quasi-isogeny, i.e., an isomorphism between X and Y in the isogeny category of abelian varieties with coefficients in $\mathbb{Z}_\ell$. This bijection shows that, any isomorphism between Tate modules (with Galois actions) of abelian varieties over k is uniquely induced by a $\mathbb{Z}_\ell$-quasi-isogeny. Therefore such bijectivity is also known as *Tate's isogeny theorem*, proved by Tate for finite fields, by Zarhin and Faltings for finitely generated fields (*cf.* [49, 53, 14]).

On the other hand, the derived equivalences provide a natural source for isogenies between abelian varieties (*cf.* [39, 40]). A natural question is to determine when two isogenous abelian varieties are derived equivalent. More generally, following the work in [23] and [17], one can introduce the notion of *twisted derived equivalence* or so called *derived isogenous* as follows: we say X and Y are derived isogenous if they can be connected by derived equivalence between twisted abelian varieties, i.e., there exist twisted abelian varieties (X_i, α_i) and (X_i, β_i) such that there is a zig-zag of derived equivalences

$$\begin{aligned}
\mathrm{D}^b(X, \alpha) &\xrightarrow{\simeq} \mathrm{D}^b(X_1, \beta_1) \\
\mathrm{D}^b(X_1, \alpha_2) &\xrightarrow{\simeq} \mathrm{D}^b(X_2, \beta_2) \\
&\vdots \\
\mathrm{D}^b(X_n, \alpha_{n+1}) &\xrightarrow{\simeq} \mathrm{D}^b(Y, \beta_n)
\end{aligned} \tag{1.0.1}$$

where $\mathrm{D}^b(X, \alpha)$ is the bounded derived category of α-twisted sheaves on X. We may denote by $\mathrm{D}^b(X) \sim \mathrm{D}^b(Y)$ for a derived isogeny between X and Y.

In this paper, we try to classify the derived isogenies between abelian surfaces over arbitrary algebraically closed fields, where it should behave similarly as the case of K3 surfaces. In this case, we will show that the derived isogeny also has natural cohomological and motivic realizations, i.e., such equivalence can be read off from the conditions on cohomology groups or motives. A notable fact for abelian surfaces is that besides the Tate modules, their 2^{nd} cohomology groups also carry rich structures. For instance, in the untwisted case, Orlov's derived Torelli theorem states that two complex abelian surfaces X and Y are derived equivalent if and only if there is a symplectic Hodge isometry between 1^{st} cohomology groups:

$$\mathrm{H}^1(X,\mathbb{Z}) \oplus \mathrm{H}^1(\widehat{X},\mathbb{Z}) \cong_{\mathrm{Hdg}} \mathrm{H}^1(Y,\mathbb{Z}) \oplus \mathrm{H}^1(\widehat{Y},\mathbb{Z}),$$

where $\widehat{X}$ and $\widehat{Y}$ are the dual abelian surfaces. On the other hand, the Orlov–Shioda derived Torelli theorem shows that there is another Hodge theoretical realization of derived equivalences:

$$\mathrm{D}^b(X) \cong \mathrm{D}^b(Y) \Leftrightarrow \widetilde{\mathrm{H}}(X,\mathbb{Z}) \cong_{\mathrm{Hdg}} \widetilde{\mathrm{H}}(Y,\mathbb{Z}) \Leftrightarrow \mathrm{T}(X) \cong_{\mathrm{Hdg}} \mathrm{T}(Y),$$

where $\widetilde{\mathrm{H}}(X,\mathbb{Z})$ and $\widetilde{\mathrm{H}}(Y,\mathbb{Z})$ are the Mukai lattices, $\mathrm{T}(X) \subseteq \mathrm{H}^2(X,\mathbb{Z})$ and $\mathrm{T}(Y) \subseteq \mathrm{H}^2(Y,\mathbb{Z})$ denote the transcendental lattices, $\cong_{\mathrm{Hdg}}$ means integral Hodge isometries. Inspired from this result and the work of [44], we find that the twisted derived equivalence and certain classes of isogenies also have the same cohomological realization on the 2^{nd}-cohomology. This also implies that there is a motivic realization of derived isogenous between abelian surfaces, which can be viewed as an analogy of the motivic global Torelli theorem on K3 surfaces (*cf.* [23, Conjecture 0.3] and [17, Theorem 1]). The first main result of this paper is formulated as follows.

Theorem 1.0.1 (Twisted derived Torelli). *Let X and Y be smooth abelian surfaces over $\bar{k}$ with* $\mathrm{char}(k) = 0$. *Then the following conditions are equivalent*

(i) *X and Y are principally isogenous (See Definition 5.2.4);*
(ii) *X and Y are derived isogenous;*
(iii) *the associated Kummer surfaces* $\mathrm{Km}(X)$ *and* $\mathrm{Km}(Y)$ *are derived isogenous;*
(iv) *Chow motives $\mathfrak{h}(X) \cong \mathfrak{h}(Y)$ are isomorphic as Frobenius exteria algebras;*
(v) *even degree Chow motives $\mathfrak{h}^{even}(X) \cong \mathfrak{h}^{even}(Y)$ are isomorphic as Frobenius algebras in the sense of* [17].

When $k = \mathbb{C}$, then the conditions above are also equivalent to

(vi) $\mathrm{H}^2(X,\mathbb{Q}) \cong \mathrm{H}^2(Y,\mathbb{Q})$ *as a rational Hodge isometry;*
(vii) $\widetilde{\mathrm{H}}(X,\mathbb{Q}) \cong \widetilde{\mathrm{H}}(Y,\mathbb{Q})$ *as a rational Hodge isometry;*
(viii) $\mathrm{T}(X) \otimes \mathbb{Q} \cong \mathrm{T}(Y) \otimes \mathbb{Q}$ *as a rational Hodge isometry.*

Moreover, the Hodge isometry in (vi)-(viii) are induced by a derived isogeny.

Here a motive $\mathfrak{h}$ is a Frobenius exteria algebra object if there is an isomorphism of algebra objects

$$\mathfrak{h} \cong \bigoplus_{i \geq 0} \wedge^i \mathfrak{h}^1$$

such that the even degree part $\mathfrak{h}^{even} = \oplus_{k \geq 0} \wedge^{2k} \mathfrak{h}^1$ forms a Frobenius algebra object. An isomorphism between Frobenius exterior algebras means an isomorphism of algebra objects which preserves the Frobenius structure on their even part.

As mentioned above, the equivalences $(ii) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (viii)$ provide a Hodge-theoretic realization of derived isogenies. The proof follows a similar strategy of [23, Theorem 0.1], which makes use of Shioda's period map and Cartan-Dieudonné decomposition of a rational isometry. For equivalences $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$, it follows easily from $(i) \Leftrightarrow (ii)$ (*cf.* [17, Proposition 4.5]) and it can be viewed as a motivic global Torelli theorem for abelian surfaces. The equivalence $(i) \Leftrightarrow (ii)$ is concluded by so called rational Shioda's trick on abelian surfaces. The original Shioda's trick in [44] plays a key role in the proof of Shioda's global Torelli theorem for abelian surfaces, which links the weight-1 integral Hodge structure to the weight-2 integral Hodge structure of an abelian surface. In fact, we prove the following

Theorem 1.0.2 (Shioda's trick, see §4). *Let X and Y be two abelian surfaces over k with* $\mathrm{char}(k) \neq 2$. *Then we have*

- *If $k=\mathbb{C}$, then for any admissible Hodge isometry*
$$\psi\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})$$
we can find an isogeny $f\colon Y\to X$ of degree d^2 such that $\psi=\frac{f^*}{d}$.
- *For $\ell\nmid \mathrm{char}(k)$, any admissible isometry of $\mathrm{Gal}(\bar{k}/k)$-modules*
$$\psi\colon \mathrm{H}^2_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}},\mathbb{Z}_\ell)$$
is induced by a $\mathbb{Z}_\ell$-quasi-isogeny $f_\ell\colon Y_{k'}\to X_{k'}$ for some finite extension k'/k, i.e., $f_\ell^*=\psi$ or $-\psi$.
- *If $\mathrm{char}(k)=p>2$, any admissible isometry of F-crystals*
$$\psi\colon \mathrm{H}^2(X_{k^{\mathrm{perf}}}/W(k^{\mathrm{perf}})) \xrightarrow{\sim} \mathrm{H}^2(Y_{k^{\mathrm{perf}}}/W(k^{\mathrm{perf}}))$$
is induced by a $\mathbb{Z}_{p^2}$-quasi-isogeny $f_p\colon Y_{k'}\to X_{k'}$ for some finite extension k'/k, i.e., $f_p^*=\psi$ or $-\psi$. Here $\mathbb{Z}_{p^2}=W(\mathbb{F}_{p^2})$.

More generally, one can consider the Hodge isometries of the form
$$\mathrm{H}^2(X,\mathbb{Q})\cong \mathrm{H}^2(Y,\mathbb{Q})(d),$$
called *Hodge isogenies*. Due to the Hodge conjecture on product of abelian surfaces, we know that every Hodge isogeny is algebraic. Our generalized Shioda's trick actually shows that it is induced by certain quasi-isogeny. Similarly, the ℓ-adic and p-adic Shioda's trick gives a proof of Tate conjecture for isometries (as either Galois-modules or crystals) between the 2^{nd}-cohomology group of abelian surfaces over finite fields. See Corollary 4.5.5 for more details.

When $\mathrm{char}(k)>0$, due to the absence of a satisfactory global Torelli theorem, one can not follow the argument in characteristic zero directly. Our second main contribution of this paper is the following

Theorem 1.0.3. *Let X and Y be two abelian surfaces over an $\bar{k}$ with $\mathrm{char}(k)=p>2$. Then the following statements are equivalent.*

(i′) *X and Y are prime-to-p derived isogenous.*
(ii′) *X and Y are prime-to-p principally isogenous.*

Moreover, in case that X is supersingular, then Y is derived isogenous to X if and only if Y is supersingular.

Here, we say a derived isogeny as (1.0.1) is *prime-to-p* if its crystalline realization is integral (See Definition 3.1.2 for details). The main ingredients in the proof of Theorem 1.0.3 are the lifting-specialization technique. This method actually shows that there is an implication $(i')\Rightarrow(ii')$ for non prime-to-p derived isogeny (See Theorem 6.5.1) and we believe that the existence of quasi-liftable isogenous will imply the existence of derived isogeny (See Conjecture 6.5.1). Another natural question is whether two abelian surfaces are derived isogenous if and only if their associated Kummer surfaces are derived isogenous over positive characteristic fields. Unfortunately, we can not fully prove the equivalence. Instead, we provide a partial solution of this question. See Theorem 6.5.2 for more details.

One may ask whether the similar results also hold for K3 surfaces. Recall that two K3 surfaces S and S' over a finite field $\mathbb{F}_q$ are (geometrically) isogenous in the sense of [51] if there exists an algebraic correspondence Γ which induces an isometry of $\mathrm{Gal}(\bar{\mathbb{F}}_p/k)$-modules
$$\Gamma_\ell^*\colon \mathrm{H}^2_{\text{ét}}(S_{\bar{\mathbb{F}}_p},\mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(S'_{\bar{\mathbb{F}}_p},\mathbb{Q}_\ell),$$
for all $\ell\nmid p$ and an isometry of isocrystals
$$\Gamma_p^*\colon \mathrm{H}^2_{\mathrm{crys}}(S_k/K) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{crys}}(S'_k/K),$$
for some finite extension $k/\mathbb{F}_q$ and the fraction field K of $W=W(k)$. Then we say the isogeny is prime-to-p if the isometry Γ_p^* is integral, i.e., $\Gamma_p^*\left(\mathrm{H}^2_{\mathrm{crys}}(S_k/W)\right)=\mathrm{H}^2_{\mathrm{crys}}(S'_k/W)$. Then we have a formulation of the twisted derived Torelli conjecture for K3 surfaces.

Conjecture 1.0.1. *For two K3 surfaces S and S' over a finite field k with $\mathrm{char}(k)=p>0$, then the following are equivalent.*

(a) *There exists a prime-to-p derived isogeny* $\mathrm{D}^b(S) \sim \mathrm{D}^b(Y)$.
(b) *There exists a prime-to-p isogeny between* S *and* S'.

The implication $(a) \Rightarrow (b)$ is clear, while the converse remains open. In the case of Kummer surfaces, our results provide some evidence of Conjecture 1.0.1. We shall also mention that recently Bragg and Yang have studied the derived isogeny between K3 surfaces in [8] and they provided a weaker version of the statement in Conjecture 1.0.1 (*cf.* [8, Theorem 1.2]).

Organization of the paper. In Section 2, we perform the computations of the Brauer group of abelian surfaces via Kummer construction. This allows us to prove the lifting theorem for twisted abelian surfaces of finite height and the representability of flat cohomologies on supersingular abelian surfaces.

In Section 3, we collect some facts on derived isogenies between abelian surfaces and their cohomological realizations, which including the twisted Mukai lattices, **B**-field theory, a filtered Torelli theorem and its relation to the moduli space of twisted sheaves.

In Section 4, we revise Shioda's work and extend it to rational Hodge isogenies. This is also the key ingredient for proving Theorem 1.0.1. Furthermore, after introducing the admissible ℓ-adic and p-adic bases, we prove the ℓ-adic and p-adic Shioda's trick for admissible isometries on abelian surfaces. As an application, we confirm the Tate conjecture for Galois and K3-crystal isometries between abelian surfaces over finitely generated fields.

Section 5 and 6 are devoted to proving Theorem 1.0.1 and Theorem 1.0.3. Theorem 1.0.1 is essentially Theorem 5.1.3 and Theorem 5.2.5. The proof of Theorem 1.0.3 is much more subtle. We establish the lifting and the specialization theorem for prime-to-p derived isogeny. Then one can conclude $(i') \Leftrightarrow (ii')$ from Theorem 1.0.1. At the end of Section 6, we follow Bragg and Lieblich's twistor line argument in [6] to conclude the last assertion of Theorem 1.0.3.

Acknowledgement. The authors are grateful to the useful comments by Ziquan Yang. The authors are supported by NKRD Program of China (No. 2020YFA0713200), NSFC grants for Innovative Research Groups (No.12121001), General Program (12171090).

2. Twisted abelian surface

In this section, we give some basic results in the theory of twisted abelian surfaces. Many results looks parallel to the case of twisted K3 surfaces, but the proof requests additional attentions in many places (see Remark 2.2.2, 2.3.3). At the end of this section, we will introduce the prime-to-ℓ derived isogenous for abelian surfaces.

2.1. **Notations and Conventions.** Throughout this section, k is a field with $\mathrm{char}(k) = p \geq 0$. When k is perfect and $p > 0$, we let $W := W(k)$ be the p-typical Witt ring of k equipped with a morphism $\sigma\colon W \to W$ induced by the Frobenius map. If k is imperfect, then we will denote k^{perf} for its perfect closure in a fixed algebraic clousre $k \subset \bar{k}$.

Let X be a smooth projective variety over k. We denote by $\mathrm{H}^{\bullet}_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$ the ℓ-adic étale cohomology group of $X_{\bar{k}}$. The $\mathbb{Z}_\ell$-module $\mathrm{H}^{\bullet}_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$ has been endowed with a canonical $G_k = \mathrm{Gal}(\bar{k}/k)$-action. If k is a perfect field, we use $\mathrm{H}^i_{\mathrm{crys}}(X/W)$ to denote the i-th crystalline cohomology group of X over the p-adic base $W \twoheadrightarrow k$, which is a W-module. It is endowed with a natural σ-linear map

$$\varphi : \mathrm{H}^i_{\mathrm{crys}}(X/W) \to \mathrm{H}^i_{\mathrm{crys}}(X/W)$$

induced from the absolute Frobenius morphism $F_X\colon X \to X$.

When X is an abelian variety over k, we denote $\widehat{X}$ for its dual abelian variety and $X[p^\infty]$ for the associated p-divisible group. There is a natural identification of its contravariant Dieudonné module with its first crystalline cohomology:

$$\mathbb{D}(X[p^\infty]) := M(X[p^\infty]^\vee) \cong \mathrm{H}^1_{\mathrm{crys}}(X/W),$$

where $M(-)$ is the Dieudonné module functor on p-divisible groups defined in [32].

Let us recall some basic notions on the motivic decomposition of abelian surfaces. Deninger–Murre [13] produced a canonical motivic decomposition for any abelian surface:

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4} \mathfrak{h}^i(X)$$

such that $\mathrm{H}^*(\mathfrak{h}^i) \cong \mathrm{H}^i(X)$ for any Weil cohomology $\mathrm{H}^*(-)$.Furthermore, Künnemann [27] showed that $\mathfrak{h}^i(X) = \bigwedge^i \mathfrak{h}^1(X)$ for all i, $\mathfrak{h}^4(X) \simeq \mathbb{1}(-4)$ and $\bigwedge^i \mathfrak{h}^1(A) = 0$ for $i > 2g$, where the wedge product $\wedge$ is given by the natural tensor structure in the category of motives.

For any abelian group G and an integer n, we denote $G[n]$ for the subgroup of n-torsions in G and $G\{n\}$ for the union of all n-power torsions.

2.2. **Gerbes on abelian surfaces and associated Kummer surfaces.** Let X be an abelian surface over a field k and let $\mathscr{X} \to X$ be a μ_n-gerbe over X. This corresponds to a pair (X, α) for some $\alpha \in \mathrm{H}^2_{\mathrm{fl}}(X, \mu_n)$, where the cohomology group is with respect to the flat topology. Since μ_n is commutative, there is a bijection of sets

$$\{\mu_n\text{-gerbes on } X\}/\simeq \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{fl}}(X, \mu_n),$$

where $\simeq$ is the μ_n-equivalence defined as in [18, IV.3.1.1]. We may write $\alpha = [\mathscr{X}]$. The Kummer exact sequence induces a surjective map

$$\mathrm{H}^2_{\mathrm{fl}}(X, \mu_n) \to \mathrm{Br}(X)[n] \tag{2.2.1}$$

where the right-hand side is the *cohomological Brauer group* $\mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. For any μ_n gerbe $\mathscr{X}$ on X, there is an associated $\mathbb{G}_m$-gerbe on X via (2.2.1), denoted by $\mathscr{X}_{\mathbb{G}_m}$. Let $\mathscr{X}^{(n)}$ be the gerbe corresponding to cohomological class $n[\mathscr{X}] \in \mathrm{H}^2_{\mathrm{fl}}(X, \mu_n)$. If $[\mathscr{X}_{\mathbb{G}_m}] = 0$, then we will call $\mathscr{X}$ an essentially-trivial μ_n-gerbe.

If k has characteristic $p \neq 2$, there is an associated Kummer surface $\widetilde{X}$ constructed as follows:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\sigma}"] \arrow[d, "\pi"] & X \arrow[d] \\ \mathrm{Km}(X) \arrow[r, "\sigma"] & X/\iota \end{tikzcd} \tag{2.2.2}$$

where

- ι is the involution of X;
- σ is the crepant resolution of quotient singularities;
- $\widetilde{\sigma}$ is the blow-up of X along the closed subscheme $X[2] \subset X$. Its birational inverse is denoted by $\widetilde{\sigma}^{-1}$.

Let $E \subset \widetilde{X}$ be the exceptional locus of $\widetilde{\sigma}$. Then we have a composition of the sequence of morphisms

$$(\widetilde{\sigma}^{-1})^* : \mathrm{Br}(\widetilde{X}) \to \mathrm{Br}(\widetilde{X} \setminus E) \cong \mathrm{Br}(X \setminus X[2]) \cong \mathrm{Br}(X).$$

Here, the last isomorphism $\mathrm{Br}(X) \to \mathrm{Br}(X \setminus X[2])$ is due to Grothendieck's purity theorem (*cf.* [20, 50]).

Proposition 2.2.1. *When $k = \bar{k}$, the $(\widetilde{\sigma}^{-1})^*\pi^*$ induces an isomorphism between cohomological Brauer groups*

$$\Theta\colon \ \mathrm{Br}(\mathrm{Km}(X)) \to \mathrm{Br}(X). \tag{2.2.3}$$

In particular, when X is supersingular over $\bar{k}$, then $\mathrm{Br}(X)$ *is isomorphic to the additive group $\bar{k}$.*

Proof. For coprime to p torsion part of (2.2.3), the proof is essentially the same as [45, Proposition 1.3]. See also [47, Lemma 4.1] for the case $k = \mathbb{C}$. For p-primary torsion part, we have

$$\mathrm{Br}(\mathrm{Km}(X))\{p\} \cong \mathrm{Br}(X)^{\iota}\{p\}$$

from the Hochschild–Serre spectral sequence, where $\mathrm{Br}(X)^{\iota}$ is the ι-invariant subgroup. Hence it suffices to prove that ι acts trivially on $\mathrm{Br}(X)$.

In fact, $\mathrm{H}^2_{\mathrm{fl}}(X, \mu_p)$ can be ι-equivariantly embedded to $\mathrm{H}^2_{\mathrm{dR}}(X/k)$ by de-Rham-Witt theory (*cf.* [37, Proposition 1.2]). The action of ι on $\mathrm{H}^2_{\mathrm{dR}}(X/k) = \wedge^2 \mathrm{H}^1_{\mathrm{dR}}(X/k)$ is the identity, as its action on $\mathrm{H}^1_{\mathrm{dR}}(X/k)$ is given by $x \mapsto -x$. Thus the involution on $\mathrm{H}^2_{\mathrm{fl}}(X, \mu_p)$ is trivial. Then by the exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}/p \to \mathrm{H}^2_{\mathrm{fl}}(X, \mu_p) \to \mathrm{Br}(X)[p] \to 0,$$

we can deduce that $\mathrm{Br}(X)[p]$ is invariant under the involution. Furthermore, for p^n-torsions with $n \geq 2$, we can proceed by induction on n. Assume that all elements in $\mathrm{Br}(X)[p^d]$ are ι-invariant if $1 \leq d < n$. By abuse of notation, we still use ι to denote the induced map $\mathrm{Br}(X) \to \mathrm{Br}(X)$. For $\alpha \in \mathrm{Br}(X)[p^n]$, $p\alpha \in \mathrm{Br}(X)[p^{n-1}]$ is ι-invariant. This gives

$$p\alpha = \iota(p\alpha) = p\iota(\alpha),$$

which implies $\alpha - \iota(\alpha) \in \mathrm{Br}(X)[p]$. Applying ι on $\alpha - \iota(\alpha)$, we can obtain

$$\alpha - \iota(\alpha) = \iota(\alpha) - \alpha.$$

It implies $\alpha - \iota(\alpha)$ is also a 2-torsion element. Since p is coprime to 2, we can conclude that $\alpha = \iota(\alpha)$.

If X is supersingular, then $\mathrm{Km}(X)$ is also supersingular. We have already known that the Brauer group of a supersingular K3 surface is isomorphic to k by [2]. Thus $\mathrm{Br}(X) \cong k$.

□

Remark 2.2.2. In the case A being supersingular, the method of [2] can not be directly applied as $\mathrm{H}^1_{\mathrm{fl}}(X, \mu_{p^n})$ is not trivial in general for an abelian surface X.

Remark 2.2.3. For abelian surfaces over a non-closed field or a ring, we still have the canonical map (2.2.3), but it is not necessarily an isomorphism.

In [5], Bragg has shown that a twisted K3 surface can be lifted to characteristic 0. Though his method can not be directly applied to twisted abelian surfaces, one can still obtain a lifting result for twisted abelian surfaces via using the Kummer construction. The following result will be frequently used in this paper.

Lemma 2.2.4. *Let $\mathscr{X} \to X$ be a $\mathbb{G}_m$-gerbe on an abelian surface X over $k = \bar{k}$. Suppose* $\mathrm{char}(k) > 2$ *and X has finite height. Then there exists a lifting $\mathfrak{X} \to \mathcal{X}$ of $\mathscr{X} \to X$ over some discrete valuation ring W' whose residue field is k such that the specialization map*

$$\mathrm{NS}(\mathcal{X}_{K'}) \to \mathrm{NS}(X)$$

on Néron-Severi groups is an isomorphism. Here, K' is the fraction field of W' and $\mathcal{X}_{K'}$ is the generic fiber of $\mathcal{X} \to \operatorname{Spec} W'$.

Proof. The existence of such lifting is ensured by [5, Theorem 7.10], [28, Lemma 3.9] and Proposition 2.2.1. Roughly speaking, let $\mathscr{S} \to \mathrm{Km}(X)$ be the associated twisted Kummer surface via the isomorphism (2.2.3) in Proposition 2.2.1. Then [5, Theorem 7.10] asserts that there exists a lifting $\mathfrak{S} \to \mathcal{S}$ of $\mathscr{S} \to \mathrm{Km}(X)$ such that the specialization map of Néron-Severi groups is an isomorphism

$$\mathrm{NS}(\mathcal{X}_{K'}) \xrightarrow{\sim} \mathrm{NS}(X). \tag{2.2.4}$$

Then [28, Lemma 3.9] says that one can find a lifting $\mathcal{X}/W'$ of X such that $\mathrm{Km}(\mathcal{X}) \cong \mathcal{S}$ over W'. According to Remark 2.2.3, there is a canonical map

$$\Theta : \mathrm{Br}(\mathrm{Km}(\mathcal{X})) \to \mathrm{Br}(\mathcal{X})$$

as in (2.2.3). Consider the image $\Theta([\mathfrak{S}]) \in \mathrm{Br}(\mathcal{X})$, one can take $\mathfrak{X} \to \mathcal{X}$ to be the associated twisted abelian surface. Then $\mathfrak{X} \to \mathcal{X}$ will be a lifting of $\mathscr{X} \to X$ as the restriction of the Brauer class $[\mathfrak{X}]$ to X is $[\mathscr{X}]$. □

2.3. **Representability of flat cohomology.** Let us focus on the case $\mathrm{char}(k) = p > 0$ and study the flat cohomology of μ_p on abelian surfaces. This will play an important role in the study of supersingular twisted abelian surfaces.

Let $f\colon X \to S$ be a flat and proper morphism of schemes of finite type over k. Consider the sheaf of abelian groups $\mathrm{R}^i f_* \mu_p$ on the big fppf site $(\mathtt{Sch}/S)_{\mathrm{fl}}$, which can be expressed as the fppf-sheafification of

$$S' \mapsto \mathrm{H}^i_{\mathrm{fl}}(X_{S'}, \mu_p)$$

for any S-scheme S'. In general, the representability of $\mathrm{R}^i f_* \mu_p$ is not easy to see by the "wildness" of flat cohomology with p-torsion coefficients. In this part, we will prove the representability for supersingular abelian surfaces.

Suppose S is perfect. Consider the auxiliary big fppf site $(\mathtt{Perf}/S)_{\mathrm{fl}}$ for the full subcategory $\mathtt{Perf}/S \subset \mathtt{Sch}/S$ whose objects are perfect schemes over S. There is a functor between category of flat sheaves

$$(-)^{\mathrm{perf}}\colon\ \mathrm{Sh}\left((\mathtt{Sch}\,/S)_{\mathrm{fl}}\right) \to \mathrm{Sh}\left((\mathtt{Perf}\,/S)_{\mathrm{fl}}\right). \tag{2.3.1}$$

induced by the natural inclusion $(\mathtt{Perf}\,/S)_{\mathrm{fl}} \hookrightarrow (\mathtt{Sch}\,/S)_{\mathrm{fl}}$, called *perfection*.

Proposition 2.3.1. *Let $f\colon X \to S$ be an abelian S-scheme of relative dimension 2, whose geometric fibers are all supersingular. Then*

(1) $\mathrm{R}^1 f_* \mu_p \cong X[p]$ is a finite flat S-group scheme of local-local type (i.e. being self-dual under Cartier duality).

(2) For any $\pi\colon \mathrm{Spec}(A) \to S$ with A being perfect, we have $\mathrm{H}^i_{\mathrm{fl}}(A, \pi^* \mathrm{R}^1 f_* \mu_p) = 0$ for $i \geq 1$. In particular, if S is perfect, then $(\mathrm{R}^1 f_* \mu_p)^{\mathrm{perf}} = 0$.

Proof. For (1), it suffices to check them affine locally on the base. Assume S is an affine scheme of finite type over k. By taking the Stein factorization, we can further assume $f_* \mathcal{O}_X \cong \mathcal{O}_S$. Then $f_* \mu_p \cong \mu_p$ also holds universally. Under this assumption, we have an exact sequence of fppf-sheaves by Kummer theory:

$$0 \to \mathrm{R}^1 f_* \mu_p \to \mathrm{R}^1 f_* \mathbb{G}_m \to \mathrm{R}^1 f_* \mathbb{G}_m. \tag{2.3.2}$$

Since $\mathrm{R}^1 f_* \mathbb{G}_m$ computes the relative Picard scheme $\mathrm{Pic}_{X/S}$ and the Néron-Severi group of X is torsion-free, we can see

$$\mathrm{R}^1 f_* \mu_p \cong \ker\left(\mathrm{Pic}_{X/S} \xrightarrow{\cdot p} \mathrm{Pic}_{X/S}\right) \cong \ker\left(\mathrm{Pic}^0_{X/S} \xrightarrow{\cdot p} \mathrm{Pic}^0_{X/S}\right).$$

On the other hand, it is well-known that $\mathrm{Pic}^0_{X/S}$ is represented by the dual abelian S-scheme $\widehat{X}$ (*cf.* [36, Corollay 6.8]). Thus $\mathrm{R}^1 f_* \mu_p$ is representable by the commutative finite group S-scheme $\widehat{X}[p]$. Since the geometric fiber of f is supersingular, there is an isomorphism

$$X[p] \cong \widehat{X}[p].$$

It follows that $X[p]$ is a finite group S-scheme of local-local type.

For (2), take the following smooth group resolution of α_p,

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0,$$

we can see that $\mathrm{H}^i_{\mathrm{fl}}(A, \alpha_p) = 0$ for $i \geq 2$ for any ring A. For any finite flat group scheme G of local-local type, we can fill it in an exact sequence

$$0 \to G' \to G \to G'' \to 0$$

such G' and G'' are of smaller p-ranks. Thus by induction, we can prove that $\mathrm{H}^i_{\mathrm{fl}}(A, \mathrm{R}^1 f_* \mu_p) = 0$ for $i \geq 2$ and any finite flat group scheme G of local-local type.

For $i = 1$ and A being perfect, we can see

$$\mathrm{H}^1_{\mathrm{fl}}(A, \alpha_p) = A/A^p = 0.$$

Thus $\mathrm{H}^1_{\mathrm{fl}}(A, \mathrm{R}^1 f_* \mu_p) = 0$ by the same induction as before. ∎

Theorem 2.3.2. *Let $f\colon \mathcal{X} \to S$ be a family of supersingular abelian surfaces over an algebraic space S. Then $\mathrm{R}^2 f_* \mu_p$ is representable by an algebraic space, which is separated and locally of finite presentation over S.*

Proof. This is from a standard argument by checking the list of conditions [0']-[5'] in Artin's criterion [1, Theorem 5.3]. For this, we may use the tangent-obstruction theory of flat cohomology classes (see [5, §2] for example). The difficult part is the separateness of $\mathrm{R}^2 f_* \mu_p$, i.e.,

$$\Delta_{\mathrm{R}^2 f_* \mu_p}\colon \mathrm{R}^2 f_* \mu_p \to \mathrm{R}^2 f_* \mu_p \times_S \mathrm{R}^2 f_* \mu_p$$

is representable by closed immersion, which corresponds to the Artin's condition [3']. Roughly speaking, our strategy is to show that $\mathrm{R}^2 f_* \mu_p$ is representable after taking the perfection and then descends to the general case.

Let us check the separateness of $\mathrm{R}^2 f_* \mu_p$. Since the separateness is local in fppf-topology, we can take the same reduction for the base S as in the proof of Lemma 2.3.1. Let $\mathrm{Spec}(A)$ be an affine scheme over S of finite type and denote by $\mathcal{F}$ the base change $(R^2 f_* \mu_p)_A$. It suffices to prove the diagonal morphism

$$\Delta_{\mathcal{F}/A}\colon \mathcal{F} \to \mathcal{F} \times_A \mathcal{F}$$

is representable by a closed immersion.

As a first step, we assume that A is perfect, in which case $A/A^p = 0$. By Lemma 2.3.1, we have $(\mathrm{R}^1 f_* \mu_p)^{\mathrm{perf}} = 0$. Then we can proceed the proof as in [6, Proposition 2.17] to show that the diagonal morphism

$$\Delta_{\mathcal{F}/A}\colon \mathcal{F} \to \mathcal{F} \times_A \mathcal{F} \tag{2.3.3}$$

is representable by a closed immersion in $(\mathtt{Perf}/A)_{\mathrm{fl}}$. Restrict the perfection functor (2.3.1) on the full subcategory of algebraic spaces over A, we can obtain a fully faithful functor

$$\mathtt{PerfAs}/A \hookrightarrow \mathtt{As}/A$$

from the category of algebraic spaces defined using $(\mathtt{Perf}/A)_{\mathrm{fl}}$ to the category of algebraic spaces defined using $(\mathtt{Sch}/A)_{\mathrm{fl}}$. The essential image of this inclusion consists of perfect algebraic spaces over S (*cf.* [46, Lemma 04W1]). Thus $(\mathcal{F})^{\mathrm{perf}}$ is representable by a separated perfect algebraic space over $\mathrm{Spec}(A)$ defined using $(\mathtt{Perf}/A)_{\mathrm{fl}}$. Moreover, it also implies $\mathrm{R}^2 f_* \mu_p$ is representable by (at least) quasi-separated algebraic space over A by [46, Lemma 02YS] as the adjunction $i_*(\mathrm{R}^2 f_* \mu_p)^{\mathrm{perf}} \to \mathrm{R}^2 f_* \mu_p$ is representable by schemes.

If A is not perfect, then we can consider the directed inverse system

$$\mathrm{Spec}(A^{1/p^\infty}) \to \cdots \to \mathrm{Spec}(A^{1/p^2}) \to \mathrm{Spec}(A^{1/p}) \to \mathrm{Spec}(A).$$

Clearly, the transitions are affine morphisms. By previous discussion, we have already known that $\Delta_{\mathcal{F}/A}$ is locally of finite type ($\mathcal{F}$ is quasi-separated algebraic space) and its base change

$$\Delta_{\mathcal{F}/A} \times \mathrm{Spec}(A^{1/p^\infty}) = \Delta_{\widetilde{\mathcal{F}}/A^{1/p^\infty}} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}} \times_{A^{1/p^\infty}} \widetilde{\mathcal{F}}$$

is a closed immersion. Then by passing to limit as [46, Lemma 0850], we can see $\Delta_{\mathcal{F}/A}$ is also a closed immersion. Thus we conclude that $\mathrm{R}^2 f_* \mu_p$ is representable by a separated algebraic space over S. □

Remark 2.3.3. The case that $X \to S = \mathrm{Spec}(k)$ being a smooth surface for some field k is claimed by Artin in [2, Theorem 3.1] without proof.

For relative K3 surfaces, there is a moduli-theoretic proof given by Bragg and Lieblich using the stack of Azumaya algebras (*cf.* [6, Theorem 2.1.6]). Their proof can not be directly used for relative abelian surfaces as the essential assumption $\mathrm{R}^1 f_* \mu_p = 0$ fails in fppf site $(\mathtt{Sch}/S)_{\mathrm{fl}}$.

Recently, Bragg and Olsson have proved that, when f is smooth and proper, $\mathrm{R}^2 f_* \mu_p$ is representable if $\mathrm{R}^1 f_* \mu_p$ is representable and flat over the base, e.g. for a S-abelian scheme $f\colon \mathcal{A} \to S$ (*cf.* [7, Corollary 5.8, Example 5.9]). Thus Theorem 2.3.2 can be viewed as a special case of their result. This also provides a proof for Artin's claim (see Corollary 1.4 in loc.cit.).

The following observation is essential in the construction of twistor space of supersingular abelian or K3 surfaces.

Corollary 2.3.4 ([6, Proposition 2.2.4]). *Keep the assumptions same as in Theorem 2.3.2. The connected components of any geometric fiber of* $\mathrm{R}^2 f_*\mu_p \to S$ *is isomorphic to the additive group scheme* $\mathbb{G}_a$.

Proof. Note that the completion of each geometric fiber of $\mathrm{R}^2 f_*\mu_p$ at $s \in S$, along the identity section, is isomorphic to the formal Brauer group $\widehat{\mathrm{Br}}_{X_s/k(s)}$, which is isomorphic to $\widehat{\mathbb{G}}_a$. The only smooth group scheme at $k(s)$ with this property is $\mathbb{G}_a$. □

3. Derived isogeny between abelian surfaces

In this section, we consider the action of derived isogenies on the cohomology group of abelian surfaces and define the the prime-to-ℓ derived isogenies. Following the work on K3 surfaces in [25, 5, 8], we introduce all the versions of B-field theory and establish the filtered Torelli theorem.

3.1. **Realization of derived isogeny on cohomology groups.** Similar as K3 surfaces, the action of derived isogeny on abelian surfaces are isometries between the twisted Mukai lattices (*cf.* [8, Theorem 3.6]). Moreover, following [17, §1.2.3] (whose argument works for any algebraic surface), we will see that the twisted derived equivalence induces isomorphism between the second component of Chows motives.

For any abelian surface X over k, let

$$\mathfrak{h}^2(X) = \mathfrak{h}_{\mathrm{alg}}(X) \oplus \mathfrak{h}_{\mathrm{tr}}(X)$$

be the decomposition as in [loc.cit. Theorem 1.4], which corresponds to idempotent correspondences $\pi^2_{\mathrm{alg},X}$ and $\pi^2_{\mathrm{tr},X}$ in $\mathrm{CH}^2(X \times X)_{\mathbb{Q}}$. For any derived equivalence $\mathrm{D}^b(X,\alpha) \xrightarrow{\sim} \mathrm{D}^b(Y,\beta)$, it can be uniquely (up to isomorphism) written as a Fourier-Mukai transform with kernel $\mathcal{P} \in \mathrm{D}^b(X \times Y, \alpha^{-1} \boxtimes \beta)$

$$\Phi^{\mathcal{P}} \colon \mathrm{D}^b(X,\alpha) \xrightarrow{\sim} \mathrm{D}^b(Y,\beta).$$

Consider the cycle class

$$[\Gamma_{\mathrm{tr}}] = v(\mathcal{P})_2 \in \mathrm{CH}^2(\mathscr{X} \times \mathscr{Y})_{\mathbb{Q}} \cong \mathrm{CH}^2(X \times Y)_{\mathbb{Q}},$$

where $v(\mathcal{P})_2$ is the dimension two component of the Mukai vector of $\mathcal{P}$. It will induce an isomorphism of motives

$$[\Gamma_{\mathrm{tr}}]_2 \coloneqq \pi^2_{\mathrm{tr},Y} \circ [\Gamma_{\mathrm{tr}}] \circ \pi^2_{\mathrm{tr},X} \colon \mathfrak{h}^2_{\mathrm{tr}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(Y).$$

This also induces an isomorphism of the twisted Néron-Severi lattices of $X_{\bar{k}}$ and $Y_{\bar{k}}$, and thus an isomorphism of their rational extended Néron-Severi lattices. By applying Witt's cancellation theorem, one can obtain a correspondence $[\Gamma_{\mathrm{alg}}] \in \mathrm{CH}^2(X \times Y)_{\mathbb{Q}}$ such that

$$[\Gamma_{\mathrm{alg}}]_2 \coloneqq \pi^2_{\mathrm{alg},Y} \circ [\Gamma_{\mathrm{alg}}] \circ \pi^2_{\mathrm{alg},X} \colon \mathfrak{h}^2_{\mathrm{alg}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{alg}}(Y).$$

This gives an isomorphism

$$[\Gamma] \coloneqq [\Gamma_{\mathrm{tr}}]_2 + [\Gamma_{\mathrm{alg}}]_2 \colon \mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y).$$

Any cohomoglical realization of such isomorphism clearly preserves the Poincaré pairing. Therefore, by taking the corresponding cohomological realization, we obtain

Proposition 3.1.1. *Assume* $\mathrm{char}(k) = p \neq 2$. *Let* ℓ *be a prime not equal to* p. *If* X *and* Y *are twisted derived equivalent over* k, *then* $[\Gamma]$ *will induce a* $\mathrm{Gal}(\bar{k}/k)$*-equivariant isometry*

$$\varphi_\ell \colon \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Q}_\ell). \tag{3.1.1}$$

Suppose k *is perfect, it will induce an isometry between* F*-isocrystals*

$$\varphi_K \colon \mathrm{H}^2_{\mathrm{crys}}(X/K) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{crys}}(Y/K). \tag{3.1.2}$$

The cohomological realizations in Proposition 3.1.1 are not integral in general. We can introduce the prime-to-ℓ derived isogeny via the integral cohomological realizations, which will be used in the rest of the paper.

Definition 3.1.2. Let ℓ be a prime and $\mathrm{char}(k)=p$. When $\ell\neq p$, a derived isogeny $\mathrm{D}^b(X)\sim \mathrm{D}^b(Y)$ given by

$$\begin{aligned}
\mathrm{D}^b(X,\alpha) &\xrightarrow{\simeq} \mathrm{D}^b(X_1,\beta_1)\\
&\quad \mathrm{D}^b(X_1,\alpha_2) \xrightarrow{\simeq} \mathrm{D}^b(X_2,\beta_2)\\
&\qquad\qquad\qquad \vdots\\
&\qquad\qquad \mathrm{D}^b(X_n,\alpha_{n+1}) \xrightarrow{\simeq} \mathrm{D}^b(Y,\beta_n)
\end{aligned}$$

is called *prime-to-ℓ* if each cohomological realization in the zig-zag sequence

$$\varphi^i_\ell\colon \mathrm{H}^2_{\text{ét}}(X_{i-1,\bar{k}},\mathbb{Q}_\ell)\xrightarrow{\sim}\mathrm{H}^2_{\text{ét}}(X_{i,\bar{k}},\mathbb{Q}_\ell)$$

is integral, i.e. $\varphi_\ell\left(\mathrm{H}^2_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)\right)=\mathrm{H}^2_{\text{ét}}(Y_{\bar{k}},\mathbb{Z}_\ell)$. In the case $\ell=p$, it is called *prime-to-p* if each $\varphi^i_p\colon \mathrm{H}^2_{\text{crys}}(X_{i-1}/K)\xrightarrow{\sim}\mathrm{H}^2_{\text{crys}}(X_{i+1}/K)$ is integral.

3.2. **Mukai lattices and B-fields.** If X is a complex abelian surface, the *Mukai lattice* is defined as

$$\widetilde{\mathrm{H}}(X,\mathbb{Z})\coloneqq \mathrm{H}^0(X,\mathbb{Z})\oplus\mathrm{H}^2(X,\mathbb{Z})\oplus\mathrm{H}^4(X,\mathbb{Z})$$

with the Mukai pairing

$$\langle (r,c,\chi),(r',c',\chi')\rangle = r\chi'-cc'+r\chi'. \tag{3.2.1}$$

For any $B\in\mathrm{H}^2(X,\mathbb{Q})$, we define the *twisted Mukai lattice* as

$$\widetilde{\mathrm{H}}(X,\mathbb{Z};B)\coloneqq \exp(B)\cdot\widetilde{\mathrm{H}}(X,\mathbb{Z})\subset\widetilde{\mathrm{H}}(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}.$$

It is equipped with a natural Hodge structure, see [25, Proposition 1.2] for details. For such B, we can associate a Brauer class α_B via the exponential sequence

$$\mathrm{H}^2(X,\mathbb{Q})\to\mathrm{H}^2(X,\mathcal{O}_X)\to\mathrm{H}^2(X,\mathbb{G}_m)=\mathrm{Br}(X).$$

Conversely, given $\alpha\in\mathrm{Br}(X)$, one can find a lift B of α in $\mathrm{H}^2(X,\mathbb{Q})$ because $\mathrm{Br}(X)$ is torsion and $\mathrm{H}^2(X,\mathcal{O}_X)\twoheadrightarrow\mathrm{Br}(X)$ is surjective. B is called a **B**-field lift of α.

For general base field k, we also have the following notion of Mukai lattices [30, §2].

- if it is separably closed with $\mathrm{char}(k)\geq 0$, then the ℓ-adic Mukai lattice is defined on the even degrees of integral ℓ-adic cohomology of X for ℓ coprime to $\mathrm{char}(k)$, denoted by $\widetilde{\mathrm{H}}(X,\mathbb{Z}_\ell)$; or
- if it is perfect with $\mathrm{char}(k)=p>0$, then the p-adic Mukai lattice is defined on the even degrees of crystalline cohomology of X with coefficients in $W(k)$, denoted by $\widetilde{\mathrm{H}}(X,W)$.
- We also have the algebraic Mukai lattice

  $$\widetilde{N}(X)\coloneqq\mathbb{Z}\oplus\mathrm{NS}(X)\oplus\mathbb{Z},$$

  which is well-defined for any base field.

ℓ-adic and crystalline B-field. In addition, we need the following generalized notions of **B**-fields in both ℓ-adic cohomology and crystalline cohomology as an analogue to that in Betti cohomology. In this part, we assume that $k=\bar{k}$ for simplicity of notations.

For a prime $\ell\neq p$ and $n\in\mathbb{N}$, the Kummer sequence of étale sheaves

$$1\to\mu_{\ell^n}\to\mathbb{G}_m\xrightarrow{(\cdot)^n}\mathbb{G}_m\to 1, \tag{3.2.2}$$

induces a long exact sequence

$$\cdots\mathrm{Pic}(X)\xrightarrow{\cdot l^n}\mathrm{Pic}X\to\mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n})\to\mathrm{Br}(X)[\ell^n]\to 0.$$

Taking the inverse limit $\varprojlim_n$, we get a map

$$\pi_\ell\colon \mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}_\ell(1))=\varprojlim_n\mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n})\to\mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n})\twoheadrightarrow\mathrm{Br}(X)[\ell^n].$$

Lemma 3.2.1. *The map π_ℓ is surjective.*

Proof. We have a short exact sequence (*cf.* [33, Chap.V, Lemma 1.11])

$$0 \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))/\ell^n \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}_\ell(1))[\ell^n] \to 0.$$

As $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}_\ell(1))$ is torsion-free for any abelian surface X, we have an isomorphism

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))/\ell^n \cong \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}).$$

Therefore, the reduction morphism $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n})$ can be identified with

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \twoheadrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))/\ell^n,$$

which is surjective. The assertion then follows from it. □

Let $B_\ell(\alpha) := \pi_\ell^{-1}(\alpha)$, which is non-empty for any $\alpha \in \mathrm{Br}(X)[\ell^n]$ by Lemma 3.2.1. For crystalline cohomology, we have the following commutative diagram via the de Rham-Witt theory (*cf.* [26, I.3.2, II.5.1, Théorème 5.14])

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^2(X, \mathbb{Z}_p(1)) & \longrightarrow & \mathrm{H}^2_{\text{crys}}(X/W) & \longrightarrow & \mathrm{H}^2_{\text{crys}}(X/W) \\ & & \downarrow & & \downarrow{\scriptstyle p_n := (\otimes W_n)} & & \\ & & \mathrm{H}^2_{\text{fl}}(X, \mu_{p^n}) & \xrightarrow{d\log} & \mathrm{H}^2_{\text{crys}}(X/W_n) & & \end{array} \tag{3.2.3}$$

where $\mathrm{H}^2(X, \mathbb{Z}_p(1)) := \varprojlim_n \mathrm{H}^2_{\text{fl}}(X, \mu_{p^n})$. The $d\log$ map is known to be injective the flat duality (*cf.* [37, Proposition 1.2]). Since the crystalline cohomology groups of an abelian surface are torsion-free, the mod p^n reduction map p_n is surjective. Consider the canonical map

$$\pi_p \colon \mathrm{H}^2_{\text{fl}}(X, \mu_{p^n}) \twoheadrightarrow \mathrm{Br}(X)[p^n],$$

induced by the Kummer sequence, which is surjective for the similar reason as in the ℓ-adic case. We set

$$B_p(\alpha) := \left\{ b \in \mathrm{H}^2_{\text{crys}}(X/W) | p_n(b) = d\log(t) \text{ such that } \pi_p(t) = \alpha \right\}.$$

Following [8, Definition 2.17], we can introduce the B-fields for twisted abelian surfaces.

Definition 3.2.2. Let ℓ be a prime and let $\alpha \in \mathrm{Br}(X)[\ell^n]$ be a Brauer class of X of order ℓ^n.

- If $\ell \neq p$, an *ℓ-adic **B**-field lift* of α on X is an element $B = \frac{b}{\ell^n} \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ for some $b \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell)$ such that $b \in B_\ell(\alpha)$.
- If $\ell = p$, a *crystalline **B**-field* lift of α is an element $B = \frac{b}{p^n} \in \mathrm{H}^2_{\text{crys}}(X/W)[\frac{1}{p}]$ with $b \in \mathrm{H}^2_{\text{crys}}(X/W)$ such that $b \in B_p(\alpha)$.

More generally, for any $\alpha \in \mathrm{Br}(X)$, a mixed **B**-field lift of α is a set $B = \{B_\ell\} \cup \{B_p\}$ consisting of a choice of an ℓ-adic **B**-field lift B_ℓ of α for each $\ell \neq p$ and a crystalline **B**-field lift B_p of α.

Remark 3.2.3. Not all elements in $\mathrm{H}^2_{\text{crys}}(X/W)[\frac{1}{p}]$ are crystalline **B**-fields since the map $d\log$ is not surjective. From the first row in the diagram (3.2.3), we can see $B \in \mathrm{H}^2_{\text{crys}}(X/W)[\frac{1}{p}]$ is a **B**-field lift of some Brauer class if and only if $F(B) = pB$.

For a ℓ-adic or crystalline B-field $B = \frac{b}{m}$, let $\exp(B) = 1 + B + \frac{B^2}{2}$. We define the twisted Mukai lattice as

$$\widetilde{\mathrm{H}}(X, B) = \begin{cases} \exp(B)\widetilde{\mathrm{H}}(X, \mathbb{Z}_\ell) & \text{if } p \nmid m \\ \\ \exp(B)\widetilde{\mathrm{H}}(X, W) & \text{if } m = p^n \end{cases} \tag{3.2.4}$$

under the Mukai pairing (3.2.1). Moreover, $\widetilde{\mathrm{H}}(X, B)$ is a Frobenius action stable W-lattice in $\widetilde{\mathrm{H}}(X, K)$.

Lemma 3.2.4. *([5, Lemma 3.2.4]) For any $\alpha \in \mathrm{Br}(X)[\ell^n]$, $\widetilde{\mathrm{H}}(X, B)$ is independent of the choice of the lift B up to an isomorphism.*

Now let $\mathscr{X} \to X$ be a μ_n-gerbe over X whose associated Brauer class is $[\mathscr{X}]$. The category $\mathtt{Coh}(\mathscr{X})$ of *α-twisted coherent sheaves* consists of 1-fold $\mathscr{X}$-twisted coherent sheaves in the sense of Lieblich (*cf.* [29]), which is proven to be a Grothendieck category. Let $\mathrm{D}^b(\mathscr{X})$ be the bounded derived category of $\mathtt{Coh}(\mathscr{X})$. Consider the Grothendieck group $\mathrm{K}_0(\mathscr{X})$ of $\mathtt{Coh}(\mathscr{X})$. There is a *twisted Chern character* map

$$\mathrm{ch}_B \colon \mathrm{K}_0(\mathscr{X}) \to \widetilde{\mathrm{H}}(X, B)_K,$$

see [25, Proposition 1.2] and [6, Definition 4.1.1] for (ℓ-adic) and crystalline cases respectively. The twisted Chern character ch_B factors througth the rational extended Néron-Severi lattice $\widetilde{N}(X)_{\mathbb{Q}}$:

$$\begin{tikzcd} \mathrm{K}_0(\mathscr{X}) \arrow[rr, "\mathrm{ch}_B"] \arrow[dr, "\mathrm{ch}_{\mathscr{X}}"'] & & \widetilde{\mathrm{H}}(X, B)_K \\ & \widetilde{N}(X)_{\mathbb{Q}}, \arrow[ur, "\exp(B)\,\mathrm{cl}_{\mathrm{H}}"'] & \end{tikzcd}$$

where cl_{H} is the cycle class map. The image of $\mathrm{K}_0(\mathscr{X})$ in $\widetilde{N}(X)_{\mathbb{Q}}$ under ch_B is denoted by $\widetilde{\mathrm{N}}(\mathscr{X})$. For any $\mathscr{X}$-twisted sheaf $\mathcal{E}$ on X, the Mukai vector $v_B(\mathcal{E})$ is defined to be

$$\mathrm{ch}_B([\mathcal{E}])\sqrt{\mathrm{Td}(X)} \in \widetilde{\mathrm{H}}(X, B)_K.$$

Since the Todd class $\mathrm{Td}(X)$ is trivial when X is an abelian surface, $v_B(\mathcal{E}) = \mathrm{ch}_B([\mathcal{E}]) \in \widetilde{\mathrm{H}}(X, B)_K$.

3.3. **A filtered Torelli Theorem.** The rational numerical Chow ring $\mathrm{CH}^*_{\mathrm{num}}(X)_{\mathbb{Q}}$ is equipped with a codimension filtration

$$\mathrm{Fil}^i\, \mathrm{CH}^*_{\mathrm{num}}(X)_{\mathbb{Q}} \coloneqq \bigoplus_{i \geq k} \mathrm{CH}^k_{\mathrm{num}}(X)_{\mathbb{Q}}.$$

As X is a surface, we have a natural identification $\widetilde{N}(X)_{\mathbb{Q}} \cong \mathrm{CH}^*_{\mathrm{num}}(X)_{\mathbb{Q}}$, which gives a filtration of the rational extended Néron-Severi lattice. Let $\Phi^{\mathcal{P}}$ be a Fourier-Mukai transform with respect to $\mathcal{P} \in \mathrm{D}^b(X \times Y)$. The equivalence $\Phi^{\mathcal{P}}$ is called *filtered* if the induced numerical Chow realization $\Phi^{\mathcal{P}}_{\mathrm{CH}}$ preserves the codimension filtration. A filtered Fourier-Mukai transform is defined in a same way since the twisted Chern character $\mathrm{ch}_{\mathscr{X}}$ maps onto $\widetilde{N}(\mathscr{X}) \subset \widetilde{N}(X)_{\mathbb{Q}}$.

At the cohomological level, the codimension filtration on $\widetilde{\mathrm{H}}(X)[\frac{1}{\ell}]$ (the prime ℓ depends on the choice of ℓ-adic or crystalline twisted Mukai lattice) is given by $F^i = \oplus_{r \geq i} \mathrm{H}^{2r}(X)[\frac{1}{\ell}]$. Let B be a $\mathbf{B}$-field lift of $[\mathscr{X}]$. The filtration on $\widetilde{\mathrm{H}}(X, B)$ is defined by

$$F^i\widetilde{\mathrm{H}}(X, B) = \widetilde{\mathrm{H}}(X, B) \cap F^i\widetilde{\mathrm{H}}(X)[\frac{1}{\ell}].$$

A direct computation shows that the graded pieces of $F^\bullet$ are

$$\begin{aligned} \mathrm{Gr}^0_F\widetilde{\mathrm{H}}(X, B) &= \left\{(r, rB, \frac{rB^2}{2})\middle| r \in \mathrm{H}^0(X)\right\}, \\ \mathrm{Gr}^1_F\widetilde{\mathrm{H}}(X, B) &= \big\{(0, c, c \cdot B) | c \in \mathrm{H}^2(X)\big\} \cong \mathrm{H}^2(X), \\ \mathrm{Gr}^2_F\widetilde{\mathrm{H}}(X, B) &= \big\{(0, 0, s) | s \in \mathrm{H}^4(X)\big\} \cong \mathrm{H}^4(X)(1). \end{aligned} \tag{3.3.1}$$

Lemma 3.3.1. *A twisted Fourier-Mukai transform $\Phi^{\mathcal{P}} : \mathrm{D}^b(\mathscr{X}) \to \mathrm{D}^b(\mathscr{Y})$ is filtered if and only if its cohomological realization is filtered for certian $\boldsymbol{B}$-field lifts.*

Proof. It is clear that being filtered implies being cohomological filtered. This is because the map $\exp(B) \cdot \mathrm{cl}_H \colon \widetilde{N}(X, \mathbb{Q}) \to \widetilde{\mathrm{H}}(X, B)$ preserves the filtrations for any $\mathbf{B}$-field lift B of $[\mathscr{X}]$.

For the converse, just notice that $\Phi^{\mathcal{P}}$ is filtered if and only if the induced map $\Phi^{\mathcal{P}}_{\mathrm{CH}}$ takes the vector $(0, 0, 1)$ to $(0, 0, 1)$. As $\Phi^{\mathcal{P}}$ is cohomological filtered for B, we can see the cohomological realization of $\Phi^{\mathcal{P}}$ preserves the graded piece Gr^2_F in (3.3.1). This implies that $\Phi^{\mathcal{P}}_{\mathrm{CH}}$ takes $(0, 0, 1)$ to $(0, 0, 1)$. $\square$

Proposition 3.3.2 (filtered Torelli theorem for twisted abelian surfaces). *Suppose $k = \bar{k}$. Let $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ be μ_n-gerbes on abelian surfaces. Then following statements are equivalent*

(1) There is an isomorphism between associated $\mathbb{G}_m$-gerbes $\mathscr{X}_{\mathbb{G}_m}$ and $\mathscr{Y}_{\mathbb{G}_m}$.
(2) There is a filtered Fourier-Mukai transform $\Phi^{\mathcal{P}}$ from $\mathscr{X}$ to $\mathscr{Y}$.

Proof. For untwisted case, i.e. $\mathscr{X} = X$ and $\mathscr{Y} = Y$, this is exactly [21, Proposition 3.1]. Here we extend it to the twisted case. As one direction is obvious, it suffices to show that (2) can imply (1). Set

$$\mathcal{P}_x := \Phi^{\mathcal{P}}(\mathcal{O}_x) = \mathcal{P}|_{\{x\}\times Y}.$$

Since $\mathtt{Coh}(\mathscr{X})$ and $\mathtt{Coh}(\mathscr{Y})$ have no spherical objects, there is an integer m and a $\mathscr{X} \times \mathscr{Y}^{(-1)}$-twisted sheaf $\mathcal{E} \in \mathtt{Coh}(\mathscr{X} \times \mathscr{Y}^{(-1)})$ such that $\mathcal{P} \cong \mathcal{E}[m]$ by [24, Proposition 3.18]. Since $\Phi^{\mathcal{P}}_{\mathscr{X}\to\mathscr{Y}}$ sends $(0, 0, 1)$ to $(0, 0, 1)$, $\mathcal{E}_x$ is just a skyscraper sheaf on $\{x\} \times Y$, which is naturally non-twisted as $k = \bar{k}$. Thus $\mathcal{E}$ can be viewed as an invertible sheaf on its support and also as a line bundle $\mathcal{L}$ on X.

Let $\mathcal{O}_\Gamma$ be the structure sheaf of the schematic support of $\mathcal{E}$ on $X \times Y$. Then there is a morphism $f\colon X \to Y$ such that

$$\Phi^{\mathcal{E}}_{\mathscr{X}\to\mathscr{Y}} \simeq f_*(\mathcal{L} \otimes (-)).$$

The induced integral functor $\Phi^{\mathcal{O}_\Gamma}_{X\to Y}$ takes skyscraper sheaves to skyscraper sheaves from the construction. Hence $\Phi^{\mathcal{E}}_{X\to Y}$ is fully faithful as skyscraper sheaves form a spanning class of the bounded derived category of a smooth projective variety. Moreover, it is an equivalence as abelian surfaces have trivial canonical class, by [22, Corollary 7.8]. By [24, Corollary 5.23]), the morphism $f\colon X \to Y$ is an isomorphism. Let B be a B-field lift of $[\mathscr{X}_{\mathbb{G}_m}]$. Since $\Phi^{\mathcal{E}}_{\mathscr{X}\to\mathscr{Y}}$ is filtered, it takes $(1, B, \frac{B^2}{2})$ to $(1, B', \frac{B'^2}{2})$ for some $B \in \mathrm{H}^2(X)_{\mathbb{Q}}$ by Lemma 3.3.1. Thus B' is a **B**-field lift of $[\mathscr{Y}_{\mathbb{G}_m}]$ and $B = f^*B'$, which implies $f^*[\mathscr{Y}_{\mathbb{G}_m}] = [\mathscr{X}_{\mathbb{G}_m}]$. □

3.4. **Twisted FM partners via moduli space of twisted sheaves.** Keep the notations as before, we denote by $\mathscr{M}_H(\mathscr{X}, v)$ (or $\mathscr{M}^\alpha_H(X, v)$) the moduli stack of H-semistable $\mathscr{X}$-twisted sheaves with Mukai vector $v \in \widetilde{\mathrm{N}}(\mathscr{X})$, where H is a v-generic ample divisor on X and $\alpha = [\mathscr{X}]$ the associated Brauer class of X (*cf.* [29] or [52]). To characterize the Fourier-Mukai partners of twisted abelian surfaces via the moduli space of twisted sheaves, we first need the following criterion on non-emptiness of moduli space of (twisted) sheaves on an abelian surface X.

Proposition 3.4.1 (Yoshioka, Bragg-Lieblich). *Let n be a positive integer. Assume that either $p \nmid n$ or X is supersingular. Let $\mathscr{X} \to X$ be a μ_n-gerbe on X. Let $v = (r, \ell, s) \in \widetilde{\mathrm{N}}(\mathscr{X})$ be a primitive Mukai vector such that $v^2 = 0$. Fix a v-generic ample divisor H. If one of the following holds:*

(1) $r > 0$.
(2) $r = 0$ and ℓ is effective.
(3) $r = \ell = 0$ and $s > 0$.

then the coarse moduli space $M_H(\mathscr{X}, v) \neq \emptyset$ and the moduli stack $\mathscr{M}_H(\mathscr{X}, v)$ is a $\mathbb{G}_m$-gerbe on $M_H(\mathscr{X}, v)$. Moreover, its coarse moduli space $M_H(\mathscr{X}, v)$ is an abelian surface.

Proof. If $\mathscr{X} \to X$ is a μ_n-gerbe such that $p \nmid n$, then the statements are proven in [34, Proposition A.2.1]. It is based on a statement of lifting a Brauer classes on A to characteristic 0 which requires the condition $p \nmid n$.

When X is supersingular and $\mathscr{X} \to X$ is a μ_p-gerbe, provided Corollary 2.3.4, the assertion will follow from a similar argument in [6, Proposition 4.1.20] for supersingular K3 surfaces. Roughly speaking, in the case $\mathscr{X} \to X$ is a trivial μ_p-gerbe, this can be proved by the same lifting argument (see also [16, Proposition 6.9]). When $\mathscr{X} \to X$ is non-trivial, Corollary 2.3.4 results in a universal twistor family $f : \mathfrak{X} \to \mathbb{A}^1$ of twisted supersingular abelian surfaces whose fibers contain $\mathscr{X} \to X$ and the trivial μ_p-gerbe over X. By taking the relative moduli space of twisted sheaves (with suitable v-generic polarization) on $\mathfrak{X} \to \mathbb{A}^1$, we can see the non-emptiness of $M_H(\mathscr{X}, v)$ from the case of essentially trivial gerbes. □

Let $\mathscr{X} \to X$ be a $\mathbb{G}_m$-gerbe on X such that $[\mathscr{X}] \in \mathrm{Br}(X)[\ell^n]$. Take a **B**-field lift $B = \frac{b}{\ell^n}$ of $[\mathscr{X}]$. Under the isomorphism $\mathrm{H}^2(X, \mathbb{Z}_\ell(1)) \cong \mathrm{H}^2(\widehat{X}, \mathbb{Z}_\ell(1))$ induced by the Poincaré bundle, there is an element $\widehat{b} \in \mathrm{H}^2(\widehat{X}, \mathbb{Z}_\ell(1))$ being the image of b and $\frac{\widehat{b}}{\ell^n}$ is a ℓ-adic (or crystalline) **B**-field on $\widehat{X}$. Thus we have a unique Brauer class $[\widehat{\mathscr{X}}]$ corresponding to $\frac{\widehat{b}}{\ell^n}$. We will denote by $\widehat{\mathscr{X}}$ the $\mathbb{G}_m$-gerbe on $\widehat{X}$ with respect to $\widehat{\alpha}$.

Theorem 3.4.2. *With the same assumptions as in Proposition 3.4.1. Let $\mathscr{X} \to X$ be μ_n-gerbe on an abelian surface X. Then the associated $\mathbb{G}_m$-gerbe of any Fourier-Mukai partner of $\mathscr{X}$ is isomorphic to a $\mathbb{G}_m$-gerbe on the moduli space of $\mathscr{X}$-twisted sheaves $M_H(\mathscr{X}, v)$ with $\mathscr{X}$ being $\mathscr{X}$ or $\widehat{\mathscr{X}}$.*

Proof. Let $\mathscr{M}$ be a Fourier-Mukai partner of $\mathscr{X}$. Let $\Phi^{\mathcal{P}}_{\mathscr{M}\to\mathscr{X}}$ be the Fourier-Mukai transform. Let v be the image of $(0,0,1)$ under $\Phi^{\mathcal{P}}_{\mathscr{M}\to\mathscr{X}}$. We can assume v satisfying one of the conditions in Proposition 3.4.1 by changing $\mathscr{X}$ to $\widehat{\mathscr{X}}$ if necessary. Denote $\mathscr{X}$ by $\mathscr{X}$ or $\widehat{\mathscr{X}}$. It is proved that the moduli stack $\mathscr{M}_H(\mathscr{X}, v)$ is a $\mathbb{G}_m$-gerbe on $M_H(A, v)$ in Proposition 3.4.1. Then there is a Fourier-Mukai transform

$$\Phi^{\mathcal{P}} \colon \mathrm{D}^b(\mathscr{M}_H(\mathscr{X}, v)^{(-1)}) \to \mathrm{D}^b(\mathscr{X}^{(1)}) \tag{3.4.1}$$

induced by the tautological sheaf $\mathcal{P}$ on $\mathscr{M}_H(\mathscr{X}, v) \times \mathscr{X}$, whose cohomological realization maps the Mukai vector $(0,0,1)$ to v. Combining it with the derived equivalence

$$\Phi \colon \mathrm{D}^b(\mathscr{X}) \to \mathrm{D}^b(\mathscr{M}),$$

we will obtain a filtered derived equivalence from $\mathscr{M}_H(\mathscr{X}, v)^{(-1)}$ to $\mathscr{M}^{(1)}$. This induces an isomorphism from $\mathscr{M}_H(\mathscr{X}, v)^{(-1)}$ to $\mathscr{M}^{(1)}_{\mathbb{G}_m}$ by Theorem 3.3.2. □

4. Shioda's Torelli theorem for abelian surfaces

In [44], Shioda noticed that there is a way to extract the information of the 1^{st}-cohomology of a complex abelian surface from its 2^{nd}-cohomology, called Shioda's trick. This established a global Torelli theorem for complex abelian surfaces via the 2^{nd}-cohomology. The aim of this section is to generalize Shioda's method to all fields and establish an isogenous theorem for abelian surfaces via the 2^{nd}-cohomology. We will deal with Shioda's trick for étale cohomology and crystalline cohomology separately. Throughout this section, we let X and Y be abelian surfaces over a field k.

4.1. **Recap of Shioda's trick for Hodge isometry.** We first recall Shioda's construction. Suppose X is a complex abelian surface. Its singular cohomology ring $\mathrm{H}^\bullet(X, \mathbb{Z})$ is canonically isomorphic to the exterior algebra $\wedge^\bullet \mathrm{H}^1(X, \mathbb{Z})$. Let V be a free $\mathbb{Z}$-module of rank 4. We denote by Λ the lattice $(\wedge^2 V, q)$ where $q : \wedge^2 V \times \wedge^2 V \to \mathbb{Z}$ is the wedge product. After choosing a $\mathbb{Z}$-basis $\{v_i\}_{1 \le i \le 4}$ for $\mathrm{H}^1(X, \mathbb{Z})$, we have an isometry of $\mathbb{Z}$-lattice $\Lambda \xrightarrow{\sim} \mathrm{H}^2(X, \mathbb{Z})$. The set of vectors

$$\{v_{ij} \coloneqq v_i \wedge v_j\}_{0 \le i < j \le 4}$$

clearly forms a basis of $\mathrm{H}^2(X, \mathbb{Z})$, which will be called an *admissible basis* of A for its second singular cohomology. For another complex abelian surface Y, a Hodge isometry

$$\psi \colon \mathrm{H}^2(Y, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(X, \mathbb{Z})$$

will be called *admissible* if $\det(\psi) = 1$, with respect to some admissible bases on X and Y. It is clear that the admissibility of a morphism is independent of the choice of admissible bases.

In terms of admissible basis, we can view ψ as an element in $\mathrm{SO}(\Lambda)$. On the other hand, we have the following exact sequence of groups

$$1 \to \{\pm 1\} \to \mathrm{SL}_6(\mathbb{Z}) \xrightarrow{\wedge^2} \mathrm{SO}(\Lambda) \tag{4.1.1}$$

Shioda observed that the image of $\mathrm{SL}_6(\mathbb{Z})$ in $\mathrm{SO}(\Lambda)$ is a subgroup of index two and does not contain $-\mathrm{id}_\Lambda$. From this, he proved the following

Theorem 4.1.1 (Shioda). *For any admissible integral Hodge isometry ψ, there is an isomorphism of integral Hodge structures*

$$\varphi\colon \mathrm{H}^1(Y,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(X,\mathbb{Z})$$

such that $\wedge^2(\varphi)=\psi$ or $-\psi$.

This is what we call "Shioda's trick". As we can assume a Hodge isometry being admissible after taking the dual abelian variety for one of them, we can obtain the Torelli theorem for complex abelian surfaces by using the weight two Hodge structures, i.e., X is isomorphic to Y or its dual $\widehat{Y}$ if and only if there is an integral Hodge isometry $\mathrm{H}^2(X,\mathbb{Z})\cong \mathrm{H}^2(Y,\mathbb{Z})$ (*cf.* [44, Theorem 1]).

4.2. **Admissible basis.** In order to extend Shioda's work to arbitrary fields, we need to define admissibility for various cohomology theories (e.g. étale cohomology and crystalline cohomology).

Let k be a perfect field with $\mathrm{char}(k)=p\geq 0$. Suppose X is an abelian surface over k and $\ell\nmid p$ is a prime. For simplicity of notations, we will denote $\mathrm{H}^\bullet(-)_R$ for one of the following cohomology theories:

(1) if $k\hookrightarrow\mathbb{C}$ and $R=\mathbb{Z}$ or any number field E, then $\mathrm{H}^\bullet(X)_R=\mathrm{H}^\bullet(X(\mathbb{C}),R)$ the singular cohomology.
(2) if $R=\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, then $\mathrm{H}^\bullet(X)_R=\mathrm{H}^\bullet_{\text{ét}}(X_{\bar{k}},R)$, the ℓ-adic étale cohomology.
(3) if $\mathrm{char}(k)=p>0$ and $R=W$ or K, then $\mathrm{H}^\bullet(X)_R=\mathrm{H}^\bullet_{\text{crys}}(X_{k^{\text{perf}}}/W)$ or $\mathrm{H}^\bullet_{\text{crys}}(X_{k^{\text{perf}}}/W)\otimes K$, the crystalline cohomology.

There is an isomorphism between the cohomology ring $\mathrm{H}^\bullet(X)_R$ and the exterior algebra $\wedge^\bullet\mathrm{H}^1(X)_R$. We denote by $\mathrm{tr}_X\colon \mathrm{H}^4(X)_R\xrightarrow{\sim} R$ the corresponding trace map. Then the Poincaré pairing $\langle -,-\rangle$ on $\mathrm{H}^2(X)_R$ can be realized as

$$\langle\alpha,\beta\rangle=\mathrm{tr}_X(\alpha\wedge\beta).$$

Analogous to §4.1, a R-basis $\{v_i\}$ of $\mathrm{H}^1(X)_R$ will be called a *d-admissible basis* if it satisfies

$$\mathrm{tr}_X(v_1\wedge v_2\wedge v_3\wedge v_4)=d$$

for some $d\in R^*$. When $d=1$, it will be called an *admissible basis.* For any d-admissible (resp. admissible) basis $\{v_i\}$, the associated R-basis $\{v_{ij}\coloneqq v_i\wedge v_j\}_{i<j}$ of $\mathrm{H}^2(X)_R$ will also be called d-admissible (resp. admissible).

Example 4.2.1. Let $\{v_1,v_2,v_3,v_4\}$ be a R-linear basis of $\mathrm{H}^1(X)_R$. Suppose

$$\mathrm{tr}_X(v_1\wedge v_2\wedge v_3\wedge v_4)=t\in R^*.$$

For any $d\in R^*$, there is a natural d-admissible R-linear basis $\{\frac{d}{t}v_1,v_2,v_3,v_4\}$

Definition 4.2.2. Let X and Y be abelian surfaces over k.

- a R-linear isomorphism $\psi\colon \mathrm{H}^1(X)_R\to\mathrm{H}^1(Y)_R$ is d-admissible if it takes an admissible basis to a d-admissible basis.
- a R-linear isomorphism $\varphi\colon \mathrm{H}^2(X)_R\to\mathrm{H}^2(Y)_R$ is d-admissible if

$$\mathrm{tr}_Y\circ\wedge^2(\varphi)=d\,\mathrm{tr}_X$$

for some $d\in R^*$, or equivalently, it sends an admissible basis to a d-admissible basis. When $d=1$, it will also be called admissible.

The set of d-admissible isomorphisms will be denoted by $\mathrm{Iso}^{\mathrm{ad},(d)}(\mathrm{H}^1(X)_R,\mathrm{H}^1(Y)_R)$ and $\mathrm{Iso}^{\mathrm{ad},(d)}(\mathrm{H}^2(X)_R,\mathrm{H}^2(Y)_R)$ respectively.

For any isomorphism $\varphi\colon \mathrm{H}^2(X)_R\xrightarrow{\sim}\mathrm{H}^2(Y)_R$, let $\det(\varphi)$ be the determinant of the matrix with respect to some admissible bases. It is not hard to see $\det(\varphi)$ is independent of the choice of admissible bases, and φ is admissible if and only if $\det(\varphi)=1$.

Example 4.2.3. For the dual abelian surface $\widehat{X}$, the dual basis $\{v_i^*\}$ with respect to Poincaré pairing naturally forms an admissible basis, under the identification $\mathrm{H}^1(X)_R^\vee \cong \mathrm{H}^1(\widehat{X})_R$. Let

$$\psi_{\mathcal{P}} \colon \mathrm{H}^2(X)_R \to \mathrm{H}^2(\widehat{X})_R$$

be the isomorphism induced by the Poincaré bundle $\mathcal{P}$ on $X \times \widehat{X}$. A direct computation shows that $\psi_{\mathcal{P}}$ is nothing but

$$-\mathrm{D} \colon \mathrm{H}^2(X)_R \xrightarrow{\sim} \mathrm{H}^2(X)_R^\vee \cong \mathrm{H}^2(\widehat{X})_R,$$

where D is the Poincaré duality. For an admissible basis $\{v_i\}$ of X, its R-linear dual $\{v_i^*\}$ with respect to Poincaré pairing forms an admissible basis of $\widehat{X}$. By our construction, we can see

$$\mathrm{D}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) = (v_{34}^*, -v_{24}^*, v_{23}^*, v_{14}^*, -v_{13}^*, v_{12}^*),$$

which implies that D is of determinant -1 under these admissible bases. Thus the determinant of $\psi_{\mathcal{P}}$ is not admissible.

Example 4.2.4. Let $f \colon X \to Y$ be an isogeny of degree d for some $d \in \mathbb{Z}_{\geq 0}$ between two abelian surfaces. If d is coprime to ℓ, then it will induce an isomorphism

$$f^* \colon \mathrm{H}^2(Y)_{\mathbb{Z}_\ell} \xrightarrow{\sim} \mathrm{H}^2(X)_{\mathbb{Z}_\ell},$$

which is d-admissible. If $d = n^4$, then $(\frac{1}{n} f)^*$ will be an admissible $\mathbb{Z}_\ell$-integral isometry with respect to the Poincaré pairing.

If $\ell \neq 2$, then d is always a square in $\mathbb{Z}_\ell$. Thus there is some $\xi \in \mathbb{Z}_\ell^*$ such that $d = \xi^4$. Therefore, we can always find an admissible $\mathbb{Z}_\ell$-integral isomorphism $\frac{1}{\xi} f^* \colon \mathrm{H}^1(Y)_{\mathbb{Z}_\ell} \to \mathrm{H}^1(X)_{\mathbb{Z}_\ell}$.

Example 4.2.5. Suppose X is an abelian surface over a perfect field k with $\mathrm{char}(k) = p > 0$. Then F-crystal $\mathrm{H}^1(X)_W$ together with the trace map

$$\mathrm{tr}_X \colon \mathrm{H}^4(X)_W \xrightarrow{\sim} W$$

form an abelian crystal, in the sense of [37, §6]. We can see $\mathrm{H}^1(X)_W \cong \mathrm{H}^1(Y)_W$ as abelian crystals if and only if there is an admissible isomorphism $\mathrm{H}^1(X)_W \xrightarrow{\sim} \mathrm{H}^1(Y)_W$.

4.3. **More on admissible basis of F-crystals.** In contrast to ℓ-adic étale cohomology, the semilinear structure on crystalline cohomology from its Frobenius is more tricky to work with. Therefore, it seems necessary for us to spend more words on the interaction of Frobenius with admissible bases.

We have the following Frobenius pull-back diagram:

$$\begin{tikzcd}
X \arrow[dr, "F_X^{(1)}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\
& X^{(1)} \arrow[r] \arrow[d] & X \arrow[d] \\
& \mathrm{Spec}(k) \arrow[r, "\sigma"] & \mathrm{Spec}(k)
\end{tikzcd}$$

Via the natural identification $\mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W) \cong \mathrm{H}^1_{\mathrm{crys}}(X/W) \otimes_\sigma W$, the σ-linearization of Frobenius action on $\mathrm{H}^1_{\mathrm{crys}}(X/W)$ can be viewed as the injective W-linear map

$$F^{(1)} \coloneqq \left(F_X^{(1)}\right)^* \colon \mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W) \hookrightarrow \mathrm{H}^1_{\mathrm{crys}}(X/W).$$

There is a decomposition $\mathrm{H}^1_{\mathrm{crys}}(X/W) = H_0(X) \oplus H_1(X)$ such that

$$F^{(1)}\left(\mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W)\right) \cong H_0(X) \oplus pH_1(X), \tag{4.3.1}$$

and $\mathrm{rank}_W H_i = 2$ for all $i = 0, 1$, which is related to the Hodge decomposition of the de Rham cohomology of X/k by Mazur's theorem; see [4, §8, Theorem 8.26].

The Frobenius map can be expressed in terms of admissible basis. We can choose an admissible basis $\{v_i\}$ of $\mathrm{H}^1_{\mathrm{crys}}(X/W)$ such that

$$v_1, v_2 \in H_0(X) \quad \text{and} \quad v_3, v_4 \in H_1(X).$$

Then $\{p^{\alpha_i} v_i\} \coloneqq \{v_1, v_2, pv_3, pv_4\}$ forms an admissible basis of $\mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W)$ under the identification (4.3.1), since $\mathrm{tr}_p \circ \wedge^4 F^{(1)} = p^2 \sigma_W \circ \mathrm{tr}_p$. In term of these basis, the Frobenius map can be written as

$$F^{(1)}(p^{\alpha_i} v_i) = \sum_j c_{ij} p^{\alpha_j} v_j,$$

where $C_X = (c_{ij})$ forms an invertible 4×4-matrix with coefficients in W.

Suppose Y is another abelian surface over k and $\rho \colon \mathrm{H}^1_{\mathrm{crys}}(X/W) \to \mathrm{H}^1_{\mathrm{crys}}(Y/W)$ is an admissible map. Denote $\rho^{(1)}$ for the induced map $\rho \otimes_\sigma W \colon \mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W) \to \mathrm{H}^1_{\mathrm{crys}}(Y^{(1)}/W)$. The following lemma is clear.

Lemma 4.3.1. *The map ρ is a morphism between F-crystals if and only if $C_Y^{-1} \cdot \rho^{(1)} \cdot C_X = \rho$, where "$\cdot$" denotes by the action of matrix with respect to the chosen admissible bases.*

4.4. **Generalized Shioda's trick.** Let us review some basic properties of the special orthogonal group scheme over an integral domain. Let Λ be an even $\mathbb{Z}$-lattice of rank $2n$. Then we can associate it with a vector bundle $\underline{\Lambda}$ on $\mathrm{Spec}(\mathbb{Z})$ with constant rank $2n$ equipped with a quadratic form q over $\mathrm{Spec}(\mathbb{Z})$ obtained from Λ. Then the functor

$$A \mapsto \left\{ g \in \mathrm{GL}(\Lambda_A) \middle| q_A(g \cdot x) = g \cdot q_A(x) \text{ for all } x \in \Lambda_A \right\}$$

represents a $\mathbb{Z}$-subscheme of $\mathrm{GL}(\Lambda)$, denoted by $\mathrm{O}(\Lambda)$. There is a homomorphism between $\mathbb{Z}$-group schemes

$$D_\Lambda \colon \mathrm{O}(\Lambda) \to \underline{\mathbb{Z}/2\mathbb{Z}},$$

which is called the Dickson morphism. It is surjective as Λ is even, and its formation commutes with any base change. The *special orthogonal group scheme* over $\mathbb{Z}$ with respect to Λ is defined to be the kernel of D_Λ, which is denoted by $\mathrm{SO}(\Lambda)$. Moreover, we have

$$\mathrm{SO}(\Lambda)_{\mathbb{Z}[\frac{1}{2}]} \cong \ker\left(\det \colon \mathrm{O}(\Lambda) \to \mathbb{G}_m\right)_{\mathbb{Z}[\frac{1}{2}]}.$$

It is well-known that $\mathrm{SO}(\Lambda) \to \mathrm{Spec}(\mathbb{Z})$ is smooth of relative dimension 15 and with connected fibers; see [11, Theorem C.2.11] for instance. For any ℓ, the special orthogonal group scheme

$$\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell}) \cong \mathrm{SO}(\Lambda)_{\mathbb{Z}_\ell}$$

is smooth over $\mathbb{Z}_\ell$ with connected fibers, which implies its generic fiber $\mathrm{SO}(\Lambda_{\mathbb{Q}_\ell})$ is connected. Thus $\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ is clearly connected as a group scheme over $\mathbb{Z}_\ell$ as $\mathrm{SO}(\Lambda_{\mathbb{Q}_\ell}) \subset \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ is dense. More generally, this is also the case of a Dedekind domain R.

Assume $\Lambda = U^{\oplus 3}$, where U is the hyperbolic lattice. Then we have

Lemma 4.4.1. *If $\frac{1}{2} \in R$, then there is an exact sequence of smooth R-group schemes*

$$1 \to \mu_{2,R} \to \mathrm{SL}(V)_R \xrightarrow{\wedge^2(-)_R} \mathrm{SO}(\Lambda)_R \to 1.$$

(as fppf-sheaves if $\frac{1}{2} \notin R$.) Moreover, there is an exact sequence

$$1 \to \{\pm \mathrm{id}_4\} \to \mathrm{SL}(V)(R) \xrightarrow{\wedge^2(-)_R} \mathrm{SO}(\Lambda)(R) \xrightarrow{\mathrm{SN}} R^*/(R^*)^2, \tag{4.4.1}$$

where SN *is the map of spinor norm (see* [3, §3.3] *for the definition).*

Proof. For the first statement, it suffices to assume $R = \mathrm{Spec}(\bar{k})$ for an algebraically closed field $\bar{k}$. It is clearly from computation.

Note that we have an exact sequence on rational points (*cf.* [18, Proposition 3.2.2])

$$1 \to \mu_2(R) \to \mathrm{SL}(V)(R) \to \mathrm{SO}(\Lambda)(R) \to \mathrm{H}^1(\mathrm{Spec}(R), \mu_2).$$

From the Kummer sequence for μ_2, we can see

$$\mathrm{H}^1(\mathrm{Spec}(R), \mu_2) \cong \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}(R), \mu_2) \cong R^*/(R^*)^2.$$

Let K be the fraction field of R. For the last statement, it is sufficient to see that there is an isomorphism of K-group schemes $\mathrm{SL}(V)_K \xrightarrow{\sim} \mathrm{SO}(\Lambda)_K$ such that the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Spin}(\Lambda)_K \arrow[d] & \\
\mathrm{SL}(V)_K \arrow[ur, "\sim"] \arrow[r] & \mathrm{SO}(\Lambda)_K \arrow[r] \arrow[d, "\mathrm{SN}"] & K^*/(K^*)^2 \\
 & K^*/(K^*)^2 \arrow[ur, "\sim"] &
\end{tikzcd}$$

Such an isomorphism exists as the both two central isogenies are universal coverings of $\mathrm{SO}(\Lambda)_K$. $\square$

Remark 4.4.2. For instance, when $R = \mathbb{Z}_\ell$, we have

$$\mathbb{Z}_\ell^*/(\mathbb{Z}_\ell^*)^2 \cong \begin{cases} \{\pm 1\} & \text{if } \ell \neq 2, \\ \{\pm 1\} \times \{\pm 5\} & \text{if } \ell = 2. \end{cases}$$

Thus the image of $\mathrm{SL}(V)(\mathbb{Z}_\ell)$ is a finite index subgroup in $\mathrm{SO}(\Lambda)(\mathbb{Z}_\ell)$. This also holds for the W-value points, since $W^*/(W^*)^2 \subseteq \{\pm 1\}$ by Hensel's lemma, the image of $\mathrm{SL}(V)(W)$ is of finite index ≤ 2.

Let $V_R = \mathrm{H}^1(X)_R$. We can see the set

$$\mathrm{Iso}^{\mathrm{ad},(d)}(\mathrm{H}^1(X)_R, \mathrm{H}^1(Y)_R)$$

is a naturally (left) $\mathrm{SL}(V_R)$-torsor, and the set

$$\mathrm{Iso}^{\mathrm{ad},(d)}\left(\mathrm{H}^2(X)_R, \mathrm{H}^2(Y)_R\right)$$

is a (left) $\mathrm{SO}(\Lambda_R)$-torsor. The wedge product provides a natural map

$$\wedge^2 \colon \mathrm{Iso}^{\mathrm{ad},(d)}\left(\mathrm{H}^1(X)_R, \mathrm{H}^1(Y)_R\right) \to \mathrm{Iso}^{\mathrm{ad},(d)}\left(\mathrm{H}^2(X)_R, \mathrm{H}^2(Y)_R\right).$$

Let $\{v_i\}$ be an admissible basis of $\mathrm{H}^1(X)_R$ and let $\{v_i'\}$ be a d-admissible basis of $\mathrm{H}^1(Y)_R$ respectively. There is an admissible isomorphism $\psi_0 \in \mathrm{Iso}^{\mathrm{ad},(d)}(\mathrm{H}^1(X)_R, \mathrm{H}^1(Y)_R)$ such that $\psi_0(v_i) = v_i'$. For an admissible isometry $\varphi \colon \mathrm{H}^2(X, R) \to \mathrm{H}^2(Y, R)$, we can see

$$\varphi = g \circ \wedge^2(\psi_0^{-1}), \text{ for some } g \in \mathrm{SO}(\Lambda_R).$$

In this way, any d-admissible isomorphism φ can be identified with an element $g \in \mathrm{SO}(\Lambda)(R)$ when the admissible bases are fixed. This allows us to deal with d-admissible isomorphisms group-theoretically. In particular, we have the following notion of spinor norm.

Definition 4.4.3. The *spinor norm* of the d-admissible isomorphism φ is defined to the image of g under $\mathrm{SN} \colon \mathrm{SO}(\Lambda)(R) \to R^*/(R^*)^2$, denoted by $\mathrm{SN}(\varphi)$.

Lemma 4.4.4. *The spinor norm* $\mathrm{SN}(\varphi)$ *is independent of the choice of admissible bases.*

Proof. For different choice of admissible bases, we can see the resulted $\widetilde{g} = KgK^{-1}$ for some $K \in \mathrm{SO}(\Lambda_R)$. Therefore $\mathrm{SN}(\widetilde{g}) = \mathrm{SN}(g)$. $\square$

Remark 4.4.5. When R is a field, the spinor norm can be computed by the Cartan-Dieudonné decomposition. That means, we can write any $g \in \mathrm{SO}(\Lambda)(R)$ into the composition of reflections:

$$\varphi_{b_n} \circ \varphi_{b_{n-1}} \circ \cdots \circ \varphi_{b_1}$$

for some non-isotropic vectors $b_1, \cdots, b_n \in \Lambda_R$, and $\mathrm{SN}(g) = \left[(b_1)^2 \cdots (b_{n-1})^2 (b_n)^2\right]$.

Lemma 4.4.6. *The d-admissible isomorphism φ is a wedge of some d-admissible isomorphism* $\psi \colon \mathrm{H}^1(X, R) \to \mathrm{H}^1(Y, R)$ *if and only if* $\mathrm{SN}(\varphi) = 1$.

Proof. The exact sequence (4.4.1) shows that if $\mathrm{SN}(\varphi) = \mathrm{SN}(g) = 1$, then there is some $h \in \mathrm{SL}(V_R)$ such that $\wedge^2(h) = g$. Thus we can take $\psi = h \circ \psi_0$ when $\mathrm{SN}(\varphi) = 1$, and see that

$$\wedge^2(\psi) = \wedge^2(h) \circ \wedge^2(\psi_0) = \varphi.$$

The converse is clear. □

4.5. **Shioda's trick for Hodge isogenies.** When $k = \mathbb{C}$ and d is an integer, we say an isometry

$$\varphi\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})(d)$$

a *Hodge isogeny of degree* d if it is also a morphism of Hodge structures. In particular, if $d = 1$, then it is the classical Hodge isometry we ususally talk about. Clearly, a d-admissible rational Hodge isomorphism is a Hodge isogeny of degree d. In terms of spinor norms, we can generalize Shioda's theorem 4.1.1 to admissible rational Hodge isogenies.

Proposition 4.5.1 (Shioda's trick on admissible Hodge isogenies)**.**

(1) A d-admissible Hodge isogeny of degree d

$$\varphi\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})(d)$$

is a wedge of some rational Hodge isomorphism $\psi\colon \mathrm{H}^1(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^1(Y,\mathbb{Q})$, if its spinor norm is a square in $\mathbb{Q}^*$. In this case, the Hodge isogeny is induced by a quasi-isogeny of degree d^2.

(2) When $d = 1$, any admissible Hodge isometry $\varphi : \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})$ is induced by an isogeny $f\colon Y \to X$ of degree n^2 for some integer n such that $\varphi = \frac{f^*}{n}$.

Proof. Under the assumption of (1), we can find a d-admissible isomorphism ψ by applying the Lemma 4.4.6 and Remark 4.4.5. It remains to prove that ψ preserves the Hodge structure, which is essentially the same as in [44, Theorem 1].

For (2), we shall suppose the spinor norm $\mathrm{SN}(\varphi) = n_0\mathbb{Q}^{*2} \in \mathbb{Q}^*/\mathbb{Q}^{*2}$. Let $E = \mathbb{Q}(\sqrt{n_0})$. We can see the base-change $\mathrm{H}^2(X,E) \xrightarrow{\sim} \mathrm{H}^2(Y,E)$ is a Hodge isometry with coefficients in E such that $\mathrm{SN}(\varphi) = 1 \in E^*/(E^*)^2$. Then by applying Lemma 4.4.6, we will obtain an admissible (fixing admissible bases for $\mathrm{H}^1(X,\mathbb{Q})$ and $\mathrm{H}^1(Y,\mathbb{Q})$) Hodge isomorphism $\psi\colon \mathrm{H}^1(X,E) \xrightarrow{\sim} \mathrm{H}^1(Y,E)$. Let

$$\sigma\colon a + b\sqrt{n_0} \rightsquigarrow a - b\sqrt{n_0}$$

be the generator of $\mathrm{Gal}(E/\mathbb{Q})$. As we have fixed the $\mathbb{Q}$-linear admissible bases, the wedge map

$$\mathrm{SL}_4(E) \xrightarrow{\wedge^2} \mathrm{SO}(\Lambda)(E)$$

is defined over $\mathbb{Q}$, and so is σ-equivariant. Let g be the element in $\mathrm{SO}(\Lambda_E)$ corresponding to ψ. As $\wedge^2(g) \in \mathrm{SO}(\Lambda) \subset \mathrm{SO}(\Lambda_E)$, we can see

$$(\wedge^2(\sigma(g)) = \sigma(\wedge^2(g)) = \wedge^2(g).$$

which implies that $\sigma(g)g^{-1} = -\mathrm{id}_4$ since $\ker(\wedge^2) = \{\pm \mathrm{id}_4\}$. This means that $g_0 = \sqrt{n_0}g$ is lying in $\mathrm{GL}_4(\mathbb{Q})$. Let

$$\psi_0\colon \mathrm{H}^1(X,\mathbb{Q}) \to \mathrm{H}^1(Y,\mathbb{Q})$$

be the corresponding element of g_0 in $\mathrm{Iso}^{\mathrm{ad},(n_0^2)}\left(\mathrm{H}^1(X,\mathbb{Q}),\mathrm{H}^1(Y,\mathbb{Q})\right)$. As $\wedge^2\psi_0 = n_0\psi$ is a Hodge isogeny, the Shioda's trick then implies that ψ_0 is a Hodge isomorphism as well. Thus ψ_0 lifts to a quasi-isogeny $f_0\colon Y \to X$ and we have

$$\varphi = \wedge^2(\psi) = \frac{f_0^*}{n_0}\colon \mathrm{H}^2(X,\mathbb{Q}) \to \mathrm{H}^2(Y,\mathbb{Q}).$$

□

Remark 4.5.2. If a Hodge isometry $\psi\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})$ is not admissible, i.e., its determinant is -1 with respect to some admissible bases, then we can take its composition with the isometry $\psi_{\mathcal{P}}$ induced by the Poincaré bundle as in Example 4.2.3. After that, we can see $\psi_{\mathcal{P}} \circ \psi$ is admissible and is induced by an isogeny $f\colon \widehat{Y} \to X$.

ℓ-adic and p-adic Shioda's trick. For the integral ℓ-adic étale cohomology, we have the following statement similar to Shioda's trick for integral Betti cohomology.

Proposition 4.5.3 (ℓ-adic Shioda's trick)**.** *Suppose $\ell \neq 2$. For any d-admissible*

$$\varphi_\ell \colon \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell),$$

we can find a d-admissible $\mathbb{Z}_\ell$-isomorphism ψ_ℓ such that $\wedge^2(\psi_\ell) = \varphi_\ell$ or $-\varphi_\ell$. Moreover, if φ_ℓ is G_k-equivariant, then ψ_ℓ is also G_k-equivariant after replacing k by some finite extension.

Proof. As $\mathbb{Z}_\ell^*/(\mathbb{Z}_\ell^*)^2 = \{\pm 1\}$ for any $\ell \neq 2$, the spinor norm of φ_ℓ is equal to ± 1. Thus φ_ℓ or $-\varphi_\ell$ is of spinor norm one. Then the first statement follows from Lemma 4.4.6.

Suppose φ_ℓ is G_k-equivariant. We may assume $\wedge^2(\psi_\ell) = \varphi_\ell$ for simplicity. For any $g \in G_k$, we have

$$\wedge^2(g^{-1}\psi_\ell g) = g^{-1} \wedge^2 (\psi_\ell) g = \varphi_\ell = \wedge^2(\psi_\ell).$$

Therefore, $g^{-1}\psi_\ell g = \pm\psi_\ell$. By passing to a finite extension k'/k, we always have $g^{-1}\psi_\ell g = \psi_\ell$ for all $g \in G_{k'}$ which proves the assertion. □

For F-crystals attached to abelian surfaces, we can also play the Shioda's trick.

Proposition 4.5.4 (p-adic Shioda's trick)**.** *Suppose k is perfect such that* $\mathrm{char}(k) = p > 2$. *For any d-admissible W-linear isomorphism*

$$\varphi_W \colon \mathrm{H}^2_{\text{crys}}(Y/W) \xrightarrow{\sim} \mathrm{H}^2_{\text{crys}}(X/W),$$

we can find an d-admissible W-linear isomorphism $\rho : \mathrm{H}^1_{\text{crys}}(Y/W) \xrightarrow{\sim} \mathrm{H}^1_{\text{crys}}(X/W)$ such that $\wedge^2(\rho) = \varphi_W$ or $-\varphi_W$. Moreover, if φ_W is a morphism of F-crystals, then ρ is an isomorphism of 2^{nd}-iterate of F-crystals.

Proof. The first statement follows from a similar reason as in Proposition 4.5.3.

For the second statement, we assume $\wedge^2(\rho) = \varphi_W$. If φ_W commutes with the Frobenius action, then we have

$$\wedge^2(C_Y^{-1} \cdot \rho^{(1)} \cdot C_X) = \varphi_W.$$

as in §4.3. Thus $C_Y^{-1} \cdot \rho^{(1)} \cdot C_X = \pm\rho$, which implies

$$\rho \circ F_X = \pm F_Y \circ \rho$$

by Lemma 4.3.1. Therefore, ρ commutes with the 2^{nd}-iterate Frobenius F_X^2 and F_Y^2. □

Combined with Tate's isogeny theorem as referred in the introduction, we have the following direct consequences of Propositions 4.5.3 and 4.5.4. It includes a special case of Tate's conjecture.

Corollary 4.5.5. *If $\ell \neq 2$ and k is finitely generated, then*

(1) for any admissible isometry of $\mathrm{Gal}(\bar{k}/k)$*-modules*

$$\varphi_\ell \colon \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell),$$

we can find a $\mathbb{Z}_\ell$-quasi-isogeny $f_\ell \in \mathrm{Hom}_{k'}(X_{k'}, Y_{k'}) \otimes \mathbb{Z}_\ell$ for some finite extension k'/k, which induces φ_ℓ or $-\varphi_\ell$. In particular, φ_ℓ is algebraic.

(2) for any admissible isometry of F-crystals

$$\varphi_W \colon \mathrm{H}^2_{\text{crys}}(Y_{k^{\text{perf}}}/W(k^{\text{perf}})) \xrightarrow{\sim} \mathrm{H}^2_{\text{crys}}(X_{k^{\text{perf}}}/W(k^{\text{perf}})),$$

we can find a $\mathbb{Z}_{p^2}$-quasi-isogeny $f_p \in \mathrm{Hom}_{k'}(X_{k'}, Y_{k'}) \otimes \mathbb{Z}_{p^2}$ which induces φ_W or $-\varphi_W$ for some finite extension k'/k, where $\mathbb{Z}_{p^2} = W(\mathbb{F}_{p^2})$. In particular, φ_W is algebraic.

Proof. For (1), Proposition 4.5.3 implies there is an $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism

$$\psi_\ell \colon \mathrm{H}^1_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell),$$

inducing φ_ℓ or $-\varphi_\ell$ after a finite extension of k. Then f_ℓ exists by the canonical bijection (*cf.* [15, VI, §3 Theorem 1])

$$\mathrm{Hom}_k(X, Y) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_k\left(\mathrm{H}^1_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell), \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)\right).$$

For (2), Proposition 4.5.4 implies that there is an isomorphism

$$\rho\colon \mathrm{H}^1_{\mathrm{crys}}(Y_{k^{\mathrm{perf}}}/W(k^{\mathrm{perf}})) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{crys}}(X_{k^{\mathrm{perf}}}/W(k^{\mathrm{perf}}))$$

such that $F_X \circ \rho = \pm \rho \circ F_Y$.

If $F_X \circ \rho = \rho \circ F_Y$ then it can be conclude by the canonical isomorphisms

$$\mathrm{Hom}_k(X,Y) \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathrm{Hom}_k\left(X[p^\infty], Y[p^\infty]\right) \xrightarrow{\sim} \mathrm{Hom}_{k^{\mathrm{perf}}}\left(X_{k^{\mathrm{perf}}}[p^\infty], Y_{k^{\mathrm{perf}}}[p^\infty]\right), \tag{4.5.1}$$

where the bijectivity of the first arrow is from de Jong's theorem (*cf.* [12, Theorem 2.6]) and the second one is from the faithfully flat descent of p-divisible groups.

It remains to consider the case $F_X \circ \rho = -\rho \circ F_Y$. After taking a finite extension of k, we may assume that $\mathbb{Z}_{p^2} \subset W(k^{\mathrm{perf}})$. Hence there is $\xi \in W(k^{\mathrm{perf}})$ such that $\xi^{p-1} + 1 = 0$. We can see

$$F_X \circ (\xi\rho) = \xi^p F_X \circ \rho = (\xi\rho) \circ F_Y.$$

Again, the bijection (4.5.1) implies that $\xi\rho$ is induced by a $\mathbb{Z}_p$-quasi-isogeny $f_0 \in \mathrm{Hom}_k(X,Y) \otimes \mathbb{Z}_p$. Note that $\xi \in \mathbb{Z}_{p^2}^*$. We can take

$$f_p = \frac{f_0}{\xi} \in \mathrm{Hom}_k(X,Y) \otimes \mathbb{Z}_{p^2}.$$

□

Remark 4.5.6. In [54], Zarhin introduces the notion of *almost isomorphism.* Two abelian varieties over k are called almost isomorphic if their Tate modules T_ℓ are isomorphic as Galois modules (replaced by Tate modules of p-divisible groups when $\ell = p$). Proposition 4.5.3 and 4.5.4 imply that it is possible to characterize almost isomorphic abelian surfaces by their 2^{nd}-cohomology groups.

5. DERIVED ISOGENY IN CHARACTERISTIC ZERO

In this section, we follow [17] and [23] to prove the twisted Torelli theorem for abelian surfaces over algebraic closed fields in characteristic zero.

5.1. Over $\mathbb{C}$: Hodge isogeny versus derived isogeny.

Let X and Y be complex abelian surfaces.

Definition 5.1.1. A rational Hodge isometry $\psi_b\colon \mathrm{H}^2(X,\mathbb{Q}) \to \mathrm{H}^2(Y,\mathbb{Q})$ is called *reflexive* if it is induced by a reflection on Λ along a vector $b \in \Lambda$:

$$\varphi_b\colon \Lambda_{\mathbb{Q}} \xrightarrow{\sim} \Lambda_{\mathbb{Q}} \quad x \mapsto x - \frac{2(x,b)}{(b,b)}b.$$

Lemma 5.1.2. *Any reflexive Hodge isometry ψ_b induces a Hodge isometry on twisted Mukai lattices*

$$\widetilde{\psi}_b\colon \widetilde{\mathrm{H}}(X,\mathbb{Z};B) \to \widetilde{\mathrm{H}}(Y,\mathbb{Z};B'),$$

where $B = \frac{2}{(b,b)}b \in \mathrm{H}^2(X,\mathbb{Q})$ (via some marking $\Lambda \cong \mathrm{H}^2(X,\mathbb{Z})$) and $B' = -\psi_b(B)$.

Proof. The proof can be found in [23, §1.2]. □

In analogy to [23, Theorem 1.1], the following result characterizes the reflexive Hodge isometries between abelian surfaces.

Theorem 5.1.3. *Let X and Y be two complex abelian surfaces. If there is a reflexive Hodge isometry*

$$\psi_b\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q}),$$

for some $b \in \Lambda$, then there exist $\alpha \in \mathrm{Br}(X)$ and $\beta \in \mathrm{Br}(Y)$ such that ψ_b is induced by a derived equivalence

$$\mathrm{D}^b(X,\alpha) \simeq \mathrm{D}^b(Y,\beta).$$

Equivalently, X or $\widehat{X}$ is isomorphic to the coarse moduli space of twisted coherent sheaves on Y, and ψ_b is induced by the twisted Fourier-Mukai transform associated to the universal twisted sheaf.

Proof. According to Lemma 5.1.2, there is a Hodge isometry

$$\widetilde{\psi}_b\colon \widetilde{\mathrm{H}}(X,\mathbb{Z};B) \xrightarrow{\sim} \widetilde{\mathrm{H}}(Y,\mathbb{Z};B').$$

Let $v_{B'}$ be the image of Mukai vector $(0,0,1)$ under $\widetilde{\psi}_b$. From our construction, there is a Mukai vector

$$v = (r,c,\chi) \in \widetilde{\mathrm{H}}(Y,\mathbb{Z})$$

such that $v_{B'} = \exp(B')\cdot v$. We can assume that v is positive by some suitable autoequivalence of $\mathrm{D}^b(Y)$. Let β be the Brauer class on Y with respect to B' and $\mathscr{Y} \to Y$ be the corresponding $\mathbb{G}_m$-gerbe. For some $v_{B'}$-generic polarization H, the moduli stack $\mathscr{M}_H(\mathscr{Y}, v_{B'})$ of β-twisted sheaves on Y with Mukai vector $v_{B'}$ forms a $\mathbb{G}_m$-gerbe on its coarse moduli space $M_H(\mathscr{Y}, v_{B'})$ such that

$$[\mathscr{M}_H(\mathscr{Y}, v_{B'})] \in \mathrm{Br}(M_H(\mathscr{Y}, v_B))[r]$$

(*cf.* [29, Proposition 2.3.3.4, Corollary 2.3.3.7]).

The kernel $\mathscr{P}$ is the tautological twisted sheaf on $\mathscr{Y} \times \mathscr{M}_H(\mathscr{Y}, v_{B'})$ which induces a twisted Fourier-Mukai transform

$$\Phi_{\mathscr{P}}\colon\ \mathrm{D}^b(Y,\beta) \to \mathrm{D}^b(\mathscr{M}_H(\mathscr{Y}, v_{B'})) \simeq \mathrm{D}^b(M_{H'}(\mathscr{Y}, v_{B'}),\alpha),$$

where $\alpha = [\mathscr{M}_H(\mathscr{Y}, v_{B'})] \in \mathrm{Br}(M_{H'}(\mathscr{Y}, v_{B'}))$ (*cf.* [52, Theorem 4.3]). It induces a Hodge isometry

$$\widetilde{\mathrm{H}}(Y,\mathbb{Z};B') \xrightarrow{\sim} \widetilde{\mathrm{H}}(M_H(\mathscr{Y}, v_{B'}),\mathbb{Z};B''),$$

where B'' is a **B**-field lift of α. Its composition with $\widetilde{\psi}_b$ is a Hodge isometry

$$\widetilde{\mathrm{H}}(X,\mathbb{Z};B) \xrightarrow{\sim} \widetilde{\mathrm{H}}(M_H(\mathscr{Y}, v_{B'}),\mathbb{Z};B''), \tag{5.1.1}$$

sending the Mukai vector $(0,0,1)$ to $(0,0,1)$ and preserving the Mukai pairing. We can see $(1,0,0)$ is mapping to $(1,b,\frac{b^2}{2})$ for some $b \in \mathrm{H}^2(Y,\mathbb{Z})$ via (5.1.1). Thus we can replace B'' by $B''+b$, which will not change the corresponding Brauer class, to obtain a Hodge isometry which takes $(1,0,0)$ to $(1,0,0)$ and $(0,0,1)$ to $(0,0,1)$ at the same time. This yields a Hodge isometry

$$\mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(M_{H'}(\mathscr{Y}, v_{B'}),\mathbb{Z}).$$

Then we can apply Shioda's Torelli Theorem of abelian surfaces [44] to conclude that

$$M_{H'}(\mathscr{Y}, v_{B'}) \cong X \text{ or } \widehat{X}.$$

When $X \cong M_{H'}(\mathscr{Y}, v_{B'})$, $\Phi_{\mathscr{P}}$ gives the derived equivalence as desired. When $\widehat{X} \cong M_{H'}(\mathscr{Y}, v_{B'})$, we can prove the assertion by using the fact X and $\widehat{X}$ are derived equivalent. □

Next, we are going to show that any rational Hodge isometry can be decomposed into a chain of reflexive Hodge isometries. This is a special case of Cartan-Dieudonné theorem which says that any element $\varphi \in \mathrm{SO}(\Lambda_{\mathbb{Q}})$ can be decomposed as products of reflections:

$$\varphi = \varphi_{b_1} \circ \varphi_{b_2} \circ \cdots \circ \varphi_{b_n}, \tag{5.1.2}$$

such that $b_i \in \Lambda$, and $(b_i)^2 \neq 0$. Then from the surjectivity of period map [44, Theorem ii], for any rational Hodge isometry

$$\mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q}),$$

we can find a sequence of abelian surfaces $\{X_i\}$ and Hodge isometries $\psi_{b_i}\colon \mathrm{H}^2(X_{i-1},\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_i,\mathbb{Q})$, where $X_0 = X$ and $X_n = Y$, such that ψ_{b_i} induces φ_{b_i} on $\Lambda_{\mathbb{Q}}$. We can arrange them as (1.0.1):

$$\begin{array}{llll}
\mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\psi_{b_1}} \mathrm{H}^2(X_1,\mathbb{Q}) & & & \\
& \mathrm{H}^2(X_1,\mathbb{Q}) \xrightarrow{\psi_{b_2}} \mathrm{H}^2(X_2,\mathbb{Q}) & & \\
& & \vdots & \\
& & & \mathrm{H}^2(X_{n-1},\mathbb{Q}) \xrightarrow{\psi_{b_n}} \mathrm{H}^2(Y,\mathbb{Q}).
\end{array} \tag{5.1.3}$$

Finally, this yields

Corollary 5.1.4. *If there is a rational Hodge isometry $\varphi\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})$, then there is a derived isogeny from X to Y, whose Hodge realization is φ.*

As a consequence, we get

Corollary 5.1.5. *There is a rational Hodge isometry $\mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})$ if and only if there is a derived isogeny from $\mathrm{Km}(X)$ to $\mathrm{Km}(Y)$.*

Proof. Witt's cancellation theorem implies that

$$\mathrm{H}^2(X,\mathbb{Q}) \simeq \mathrm{H}^2(Y,\mathbb{Q}) \Leftrightarrow T(X)_{\mathbb{Q}} \simeq T(Y)_{\mathbb{Q}},$$

as Hodge isometries. According to [23, Theorem 0.1], $\mathrm{Km}(X)$ is derived isogenous to $\mathrm{Km}(Y)$ if and only if there is a Hodge isometry $T(\mathrm{Km}(X))_{\mathbb{Q}} \simeq T(\mathrm{Km}(Y))_{\mathbb{Q}}$. Then the statement is clear from the fact there is a canonical integral Hodge isometry $T(X)(2) \simeq T(\mathrm{Km}(X))$ (*cf.* [35, Proposition 4.3(i)]). □

Remark 5.1.6. A consequence of Theorem 5.1.4 is that any rational Hodge isometry between abelian surfaces is algebraic, which is a special case of Hodge conjecture on product of two abelian surface. Unlike the case of K3 surfaces, the Hodge conjecture for product of abelian surfaces were known for a long time. See [42, Theorem 3.15] for example.

Moreover, we may call a reflexive Hodge isometry

$$\psi_b\colon \mathrm{H}^2(X_{\mathbb{C}},\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y_{\mathbb{C}},\mathbb{Q})$$

induced by a primitive vector $b \in \Lambda$ *prime-to-ℓ* if $\ell \nmid n = \frac{(b)^2}{2}$. The following results imply that the Hodge realization of prime-to-ℓ derived isogeny is a composition of finitely many prime-to-ℓ reflexive Hodge isometries.

Lemma 5.1.7. *If the induced derived isogeny $\mathrm{D}^b(X_{\mathbb{C}}) \sim \mathrm{D}^b(Y_{\mathbb{C}})$ in Corollary 5.1.4 is prime-to-ℓ, then each reflexive Hodge isometry ψ_b in (5.1.3) is prime-to-ℓ.*

Proof. Otherwise, we can take ℓ^k to be the ℓ-factor of n. As ψ_b restricts to an isomorphism

$$\mathrm{H}^2(X_{\mathbb{C}},\mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} \mathrm{H}^2(Y_{\mathbb{C}},\mathbb{Z}) \otimes \mathbb{Z}_{(\ell)},$$

we have $\ell^k \mid (x,b)$ for any $x \in \Lambda$. This means ℓ^k divides the divisibility of b, which is impossible as Λ is unimodular. □

Remark 5.1.8. With notations in Theorem 5.1.3, if ψ_b is prime-to-ℓ with $n = \frac{b^2}{2}$, then the Fourier-Mukai equivalence $\mathrm{D}^b(X,\alpha) \xrightarrow{\sim} \mathrm{D}^b(Y,\beta)$ in Theorem 5.1.3 satisfies

$$\alpha^n = \exp(nB) = 1 \in \mathrm{Br}(X),$$

which implies $\alpha \in \mathrm{Br}(X)[n]$. Similarly, n divides the order of $\beta = \exp(B') \in \mathrm{Br}(Y)$.

5.2. **Isogeny versus derived isogeny.** Let us now describe derived isogenies via suitable isogenies.

Recall that the isogeny category of abelian varieties $\mathtt{AV}_{\mathbb{Q},k}$ consists of all abelian varieties over a field k as objects, and the Hom-sets are

$$\mathrm{Hom}_{\mathtt{AV}_{\mathbb{Q},k}}(X,Y) \coloneqq \mathrm{Hom}_{\mathtt{AV}_k}(X,Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We may also denote by $\mathrm{Hom}^0(X,Y)$ for $\mathrm{Hom}_{\mathtt{AV}_{\mathbb{Q},k}}(X,Y)$ if there are no confusion on the defining field k. An isomorphism f from X to Y in the isogeny category $\mathtt{AV}_{\mathbb{Q},k}$ is called a quasi-isogeny from X to Y. For any quasi-isogeny f, we can find a minimal integer n such that

$$nf\colon X \to Y$$

is an isogeny, i.e., a finite surjective morphism of abelian varieties. When $k = \mathbb{C}$, with the uniformization of complex abelian varieties, we have a canonical bijection

$$\mathrm{Hom}_{\mathtt{AV}_{\mathbb{Q},\mathbb{C}}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Hdg}}\left(\mathrm{H}^1(Y,\mathbb{Q}),\mathrm{H}^1(X,\mathbb{Q})\right),$$

where the right-hand side is the set of $\mathbb{Q}$-linear morphisms preserving Hodge structures. Then the integer n for f is also the minimal integer such that $(nf)^*(\mathrm{H}^1(Y,\mathbb{Z})) \subseteq \mathrm{H}^1(X,\mathbb{Z})$.

It is well-known that the Hom-sheaf $\underline{\mathrm{Hom}}(X,Y)$ is representable by an étale group scheme over k. Therefore, via Galois descent, we have

$$\mathrm{Hom}_{\mathtt{AV}_{\bar{k}}}(X_{\bar{k}}, Y_{\bar{k}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathtt{AV}_{\bar{K}}}(X_{\bar{K}}, Y_{\bar{K}}), \tag{5.2.1}$$

for any algebraically closed field $\overline{K} \supset k$. A similar statement holds for derived isogenies.

Lemma 5.2.1. *Let X and Y are abelian surfaces defined over k with $\mathrm{char}(k) = 0$. Let $\overline{K} \supseteq k$ be an algebraically closed field containing k. Let $\bar{k}$ be the algebraically closure of k in $\bar{K}$. Then if $X_{\bar{K}}$ and $Y_{\bar{K}}$ are twisted derived equivalent, so is $X_{\bar{k}}$ and $Y_{\bar{k}}$.*

Proof. As $X_{\bar{K}}$ is twisted derived equivalent to $Y_{\bar{K}}$, by Theorem 3.4.2, there exist finitely many abelian surfaces $X_0, X_1, \ldots, X_n$ defined over $\bar{K}$ with $X_0 = X_{\bar{K}}$ and

$$X_i \text{ or } \widehat{X_i} = M_{H_i}(\mathscr{X}_{i-1}, v_i) \quad Y_{\bar{K}} \text{ or } \widehat{Y}_{\bar{K}} \cong M_{H_n}(\mathscr{X}_n, v_n)$$

for some $[\mathscr{X}_{i-1}] \in \mathrm{Br}(X_{i-1})$. Let us construct abelian surfaces over $\bar{k}$ to connect $X_{\bar{k}}$ and $Y_{\bar{k}}$ as follows:

Set $X_0' = X_{\bar{k}}$, then we take $X_1' = M_{H_1'}(\mathscr{X}_0', v_1')$ where $\mathscr{X}_0', H_1'$ and v_1' are the descent of $\mathscr{X}_0, H_1$ and v via the isomorphisms $\mathrm{Br}(X_{\bar{K}}) \cong \mathrm{Br}(X_{\bar{k}})$, $\mathrm{Pic}(X_K) \cong \mathrm{Pic}(X_{\bar{k}})$ and $\widetilde{\mathrm{H}}(X_K) \cong \widetilde{\mathrm{H}}(X_{\bar{k}})$. Then inductively, we can define X_i' as the moduli space of twisted sheaves $M_{H_i'}(\mathscr{X}_{i-1}', v_i')$ (or its dual respectively) over $\bar{k}$. Note that we have natural isomorphisms

$$(M_{H_i'}(\mathscr{X}_{i-1}', v_i'))_{\bar{K}} \cong M_{H_i}(\mathscr{X}_{i-1}, v_i)$$

over $\bar{K}$. In particular, $(M_{H_i'}(\mathscr{X}_n', v_i'))_{\bar{K}} \cong Y_{\bar{K}}$. It follows that $M_{H_i'}(\mathscr{X}_n', v_i') \cong Y_{\bar{k}}$. □

More generally, we can replace $\mathbb{Q}$ in $\mathtt{AV}_{\mathbb{Q},k}$ by any ring R. Then any isomorphism from X to Y in $\mathtt{AV}_{R,k}$ will be called a *R-(quasi)-isogeny*. In particular,

Definition 5.2.2. An element $f \in \mathrm{Hom}_k(X,Y) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}$ which has an inverse in $\mathrm{Hom}_k(Y,X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}$ is called a prime-to-ℓ quasi-isogeny, where $\mathbb{Z}_{(\ell)}$ is the localization of $\mathbb{Z}$ at (ℓ).

For any abelian surface $X_{\mathbb{C}}$ over $\mathbb{C}$, a standard argument shows that there is a finitely generated field $k \subset \mathbb{C}$ and an abelian surface X over k such that $X \times_k \mathbb{C} \cong X_{\mathbb{C}}$. We have the following Artin comparison

$$\mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \cong \mathrm{H}^i(X_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell, \tag{5.2.2}$$

for any $i \in \mathbb{Z}$ and ℓ a prime. Suppose Y is another abelian surface defined over k. Suppose $f \colon Y_{\mathbb{C}} \to X_{\mathbb{C}}$ is a prime-to-ℓ quasi-isogeny. By definition, it induces an isomorphism of $\mathbb{Z}_{(\ell)}$-modules

$$f^* \colon \mathrm{H}^1(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} \mathrm{H}^1(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)},$$

such that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{H}^i(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \arrow[d, hook] \\
\mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \arrow[r, "\sim"] & \mathrm{H}^i_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell)
\end{tikzcd}$$

for any i, under the comparison (5.2.2). For the converse, we have the following simple fact.

Lemma 5.2.3. *A (quasi)-isogeny $f \colon Y_{\mathbb{C}} \to X_{\mathbb{C}}$ is prime-to-ℓ if and only it induces an integral ℓ-adic realization*

$$f^* \colon \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell).$$

Inspired by Shioda's trick for Hodge isogenies 4.5.1, we introduce the following notions.

Definition 5.2.4. Let X and Y be g-dimensional abelian varieties over k. We say X and Y are (prime-to-ℓ) *principally isogenous* if there is a (prime-to-ℓ) isogeny f from X or $\widehat{X}$ to Y of square degree, i.e., $\deg(f) = d^2$ for some $d \in \mathbb{Z}$. In this case, we may call f a *principal isogeny*.

Furthermore, we say f is *quasi-liftable* if f can be written as the composition of finitely many liftable isogenies.

Now, we can state the main result in this section.

Theorem 5.2.5. *Suppose* $\mathrm{char}(k) = 0$. *The following statements are equivalent:*

(1) X is (prime-to-ℓ) principally isogenous to Y over $\bar{k}$.
(2) X and Y are (prime-to-ℓ) derived isogenous over $\bar{k}$.

Proof. (1) $\Rightarrow$ (2): we can assume that $f : X \to Y$ is a principal isogeny defined over a finitely generated field k'. By embedding k' into $\mathbb{C}$, two complex abelian surfaces $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are derived isogenous since there is a rational Hodge isometry

$$\frac{1}{n} f^* \otimes \mathbb{Q} : \mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Q} \cong \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Q}$$

where $\deg(f) = n^2$. By Lemma 5.2.1, one can conclude $X_{\bar{k}}$ and $Y_{\bar{k}}$ are derived isogenous, with the rational Hodge realization $\frac{1}{n} f^* \otimes \mathbb{Q}$.

If f is a prime-to-ℓ isogeny, the map $\frac{1}{n} f^*$ restricts to an isomorphism

$$\mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)}.$$

Then one can take a prime-to-ℓ Cartan-Dieudonné decomposition (see Lemma 5.2.6 below), which decomposes $\frac{1}{n} f^* \otimes \mathbb{Q}$ into a sequence of prime-to-ℓ reflexive Hodge isometries. The assertion follows immediately.

To deduce (2) $\Rightarrow$ (1), we may assume X and Y are derived isogenous over a finitely generated field k'. Embed k' into $\mathbb{C}$, $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are derived isogenous as well. According to Proposistion 4.5.1, they are principally isogenous over $\mathbb{C}$. It follows that X and Y are principally isogenous over $\bar{k}$ by the bijection (5.2.1).

If $\mathrm{D}^b(X) \sim \mathrm{D}^b(Y)$ is prime-to-ℓ, then each reflexive Hodge isometry ψ_b in (5.1.3) is prime-to-ℓ by Lemma 5.1.7. The principal isogeny which induces ψ_b is prime-to-ℓ by Lemma 5.2.3. This proves the assertion. $\square$

Lemma 5.2.6 (prime-to-ℓ Cartan-Dieudonné decomposition)**.** *Let Λ be an integral lattice over $\mathbb{Z}$. Any orthogonal matrix $A \in \mathrm{O}(\Lambda)(\mathbb{Z}_{(\ell)}) \subset \mathrm{O}(\Lambda)(\mathbb{Q})$, with $(\ell > 2)$, can be decomposed into a sequence of prime-to-ℓ reflections.*

Proof. To prove the assertion, we will follow the proof of [43] to refine Cartan-Dieudonné decomposition for any field. In general, if Λ_k is quadratic space over a field k with the Gram matrix G. Let I be the identity matrix and let R_b be the reflection with respect to $b \in \Lambda_k$. The proof of Cartan-Dieudonné decomposition in [43] relies on the following facts: for any element $A \in \mathrm{O}(\Lambda_k)$, we have

i) A is a reflection if $\mathrm{rank}(A - I) = 1$ (*cf.* [43, Lemma 2])
ii) if $S = G(A - I)$ is not skew symmetric and $a \in \Lambda$ satisfying $a^t S a \neq 0$ and

$$S + S^t \neq \frac{1}{a^t S a}(Sb \cdot b^t S + S^t b \cdot b^t S^t),$$

then $\mathrm{rank}(AR_b - I) = \mathrm{rank}(A - I) - 1$ and $G(AR_b - I)$ is not skew symmetric with $b = (A - I)a$ satisfying $b^2 = -2a^t S a$. Such a always exists. (*cf.* [43, Lemma 4, Lemma 5]).
iii) if $S = G(A - I)$ is skew symmetric, then there exists $b \in \Lambda$ such that $G(AR_b - I)$ is not skew symmetric (*cf.* [43, Theorem 2]).

Then one can decompose A as a series of reflections by repeatedly using ii). In our case, it suffices to show that if A is coprime to ℓ, i.e. nA is integral for some n coprime to ℓ, then

i') A is a corpime to ℓ reflection if $\mathrm{rank}(A - I) = 1$;
ii') if $S = G(A - I)$ is not skew symmetric and there exists $a \in \Lambda$ satisfying $p \nmid a^t S a$ and

$$S + S^t \neq \frac{1}{a^t S a}(Sb \cdot b^t S + S^t b \cdot b^t S^t),$$

such that AR_b is coprime to ℓ and $G(AR_b - I)$ is not skew symmetric with b constructed above;

iii') if $S = G(A - I)$ is skew symmetric, then there exists $b \in \Lambda$ such that AR_b is coprime to ℓ and $G(AR_b - I)$ is not skew symmetric.

This means that we only need to find some prime-to-ℓ reflections satisfying the conditions as above. By our assumption, the modulo ℓ reduction $\Lambda_{\mathbb{F}_\ell}$ of Λ remains non-degenerate. If A is coprime to ℓ, then we can consider the reduction $A \mod \ell$ and apply i)-iii) to $A \mod \ell \in \mathrm{O}(\Lambda_{\mathbb{F}_\ell})$ to obtain reflections over $\mathbb{F}_\ell$. We can lift the reflections to $\mathrm{O}(\Lambda_{\mathbb{Q}})$, which are obviously coprime to ℓ. One can easily check such reflections satisfy ii') and iii'). $\square$

Remark 5.2.7. We shall note that under the explicit construction in Theorem 5.1.3, the corresponding Fourier-Mukai transform $\mathrm{D}^b(X, \alpha) \xrightarrow{\simeq} \mathrm{D}^b(Y, \beta)$ of Ψ_b satisfies

$$\alpha^n = \exp(nB) = \exp(b) = 1 \in \mathrm{Br}(X).$$

It follows that $\alpha \in \mathrm{Br}(X)[n]$. Similarly, we have $\beta = \exp(B') \in \mathrm{Br}(Y)[n]$. Therefore, if $\mathrm{D}^b(X) \sim \mathrm{D}^b(Y)$ is prime-to-ℓ, then we may assume α and β are both prime-to-ℓ Brauer classes.

5.3. **Proof of Theorem 1.0.1.** Let us summarize all the results which conclude Theorem 1.0.1. By a similar argument in Theorem 5.2.5, we can reduce them to the case $k = \mathbb{C}$.

Proof of (i) $\Leftrightarrow$ (ii). This is Theorem 5.2.5.

Proof of (ii) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii). This follows from Corollary 5.1.4 and Witt cancellation.

Proof of (ii) $\Leftrightarrow$ (iii). This is Corollary 5.1.5.

Proof of (i) $\Rightarrow$ (iv) $\Rightarrow$ (v). This is obvious. See [17, Proposition 4.6] for example.

Proof of (v) $\Rightarrow$ (i). Let $\Gamma\colon \mathfrak{h}^{even}(X) \xrightarrow{\sim} \mathfrak{h}^{even}(Y)$ be the isomorphism of Frobenius algebra objects. The Betti realization of its second component is a Hodge isometry by the Frobenius condition. Thus X and Y are derived isogenous, and hence principally isogenous.

6. Isogeny over positive characteristic fields

In this section, we will prove the twisted derived Torelli theorem for abelian surfaces over odd characteristic fields.

6.1. **Prime-to-p derived isogeny in mixed characteristic.** Let us start with an important lemma for prime-to-p derived isogenies.

Lemma 6.1.1. *Let K be a complete discrete valuated field in characteristic zero, whose residue field is perfect with characteristic p. Denote by $\mathcal{O}_K$ the ring of integers. Let $\mathfrak{X} \to \mathcal{X}$ and $\mathfrak{Y} \to \mathcal{Y}$ be twisted abelian surfaces over $\mathcal{O}_K$ whose special fibers are $\mathscr{X}_0 \to X_0$ and $\mathscr{Y}_0 \to Y_0$, and generic fibers are $\mathfrak{X}_K \to \mathcal{X}_K$ and $\mathfrak{Y}_K \to \mathcal{Y}_K$. Suppose $f_0\colon \mathrm{D}^b(\mathscr{X}_0) \xrightarrow{\sim} \mathrm{D}^b(\mathscr{Y}_0)$ is a prime-to-p derived equivalence and $f\colon \mathrm{D}^b(\mathfrak{X}) \xrightarrow{\sim} \mathrm{D}^b(\mathfrak{Y})$ is a lifting of f_0, then $f_K\colon \mathrm{D}^b(\mathfrak{X}_K) \xrightarrow{\sim} \mathrm{D}^b(\mathfrak{Y}_K)$ is also prime-to-p.*

Proof. It suffices to prove that the p-adic realization of f_K is integral. This can be deduced from an argument from the integral p-adic Hodge theory, as mentioned in the proof of [8, Theorem 1.4].

Let us sketch the proof. As f_0 is prime-to-p, its cohomological realization restricts to an isometry of F-crystals

$$\varphi_p\colon \mathrm{H}^2_{\mathrm{crys}}(X_0/W) \simeq \mathrm{H}^2_{\mathrm{crys}}(Y_0/W)$$

by our definition. The base-extension $\varphi_p \otimes K$ can be identified with the cohomological realization of f_K on the de Rham cohomology

$$\varphi_K\colon \mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}_K/K) \simeq \mathrm{H}^2_{\mathrm{dR}}(\mathcal{Y}_K/K)$$

by Berthelot-Ogus comparison. It also preserves the Hodge filtration. Let S be the divided power envelope of pair $(W[\![u]\!], \ker(W[\![u]\!] \to \mathcal{O}_K))$. Then the map

$$(\varphi_p, \varphi_K) \otimes_W S\colon \mathrm{H}^2_{\mathrm{crys}}(X_0/S) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{crys}}(Y_0/S)$$

is an isomorphism of strongly divisible S-lattices in the sense of Breuil (see [9, §4] for example). One can apply functor $\mathrm{Fil}^0(-\otimes_S A_{\mathrm{crys}})^{\varphi=1}$ on it to see that φ_K restricts to an $\mathbb{Z}_p$-integral $\mathrm{Gal}(\bar{K}/K)$-equivariant isometry

$$\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(\mathcal{Y}_{\bar{K}}, \mathbb{Z}_p)$$

(*cf.* [9, Theorem 5.2]). □

6.2. **Serre–Tate theory and lifting of prime-to-p quasi-isogeny.** The Serre–Tate theorem says that the deformation theory of an abelian scheme in characteristic p is equivalent to the deformation theory of its p-divisible group (*cf.* [32, Chapter V.§2, Theorem 2.3]). Let $S_0 \hookrightarrow S$ be an infinitesimal thickening of schemes such that p is locally nilpotent on S. Let $\mathcal{D}(S_0, S)$ be the category of pairs $(\mathcal{X}_0, \mathcal{G})$, where $\mathcal{X}_0$ is an abelian scheme over S_0 and $\mathcal{G}$ is a lifting of p-divisible group $\mathcal{X}_0[p^\infty]$ to S. There is an equivalence of categories

$$\begin{aligned} \{\text{abelain schemes over } S\} &\xrightarrow{\sim} \mathcal{D}(S_0, S) \\ \mathcal{X} &\mapsto (\mathcal{X}\times_S S_0, \mathcal{X}[p^\infty]). \end{aligned}$$

Now we set $S_0 = \mathrm{Spec}(k)$ and $S = \mathrm{Spec}(V/(\pi^{n+1}))$ for a perfect field k, V is a totally ramified finite extension of $W(k)$ and an integer $n \geq 1$. Since there is an equivalence between the category of p-divisible groups over V and the category of inductive systems of p-divisible groups over $V/(\pi^n)$, we have an identification

$$\mathcal{D}(k, V) = \varprojlim_n \mathcal{D}(k, V/(\pi^n)).$$

As a consequence, we get

Lemma 6.2.1. *There is an equivalence of categories*

$$\begin{aligned} \{\textit{formal abelain schemes over } V\} &\xrightarrow{\sim} \mathcal{D}(k, V) \\ A &\mapsto (A\times_V k, A[p^\infty]) . \end{aligned}$$

One can lift separable isogenies between abelian surfaces, which is well-known to experts.

Proposition 6.2.2. *Suppose $p > 2$. Let $f\colon X \to Y$ be a separable isogeny. There are liftings $\mathcal{X} \to \mathrm{Spec}(V)$ and $\mathcal{Y} \to \mathrm{Spec}(V)$ of X and Y respectively, such that the isogeny f can be lifted to an isogeny $f_V\colon \mathcal{X} \to \mathcal{Y}$. In particular, every prime-to-p isogeny is liftable.*

Proof. Suppose we are given a lifting

$$\widetilde{f}[p^\infty]\colon \mathcal{G}_X \to \mathcal{G}_Y$$

of the isogeny of p-divisible groups $f[p^\infty]\colon X[p^\infty] \to Y[p^\infty]$. Then we can apply Lemma 6.2.1 to get a formal lifting of f to $\mathrm{Spec}(V)$:

$$\widetilde{f}\colon \mathcal{X} \to \mathcal{Y},$$

such that $\widetilde{f}$ is finite and $\mathcal{Y}$ admits an algebraization. It suffices to show $\widetilde{f}$ also admits an algebraization, which can be done by [19, Proposition (5.4.4)].

The required lifting of p-divisible groups can be constructed as follows. Since $f[p^\infty]$ is separable, its kernel is a finite étale group scheme, which is freely liftable. Assume that $f[p^\infty]$ is an isomorphism. Then the statement follows from a standard inductive construction via Grothendieck–Messing theory. □

6.3. **Specialization of derived isogenies.** Next, we shall show that prime-to-p geometric derived isogenies are preserved under reduction.

Theorem 6.3.1. *Let V be a discrete valuation ring with residue field k and let η be its generic point. Assume that $\mathrm{char}(k) = p > 2$. Let $\mathcal{X} \to \mathrm{Spec}(V)$ and $\mathcal{Y} \to \mathrm{Spec}(V)$ be two projective families of abelian surfaces or K3 surfaces over $\mathrm{Spec}(V)$. If there is a derived equivalence*

$$\Psi^{\mathcal{P}}\colon\ \mathrm{D}^b(X_{\bar{\eta}}, \alpha) \xrightarrow{\sim} \mathrm{D}^b(Y_{\bar{\eta}}, \beta)$$

between geometric generic fibers such that $\mathrm{ord}(\alpha)$ *and* $\mathrm{ord}(\beta)$ *are prime-to-p, then their special fibers are twisted derived equivalent.*

Proof. We denote by X_0 and Y_0 the geometric special fibers of $\mathcal{X}/V$ and $\mathcal{Y}/V$ respectively. It is known that there is an isomorphism

$$\mathcal{Y}_{\bar{\eta}} \cong M^{\alpha}_{\mathcal{H}_\eta}(\mathcal{X}_{\bar{\eta}}, v_{\bar{\eta}}),$$

for some twisted Mukai vector $v_{\bar{\eta}} \in \widetilde{\mathrm{N}}(\mathcal{X}_{\bar{\eta}}, \alpha)$. Up to taking a finite extension of V, we may assume that α can be defined over η.

We claim that one can lift α to a class in $\mathrm{Br}(\mathcal{X})$ if $p \nmid \mathrm{ord}(\alpha)$. For simplicity, we assume $d = \ell^n$ for some prime ℓ. In this case, the Gysin's sequence and Gabber's absolute purity gives an exact sequence

$$0 \to \mathrm{Br}(\mathcal{X})\{\ell\} \to \mathrm{Br}(\mathcal{X}_\eta)\{\ell\} \to \varinjlim_n \mathrm{H}^1_{\text{ét}}(X_0, \mathbb{Z}/\ell^n). \tag{6.3.1}$$

(*cf.* [10, Theorem 3.7.1 (ii)]). If $\mathcal{X}$ is a K3 surface, then we have $\mathrm{H}^1_{\text{ét}}(X_0, \mathbb{Z}/\ell^n) = 0$ for all n, and thus one can find a lift $\tilde{\alpha} \in \mathrm{Br}(\mathcal{X})$ of α by (6.3.1). When $\mathcal{X}$ is an abelian surface over $\mathrm{Spec}(V)$, the Gysin sequence can not directly give the existence of liftings of α. Again, one can use the trick of Kummer surfaces. Consider the relative Kummer surface $\mathrm{Km}(\mathcal{X}) \to \mathrm{Spec}(V)$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Br}(\mathrm{Km}(\mathcal{X}))\{\ell\} \arrow[r, "\sim"] \arrow[d, "\Theta"] & \mathrm{Br}(\mathrm{Km}(\mathcal{X}_\eta))\{\ell\} \arrow[d, "\Theta_\eta"] \\
\mathrm{Br}(\mathcal{X})\{\ell\} \arrow[r, hook] & \mathrm{Br}(\mathcal{X}_\eta)\{\ell\}
\end{tikzcd}$$

from Proposition 2.2.1. After passing to a finite extension, we can assume α lies in the image of Θ_η. As the top arrow is surjective and Θ_η is an isomorphism, we may find a lift $\widetilde{\alpha} \in \mathrm{Br}(\mathcal{X})\{\ell\}$ whose restriction on $\mathcal{X}_\eta$ is α.

Now, we can pick liftings $v \in \widetilde{\mathrm{N}}(\mathcal{X})$ and $\mathcal{H} \in \mathrm{H}^0(\mathcal{X}, \mathrm{Pic}_{\mathcal{X}/V})$ so that $v|_{\mathcal{X}_\eta} = v_\eta$ and $\mathcal{H}|_{\mathcal{X}_\eta} = \mathcal{H}_\eta$. Then we let $M^{\widetilde{\alpha}}_{\mathcal{H}}(\mathcal{X}, v)$ be the corresponding relative moduli space of twisted sheaves and set $\alpha_0 = \widetilde{\alpha}|_{X_0} \in \mathrm{Br}(X_0)$. The generic fiber of $M^{\widetilde{\alpha}}_{\mathcal{H}}(\mathcal{X}, v) \to \mathrm{Spec}(V)$ is isomorphic to $M^{\alpha}_{\mathcal{H}_\eta}(\mathcal{X}_\eta, v_\eta)$ after a finite base extension. Note that its special fiber is also isomorphic to $M^{\alpha_0}_{H_0}(X_0, v_0)$ after some finite field extension, we have the following commutative diagram after taking a finite ring extension of V:

$$\begin{tikzcd}
M^{\widetilde{\alpha}}_{\mathcal{H}}(\mathcal{X}, v) \arrow[d] & M^{\alpha}_{\mathcal{H}_\eta}(\mathcal{X}_\eta, v_\eta) \arrow[l] \arrow[r, "\cong"] \arrow[d] & \mathcal{Y}_\eta \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\mathrm{Spec}(V) & \mathrm{Spec}(k(\eta)) \arrow[l] \arrow[r] & \mathrm{Spec}(k(\eta)) \arrow[r] & \mathrm{Spec}(V)
\end{tikzcd}$$

According to Matsusaka-Mumford [31, Theorem 1], the isomorphism can be extended to the special fiber. Thus Y_0 is isomorphic to $M^{\alpha_0}_{H_0}(X_0, v_0)$. It follows that $\mathrm{D}^b(X_0, \alpha_0) \simeq \mathrm{D}^b(Y_0, \beta_0)$. □

Using Remark 5.1.8, one can easily deduce that every prime-to-p derived isogeny can be specialized.

Remark 6.3.2. Our proof fails when k is imperfect and the twisted derived equivalence is not prime-to-p. This is because if the associated Brauer class α has order p^n, the map

$$\mathrm{Br}(\mathcal{X})[p^n] \to \mathrm{Br}(\mathcal{X}_\eta)[p^n]$$

may not be surjective (*cf.* [41, 6.8.2]).

6.4. **Proof of Theorem 1.0.3.** When X or Y is supersingular, the assertion will be will be proved in Proposition 6.6.3. So we can assume that X and Y both have finite height.

Proof of $(i') \Rightarrow (ii')$. With p-adic Shioda's trick, we can conclude that X and Y are prime-to-p isogenous. It remains to see they are principally isogenous. The easiest way for proving this is by lifting to characteristic 0.

As the composition of prime-to-p isogenies remains prime-to-p, it suffices to consider a single derived equivalence

$$\mathrm{D}^b(\mathscr{X}) = \mathrm{D}^b(X,\alpha) \simeq \mathrm{D}^b(Y,\beta) = \mathrm{D}^b(\mathscr{Y})$$

for some gerbes $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ satisfying that the orders of $\alpha = [\mathscr{X}]$ and $\beta = [\mathscr{Y}]$ are both prime-to-p. As indicated in the proof of Proposition 3.1.1, we can lift the derived equivalence between $\mathscr{X}$ and $\mathscr{Y}$ to some finite extension W' of $W = W(k)$, i.e., there exists $\mathfrak{X}/W'$ and $\mathfrak{Y}/W'$ as liftings of $\mathscr{X}$ and $\mathscr{Y}$ such that $\mathfrak{X}$ and $\mathfrak{Y}$ are derived equivalent, which is prime-to-p by Lemma 6.1.1. By Theorem 1.0.1, the generic fibers of the $\mathcal{X}$ and $\mathcal{Y}$ are geometrically prime-to-p principally isogenous. Then we can conclude the special fibers are prime-to-p principally isogenous by using Tate's spreading theorem (*cf.* [48, Theorem 4]).

Proof of $(ii') \Rightarrow (i')$. Suppose that we are given an isogeny $\phi\colon Y \to X$, which is prime-to-p of degree d^2. By Proposition 6.2.2, we can lift it to a prime-to-p isogeny of degree d^2 over some finite flat extension V of W:

$$\Phi\colon \mathcal{Y} \to \mathcal{X}.$$

The isogeny Φ_K between the generic fibers induces a G_K-equivariant isometry

$$\frac{\Phi_K^*}{d}\colon \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{K}},\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(\mathcal{Y}_{\bar{K}},\mathbb{Z}_p).$$

By Theorem 5.2.5, there exists a geometric prime-to-p derived isogeny $\mathrm{D}^b(\mathcal{X}_K) \sim \mathrm{D}^b(\mathcal{Y}_K)$ whose p-adic cohomological realization is $\frac{\Phi_K^*}{d}$. The assertion then follows from Theorem 6.3.1.

6.5. **Further remarks.** From the proof of Theorem 1.0.3 $(i') \Rightarrow (ii')$, we can see that the lifting-specialization argument also works for non prime-to-p derived isogenies. Thus we have

Theorem 6.5.1. *Suppose X and Y are abelian surfaces over $\bar{k}$ with finite height and* $\mathrm{char}(k) \neq 2$. *If X and Y are derived isogenous, then they are quasi-liftable principal isogenous.*

Moreover, we believe that the converse of Theorem 6.5.1 also holds.

Conjecture 6.5.1. *With the same assumptions in Proposition 6.5.1. Then X and Y are derived isogenous if and only if they are quasi-liftable isogenous.*

Our approach remains valid once there is a specialization theorem for non prime-to-p derived isogenies. According to the proof of Theorem 6.3.1, it suffices to know the existence of specialization of Brauer classes of order p. Following the notations in Theorem 6.3.1, this means that the restriction map $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_\eta)$ is surjective. See Remark 6.3.2.

Now we discuss the connections between the derived isogenies of abelian surfaces and their associated Kummer surfaces. Using the lifting argument, the following theorem is an immediate consequence of the known result in characteristic 0.

Theorem 6.5.2. *With the assumption as in Theorem 6.5.1. If X and Y are prime-to-p derived isogenous, then the associated Kummer surfaces* $\mathrm{Km}(X)$ *and* $\mathrm{Km}(Y)$ *are prime-to-p derived isogenous. Moreover, if two twisted surfaces* $(\mathrm{Km}(X),\alpha)$ *and* $(\mathrm{Km}(Y),\beta)$ *are derived equivalent with* $p \nmid \mathrm{ord}(\alpha)$ *and* $p \nmid \mathrm{ord}(\beta)$, *then X and Y are prime-to-p derived isogenous.*

Proof. For the first assertion, as before, we can quasi-lift the prime-to-p derived isogeny between X and Y to characteristic 0. By Theorem 1.0.3 and Lemma 6.1.1, their liftings are geometrically prime-to-p derived isogenous. According to [47, Corollary 4.3], we get that the associated Kummer surfaces are prime-to-p derived isogenous. It follows from Theorem 6.3.1 that $\mathrm{Km}(X)$ and $\mathrm{Km}(Y)$ are prime-to-p derived isogenous.

For the last assertion, according to [8, Theorem 4.6], we can find liftings

$$(\mathcal{S}_1,\widetilde{\alpha}) \to \mathrm{Spec}\ W', (\mathcal{S}_2,\widetilde{\beta}) \to \mathrm{Spec}\ W'$$

of $(\mathrm{Km}(X), \alpha)$ and $(\mathrm{Km}(Y), \beta)$ over discrete valuation ring W' with residue field k and fraction field K' such that

(1) the generic fibers $(\mathcal{S}_{1,K'}, \widetilde{\alpha}|_{K'})$ and $(\mathcal{S}_{2,K'}, \widetilde{\beta}|_{K'})$ are geometrically derived equivalent.
(2) $\mathrm{NS}(\mathcal{S}_{1,K'}) \cong \mathrm{NS}(\mathrm{Km}(X))$ and $\mathrm{NS}(\mathcal{S}_{2,K'}) \cong \mathrm{NS}(\mathrm{Km}(Y))$ via the specialization map,

As seen in the proof of Lemma 2.2.4, with condition (2), we know that $\mathcal{S}_1$ is isomorphic to $\mathrm{Km}(\mathcal{X})$ and $\mathcal{S}_2$ is isomorphic to $\mathrm{Km}(\mathcal{Y})$ for some liftings of X and Y respectively. By Theorem 1.0.1, the generic fibers of $\mathcal{X}$ and $\mathcal{Y}$ are geometrically prime-to-p derived isogeny. Again, the assertion follows from Theorem 6.3.1. □

Remark 6.5.3. It is natural to ask if one can apply the lifting method to prove the converse of Theorem 6.5.2, i.e. if $\mathrm{Km}(X)$ and $\mathrm{Km}(Y)$ are prime-to-p derived isogenous, so is X and Y. The issue is that the derived isogeny between $\mathrm{Km}(X)$ and $\mathrm{Km}(Y)$ is only quasi-liftable, not liftable. In other words, although we can lift every derived equivalence between twisted abelian surfaces or K3 surface to characteristic 0, we can not necessarily find some liftings of X and Y respectively such that the generic fibers of their associated Kummer surfaces are prime-to-p geometrically derived isogenous.

6.6. **Supersingular twisted abelian surfacces.** At last, we come to the case X is supersingular over an algebraically closed field k such that $\mathrm{char}(k) = p > 2$. We obtain a supersingular twisted derived Torelli theorem via Ogus's supersingular Torelli theorem.

Theorem 6.6.1. *Let X an Y be two supersingular abelian surfaces over k. For $\mathbb{G}_m$-gerbes $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$, the following statements are equivalent:*

(1) There is a Fourier-Mukai transform $\mathrm{D}^b(\mathscr{X}) \simeq \mathrm{D}^b(\mathscr{Y})$.
(2) There is an isomorphism between K3 crystals $\widetilde{\mathrm{H}}(\mathscr{X}, W) \cong \widetilde{\mathrm{H}}(\mathscr{Y}, W)$.

Proof. The proof is similar to the case of K3 surfaces which is given in [5]. We sketch it here.

For (1) ⇒ (2), we only need to show the cohomological Fourier-Mukai transform induces an isomorphism on twisted Mukai lattices. When $p > 3$, this is due to a direct Chern character computation (*cf.* [5, Appendix A]). When $p = 3$, one can follow [6, Proposition 4.2.4] using the twistor lines and lifting argument. As the proof is similar, we omit the details here.

To prove that (2) implies (1), let us take $v = \rho(0, 0, 1)$, there is a filtered isomorphism

$$\gamma \colon \widetilde{\mathrm{H}}(\mathscr{X}, W) \xrightarrow{\rho} \widetilde{\mathrm{H}}(\mathscr{Y}, W) \xrightarrow{\phi^{\mathcal{E}}} \widetilde{\mathrm{H}}(\mathscr{M}_H(\mathscr{Y}, v), W) \tag{6.6.1}$$

where $\phi^{\mathcal{E}}$ is the cohomological Fourier-Mukai transform induced by the universal twisted sheaf $\mathcal{E}$ on $Y \times \mathscr{M}_H(\mathscr{Y}, v)$. Then there is an isomorphism

$$f \colon X \xrightarrow{\sim} M_H(\mathscr{Y}, v)$$

since γ induces an isomorphism between supersingular K3 crystals (*cf.* (3.3.1)). The equality $[\mathscr{X}] = f^*[\mathscr{M}_H(\mathscr{Y}, v)]$ is from a direct computation. □

In [6], Bragg and Lieblich have developed the theory of twistor space for supersingular K3 surfaces. In terms of it, they are able to constructed the twisted period space of supersingular K3 surfaces. One can recap their construction and extend it to abelian surfaces as below:

Fix a supersingular K3 lattice Λ, which is a free $\mathbb{Z}$-lattice whose discriminant $\mathrm{disc}(\Lambda \otimes \mathbb{Q}) = -1$, signature $(1, n)$ ($n = 5$ or 21)and the $\Lambda^\vee/\Lambda$ is p-torsion. Then $|\Lambda^\vee/\Lambda| = p^{2\sigma_0(\Lambda)}$ for $1 \leq \sigma_0(\Lambda) \leq \frac{(n-1)}{2}$. The lattice Λ is also determined by $\sigma_0(\Lambda)$, called the *Artin invariant* of Λ. Set

$$\widetilde{\Lambda} = \Lambda \oplus U(p) \text{ and } \widetilde{\Lambda}_0 = p\widetilde{\Lambda}^\vee / p\widetilde{\Lambda}.$$

where $U(p)$ is the twisted hyperbolic plane generated by vectors e and f such that $e^2 = f^2 = 0$ and $e \cdot f = -p$. Let M_{Λ_0} be the moduli space of characteristic subspaces of Λ_0 and let $M^{\langle e \rangle}_{\widetilde{\Lambda}_0}$ be the moduli space of characteristic subspaces of $\widetilde{\Lambda}_{K3,0}$ which don't contain e.

Definition 6.6.2. The twistor line in $M_{\widetilde{\Lambda}_0}$ is the subvariety $\mathbb{A}^1$ that is a connected component of a fiber of π_v over a k-point $[K] \in M_{v^\perp/v}(k)$ for some isotropic $v \in \widetilde{\Lambda}_0$.

To emphasis that we are at either the case $n = 21$ or $n = 5$, we may write $\Lambda = \Lambda_{K3}$ and $\Lambda = \Lambda_{Ab}$ respectively. For K3 surfaces, it has been shown that the moduli functor $S_{\Lambda_{K3}}$ of Λ_{K3}-marked supersingular K3 is representable by a locally of finite presentation, locally separated and smooth algebraic space of dimension $\sigma_0(\Lambda_{K3}) - 1$. There is a universal family

$$u\colon \mathcal{X} \to S_{\Lambda_{K3}}$$

(as algebraic spaces), which is smooth with relative dimension 1. The higher direct image $\mathrm{R}^2 u_*^{\mathrm{fl}} \mu_p$ is representable by an algebraic group space over $S_{\Lambda_{K3}}$ after perfection, denoted by

$$\pi\colon \mathscr{S}_{\Lambda_{K3}} \to S_{\Lambda_{K3}}$$

(see loc.cit. Theorem 2.1.6). The connected component of the identity $\mathscr{S}^o_{\Lambda_{K3}} \subset \mathscr{S}_{\Lambda_{K3}}$ parameterizes the μ_p-gerbes which are not essentially-trivial except the identity, at each Λ-marked K3 surface in $S_{\Lambda_{K3}}(k)$. Then there are *(twisted) period morphisms*

$$\rho : S_{\Lambda_{K3}} \to M_{\Lambda_{K3,0}} \text{ and } \widetilde{\rho}\colon \mathscr{S}^o_{\Lambda_{K3}} \to M^{\langle e\rangle}_{\widetilde{\Lambda}_0},$$

(*cf.* [38, §3] and [6, Definition 3.5.7]). Then the method in *loc.cit.* shows that there is a Cartisian diagram

$$\begin{tikzcd} \mathscr{S}^o_{\Lambda_{K3}} \arrow[r, "\pi"] \arrow[d, "\widetilde{\rho}"] & S_{\Lambda_{K3}} \arrow[d, "\rho"] \\ M^{\langle e\rangle}_{\widetilde{\Lambda}_{K3,0}} \arrow[r] & M_{\Lambda_{K3,0}}, \end{tikzcd} \tag{6.6.2}$$

and ρ is étale surjective. The twisted period map $\widetilde{\rho}$ factors as

$$\begin{tikzcd} \mathscr{S}^o_{\widetilde{\Lambda}_{K3}} \arrow[r, "\widetilde{\rho}'"] \arrow[rr, bend left, "\widetilde{\rho}"] & \mathscr{P}_{\Lambda_{K3}} \arrow[r] & M^{\langle e\rangle}_{\widetilde{\Lambda}_{K3,0}}, \end{tikzcd}$$

where $\mathscr{P}_{\widetilde{\Lambda}_{K3}}$ is the moduli space of ample cones of characteristic subspaces defined by Ogus ([38]). It has been shown that $\widetilde{\rho}'$ is an isomorphism (*cf.* [6, Theorem 5.1.7]). In particular, this implies that the moduli space of supersingular K3 surfaces of Artin invariant ≤ 2 is rationally fibered over the moduli space of supersingular K3 surfaces of Artin invariant 1, whose fiber is a twistor line, corresponding to the relative moduli spaces of twisted sheaves on universal gerbes associated to the Brauer groups of the superspecial K3 surface.

For supersingular abelian surfaces, everything works by replacing Λ_{K3} with Λ_{Ab}. Indeed, the proof in [6, Proposition 5.1.5] already shows that the twisted period map $\widetilde{\rho}'$ for abelian surfaces will be an isomorphism. Another simple way to see this is via the Kummer construction. One just notice that the moduli space of supersingular abelian surfaces is isomorphic to the moduli space of supersingular Kummer surfaces and they have isomorphic period spaces, i.e.,

$$M^{\langle e\rangle}_{\widetilde{\Lambda}_{K3,0}} \cong M^{\langle e\rangle}_{\widetilde{\Lambda}_{Ab},0}$$

when $\sigma_0(\Lambda_{K3}) = \sigma_0(\Lambda_{Ab}) \leq 2$. This gives

Proposition 6.6.3. *For non-superspecial supersingular abelian surface X', there exists a Brauer class $[\mathscr{X}] \in \mathrm{Br}(X)$ such that $\mathrm{D}^b(\mathscr{X}) \simeq \mathrm{D}^b(X')$. In particular, X' is a moduli space of twisted sheaves on X.*

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