UNPOLARIZED SHAFAREVICH CONJECTURES FOR HYPER-KÄHLER VARIETIES

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ABSTRACT. Shafarevich conjecture/problem is about the finiteness of isomorphism classes of a family of varieties defined over a number field with good reduction outside a finite collection of places. For K3 surfaces, such a finiteness result was proved by Y. She. For hyper-Kähler varieties, which are higher dimensional analogues of K3 surfaces, Y. André has verified the Shafarevich conjecture for hyper-Kähler varieties of a given dimension and admitting a very ample polarization of bounded degree. In this paper, we provide a unification of both results by proving the (unpolarized) Shafarevich conjecture for hyper-Kähler varieties in a given deformation type. In a similar fashion, generalizing a result of Orr and Skorobogatov on K3 surfaces, we prove the finiteness of geometric isomorphism classes of hyper-Kähler varieties of CM type in a given deformation type defined over a number field with bounded degree. A key to our approach is a uniform Kuga–Satake map, inspired by She's work, and we study its arithmetic properties, which are of independent interest.

1. INTRODUCTION

1.1. **Background.** Let F be a number field and S a finite set of places of F. The classical Hermite–Minkowski theorem says that there are only finitely many extensions of F with a fixed degree that is unramified outside S. The geometric generalization is the so-called *Shafarevich question*: given a family $\mathcal{M}$ of smooth projective varieties defined over F, we ask the following:

Question 1.1 (Finiteness of varieties)**.** Is the set $\mathrm{Shaf}_{\mathcal{M}}(F, S)$ of F-isomorphism classes of varieties in $\mathcal{M}$ defined over F and with good reduction outside S finite?

As the notation indicates, such question can be traced back to a famous conjecture of Shafarevich [54] for the family of smooth curves of a given genus, which plays an important role in Faltings' proof of the Mordell conjecture in [17]. Such Shafarevich-type question has been investigated in various situations, where results fall into two categories: polarized versus unpolarized.

For *polarized varieties*, i.e. varieties equipped with an ample line bundle, the question has an affirmative answer in a number of cases:

- (Faltings [17]) Curves of a fixed genus $g \geq 2$, i.e. $\mathrm{Shaf}_{\mathcal{M}_g}(F, S)$ is a finite set.
- (Faltings [17]) Abelian varieties of a fixed dimension and admitting a polarization of a given degree, i.e. $\mathrm{Shaf}_{\mathcal{A}_{g,d}}(F, S)$ is a finite set.
- (Scholl [53]) del Pezzo surfaces.
- (André [4]) K3 surfaces admitting a polarization of a given degree.
- (André [4]) Hyper-Kähler varieties (with $b_2 > 3$) of given dimension with a very ample polarization of bounded degree; and similarly for more general "K3-type" varieties such as cubic fourfolds.

Key words and phrases. K3 surfaces, Hyper-Kähler varieties, Shafarevich conjectures, Shimura varieties, uniform Kuga–Satake construction, arithmetic period map, CM type.

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- (Javanpeykar–Loughran [24]) Flag varieties.
- (Javanpeykar [26]) Canonically polarized surfaces fibered over a curve.
- (Javanpeykar–Loughran [25], [28]) Certain varieties "controlled" by abelian varieties (via e.g. intermediate Jacobian): complete intersections of Hodge niveau ≤ 1, prime Fano threefolds of index 2, sextic surfaces etc.

These results can often be reinterpreted as the finiteness of $\mathcal{O}_{F,S}$-points in certain moduli spaces; see works of Javanpeykar and his coauthors [29], [27], [30], [31] for related studies from this point of view of arithmetic hyperbolicity. More recently, Lawrence and Sawin [37] proved some analogous finiteness result for hypersurfaces in abelian varieties based on the techniques in [38].

The *unpolarized* Shafarevich conjecture is a much stronger statement which bypasses the restriction on polarizations (e.g. degree or natural embedding) in the finiteness statements in the above polarized version. The first example is Zarhin's result [64] which gives a positive answer to Question 1.1 for abelian varieties of a given dimension, i.e. the following set is finite:

$$\bigcup_d \mathrm{Shaf}_{\mathcal{A}_{g,d}}(F, S),$$

generalizing the aforementioned theorem of Faltings. As for K3 surfaces, Y. She [56] established the unpolarized Shafarevich conjecture, strengthening André's result:

Theorem 1.2 (She)**.** *The following set is finite:*

$$\mathrm{Shaf}_{\mathrm{K3}}(F, S) = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is a K3 surface over } F \\ \text{having good reduction outside } S \end{array} \right\} / \cong_F .$$

More recently, Takamatsu [58] proved an even stronger version of the previous theorem by releasing the good reduction condition to a condition on unramified places of the second cohomology as Galois modules, hence establishing the so-called *cohomological (unpolarized) Shafarevich conjecture* for K3 surfaces. He also established the analogous results for bielliptic surfaces [60] and Enriques surfaces [59].

1.2. **Finiteness of (unpolarized) hyper-Kähler varieties with bounded bad reduction.** In this paper, we aim to generalize She's result (Theorem 1.2) to higher-dimensional analogues of K3 surfaces, which are hyper-Kähler varieties (see Section 2 for generalities on this type of varieties). We propose the following conjecture. As before, we fix a number field F and a finite set of places S. A smooth projective F-variety is called hyper-Kähler if the associated complex variety is hyper-Kähler (for any/all embeddings of F into $\mathbb{C}$).

Conjecture 1.3 (Unpolarized Shafarevich conjecture for hyper-Kähler varieties)**.** *Given a positive integer n, there are only finitely many F-isomorphism classes of $2n$-dimensional hyper-Kähler varieties defined over F and having good reduction outside S.*

This conjecture in full seems out of reach in view of the topological difficulty that even the finiteness of deformation classes of complex hyper-Kähler manifolds in a given dimension is unknown. It is therefore natural to restrict ourselves to a given deformation type. Let M be a deformation class of complex hyper-Kähler manifolds (e.g. M can be $K3^{[n]}$, Kum_n, OG6, or OG10). A hyper-Kähler variety X defined over F is in M if $X \times_F \mathrm{Spec}(\mathbb{C})$ is so for some embedding $F \hookrightarrow \mathbb{C}$. The following *Shafarevich set* is the central object of study in this paper:

$$\mathrm{Shaf}_M(F, S) = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is a hyper-Kähler variety over } F \text{ of deformation type } M \\ \text{having good reduction outside } S \end{array} \right\} / \cong_F . \tag{1.1}$$

Inspired by Takamatsu's cohomological Shafarevich conjecture for K3 surfaces in [58], let us also consider the following larger *cohomological Shafarevich set*:

$$\mathrm{Shaf}_M^{\mathrm{hom}}(F,S)=\left\{X \;\middle|\; \begin{array}{l} X \text{ is a hyper-Kähler variety over } F \text{ of deformation type } M \\ \text{with } 2^{nd}\text{-cohomology group unramified outside } S \end{array}\right\} / \cong_F . \tag{1.2}$$

Comparing with the case of K3 surfaces, there are two main differences for higher-dimensional hyper-Kähler varieties which cause substantial difficulties. The first one is the failure of the weak Torelli theorem (i.e. whether $\mathrm{Aut}(X)$ acts faithfully on $\mathrm{H}^2(X)$, see Example 2.9), which actually leads to the failure of the cohomological Shafarevich conjecture for (polarized or unpolarized) hyper-Kähler varieties in general, see Remark 2.11.

The second one is the existence of non-isomorphic birational transformations between higher-dimensional hyper-Kähler varieties. Birational maps clearly preserve the cohomological Shafarevich condition (see Proposition 2.12) and hence $\mathrm{Shaf}_M^{\mathrm{hom}}(F,S)$ contains the entire F-birational class of X for each $X\in\mathrm{Shaf}_M^{\mathrm{hom}}(F,S)$. One needs therefore finiteness result for birational classes of hyper-Kähler varieties.

The first main result of the paper is a positive answer to Conjecture 1.3 for all hyper-Kähler varieties, upon fixing the deformation type.

Theorem 1.4 (**Unpolarized Shafarevich conjecture in a deformation class).** *Let F be a number field and S a finite set of places of F. Let M be a deformation type of hyper-Kähler varieties with $b_2\neq 3$.*

(i) There are only finitely many F-birational isomorphism classes in $\mathrm{Shaf}_M(F,S)$.
If $b_2\geq 5$, then $\mathrm{Shaf}_M(F,S)$ is a finite set.

(ii) There are only finitely many geometrically birational isomorphism classes in $\mathrm{Shaf}_M^{\mathrm{hom}}(F,S)$.
If $b_2\geq 5$, then there are only finitely many geometric isomorphism classes in $\mathrm{Shaf}_M^{\mathrm{hom}}(F,S)$.

(iii) Suppose that the weak Torelli theorem holds for the deformation type M. Then there are only finitely many F-birational isomorphism class in $\mathrm{Shaf}_M^{\mathrm{hom}}(F,S)$.
If $b_2\geq 5$, then $\mathrm{Shaf}_M^{\mathrm{hom}}(F,S)$ is a finite set.

The finiteness of deformation classes of hyper-Kähler manifolds in a given dimension is unknown. However, in [22], Huybrechts proved such a finiteness upon fixing the Beauville–Bogolomov form. This has been strengthened by Kamenova in [32]:

Theorem 1.5 (Huyberchts, Kamenova)**.** *There are only finitely many deformation classes of hyper-Kähler manifolds, with given Fujiki invariant and discriminant of the Beauville–Bogolomov form.*

The definitions of the Fujiki invariant and the discriminant, which are natural deformation invariants for hyper-Kähler varieties, will be recalled in Section 2. Combining this theorem with Theorem 1.4 we obtain the following result in the direction of Conjecture 1.3:

Theorem 1.6. *Let F be a number field and S a finite set of places of F. For any positive integers n, c and Δ, there are only finitely many F-isomorphism classes of $2n$-dimensional hyper-Kähler varieties defined over F, with Fujiki constant c, discriminant of Beauville–Bogomolov form Δ, $b_2\geq 5$, and with good reduction outside S.*

The gap between Theorem 1.6 and Conjecture 1.3 can be resolved if there are only finitely many possibilities for Fujiki invariants and discriminants of the Beauville–Bogomolov forms of hyper-Kähler manifolds in a given dimension.

1.3. **Finiteness of geometric hyper-Kähler varieties of CM type.** The key ingredient in the proof of Theorem 1.4 is the construction of a *uniform Kuga–Satake map* and its arithmetic properties (Sections 5 and 6), which allow us to show, as an intermediate step towards Theorem 1.4, that there are only finitely many isomorphism classes of (geometric) Picard lattices of hyper-Kähler varieties arising from $\mathrm{Shaf}_M^{\mathrm{hom}}(F, S)$ (see Theorem 7.4).

More generally, we propose the following conjecture, inspired by yet another question of Shafarevich [55].

Conjecture 1.7 (Finiteness of geometric Picard lattices)**.** *Let d be an integer and M a deformation type of hyper-Kähler manifolds. Then there are only finitely many isometry classes in the following set of lattices:*

$$\left\{ \mathrm{Pic}(X_{\mathbb{C}}) \middle| \begin{array}{l} X \text{ is a hyper-Kähler over a number field } F \text{ of degree} \leq d \\ X_{\mathbb{C}} := X \times_F \mathrm{Spec}(\mathbb{C}) \text{ is of deformation type } M \text{ for some } F \hookrightarrow \mathbb{C} \end{array} \right\}. \tag{1.3}$$

Here the Picard group is equipped with the restriction of the Beauville–Bogomolov quadratic form.

As a special case, if we fix the number field F in (1.3), the finiteness of isometry classes in (1.3) can be regarded as a strengthening of the Bombieri–Lang conjecture for moduli spaces of lattice-polarized hyper-Kähler varieties, which predicts that F-rational points in a moduli space $\mathscr{F}$ of lattice-polarized hyper-Kähler varieties are contained in the Noether–Lefschetz loci if $\mathscr{F}$ is of "sufficiently" general type.

One could also speculate the more ambitious form without the restriction on deformation type. In the case of K3 surfaces, Conjecture 1.7 has been confirmed for K3 surfaces of CM type by Orr–Skorobogatov in [49, Corollary B.1]. In fact, they proved the following stronger result [49, Theorem B]:

Theorem 1.8 (Orr–Skorobogatov)**.** *There are only finitely many geometric isomorphism classes of K3 surfaces of CM type which can be defined over a number field of a given degree.*

As another application of the uniform Kuga–Satake construction, our second main result generalizes Theorem 1.8 to hyper-Kähler varieties of CM type.

Theorem 1.9. *Let d be a positive integer and M a fixed deformation type of hyper-Kähler varieties with $b_2 \geq 4$. Then there are only finitely many geometrically birational equivalence classes of hyper-Kähler varieties of CM type and of deformation type M which can be defined over a number field of degree $\leq d$. When $b_2 \geq 5$, there are only finitely many geometric isomorphism classes of hyper-Kähler varieties of CM type and of deformation type M which can be defined over a number field of degree $\leq d$.*

Note that (geometrically) birational transformations preserve the (geometric) Picard lattice. In particular, Conjecture 1.7 holds for hyper-Kähler varieties of CM type with $b_2 \geq 4$:

Corollary 1.10. *In the same setting as Theorem 1.9, there are only finitely many isometry classes of the geometric Picard lattices of hyper-Kähler varieties of CM type and of deformation type M which can be defined over a number field of degree $\leq d$.*

As a consequence, we get the uniform boundedness of Brauer groups for hyper-Kähler varieties of CM type.

Corollary 1.11. *Fix a deformation type M of hyper-Kähler varieties. For any positive integer d, there exists a constant N such that*

$$|\operatorname{Br}(X)/\operatorname{Br}_0(X)| < N \textit{ and } |\operatorname{Br}(X_{\bar{\mathbb{Q}}})^{\mathrm{Gal}(\bar{\mathbb{Q}}/F)}| < N$$

for any hyper-Kähler variety X of CM type, of deformation type M with $b_2 \geq 4$, and defined over a number field F of degree at most d.

Structure of the paper. In Section 2, we recall some basic notions on hyper-Kähler varieties over a field of characteristic zero, including the Beauville–Bogomolov form, (birational) automorphisms, and various Shafarevich sets, etc. In Section 3, we treat wall divisors, (birational) ample cones, and state Takamatsu's result on finiteness of birational models over arbitrary field of characteristic zero (Theorem 3.5). In Section 4, we define the moduli stacks/spaces of (oriented) polarized hyper-Kähler varieties (with level structures). In Section 5, we define the period map from the moduli spaces in Section 4 to Shimura varieties of orthogonal type, and state (Proposition 5.8) that it is defined over $\mathbb{Q}$, a result due to Bindt. In Section 6, we develop the uniform Kuga–Satake construction. In Section 7, we prove the main results: Theorem 1.4 in the end of Section 7.3, Theorem 1.9 and Corollary 1.11 in Section 7.4.

Remark 1.12. In the final stage of the preparation of this paper, the preprint of Teppei Takamatsu [62] appeared on arXiv, proving independently essentially the same result as our Theorem 1.4 (the unpolarized Shafarevich conjecture). We thank him for enriching communications regarding two papers.

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2. Hyper-Kähler varieties

We collect in this section some basics on hyper-Kähler varieties over fields of characteristic zero.

2.1. Beauville–Bogomolov form.

A *hyper-Kähler* (or *irreducible symplectic*) variety over a field k of characteristic 0 is defined as a geometrically connected smooth projective k-variety X such that it is geometrically simply connected $\pi_1^{\text{ét}}(X_{\bar{k}}) = 1$, and $\mathrm{H}^0(X, \Omega_X^2)$ is spanned by a nowhere degenerate algebraic 2-form. Note that the 2-form is automatically closed (hence symplectic) by the degeneration of the Hodge-de Rham spectral sequence. When $k = \mathbb{C}$, $\mathrm{H}^2(X, \mathbb{Z})$ carries an integral, primitive quadratic form q_X, called the Beauville–Bogomolov (BB) form, which satisfies the following conditions:

(1) q_X is non-degenerate and of signature $(3, b_2(X) - 3)$.
(2) There exists a positive rational number c, called the *Fujiki invariant*, such that $q_X^n(\alpha) = c \int_X \alpha^{2n}$ for all classes $\alpha \in \mathrm{H}^2(X, \mathbb{Z})$.
(3) The Hodge decomposition $\mathrm{H}^2(X, \mathbb{C}) = \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus H^{0,2}(X)$ is orthogonal with respect to $q_X \otimes \mathbb{C}$.

See [5] and [21] for basic results and typical examples of hyper-Kähler varieties over the field of complex numbers. By a result of Bogomolov [8], the deformation space of X is unobstructed and its smooth deformations remain hyper-Kähler. The property (2) ensure that the BB form q_X and the Fujiki invariant are invariant under deformations. Over complex numbers, for hyper-Kähler varieties in a fixed deformation class, the lattice realized by the BB form on $\mathrm{H}^2(X, \mathbb{Z})$ is constant. From now on, we may assume k is finitely generated.

Definition 2.1. Let M be a fixed hyper-Kähler manifold. Given an embedding $\iota : k \hookrightarrow \mathbb{C}$, we say that a hyper-Kähler variety X over k is of (complex) deformation type M with respect to ι if the complex variety $X_{\mathbb{C}} := X \times_k \mathrm{Spec}(\mathbb{C})$ is deformation equivalent to M. We denote by Λ_M the lattice realized by the BB form on $\mathrm{H}^2(M, \mathbb{Z})$ and c_M its Fujiki invariant.

In our definition, the geometric deformation type M depends on the embedding $\iota : k \hookrightarrow \mathbb{C}$. It is an interesting open question whether the deformation type is independent of the choice of ι.

Remark 2.2. All known examples of hyper-Kähler varieties have an even BB form, i.e. Λ_M is an even lattice. It is still unknown whether the evenness of Λ_M is a general phenomenon (see also [7]). However, let us mention that there are examples of hyper-Kähler *orbifolds* with odd BB form [33].

Convention: In this paper, if we do not fix the embedding of a number field $F \hookrightarrow \mathbb{C}$, we say a hyper-Kähler variety X over F is of deformation type M if there exists an embedding $F \hookrightarrow \mathbb{C}$ such that $X_{\mathbb{C}} = X \times_F \operatorname{Spec}(\mathbb{C})$ is of deformation type M. Clearly, X can have at most finitely many deformation types.

For a hyper-Kähler variety X defined over a field k of characteristic zero, upon fixing an embedding $\bar{k} \hookrightarrow \mathbb{C}$, Artin's comparison isomorphism

$$\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \widehat{\mathbb{Z}} \cong \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \widehat{\mathbb{Z}}(1)) \tag{2.1}$$

allows us to transport the BB form on $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ to a $\widehat{\mathbb{Z}}$-BB form:

$$q\colon \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \widehat{\mathbb{Z}}(1)) \times \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \widehat{\mathbb{Z}}(1)) \to \widehat{\mathbb{Z}}.$$

As shown in [7, Lemma 4.2.1], this $\widehat{\mathbb{Z}}$-BB form is unique and independent of the embedding. Hence if X is of deformation type M, we may call $\Lambda_M \otimes \widehat{\mathbb{Z}}$ the $\widehat{\mathbb{Z}}$-BB form on X.

2.2. **Polarization of hyper-Kähler varieties.** For any smooth family of projective hyper-Kähler varieties $\pi\colon \mathfrak{X} \to T$ over a $\mathbb{Q}$-scheme T, the relative Picard functor $\mathrm{Pic}_{X/T}$ is known to be represented by a separated algebraic space over $\mathbb{Q}$. In particular, if X is a hyper-Kähler variety over a field k of characteristic zero, then the relative Picard functor $\mathrm{Pic}_{X/k}$ is represented by a separated, smooth, 0-dimensional scheme over k (*cf.* [11, §8.4, Theorem 1]). Let G_k be the absolute Galois group of k, the group of k-rational points $\mathrm{Pic}_{X/k}(k)$ is isomorphic to the subgroup $\mathrm{Pic}_{\overline{X}/\bar{k}}(\bar{k})^{G_k} \subset \mathrm{Pic}_{\overline{X}/\bar{k}}(\bar{k})$, hence is torsion free. Denote $\mathrm{Pic}_{X/k}(k)$ by Pic_X in the sequel.

Definition 2.3 (cf. [4, 51])**.** Let B be a $\mathbb{Q}$-scheme and let $\mathfrak{X} \to B$ be a smooth proper morphism of algebraic spaces whose fibers are hyper-Kähler varieties. A polarization on $\mathfrak{X}/B$ is a global section $\mathcal{H} \in \mathrm{Pic}_{\mathfrak{X}/B}(B)$ such that for every geometric point b of B, the fiber $\mathcal{H}_b \in \mathrm{Pic}_{\mathfrak{X}_b}$ is a primitive polarization of $\mathfrak{X}_b$.

Combining the Leray–Serre spectral sequence and Hilbert's theorem 90, we get an exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}_X \cong \mathrm{Pic}_{\overline{X}/\bar{k}}(\bar{k})^{G_k} \xrightarrow{\delta} \mathrm{Br}(k).$$

Thus $\mathrm{Pic}(X) \subset \mathrm{Pic}_X$ as a finite index subgroup (see also [4, Lemma 2.3.1]). For each prime ℓ, the ℓ-adic cycle class map defines an embedding

$$c_1^{\ell}\colon \mathrm{Pic}(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(1)) \tag{2.2}$$

whose image lies in $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)^{G_k}$. The G_k-invariant elements in $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ are called *Tate classes*. The Kummer sequence induces a long exact sequence of étale cohomology groups

$$0 \to \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{G}_m) \xrightarrow{(\cdot)^{\ell^n}} \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mu_{\ell^n}) \to \mathrm{Br}(X_{\bar{k}}) \to \cdots \tag{2.3}$$

(here we use the assumption that $\mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mu_{\ell^n}) = 0$). By passing to $\varprojlim_n$, we will obtain a morphism

$$\mathrm{Pic}(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(1)), \tag{2.4}$$

through which the cycle class map c_1^ℓ factors. Therefore, we can extend c_1^ℓ to a map

$$\mathrm{Pic}_X \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(1)),$$

which is still denoted by c_1^ℓ. This isomorphism is compatible with the cycle class map (2.2) in that it restricts to isomorphisms of $\mathbb{Z}_\ell$-lattices

$$c_1^\ell(\mathrm{Pic}(X_{\bar{k}}) \otimes \mathbb{Z}_\ell) \cong c_1(\mathrm{Pic}(X_{\bar{k}})) \otimes \mathbb{Z}_\ell, \tag{2.5}$$

for all ℓ. From the exponential sequence, we can see the cokernel of the first Chern class map $\mathrm{Pic}(X_{\bar{k}}) \cong \mathrm{Pic}(X_{\mathbb{C}}) \subseteq \mathrm{H}^2(X(\mathbb{C}), \mathbb{Z})$ lies in $\mathrm{H}^2(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}})$, which is torsion-free. Thus $\mathrm{Pic}(X_{\bar{k}}) \subseteq \mathrm{H}^2(X(\mathbb{C}), \mathbb{Z})$ is saturated. Moreover, as $\mathrm{Pic}_X \subseteq \mathrm{Pic}(X_{\bar{k}})$ is also saturated by definition, we have the following result.

Proposition 2.4. *Under the embedding*

$$\prod_\ell c_1^\ell \colon \mathrm{Pic}_X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \hookrightarrow \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \widehat{\mathbb{Z}}(1)),$$

the $\widehat{\mathbb{Z}}$-lattice $\mathrm{Pic}_X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ forms a saturated sublattice.

Remark 2.5. For ℓ-adic cohomology, the cokernel of $\mathrm{Pic}(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$ is isomorphic to $\mathrm{Br}(X_{\bar{k}})[\ell^\infty]$, which is a torsion group. In other words, for any $\mathcal{Z} \in \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(1))$, there is an integer n such that $\ell^n \mathcal{Z} = [\mathcal{L}]$ and $[\mathcal{L}]$ is the image of some $\mathcal{L} \in \mathrm{Pic}(X_{\bar{k}}) \otimes \mathbb{Z}_\ell$. Thus $\mathrm{Pic}(X_{\bar{k}}) \otimes \mathbb{Z}_\ell$ is not saturated in $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$ when $\mathrm{Br}(X_{\bar{k}})[\ell^\infty] \neq \{1\}$.

When $b_2(X) > 3$, André has proved the Tate conjecture for divisors on polarized hyper-Kähler varieties over number fields.

Theorem 2.6. [4, Theorem 1.6.1 (2)] *Let (X, H) be a polarized hyper-Kähler variety over a number field F. Let $\mathrm{P}^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell(1)) = c_1^\ell(H)^\perp \subseteq \mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell(1))$ be the primitive part. Then every Tate class in $\mathrm{P}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ is algebraic, i.e., induced by a $\mathbb{Q}_\ell$-linear combination of algebraic cycles.*

Corollary 2.7. *If k is a number field, then*

$$\prod_\ell c_1^\ell \colon \mathrm{Pic}_X \otimes \widehat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \widehat{\mathbb{Z}}(1))^{G_k}.$$

Proof. Theorem 2.6 implies that there is an isomorphism of $\mathbb{Q}_\ell$-vector spaces:

$$c_1^\ell \otimes \mathbb{Q}_\ell \colon \mathrm{Pic}_X \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)^{G_k}.$$

Hence $\mathrm{rk}(\mathrm{Pic}_X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) = \mathrm{rk}\left(\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \widehat{\mathbb{Z}}(1))^{G_k}\right)$. Then the isomorphism follows from the saturatedness of $\mathrm{Pic}_X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ proved in Proposition 2.4. □

2.3. **Automorphisms and birational self-maps.** Let X be a hyper-Kähler variety over a finitely generated field k of characteristic zero. For any field extension k'/k, denote by $\mathrm{Aut}(X_{k'})$ the group of k'-automorphisms of $X_{k'}$ and $\mathrm{Bir}(X_{k'})$ the group of k'-birational self-maps. The group $\mathrm{Aut}(X_{k'})$ is discrete and it acts naturally on the cohomology groups of $X_{k'}$. In particular, its action on the second cohomology preserves the Beauville–Bogomolov form. Define

$$\mathrm{Aut}_0(X_{\mathbb{C}}) := \ker\left(\mathrm{Aut}(X_{\mathbb{C}}) \to \mathrm{O}(\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}))\right). \tag{2.6}$$

Theorem 2.8 ([20, Theorem 2.1]). *The group* $\mathrm{Aut}_0(X_{\mathbb{C}})$ *is a finite group and it is deformation invariant.*

When $\mathrm{Aut}_0(X_{\mathbb{C}})$ is trivial for some/any X of deformation type M, we say that the *weak Torelli theorem* holds for hyper-Kähler varieties of deformation type M. If we restrict the action of $\mathrm{Aut}(X_k)$ to Pic_X, we obtain a morphism

$$\mathrm{Aut}(X_k) \to \mathrm{O}(\mathrm{Pic}_X) \tag{2.7}$$

whose kernel is again finite.

Example 2.9. Let us recall the kernels of $\mathrm{Aut}(X_{\mathbb{C}}) \to O(\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}))$ for complex hyper-Kähler varieties $X_{\mathbb{C}}$ of known deformation types.

(1) $K3^{[n]}$-type: Beauville has shown that $\mathrm{Aut}_0(S_{\mathbb{C}}^{[n]})$ is trivial in [5].
(2) Generalized Kummer type: we have
$$\mathrm{Aut}_0(K_n(A)_{\mathbb{C}}) \cong A[n] \rtimes \mathbb{Z}/2\mathbb{Z},$$
by [9, Corollary 5]. Here, A is an abelian surface and $A[n]$ is the group of its n-torsion points.
(3) OG6-type: Mongardi and Wandel showed that $\mathrm{Aut}_0(X_{\mathbb{C}}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$ in [46, Theorem 5.2].
(4) OG10-type: $\mathrm{Aut}_0(X_{\mathbb{C}})$ is trivial by [46, Theorem 3.1].

Moreover, we can consider the automorphism group of polarized hyper-Kähler varieties. For a polarized pair (X, H), we define

$$\mathrm{Aut}(X_{k'}, H_{k'}) = \{f \in \mathrm{Aut}(X_{k'}) |\ f^*H = H\},$$

which is clearly finite. Similarly, one could also consider the group $\mathrm{Bir}(X_{k'}, H_{k'})$ of k'-birational self maps of $(X_{k'}, H_{k'})$. However, by a result of Fujiki [19, Thm 4.4], any birational map preserving an ample class is an isomorphism, i.e. $\mathrm{Bir}(X_{k'}, H_{k'}) = \mathrm{Aut}(X_{k'}, H_{k'})$.

2.4. **Shafarevich sets for hyper-Kähler varieties.** We introduce some variants of the Shafarevich sets defined in (1.1) and (1.2):

Definition 2.10 (Polarized and unpolarized Shafarevich sets)**.** Let R be a finite type algebra over $\mathbb{Z}$ which is a normal domain with fraction field F. Let M be a deformation type of $2n$-dimensional hyper-Kähler varieties and d a positive integer. We define the following *Shafarevich set* and *polarized Shafarevich set*:

$$\mathrm{Shaf}_M(F, R) = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is a hyper-Kähler variety of type } M \text{ over } F, \\ \text{for any height 1 prime ideal } \mathfrak{p} \in \mathrm{Spec}\, R, \\ X \text{ has good reduction at } \mathfrak{p} \end{array} \right\} / \cong_F \tag{2.8}$$

$$\mathrm{Shaf}_{M,d}(F, R) = \left\{ (X, H) \;\middle|\; \begin{array}{l} (X, H) \text{ is a polarized hyper-Kähler variety of type } M \text{ over } F, \\ (H)^{2n} = d\,, \\ \text{for any height 1 prime ideal } \mathfrak{p} \in \mathrm{Spec}\, R, \\ X \text{ has good reduction at } \mathfrak{p} \end{array} \right\} / \cong_F$$

Here, X has good reduction at $\mathfrak{p}$ means there is a regular algebraic space $\mathfrak{X}$ which is smooth and proper over the local ring $R_{\mathfrak{p}}$ such that $\mathfrak{X}_F \cong X$ as F-schemes.

Following [58], we define the following *cohomological Shafarevich set* and its polarized version:

$$\mathrm{Shaf}^{\mathrm{hom}}_M(F, R) = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is a hyper-Kähler variety of type } M \text{ over } F, \\ \text{for any height 1 prime ideal } \mathfrak{p} \in \mathrm{Spec}\, R, \\ \mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell) \text{ is unramified at } \mathfrak{p} \text{ for some } \ell \end{array} \right\} / \cong_F \tag{2.9}$$

$$\mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F,R)=\left\{(X,H)\;\middle|\;\begin{array}{l}(X,H)\text{ is a polarized hyper-Kähler variety of type }M\text{ over }F,\\ (H)^{2n}=d\,,\\ \text{for any height 1 prime ideal }\mathfrak{p}\in\operatorname{Spec}R,\\ \mathrm{H}^2_{\text{ét}}(X_{\overline{F}},\mathbb{Q}_\ell)\text{ is unramified at }\mathfrak{p}\text{ for some }\ell\end{array}\right\}/\cong_F \tag{2.10}$$

Here, $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}},\mathbb{Q}_\ell)$ is unramified at $\mathfrak{p}$ if the inertia group acts trivially. The cohomological Shafarevich condition in (2.10) and (2.9) is independent of the choice of ℓ, see [58, Section 5].

Remark 2.11. We point out that the cohomological Shafarevich sets $\mathrm{Shaf}^{\mathrm{hom}}_{M}(F,R)$ and $\mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F,R)$ are *not* necessarily finite. This is due to the possible failure of the weak Torelli theorem (see Section 2.3). More precisely, there is a bijection between the set of all F-forms of $X_{\bar{F}}$ and the Galois cohomology

$$\mathrm{H}^1(\mathrm{Gal}(\bar{F}/F),\mathrm{Aut}(X_{\bar{F}})), \tag{2.11}$$

via the construction of twisting [10]. Since $\mathrm{Aut}(X_{\bar{F}})$ is finitely generated ([14]), up to replacing F by a finite extension, we can assume that $\mathrm{Gal}(\bar{F}/F)$ acts trivially on $\mathrm{Aut}(X_{\bar{F}})$. As a result, $\mathrm{H}^1(\mathrm{Gal}(\bar{F}/F),\mathrm{Aut}_0(X_{\bar{F}}))$ identifies with a subset of (2.11). Twisting by distinct elements in $\mathrm{H}^1(\mathrm{Gal}(\bar{F}/F),\mathrm{Aut}_0(X_{\bar{F}}))$ give rise to different F-forms of $(X_{\bar{F}},H_{\bar{F}})$ (resp. $X_{\bar{F}}$), while the unramifiedness condition is preserved, i.e. all those F-forms are in $\mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F,R)$ (resp. $\mathrm{Shaf}^{\mathrm{hom}}_{M}(F,R)$). However, the cohomology $\mathrm{H}^1(\mathrm{Gal}(\bar{F}/F),\mathrm{Aut}_0(X_{\bar{F}}))\simeq\mathrm{Hom}(\mathrm{Gal}(\bar{F}/F),\mathrm{Aut}_0(X_{\bar{F}}))$ is infinite in general, for example, when X is of generalized Kummer or OG6 deformation type; see Example 2.9.

The result of André in [4] actually shows that the set $\mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F,R)$ is finite. As indicated in [39] and [58], the unramifiedness condition in (2.10) is strictly stronger than the condition that X has good reduction. For K3 surfaces, Liedtke and Matsumoto [39] have shown that the unramifiedness at $\mathfrak{p}$ is equivalent to X having a good reduction at $\mathfrak{p}$ after a finite extension if X admits a polarization of small degree.

Note that the unpolarized Shafarevich sets $\mathrm{Shaf}_M(F,R)$ admits a natural surjection (forgetting the polarization) from the union of polarized Shafarevich sets:

$$\coprod_d \mathrm{Shaf}_{M,d}(F,R)\twoheadrightarrow\mathrm{Shaf}_M(F,R) \tag{2.12}$$

Similarly, for the cohomological Shafarevich sets:

$$\coprod_d \mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F,R)\twoheadrightarrow\mathrm{Shaf}^{\mathrm{hom}}_M(F,R) \tag{2.13}$$

As one can see from the definition, there is an inclusion

$$\mathrm{Shaf}_M(F,R)\subseteq\mathrm{Shaf}^{\mathrm{hom}}_M(F,R).$$

As mentioned in the introduction, the set $\mathrm{Shaf}^{\mathrm{hom}}_M(F,R)$ and even $\mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F,R)$ in general may not necessarily be finite because of the possible existence of non-trivial automorphisms which act trivially on the second cohomology group (see Section 2.3 and Remark 2.11). Furthermore, even in the case where the automorphism group acts on the second cohomology faithfully, the set $\mathrm{Shaf}_M(F,R)$ remains a proper subset of $\mathrm{Shaf}^{\mathrm{hom}}_M(F,R)$. The obstruction comes from the following result.

Proposition 2.12. *Let X and Y be hyper-Kähler varieties defined over F. Suppose that X and Y are birationally equivalent over F. Then $X\in\mathrm{Shaf}^{\mathrm{hom}}_M(F,R)$ if and only if $Y\in\mathrm{Shaf}^{\mathrm{hom}}_M(F,R)$.*

Proof. The birational isomorphism between X and Y is an isomorphism in codimension 1, hence we have an isomorphism
$$\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell(1)) \cong \mathrm{H}^2_{\text{ét}}(Y_{\bar{F}}, \mathbb{Q}_\ell(1))$$
as $\mathrm{Gal}(\bar{F}/F)$-modules for all ℓ and $\mathfrak{p}$. It follows that $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell)$ is unramified at $\mathfrak{p}$ if and only if $\mathrm{H}^2_{\text{ét}}(Y_{\bar{F}}, \mathbb{Q}_\ell)$ is unramified at $\mathfrak{p}$. □

According to Proposition 2.12, the cohomological Shafarevich condition can not distinguish between birationally isomorphic elements in $\mathrm{Shaf}^{\mathrm{hom}}_M(F, R)$. This suggests our approach towards the Shafarevich conjecture: on one hand, we need the finiteness of isomorphism classes within a birational isomorphism class (this is guaranteed by a result of Takamatsu [61], recalled below as Theorem 3.5); on the other hand, we study the finiteness of birational isomorphism classes in various Shafarevich sets, which is the main goal of the paper, accomplished in Section 7.3).

3. Wall divisors on hyper-Kähler varieties

Let X be a hyper-Kähler variety over k of a given deformation type. Let $N^1(X) = \mathrm{Pic}(X)$ be the Néron–Severi lattice of X, equipped with the Beauville–Bogomolov form. Inside the *Néron–Severi space*
$$N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R},$$
we have the positive cone $\mathrm{Pos}(X)$ defined as the connected component of $\{v \in N^1(X)_{\mathbb{R}} \mid v^2 > 0\}$ containg the ample classes, and the nef cone $\mathrm{Nef}(X)$ defined as the closure of the ample cone $\mathrm{Amp}(X)$. The *birational ample cone*, denoted by $\mathrm{BA}(X)$, is defined as the union
$$\bigcup_f f^*\mathrm{Amp}(Y),$$
where f runs through all k-birational isomorphisms between X and other hyper-Kähler varieties over k. It is known that the closure of $\mathrm{BA}(X)$ is the movable cone $\overline{\mathrm{Mov}}(X)$ (*cf.* [20, 61]).

Definition 3.1. ([1], [45]) Let D be a divisor class on a hyper-Kähler variety $X_{\mathbb{C}}$. Then D is called a *wall divisor* or a *monodromy birationally minimal* (MBM) class if $q(D) < 0$ and
$$\Phi(D^\perp) \cap \mathrm{BA}(X_{\mathbb{C}}) = \emptyset,$$
for any Hodge monodromy operator Φ. We denote the set of wall divisors on $X_{\mathbb{C}}$ by $\mathcal{W}(X_{\mathbb{C}})$.

Remark 3.2. There are other equivalent definitions for wall divisors. In [1], the MBM classes are defined as certain classes in $\mathrm{H}^2(X, \mathbb{Z})$ for all X in a given deformation family and the collection of MBM classes is monodromy invariant. The notion in Definition 3.1 corresponds to those MBM classes that are of Hodge type $(1,1)$ on X. See [1] for more details.

The collection of wall divisors are deformation invariant where they stay of $(1,1)$-type. Almost by construction, we have the following connection between wall divisors and the ample cone of hyper-Kähler varieties.

Lemma 3.3 (cf. [1], [20], [45]). *Let X be a hyper-Kähler variety and $\mathcal{W}(X_{\mathbb{C}})$ be the collection of wall divisors of $X_{\mathbb{C}}$. Let $\mathcal{W}(X_{\mathbb{C}})^\perp$ be the union of all orthogonal complement to all $D \in \mathcal{W}(X_{\mathbb{C}})$. Then the ample cone $\mathrm{Amp}(X)$ is a connected component of $\mathrm{Pos}(X)\backslash\mathcal{W}(X_{\mathbb{C}})^\perp$.*

Proof. For complex hyper-Kähler varieties, the result is a reformulation of the definition of wall divisors. Now for X defined over k, the Néron–Severi space $N^1(X)_{\mathbb{R}}$ is the Galois-invariant space of $N^1(X_{\mathbb{C}})_{\mathbb{R}}$. Clearly, a divisor is ample on X if and only if it is ample when pulled back to $X_{\mathbb{C}}$. Therefore, $\mathrm{Amp}(X)$ is the intersection of $\mathrm{Amp}(X_{\mathbb{C}})$ with $N^1(X)_{\mathbb{R}}$. Since $\mathrm{Amp}(X_{\mathbb{C}})$ is a connected component of the complement of $\mathcal{W}(X_{\mathbb{C}})^{\perp}$ in $N^1(X_{\mathbb{C}})_{\mathbb{R}}$, and it is convex, $\mathrm{Amp}(X)$ must also be a connected component of the complement of $\mathcal{W}(X_{\mathbb{C}})^{\perp}$ inside $N^1(X)_{\mathbb{R}}$. □

In [2, 3], Verbitsky and Amerik proved the following boundedness results on wall divisor classes and confirmed the Kawamata–Morrison cone conjecture for hyper-Kähler varieties.

Theorem 3.4 ([2, Theorem 1.5, Theorem 1.7])**.** *Let X be a smooth hyper-Kähler variety with $b_2 \geq 5$ over a field k of characteristic zero. Then the Beauville–Bogomolov square of the elements in $\mathcal{W}(X_{\mathbb{C}})$ are bounded (from below).*

Using Markman and Yoshioka's argument in [43, Corollary 2.5], the movable cone conjecture for complex hyper-Kähler varieties (proved by Markman [42]) and the boundedness of squares of wall divisors allow Amerik and Verbitsky to conclude the finiteness of the set of birational hyper-Kähler models of a given hyper-Kähler variety over the complex numbers. Over an arbitrary field of characteristic zero, using the method of Bright–Logan–van Luijk [12] for K3 surfaces, Takamatsu [61, Theorem 4.2.7] generalized Markman's work [42] and deduced in the similar way the finiteness of birational hyper-Kähler models:

Theorem 3.5 (Takamatsu)**.** *Let k be a field of characteristic 0. Let X be a hyper-Kähler variety defined over k with $b_2 \geq 5$. Then there are only finitely many k-isomorphism classes of hyper-Kähler varieties defined over k which are k-birational to X.*

4. Moduli spaces of polarized hyper-Kähler varieties

In this section, we review the moduli theory of polarized hyper-Kähler varieties.

4.1. **Moduli spaces of polarized hyper-Kähler varieties.** Given a positive integer d, let us consider the moduli functor

$$\mathscr{F}_d\colon (\mathtt{Sch}/\mathbb{Q})^{op} \to \mathtt{Groupoids}$$

defined by

$$\mathscr{F}_d(T) = \left\{ (f\colon \mathfrak{X} \to T, \xi) \;\middle|\; \begin{array}{l} \mathfrak{X} \to T \text{ is a family of } 2n\text{-dimensional smooth polarized varieties,} \\ \xi \in \mathrm{Pic}_{\mathfrak{X}/T}(T) \text{ is a polarization, } (\mathfrak{X}_t, \xi_t) \text{ is a polarized} \\ \text{hyper-Kähler variety with } \xi_t^{2n} = d \text{ for all } t \in T(\bar{k}), \end{array} \right\} \tag{4.1}$$

for any k-scheme T. Here, $(f_1 : \mathfrak{X}_1 \to T, \xi_1)$ is equivalent to $(f_2 : \mathfrak{X}_2 \to T, \xi_2)$ if there exists an isomorphism

$$\psi : \mathfrak{X}_1 \to \mathfrak{X}_2$$

over T and a line bundle λ on T such that $\psi^*(\xi_2) \cong \xi_1 \otimes f_1^*(\lambda)$.

Let $(\mathscr{F}_d)_{\text{ét}}$ be the étale site of stack $\mathscr{F}_d$ associated to $(\mathtt{Sch}_{/\mathbb{Q}})_{\text{ét}}$, i.e., the class of objects in $(\mathscr{F}_d)_{\text{ét}}$ consists of all hyper-Kähler spaces $(f\colon \mathfrak{X} \to T, \xi)$ and the coverings are of the form

$$\begin{array}{ccc} \mathfrak{X}_i & \longrightarrow & \mathfrak{X} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ T_i & \xrightarrow{g_i} & T \end{array}$$

where g_i is an étale surjective morphism.

Proposition 4.1 ([7, Theorem 3.3.2])**.** *The moduli stack $\mathscr{F}_d$ is a separated smooth Deligne–Mumford stack of finite type over $\mathbb{Q}$.*

We shall introduce the moduli stack $\widetilde{\mathscr{F}_d}$ of oriented polarized hyper-Kähler spaces of degree d, as a finite covering of $\mathscr{F}_d$.

Definition 4.2 (Orientation)**.** Let S be a $\mathbb{Q}$-scheme, and let $f\colon \mathfrak{X} \to S$ be a smooth proper morphism of algebraic spaces whose fibers are hyper-Kähler varieties. An *orientation* on $\mathfrak{X}/S$ is an isomorphism of lisse sheaves over $S_{\text{ét}}$:

$$\omega\colon \underline{\mathbb{Z}_2} \to \det R^2_{\text{ét}} f_* \mathbb{Z}_2(1). \tag{4.2}$$

The pair $(\mathfrak{X}/S, \omega)$ will be called an *oriented hyper-Kähler space* over S.

The moduli stack of oriented polarized hyper-Kähler varieties $\widetilde{\mathscr{F}_d}$ is defined to be the stack parameterizing tuples $(\mathfrak{X}/S, \lambda, \omega)$, where $(\mathfrak{X}/S, \lambda) \in \mathscr{F}_d(S)$ is an element such that the fibers of $\mathfrak{X}/S$ have second Betti number greater than 3, and ω is an orientation on $\mathfrak{X}/S$.

Proposition 4.3. *The natural forgetful functor $\widetilde{\mathscr{F}_d} \to \mathscr{F}_d$ forms a finite étale covering of degree 2.*

Proof. Suppose ω_1 and ω_2 are two orientations on $\mathfrak{X}/S$. It gives an automorphism

$$\omega_2^{-1} \circ \omega_1 \in \mathrm{Aut}(\underline{\mathbb{Z}_2}) \cong \{\pm 1\}.$$

Thus the orientations on $(\mathfrak{X}/S, \lambda) \in \mathscr{F}_d(S)$ forms a $\{\pm 1\}$-torsor. It means the forgetful functor is representable by an $\{\pm 1\}$-torsor, which is finite and étale. $\square$

Corollary 4.4. *The stack $\widetilde{\mathscr{F}_d}$ is a smooth Deligne–Mumford stack of finite type over $\mathbb{Q}$*

The following observation in [7, Lemma 4.3.2] shows that the "orientation" defined here is equivalent to that in [57], at least on a normal base.

Lemma 4.5. *Let S be a normal scheme of finite type over $\mathbb{Q}$. For any oriented hyper-Kähler space $(\mathfrak{X}/S, \omega)$ with $b_2(\mathfrak{X}_s) > 3$ for each fiber $\mathfrak{X}_s$, there is a unique isomorphism of lisse $\widehat{\mathbb{Z}}$-local systems*

$$\omega_{\text{ét}}\colon \underline{\widehat{\mathbb{Z}}} \xrightarrow{\sim} \det R^2_{\text{ét}} f_* \widehat{\mathbb{Z}}(1),$$

whose reduction on $\mathbb{Z}_2$ is ω.

Proof. The uniqueness of $\omega_{\text{ét}}$ is clear. It suffices to show the existence of isomorphism of lisse $\mathbb{Z}_\ell$-sheaves

$$\omega_{\text{ét},\ell}\colon \mathbb{Z}_\ell \to \det R^2_{\text{ét}} f_* \mathbb{Z}_\ell(1),$$

for any prime ℓ. Suppose S is geometrically connected. Let $\eta \in S$ be a generic point of am irreducible component. As S is normal, the specialization map $\pi_1^{\text{ét}}(\eta, \bar\eta) \to \pi_1^{\text{ét}}(S, \bar\eta)$ is surjective. Thus we can reduce the statement to the case that $S = \mathrm{Spec}(k(\eta))$ for some finitely generated field $k(\eta)$, that is, the $\mathrm{Gal}(k(\bar\eta)/k(\eta))$-action on $\det\left(\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar\eta}, \mathbb{Z}_\ell)\right)$ is trivial for any ℓ.

As $b_2(\mathfrak{X}_{\bar\eta}) > 3$, we can see the image of $\mathrm{Gal}(k(\bar\eta)/k(\eta))$ in $\mathrm{GL}(\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar\eta}, \widehat{\mathbb{Z}}))$ lies in the special orthogonal group $\mathrm{SO}(\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar\eta}, \widehat{\mathbb{Z}}))$ by the argument in [4, Lemma 8.4.1]. Thus the Galois action on $\det\left(\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar\eta}, \widehat{\mathbb{Z}})\right)$ is trivial. $\square$

As $\widetilde{\mathscr{F}_d}$ is a smooth Deligne-Mumford stack, for any étale morphism $S \to \widetilde{\mathscr{F}_d}$ from a $\mathbb{Q}$-scheme S, we can see S is also smooth over $\mathbb{Q}$, and in particular is normal. Thus for the oriented universal family $(f_S\colon \mathfrak{X}_S \to S, \omega_S)$, we have a unique isomorphism of lisse $\widehat{\mathbb{Z}}$-local systems

$$\omega_{\text{ét},S}\colon \underline{\widehat{\mathbb{Z}}}_S \xrightarrow{\sim} R^2_{\text{ét}} f_{S,*} \widehat{\mathbb{Z}}(1)$$

by applying Lemma 4.5. Denote $R^2_{\text{ét}} f_* \widehat{\mathbb{Z}}(1)$ for the sheafification of

$$S \mapsto R^2_{\text{ét}} f_{S,*} \widehat{\mathbb{Z}}(1)$$

on the site $(\widetilde{\mathscr{F}_d})_{\text{ét}}$. We have

Proposition 4.6. *There is an isomorphism of lisse $\widehat{\mathbb{Z}}$-local systems on $(\widetilde{\mathscr{F}_d})_{\text{ét}}$:*

$$\omega_{\text{ét}} : \underline{\widehat{\mathbb{Z}}} \xrightarrow{\sim} R^2_{\text{ét}} f_* \widehat{\mathbb{Z}}(1). \tag{4.3}$$

Different from the case of K3 surfaces, the moduli stack $\mathscr{F}_d$ is not necessarily (geometrically) connected. As there are only finitely many connected components, we may assume that the moduli stack $\mathscr{F}_d$ or $\widetilde{\mathscr{F}_d}$ is connected otherwise we choose a connected component.

Let $\mathscr{F}_d^\dagger$ be a connected component of $\mathscr{F}_d$. Let $(X_0, H_0) \in \mathscr{F}_d^\dagger(\mathbb{C})$. Let Λ be the BB-lattice for X_0. Let Λ_h be the orthogonal complement of $h = c_1(H_0)$ in Λ. For any geometric point (X, ξ) in $\mathscr{F}_d^\dagger(\mathbb{C})$, there is an isomorphism of $\widehat{\mathbb{Z}}$-lattices

$$\phi \colon \Lambda \otimes \widehat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X, \widehat{\mathbb{Z}})$$

such that $\phi(h) = \xi$ by the connectedness of $\mathscr{F}_d^\dagger$ (*cf.* [7, Lemma 4.5.1]).

Remark 4.7. It is unclear to us whether the hyper-Kähler varieties in $\mathscr{F}_d^\dagger(\mathbb{C})$ are deformation equivalent.

Remark 4.8. For K3 surfaces, the moduli stack $\mathscr{F}_d$ can be defined over integers and it is smooth over $\mathbb{Z}[\frac{1}{2d}]$. It is natural to ask whether the same assertion holds for higher dimensional hyper-Kähler varieties. This is known to be true for some families, such as the Fano varieties of lines on cubic fourfolds. However, this is a rather difficult problem in general because the deformation theory and even the "right" definition of hyper-Kähler varieties over mixed characteristic fields are unclear at present.

4.2. **Level structures.** Following the work in [51] and [6], we write $G = \mathrm{SO}(\Lambda_h)_{\mathbb{Q}}$ and let $K \to G(\mathbb{A}_f)$ be a continuous group homomorphism from a profinite group, with open compact image and finite kernel. We say that K is *admissible* if every element of K can be viewed as an isometry of $\Lambda \otimes \mathbb{A}_f$ fixing h and stabilizing $\Lambda_h \otimes \widehat{\mathbb{Z}}$. Recall that there is an injection

$$K_h \coloneqq \left\{ g \in \mathrm{SO}(\Lambda)(\widehat{\mathbb{Z}}) \mid g(h) = h \right\} \to G(\mathbb{A}_f). \tag{4.4}$$

By definition, K_h is admissible, and K is admissible if and only if the image of K lies in K_h.

Consider $\mathcal{J}_d$, the sheafification of presheaf of sets on $(\widetilde{\mathscr{F}_d^\dagger})_{\text{ét}}$, with sections over a connected $\mathbb{Q}$-scheme T given by

$$(f \colon \mathfrak{X} \to T, \xi; \omega) \rightsquigarrow \left\{ \alpha \colon \Lambda \otimes \widehat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{t}}, \widehat{\mathbb{Z}}(1)) \,\middle|\, \begin{array}{c} \alpha \text{ is an isometry} \\ \text{such that } \alpha(h) = \widehat{c}_1(\xi_{\bar{t}}) \\ \text{and } \det(\alpha) = \omega_{\text{ét}}. \end{array} \right\}^{\pi_1^{\text{ét}}(T, \bar{t})} \tag{4.5}$$

Here $\widehat{c}_1$ is the product of all ℓ-adic first Chern maps and $\pi_1^{\text{ét}}(T, \bar{t})$ naturally acts on the right factor of α.

By construction, any admissible subgroup K naturally acts on $\mathcal{J}_h(T)$ (on the left factor of each α). Denote $K \backslash \mathcal{J}_h$ for the quotient of $\mathcal{J}_h$ by K on the left, which is well-defined in the category of étale sheaves on $(\widetilde{\mathscr{F}_d})^\dagger_{\text{ét}}$.

Definition 4.9. A K-level structure on a hyper-Kähler space $f \colon \mathfrak{X} \to S$ is a section $\alpha \in K \backslash \mathcal{J}_h(S)$.

Let $\widetilde{\mathscr{F}}_{d,K}$ be the moduli stack of oriented polarized hyper-Kähler varieties of degree d with a K-level structure. It can be viewed as a finite covering of the original moduli stack (*cf.* [40, Proposition 3.11]), i.e.,

Proposition 4.10. *The stack $\widetilde{\mathscr{F}}^{\dagger}_{d,K}$ is a smooth Deligne-Mumford stack and the forgetful map*

$$\pi_{d,K}\colon \widetilde{\mathscr{F}}^{\dagger}_{d,K} \to \widetilde{\mathscr{F}}^{\dagger}_{d}$$

is finite and étale.

For any field F in characteristic zero, $\widetilde{\mathscr{F}}^{\dagger}_{d,K}(F)$ consists of all $(X, H, \omega; \alpha)$ such that

- (X, H, ω) is a oriented hyper-Kähler vareity over F, and
- K-level structure $\alpha\colon \Lambda \otimes \widehat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \widehat{\mathbb{Z}}(1))$ satisfying
$$\alpha(\rho)(\mathrm{Gal}(\bar{F}/F)) \subseteq K,$$
where ρ is the Galois action $\mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}\,(\mathrm{H}^2_{\text{ét}}(X, \widehat{\mathbb{Z}}(1))$ and $\alpha(\rho)(\sigma) = \alpha^{-1} \circ \sigma \circ \alpha$ for any $\sigma \in \mathrm{Gal}(\bar{F}/F)$.

In this paper, we mainly consider the following level structures on hyper-Kähler varieties.

Definition 4.11. (1) The *full-level-n structure* is defined by the admissible group
$$K_{h,n} \coloneqq \left\{ g \in \mathrm{SO}(\Lambda)(\widehat{\mathbb{Z}}) \middle| gh = h, g \equiv 1 \mod n \right\} \to G(\mathbb{A}_f).$$

(2) As in [52], we introduce the *spin level structures* defined as follows: Let $\mathrm{Cl}^+(\Lambda_h)$ be the even Clifford algebra of $\Lambda_{h,\mathbb{Q}}$. Let
$$\mathrm{CSpin}(\Lambda_h)(\widehat{\mathbb{Z}}) = \mathrm{CSpin}(\Lambda_h)(\mathbb{A}_f) \cap \mathrm{Cl}^+(\Lambda_h \otimes \widehat{\mathbb{Z}}),$$
and
$$K^{\mathrm{sp}}_n = \left\{ g \in \mathrm{CSpin}(\Lambda_h)(\widehat{\mathbb{Z}}) \middle| g \equiv 1 \mod n \right\}.$$
Recall that one has an adjoint representation
$$\mathrm{ad}\colon \mathrm{GSpin}(\Lambda_h) \to G$$
defined by $\mathrm{ad}(x) = (v \mapsto xvx^{-1})$. The spin level-$n$ structure is the map
$$K^{\mathrm{sp}}_n \xrightarrow{\mathrm{ad}} G(\mathbb{A}_f).$$
We set $\mathbf{K}_n \subseteq G(\widehat{\mathbb{Z}})$ to be the image K^{sp}_n under adjoint representation: it is an open compact subgroup of $G(\mathbb{A}_f)$.

For the following usage, we denote $K^{\mathrm{ad}}_{L,n}$ and $K^{\mathrm{sp}}_{L,n}$ for the corresponding level-structure for a different lattice L.

It is known that the weak Torelli theorem may fail for hyper-Kähler varieties (Example 2.9), hence $\widetilde{\mathscr{F}}_{d,K}$ is not represented by a scheme (only a stack) even when K is very small, but we have the following remedy.

Corollary 4.12. *If the very general member of hyper-Kähler varieties has only the trivial automorphism, then $\widetilde{\mathscr{F}}^{\dagger}_{d,\mathbf{K}_n}$ is represented by a $\mathbb{Q}$-scheme $\mathcal{F}^{\dagger}_{d,\mathbf{K}_n}$ for $n \geq 3$.*

Proof. It suffices to show that any geometric point (X, ξ, α) in $\mathscr{F}^{\dagger}_{d,K^{\mathrm{ad}}_n}(\mathbb{C})$ has only trivial automorphism. Under our assumption, this follows from the fact the automorphism group of X acts faithfully on $\mathrm{H}^2(X, \mathbb{Z}(1))$ and any finite order automorphism of the pair $(\mathrm{P}^2(X, \mathbb{Z}(1)), \alpha)$ is trivial for $n \geq 3$ (*cf.* [51, Lemma 1.5.12]). □

Convention. By abuse of notations, we may omit the orientations in the tuple (X, ξ, ω, α) for simplicity.

5. Arithmetic period map for polarized hyper-Kähler varieties

A standard approach to study the moduli space $\widetilde{\mathscr{F}}_d$ introduced in the previous section is via the period map, which serves in a canonical way as a connection between $\widetilde{\mathscr{F}}_d$ and the Shimura varieties of orthogonal type.

5.1. **Orthogonal Shimura varieties.** Let us first recall the standard notions of Shimura varieties attached to a lattice. Let L be the lattice over $\mathbb{Z}$ of signature $(2,m)$ with $m \geq 1$ as in §4.2 and let $G = \mathrm{SO}(L \otimes \mathbb{Q})$ be the orthogonal group scheme over $\mathbb{Q}$ associated to $L_{\mathbb{Q}}$.

Consider the pair (G, D), where D is the space of oriented negative definite planes in $\Lambda_{\mathbb{R}}$:

$$D = \mathrm{SO}(2,m)/\mathrm{SO}(2) \times \mathrm{SO}(m)$$

where $\mathrm{SO}(2,m) = \mathrm{SO}(L)(\mathbb{R})$. For any admissible level structure $K \to G(\mathbb{A}_f)$, one can associated a Shimura stack $\mathrm{Sh}_K(G,D)$, which is in general a smooth Deligne-Mumford stack over $\mathbb{Q}$. When K is neat, $\mathrm{Sh}_K(G,D)$ is a smooth quasi-projective variety. Its complex points can be expressed as the double coset (quotient stack)

$$\mathrm{Sh}_K(G,D)(\mathbb{C}) = G(\mathbb{Q})\backslash(D \times G(\mathbb{A}_f))/K. \tag{5.1}$$

The pair (G,D) is called the Shimura data. It is well-known that the reflex field of the Shimura datumn (G,D) is equal to $\mathbb{Q}$. Let $\mathrm{Sh}_K(G,D)$ be the canonical model of $\mathrm{Sh}_K(G,D)(\mathbb{C})$ over $\mathbb{Q}$.

Remark 5.1. In [57] and [7], they denote $\mathrm{Sh}_K[G,D]$ for Shimura stack in order to distinguish the usual Shimura variety. For simplicity of notation, we will not do it here.

For simplicity, we denote by K_L the **discriminant kernel** of $G(\mathbb{A}_f)$, which is the largest subgroup of $G(\widehat{\mathbb{Z}})$ that acts trivially on the discriminant of L. It is just the image of $\mathrm{CSpin}(L)(\widehat{\mathbb{Z}})$. In particular, we have

$$K^{\mathrm{ad}}_{L,n} \subseteq K_L$$

as open compact subgroup for $n \geq 2$.

We write

$$\mathrm{Sh}(L) = \mathrm{Sh}_{K_L}(G,D)$$

for the orthogonal Shimura variety with level K_L, and

$$\mathrm{Sh}_K(L) = \mathrm{Sh}_K(G,D)$$

for any open compact subgroup $K \subset K_L$. If L contains a hyperbolic lattice, the Shimura variety $\mathrm{Sh}(L)(\mathbb{C}) = \Gamma_L\backslash D$ is irreducible and Γ_L is the largest subgroup of $G(\mathbb{Z})$ that acts trivially on the discriminant (*cf.* [40, (4.1)]). For any $n \geq 2$, there is a finite cover

$$\mathrm{Sh}_{K^{\mathrm{ad}}_{L,n}}(L) \to \mathrm{Sh}_{K_L}(L)$$

of degree $[K_L : K^{\mathrm{ad}}_{L,n}]$.

Let $\underline{L}$ be the local system on $\mathrm{Sh}(L)(\mathbb{C})$ attached to the tautological representation L of $\mathrm{O}(L)$. Note that L is equipped with a canonical symmetric bilinear form, which equips it with an injective map

$$\underline{L} \to \underline{L}^{\vee}$$

into its dual local system. The finite local system $\underline{L}^{\vee}/\underline{L}$ with its $\mathbb{Q}/\mathbb{Z}$-valued quadratic form is canonically isomorphic to the constant sheaf $\underline{\mathrm{disc}(L)}$ over $\mathrm{Sh}(L)(\mathbb{C})$.

The Shimura variety $\mathrm{Sh}(\Lambda_h)$ has the following modular description (*cf.* [44, §3] or [7, Lemma 4.4.4]).

Proposition 5.2. *The groupoid* $\mathrm{Sh}(L)(S)$ *is the solution of the moduli problem of tuples*

$$(H, q, \xi, \omega, \alpha_{K_L}),$$

where

- H *is a variation of* $\mathbb{Z}$*-Hodge structures over* S *whose fibers are Hodge lattices of K3-type with signature* $(3, m)$*;*
- $q\colon H \times H \to \underline{\mathbb{Z}}$ *is a bilinear form of variation Hodge structures;*
- ξ *is a global section of* H *of type* $(0,0)$ *such that* $q(\xi, \xi) > 0$*;*
- $\omega\colon \underline{\mathbb{Z}} \to \det(H)$ *is an isomorphism of local systems;*
- α_{K_L} *is a* K_L*-level structure on* (H, q, ξ)*;*

such that for any point $s \in S$ *there is a rational Hodge isometry* $\beta_s\colon H_s \otimes \mathbb{Q} \xrightarrow{\sim} \Lambda \otimes \mathbb{Q}$ *such that* $\beta_s(\xi_s) = h$ *and preserving the determinant.*

5.2. **Integral model of** $\mathrm{Sh}(L)$**.** Keep the notations same as previous section. In this part, we will recall some results on the existence of integral canonical models of the orthogonal Shimura varieties.

We refer to [41, Definition 4.2] for the definition of (smooth) integral canonical model. Among the requirements of this definition, the most important one for our usage is the smooth extension property. For the convenience of readers, we record it here.

Definition 5.3. A (pro)-scheme $\mathscr{S}$ over $\mathbb{Z}_{(p)}$ is with *(smooth) extension property* if for any regular and locally-healthy (formally smooth) scheme $\mathscr{X}$ over $\mathbb{Z}_{(p)}$, any morphism $\mathscr{X}_{\mathbb{Q}} \to \mathscr{S}$ can be extended to $\mathscr{X} \to \mathscr{S}$.

Then we have

Theorem 5.4. *Suppose the discriminant group* $\mathrm{disc}(L \otimes \mathbb{Z}[\frac{1}{N}])$ *is cyclic. Then the Shimura varieties* $\mathrm{Sh}(L)$ *admits a canonical regular integral models* $\mathscr{S}_L$ *over* $\mathbb{Z}[\frac{1}{2N}]$*. Moreover, for any neat* $K \subset K_L$ *such that the* p*-primary component* $K_p = K_{L,p}$ *for all* $p \nmid 2N$*, there is an étale extension*

$$\mathscr{S}_{L,K} \to \mathscr{S}_L.$$

Proof. For any $p \nmid 2N$, the $\mathbb{Z}_{(p)}$-lattice $L \otimes \mathbb{Z}_{(p)}$ is with cyclic discriminant group. Let $K_p \subset G(\mathbb{Z}_p)$ be the p-primary component of the decomposition $K_L = K^p K_p$. Applying [41, Theorem 4.4], we will obtain an integral model $\mathscr{S}_{L,K_p} = \{\mathscr{S}_{L,K_p,K}\}_{K \subset K^p}$ over $\mathbb{Z}_{(p)}$ as a pro-scheme with étale connect morphisms

$$\mathrm{Sh}(L)_{(p)} := \mathrm{Sh}_{K_p}(G, D) = \varprojlim_{K \subset K^p} \mathrm{Sh}_{KK_p}(G, D),$$

which is formally smooth and with smooth extension property. We can consider

$$\mathscr{S}_L = \bigcup_{p \notin S} \mathscr{S}_{L,K_p}/K_p,$$

which is an integral canonical model of $\mathrm{Sh}(L)$ over $\mathbb{Z}[\frac{1}{2N}]$ as required. □

Given L of signature $(2, n)$, suppose we can find a primitive embedding of $\mathbb{Z}$-lattices

$$L \hookrightarrow L_0$$

such that L_0 has signature $(2, n+1)$. Let N be the product of all primes in $|\mathrm{disc}(L_0)|$. Then L_0 is unimodular over $\mathbb{Z}[\frac{1}{N}]$ and the discrimiant group $\mathrm{disc}\left(L \otimes \mathbb{Z}[\frac{1}{N}]\right)$ is cyclic. As a consequence of Theorem 5.4, we get

Corollary 5.5. *Let* $m \geq 3$ *be an integer. Let* N *be the product of all primes in and* $2m|\mathrm{disc}(\Lambda)|$*. Then* $\mathrm{Sh}_{K_n^{\mathrm{ad}}}(\Lambda_h)$ *admits a regular integral canonical model over* $\mathbb{Z}[\frac{1}{2N}]$ *for all* Λ_h*.*

5.3. **Period map over $\mathbb{C}$.** Let $\widetilde{\mathscr{F}}_d^\dagger$ be a fixed connected component which contains a geometric point (X_0, H) as in §4.2. There is a $\widehat{\mathbb{Z}}$-isometry

$$\phi\colon \Lambda \otimes \widehat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X, \widehat{\mathbb{Z}})$$

such that $\phi(h) = \widehat{c}_1(\xi)$ for any $(X, \xi) \in \widetilde{\mathscr{F}}_d^\dagger(\mathbb{C})$. This $\widehat{\mathbb{Z}}$-isometry also induces an isomorphism

$$\Lambda_h^\vee/\Lambda_h \xrightarrow{\sim} \mathrm{disc}(c_1(\xi)^\perp).$$

With the modular description of the orthogonal Shimura stack given in Proposition 5.2, we thus obtain an analytic map

$$\widetilde{\mathscr{F}}_{d,K}^\dagger(\mathbb{C}) \to \mathrm{Sh}_K(\Lambda_h)(\mathbb{C}),$$

by sending (X, ξ, ω, α) to $\Big((\mathrm{P}^2_B(X, \mathbb{Z}), F^\bullet_{\mathrm{Hdg}}), q_X, \omega, c_1(\xi), \alpha\Big)$. Such morphism is called the **period map** of primitively polarized hyper-Kähler varieties of degree d, denoted by $\mathcal{P}_{h,K,\mathbb{C}}$. The global Torelli theorem for polarized hyper-Kähler varieties shows

Theorem 5.6 ([63, Theorem 8.4])**.** *The period map*

$$\mathcal{P}_{h,K,\mathbb{C}}\colon \widetilde{\mathscr{F}}_{d,K}^\dagger(\mathbb{C}) \to \mathrm{Sh}_K(\Lambda_h)(\mathbb{C}) \tag{5.2}$$

is étale.

5.4. **Descent of (local) period map.** Let F be a number field. Fix a complex embedding $F \subset \mathbb{C}$. Let (X, H) be a polarized hyper-Kähler variety over F. Let V be its deformation space. Consider its universal deformation $f\colon \mathcal{X} \to V$.

Proposition 5.7 ([15, Proposition 16])**.** *There is a finite extension F' of F and a formally étale morphism of F'-schemes $V \to \mathrm{Sh}(\Lambda_h)_{F'}$ which factors through $\mathcal{P}_{h,\mathbb{C}}$ after taking field extension along $F' \subset \mathbb{C}$.*

Globally, we have the following result of Bindt.

Proposition 5.8 ([7, Theorem 4.5.2])**.** *For any admissible level $K \subset K_{\Lambda_h}$, there is an étale morphism over $\mathbb{Q}$*

$$\mathcal{P}_{d,K}\colon \widetilde{\mathscr{F}}_{d,K}^\dagger \to \mathrm{Sh}_K(\Lambda_h),$$

such that $\mathcal{P}_{d,K} \otimes \mathbb{C} = \mathcal{P}_{d,K,\mathbb{C}}$. Moreover, we have a 2-Cartisian diagram

$$\begin{tikzcd}
\widetilde{\mathscr{F}}_{d,K}^\dagger \arrow[r, "\mathcal{P}_{d,K}"] \arrow[d, "\pi_{d,K}"'] & \mathrm{Sh}_K(\Lambda_h) \arrow[d] \\
\widetilde{\mathscr{F}}_d^\dagger \arrow[r, "\mathcal{P}_d"] & \mathrm{Sh}_{K_h}(\mathrm{SO}(\Lambda_h), D_{\Lambda_h}).
\end{tikzcd}$$

6. Kuga–Satake morphism for hyper-Kähler varieties

The Kuga–Satake map plays a central role in our approach. In this section, we give an overview of the Kuga–Satake construction for polarized hyper-Kähler varieties, and construct a uniform Kuga–Satake map for all polarization types.

6.1. **Kuga–Satake construction in general.** Given a lattice L of signature $(2, b-2)$ for $b \geq 3$ and an element $e \in \mathrm{Cl}^+(L)$ such that $\iota(e) = -e$ under the canonical involution, the spin representation defines a morphism of Shimura datum

$$\mathrm{sp}_e \colon (\mathrm{GSpin}(L), \widetilde{D}_L) \to (\mathrm{GSp}(W), \Omega^{\pm})$$

where $W = (\mathrm{Cl}^+(L), \varphi_e)$ and $\Omega^{\pm}$ is the Siegel half space. This induces morphisms

$$\begin{tikzcd} \mathrm{Sh}_K(\mathrm{GSpin}(L), \widetilde{D}_L) \arrow[d, "\mathbf{ad}"] \arrow[dr, "\mathbf{sp}_e"] & \\ \mathrm{Sh}_{K''}(\mathrm{SO}(L), D_L) & \mathrm{Sh}_{K'}(\mathrm{GSp}(W), \Omega^{\pm}) \end{tikzcd} \tag{6.1}$$

between the canonical models of Shimura varieties, where $\mathrm{sp}_e(K) \subseteq K'$ and $\mathrm{ad}(K) \subseteq K''$.

If we take the levels K and K'' to be so called spin levels as in Definition 4.11, then there exists a (non-canonical) section

$$\gamma_{\mathbb{C}} \colon \mathrm{Sh}_{K''}(\mathrm{SO}(L), D_L)(\mathbb{C}) \to \mathrm{Sh}_K(\mathrm{GSpin}(L), \widetilde{D}_L)(\mathbb{C}) \tag{6.2}$$

of $\mathbf{ad}$ (*cf.* [52, §5.5]). It has a descent over a number field E, denoted by γ_E. The field E only depends on the group K. The composition

$$\mathbf{sp}_e(\mathbb{C}) \circ \gamma_{\mathbb{C}} \colon \mathrm{Sh}_K(L)(\mathbb{C}) \to \mathrm{Sh}_{K'}(\mathrm{GSp}(W), \Omega^{\pm})(\mathbb{C})$$

is called the **Kuga–Satake map**.

Geometrically, write $W_R = W \otimes R$. Then $W_{\mathbb{R}}$ is naturally a G-module (action by left multiplication), gives rise to a polarizable Hodge structure of weight one on $W_{\mathbb{Z}}$. This defines a complex abelian variety $A(W)$ of dimension 2^N, called the **Kuga–Satake variety** attached to $(W_{\mathbb{Z}}, \varphi)$, by the condition that $H_1(A, \mathbb{Z}) = W_{\mathbb{Z}}$ as a Hodge structure. André has shown that ψ_L can be defined over some number field and this construction is the main ingredient to prove the polarized Shafarevich conjecture.

If one takes L to be Λ_h, the composition

$$\psi_h \colon \widetilde{\mathscr{F}}^{\dagger}_{d,K} \to \mathrm{Sh}_K(L) \to \mathrm{Sh}_{K'}(\mathrm{GSp}(W), \Omega^{\pm})$$

is the classical Kuga–Satake morphism constructed by Deligne [16] and André [4].

6.2. **Uniform Kuga–Satake for hyper-Kähler varieties.** Given a deformation type M of hyper-Kähler varieties with $\Lambda_M = \Lambda$, we can define the so-called **uniform Kuga–Satake construction**, which extends the construction in [56]. The first step is to embed the lattice Λ_h of signature $(2, n)$ primitively into a fixed unimodular lattice of signature $(2, m)$ for any h (up to multiplying the quadratic form by 2).

Lemma 6.1 (Uniform lattice)**.** *Given an even lattice (Λ, q) of signature $(3, m)$, there exists an even unimodular lattice Σ (depending only on $\mathrm{rank}(\Lambda)$) such that there is a primitive embedding*

$$h^{\perp} = \Lambda_h \hookrightarrow \Sigma, \tag{6.3}$$

for any $h \in \Lambda$ with $q(h) > 0$.

Proof. By Nikulin's result [48, Theorem 1.12.4], the lattice Λ admits a primitive embedding into an even unimodular lattice Σ of signature $(2, N)$ provided $N \geq 2\mathrm{rank}(\Lambda) - 2$. One can thus choose $N > 2m$ to be the smallest integer satisfying 8 divides $(N-2)$ and let

$$\Sigma = U^{\oplus 2} \oplus E_8^{\oplus d}$$

where $d = \frac{N-2}{8}$. $\square$

Remark 6.2. If the Beauville–Bogolomov lattice Λ is not even, Lemma 6.1 may fail. Nevertheless, we can replace it by a twisting quadratic form on $\mathrm{H}^2(X, \mathbb{Z})$ obtained by multiplying the Beauville-Bogolomov form by 2 and the same construction still applies.

According to Remark 6.2, we can alwasy assume Λ to be even after possibly taking a 2-twist. For a lattice L, we write $G_L = \mathrm{SO}(L)$ for short. The inclusion (6.3) defines a map of Hodge structures $D_{\Lambda_h} \to D_\Sigma$, which gives an embedding of Shimura datum

$$\iota_h : (G_{\Lambda_h}, D_{\Lambda_h}) \to (G_\Sigma, D_\Sigma).$$

Let $K \subseteq G_\Sigma(\mathbb{A}_f)$ be a compact open subgroup. Then for any open compact subgroup $K_h \subseteq G_{\Lambda_h}(\mathbb{A}_f)$ with $\iota_h(K_h) \subseteq K$, we have a finite and unramified map

$$\mathrm{Sh}_{K_h}(G_{\Lambda_h}, D_{\Lambda_h}) \to \mathrm{Sh}_K(G_\Sigma, D_\Sigma). \tag{6.4}$$

defined over $\mathbb{Q}$.

If K is contained in the discriminant kernel K_Σ, then we will get a section as (6.2) for

$$\mathbf{ad}(\mathbb{C})\colon \mathrm{Sh}_{K^{\mathrm{sp}}}(\mathrm{GSpin}(\Sigma), \widetilde{D}_L)(\mathbb{C}) \to \mathrm{Sh}_K(G_\Sigma, D_L)(\mathbb{C}) = \mathrm{Sh}_K(\Sigma)(\mathbb{C}).$$

In particular, we will take the following level-n structures

$$\mathbf{K}'_n := K^{\mathrm{ad}}_{\Sigma,n} \quad \text{and} \quad \mathbf{K}''_n := \Big\{ g \in \mathrm{GSp}(W)(\widehat{\mathbb{Z}}) \big| g \equiv 1 \bmod n \Big\}, \tag{6.5}$$

where $W = (\mathrm{Cl}^+(\Sigma_\mathbb{Q}), \varphi_e)$ the symplectic space described in §6.1. For any $d \in \mathbb{Z}_{\geq 1}$, we have the following commutative diagram of stacks over $\mathbb{Q}$

$$\begin{tikzcd}
 & & \mathrm{Sh}_{K^{\mathrm{sp}}_{\Sigma,n}}(\mathrm{GSpin}(\Sigma), \widetilde{D}_\Sigma) \arrow[dl, dashed, shift left, "\mathbf{ad}"] \arrow[d, "\mathbf{sp}"] \\
\widetilde{\mathscr{F}}^\dagger_{d,\mathbf{K}_n} \arrow[r] \arrow[d] & \mathrm{Sh}_{\mathbf{K}'_n}(\Sigma) \arrow[d] \arrow[ur, dashed, shift left, "\gamma_{E_n}"] & \mathrm{Sh}_{\mathbf{K}''_n}(\mathrm{GSp}(W), \Omega^\pm) \\
\widetilde{\mathscr{F}}^\dagger_d \arrow[r] & \mathrm{Sh}(\Sigma) &
\end{tikzcd} \tag{6.6}$$

where γ_{E_n} is the descent of a chosen section $\gamma_\mathbb{C}$ to the finite abelian extension E_n of $\mathbb{Q}$ corresponding to $\mathbb{R}_{>0}\mathbb{Q}^\times \mathbf{N}(K^{\mathrm{sp}}_{\Sigma,n}) \subseteq \mathbb{A}^\times_\mathbb{Q}$ via the class field theory. We now obtain a map

$$\Psi^{\mathrm{KS}}_{d,\mathbf{K}_n,\mathbb{C}}\colon \widetilde{\mathscr{F}}^\dagger_{d,\mathbf{K}_n}(\mathbb{C}) \to \mathrm{Sh}_{\mathbf{K}''_n}(\mathrm{GSp}(W), \Omega^\pm)(\mathbb{C}) \tag{6.7}$$

as the composition, which is quasi-finite. It is called the **uniform Kuga–Satake map**. By the construction, we have the following descent theorem.

Theorem 6.3. *The uniform Kuga–Satake map $\Psi^{\mathrm{KS}}_{d,\mathbf{K}_n,\mathbb{C}}$ descends to map*

$$\Psi^{\mathrm{KS}}_{d,\mathbf{K}_n} : \widetilde{\mathscr{F}}^\dagger_{d,\mathbf{K}_n} \to \mathrm{Sh}_{\mathbf{K}''_n}(\mathrm{GSp}(W), \Omega^\pm)$$

over the number field E_n. The field E_n is clearly independent of d and h.

Proof. The $\Psi^{\mathrm{KS}}_{d,\mathbf{K}_n,\mathbb{C}}$ has a descent over E_n as the period map $\widetilde{\mathscr{F}}^\dagger_{d,\mathbf{K}_n}(\mathbb{C}) \to \mathrm{Sh}_{\mathbf{K}_n}(\Lambda_h)(\mathbb{C})$ has a descent over $\mathbb{Q}$ by Proposition 5.8. It is clear that E_n is independent of h. $\square$

Definition 6.4. If $f\colon T \to \widetilde{\mathscr{F}}^{\dagger}_{d,\mathbf{K}_n}$ is a map corresponding to a polarized oriented hyper-Kähler space with level structure $(\mathfrak{X}, \xi, \alpha)$ over T. We define the uniform Kuga–Satake abelian space $\mathcal{A}_T \to T$ associated to f as the pull back of the universal polarized abelian scheme on $\mathrm{Sh}_{\mathbf{K}''_n}(\mathrm{GSp}(W), \Omega)$ under the composition

$$T \to \widetilde{\mathscr{F}}^{\dagger}_{d,\mathbf{K}_n} \to \mathrm{Sh}_{\mathbf{K}''_n}(\mathrm{GSp}(W), \Omega^{\pm}). \tag{6.8}$$

Suppose $f\colon \operatorname{Spec} F \to \mathrm{Sh}_{\mathbf{K}_n}(\Lambda_h)$ comes from a F-point of $\widetilde{\mathscr{F}}^{\dagger}_{d,\mathbf{K}_n}$, which represents the tuple of polarized hyper-Kähler variety (X, H, α) with a $\mathbf{K}_n$-level structure.

Proposition 6.5. *Let (A, L) be the associated uniform Kuga–Satake polarized abelian variety of (X, H, α). Then there is a Galois equivariant lattice embedding*

$$\mathrm{P}^2_{\text{ét}}(X_{\bar{F}}, \widehat{\mathbb{Z}}(1)) \cong f^*(\underline{\Lambda_h}) \hookrightarrow \mathrm{End}_{\mathrm{Cl}^+(\Sigma)}\left(\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \widehat{\mathbb{Z}})\right)$$

such that $\mathrm{Gal}(\overline{F}/F)$ acts trivially on the orthogonal complement.

Proof. This was proved in [40] and [56]. □

7. Finiteness results

In this section we prove all the finiteness results claimed in the introduction.

7.1. **Reduction of the base field.** Throughout this section, we shall fix an embedding $F \hookrightarrow \mathbb{C}$. In order to apply the uniform Kuga–Satake construction, we need to suitably change the base field. This is allowed due to the following reduction results.

Lemma 7.1. *Let R_E/R be a finite Galois extension between finitely generated normal domains whose fraction field is E/F. We define*

$$\mathrm{Shaf}_M(E/F, R_E) = \left\{X' \in \mathrm{Shaf}_M(E, R_E)\ |X' \cong X \times_F \mathrm{Spec}(E) \textit{ for some } X \in \mathrm{Shaf}_M(F, R)\right\}$$

For any $X \in \mathrm{Shaf}(F, R)$, the set

$$\{Y \in \mathrm{Shaf}_M(F, R)|\ Y_E \cong X_E\}$$

is finite. If $\mathrm{Shaf}_M(E/F, R_E)$ is finite, then $\mathrm{Shaf}_M(F, R)$ is finite. Similar result holds for the finiteness of birational isomorphism classes of the Shafarevich sets.

Proof. See [56, Lemma 4.1.4]. □

To apply the uniform Kuga–Satake map, we need to associate the element in $\mathrm{Shaf}_M(E/F, S_E)$ a level structure defined over a common field extension E/F which does not depend on h. This can be done by applying the theory of integral models of Shimura varieties $\mathrm{Sh}_K(\Lambda)$, due to Kisin [34] and Madapusi Pera [41].

Lemma 7.2. *With notations in (6.5), there exists a finite field extension E/F such that for every polarized hyper-Kähler variety (X, H) of deformation type M defined over F, it can be equipped with a $\mathbf{K}_4$-level structure over E.*

Proof. It suffices to show that there exists E/F such that for any map $\operatorname{Spec} F \to \mathrm{Sh}(\Lambda_h)$, it admits a lift to $\mathrm{Sh}_{\mathbf{K}_4}(\Lambda_h)$ after a finite base change to E. By Theorem 5.4 and Corollary 5.5, the forgetful map

$$\mathrm{Sh}_{\mathbf{K}_4}(\Lambda_h) \to \mathrm{Sh}(\Lambda_h)$$

is a finite étale covering which can be extended integrally over $\mathbb{Z}[\frac{1}{2N}]$. Its degree is at most $|\mathrm{SO}(\Lambda)(\widehat{\mathbb{Z}}/4\widehat{\mathbb{Z}})|$.

Let E be composition of all field extensions of degree at most $|\mathrm{SO}(\widehat{\Lambda})(\mathbb{Z}/4\mathbb{Z})|$, and unramified away from the primes dividing $2|\mathrm{disc}(\Lambda)|$. There are finitely many such field extensions by the Hermite-Minkowski theorem for finitely generated field ([18, Chap. VI, Sect. 2]). Thus E/F is a finite extension. For such E, the image of $\mathrm{Gal}(\bar{F}/E)$ in $\mathrm{SO}(\Lambda_h)(\widehat{\mathbb{Z}})$ lies in $\mathbf{K}_4$. Thus the composition $\mathrm{Spec}(E) \to \mathrm{Spec}(F) \to \mathrm{Sh}(\Lambda_h)$ admits a lift to $\mathrm{Sh}_{\mathbf{K}_4}(\Lambda_h)$. □

7.2. **Finiteness of Picard lattices.** By Lemma 7.1, we are allowed to have finite field extensions and hence we may always assume that the uniform Kuga-Satake map $\Psi^{\mathrm{KS}}_{d,\mathbf{K}_4}$ is defined over F. The following fact is standard and well-known to experts, which says that the cohomological Shafarevich condition is preserved under (uniform) Kuga–Satake constructions.

Lemma 7.3. *Let (X, H, α) be a polarized hyper-Kähler variety with a $\mathbf{K}_4$-level structure defined over F such that $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell)$ is unramified. Then the associated uniform Kuga–Satake abelian variety $\Psi^{\mathrm{KS}}_{d,\mathbf{K}_4}(X, H, \alpha)$ has good reduction at $\mathfrak{p}$.*

Proof. See [4, Lemma 9.3.1] and also [58, Proposition 4.2.4]. □

The key finiteness result is the following:

Theorem 7.4. *There exist only finitely many couples of lattices $(\Sigma, \widehat{\Sigma})$ (depending on Λ_M) such that*

$$\mathrm{Pic}_X \cong \Sigma \ \textit{and} \ \mathrm{Pic}(X_{\mathbb{C}}) \cong \widehat{\Sigma}$$

for some $X \in \mathrm{Shaf}^{\mathrm{hom}}_M(F, R)$.

Proof. The argument is similar to the case of K3 surfaces in [56, Corollary 4.1.13]. We nevertheless give some details of the proof for hyper-Kähler varieties. The main difference to the case of K3 surfaces is that Λ_M is now not unimodular in general.

For a polarized hyper-Kähler variety (X, H) of deformation type M defined over F, it has a $\mathbf{K}_4$-level structure α_E over E by Lemma 7.2. We may assume E/F is Galois. By Lemma 7.3 and Faltings' theorem, there are only finitely many uniform Kuga–Satake abelian varieties $A_i, i = 1, \dots, m$ defined over E associated to (X_E, H_E, α_E) with $X \in \mathrm{Shaf}_M(F, R)$. Consider the transcendental lattice of X_E

$$T(X_E) := \mathrm{Pic}^{\perp}_{X_E} \subseteq \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}),$$

then from Theorem 2.6, we know that

$$T(X_E) \otimes \widehat{\mathbb{Z}} \cong (\mathrm{P}^2_{\text{ét}}(X_{\bar{E}}, \widehat{\mathbb{Z}}(1))^{\mathrm{Gal}(\bar{E}/E)})^{\perp} \subseteq \mathrm{P}^2_{\text{ét}}(X_{\bar{E}}, \widehat{\mathbb{Z}}(1)).$$

According to Proposition 6.5, we have a $\mathrm{Gal}(\bar{E}/E)$-equivariant isometric embedding

$$\mathrm{P}^2_{\text{ét}}(X_{\bar{E}}, \widehat{\mathbb{Z}}(1)) \to \mathrm{End}_{\mathrm{Cl}^+(\Sigma)}(\mathrm{H}^1_{\text{ét}}((A_i)_{\bar{E}}, \widehat{\mathbb{Z}})) \tag{7.1}$$

for some A_i and its orthogonal complement is a trivial $\mathrm{Gal}(\bar{E}/E)$-submodule. Via (7.1), $T(X_E) \otimes \widehat{\mathbb{Z}}$ is isomorphic to $(\mathrm{P}^2_{\text{ét}}(X_E, \widehat{\mathbb{Z}}(1))^{\mathrm{Gal}(\bar{E}/E)})^{\perp}$ in $\mathrm{End}_{\mathrm{Cl}^+(\Sigma)}(\mathrm{H}^1_{\text{ét}}((A_i)_{\bar{E}}, \widehat{\mathbb{Z}}))$. The same argument in [56, Proposition 4.1.11]) shows that there is a finite set of $\mathrm{Gal}(\bar{E}/E)$-module V_j with coefficients in $\widehat{\mathbb{Z}}$ such that

$$T(X_E) \otimes \widehat{\mathbb{Z}} \simeq V_j \text{ and } V_j \subset \mathrm{End}_{C(\Sigma)}(\mathrm{H}^1_{\text{ét}}((A_i)_{\bar{E}}, \widehat{\mathbb{Z}}))$$

for some A_i. This implies $\mathrm{disc}(T(X_E)) = \mathrm{disc}(V_j)$ is bounded. As the rank of $T(X_E)$ is automatically bounded by the Betti number, the isomorphism classes of $T(X_E)$ has to be finite.

Now, using [35, San 30.2], we know that the collection of primitive embeddings

$$T(X_E) \to \Lambda$$

is finite. It follows that the set of lattices $\mathrm{Pic}_{X_E} = T(X_E)^{\perp}$ is finite. As there are only finitely many conjugacy classes of homomorphisms $\mathrm{Gal}(E/F) \to \mathrm{O}(\mathrm{Pic}_{X_E})$ (see [50, Theorem 4.3]), we get the finiteness of Pic_X.

Similarly, the same argument as above shows that the geometric transcendental lattice

$$T(X_{\mathbb{C}}) = \mathrm{Pic}(X_{\mathbb{C}})^{\perp}$$

has bounded discriminant and one can conclude that the set of isomorphism classes of $\mathrm{Pic}(X_{\mathbb{C}})$ is finite. □

7.3. **Finiteness of birational classes.** We now prove the finiteness of birational isomorphism classes in various Shafarevich sets. Note that in this step we make no use of the Kawamata–Morrison cone conjecture (or equivalently, boundedness of Beauville–Bogomolov squares of wall divisors) over non algebraically closed fields (Theorem 3.5); the latter only comes into play when we deduce the finiteness of Shafarevich sets from their birational finiteness.

The first step is the existence of birational ample classes with uniformly bounded Beauville–Bogomolov square on hyper-Kähler varieties in $\mathrm{Shaf}^{\mathrm{hom}}_M(F, R)$.

Proposition 7.5. *There exists an integer N such that for any $X \in \mathrm{Shaf}^{\mathrm{hom}}_M(F, R)$, there is a polarized hyper-Kähler variety $(Y, H) \in \mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F, R)$ such that Y is F-birational to X and $d \leq N$.*

Proof. By Theorem 7.4, there are only finitely many isomorphism classes of lattices Pic_X and $\mathrm{Pic}(X_{\mathbb{C}})$ for $X \in \mathrm{Shaf}^{\mathrm{hom}}_M(F, R)$. Therefore, we may fix the geometric Picard lattice $\Xi_{\mathbb{C}}$ and the Picard lattice Ξ. Since the moduli space of $\Xi_{\mathbb{C}}$-lattice polarized hyper-Kähler varieties of given deformation type M is of finite type, it is enough to consider its irreducible components one by one.

Fix such a component and take an $X \in \mathrm{Shaf}^{\mathrm{hom}}_M(F, R)$ in this component such that $\mathrm{Pic}(X_{\mathbb{C}}) \cong \Xi_{\mathbb{C}}$, $\mathrm{Pic}_X \cong \Xi$ (if there is no such X, then the component is irrelevant to the question). Denote by $i : \Xi \hookrightarrow \Xi_{\mathbb{C}}$ the natural primitive embedding coming from $\mathrm{Pic}_X \hookrightarrow \mathrm{Pic}(X_{\mathbb{C}})$. It suffices to show that there exists an integer N, such that for each hyper-Kähler variety X' defined over F satisfying the following conditions:

- $\mathrm{Pic}(X'_{\mathbb{C}}) \cong \Xi_{\mathbb{C}}$ and $\mathrm{Pic}_{X'} \cong \Xi$, as lattices,
- the $\Xi_{\mathbb{C}}$-lattice-polarized hyper-Kähler manifolds $(X_{\mathbb{C}}, \Xi_{\mathbb{C}} \cong \mathrm{Pic}(X_{\mathbb{C}})$ and $(X'_{\mathbb{C}}, \Xi_{\mathbb{C}} \cong \mathrm{Pic}(X'_{\mathbb{C}}))$ are deformation equivalent;

there exists a hyper-Kähler variety Y defined over F which is F-birational to X' and carries a polarization H of Beauville–Bogomolov square $\leq N$.

Let $\mathcal{W}(X_{\mathbb{C}}) \subseteq \Xi_{\mathbb{C}}$ be the collection of wall divisors on $X_{\mathbb{C}}$. We define N as follows:

$$N := \max_{\varphi:\Xi\hookrightarrow\Xi_{\mathbb{C}}} \inf\{v^2 > 0 |\, v \in \Xi, \varphi(v) \cdot w \neq 0,\ \forall w \in \mathcal{W}(X_{\mathbb{C}})\}, \tag{7.2}$$

where the maximum is running over all the primitive embeddings of Ξ into $\Xi_{\mathbb{C}}$. As there are only finitely many primitive embeddings $\Xi \hookrightarrow \Xi_{\mathbb{C}}$ up to the action of the Hodge monodromy group (whose image in $O(\Xi_{\mathbb{C}})$ is a finite index subgroup and its action on $\Xi_{\mathbb{C}}$ does not change $\mathcal{W}(X_{\mathbb{C}})$), the maximum exists once the set $\{v^2 > 0 |\, v \in \Xi, \varphi(v) \cdot w \neq 0\ \forall w \in \mathcal{W}(X_{\mathbb{C}})\}$ is nonempty for some φ. This is clear as one can take $\varphi = i$ and a polarization of X. Hence N is well-defined and finite.

Now by the assumption that $(X_{\mathbb{C}}, \Xi_{\mathbb{C}} \cong \mathrm{Pic}(X_{\mathbb{C}})$ and $(X'_{\mathbb{C}}, \Xi_{\mathbb{C}} \cong \mathrm{Pic}(X'_{\mathbb{C}}))$ are deformation equivalent as lattice polarized hyper-Kähler varieties, and the deformation invariance of wall divisors (see [1]), we know that the isometries $\mathrm{Pic}(X'_{\mathbb{C}}) \cong \Xi \cong \mathrm{Pic}(X_{\mathbb{C}})$ induce an identification between $\mathcal{W}(X'_{\mathbb{C}})$ and $\mathcal{W}(X_{\mathbb{C}})$. Recall that the set of birational ample classes in $\mathrm{Pic}_{X'}$ is given by

$$\{v \in \mathrm{Pic}_{X'} |\, v \cdot w \neq 0\ \forall w \in \mathcal{W}(X'_{\mathbb{C}})\}.$$

It follows that $d := \inf\{v^2 > 0 |\ v \in \mathrm{Pic}_{X'} \text{ and } v \cdot w \neq 0\ \forall w \in \mathcal{W}(X'_{\mathbb{C}})\}$ is no greater than N. Take $H' \in \mathrm{Pic}_{X'}$ with $(H')^2 = d$, then consider the polarized variety over F:

$$(Y, H) := \left(\mathrm{Proj} \bigoplus_m H^0(X', \mathcal{O}(mH')), \mathcal{O}(1)\right).$$

By the minimal model theory for hyper-Kähler varieties, $(Y_{\mathbb{C}}, H)$ is a polarized hyper-Kähler variety, which is F-birational isomorphic to X, hence of deformation type M by [21, Theorem 4.6]. Hence (Y, H) is also hyper-Kähler and satisfies all the desired properties. □

In case that the weak Torelli theorem holds, André's proof in [4] has shown the finiteness of the set $\mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F, R)$. However, in general, $\mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F, R)$ is not necessarily finite. To proceed, we need the following enhancement of the André's finiteness of polarized Shafarevich sets in [4].

Theorem 7.6. *The subset*

$$\widetilde{\mathrm{Shaf}}^{\mathrm{hom}}_{M,d}(F, R) = \{(X, H) \in \mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F, R) |\ \exists\, Y \in \mathrm{Shaf}_M(F, R) \text{ such that } X \sim_{bir} Y\} \tag{7.3}$$

is finite.

Proof. According to [4], the usual Kuga–Satake construction already shows that there are only finitely many geometric isomorphism classes in $\mathrm{Shaf}_{M,d}(F, R)$ is finite. For every $(X, H) \in \mathrm{Shaf}^{\mathrm{hom}}_{M,d}(F, R)$ equipped with a $\mathbf{K}_4$-level structure α defined over F, André's result shows that the geometric isomorphism classes of (X, H, α) is finite, because the isomorphism classes of their Kuga–Satake varieties are finite. It suffices to show the set

$$\Pi_{(X,H)} = \left\{(X', H', \alpha') |\ (X', H') \in \widetilde{\mathrm{Shaf}}^{\mathrm{hom}}_{M,d}(F, R), (X'_{\bar{F}}, H'_{\bar{F}}, \alpha'_{\bar{F}}) \cong (X_{\bar{F}}, H_{\bar{F}}, \alpha_{\bar{F}})\right\} \tag{7.4}$$

is finite. We proceed a similar argument in [4, Lemma 9.5.1] by using Hermite–Minkowski theorem. Note that $\Pi_{(X,H)}$ can be identified as a subset of the Galois cohomology

$$\mathrm{H}^1(\mathrm{Gal}(\bar{F}/F), \mathrm{Aut}(X_{\bar{F}}, H_{\bar{F}}, \alpha_{\bar{F}})), \tag{7.5}$$

consisting of elements lying in the image of $\mathrm{H}^1(\mathrm{Gal}(\bar{R}_{\mathfrak{p}}/R_{\mathfrak{p}}), \mathrm{Aut}(X_{\bar{F}}, H_{\bar{F}}, \alpha_{\bar{F}}))$ for every $\mathfrak{p}$. Where $\bar{R}_{\mathfrak{p}}$ is the normal closure of $R_{\mathfrak{p}}$ in $\bar{F}$.

First, we have

$$\mathrm{Aut}(X_{\bar{F}}, H_{\bar{F}}, \alpha_{\bar{F}}) = \mathrm{Aut}(X_{\bar{F}}, \alpha_{\bar{F}}) = \mathrm{Aut}_0(X_{\bar{F}}).$$

The second equality is because if we fix a K-level structure with $K \subset \mathbf{K}_n$ for $n \geq 3$, the automorphism of X acts trivially on $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z})$ by [47, p. 207, Lemma].

Next, from the definition of (7.3), there is a F-birational map $\varphi : X \dashrightarrow Y$ for some $Y \in \mathrm{Shaf}_M(F, R)$. The birational map induces a canonical $\mathrm{Gal}(\bar{F}/F)$-equivariant isomorphism

$$\begin{aligned} \mathrm{Aut}(X_{\bar{F}}, \alpha_{\bar{F}}) &\cong \mathrm{Bir}(Y_{\bar{F}}, \varphi^*\alpha_{\bar{F}}) = \mathrm{Aut}(Y_{\bar{F}}, \varphi^*\alpha_{\bar{F}}) = \mathrm{Aut}_0(Y_{\bar{F}}) \\ f &\mapsto \varphi \circ f \circ \varphi^{-1} \end{aligned} \tag{7.6}$$

Let $Y_{\mathfrak{p}}$ be the smooth proper model of Y (as an algebraic space) over the discrete valuation ring $R_{\mathfrak{p}}$. Set $\beta = \varphi^*\alpha$ and denote by $\beta_{\mathfrak{p}}$ the composition

$$\Lambda \otimes \widehat{\mathbb{Z}} \xrightarrow{\beta} \mathrm{H}^2_{\text{ét}}(Y_{\mathfrak{p},\bar{F}}, \widehat{\mathbb{Z}}(1)) \cong \mathrm{H}^2_{\text{ét}}(Y_{\mathfrak{p},\overline{R}_{\mathfrak{p}}}, \widehat{\mathbb{Z}}(1)).$$

We consider the following group

$$\mathcal{G}_{\mathfrak{p}}(\overline{R}_{\mathfrak{p}}) := \left\{g \in \mathrm{Bir}(Y_{\mathfrak{p}} \times \overline{R}_{\mathfrak{p}}, \beta_{\mathfrak{p}}) \middle| g|_{Y_{\bar{F}}} \in \mathrm{Aut}_0(Y_{\bar{F}})\right\}/\sim$$

where $g_1 \sim g_2$ if $g_1|_{Y_{\bar{F}}} = g_2|_{Y_{\bar{F}}}$. It is non-empty since one can always specialize an automorphism to a birational self-map (see for example [36, Chapter IV, Exercise 1.17]). We may assume $|\mathcal{G}_{\mathfrak{p}}(\overline{R}_{\mathfrak{p}})|$ (which divides $|\operatorname{Aut}_0(Y_{\bar{F}})|$) is invertible in the residue field of $R_{\mathfrak{p}}$ by taking localization of R away from finitely many $\mathfrak{p}$. Then it forms a finite étale group scheme $\mathcal{G}_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ under the natural $\pi_1^{\text{ét}}(R_{\mathfrak{p}}, \bar{F})$-action, and we have $\mathcal{G}_{\mathfrak{p}}(\overline{R}_{\mathfrak{p}}) = \operatorname{Aut}_0(Y_{\bar{F}})$ by construction.

Then from the previous discussion, we can see that the elements in $\Pi_{(X,H)}$ can be viewed as $\operatorname{Aut}_0(X_{\bar{F}})$-torsors over F which can specialize to a $\mathcal{G}_{\mathfrak{p}}$-torsor for any $\mathfrak{p}$. Thus any one of such Galois action of $\operatorname{Gal}(\bar{F}/F)$ on $\operatorname{Aut}(X_{\bar{F}}, H_{\bar{F}}, \alpha_{\bar{F}})$ factors through a continuous action of $\pi_1^{\text{ét}}(R, \bar{F})$. There are finitely many such continuous actions by Hermite–Minkowski theorem for $\pi_1^{\text{ét}}(R, \bar{F})$ (*cf.* [18, Chap. VI, Sect. 2]). Thus we can obtain the finiteness of $\Pi_{(X,H)}$. □

Now we are ready to prove our main results on the unpolarized Shafarevich conjecture:

Proof of Theorem 1.4. (i) By Proposition 7.5, there exists an integer $N > 0$, such that the subset

$$\operatorname{Shaf}_M(F,S)/\sim_{F\text{-}\mathrm{bir}} \subset \operatorname{Shaf}_M^{\mathrm{hom}}(F,S)/\sim_{F\text{-}\mathrm{bir}}$$

is contained in the image of the following composition of natural maps

$$\coprod_{d\leq N} \widetilde{\operatorname{Shaf}}^{\mathrm{hom}}_{M,d}(F,S) \to \operatorname{Shaf}_M^{\mathrm{hom}}(F,S) \twoheadrightarrow \operatorname{Shaf}_M^{\mathrm{hom}}(F,S)/\sim_{F\text{-}\mathrm{bir}}.$$

However, by Theorem 7.6, each $\widetilde{\operatorname{Shaf}}^{\mathrm{hom}}_{M,d}(F,S)$ is a finite set. Hence $\operatorname{Shaf}_M(F,S)/\sim_{F\text{-}\mathrm{bir}}$ is finite. When $b_2 \geq 5$, the fibers of the map $\operatorname{Shaf}_M(F,S) \twoheadrightarrow \operatorname{Shaf}_M(F,S)/\sim_{F\text{-}\mathrm{bir}}$ are finite thanks to Theorem 3.5. Therefore, the finiteness of $\operatorname{Shaf}_M(F,S)/\sim_{F\text{-}\mathrm{bir}}$ implies the finiteness of $\operatorname{Shaf}_M(F,S)$.
(ii) and (iii): The argument is similar to (i). By Proposition 7.5, there exists an integer $N > 0$, such that the following composition map is surjective:

$$\coprod_{d\leq N} \operatorname{Shaf}^{\mathrm{hom}}_{M,d}(F,S) \hookrightarrow \operatorname{Shaf}_M^{\mathrm{hom}}(F,S) \twoheadrightarrow \operatorname{Shaf}_M^{\mathrm{hom}}(F,S)/\sim_{F\text{-}\mathrm{bir}}. \tag{7.7}$$

In [4, Section 9], André has actually proved the following finiteness result via his Kuga-Satake construction

- the geometric isomorphism classes in $\operatorname{Shaf}^{\mathrm{hom}}_{M,d}(F,S)$ is finite.
- if the weak Torelli theorem holds, then $\operatorname{Shaf}^{\mathrm{hom}}_{M,d}(F,S)$ is finite.

Therefore, one can conclude the finiteness of $\operatorname{Shaf}_M^{\mathrm{hom}}(F,S)/\sim_{\bar{F}\text{-}\mathrm{bir}}$, as well as the finiteness of $\operatorname{Shaf}_M^{\mathrm{hom}}(F,S)/\sim_{F\text{-}\mathrm{bir}}$ once the weak Torelli holds. When $b_2 \geq 5$, again, we can get the finiteness for isomorphism classes from Theorem 3.5. □

7.4. **Finiteness of CM type hyper-Kähler varieties.** Recall that a hyper-Kähler variety X over $\mathbb{C}$ is called of *CM type* if the Mumford–Tate group $\mathrm{MT}(T(X))$ is abelian, where $T(X) \subset \mathrm{H}^2(X,\mathbb{Q})$ is the transcendental part. The following results allows us to study hyper-Kähler varieties of CM type via abelian varieties of CM type.

Lemma 7.7. *The Kuga–Satake variety of a complex hyper-Kähler variety X is of CM type if and only if X is of CM type. In particular, the point*

$$\Psi^{\mathrm{KS}}_{d,\mathbf{K}_4}(X,H,\alpha) \in \operatorname{Sh}_{\mathbf{K}_4''}(\operatorname{GSp}(W),\Omega^{\pm})(\mathbb{C})$$

is a CM-point if X is of CM type.

Proof. A proof for the statement for K3 surfaces can be found in for example [23, Proposition 9.3], which works for general Hodge structures of K3-type. We sketch it here for the convenience of readers.

Let A be the (uniform) Kuga–Satake abelian variety of X. Let $\mathrm{MT}(A)$ be the Mumford-Tate group of $\mathrm{H}_1(A,\mathbb{Q})$. The Hodge structure of $\mathrm{H}_1(A,\mathbb{Q})$ factors through $\widetilde{h}$ in the commutative diagram

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\widetilde{h}"] \arrow[dr, "h"'] & \mathrm{GSpin}(\Sigma)_{\mathbb{R}} \arrow[d, "\mathbf{ad}"] \\ & G_{\Sigma,\mathbb{R}} \end{tikzcd}$$

the Mumford-Tate group is contained in $\mathrm{GSpin}(\Sigma)_{\mathbb{Q}}$. The commutativity of the diagram implies that $\mathrm{MT}(T(X))$ is contained in $\mathbf{ad}(\mathrm{MT}(A))$.

If A is of CM type, then $\mathrm{MT}(A)$ is abelian, and so is its image in G_Σ under $\mathbf{ad}$. Therefore $\mathrm{MT}(T(X))$ is also abelian. On the contrary, we note that $\widetilde{h}(\mathbb{S}) \subseteq \mathbf{ad}^{-1}(\mathrm{MT}(T(X))_{\mathbb{R}}$. Thus $\mathrm{MT}(A)$ is a subgroup of $\mathbf{ad}^{-1}(\mathrm{MT}(T(X))$, which are both solvable if $\mathrm{MT}(T(X))$ is abelian. Since $\mathrm{MT}(A)$ is reductive, $\mathrm{MT}(A)$ is abelian if $\mathrm{MT}(T(X))$ is abelian. □

Remark 7.8. Lemma 7.7 also implies that a hyper-Kähler variety X over a field F of characteristic zero being of CM type is independent of the embedding $F \hookrightarrow \mathbb{C}$.

The following finiteness result for abelian varieties of CM type is obtained by Orr–Skorobogatov.

Theorem 7.9 ([49, Theorem 2.5])**.** *The isomorphism classes of abelian varieties of CM type defined over a number field of bounded degree form a finite set.*

Moreover, they used this to deduce a finiteness result of CM-points on a Shimura variety of abelian type.

Theorem 7.10 ([49, Proposition 3.1])**.** *Let X be a component of a Shimura variety of abelian type. The set of CM-points in X defined over a number field of bounded degree is finite.*

We now give the proof of the generalizations of these results for hyper-Kähler varieties of CM type.

Proof of Theorem 1.9. We need to show the finiteness of the following set:

$$\mathtt{CM}_d(M) = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is a hyper-Kähler variety over a number field } F \text{ of degree} \leq d \\ X_{\mathbb{C}} \text{ is of deformation type } M \text{ and of CM type for some } F \hookrightarrow \mathbb{C} \end{array} \right\}.$$

We first prove that the isometry class of $\{\mathrm{NS}(X_{\mathbb{C}}) |\, \mathtt{CM}_d(M)\}$ is finite. By Lemma 7.2, we can find an integer n such that for every $X/F \in \mathtt{CM}_d(M)$ with $[F:\mathbb{Q}] \leq d$, it has a $\mathbf{K}_4$-level structure over a finite extension F'/F with $[F':F] \leq n$. Note that n depends only on the $\widehat{\mathbb{Z}}$-BB form and hence is independent of the field F and the embedding $F \hookrightarrow \mathbb{C}$.

By Lemma 7.7 and Theorem 7.10, the collection

$$\left\{ \Psi^{\mathrm{KS}}_{d,\mathbf{K}_4}(X_{\mathbb{C}}, H_{\mathbb{C}}, \alpha_{\mathbb{C}}) \;\middle|\; \begin{array}{l} (X,H,\alpha) \text{ is a polarized hyper-Kähler with a } \mathbf{K}_4\text{-level structure} \\ \text{defined over } F' \text{ with } [F':\mathbb{Q}] \leq nd;\ X \in \mathtt{CM}_d(M) \end{array} \right\} \tag{7.8}$$

is finite. By the same argument in Theorem 7.4, one can get the finiteness of the isometry classs of geometric transcendental lattices $T(X_{\mathbb{C}})$, which implies the finiteness of the isometry classes of $\mathrm{Pic}(X_{\mathbb{C}})$.

According to the proof in Theorem 7.5, up to a birational transformation, there exists a polarization on $X_{\mathbb{C}}$ of degree $\leq N$ for some N. Note that the geometric isomorphism classes of polarized hyper-Kähler varieties (X, H) with $X \in \mathtt{CM}_d(M)$ and $H^2 \leq N$ is finite because the Kuga–Satake map over $\mathbb{C}$ is quasi-finite. Thus the geometric birational isomorphism class of $\mathtt{CM}_d(M)$ is finite.

When $b_2 \geq 5$, the last assertion follows from Theorem 3.5. □

Proof of Corollary 1.11. According to [49, Theorem 5.1], for any positive integer d, there is a constant $C(d, X)$ such that, for any L-form Y of X with $[L : F] \leq d$,

$$|\mathrm{Br}(Y_{\bar{F}})|^{\mathrm{Gal}(\bar{F}/L)} < C(d, X).$$

if the integral Mumford–Tate conjecture in codimension one holds. It is true for hyper-Kähler varieties: From [13, Proposition 6.2], we can see the Hodge structure on $\mathrm{H}^2(X, \mathbb{Z})$ is Hodge-maixmal. Then we can follow the proof of Theorem 6.6 in *loc.cit.* to show that the classical Mumford–Tate conjecture of X implies the integral version, where we need to use the period map in Proposition 5.8. However, the Mumford–Tate conjecture in codimension one holds true for all hyper-Kähler varieties of $b_2 \geq 4$ [4, Corollary 1.5.2].

Finally we can conclude by the finiteness of X with CM type and bounded defining field degree. □

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