TWISTED DERIVED EQUIVALENCES AND ISOGENIES FOR ABELIAN SURFACES

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ABSTRACT. In this paper, we study the twisted Fourier-Mukai partners of abelian surfaces. Following the work of Huybrechts [19], we introduce the twisted derived equivalence between abelian surfaces. We show that there is a twisted derived Torelli theorem for abelian surfaces over fields with characteristic $\neq 2$. Over complex numbers, twisted derived equivalence corresponds to rational Hodge isometries between the second cohomology groups, which is in analogy to the work of Huybrechts and Fu-Vial on K3 surfaces. Their proof relies on the global Torelli theorem over $\mathbb{C}$, which is missing in positive characteristics. To overcome this issue, we extend Shioda's trick [39] on singular cohomology groups to étale and crystalline cohomology groups and make use of Tate's isogeny theorem to give a characterization of twisted derived equivalence on abelian surfaces via using so called principal quasi-isogenoy.

1. INTRODUCTION

Let X and Y be abelian varieties of dimension g over a field k of characteristic p. We say X and Y are isogenous over k if there exists a surjective map $\phi : X \to Y \in \mathrm{Hom}_k(X, Y)$ with finite kernel. When k is finitely generated over its prime field, there is a canonical bijection for each $\ell \neq p$:

$$\mathrm{Hom}_k(X, Y) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(k^s/k)}(T_\ell(X), T_\ell(Y)),$$

where $\mathrm{Hom}_{\mathrm{Gal}(k^s/k)}(T_\ell(X), T_\ell(Y))$ denotes the Galois-equivariant maps between the Tate modules of X and Y. This gives a cohomological realization of isogenies, and is also known as *Tate's isogeny theorem*, proved by Tate for finite fields, by Zarhin and Faltings for finitely generated fields (*cf.* [44, 48, 12]). For $\ell = p$, we can replace the Tate modules by p-divisible groups (also called Barsotti-Tate groups). The bijectivity also holds, which is known as a theorem of de Jong proved in [10].

On the other hand, the (twisted) Fourier-Mukai equivalences provide a natural source for isogenies bewteen abelian varieties (*cf.* [34, 35]). A natural question is to determine when two isogenous abelian varieties are Fourier-Mukai partners or more generally, twisted Fourier-Mukai partners. Following the work in [19] and [14], one can introduce the notion of *twisted derived equivalence* as follows: we say X and Y are twisted derived equivalent if they can be connected by twisted derived equivalences, i.e. there exists twisted abelian varieties (X_i, α_i) and (X_i, β_i) such that

$$\begin{aligned} \mathrm{D}^b(X, \alpha) &\xrightarrow{\simeq} \mathrm{D}^b(X_1, \beta_1) \\ \mathrm{D}^b(X_1, \alpha_2) &\xrightarrow{\simeq} \mathrm{D}^b(X_2, \beta_2) \\ &\vdots \\ \mathrm{D}^b(X_n, \alpha_n) &\xrightarrow{\simeq} \mathrm{D}^b(Y, \beta_n) \end{aligned} \tag{1.0.1}$$

where $\mathrm{D}^b(X, \alpha)$ is the bounded derived category of α-twisted sheaves on X.

In this paper, we try to classify the twisted derived equivalence of abelian surfaces over arbitrary algerbaically closed fields, where it should behave similarly as the case of K3 surfaces. In this case, we will show that the twisted derived equivalence also has natural cohomological and motivic realizations, i.e. such equivalence can be read off from the conditions on cohomology groups or motives. A notable fact for abelian surfaces is that besides the Tate modules, their 2^{nd} cohomology groups also carry rich structures. For instance, Orlov's derived Torelli theorem states that two complex abelian surfaces X and Y are derived equivalence if and only if there is a symplectic Hodge isometry

$$\mathrm{H}^1(X,\mathbb{Z})\oplus\mathrm{H}^1(\widehat{X},\mathbb{Z})\cong\mathrm{H}^1(Y,\mathbb{Z})\oplus\mathrm{H}^1(\widehat{Y},\mathbb{Z})$$

where $\widehat{X}$ and $\widehat{Y}$ are the dual abelian surfaces. On the other hand, the Orlov–Shioda derived Torelli theorem shows that there is another Hodge theoretic realization of derived equivalences:

$$\mathrm{D}^b(X)\cong\mathrm{D}^b(Y)\Leftrightarrow\widetilde{\mathrm{H}}(X,\mathbb{Z})\cong\widetilde{\mathrm{H}}(Y,\mathbb{Z})\Leftrightarrow\mathrm{T}(X)\cong\mathrm{T}(Y)$$

Here, $\widetilde{\mathrm{H}}(X,\mathbb{Z})$ and $\widetilde{\mathrm{H}}(Y,\mathbb{Z})$ are the Mukai lattices, $\mathrm{T}(X)\subseteq\mathrm{H}^2(X,\mathbb{Z})$ (resp. $\mathrm{T}(Y)$) denotes the transcendental lattice and the isomorphisms are integral Hodge isometries. Inspired from this result and the work of [39], we find that the twisted derived equivalence and certain classes of isogenies also have the same cohomological realization on the 2^{nd}-cohomology. Our first result is

Theorem 1.0.1. *Let X and Y be smooth abelian surfaces over $\bar{k}$ with $\mathrm{char}(k)=0$. Then the following two conditions are equivalent*

(i) *$X\simeq Y$ are principally quasi-isogenous;*
(ii) *X and Y are twisted derived equivalent;*
(iii) *Chow motives $\mathfrak{h}(X)\cong\mathfrak{h}(Y)$ are isomorphic as exteria algebra;*
(iv) *even degree Chow motives $\mathfrak{h}^{even}(X)\cong\mathfrak{h}^{even}(Y)$ are isomorphic as Frobenius algebra.*

When $k=\mathbb{C}$, then the conditions above are also equivalent to

(v) *$\mathrm{H}^2(X,\mathbb{Q})\cong\mathrm{H}^2(Y,\mathbb{Q})$ as a rational Hodge isometry;*
(vi) *$\widetilde{\mathrm{H}}(X,\mathbb{Q})\cong\widetilde{\mathrm{H}}(Y,\mathbb{Q})$ as a rational Hodge isometry;*
(vii) *$\mathrm{T}(X)\otimes\mathbb{Q}\cong\mathrm{T}(Y)\otimes\mathbb{Q}$ as a rational Hodge isometry.*

Here, X and Y are principally quasi-isogenous means that there is a quasi-isogeny of degree one from X to Y or $\widehat{Y}$ (*cf.* Definition 4.2.2). The equivalences $(ii)\Leftrightarrow(v)\Leftrightarrow(vi)\Leftrightarrow(vii)$ provide a Hodge-theoretic realization of twisted derived equivalences. The proof follows a similar strategy of [19, Theorem 0.1], which makes use of Shioda's global Torelli theorem and Cartan-Dieudonné decomposition of rational isometries.

The equivalence $(i)\Leftrightarrow(ii)$ are concluded by so called ℓ-adic Shioda's trick on abelian surfaces. The original Shioda's trick in [39] plays a key role in the proof of Shioda's global Torelli theorem for abelian surfaces, which links the (integral) 1^{st}- Betti cohomology with the 2^{nd}-Betti cohomology of an abelian surface. For equivalences $(ii)\Leftrightarrow(iii)\Leftrightarrow(iv)$, it follows easily from $(i)\Leftrightarrow(ii)$ and it can be viewed as a motivic global Torelli theorem for abelian surfaces. We refer readers to see [19, Conjecture 0.3] and [14, Theorem 1] for the motivic global Torelli theorem on K3 surfaces.

When $\mathrm{char}(k)>0$, due to the absence of a satisfactory global Torelli theorem, one can not follow the argument in characteristic zero directly. To overcome this problem, we make use of Tate's isogeny theorem to extend Theorem 1.0.1 to positive characteristic fields. The main contribution of this paper is the following

Theorem 1.0.2. *Let X and Y be two abelian surfaces over k with $\mathrm{char}(k)=p>2$. Then the following statements are equivalent:*

(i') *$X_{\bar{k}}$ and $Y_{\bar{k}}$ are prime-to-p twisted dervied equivalent,*
(ii') *$X_{\bar{k}}$ and $Y_{\bar{k}}$ are prime-to-p principally quasi-isogenous,*

If k is finite, then (i) and (ii) are equivalent to the condition

(iii') *$\mathrm{H}^2_{\mathrm{crys}}(X_{k'}/W')\cong\mathrm{H}^2_{\mathrm{crys}}(Y_{k'}/W')$ as K3-crystals for some finite field extension k'/k and $W'=W(k')$.*

Moreover, in case that $X_{\bar{k}}$ is supersingular, then $Y_{\bar{k}}$ is twisted derived equivalent to $X_{\bar{k}}$ if and only if $Y_{\bar{k}}$ is supersingular.

Here, we say a twisted derived equivalence as (1.0.1) is *prime-to-p* if the order of every Brauer class α_i is coprime to p. (*cf.* §4 for details).

The main ingredients in the proof of Theorem 1.0.2 are the p-adic Shioda's trick and the lifting-specialization technique. We shall remark that the ℓ-adic and p-adic Shioda's trick gives a proof of Tate conjecture for isometries (as either Galois-modules or crystals) between the 2^{nd}-cohomology group of abelian surfaces over finite fields (See Corollary 3.4.7). Moreover, our method also shows that there is an implication $(i') \Rightarrow (ii')$ for non prime-to-p twisted derived equivalence (See Theorem 5.3.2) and we believe that the existence of quasi-liftable principal quasi-isogenous will imply the existence of twisted derived equivalence (See Conjecture 5.3.1).

One may ask whether the similar results also hold for K3 surfaces. Recall that two K3 surfaces S and S' over a finite field k are (geometrically) isogenous in the sense of [46] if there exists an algebraic correspondence $\mathcal{Z} \subseteq S_{\bar{k}} \times S'_{\bar{k}}$ which induces an isometry

$$\mathcal{Z}_\ell^* : \mathrm{H}^2_{\text{ét}}(S_{\bar{k}}, \mathbb{Q}_\ell) \cong \mathrm{H}^2_{\text{ét}}(S'_{\bar{k}}, \mathbb{Q}_\ell)$$

for all $\ell \nmid p$ and a crystalline isometry $\mathcal{Z}_p^* : \mathrm{H}^2_{\text{crys}}(S_{k'}/W')[\frac{1}{p}] \cong \mathrm{H}^2_{\text{crys}}(S'_{k'}/W)'[\frac{1}{p}]$ for some finite extension k'/k and $W' = W(k')$. Then we say the isogeny $S \rightsquigarrow S'$ is prime-to-p if the isometry $\mathcal{Z}_p^*$ is integral, i.e. $\mathrm{H}^2_{\text{crys}}(S_{k'}/W) \cong \mathrm{H}^2_{\text{crys}}(S'_{k'}/W)$. Then we have a formulation of the twisted derived Torelli conjecture for K3 surfaces.

Conjecture 1.0.1. *For two K3 surfaces S and S' over a finite field k with* $\mathrm{char}(k) = p > 0$, *then the following are equivalent*

(a) *there exists a prime-to-p derived isogeny* $\mathfrak{h}^2(S) \cong \mathfrak{h}^2(S')$,
(b) *there exists a prime-to-p isogeny between S and S',*
(c) $\mathrm{H}^2_{\text{crys}}(S_{k'}/W') \cong \mathrm{H}^2_{\text{crys}}(S'_{k'}/W')$ *as a K3-crystals isometry for some finite field extension k'/k and $W' = W(k')$.*

The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear, while the converses remain open. We will investigate this problem in a sequel to this paper. We shall mention that recently Bragg and Yang have studied the twisted derived equivalence between K3 surfaces in [7] and they provided a weaker version of the statement in Conjecture 1.0.1 (*cf.* [7, Theorem 1.2]).

Organization of the paper. In Section 2, we will recollect some facts on twisted abelian surfaces, including the twisted Mukai lattices, **B**-field theory and twisted derived categories. We perform the computations of the Brauer group of abelian surfaces via Kummer construction. This allows us to prove the lifting theorem for twisted abelian surfaces of finite height and the representability of flat cohomologies on supersingular abelian surfaces.

In Section 3, we revise Shioda's work and generalize it to various cohomology theory of abelian surfaces. After introducing the admissible ℓ-adic and p-adic bases, we prove the ℓ-adic and p-adic Shioda's trick for admissible isometries on abelian surfaces. As an application, we confirm the Tate conjecture for Galois and K3-crystal isometries between abelian surfaces over finite fields.

Section 4 and 5 are devoted to proving Theorem 1.0.1 and Theorem 1.0.2. Theorem 1.0.1 is essentially Theorem 4.1.3 and Theorem 4.2.6. The proof of Theorem 1.0.2 is much more subtle. We establish the lifting and the specialization theorem for prime-to-p twisted derived equivalence. Then one can conclude $(i') \Leftrightarrow (ii')$ from Theorem 1.0.1. The proof of $(i') \Leftrightarrow (iii')$ is a combination of p-adic Shioda's trick and Tate's isogenous theorem. At the end of Section 5, we follow Bragg and Lieblich's twistor line argument in [5] to conclude the last assertion of Theorem 1.0.2.

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2. Twisted abelian surface

In this section, we give an overview of the theory of twisted abelian surfaces and their derived categories. Although the results looks parallel to the case of twisted K3 surfaces, the proof requests additional attentions in many places (see Remark 2.2.2, 2.5.3).

2.1. **Notations and Conventions.** Throughout this section, k is a field with $\mathrm{char}(k) = p \geq 0$. When k is perfect and $p > 0$, we let $W := W(k)$ be the p-typical Witt ring of k equipped with a morphism $\sigma\colon W \to W$ induced by the Frobenius map.

Let X be a smooth projective variety over k. We denote by $\mathrm{H}^\bullet_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$ the ℓ-adic étale cohomology group of $X_{\bar{k}}$. The $\mathbb{Z}_\ell$-module $\mathrm{H}^\bullet_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$ are endowed with a canonical $G_k = \mathrm{Gal}(\bar{k}/k)$-action. If k is a perfect field, we use $\mathrm{H}^i_{\mathrm{crys}}(X/W)$ to denote the i-th crystalline cohomology group of X over the p-adic base $W \twoheadrightarrow k$, which is a W-module. It is endowed with a natural σ-linear map

$$\varphi : \mathrm{H}^i_{\mathrm{crys}}(X/W) \to \mathrm{H}^i_{\mathrm{crys}}(X/W)$$

induced from the absolute Frobenius morphism $F_X\colon X \to X$.

When X is an abelian variety over k, we use $X[p^\infty]$ to denote the associated p-divisible group. There is a natural identification of its contravariant Dieudonné module with its first crystalline cohomology:

$$\mathbb{D}(X[p^\infty]) := M(X[p^\infty]^\vee) \cong \mathrm{H}^1_{\mathrm{crys}}(X/W),$$

where $M(-)$ is the Dieudonné module functor on p-divisible groups defined in [29].

Let us recall some basic notions on the motivic decomposition of abelian surfaces. Deninger–Murre [11] produced a canonical motivic decomposition for any abelian surface:

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4} \mathfrak{h}^i(X).$$

Furthermore, Künnemann [24] showed that $\mathfrak{h}^i(X) = \bigwedge^i \mathfrak{h}^1(X)$ for all i, $\mathfrak{h}^4(X) \simeq \mathbb{1}(-4)$ and $\bigwedge^i \mathfrak{h}^1(A) = 0$ for $i > 2g$.

2.2. **Gerbes on abelian surfaces.** Let X be an abelian surface over an algeraically closed field k and let $\mathscr{X} \to X$ be a μ_n-gerbe over X. This corresponds to a pair (X, α) for some $\alpha \in \mathrm{H}^2_{\mathrm{fl}}(X, \mu_n)$, where the cohomology group is with respect to the flat topology. Since μ_n is commutative, there is a bijection of sets

$$\{\mu_n\text{-gerbes on } X\}/\simeq \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{fl}}(X, \mu_n),$$

where $\simeq$ is the μ_n-equivalence defined as in [15, IV.3.1.1]. The corresponding cohomology class α is also denoted by $[\mathscr{X}]$. The Kummer exact sequence induces a surjective map

$$\mathrm{H}^2_{\mathrm{fl}}(X, \mu_n) \to \mathrm{Br}(X)[n] \tag{2.2.1}$$

where the right-hand side is the *cohomological Brauer group* $\mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. For any μ_n gerbe $\mathscr{X}$ on X, there is an associated $\mathbb{G}_m$-gerbe on X via (2.2.1), denoted by $\mathscr{X}_{\mathbb{G}_m}$. Let $\mathscr{X}^{(i)}$ be the gerbe corresponding to cohomological class $i[\mathscr{X}] \in \mathrm{H}^2_{\mathrm{fl}}(X, \mu_n)$. If ℓ is a prime factor of n such that $n = \ell^d m$ with $(m, \ell) = 1$, then $[\mathscr{X}^{(m)}_{\mathbb{G}_m}] \in \mathrm{Br}(X)[\ell^d]$. Thus the gerbe $\mathscr{X}^{(m)}$ can be viewed as a μ_{ℓ^n}-gerbe on X.

If k has characteristic $p \neq 2$, there is an associated Kummer surface $\widetilde{X}$ constructed as follows:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\sigma}"] \arrow[d, "\pi"] & X \arrow[d] \\ \mathrm{Km}(X) \arrow[r, "\sigma"] & X/\iota \end{tikzcd} \tag{2.2.2}$$

where

- ι is the involution of X;
- σ is the crepant resolution of quotient singularities;
- $\widetilde{\sigma}$ is the blow-up of X along the closed subscheme $X[2] \subset X$. Its birational inverse is denoted by $\widetilde{\sigma}^{-1}$.

Let $E \subset \widetilde{X}$ be the exceptional locus of $\widetilde{\sigma}$. Consider the sequence of morphisms

$$\mathrm{Br}(\widetilde{X}) \to \mathrm{Br}(\widetilde{X} \setminus E) \xrightarrow[\sim]{(\widetilde{\sigma}|_{\widetilde{X}\setminus E})^{-1}} \mathrm{Br}(X \setminus X[2]).$$

By the Grothendieck's purity theorem, the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(X \setminus X[2])$ is an isomorphism. We can take the composition of the previous sequence of morphisms with the inverse of this isomoprhism, denoted by $(\widetilde{\sigma}^{-1})^*$.

Proposition 2.2.1. *The $(\widetilde{\sigma}^{-1})^*\pi^*$ induces a morphism between cohomological Brauer groups*

$$\Theta\colon \mathrm{Br}(\mathrm{Km}(X)) \to \mathrm{Br}(X), \tag{2.2.3}$$

which is an isomorphism when $k = \bar{k}$. In particular, when X is supersingular over $\bar{k}$, then $\mathrm{Br}(X)$ *is isomorphic to the additive group $\bar{k}$.*

Proof. Clearly, it suffices to show Θ is an isomorphism when $k = \bar{k}$. The comparison between these two ℓ-torsion parts is essentially same as [40, Proposition 1.3]. See also [42, Lemma 4.1] for the case $k = \mathbb{C}$.

For p-primary torsions, we have

$$\mathrm{Br}(\mathrm{Km}(X))[p^\infty] \cong \mathrm{Br}(X)^\iota[p^\infty]$$

from the Hochschild–Serre spectral sequence. Thus it suffices to prove that the involution on $\mathrm{Br}(X)$ is trivial. In fact, $\mathrm{H}^2_{\mathrm{fl}}(A, \mu_p)$ can be ι-equivariantly embedded to $\mathrm{H}^2_{\mathrm{dR}}(A/k)$ by de-Rham-Witt theory (cf. [32, Proposition 1.2]). Thus the involution on $\mathrm{H}^2_{\mathrm{fl}}(X, \mu_p)$ is trivial.

Then by the exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}/p \to \mathrm{H}^2_{\mathrm{fl}}(X, \mu_p) \to \mathrm{Br}(X)[p] \to 0,$$

we can deduce that $\mathrm{Br}(X)[p]$ is invariant under the involution. Furthermore, for p^n-torsions with $n \geq 2$, we can proceed by induction on n. Assume that all elements in $\mathrm{Br}(X)[p^d]$ are ι-invariant if $1 \leq d < n$. By abuse of notation, we still use ι to denote the induced map $\mathrm{Br}(X) \to \mathrm{Br}(X)$. For $\alpha \in \mathrm{Br}(X)[p^n]$, $p\alpha \in \mathrm{Br}(X)[p^{n-1}]$ is ι-invariant. This gives

$$p\alpha = \iota(p\alpha) = p\iota(\alpha),$$

which implies $\alpha - \iota(\alpha) \in \mathrm{Br}(X)[p]$. Applying ι on $\alpha - \iota(\alpha)$, we can obtain

$$\alpha - \iota(\alpha) = \iota(\alpha) - \alpha.$$

It implies $\alpha - \iota(\alpha)$ is also a 2-torsion element. Since p is coprime to 2, we can conclude that $\alpha = \iota(\alpha)$.

If X is supersingular, then $\mathrm{Km}(X)$ is also supersingular. We have already known that the Brauer group of a supersingular K3 surface is isomorphic to k by [2]. Thus $\mathrm{Br}(X) \cong k$. □

Remark 2.2.2. In the case A being supersingular, the method of [2] can not be directly applied as $\mathrm{H}^1_{\mathrm{fl}}(X, \mu_{p^n})$ is not trivial in general for an abelian surface X.

Remark 2.2.3. For the abelian surface over a non-closed field or a ring, we still have a canonical map (2.2.3) by using the purity of Brauer groups (*cf.* [17, 45]). But it is not necessarily an isomorphism.

In [4], Bragg has shown that a twisted K3 surface can be lifted to characteristic 0. Though his method can not be directly applied to twisted abelian surfaces, one can still obtain a lifting result for twisted abelian surfaces via using the Kummer construction. The following result will be frequently used in this paper.

Lemma 2.2.4. *Let $\mathscr{X} \to X$ be a $\mathbb{G}_m$-gerbe on an abelian surface X over $k = \bar{k}$. Suppose* $\mathrm{char}(k) > 2$ *and X has finite height. Then there exists a lifting $\mathfrak{X} \to \mathcal{X}$ of $\mathscr{X} \to X$ over a finite extension W' of W such that specialization map*

$$\mathrm{NS}(\mathcal{X}_{K'}) \to \mathrm{NS}(X)$$

on Néron-Severi groups is an isomorphism. Here, K' is the fraction field of W' and $\mathcal{X}_{K'}$ is the generic fiber of $\mathcal{X} \to \operatorname{Spec} W'$.

Proof. The existence of such lift is ensured by [4, Theorem 7.3], [25, Lemma 3.9] and Proposition 2.2.1. Roughly speaking, let $\mathscr{S} \to \mathrm{Km}(X)$ be the associated twisted Kummer surface via the isomorphism (2.2.3). Then [4, Theorem 7.3] asserts that there exists a lifting $\mathfrak{S} \to \mathcal{S}$ of $\mathscr{S} \to \mathrm{Km}(X)$ such that the specialization map of Néron-Severi groups is an isomorphism

$$\mathrm{NS}(\mathcal{X}_{K'}) \xrightarrow{\sim} \mathrm{NS}(X). \tag{2.2.4}$$

Then [25, Lemma 3.9] says that one can find a lifting $\mathcal{X}/W'$ of X such that $\mathrm{Km}(\mathcal{X}) \cong \mathcal{S}$ over W'. By Remark 2.2.3, we have a map $\Theta\colon \mathrm{Br}(\mathrm{Km}(\mathcal{X})) \to \mathrm{Br}(\mathcal{X})$ as in (2.2.3). Consider image of $[\mathfrak{S}]$ in $\mathrm{Br}(\mathcal{X})$ via Θ, then we can take $\mathfrak{X} \to \mathcal{X}$ to be the associated twisted abelian surface. Clearly, $\mathfrak{X} \to \mathcal{X}$ is a lift of $\mathscr{X} \to X$ as the restriction of the Brauer class $[\mathfrak{X}]$ to X is $[\mathscr{X}]$ from Proposition 2.2.1. □

2.3. **Mukai lattices and B-fields.** If X is a complex abelian surface, the *Mukai lattice* is defined as

$$\widetilde{\mathrm{H}}(X,\mathbb{Z}) \coloneqq \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z})$$

with the Mukai pairing

$$\langle (r,c,\chi),(r',c',\chi')\rangle = r\chi' - cc' + r\chi'. \tag{2.3.1}$$

For any $B \in \mathrm{H}^2(X,\mathbb{Q})$, we can define the *twisted Mukai lattice*

$$\widetilde{\mathrm{H}}(X,\mathbb{Z};B) \coloneqq \exp(B)\cdot \widetilde{\mathrm{H}}(X,\mathbb{Z}) \subset \widetilde{\mathrm{H}}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

It is equipped with a Hodge structure, see [21, Proposition 1.2] for details. For such B, we can associate a Brauer class α_B via the exponential sequence

$$\mathrm{H}^2(X,\mathbb{Q}) \to \mathrm{H}^2(X,\mathcal{O}_X) \to \mathrm{H}^2(X,\mathbb{G}_m) = \mathrm{Br}(X).$$

Conversely, given $\alpha \in \mathrm{Br}(X)$, one can find a lift B of α in $\mathrm{H}^2(X,\mathbb{Q})$ because $\mathrm{Br}(X)$ is torsion and $\mathrm{H}^2(X,\mathcal{O}_X) \twoheadrightarrow \mathrm{Br}(X)$ is surjective and B is called a **B**-field lift of α.

For general base field k, we also have the following notion of Mukai lattices [27, §2].

- if it is separably closed with $\mathrm{char}(k) \geq 0$, then the ℓ-adic Mukai lattice is defined on the even degrees of integral ℓ-adic cohomology of X for ℓ coprime to $\mathrm{char}(k)$, denoted by $\widetilde{\mathrm{H}}(X,\mathbb{Z}_\ell)$; or
- if it is perfect with $\mathrm{char}(k) = p > 0$, then the p-adic Mukai lattice is defined on the even degrees of crystalline cohomology of X with coefficients in $W(k)$, denoted by $\widetilde{\mathrm{H}}(X,W)$.
- We also have the algebraic Mukai lattice

  $$\widetilde{N}(X) \coloneqq \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z},$$

  which is well-defined for any base field.

ℓ-adic and crystalline B-field. In addition, we need the following generalized notions of **B**-fields in both ℓ-adic cohomology and crystalline cohomology as an analogue to that in Betti cohomology.In this part, we assume $k = \bar{k}$ for simplicity of notation.

For a prime $\ell \neq p$ and $n \in \mathbb{N}$, the Kummer sequence of étale sheaves

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 1, \tag{2.3.2}$$

induces a long exact sequence

$$\cdots \mathrm{Pic}(X) \xrightarrow{\cdot l^n} \mathrm{Pic} X \to \mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n}) \to \mathrm{Br}(X)[\ell^n] \to 0.$$

Taking the inverse limit $\varprojlim_n$, we get a map

$$\pi_\ell\colon \mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}_\ell(1)) = \varprojlim_n \mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n}) \to \mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n}) \twoheadrightarrow \mathrm{Br}(X)[\ell^n].$$

Lemma 2.3.1. *The map π_ℓ is surjective.*

Proof. Note that the reduction mod ℓ^n map $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n})$ fits into a short exact sequence after passing to the inverse limit:

$$0 \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))/\ell^n \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}_\ell(1))[\ell] \to 0.$$

As $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}_\ell(1)) = \left(\wedge^3 \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}_\ell)\right)(1)$ is torsion-free for any abelian surface X, $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}_\ell(1))[\ell] = 0$. Thus the reduction induces an isomorphism

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))/\ell^n \cong \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}).$$

The assertion then follows. $\square$

Let $B_\ell(\alpha) := \pi_\ell^{-1}(\alpha)$, which is non-empty for any $\alpha \in \mathrm{Br}(X)[\ell^n]$ by Lemma 2.3.1. For the crystalline cohomology, we have the following commutative diagram via the de Rham-Witt theory (cf. [22, I.3.2, II.5.1, Théorème 5.14])

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^2(X, \mathbb{Z}_p(1)) \arrow[r] \arrow[d] & \mathrm{H}^2_{\text{crys}}(X/W) \arrow[r, "p-F"] \arrow[d, "p_n := (-\otimes W_n)"] & \mathrm{H}^2_{\text{crys}}(X/W) \\ & \mathrm{H}^2_{\text{fl}}(X, \mu_{p^n}) \arrow[r, "d\log"] & \mathrm{H}^2_{\text{crys}}(X/W_n) & , \end{tikzcd} \tag{2.3.3}$$

where $\mathrm{H}^2(X, \mathbb{Z}_p(1)) := \varprojlim_n \mathrm{H}^2_{\text{fl}}(X, \mu_{p^n})$. The $d\log$ map is known to be injective since the Hodge-de Rham spectral sequence of X degenerates at E_1, by the flat duality $\mathrm{H}^2_{\text{fl}}(X, \mu_{p^n}) \cong \mathrm{H}^1_{\text{ét}}(X, \mathcal{O}_X^*/\mathcal{O}_X^{*p^n})$ of surfaces (cf. [32, Proposition 1.2]). Then we can associate a p^n-torsion Brauer class α_b via the map. Since the crystalline groups of an abelian surface are torsion-free, the mod p^n reduction p_n is surjective. Consider the surjective map

$$q_n \colon \mathrm{H}^2_{\text{fl}}(X, \mu_{p^n}) \twoheadrightarrow \mathrm{Br}(X)[p^n],$$

we let $B_p(\alpha)$ be the set

$$\left\{ b \in \mathrm{H}^2_{\text{crys}}(X/W) | p_n(b) = d\log(t) \text{ such that } q_n(t) = \alpha \right\}.$$

Following [7, Definition 2.17], we can introduce the B-fields for twisted abelian surfaces.

Definition 2.3.2. Let ℓ be a prime and let $\alpha \in \mathrm{Br}(X)[\ell^n]$ be a Brauer class of X of order ℓ^n.

- If $\ell \neq p$, an *ℓ-adic $\mathbf{B}$-field lift* of α on X is an element $B = \frac{b}{\ell^n} \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ for some $b \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell)$ such that $b \in B_\ell(\alpha)$.
- If $\ell = p$, a *crystalline $\mathbf{B}$-field* lift of α is an element $B = \frac{b}{p^n} \in \mathrm{H}^2_{\text{crys}}(X/W)[\frac{1}{p}]$ with $b \in \mathrm{H}^2_{\text{crys}}(X/W)$ such that $b \in B_p(\alpha)$.

More generally, for any $\alpha \in \mathrm{Br}(X)$, a mixed $\mathbf{B}$-field lift of α is a set $B = \{B_\ell\} \cup \{B_p\}$ consisting of a choice of an ℓ-adic $\mathbf{B}$-field lift B_ℓ of α for each $\ell \neq p$ and a crystalline $\mathbf{B}$-field lift B_p of α.

Remark 2.3.3. Not all elements in $\mathrm{H}^2_{\text{crys}}(X/W)[\frac{1}{p}]$ are crystalline $\mathbf{B}$-fields since the map $d\log$ is not surjective. From the first row in diagram (2.3.3), we can see $B \in \mathrm{H}^2_{\text{crys}}(X/W)[\frac{1}{p}]$ is a $\mathbf{B}$-field lift of some Brauer class if and only if $F(B) = pB$.

For a ℓ-adic or crystalline B-field $B = \frac{b}{m}$, let $\exp(B) = 1 + B + \frac{B^2}{2}$. We define the twisted Mukai lattice as

$$\widetilde{\mathrm{H}}(X, B) = \begin{cases} \exp(B)\widetilde{\mathrm{H}}(X, \mathbb{Z}_\ell) & \text{if } p \nmid m \\ \\ \exp(B)\widetilde{\mathrm{H}}(X, W) & \text{if } m = p^n \end{cases} \tag{2.3.4}$$

under the Mukai pairing (2.3.1). Moreover, $\widetilde{\mathrm{H}}(X, B)$ is a Frobenius action stable W-lattice in $\widetilde{\mathrm{H}}(X, K)$.

Lemma 2.3.4. *([4, Lemma 3.2.4]) For any $\alpha \in \mathrm{Br}(X)[\ell^n]$, $\widetilde{\mathrm{H}}(X, B)$ is independent of the choice of $\mathbf{B}$-field lift B up to an isomorphism.*

Now let $\mathscr{X} \to X$ be a μ_n-gerbe over X whose associated Brauer class is $[\mathscr{X}]$. The category of *α-twisted coherent sheaves* is denoted by $\mathtt{Coh}(\mathscr{X})$ consisting of 1-fold $\mathscr{X}$-twisted coherent sheaves in the sense of Lieblich (*cf.* [26]), which is proven to be a Grothendieck category. Let $\mathrm{D}^b(\mathscr{X})$ be the bounded derived category of $\mathtt{Coh}(\mathscr{X})$. Consider the Grothendieck group $\mathrm{K}_0(\mathscr{X})$ of the abelian category $\mathtt{Coh}(\mathscr{X})$, there is a *twisted Chern character* map

$$\mathrm{ch}_B \colon \mathrm{K}_0(\mathscr{X}) \to \widetilde{\mathrm{H}}(X, B)_K,$$

see [21, Proposition 1.2] and [5, Definition 4.1.1] for (ℓ-adic) and crystalline cases respectively. The twisted Chern character ch_B factors througth the rational extended Néron-Severi lattice $\widetilde{N}(X)_{\mathbb{Q}}$:

$$\begin{tikzcd}
\mathrm{K}_0(\mathscr{X}) \arrow[rr, "\mathrm{ch}_B"] \arrow[dr, "\mathrm{ch}_{\mathscr{X}}"'] & & \widetilde{\mathrm{H}}(X,B)_K \\
& \widetilde{N}(X)_{\mathbb{Q}}, \arrow[ur, "\exp(B)\,\mathrm{cl}_{\mathrm{H}}"'] &
\end{tikzcd}$$

where cl_{H} is the cycle class map. The image of $\mathrm{K}_0(\mathscr{X})$ in $\widetilde{N}(X)_{\mathbb{Q}}$ under ch_B is denoted by $\widetilde{\mathrm{N}}(\mathscr{X})$. For any $\mathscr{X}$-twisted sheaf $\mathcal{E}$ on X, the Mukai vector $v_B(\mathcal{E})$ is defined to be

$$\mathrm{ch}_B([\mathcal{E}])\sqrt{\mathrm{Td}(X)} \in \widetilde{\mathrm{H}}(X, B)_K.$$

Since the Todd class $\mathrm{Td}(X)$ is trivial when X is an abelian surface, $v_B(\mathcal{E}) = \mathrm{ch}_B([\mathcal{E}]) \in \widetilde{\mathrm{H}}(X, B)_K$.

2.4. **A filtered Torelli Theorem.** The rational numerical Chow ring $\mathrm{CH}^*_{\mathrm{num}}(X)_{\mathbb{Q}}$ is equipped with a codimension filtration

$$\mathrm{Fil}^i\,\mathrm{CH}^*_{\mathrm{num}}(X)_{\mathbb{Q}} \coloneqq \bigoplus_{i\geq k} \mathrm{CH}^k_{\mathrm{num}}(X)_{\mathbb{Q}}.$$

As X is a surface, we have natural identification $\widetilde{N}(X)_{\mathbb{Q}} \cong \mathrm{CH}^*_{\mathrm{num}}(X)_{\mathbb{Q}}$, which gives the rational extended Néron-Severi lattice a filtration. Let $\Phi^{\mathcal{P}}$ be a Fourier-Mukai transform with respect to $\mathcal{P} \in \mathrm{D}^b(X \times Y)$. The equivalence $\Phi^{\mathcal{P}}$ is called *filtered* if the induced numerical Chow realization $\Phi^{\mathcal{P}}_{\mathrm{CH}}$ preserves the codimension filtration. A filtered Fourier-Mukai transform is defined in a same way since the twisted Chern character $\mathrm{ch}_{\mathscr{X}}$ maps onto $\widetilde{N}(\mathscr{X}) \subset \widetilde{N}(X)_{\mathbb{Q}}$.

At the cohomological level, the codimension filtration on $\widetilde{\mathrm{H}}(X)[\frac{1}{\ell}]$ (the prime ℓ depends on the choice of ℓ-adic or crystalline twisted Mukai lattice) is given by $F^i = \oplus_{r\geq i}\mathrm{H}^{2r}(X)[\frac{1}{\ell}]$. Let B be a **B**-field lift of $[\mathscr{X}]$. The filtration on $\widetilde{\mathrm{H}}(X, B)$ is defined by

$$F^i\widetilde{\mathrm{H}}(X, B) = \widetilde{\mathrm{H}}(X, B) \cap F^i\widetilde{\mathrm{H}}(X)[\frac{1}{\ell}].$$

A direct computation shows that the graded pieces of $F^\bullet$ are

$$\begin{aligned}
\mathrm{Gr}^0_F\widetilde{\mathrm{H}}(X, B) &= \left\{(r, rB, \frac{rB^2}{2})\middle| r \in \mathrm{H}^0(X)\right\},\\
\mathrm{Gr}^1_F\widetilde{\mathrm{H}}(X, B) &= \big\{(0, c, c\cdot B)|c \in \mathrm{H}^2(X)\big\} \cong \mathrm{H}^2(X),\\
\mathrm{Gr}^2_F\widetilde{\mathrm{H}}(X, B) &= \big\{(0, 0, s)|s \in \mathrm{H}^4(X)\big\} \cong \mathrm{H}^4(X)(1).
\end{aligned} \tag{2.4.1}$$

Lemma 2.4.1. *A twisted Fourier-Mukai transform $\Phi^{\mathcal{P}} : \mathrm{D}^b(\mathscr{X}) \to \mathrm{D}^b(\mathscr{Y})$ is filtered if and only if its cohomological realization is filtered for certian $\boldsymbol{B}$-field lifts.*

Proof. It is clear that being filtered implies being cohomological filtered. This is because the map $\exp(B)\cdot \mathrm{cl}_H \colon \widetilde{N}(X, \mathbb{Q}) \to \widetilde{\mathrm{H}}(X, B)$ preserves the filtrations for any **B**-field lift B of $[\mathscr{X}]$.

For the converse, just notice that $\Phi^{\mathcal{P}}$ is filtered if and only if the induced map $\Phi^{\mathcal{P}}_{\mathrm{CH}}$ takes the vector $(0, 0, 1)$ to $(0, 0, 1)$. As $\Phi^{\mathcal{P}}$ is cohomological filtered for B, we can see the cohomological realization of $\Phi^{\mathcal{P}}$ preserves the graded piece Gr^2_F in (2.4.1). This implies that $\Phi^{\mathcal{P}}_{\mathrm{CH}}$ takes $(0, 0, 1)$ to $(0, 0, 1)$. ∎

Proposition 2.4.2 (filtered Torelli theorem for twisted abelian surfaces)**.** *Suppose $k = \bar{k}$. Let $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ be μ_n-gerbes on abelian surfaces. Then following statements are equivalent*

(1) There is an isomorphism between associated $\mathbb{G}_m$-gerbes $\mathscr{X}_{\mathbb{G}_m}$ and $\mathscr{Y}_{\mathbb{G}_m}$.
(2) There is a filtered Fourier-Mukai transform $\Phi^{\mathcal{P}}$ from $\mathscr{X}$ to $\mathscr{Y}$.

Proof. For untwisted case, i.e. $\mathscr{X} = X$ and $\mathscr{Y} = Y$, this is exactly [18, Proposition 3.1]. Here we extend it to the twisted case. As one direction is obvious, it suffices to show that (b) can imply (b). Let

$$\mathcal{P}_x \coloneqq \Phi^{\mathcal{P}}(\mathcal{O}_x) = \mathcal{P}|_{\{x\}\times Y}.$$

Since $\mathtt{Coh}(\mathscr{X})$ and $\mathtt{Coh}(\mathscr{Y})$ have no spherical objects, there is an integer m and a $\mathscr{X} \times \mathscr{Y}^{(-1)}$-twisted sheaf $\mathcal{E} \in \mathtt{Coh}(\mathscr{X} \times \mathscr{Y}^{(-1)})$ such that $\mathcal{P} \cong \mathcal{E}[m]$ by [20, Proposition 3.18]. Since $\Phi^{\mathcal{P}}_{\mathscr{X}\to\mathscr{Y}}$ sends $(0,0,1)$ to $(0,0,1)$, $\mathcal{E}_x$ is just a skyscraper sheaf on $\{x\} \times Y$, which is naturally non-twisted as $k = \bar{k}$. Thus $\mathcal{E}$ can be viewed as an invertible sheaf on its support and also as a line bundle $\mathcal{L}$ on X. Let $\mathcal{O}_\Gamma$ be the structure sheaf of the schematic support of $\mathcal{E}$ on $X \times Y$. Hence there is a morphism $f\colon X \to Y$ such that

$$\Phi^{\mathcal{E}}_{\mathscr{X}\to\mathscr{Y}} \simeq f_*(\mathcal{L} \otimes (-)).$$

The induced integral functor $\Phi^{\mathcal{O}_\Gamma}_{X\to Y}$ takes skyscraper sheaves to skyscraper sheaves by the construction. Hence $\Phi^{\mathcal{E}}_{X\to Y}$ is fully faithful as skyscraper sheaves form a spanning class of the bounded derived category of a smooth projective variety. Moreover, it is an equivalence as abelian surfaces have trivial canonical class. By [20, Corollary 5.23]), the morphism $f\colon X \to Y$ is an isomorphism. Let B be a B-field lift of $[\mathscr{X}_{\mathbb{G}_m}]$. Since $\Phi^{\mathcal{E}}_{\mathscr{X}\to\mathscr{Y}}$ is filtered, it takes $(1, B, \frac{B^2}{2})$ to $(1, B', \frac{B'^2}{2})$ for some $B \in \mathrm{H}^2(X)_{\mathbb{Q}}$ by Lemma 2.4.1. Thus B' is a **B**-field lift of $[\mathscr{Y}_{\mathbb{G}_m}]$ and $B = f^*B'$, which implies $f^*[\mathscr{Y}_{\mathbb{G}_m}] = [\mathscr{X}_{\mathbb{G}_m}]$. $\square$

2.5. **Representability of flat cohomology.** Let $f\colon X \to S$ be a flat and proper morphism. Consider the sheaf of abelian groups $\mathrm{R}^i f_*\mu_p$ on the big fppf site $(\mathtt{Sch}/S)_{\mathrm{fl}}$, which can be expressed as the fppf-sheafification of

$$S' \mapsto \mathrm{H}^i_{\mathrm{fl}}(X_{S'}, \mu_p)$$

for any S-scheme S'. In general, the representability of $\mathrm{R}^i f_*\mu_p$ is not easy to see by the "wildness" of flat cohomology with p-torsion coefficients. In this part, we will prove the representability for supersingular abelian surfaces.

When the base S is assumed to be perfect, it is often convenient to work with an auxiliary big fppf site $(\mathtt{Perf}/S)_{\mathrm{fl}}$ for the full subcategory $\mathtt{Perf}/S \subset \mathtt{Sch}/S$ whose objects are perfect schemes over S. The inclusion functor $i\colon (\mathtt{Perf}/S)_{\mathrm{fl}} \to (\mathtt{Sch}/S)_{\mathrm{fl}}$ induces a morphism between topoi:

$$(i_*, (-)^{\mathrm{perf}})\colon \mathrm{Sh}((\mathtt{Perf}/S)_{\mathrm{fl}}) \to \mathrm{Sh}((\mathtt{Sch}/S)_{\mathrm{fl}}),$$

where $(-)^{\mathrm{perf}}$ is the functor right adjoint to i_*, called *perfection*. For example, if A is a ring, then $(\mathrm{Spec}(A))^{\mathrm{perf}} = \mathrm{Spec}(A^{1/p^\infty})$ where A^{1/p^∞} is the classical perfect closure of ring A. Restrict it to the full subcategory of algebraic spaces, we can obtain a fully faithful functor

$$\mathtt{PerfAs}/S \hookrightarrow \mathtt{As}/S$$

from the category of algebraic spaces defined using $(\mathtt{Perf}/S)_{\mathrm{fl}}$ to the category of algebraic spaces defined using $(\mathtt{Sch}/S)_{\mathrm{fl}}$. The essential image of this inclusion consists of perfect algebraic spaces over S (*cf.* [41, Lemma 04W1]).

Proposition 2.5.1. *Let S be a scheme over k. Let $f\colon X \to S$ be an abelian S-scheme of relative dimension 2, whose geometric fibers are all supersingular. Then*

(1) $\mathrm{R}^1 f_*\mu_p \cong X[p]$ is a finite flat S-group scheme of local-local type (i.e. being self-dual under Cartier duality).
(2) For any $\pi\colon \mathrm{Spec}(A) \to S$ with A being perfect, we have $\mathrm{H}^i_{\mathrm{fl}}(A, \pi^*\mathrm{R}^1 f_*\mu_p) = 0$ for $i \geq 1$. In particular, if S is perfect, then $(\mathrm{R}^1 f_*\mu_p)^{\mathrm{perf}} = 0$.

Proof. We firstly make the following assumptions. It suffices to check them affine locally. Assume S is an affine scheme of finite type over $\mathbb{Z}$. By taking Stein factorization, we can further assume $f_*\mathcal{O}_X \cong \mathcal{O}_S$. Hence $f_*\mu_p \cong \mu_p$ also holds universally. Under this assumption, we have an exact sequence of fppf-sheaves by Kummer theory:

$$0 \to \mathrm{R}^1 f_*\mu_p \to \mathrm{R}^1 f_*\mathbb{G}_m \to \mathrm{R}^1 f_*\mathbb{G}_m. \tag{2.5.1}$$

Since $\mathrm{R}^1 f_*\mathbb{G}_m$ computes the relative Picard scheme $\mathrm{Pic}_{X/S}$ and the Néron-Severi group of X is torsion-free, we can see

$$\mathrm{R}^1 f_*\mu_p \cong \ker\left(\mathrm{Pic}_{X/S} \xrightarrow{\cdot p} \mathrm{Pic}_{X/S}\right) \cong \ker\left(\mathrm{Pic}^0_{X/S} \xrightarrow{\cdot p} \mathrm{Pic}^0_{X/S}\right).$$

On the other hand, it is well-known that $\mathrm{Pic}^0_{X/S}$ is represented by the dual abelian S-scheme $\widehat{X}$ (cf. [31, Corollay 6.8]). Thus $\mathrm{R}^1 f_*\mu_p$ is representable by the commutative finite group S-scheme $\widehat{X}[p]$. Since the geometric fiber of f is supersingular, there is an isomorphism $X[p] \cong \widehat{X}[p]$. Therefore $X[p]$ is a finite group S-scheme of local-local type.

Taking the smooth group resolution of α_p:

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0,$$

we can see that $\mathrm{H}^i_{\mathrm{fl}}(A, \alpha_p) = 0$ for $i \geq 2$ for any ring A. For any finite flat group scheme G of local-local type, we can fill it in an exact sequence

$$0 \to G' \to G \to G'' \to 0$$

such G' and G'' are of smaller p-ranks. Thus by induction, we can prove that $\mathrm{H}^i_{\mathrm{fl}}(A, \mathrm{R}^1 f_*\mu_p) = 0$ for $i \geq 2$ and any finite flat group scheme G of local-local type. For $i = 1$ and A being perfect, we can see

$$\mathrm{H}^1_{\mathrm{fl}}(A, \alpha_p) = A/A^p = 0.$$

Thus $\mathrm{H}^1_{\mathrm{fl}}(A, \mathrm{R}^1 f_*\mu_p) = 0$ by the same induction as before. $\square$

Theorem 2.5.2. *Let $f\colon \mathcal{X} \to S$ be a family of supersingular abelian surfaces over a perfect algebraic space S. Then $\mathrm{R}^2 f_*\mu_p$ is representable by an algebraic space, which is separated and locally of finite presentation over S.*

Proof. This is from a standard argument by checking the list of conditions [0']-[5'] in Artin's criterion [1, Theorem 5.3]. For this, we may use the tangent-obstruction theory of flat cohomology classes (see [4, §2] for example). The difficult part is the separateness of $\mathrm{R}^2 f_*\mu_p$, i.e.,

$$\Delta_{\mathrm{R}^2 f_*\mu_p}\colon \mathrm{R}^2 f_*\mu_p \to \mathrm{R}^2 f_*\mu_p \times_S \mathrm{R}^2 f_*\mu_p$$

is representable by closed immersion, which corresponds to the Artin's condition [3']. Roughly speaking, our strategy is to show $\mathrm{R}^2 f_*\mu_p$ is representable after taking perfection and then descend to general case.

Now we check the separateness of $\mathrm{R}^2 f_*\mu_p$. Since the separateness is local in fppf-topology, we can take the same reduction for the base S as the proof in Lemma 2.5.1. Let A be an affine scheme over S of finite type. For simplicity of notations, we denote by $\mathcal{F}$ the base change $(R^2 f_*\mu_p)_A$. It suffices to prove the diagonal morphism

$$\Delta_{\mathcal{F}/A}\colon \mathcal{F} \to \mathcal{F} \times_A \mathcal{F}$$

is representable by a closed immersion.

As a first step, we assume that A is perfect, in which case $A/A^p = 0$. By Lemma 2.5.1, we have $(\mathrm{R}^1 f_*\mu_p)^{\mathrm{perf}} = 0$. Then we can proceed the proof as in [5, Proposition 2.17] to show that the diagonal morphism

$$\Delta_{\mathcal{F}/A}\colon \mathcal{F} \to \mathcal{F} \times_A \mathcal{F} \tag{2.5.2}$$

is representable by an closed immersion in $(\mathtt{Perf}/A)_{\mathrm{fl}}$. Thus $(\mathcal{F})^{\mathrm{perf}}$ is representable by a separated algebraic space over $\mathrm{Spec}(A)$ defined using $(\mathtt{Perf}/A)_{\mathrm{fl}}$. Moreover, it also implies $\mathrm{R}^2 f_*\mu_p$ is representable by (at least) quasi-separated algebraic space over S by [41, Lemma 02YS] since it is clearly the adjunction $i_*(\mathrm{R}^2 f_*\mu_p)^{\mathrm{perf}} \to \mathrm{R}^2 f_*\mu_p$ is representable by schemes.

If A is not perfect, then we can consider the directed inverse system

$$\mathrm{Spec}(A^{1/p^\infty}) \to \cdots \to \mathrm{Spec}(A^{1/p^2}) \to \mathrm{Spec}(A^{1/p}) \to \mathrm{Spec}(A).$$

Clearly, its transitions are affine morphisms. By previous discussion, we have already known that $\Delta_{\mathcal{F}/A}$ is locally of finite type ($\mathcal{F}$ is quasi-separated algebraic space) and its base change

$$\Delta_{\mathcal{F}/A} \times \mathrm{Spec}(A^{1/p^\infty}) = \Delta_{\widetilde{\mathcal{F}}/A^{1/p^\infty}} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}} \times_{A^{1/p^\infty}} \widetilde{\mathcal{F}}$$

is a closed immersion. Then by passing to limit as [41, Lemma 0850], we can see $\Delta_{\mathcal{F}/A}$ is also a closed immersion. Thus we conclude that $\mathrm{R}^2 f_* \mu_p$ is representable by a separated algebraic space over S. □

Remark 2.5.3. The case that $X \to S = \mathrm{Spec}(k)$ being a smooth surface for some field k is claimed by Artin in [2, Theorem 3.1] without proof.

For relative K3 surfaces, there is a moduli-theoretic proof given by Bragg and Lieblich using the stack of Azumaya algebras (*cf.* [5, Theorem 2.1.6]). Their proof can not be directly used for relative abelian surfaces as the essential assumption $\mathrm{R}^1 f_* \mu_p = 0$ fails in fppf site $(\mathtt{Sch}/S)_{\mathrm{fl}}$.

Recently, Bragg and Olsson has proved that, for smooth and proper f, $\mathrm{R}^2 f_* \mu_p$ is representable if $\mathrm{R}^1 f_* \mu_p$ is representable and flat over the base, e.g. for a S-abelian scheme $f\colon \mathcal{A} \to S$ (*cf.* [6, Corollary 5.8, Example 5.9]). Thus Theorem 2.5.2 can be viewed as a special case of their result. They also provides a proof for Artin's claim (see Corollay 1.4 in loc.cit.).

The following observation is essential in the construction of twistor space of supersingular abelian or K3 surfaces.

Corollary 2.5.4 ([5, Proposition 2.2.4])**.** *Keep the notations in 2.5.2. The connected components of any geometric fiber of the algebraic group space $\mathrm{R}^2 f_* \mu_p$ are isomorphic to the additive group scheme $\mathbb{G}_a$.*

Proof. Note that the completion of each geometric fiber of $\mathrm{R}^2 f_* \mu_p$ at $s \in S$, along the identity section, is isomorphic to the formal Brauer group $\widehat{\mathrm{Br}}_{X_s/k(s)}$, which is isomorphic to $\widehat{\mathbb{G}}_a$. The only smooth group scheme at $k(s)$ with this property is $\mathbb{G}_a$. □

2.6. **Twisted FM partner via moduli space of twisted sheaves.** Keep the notations as before, we denote by $\mathscr{M}_H(\mathscr{X}, v)$ (or $\mathscr{M}_H^\alpha(X, v)$) the moduli stack of H-semistable $\mathscr{X}$-twisted sheaves with Mukai vector $v \in \widetilde{\mathrm{N}}(\mathscr{X})$, where H is a v-generic ample divisor on X and $\alpha = [\mathscr{X}]$ the associated Brauer class of X (*cf.* [26] or [47]). We can characterize the Fourier-Mukai partners of twisted abelian surfaces as moduli spaces of twisted sheaves. We first need the following criterion on non-emptiness of moduli space of (twisted) sheaves on an abelian surface X.

Proposition 2.6.1 (Yoshioka, Bragg-Lieblich)**.** *Let n be a positive integer. Assume that either $p \nmid n$ or X is supersingular. Let $\mathscr{X} \to X$ be a μ_n-gerbe on X. Let $v = (r, \ell, s) \in \widetilde{\mathrm{N}}(\mathscr{X})$ be a primitive Mukai vector such that $v^2 = 0$. Fix a v-generic ample divisor H. If one of the following holds:*

(1) $r > 0$.
(2) $r = 0$ and ℓ is effective.
(3) $r = \ell = 0$ and $s > 0$.

then the coarse moduli space $M_H(\mathscr{X}, v) \neq \emptyset$ and the moduli stack $\mathscr{M}_H(\mathscr{X}, v)$ is a $\mathbb{G}_m$-gerbe on $M_H(\mathscr{X}, v)$. Moreover, its coarse moduli space $M_H(\mathscr{X}, v)$ is an abelian surface.

Proof. If $\mathscr{X} \to X$ is a μ_n-gerbe such that $p \nmid n$, then the statements are proven in [30, Proposition A.2.1]. It is based on a statement of lifting a Brauer classes on A to characteristic 0 which requires the condition $p \nmid n$.

When X is supersingular and $\mathscr{X} \to X$ is a μ_p-gerbe, the proof of [5, Proposition 4.1.20] also proceeds. Roughly speaking, we can choose a component $\mathbb{A}^1$ of the fiber of $\mathrm{R}^2 f_* \mu_p \to SA_\Lambda$ at (X, m) (see Corollay 2.5.4), where SA_Λ is the moduli space of Λ-marked supersingular abelian surfaces and f is the universal family such that $[\mathscr{X}] \in \mathbb{A}^1(k)$. By taking the relative moduli space of twisted sheaves (with suitable v-generic polarization) on the universal family of μ_p gerbes $\widetilde{\mathscr{X}} \to \mathbb{A}^1$ restricted on this connected component, we can reduce this question to the case that $\mathscr{X} \to X$ is essentially trivial, which is equivalent to the untwisted case. Then we can conclude it by a standard lifting argument. □

Let $\mathscr{X} \to X$ be a $\mathbb{G}_m$-gerbe on X such that $[\mathscr{X}] \in \mathrm{Br}(X)[\ell^n]$. Take a **B**-field lift $B = \frac{b}{\ell^n}$ of $[\mathscr{X}]$. Under the isomorphism $\mathrm{H}^2(X, \mathbb{Z}_\ell(1)) \cong \mathrm{H}^2(\widehat{X}, \mathbb{Z}_\ell(1))$ induced by the Poincaré bundle, there is an element $\widehat{b} \in \mathrm{H}^2(\widehat{X}, \mathbb{Z}_\ell(1))$ being the image of b and $\frac{\widehat{b}}{\ell^n}$ is a ℓ-adic (or crystalline) **B**-field on $\widehat{X}$. Thus we have a unique Brauer class $[\widehat{\mathscr{X}}]$ corresponding to $\frac{\widehat{b}}{\ell^n}$. We will denote the $\mathbb{G}_m$-gerbe on $\widehat{X}$ with respect to $\widehat{\alpha}$ by $\widehat{\mathscr{X}}$.

Theorem 2.6.2. *With the same assumptions as in Proposition 2.6.1. Let $\mathscr{X} \to X$ be μ_n-gerbe on an abelian surface X. Then the associated $\mathbb{G}_m$-gerbe of any Fourier-Mukai partner of $\mathscr{X}$ is isomorphic to a $\mathbb{G}_m$-gerbe on the moduli space of $\mathscr{X}$-twisted sheaves $M_H(\mathscr{X}, v)$ with $\mathscr{X}$ being $\mathscr{X}$ or $\widehat{\mathscr{X}}$.*

Proof. Let $\mathscr{M}$ be a Fourier-Mukai partner of $\mathscr{X}$. Let $\Phi^{\mathcal{P}}_{\mathscr{M}\to\mathscr{X}}$ be the Fourier-Mukai transform. Let v be the image of $(0,0,1)$ under $\Phi^{\mathcal{P}}_{\mathscr{M}\to\mathscr{X}}$. We can assume v satisfying one of the conditions in Proposition 2.6.1 by changing $\mathscr{X}$ to $\widehat{\mathscr{X}}$ if necessary. Denote $\mathscr{X}$ by $\mathscr{X}$ or $\widehat{\mathscr{X}}$. It is proved that the moduli stack $\mathscr{M}_H(\mathscr{X}, v)$ is a $\mathbb{G}_m$-gerbe on $M_H(A, v)$ in Proposition 2.6.1. Then there is a Fourier-Mukai transform

$$\Phi^{\mathcal{P}} \colon \mathrm{D}^b(\mathscr{M}_H(\mathscr{X}, v)^{(-1)}) \to \mathrm{D}^b(\mathscr{X}^{(1)}) \tag{2.6.1}$$

induced by the tautological sheaf $\mathcal{P}$ on $\mathscr{M}_H(\mathscr{X}, v) \times \mathscr{X}$, whose cohomological realization maps the Mukai vector $(0,0,1)$ to v. Combining it with the derived equivalence

$$\Phi \colon \mathrm{D}^b(\mathscr{X}) \to \mathrm{D}^b(\mathscr{M}),$$

we will obtain a filtered derived equivalence from $\mathscr{M}_H(\mathscr{X}, v)^{(-1)}$ to $\mathscr{M}^{(1)}$. This induces an isomorphism from $\mathscr{M}_H(\mathscr{X}, v)^{(-1)}$ to $\mathscr{M}^{(1)}_{\mathbb{G}_m}$ by Theorem 2.4.2. □

Similar as K3 surfaces, the action of twisted derived equivalence on abelian surfaces are isometries between the twisted Mukai lattices (*cf.* [7, Theorem 3.6]). For our purpose, we concentrate on the action of prime-to p twisted derived equivalence on the 2^{nd} crystalline cohomology of abelian surfaces. We show that the twisted Fourier-Mukai transform between abelian surfaces induces rational isometries on the 2^{nd} étale and crystalline cohomology group.

Theorem 2.6.3. *Assume* $\mathrm{char}(k) = p \neq 2$. *Let ℓ be a prime not equal to p. If X and Y are twisted derived equivalent over k, then there is a $\mathrm{Gal}(\bar{k}/k')$-equivariant isometry*

$$\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell) \cong \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Q}_\ell). \tag{2.6.2}$$

for some finite extension k'/k. Suppose $p > 2$ and k is perfect. When the twisted derived equivalence is prime-to-p, then there is an isometry between F-crystals

$$\mathrm{H}^2_{\mathrm{crys}}(X_{k'}/W') \cong \mathrm{H}^2_{\mathrm{crys}}(Y_{k'}/W'). \tag{2.6.3}$$

for some finite extension k'/k.

Proof. This is similar to the case of K3 surfaces proved in [7]. We only need to prove the assertion for a single derived equivalence $\Phi : \mathrm{D}^b(\mathscr{X}) \xrightarrow{\sim} \mathrm{D}^b(\mathscr{Y})$ for some gerbes $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$. By Theorem 2.6.2, $\mathscr{X}$ is (geometrically) isomorphic to the moduli space of $\mathscr{Y}$-twisted sheaves on Y, i.e. $\mathscr{X}_{\bar{k}} \cong \mathscr{M}_H(\mathscr{Y}_{\bar{k}}, v)$ for some Mukai vector v. By the Witt cancellation, it suffices to show that there is an Galois equivariant isometry

$$\widetilde{\mathrm{H}}(\mathscr{X}_{\bar{k}}, \mathbb{Q}_\ell) \cong \widetilde{\mathrm{H}}(\mathscr{Y}_{\bar{k}}, \mathbb{Q}_\ell) \tag{2.6.4}$$

between the rational twisted Mukai lattices. This is straightforward. For instance, one can apply the twisted Grothendieck-Riemann-Roch formula [5, Lemma 4.1.4] to projections $\mathscr{X} \times \mathscr{Y} \to \mathscr{X}$ and $\mathscr{X} \times \mathscr{Y} \to \mathscr{Y}$ ([7, Theorem 3.6]).

The more difficult part is to prove the F-crystal isometry (2.6.3). In this case, if X or Y is supersingular, then X is actually derived equivalent to Y as the Brauer classes of supersingular abelian surfaces are p-torsion. According to [25, Theorem 4.6], X has no non-trivial Fourier-Mukai partner and hence $X \cong Y$. Hence we may assume X and Y have finite height. As before, it suffices to show that there is an integral isometry of Mukai crystals

$$\widetilde{\mathrm{H}}(X, W) \cong \widetilde{\mathrm{H}}(Y, W) \tag{2.6.5}$$

Note that the twisted Grothendieck-Riemann-Roch computation is still valid when $p \geq 5$ (*cf.* [4, Appendix A]). Here we will provide an alternative proof by lifting to characteristic 0, which also works for $p = 3$. Let us sketch the proof as below: according to Lemma 2.2.4, one can find a lifting

$$\mathfrak{X} \to \mathcal{X}$$

of $\mathscr{X} \to X$ over some finite extension W' of W with Néron-Severi group preserved. Take the relative moduli space $\mathscr{M}_{\mathcal{H}}(\mathfrak{X}, \mathbf{v})$ of twisted sheaf over W' and let $M(\mathfrak{X}, v)$ be the coarse moduli space. Here $\mathcal{H}$ and $\mathbf{v}$ are some lift of H and v over W'. Let K' be the fraction field of W' and $\mathfrak{X}_{K'} \to \mathcal{X}_{K'}$ the generic fiber. By the prime-to-p assumption, one can compute that there is a $\mathbb{Z}_p$-integral Galois equivariant (up to a finite field extension) isometry

$$\widetilde{\mathrm{H}}(\mathcal{X}_{\bar{K}'}, \mathbb{Z}_p) \cong \widetilde{\mathrm{H}}(M_{\mathcal{H}}(\mathfrak{X}_{\bar{K}'}, \mathbf{v}), \mathbb{Z}_p).$$

Note that the special fiber of the relative moduli stack $\mathscr{M}_{\mathcal{H}}(\mathfrak{X}, \mathbf{v}) \to M(\mathfrak{X}, \mathbf{v})$ is isomorphic to the gerbe $\mathscr{Y} \to Y$. Then one can apply the reduction and the comparison theorem to prove the assertion.

□

3. Shioda's Torelli theorem for abelian surfaces

In [39], Shioda noticed that there is a way to extract the information of the 1^{st}-cohomology of a complex abelian surface from its 2^{nd}-cohomology, called Shioda's trick. This established a global Torelli theorem for complex abelian surfaces via the 2^{nd}-cohomology. The aim of this section is to generalize Shioda's method to all fields and establish an isogenous theorem for abelian surfaces via the 2^{nd}-cohomology. We will deal with Shioda's trick for étale cohomology and crystalline cohomology separately.

3.1. **Recap of Shioda's trick for Hodge isometry.** We first recall Shioda's construction. Suppose X is a complex abelian surface. Its singular cohomology ring $\mathrm{H}^\bullet(X, \mathbb{Z})$ is canonically isomorphic to the exterior algebra $\wedge^\bullet \mathrm{H}^1(X, \mathbb{Z})$. Let V be a free $\mathbb{Z}$-module of rank 4. We denote by Λ the lattice $(\wedge^2 V, q)$ where $q : \wedge^2 V \times \wedge^2 V \to \mathbb{Z}$ is the wedge product. After choosing a $\mathbb{Z}$-basis $\{v_i\}_{1 \leq i \leq 4}$ for $\mathrm{H}^1(X, \mathbb{Z})$, we have an isometry of $\mathbb{Z}$-lattice $\Lambda \xrightarrow{\sim} \mathrm{H}^2(X, \mathbb{Z})$. The set of vectors

$$\{v_{ij} \coloneqq v_i \wedge v_j\}_{0 \leq i < j \leq 4}$$

clearly forms a basis of $\mathrm{H}^2(X, \mathbb{Z})$, which will be called an *admissible basis* of A for its second singular cohomology. For another complex abelian surface Y, a Hodge isometry

$$\psi \colon \mathrm{H}^2(Y, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(X, \mathbb{Z})$$

will be called *admissible* if $\det(\psi) = 1$, with respect to some admissible bases on X and Y. It is clear that the admissibility of a morphism is independent of the choice of admissible bases.

In terms of admissible basis, we can view ψ as an element in $\mathrm{SO}(\Lambda)$. On the other hand, we have the following exact sequence of groups

$$1 \to \{\pm 1\} \to \mathrm{SL}_6(\mathbb{Z}) \xrightarrow{\wedge^2} \mathrm{SO}(\lambda) \tag{3.1.1}$$

Shioda observed that the image of $\mathrm{SL}_6(\mathbb{Z})$ in $\mathrm{SO}(\Lambda)$ is a subgroup of index two and does not contain $-\mathrm{id}_\Lambda$. From this, he proved the following

Theorem 3.1.1 (Shioda). *For any admissible Hodge isometry ψ, there is an isomorphism of Hodge structure*
$$\varphi\colon \mathrm{H}^1(Y,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(X,\mathbb{Z})$$
such that $\wedge^2(\varphi)=\psi$ or $-\psi$.

This is what we call "Shioda's trick". As we can assume a Hodge isometry being admissible after taking the dual abelian variety for one of them, we can obtain the Torelli theorem for complex abelian surfaces by their weight two Hodge structures, i.e., X is isomorphic to Y or its dual $\widehat{Y}$ if and only if there is an integral Hodge isometry $\mathrm{H}^2(X,\mathbb{Z})\cong\mathrm{H}^2(Y,\mathbb{Z})$ (*cf.* [39, Theorem 1]).

3.2. **ℓ-adic and p-adic admissible basis.** In order to extend Shioda's work to an arbitrary field k with $\mathrm{char}(k)=p\geq 0$, we need to define admissibility for both étale cohomology and crystalline cohomology.

Suppose X is an abelian surface over k and $\ell\nmid p$ is a prime. There is an isomorphism between the cohomology ring $\mathrm{H}^\bullet_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)$ (resp. $\mathrm{H}^\bullet_{\text{crys}}(X_{\bar{k}}/W)$) and the exterior algebra $\wedge^\bullet\mathrm{H}^1_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)$ (resp. $\wedge^\bullet\mathrm{H}^1_{\text{crys}}(X_{\bar{k}}/W)$). We denote by
$$\mathrm{tr}_\ell\colon \mathrm{H}^4_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)\xrightarrow{\sim}\mathbb{Z}_\ell \text{ (resp. } \mathrm{tr}_p : \mathrm{H}^4_{\text{crys}}(X_{\bar{k}}/W)\xrightarrow{\sim} W)$$
the corresponding trace map. Then the Poincaré pairing $\langle -,-\rangle$ on $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)$ (resp. $\mathrm{H}^2_{\text{crys}}(X_{\bar{k}}/W)$) can be realized as
$$\langle\alpha,\beta\rangle=\mathrm{tr}_\ell(\alpha\wedge\beta) \text{ (resp. } \mathrm{tr}_p(\alpha\wedge\beta)).$$
Analogous to §3.1, a $\mathbb{Z}_\ell$-basis $\{v_i\}$ of $\mathrm{H}^1_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)$ will be called *admissible* if it satisfies
$$\mathrm{tr}_\ell(v_1\wedge v_2\wedge v_3\wedge v_4)=1.$$
The associated $\mathbb{Z}_\ell$-basis $\{v_{ij}\coloneqq v_i\wedge v_j\}_{i<j}$ of $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)$ will be called a *ℓ-adic admissible basis.* For the dual abelian surface $\widehat{X}$, the dual basis $\{v_i^*\}$ with respect to Poincaré pairing naturally forms an admissible basis, under the identification
$$\mathrm{H}^1_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)^\vee\cong\mathrm{H}^1_{\text{ét}}(\widehat{X}_{\bar{k}},\mathbb{Z}_\ell).$$
Similarly, one can define the *p-adic admissible basis* for $\mathrm{H}^1_{\text{crys}}(X_{\bar{k}}/W)$ and $\mathrm{H}^2_{\text{crys}}(X_{\bar{k}}/W)$ via using tr_p, i.e. $\mathrm{tr}_p(v_1\wedge v_2\wedge v_3\wedge v_4)=1$ for some W-basis of $\mathrm{H}^1_{\text{crys}}(X_{\bar{k}}/W)$. With these notions, we can introduce the concept of admissible morphisms and isometries between abelian surfaces.

Definition 3.2.1. Let X and Y be abelian surfaces over k.

- a $\mathbb{Z}_\ell$-linear morphism $\mathrm{H}^1_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)\to\mathrm{H}^1_{\text{ét}}(Y_{\bar{k}},\mathbb{Z}_\ell)$ is admissible if it takes an admissible basis to an admissible basis.
- a $\mathbb{Z}_\ell$-linear isometry $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)\to\mathrm{H}^2_{\text{ét}}(Y_{\bar{k}},\mathbb{Z}_\ell)$ will be called admissible if the determinant $=1$ with respect to some admissible basis.
- a W-linear map $\mathrm{H}^1_{\text{crys}}(X_{\bar{k}}/W)\to\mathrm{H}^1_{\text{crys}}(Y_{\bar{k}}/W)$ is admissible if it takes some admissible basis to an admissible basis.
- a W-linear isometry $\mathrm{H}^2_{\text{crys}}(X_{\bar{k}}/W)\to\mathrm{H}^2_{\text{crys}}(Y_{\bar{k}}/W)$ is p-adic admissible if its determinant is 1 with respect to some p-admissible basis.

Example 3.2.2. Suppose X,Y are abelian surfaces over $k=\bar{k}$. Let
$$\psi_{\mathcal{P}}\colon\mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}_\ell)\to\mathrm{H}^2_{\text{ét}}(\widehat{X},\mathbb{Z}_\ell)$$
be the isomorphism induced by the Poincaré bundle $\mathcal{P}$ on $X\times\widehat{X}$. A direct computation shows that $\psi_{\mathcal{P}}$ is nothing but
$$-\mathrm{D}\colon\mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}_\ell)\xrightarrow{\sim}\mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}_\ell)^\vee\cong\mathrm{H}^2_{\text{ét}}(\widehat{X},\mathbb{Z}_\ell),$$
where D is the Poincaré duality. For an admissible basis $\{v_i\}$ of X, its $\mathbb{Z}_\ell$-linear dual $\{v_i^*\}$ with respect to Poincaré pairing forms an admissible basis of $\widehat{X}$. By our construction, we can see
$$\mathrm{D}(v_{12},v_{13},v_{14},v_{23},v_{24},v_{34})=(v_{34}^*,-v_{24}^*,v_{23}^*,v_{14}^*,-v_{13}^*,v_{12}^*),$$

which implies that D is of determinant -1 under these ℓ-adic admissible bases. Thus the determinant of ψ_P is not admissible.

Example 3.2.3 (isogeny)**.** Let $f : Y \to X$ be an isogeny of degree n^4 between two abelian surfaces. If n is coprime to ℓ, then the induced isomorphism

$$\frac{1}{n} f^* \colon \mathrm{H}^i_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell),\ i = 1, 2$$

are admissible. In particular, $\frac{1}{n} f^*$ induces an $\mathbb{Z}_\ell$-integral isometry between their second cohomology groups.

3.3. **Admissible basis on F-crystals.** In contrast to ℓ-adic étale cohomology, the semilinear structure on crystalline cohomlogy from its Frobenius is more tricky to work with. Therefore, it seems necessary for us to spend more words on the interaction of Frobenius with admissible bases.

We have the following Frobenius pull-back diagram:

$$\begin{tikzcd}
X \arrow[dr, "F_X^{(1)}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\
& X^{(1)} \arrow[r] \arrow[d] & X \arrow[d] \\
& \mathrm{Spec}(k) \arrow[r, "\sigma"] & \mathrm{Spec}(k)
\end{tikzcd}$$

Via the natural identification $\mathrm{H}^1_{\text{crys}}(X^{(1)}/W) \cong \mathrm{H}^1_{\text{crys}}(X/W) \otimes_\sigma W$, the σ-linearization of Frobenius action on $\mathrm{H}^1_{\text{crys}}(X/W)$ can be viewed as the injective W-linear map

$$F^{(1)} \coloneqq \left(F_X^{(1)}\right)^* \colon \mathrm{H}^1_{\text{crys}}(X^{(1)}/W) \hookrightarrow \mathrm{H}^1_{\text{crys}}(X/W).$$

There is a decomposition $\mathrm{H}^1_{\text{crys}}(X/W) = H_0(X) \oplus H_1(X)$ such that

$$F^{(1)}\left(\mathrm{H}^1_{\text{crys}}(X^{(1)}/W)\right) \cong H_0(X) \oplus pH_1(X), \tag{3.3.1}$$

and $\mathrm{rank}_W H_i = 2$ for all $i = 0, 1$, which is related to the Hodge decomposition of the de Rham cohomology of X/k by Mazur's theorem; see [3, §8, Theorem 8.26].

The Frobenius map can be expressed in terms of admissible basis. We can choose an admissible basis $\{v_i\}$ of $\mathrm{H}^1_{\text{crys}}(X/W)$ such that

$$v_1, v_2 \in H_0(X) \quad \text{and} \quad v_3, v_4 \in H_1(X).$$

Then $\{p^{\alpha_i} v_i\} \coloneqq \{v_1, v_2, pv_3, pv_4\}$ forms an admissible basis of $\mathrm{H}^1_{\text{crys}}(X^{(1)}/W)$ under the identification (3.3.1), since $\mathrm{tr}_p \circ \wedge^4 F^{(1)} = p^2 \sigma_W \circ \mathrm{tr}_p$. Now the Frobenius map can be written as

$$F^{(1)}(p^{\alpha_i} v_i) = \sum_j c_{ij} p^{\alpha_j} v_j,$$

where $C_X = (c_{ij})$ forms an invertible 4×4-matrix with coefficients in W. Suppose Y is another abelian surface over k and $\rho \colon \mathrm{H}^1_{\text{crys}}(X/W) \to \mathrm{H}^1_{\text{crys}}(Y/W)$ is an admissible map. Denote $\rho^{(1)}$ for the induced map $\rho \otimes_\sigma W \colon \mathrm{H}^1_{\text{crys}}(X^{(1)}/W) \to \mathrm{H}^1_{\text{crys}}(Y^{(1)}/W)$. The following clear.

Lemma 3.3.1. *The map ρ is a morphism between F-crystals if and only if $C_Y^{-1} \cdot \rho^{(1)} \cdot C_X = \rho$, where $\cdot$ denotes for the action of matrix with respect to chosen admissible bases.*

3.4. **Generalized Shioda's trick.** Let us review some basic properties of the special orthogonal group scheme over an integral domain. Let Λ be an even $\mathbb{Z}$-lattice of rank $2n$. Then we can associate it with a vector bundle $\underline{\Lambda}$ on $\operatorname{Spec}(\mathbb{Z})$ with constant rank $2n$ equipped with a quadratic form q over $\operatorname{Spec}(\mathbb{Z})$ obtained from Λ. Then the functor

$$A \mapsto \left\{g \in \mathrm{GL}(\Lambda_A) \middle| q_A(g \cdot x) = g \cdot q_A(x) \text{ for all } x \in \Lambda_A\right\}$$

represents a $\mathbb{Z}$-subscheme of $\mathrm{GL}(\Lambda)$, denoted by $\mathrm{O}(\Lambda)$. There is a homomorphism between $\mathbb{Z}$-group schemes

$$D_\Lambda \colon \mathrm{O}(\Lambda) \to \underline{\mathbb{Z}/2\mathbb{Z}},$$

which is called the Dickson morphism. It is surjective as Λ is even, and its formation commutes with any base change. The *special orthogonal group scheme* over $\mathbb{Z}$ with respect to Λ is defined to be the kernel of D_Λ, which is denoted by $\mathrm{SO}(\Lambda)$. Moreover, we have

$$\mathrm{SO}(\Lambda)_{\mathbb{Z}[\frac{1}{2}]} \cong \ker\left(\det\colon \mathrm{O}(\Lambda) \to \mathbb{G}_m\right)_{\mathbb{Z}[\frac{1}{2}]}.$$

It is well-known that $\mathrm{SO}(\Lambda) \to \operatorname{Spec}(\mathbb{Z})$ is smooth of relative dimension 15 and with connected fibers; see [9, Theorem C.2.11] for instance. For any ℓ, the special orthogonal group scheme

$$\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell}) \cong \mathrm{SO}(\Lambda)_{\mathbb{Z}_\ell}$$

is smooth over $\mathbb{Z}_\ell$ with connected fibers, which implies its generic fiber $\mathrm{SO}(\Lambda_{\mathbb{Q}_\ell})$ is connected. Thus $\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ is clearly connected as a group scheme over $\mathbb{Z}_\ell$ as $\mathrm{SO}(\Lambda_{\mathbb{Q}_\ell}) \subset \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ is dense.

Assume $\Lambda = U^{\oplus 3}$, where U is the hyperbolic lattice. Consider the homomorphism of $\mathbb{Z}_\ell$-group schemes

$$\wedge^2 (-)_\ell \colon \mathrm{SL}_4(V_{\mathbb{Z}_\ell}) \to \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell}). \tag{3.4.1}$$

Then we have

Lemma 3.4.1. *There is an exact sequence of $\mathbb{Z}_\ell$-group schemes*

$$1 \to \mu_{2,\mathbb{Z}_\ell} \to \mathrm{SL}_4(V_{\mathbb{Z}_\ell}) \xrightarrow{\wedge^2(-)_\ell} \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell}) \to 1.$$

The image of the group homomorphism

$$\wedge^2(-)_\ell \colon \mathrm{SL}(V_{\mathbb{Z}_\ell})(\mathbb{Z}_\ell) \to \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})(\mathbb{Z}_\ell),$$

is a subgroup of $\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ *of finite index.*

Proof. Since $\mathrm{SO}(V_{\mathbb{Z}_\ell})$ is connected and $\dim(\mathrm{SL}(V_{\mathbb{Z}_\ell})) = \dim(\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell}))$, $\wedge^2(-)_\ell$ will be automatically surjective if we show that its kernel is isomorphic to the finite group scheme μ_2. It suffices to check

$$\ker(\wedge^2(-)_\ell \otimes \mathbb{Q}_\ell) \cong \mu_2(\mathbb{Q}_\ell).$$

As the kernel is stable under conjugation, we can assume that $A \in \ker\left(\wedge^2(-)_\ell \otimes \mathbb{Q}_\ell\right)$ is of the Jordan normal form. Then a direct computation shows that $A \in \{\pm \mathrm{id}_4\}$.

We have an exact sequence on rational points (*cf.* [15, Proposition 3.2.2])

$$1 \to \mu_2(\mathbb{Z}_\ell) \to \mathrm{SL}(V_{\mathbb{Z}_\ell})(\mathbb{Z}_\ell) \to \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})(\mathbb{Z}_\ell) \to \mathrm{H}^1(\operatorname{Spec}(\mathbb{Z}_\ell), \mu_2),$$

where the $\mathrm{H}^1(\operatorname{Spec}(\mathbb{Z}_\ell), \mu_2)$ is the group of μ_2-torsors on $\operatorname{Spec}(\mathbb{Z}_\ell)$. From the Kummer sequence for μ_2, we can see

$$\mathrm{H}^1(\operatorname{Spec}(\mathbb{Z}_\ell), \mu_2) \cong \mathrm{H}^1_{\text{ét}}(\operatorname{Spec}(\mathbb{Z}_\ell), \mu_2) \cong \mathbb{Z}_\ell^*/(\mathbb{Z}_\ell^*)^2.$$

Note that we have isomorphisms

$$\mathbb{Z}_\ell^*/(\mathbb{Z}_\ell^*)^2 \cong \begin{cases} \{\pm 1\} & \text{if } \ell \neq 2, \\ \{\pm 1\} \times \{\pm 5\} & \text{if } \ell = 2. \end{cases}$$

Therefore, the $\mathrm{Im}(\wedge^2(-)_\ell)$ is of finite index in $\mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$. When $\ell \neq 2$, the index of the image of $\mathrm{SL}(V_{\mathbb{Z}_\ell})(\mathbb{Z}_\ell)$ under $\lambda^2(-)_\ell$ is less equal to 2. On the other hand, we can see $-1 \notin \mathrm{Im}(\Lambda^2(-)_\ell)$. Thus $\mathrm{Im}\left(\wedge^2(-)_\ell\right) \cdot \{\pm 1\} = \mathrm{SO}(\Lambda_{\mathbb{Z}_\ell})$ if $\ell \neq 2$. $\square$

Remark 3.4.2. The statements in Lemma 3.4.1 also hold for the W-value points of these groups. Since $W^*/(W^*)^2 \subseteq \{\pm 1\}$ by Hensel lemma, the image of $\mathrm{SL}(V)(W)$ is of finite index ≤ 2.

We first consider the ℓ-adic admissible isomorphisms between abelian surfaces X and Y. Let $\{v_i\}$ and $\{v_i'\}$ be admissible bases of X and Y respectively. There is a natural admissible $\mathbb{Z}_\ell$-isomorphism:

$$\psi_0 \colon \mathrm{H}^1_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \to \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$$
$$v_i' \mapsto v_i.$$

For an admissible isometry $\varphi \colon \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \to \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell)$, the composition of $\mathbb{Z}_\ell$-automorphisms

$$\wedge^2(\psi_0) \circ \varphi \in \mathrm{SO}(\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)),$$

is an admissible automorphism. Let Λ_ℓ be the $\mathbb{Z}_\ell$-lattice $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$ with respect to Poincaré pairing. In this way, any ℓ-adic admissible isometry φ can be identified with an element $g \in \mathrm{SO}(\Lambda_\ell)(\mathbb{Z}_\ell)$. The ℓ-adic Shioda's trick reads as

Proposition 3.4.3. *Suppose $\ell \neq 2$. For any admissible isometry $\varphi_\ell \colon \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$, we can find an admissible $\mathbb{Z}_\ell$-isomorphism ψ_ℓ such that $\wedge^2(\psi_\ell) = \varphi_\ell$ or $-\varphi_\ell$. Moreover, if ψ_ℓ is G_k-equivariant, then φ_ℓ is also G_k-equivariant after replacing k by some finite extension.*

Proof. Let $\{\alpha_i\}_{i=1}^4$ and $\{\beta_j\}_{j=1}^4$ be admissible bases of $\mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$ and $\mathrm{H}^1_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell)$ respectively. Then there is an admissible isomorphism of $\mathbb{Z}_\ell$-modules:

$$\psi_0 \colon \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell)$$
$$\alpha_i \mapsto \beta_i.$$

Let $\varphi_0 = \wedge^2(\psi_0) \colon \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell)$. The composition $\varphi_\ell \circ \varphi_0 \in \mathrm{SO}(\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)) \cong G_\ell$. There is an element $\widetilde{\psi} \in \mathrm{SL}_4(\mathbb{Z}_\ell)$ and $\delta \in \Delta_\ell$ such that $\wedge^2(\widetilde{\psi}) = \varphi_\ell \circ \varphi_0$ or $-\varphi_\ell \circ \varphi_0$ by Lemma 3.4.1. Thus we have an admissible $\mathbb{Z}_\ell$-isomorphism

$$\psi_\ell \coloneqq \widetilde{\psi} \circ (\psi_0)^{-1} \colon \mathrm{H}^1_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$$

such that $\wedge^2(\psi_\ell) = \varphi_\ell$ or $-\varphi_\ell$.

Suppose ϕ_ℓ is G_k-equivariant. We may assume $\wedge^2(\psi_\ell) = \varphi_\ell$ for simplicity. For any $g \in G_k$, we have $\wedge^2(g^{-1}\psi_\ell g) = g^{-1}\wedge^2(\psi_\ell)g = \phi_\ell$. Therefore, $g^{-1}\psi_\ell g = \pm\psi_\ell$. It implies ψ_ℓ is G_k-equivariant, up to a finite extension of k, . ∎

Remark 3.4.4. Let Δ be the finite-set of 6×6 matrices

$$\left\{ \pm \mathrm{id}_6, \pm \begin{pmatrix} 5 & & \\ & \frac{1}{5} & \\ & & \mathrm{id}_4 \end{pmatrix} \right\}.$$

If $\ell = 2$, then we can find a $\delta \in \Delta$ such that $\wedge^2 \varphi_\ell = \delta \cdot \psi_\ell$. Here the δ acts with respect to the chosen admissible basis.

For F-crystals attached to abelian surfaces, we can also play the Shioda's trick.

Proposition 3.4.5. *Suppose $p > 2$. For any admissible W-linear isometry*

$$\varphi_W \colon \mathrm{H}^2_{\text{crys}}(Y/W) \xrightarrow{\sim} \mathrm{H}^2_{\text{crys}}(X/W),$$

we can find an admissible W-linear isomorphism $\rho : \mathrm{H}^1_{\text{crys}}(Y/W) \to \mathrm{H}^1_{\text{crys}}(X/W)$ such that $\wedge^2(\rho) = \varphi_W$ or $-\varphi_W$. Moreover, if φ_W is a morphism of F-crystals, then there is an isomorphism

$$\mathrm{H}^1_{\text{crys}}(Y/W) \cong \mathrm{H}^1_{\text{crys}}(X/W). \tag{3.4.2}$$

as F-crystals.

Proof. Let $\{v_i\}$ and $\{v_i'\}$ be admissible bases of $\mathrm{H}^1_{\mathrm{crys}}(X/W)$ and $\mathrm{H}^1_{\mathrm{crys}}(Y/W)$ respectively. Consider the admissible W-linear isomorphism

$$\begin{aligned} \rho_0 \colon \mathrm{H}^1_{\mathrm{crys}}(X/W) &\to \mathrm{H}^1_{\mathrm{crys}}(Y/W) \\ v_i &\mapsto v_i'. \end{aligned}$$

It is clear that $\varphi_0 \coloneqq \wedge^2(\rho_0)$ is an admissible W-linear isometry. The composition $\varphi_W \circ \varphi_0$ lies in the special orthogonal group $\mathrm{SO}(\mathrm{H}^2_{\mathrm{crys}}(X/W))$. Then there is a unique $\rho \in \mathrm{SL}\big(\mathrm{H}^1_{\mathrm{crys}}(X/W)\big)$ up to sign, such that $\wedge^2(\rho) = \varphi_W \circ \varphi_0$ or $-\varphi_W \circ \varphi_0$ by Lemma 3.4.1 and Remark 3.4.2. By abuse of notation, we use the same letter ρ for $\rho \circ \rho_0^{-1} \colon \mathrm{H}^1_{\mathrm{crys}}(Y/W) \to \mathrm{H}^1_{\mathrm{crys}}(X/W)$.

Assume $\wedge^2(\rho) = \varphi_W$. If φ_W commutes with the Frobenius action, then we have

$$\wedge^2(C_Y^{-1} \cdot \rho^{(1)} \cdot C_X) = \varphi_W.$$

as in §3.3. Thus $C_Y^{-1} \cdot \rho^{(1)} \cdot C_X = \pm\rho$, which implies $\rho \circ F_X = \pm F_Y \circ \rho$ by Lemma 3.3.1. If $\rho \circ F_X = -F_Y \circ \rho$, then we can see

$$C_{Y^{(1)}}^{-1} \cdot \rho^{(2)} \cdot C_{X^{(1)}} = -C_Y^{-1} \cdot \rho^{(1)} \cdot C_X = \rho.$$

which means that $\rho \colon \mathrm{H}^1_{\mathrm{crys}}(Y/W) \to \mathrm{H}^1_{\mathrm{crys}}(X/W)$ commutes with the 2-fold Frobenius. Let ρ' be the restriction of ρ under the W-linear embedding

$$F^{(1)} \colon \mathrm{H}^1_{\mathrm{crys}}(Y^{(1)}/W) \hookrightarrow \mathrm{H}^1_{\mathrm{crys}}(Y/W).$$

From the construction, it is clear that

$$\rho' \colon \mathrm{H}^1_{\mathrm{crys}}(Y^{(1)}/W) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{crys}}(X^{(1)}/W)$$

is a W-linear isomorphism and $\rho' \otimes_\sigma W = \rho^{(2)}$. Thus we can see

$$\rho' \circ F^*_{Y^{(1)}/k} = F^*_{X^{(1)}/k} \circ \rho^{(2)} = F^*_{X^{(1)}/k} \circ (\rho' \otimes_\sigma W), \tag{3.4.3}$$

which means ρ' is an isomorphism of F-crystals. By taking the base change of inverse Frobenius, we can conclude that there is isomorphism of F-crystals $\mathrm{H}^1_{\mathrm{crys}}(Y/W) \cong \mathrm{H}^1_{\mathrm{crys}}(X/W)$. □

Remark 3.4.6. The readers should be careful that the isomorphism (3.4.2) not necessarily induces the given admissible isometry ρ.

Combined with Tate's isogeny theorem, we can verify the Tate conjecture for isometries between the 2^{nd}-cohomology of abelian surfaces over $\overline{\mathbb{F}}_p$.

Corollary 3.4.7. *Let X and Y be abelian surfaces over $k = \mathbb{F}_{p^n}$ with $p > 2$. Then any G_k-equivariant isometry*

$$\varphi_\ell : \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \cong \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell),$$

with $\ell \nmid p$ or F-crystal isometry $\varphi_W : \mathrm{H}^2_{\mathrm{crys}}(X/W) \cong \mathrm{H}^2_{\mathrm{crys}}(Y/W)$ is algebraic after a finite field extension.

Proof. We can assume $\varphi_\bullet$ is admissible otherwise we can compose it with the dual isometry $\mathrm{H}^2(Y, -) \cong \mathrm{H}^2(\widehat{Y}, -)$. Then Proposition 3.4.3 and Proposition 3.4.5 shows that such admissible isometries are induced by quasi-isogenous between X and Y after a finite base change. The algebraicity then follows. □

Remark 3.4.8. In [49], Zarhin introduces the notion of *almost isomorphism*. Two abelian varieties over k are called almost isomorphic if their Tate modules T_ℓ are isomorphic as Galois modules (replaced by Tate modules of p-divisible groups when $\ell = p$). Proposition 3.4.3 and 3.4.5 imply that it is possible to characterize almost isomorphic abelian surfaces by their 2^{nd}-cohomology groups.

4. Twisted derived equivalence in characteristic zero

In this section, we follow [14] and [19] to prove the twisted Torelli theorem for abelian surfaces over algebraic closed fields in characteristic zero.

4.1. **Over $\mathbb{C}$: Hodge isogeny versus twisted derived equivalence.** Let X and Y be complex abelian surfaces.

Definition 4.1.1. A rational Hodge isometry $\psi_b\colon \mathrm{H}^2(X,\mathbb{Q}) \to \mathrm{H}^2(Y,\mathbb{Q})$ is called *reflexive* if it is induced by a reflection on Λ along a vector $b \in \Lambda$:

$$\varphi_b\colon \Lambda_{\mathbb{Q}} \xrightarrow{\sim} \Lambda_{\mathbb{Q}} \quad x \mapsto x - \frac{(x,b)}{(b,b)}b.$$

Lemma 4.1.2. *Any reflexive Hodge isometry ψ_b induces a Hodge isometry on twisted Mukai lattices*

$$\widetilde{\psi}_b\colon \widetilde{\mathrm{H}}(X,\mathbb{Z};B) \to \widetilde{\mathrm{H}}(Y,\mathbb{Z};B'),$$

where $B = \frac{2}{(b,b)}b \in \mathrm{H}^2(X,\mathbb{Q})$ (via some marking $\Lambda \cong \mathrm{H}^2(X,\mathbb{Z})$) and $B' = -\psi_b(B)$.

Proof. The proof can be found in [19, §1.2]. □

In analogy to [19, Theorem 1.1], the following result characterizes the reflexive Hodge isometries between abelian surfaces.

Theorem 4.1.3. *Let X and Y be two complex abelian surfaces. If there is a reflexive Hodge isometry*

$$\psi_b\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q}),$$

for some $b \in \Lambda$, then there exist $\alpha \in \mathrm{Br}(X)$ and $\beta \in \mathrm{Br}(Y)$ such that ψ_b is induced by a derived equivalence

$$\mathrm{D}^b(X,\alpha) \simeq \mathrm{D}^b(Y,\beta).$$

Equivalently, X or $\widehat{X}$ is isomorphic to the coarse moduli space of twisted coherent sheaves on Y, and ψ_b is induced by the twisted Fourier-Mukai transform associated to the universal twisted sheaf.

Proof. According to Lemma 4.1.2, there is a Hodge isometry

$$\widetilde{\psi}_b\colon \widetilde{\mathrm{H}}(X,\mathbb{Z};B) \xrightarrow{\sim} \widetilde{\mathrm{H}}(Y,\mathbb{Z};B').$$

Let $v_{B'}$ be the image of Mukai vector $(0,0,1)$ under $\widetilde{\psi}_b$. From our construction, there is a Mukai vector

$$v = (r,c,\chi) \in \widetilde{\mathrm{H}}(Y,\mathbb{Z})$$

such that $v_{B'} = \exp(B')\cdot v$. We can assume that v is positive by some suitable autoequivalence of $\mathrm{D}^b(Y)$. Let β be the Brauer class on Y with respect to B' and $\mathscr{Y} \to Y$ be the corresponding $\mathbb{G}_m$-gerbe. For some $v_{B'}$-generic polarization H, the moduli stack $\mathscr{M}_H(\mathscr{Y},v_{B'})$ of β-twisted sheaves on Y with Mukai vector $v_{B'}$ forms a $\mathbb{G}_m$-gerbe on its coarse moduli space $M_H(\mathscr{Y},v_{B'})$ such that $[\mathscr{M}_H(\mathscr{Y},v_{B'})] \in \mathrm{Br}(M_H(\mathscr{Y},v_B))[r]$ (*cf.* [26, Proposition 2.3.3.4, Corollary 2.3.3.7]).

The kernel $\mathscr{P}$ is the tautological twisted sheaf on $\mathscr{Y} \times \mathscr{M}_H(\mathscr{Y},v_{B'})$ induces a twisted Fourier-Mukai transform

$$\Phi_{\mathscr{P}}\colon \mathrm{D}^b(Y,\beta) \to \mathrm{D}^b(\mathscr{M}_H(\mathscr{Y},v_{B'})) \simeq \mathrm{D}^b(M_{H'}(\mathscr{Y},v_{B'}),\alpha),$$

where $\alpha = [\mathscr{M}_H(\mathscr{Y},v_{B'})] \in \mathrm{Br}(M_{H'}(\mathscr{Y},v_{B'}))$ (see [47, Theorem 4.3]). It induces a Hodge isometry

$$\widetilde{\mathrm{H}}(Y,\mathbb{Z};B') \xrightarrow{\sim} \widetilde{\mathrm{H}}(M_H(\mathscr{Y},v_{B'}),\mathbb{Z};B''),$$

where B'' is a $\mathbf{B}$-field lift of α. Its composition with $\widetilde{\psi}_b$ is a Hodge isometry

$$\widetilde{\mathrm{H}}(X,\mathbb{Z};B) \xrightarrow{\sim} \widetilde{\mathrm{H}}(M_H(\mathscr{Y},v_{B'}),\mathbb{Z};B''), \tag{4.1.1}$$

sending the Mukai vector $(0,0,1)$ to $(0,0,1)$ and preserving the Mukai pairing. We can see $(1,0,0)$ is mapping to $(1,b,\frac{b^2}{2})$ for some $b \in \mathrm{H}^2(Y,\mathbb{Z})$ via (4.1.1). Thus we can replace B'' by $B''+b$, which will not change the corresponding Brauer class, to obtain a Hodge isometry which takes $(1,0,0)$ to $(1,0,0)$ and $(0,0,1)$ to $(0,0,1)$ at the same time. This yields a Hodge isometry

$$\mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(M_{H'}(\mathscr{Y},v_{B'}),\mathbb{Z}).$$

Then we can apply Shioda's Torelli Theorem of abelian surfaces [39] to conclude that

$$M_{H'}(\mathscr{Y}, v_{B'}) \cong X \text{ or } \widehat{X}.$$

When $X \cong M_{H'}(\mathscr{Y}, v_{B'})$, $\Phi_{\mathscr{P}}$ gives the derived equivalence as desired. When $\widehat{X} \cong M_{H'}(\mathscr{Y}, v_{B'})$, we can prove the assertion by using the fact X and $\widehat{X}$ are derived equivalent. □

Next, we are going to show that any rational Hodge isometry can be decomposed into a chain of reflexive Hodge isometries. This is a special case of Cartan-Dieudonné theorem which says that any element $\varphi \in \mathrm{SO}(\Lambda_{\mathbb{Q}})$ can be decomposed as products of reflections:

$$\varphi = \varphi_{b_1} \circ \varphi_{b_2} \circ \cdots \circ \varphi_{b_n}, \tag{4.1.2}$$

such that $b_i \in \Lambda$, and $(b_i)^2 \neq 0$. Then from the surjectivity of period map [39, Theorem ii], for any rational Hodge isometry

$$\mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q}).$$

we can find a sequence of abelian surfaces $\{X_i\}$ and Hodge isometries $\psi_{b_i} \colon \mathrm{H}^2(X_{i-1}, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_i, \mathbb{Q})$, where $X_0 = X$ and $X_n = Y$, such that ψ_{b_i} induces φ_{b_i} on $\Lambda_{\mathbb{Q}}$. We can arrange them as (1.0.1):

$$\begin{array}{llll}
\mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\psi_{b_1}} \mathrm{H}^2(X_1, \mathbb{Q}) & & & \\
& \mathrm{H}^2(X_1, \mathbb{Q}) \xrightarrow{\psi_{b_2}} \mathrm{H}^2(X_2, \mathbb{Q}) & & \\
& & \vdots & \\
& & & \mathrm{H}^2(X_{n-1}, \mathbb{Q}) \xrightarrow{\psi_{b_n}} \mathrm{H}^2(Y, \mathbb{Q}).
\end{array}$$

Finally, this yields

Corollary 4.1.4. *If there is a rational Hodge isometry* $\mathrm{H}^2(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y, \mathbb{Q})$*, then there is a twisted derived equivalence from* X *to* Y*, whose Hodge realization is it.*

Remark 4.1.5. A consequence of Theorem 4.1.4 is that any rational Hodge isometry between abelian surfaces is algebraic, which is a special case of Hodge conjecture on product of two abelian surface. Unlike the case of K3 surfaces, the Hodge conjecture for product of abelian surfaces were known for a long time. See [37] for example.

4.2. **Quasi-isogeny versus twisted derived equivalence.** Let us now describe twisted derived equivalences via suitable quasi-isogenies. Recall that the isogeny category of abelian varieties $\mathtt{AV}_{\mathbb{Q},k}$ consists of all abelian varieties over a field k as objects, and the Hom-sets are

$$\mathrm{Hom}_{\mathtt{AV}_{\mathbb{Q},k}}(X, Y) \coloneqq \mathrm{Hom}_{\mathtt{AV}_k}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We may also denote by $\mathrm{Hom}^0(X, Y)$ for $\mathrm{Hom}_{\mathtt{AV}_{\mathbb{Q},k}}(X, Y)$ if there are no confusion on the defining field k. An isomorphism f from X to Y in the isogeny category $\mathtt{AV}_{\mathbb{Q},k}$ is called a quasi-isogeny from X to Y.

Definition 4.2.1. An element $f \in \mathrm{Hom}_k(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}$ which has an inverse in $\mathrm{Hom}_k(Y, X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}$ is called a prime-to-ℓ quasi-isogeny, where $\mathbb{Z}_{(\ell)}$ is the localization of $\mathbb{Z}$ at (ℓ).

There are canonical maps for all ℓ coprime to $\mathrm{char}(k) = p$

$$\mathrm{Hom}_k(X, Y) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{G_k}\left(\mathrm{H}^1_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell), \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)\right) \tag{4.2.1}$$

which are injective with torsion-free cokernel. When k is a finitely generated field over $\mathbb{F}_p$ or $\mathbb{Q}$, it is known that (4.2.1) is bijective (*cf.* [13, VI, §3 Theorem 1]). In the case $\ell = p > 0$, there is also a canonical map

$$\mathrm{Hom}_k(X, Y) \otimes \mathbb{Z}_p \to \mathrm{Hom}_k\left(X[p^\infty], Y[p^\infty]\right), \tag{4.2.2}$$

which is injective with torsion-free cokernel. It is also surjective when k is finitely generated over $\mathbb{F}_p$ (*cf.* [10, Theorem 2.6]).

Definition 4.2.2. Let X and Y be g-dimensional abelian varieties over k. We say X and Y are *principally quasi-isogenous* if there is a quasi-isogeny of degree one f from X to Y or its dual $\widehat{Y}$. It is called *prime to p principally quasi-isogenous* if f is prime-to-p.

For any abelian surface $X_{\mathbb{C}}$ over $\mathbb{C}$, a standard argument shows that there is a finitely generated field $k \subset \mathbb{C}$ and an abelian surface X over k such that $X \times_k \mathbb{C} \cong X$. We have the following Artin comparison

$$\mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell) \cong \mathrm{H}^i(X_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell, \tag{4.2.3}$$

for any $i \in \mathbb{Z}$ and ℓ a prime. Suppose Y is another abelian surface defined over k. Then any prime-to-ℓ $\mathbb{C}$-quasi-isogeny $f : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ will induce an isomorphism of $\mathbb{Z}_\ell$-modules

$$f^* \colon \mathrm{H}^i_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell),$$

under the comparison (4.2.3). Morover, if f is defined over a finite extension k' of k, then f^* is $G_{k'}$-equivariant.

Proposition 4.2.3. *With assumptions as above. Any rational Hodge isometry*

$$\varphi \colon \mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q})$$

is induced by a principal quasi-isogeny (up to sign) over k' for some finite extension k'/k.

Proof. It suffices to consider the case φ being admissible. Fix admissible bases for both X and Y. Then φ can be expressed as an invertible matrix with coefficients in $\mathbb{Q}$. We can choose a prime $\ell \gg 0$ so that the coefficients of φ are all in the ℓ-localization $\mathbb{Z}_{(\ell)}$, i.e., the following diagram commutes

$$\begin{tikzcd}
\mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Q}) \arrow[r, "\varphi"] & \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q}) \\
\mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \arrow[r] \arrow[u, hook] & \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} \arrow[u, hook]
\end{tikzcd}$$

Therefore it actually induces an ℓ-adic admissible $\mathbb{Z}_\ell$-isomorphism

$$\varphi_\ell \coloneqq \varphi \otimes_{\mathbb{Z}_{(\ell)}} \mathbb{Z}_\ell \colon \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell).$$

By Proposition 3.4.3, there is an admissible $\mathbb{Z}_\ell$-isomorphism

$$\psi_\ell \colon \mathrm{H}^1_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell),$$

such that $\wedge^2(\psi_\ell) = \varphi_\ell$ or $-\varphi_\ell$.

Since φ is algebraic by by Corollary 4.1.4, φ_ℓ is G_k-equivariant after a finite extension of k. Proposition 3.4.3 implies ψ_ℓ is G_k-equivariant as well (after possibly taking a further finite extension) . Thus φ is induced by a prime-to-ℓ quasi-isogeny $f_\ell \in \mathrm{Hom}_k(X, Y) \otimes \mathbb{Z}_\ell$ for some finite field extension k'/k, up to sign. It is clear that f_ℓ has degree one. ☐

Remark 4.2.4. Set $n = \frac{(b)^2}{2}$ and p a prime, we say the reflexive Hodge isometry ψ_b is *prime to* p if $p \nmid n$. We should note that under the explicit construction in Theorem 4.1.3, if ψ_b is coprime to p, then we can find a Fourier-Mukai transform

$$\Psi \colon \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$$

such that the associated Brauer class $\alpha = \exp(B) \in \mathrm{Br}(X)$ satisifies

$$\alpha^n = \exp(nB) = \exp(b) = 1 \in \mathrm{Br}(X).$$

It follows $\alpha \in \mathrm{Br}(X)[n]$. Similarly, n divides the order of $\beta = \exp(B') \in \mathrm{Br}(Y)$. By Proposition 4.2.3, we can find a quasi-isogeny $f_b \colon Y \to \widehat{X}$ of degree one such that $f_b^* = -\mathrm{D} \circ \psi_b$. Moreover, it easy to see that f_b is prime-to-p when ψ_b is coprime to p.

It is well-known that the Hom-sheaf between two abelian varieties are representable by étale group schemes, thus geometric (quasi-)isogenies between abelian varieties do not depend on the the choice of algebraically closed field. The following lemma shows that the analogue on twisted derived equivalence.

Lemma 4.2.5. *Let X and Y are abelian surfaces defined over k with* $\mathrm{char}(k) = 0$. *Let $\overline{K} \supseteq k$ be an algebraically closed field containing k. Let $\bar{k}$ be the algebraically closure of k in $\bar{K}$. Then if $X_{\overline{K}}$ and $Y_{\overline{K}}$ are twisted derived equivalent, so is $X_{\bar{k}}$ and $Y_{\bar{k}}$.*

Proof. As $X_{\bar{K}}$ is twisted derived equivalent to $Y_{\bar{K}}$, by Theorem 2.6.2, there exist finitely many abelian surfaces $X_0, X_1, \ldots, X_n$ defined over $\bar{K}$ with $X_0 = X_{\bar{K}}$ and

$$X_i \text{ or } \widehat{X_i} = M_{H_i}(\mathscr{X}_{i-1}, v_i) \quad Y_{\bar{K}} \text{ or } \widehat{Y}_{\bar{K}} \cong M_{H_n}(\mathscr{X}_n, v_n)$$

for some $[\mathscr{X}_{i-1}] \in \mathrm{Br}(X_{i-1})$. Let us construct abelian surfaces over $\bar{k}$ to connect $X_{\bar{k}}$ and $Y_{\bar{k}}$ as follows:

Set $X'_0 = X_{\bar{k}}$, then we take $X'_1 = M_{H'_1}(\mathscr{X}'_0, v'_1)$ where $\mathscr{X}'_0, H'_1$ and v'_1 are the descent of $\mathscr{X}_0, H_1$ and v via the isomorphisms $\mathrm{Br}(X_{\bar{K}}) \cong \mathrm{Br}(X_{\bar{k}})$, $\mathrm{Pic}(X_K) \cong \mathrm{Pic}(X_{\bar{k}})$ and $\widetilde{\mathrm{H}}(X_K) \cong \widetilde{\mathrm{H}}(X_{\bar{k}})$. Then inductively, we can define X'_i as the moduli space of twisted sheaves $M_{H'_i}(\mathscr{X}'_{i-1}, v'_i)$ (or its dual respectively) over $\bar{k}$. Note that we have natural isomorphisms $(M_{H'_i}(\mathscr{X}'_{i-1}, v'_i))_{\bar{K}} \cong M_{H_i}(\mathscr{X}_{i-1}, v_i)$ over $\bar{K}$ and in particular, $(M_{H'_i}(\mathscr{X}'_n, v'_i))_{\bar{K}} \cong Y_{\bar{K}}$. It follows that $M_{H'_i}(\mathscr{X}'_n, v'_i) \cong Y_{\bar{k}}$. □

Now, we can get the main result in this section.

Theorem 4.2.6. *Suppose* $\mathrm{char}(k) = 0$. *The following statements are equivalent:*

(1) X and Y are principally quasi-isogenous over $\bar{k}$.
(2) X and Y are twisted derived equivalent over $\bar{k}$.

Proof. (2) ⇒ (1) clearly follows from Proposition 4.2.3 and Corollary 4.1.4.

Conversely, suppose X and Y are principally quasi-isogenous over $\bar{k}$. Complex abelian surfaces $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are twisted derived equivalent since the quasi-isogeny induces Hodge isometry $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q}) \cong \mathrm{H}^2(Y_{\mathbb{C}}, \mathbb{Q})$. By Lemma 4.2.5, one can conclude $X_{\bar{k}}$ and $Y_{\bar{k}}$ are twisted derived equivalent. □

5. Quasi-isogeny over positive characteristic fields

In this section, we will prove the twisted derived Torelli theorem for abelian surfaces over odd characteristic fields.

5.1. **Serre–Tate theory and lifting of prime-to-p quasi-isogeny.** The Serre–Tate theorem says that the deformation theory of an abelian scheme in characteristic p is equivalent to the deformation theory of its p-divisible group (*cf.* [29, Chapter V.§2, Theorem 2.3]). More precisely, let $S_0 \hookrightarrow S$ be an infinitesimal thickening of schemes such that p is locally nilpotent on S. Let $\mathcal{D}(S_0, S)$ be the category of pairs $(\mathcal{X}_0, \mathcal{G})$, where $\mathcal{X}_0$ is an abelian scheme over S_0 and $\mathcal{G}$ is a lifting of p-divisible group $\mathcal{X}[p^\infty]$ to S.

Theorem 5.1.1 (Serre–Tate)**.** *The functor defined on the category of abelian schemes over S as*

$$\mathcal{X} \mapsto (\mathcal{X} \times_S S_0, \mathcal{X}[p^\infty])$$

is an equivalence to the category $\mathcal{D}(S_0, S)$.

To apply the theorem on the lifting problem of abelian varieties, it is the case that the $S_0 = \mathrm{Spec}(k)$ and $S = \mathrm{Spec}(V/(\pi^{n+1}))$ for a perfect field k, V is a totally ramified finite extension of $W(k)$ and an integer $n \geq 1$. Since there is an equivalence between the category of p-divisible groups over V and the category of inductive systems of p-divisible groups over $V/(\pi^n)$. There is an identification

$$\mathcal{D}(k, V) = \varprojlim_n \mathcal{D}(k, V/(\pi^n))$$

since V is complete. As a consequence, we get

Corollary 5.1.2. *There is an equivalence of categories*

$$\begin{aligned} \{\textit{formal abelain schemes over } V\} &\xrightarrow{\sim} \mathcal{D}(k, V) \\ A &\mapsto (A \times_V k, A[p^\infty])\,. \end{aligned} \tag{5.1.1}$$

Lemma 5.1.3. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an isogeny between (formal) abelian schemes over S. If $\mathcal{Y}$ admits an algebraization, then f also admits an algebraization.*

Proof. This follows directly from [16, Proposition (5.4.4)]. □

The following linear-algebraic data is used in Kisin's work on classifying p-divisible groups. Let V be a totally ramified finite extension of W and π be a uniformizer of V, whose ramification index is equal to e. Let S be p-adic completion the PD-enveloping of $W[u]$ by the Eisenstein polynomial $E(u)$ of the uniformizer $\pi \in V$, i.e.,

$$S = \widehat{W[u]\{\frac{E(u)^i}{i!}\}}.$$

The kernel of the natural reduction $S \twoheadrightarrow V$ is therefore endowed with a PD-structure, which means $(S \twoheadrightarrow V) \in \mathtt{Crys}(V)$, where $\mathtt{Crys}(V)$ is the crystalline site on V. Denote by $\mathrm{Fil}^1 S$ the closure of the ideal generated by $\{\frac{E(u)^i}{i!}\}_{i\geq 1}$ in S.

Definition 5.1.4. A filtered φ-module over S is a finite free S-module M, equipped with a submodule $\mathrm{Fil}^1 M \subset M$ such that

(1) $\mathrm{Fil}^1 S \cdot M \subset \mathrm{Fil}^1 M$,
(2) the quotient $M/\mathrm{Fil}^1 M$ is a free V-module, and
(3) the map
$$\varphi^* \mathrm{Fil}^1 M \xrightarrow{1\otimes(\varphi \bmod p)|_{\mathrm{Fil}^1 S}} M$$
is surjective.

Following [23], the category of φ-module over S is denoted by $\mathtt{BT}_S^{\varphi}$.

Note that for any p-divisible group G over V, its Dieudonné module $\mathbb{D}(G)$ actually forms a crystal on $\mathtt{Crys}(V)$ and the value $\mathbb{D}(G)(S)$ is naturally a filtered φ-module over S, whose filtration Fil^1 is defined to be the preimage of $\mathrm{Lie}(G)^* \subset \mathbb{D}(G) \cong \mathbb{D}(G_0)\otimes_W V$ under $\mathbb{D}(G)(S) \to \mathbb{D}(G)$.

Proposition 5.1.5 ([23, Proposition A.6])**.** *Suppose $p > 2$. The functor defined by*

$$\begin{aligned} \mathtt{pdiv}/V &\to \mathtt{BT}_S^{\varphi} \\ G &\mapsto (\mathbb{D}(G)(S), \mathrm{Fil}^1) \end{aligned}$$

is a category equivalence.

Now we consider the lifting of separable isogenies between abelian surfaces.

Proposition 5.1.6 (lifting of isogeny)**.** *Suppose $p > 2$. Let $f\colon X \to Y$ be a separable isogeny such that X or Y is of finite height. There are liftings $\mathcal{X} \to \mathrm{Spec}(V)$ and $\mathcal{Y} \to \mathrm{Spec}(V)$ of X and Y respectively, such that the isogeny f can be lifted to an isogeny $f_V\colon \mathcal{X} \to \mathcal{Y}$.*

Proof. For simplicity of notations, we still denote by f the map betwewn p-divisible groups of X and Y induced by the isogeny f. Combining Corollary 5.1.2 and Lemma 5.1.3, it suffices to find a lifting for f. Since f is separable, its kernel is a finite étale group scheme, which is freely liftable. Thus we may assume that $f\colon X[p^\infty] \to Y[p^\infty]$ is an isomorphism.

Suppose X is of finite height. Then Y is also of finite height. It is well-known that there is a lifting $\mathcal{Y}$ of Y to $\mathrm{Spec}(V)$ which admits an algebraization for some totally ramified finite extension V of W. Let $\mathcal{Y}_0 = \mathcal{Y} \otimes_V V/p = \mathcal{Y} \otimes_V V/(\pi^e)$, where e is the ramification index of V. By applying the Grothendieck-Messing theory along $(k[t]/(t^e) \twoheadrightarrow k)$ with trivial PD-structure, we can find isomorphisms of p-divisible groups $f_0\colon G_0 \to \mathcal{Y}_0[p^\infty]$ and $\alpha\colon G_0 \otimes k \to X[p^\infty]$, such that the following diagram is commutative:

$$\begin{tikzcd} G_0 \otimes k \arrow[r, "f_0\otimes k"] \arrow[d, "\alpha"] & \mathcal{Y}_0[p^\infty] \otimes k \arrow[d, "\simeq"] \\ X[p^\infty] \arrow[r, "f"] & Y[p^\infty] \end{tikzcd}$$

Consider the isomorphism of Dieudonné crystals $f_0^*\colon \mathbb{D}(\mathcal{Y}_0[p^\infty]) \xrightarrow{\sim} \mathbb{D}(G_0)$. Its value at S also induces an isomorphism of φ-modules over S

$$(\mathbb{D}(\mathcal{Y}[p^\infty], \mathrm{Fil}^1) \xrightarrow{\sim} (\mathbb{D}(G_0)(S), f_0^* \mathrm{Fil}^1).$$

It corresponds to an isomorphism between p-divisible groups over V functorially by Proposition 5.1.5: $f_V\colon G \xrightarrow{\sim} \mathcal{Y}[p^\infty]$, such that

$$(f, f_V)\colon (X[p^\infty], G) \to (Y[p^\infty], \mathcal{Y}[p^\infty])$$

is an isomorphism in $\mathcal{D}(k, V)$.

□

Remark 5.1.7. We should emphasize that Proposition 5.1.6 is well-known to experts as it is from a standard inductive construction via Grothendieck–Messing theory. The involvement of Proposition 5.1.5 is also unnecessary for our proof, since it also comes from an analogous construction.

5.2. **Specialization of twisted derived equivalence.** Next, we shall show that prime-to-p geometric twisted derived equivalences are preserved under reduction.

Theorem 5.2.1. *Let V be a discrete valuation ring with residue field k and let η be its generic point. Assume that $\mathrm{char}(k) = p > 2$. Let $\mathcal{X} \to \mathrm{Spec}(V)$ and $\mathcal{Y} \to \mathrm{Spec}(V)$ be two projective families of abelian surfaces or K3 surfaces over $\mathrm{Spec}(V)$. If the geometric generic fibers $\mathcal{X}_{\bar\eta}$ and $\mathcal{Y}_{\bar\eta}$ are prime-to-p twisted derived equivalent, then so are the geometric special fibers.*

Proof. We denote by X_0 and Y_0 the geometric special fibers of $\mathcal{X}/V$ and $\mathcal{Y}/V$ resepctively. Let us first assume that $\mathrm{D}^b(\mathcal{X}_{\bar\eta}, \alpha) \cong \mathrm{D}^b(\mathcal{Y}_{\bar\eta}, \beta)$ for $\alpha \in \mathrm{Br}(\mathcal{X}_{\bar\eta})$ and $\beta \in \mathrm{Br}(\mathcal{Y}_{\bar\eta})$. This is equivalent to say that there is an isomorphism

$$\mathcal{Y}_{\bar\eta} \cong M^{\alpha}_{\mathcal{H}_\eta}(\mathcal{X}_{\bar\eta}, v_{\bar\eta}),$$

for some twisted Mukai vector $v_{\bar\eta} \in \widetilde{\mathrm{N}}(\mathcal{X}_{\bar\eta}, \alpha)$. Up to taking a finite extension of V, we may assume that α can be defined over η. Now we claim that one can lift α to a class in $\mathrm{Br}(\mathcal{X})$ if $p \nmid \mathrm{ord}(\alpha)$. By assumption, $\mathrm{ord}(\alpha) = d$ is prime-to p. For simplicity, we assume $d = \ell^n$ for some prime ℓ. In this case, the Gysin's sequence and Gabber's absolute purity gives an exact sequence

$$0 \to \mathrm{Br}(\mathcal{X})\{\ell\} \to \mathrm{Br}(\mathcal{X}_\eta)\{\ell\} \to \mathrm{H}^1_{\text{ét}}(X_0, \mathbb{Q}_\ell/\mathbb{Z}_\ell). \tag{5.2.1}$$

(*cf.* [8, Theorem 3.7.1 (ii)]). If $\mathcal{X}$ is a K3 surface, then we have $\mathrm{H}^1_{\text{ét}}(X_0, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ and thus one can find a lift $\tilde\alpha \in \mathrm{Br}(\mathcal{X})$ of α by (5.2.1). When $\mathcal{X}$ is an abelian surface over $\mathrm{Spec}(V)$, the Gysin sequence can not directly give the existence of liftings of α. Again, one can use the trick of Kummer surfaces. Consider the relative Kummer surface $\mathrm{Km}(\mathcal{X}) \to \mathrm{Spec}(V)$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Br}(\mathrm{Km}(\mathcal{X})) \arrow[r] \arrow[d, "\Theta"] & \mathrm{Br}(\mathrm{Km}(\mathcal{X}_\eta)) \arrow[d, "\Theta_\eta"] \\
\mathrm{Br}(\mathcal{X}) \arrow[r] & \mathrm{Br}(\mathcal{X}_\eta)
\end{tikzcd}$$

from Proposition 2.2.1. After passing to a finite extension, we can assume $\mathrm{Br}(\mathrm{Km}(\mathcal{X}_\eta)) \cong \mathrm{Br}(\mathcal{X}_\eta)$. As the top arrow is surjective and Θ_η is an isomorphism, the bottom arrow is surjective as well.

Now, we can pick liftings $v \in \widetilde{\mathrm{H}}(\mathcal{X}, \mathbb{Z})$ and $\mathcal{H} \in \mathrm{Pic}(\mathcal{X})$ so that $v|_{\mathcal{X}_\eta} = v_\eta$ and $\mathcal{H}|_{\mathcal{X}_\eta} = \mathcal{H}_\eta$. Then we let $M^{\tilde\alpha}_{\mathcal{H}}(\mathcal{X}, v)$ be the corresponding relative moduli space of twisted sheaves and set $\alpha_0 = \widetilde{\alpha}|_{X_0} \in \mathrm{Br}(X_0)$. The generic fiber of $M^{\widetilde\alpha}_{\mathcal{H}}(\mathcal{X}, v) \to \mathrm{Spec}(V)$ is isomorphic to $M_{\mathcal{H}_\eta}(\mathcal{X}_\eta, \alpha)$ after a finite base extension. Note that its special fiber is also isomorphic to $M^{\alpha_0}_{H_0}(X_0, v_0)$ after some finite field extension, we have the following commutative diagram after taking a finite ring extension of V:

$$\begin{array}{ccccccc} M^{\widetilde{\alpha}}_{\mathcal{H}}(\mathcal{X},v) & \longleftarrow & M^{\alpha}_{\mathcal{H}_\eta}(\mathcal{X}_\eta,v_\eta) & \xrightarrow{\cong} & \mathcal{Y}_\eta & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longleftarrow & \mathrm{Spec}(k(\eta)) & \longrightarrow & \mathrm{Spec}(k(\eta)) & \longrightarrow & \mathrm{Spec}(V) \end{array}$$

According to Matsusaka-Mumford [28, Theorem 1], the isomorphism can be extended to the special fiber. Thus Y_0 is isomorphic to $M^{\alpha_0}_{H_0}(X_0,v_0)$. It follows that $\mathrm{D}^b(X_0,\alpha_0)\cong \mathrm{D}^b(Y_0,\beta_0)$. □

Remark 5.2.2. Our proof fails when k is imperfect and the twisted derived equivalence is not prime-to-p. This is because if the associated Brauer class α has order p^n, the map $\mathrm{Br}(\mathcal{X})[p^n]\to \mathrm{Br}(\mathcal{X}_\eta)[p^n]$ may not be surjective (cf. [36, 6.8.2]).

5.3. **Proof of Theorem 1.0.2.** We first show that $(i')\Rightarrow(ii')$. It suffices to consider the case $\mathrm{D}^b(\mathscr{X})\cong \mathrm{D}^b(\mathscr{Y})$ for some gerbes $\mathscr{X}\to X$ and $\mathscr{Y}\to Y$ satisfying that the orders of $[\mathscr{X}]$ and $[\mathscr{Y}]$ are both prime-to-p. The easiest way for proving this is by lifting to characteristic 0. As indicated in the proof of Theorem 2.6.3, we can lift the derived equivalence between $\mathscr{X}$ and $\mathscr{Y}$ to some finite extension W' of $W=W(k)$, i.e. there exists $\mathfrak{X}/W'$ and $\mathfrak{Y}/W'$ as liftings of $\mathscr{X}$ and $\mathscr{Y}$ such that $\mathfrak{X}$ and $\mathfrak{Y}$ are derived equivalent. By Theorem 1.0.1, the generic fibers of the $\mathcal{X}$ and $\mathcal{Y}$ are geometrically principal quasi-isogenoius. Then we can conclude the special fibers are prime-to-p principal quasi-isogenous by using Tate's spreading theorem (*cf.* [43, Theorem 4]).

Another method without using the lifting argument is via the p-adic Shioda's trick. According to Theorem 2.6.3, the crystalline realization of any prime-to-p twisted derived equivalence will induce an isometry of F-crystal

$$\mathrm{H}^2_{\mathrm{crys}}(X_{\bar k}/W(\bar k))\cong \mathrm{H}^2_{\mathrm{crys}}(Y_{\bar k}/W(\bar k)),$$

which is induced by an algebraic correspondence derived over a finitely generated field $k_0\subset \bar k$. We claim that there is a prime-to-p quasi-isogeny $f\colon Y\to X$. As in general k_0 is not perfect, the Dieudonné module does not work well for p-divisible groups over k_0. Thus we need to do flat descent as follows, in order to get back to the perfect case.

Let k^{perf} be the perfect closure of k_0 in $\bar k$ and W' be its Witt vector ring. Thus we have

$$\mathrm{H}^2_{\mathrm{crys}}(X_{k^{\mathrm{perf}}}/W')\cong \mathrm{H}^2_{\mathrm{crys}}(Y_{k^{\mathrm{perf}}}/W')$$

by taking base-change of the algebraic correspondence along $k_0\subset k^{\mathrm{perf}}$. With Shioda's trick Proposition 3.4.5, we are able to produce an isomorphism of F-crystals:

$$\mathrm{H}^1_{\mathrm{crys}}(X_{k^{\mathrm{perf}}}/W')\cong \mathrm{H}^1_{\mathrm{crys}}(Y_{k^{\mathrm{perf}}}/W').$$

This implies that $X_{k^{\mathrm{perf}}}[p^\infty]\cong Y_{k^{\mathrm{perf}}}[p^\infty]$. Moreover, since there is a canonical bijection

$$\mathrm{Hom}(X_{k_0}[p^\infty],Y_{k_0}[p^\infty])\simeq \mathrm{Hom}(X_{k^{\mathrm{perf}}}[p^\infty],Y_{k^{\mathrm{perf}}}[p^\infty])$$

by faithfully flat descent along $k_0\subset k_0^{1/p}\subset k_0^{1/p^2}\subset\cdots\subset k^{\mathrm{perf}}$ (cf. [50, Lemma 5.23]), we can get an isomorphism $X_{k_0}[p^\infty]\cong Y_{k_0}[p^\infty]$. Then the existence of prime-to-p quasi-isogeny is clearly from the surjectivity of (4.2.2) on a finitely generated field.

Now we prove $(ii')\Rightarrow(i')$. Suppose that we are given a quasi-isogeny $\phi\colon Y\to X$, which is of degree one and prime-to-p. By Proposition 5.1.6, we can lift it to a prime-to-p quasi-isogeny of degree one over some finite flat extension V of W:

$$\Phi:\mathcal{Y}\to\mathcal{X}.$$

The quasi-isogeny Φ induces an G_K-equivariant isometry Φ^* between the middle cohomolgy of $\mathcal{X}_K$ and $\mathcal{Y}_K$. By Theorem 4.1.3, there exists a geometric twisted derived equivalence whose cohomological realization is the pull-back Φ^*. According to Theorem 5.2.1, it suffices to show that this twisted derived equivalence is prime-to-p. According to Remark 4.2.4, this follows from Lemma 5.3.1 below.

Finally, it remains to prove the assertion for the case X is supersingular. This will be proved in Proposition 5.4.3.

Lemma 5.3.1 (prime-to-p Cartan-Dieudonné decomposition)**.** *Let Λ be an integral lattice over $\mathbb{Z}$. Any coprime to $p > 2$ orthogonal matrix $A \in \mathrm{O}(\Lambda_{\mathbb{Q}})$ can be decomposed into a sequence of coprime to p reflections.*

Proof. To prove the assertion, we will follow the proof of [38] to refine Cartan-Dieudonné decomposition for any field. In general, if Λ_k is quadratic space over a field k with the Gram matrix G. Let I be the identity matrix and let R_b be the reflection with respect to $b \in \Lambda_k$. The proof of Cartan-Dieudonné decomposition in [38] relies on the following facts: for any element $A \in \mathrm{O}(\Lambda_k)$, we have

i) A is a reflection if $\mathrm{rank}(A - I) = 1$ (cf. [38, Lemma 2])
ii) if $S = G(A - I)$ is not skew symmetric and $a \in \Lambda$ satisfying $a^t Sa \neq 0$ and

$$S + S^t \neq \frac{1}{a^t Sa}(Sb \cdot b^t S + S^t b \cdot b^t S^t),$$

then $\mathrm{rank}(AR_b - I) = \mathrm{rank}(A - I) - 1$ and $G(AR_b - I)$ is not skew symmetric with $b = (A - I)a$ satisfying $b^2 = -2a^t Sa$. Such a always exists. (cf. [38, Lemma 4, Lemma 5]).
iii) if $S = G(A - I)$ is skew symmetric, then there exists $b \in \Lambda$ such that $G(AR_b - I)$ is not skew symmetric (cf. [38, Theorem 2]).

Then one can decompose A as a series of reflections by repeatedly using ii). In our case, it suffices to show that if A is coprime to p, i.e. nA is integral for some n coprime to p, then

i') A is a corpime to p reflection if $\mathrm{rank}(A - I) = 1$;
ii') if $S = G(A - I)$ is not skew symmetric and there exists $a \in \Lambda$ satisfying $p \nmid a^t Sa$ and

$$S + S^t \neq \frac{1}{a^t Sa}(Sb \cdot b^t S + S^t b \cdot b^t S^t),$$

such that AR_b is coprime to p and $G(AR_b - I)$ is not skew symmetric with b constructed above;
iii') if $S = G(A - I)$ is skew symmetric, then there exists $b \in \Lambda$ with $p \nmid b^2$ such that AR_b is coprime to p and $G(AR_b - I)$ is not skew symmetric.

This means that we only need to find some prime-to-p reflections satisfying the conditions as above. By our assumption, the modulo p reduction $\Lambda_{\mathbb{F}_p}$ of Λ remains non-degenerate. If A is coprime to p, then we can consider the reduction $A \mod p$ and apply i)-iii) to $A \mod p \in \mathrm{O}(\Lambda_{\mathbb{F}_o})$ to obtain reflections over $\mathbb{F}_p$. We can lift the reflections to $\mathrm{O}(\Lambda_{\mathbb{Q}})$, which are obviously coprime to p. One can easily check such reflections satisfy ii') and iii'). □

From the proof of Theorem 1.0.2 $(i') \Rightarrow (ii')$, we can see that the lifting-specialization argument also works for non prime-to-p twisted derived equivalence. Thus we have

Theorem 5.3.2. *Suppose X and Y are abelian surfaces over $\bar{k}$ with finite height and* $\mathrm{char}(k) \neq 2$. *If X and Y are twisted derived equivalence, then they are quasi-liftable principal quasi-isogenous.*

Moreover, we believe that the converse of Theorem 5.3.2 also holds.

Conjecture 5.3.1. *With the same assumptions in Proposition 5.3.2. Then X and Y are twisted derived equivalence if and only if they are quasi-liftable principal quasi-isogenous.*

We remark that our approach remains valid once there is a specialization theorem for non prime-to-p twisted derived equivalences. According to the proof of Theorem 5.2.1, it suffices to know the existence of specialization of Brauer classes of order p. Following the notations in Theorem 5.2.1, this means that the restriction map $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_\eta)$ is surjective. See Remark 5.2.2.

5.4. **Supersingular twisted abelian surfacces.** At last, we come to the case X is supersingular over an algebraically closed field k such that $\mathrm{char}(k) = p > 2$. We obtain a supersingular twisted derived Torelli theorem via Ogus's supersingular Torelli theorem.

Theorem 5.4.1. *Let X an Y be two supersingular abelian surfaces over k. For $\mathbb{G}_m$-gerbes $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$, the following statements are equivalent:*

(1) There is a Fourier-Mukai transform $\mathrm{D}^b(\mathscr{X}) \simeq \mathrm{D}^b(\mathscr{Y})$.
(2) There is an isomorphism between K3 crystals $\widetilde{\mathrm{H}}(\mathscr{X}, W) \cong \widetilde{\mathrm{H}}(\mathscr{Y}, W)$.

Proof. For (1) $\Rightarrow$ (2), we only need to show the cohomological Fourier-Mukai transform induces an isomorphism on twisted Mukai lattices. When $p > 3$, this is due to a direct Chern character computation (*cf.* [4, Appendix A]). When $p = 3$, one can follow [5, Proposition 4.2.4] using the twistor line and lifting argument. As the proof is similar, we omit the details here.

To prove that (2) implies (1), let us take $v = \rho(0, 0, 1)$, there is a filtered isomorphism

$$\gamma\colon \widetilde{\mathrm{H}}(\mathscr{X}, W) \xrightarrow{\rho} \widetilde{\mathrm{H}}(\mathscr{Y}, W) \xrightarrow{\phi^{\mathcal{E}}} \widetilde{\mathrm{H}}(\mathscr{M}_H(\mathscr{Y}, v), W) \tag{5.4.1}$$

where $\phi^{\mathcal{E}}$ is the cohomological Fourier-Mukai transform induced by the universal twisted sheaf $\mathcal{E}$ on $Y \times \mathscr{M}_H(\mathscr{Y}, v)$. Then there is an isomorphism

$$X \cong M_H(\mathscr{Y}, v)$$

since γ induces an isomorphism between supersingular K3 crystals (*cf.* (2.4.1)). The difficulty is to verify that $[\mathscr{X}]$ is isomorphic to $[\mathscr{M}_H(\mathscr{Y}, v)]$. $\square$

In [5], Bragg and Lieblich have developed the theory of twistor space for supersingular K3 surfaces. In terms of it, they are able to constructed the twisted period space of supersingular K3 surfaces. We recap their construction and extend it to abelian surfaces as below: fix a supersingular K3 lattice Λ, which is a free $\mathbb{Z}$-lattice whose discriminant $\mathrm{disc}(\Lambda \otimes \mathbb{Q}) = -1$, signature $(1, n)$ ($n = 5$ or 21)and the $\Lambda^\vee/\Lambda$ is p-torsion. Then $|\Lambda^\vee/\Lambda| = p^{2\sigma_0(\Lambda)}$ for $1 \leq \sigma_0(\Lambda) \leq \frac{(n-1)}{2}$. The lattice Λ is also determined by $\sigma_0(\Lambda)$, called the *Artin invariant* of Λ.

Set

$$\widetilde{\Lambda} = \Lambda \oplus U(p) \text{ and } \widetilde{\Lambda}_0 = p\widetilde{\Lambda}^\vee/p\widetilde{\Lambda}.$$

where $U(p)$ is the twisted hyperbolic plane generated by vectors e and f such that $e^2 = f^2 = 0$ and $e \cdot f = -p$. Let M_{Λ_0} be the moduli space of characteristic subspaces of Λ_0 and let $M^{\langle e \rangle}_{\widetilde{\Lambda}_0}$ be the moduli space of characteristic subspaces of $\widetilde{\Lambda}_{K3,0}$ which don't contain e.

Definition 5.4.2. The twistor line in $M_{\widetilde{\Lambda}_0}$ is the subvariety $\mathbb{A}^1$ that is a connected component of a fiber of π_v over a k-point $[K] \in M_{v^\perp/v}(k)$ for some isotropic $v \in \widetilde{\Lambda}_0$.

To emphasis that we are at either the case $n = 21$ or $n = 5$, we may write $\Lambda = \Lambda_{K3}$ and $\Lambda = \Lambda_{Ab}$ respectively. For K3 surfaces, it has been shown that the moduli functor $S_{\Lambda_{K3}}$ of Λ_{K3}-marked supersingular K3 is representable by a locally of finite presentation, locally separated and smooth algebraic space of dimension $\sigma_0(\Lambda_{K3}) - 1$. There is a universal family

$$u\colon \mathcal{X} \to S_{\Lambda_{K3}}$$

(as algebraic spaces), which is smooth with relative dimension 1. The higher direct image $\mathrm{R}^2 u^{\mathrm{fl}}_* \mu_p$ is representable by an algebraic group space over $S_{\Lambda_{K3}}$ after perfection, denoted by $\pi\colon \mathscr{S}_{\Lambda_{K3}} \to S_{\Lambda_{K3}}$ (see loc.cit. Theorem 2.1.6). The connected component of the identity $\mathscr{S}^o_{\Lambda_{K3}} \subset \mathscr{S}_{\Lambda_{K3}}$ parametrizes the μ_p-gerbes which are not essentially-trivial except the identity, at each Λ-marked K3 surface in $S_{\Lambda_{K3}}(k)$. Then there are *(twisted) period morphisms*

$$\rho : S_{\Lambda_{K3}} \to M_{\Lambda_{K3,0}} \text{ and } \widetilde{\rho}\colon \mathscr{S}^o_{\Lambda_{K3}} \to M^{\langle e \rangle}_{\widetilde{\Lambda}_0},$$

(*cf.* [33, §3][5, Definition 3.5.7]). Then the method in *loc.cit.* shows that there is a Cartisian diagram

$$\begin{tikzcd} \mathscr{S}^o_{\widetilde{\Lambda}_{K3}} \arrow[r, "\pi"] \arrow[d, "\widetilde{\rho}"] & S_{\Lambda_{K3}} \arrow[d, "\rho"] \\ M^{\langle e\rangle}_{\widetilde{\Lambda}_{K3,0}} \arrow[r] & M_{\Lambda_{K3,0}}, \end{tikzcd} \tag{5.4.2}$$

and ρ is étale surjective. The twisted period map $\widetilde{\rho}$ factors as

$$\begin{tikzcd} \mathscr{S}^o_{\widetilde{\Lambda}_{K3}} \arrow[r, "\widetilde{\rho}'"] \arrow[rr, bend left, "\widetilde{\rho}"] & \mathscr{P}_{\Lambda_{K3}} \arrow[r] & M^{\langle e\rangle}_{\widetilde{\Lambda}_{K3,0}}, \end{tikzcd}$$

where $\mathscr{P}_{\widetilde{\Lambda}_{K3}}$ is the moduli space of ample cones of characteristic subspaces defined by Ogus ([33]). It has been shown that $\widetilde{\rho}'$ is an isomorphism (*cf.* [5, Theorem 5.1.7]). In particular, this implies that the moduli space of supersingular K3 surfaces of Artin invariant ≤ 2 is rationally fibered over the moduli space of supersingular K3 surfaces of Artin invariant 1, whose fiber is a twisted line, corresponding to the relative moduli spaces of twisted sheaves on universal gerbes associated to the Brauer groups of the superspecial K3 surface.

For supersingular abelian surfaces, everything works by replacing Λ_{K3} with Λ_{Ab}. Indeed, the proof in [5, Proposition 5.1.5] already shows that the twisted period map $\widetilde{\rho}'$ for abelian surfaces will be an isomorphism. A simple way to see this is via the Kummer construction. One just notice that the moduli space of supersingular abelian surfaces is isomorphic to the moduli space of supersingular Kummer surfaces and they have isomorphic period spaces, i.e.

$$M^{\langle e\rangle}_{\widetilde{\Lambda}_{K3,0}} \cong M^{\langle e\rangle}_{\widetilde{\Lambda}_{Ab,0}}$$

when $\sigma_0(\Lambda_{K3}) = \sigma_0(\Lambda_{Ab}) \leq 2$. This gives

Proposition 5.4.3. *For non-superspecial supersingular abelian surface X', there exists a Brauer class $[\mathscr{X}] \in \mathrm{Br}(X)$ such that $\mathrm{D}^b(\mathscr{X}) \simeq \mathrm{D}^b(X')$. In particular, X' is a moduli space of twisted sheaves on X.*

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