Lecture 2: Dieudonné Theory

Contents

1 Dieudonné Modules associated to p-divisible groups

Fix a perfect field k. Let W_n be the ring of Witt vectors of length n and W the Witt vector ring.

Let G be a finite group scheme over k, whose order is a power of p. Consider the W-module defined as the following:

$$M(G) \coloneqq \varinjlim_n \mathrm{Hom}(G, W_n).$$

This is a W-module since $\mathrm{Hom}(G, W_n)$ is naturally a W_n-module by

$$a\chi(t) \coloneqq a \cdot \chi(t)$$

Let R be a complete noetherian local ring. Recall that a p-divisible group over R has the following connected-étale decomposition

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0,$$

which is induced by the connected-étale decomposition of each G_i.

Proposition 1.1. Let R be a complete noetherian local ring whose residue field is k. Then there is a category equivalence

$$\Gamma \mapsto \Gamma(p)$$

from the category of commutaive divisible formal Lie groups over R to the category of p-divisible groups over R.

Let T be the torsion $W(k)$-module

$$W(k)[\frac{1}{p}]/W(k).$$

Then we define the Pointragin dual of a finite $W(k)$-module N by

$$N^* \coloneqq \mathrm{Hom}_{W(k)}(N, T).$$

It is not hard to see that $(N^*)^* \cong N$.

2 Breuil-Kisin modules

Breuil-Kisin modules can be viewed as the generalization of Dieudonné modules over discrete valuation ring in mixed characteristics.

Let R be a complete discrete valuation ring whose residue field is the perfect field k. Let π be a uniformizor of R, i.e. π is a generator of the maximal ideal of R such that $v_p(\pi) = 1$. Let $E(u)$ be the Eisenstein polynomial of π. Consider the formal power series ring $\mathfrak{S} \coloneqq W[[u]]$.

Definition 2.1. A Breuil-Kisin module is a finite free $\mathfrak{S}$-module M which is equipped with an isomorphism

$$\varphi_M \colon M \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[\frac{1}{E}] \xrightarrow{\sim} M[\frac{1}{E}].$$

Remark 2.2. The definition is similar to the definition of F-crystal over W. The φ_M is the linearized Frobenius on M.

Let $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ be the category of Breuil-Kisin modules. A fundamental result in the Breuil-Kisin theory is the following.

Theorem 2.3. There is a fully-faithful tensor (i.e. preserving the tensor structure)

$$\mathfrak{M}\colon \mathrm{Rep}_{G_K}^{\mathrm{crys},\circ} \to \mathrm{Mod}_{\mathfrak{S}}^{\varphi}$$

which is compatible with the formation of symmetric and exterior powers.

Corollary 2.4 (Crystalline and de Rham comparison)**.** Let $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be the crystalline representation induced by some $T \in \mathrm{Rep}_{G_K}^{\mathrm{crys},\circ}$. Then there are canonical isomorphisms:

$$D_{\mathrm{crys}}(V) \xrightarrow{\sim} \mathfrak{M}/u\mathfrak{M}[\frac{1}{p}] \quad D_{dR}(V) \xrightarrow{\sim} \varphi_M^*(M) \otimes_{\mathfrak{S}} K.$$

Let $\mathrm{BT}_{\mathfrak{S}}^{\varphi}$ be the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ which consists of the Breuil-Kisin modules such that $\varphi_M(M) \subset M$. The following theorem provides a classifycation for p-divisible groups in mixed characteristics.

Theorem 2.5. Let K be a totally ramified extension of $K(k)$. Let $\mathcal{O}_K$ be the ring of integers in K. There is a category equivalence

$$\mathfrak{M}\colon (p - \mathrm{div}/\mathcal{O}_K) \to \mathrm{BT}_{\mathfrak{S}}^{\varphi},$$

where the functor is applied to the p-adic Tate modules as the $\mathbb{Z}_p$-lattice of crystalline representations.

2.1 Reductive Groups in Crystalline Representations

Let $M^{\otimes}$ be the direct sum of

Proposition 2.6. Let R be a discrete valuation ring in mixed characteristics. Let $G \subset \mathrm{GL}(M)$ be a closed R-flat subgroup scheme for some finite free R-module M. Suppose that the generic fiber of G is reductive. Then G is defined by a finite collection of tensors $(s_\alpha) \subset M^{\otimes}$.