Perpetual Bounded Option with Variable Volatility

Albert Liang, Hongyi Hu, Huiping Chen, Jin Wang, Yue Wu

Abstract

The goal of this research is to price perpetual boundary options under the perpetual Black-Scholes model and the Constant Elasticity of Variance model. The specific options that we studied are one-touch options and collision options. To this end, we first set up systems of PDE/ODE using the model assumptions and boundary conditions of the options. Then, we solved for possible solutions with the aid of numerical methods. Once th general pricing formulae of one-touch options is determined, we show that as stock price tends to infinity, the price of the lower boundary one-touch option tends to a constant given sufficient volatility. Lastly, we reveal that the boundary option is a special case of the collision option with static boundaries rather than dynamic. Moreover, we verified the options price formula using Monte Carlo simulation that generates geometric asset price paths. The option pricing formula we proved is general in that it allows the asset price to have variable volatility

Introduction

A perpetual option is an option whose maturity is not fixed upon purchase. We study perpetual options because they closely resemble options with far away expiration while also being analytically simple. In this project, we will present pricing for various *One-touch* and *Collision* options in a perpetual context and discuss some of their qualities.

Definition (One-touch)

Given underlying asset S and either an upper price barrier U or lower price barrir L (or both), an option V is *one-touch* if V pays K_1 upon the first time when $S = U$ (or K_2 if $S = L$) and expires immediately afterwards.

Definition (Collision)

Given stocks S_1 and S_2, an option V is *collision* if it pays K upon the first time $S_1 = S_2$ and expires afterwards.

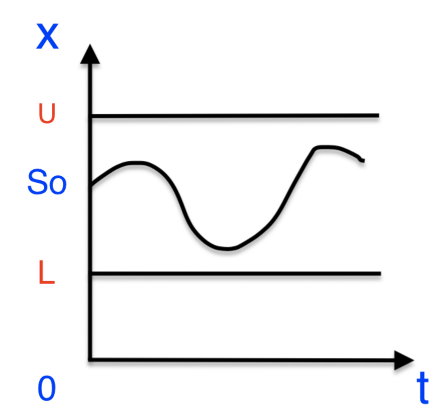
Definition (The Perpetual Black Scholes Model)

Under the Black-Scholes model, a perpetual option must satisfy

$$\frac{1}{2}\sigma(S)^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0$$

One-touch Option

The one-touch option has both an upper and lower barrier. The underlying asset price starts at some $L < S_0 < U$. The option pays K_1 when $S_t = U$ or pays K_2 when $S_t = L$. In either case, the option expires immediately.



Consider the perpetual option V on the stock of price S. We assume the perpetual Black-Scholes ODE and the following boundary conditions:

$$\begin{cases} V(U) = K_1 \\ V(L) = K_2 \end{cases}$$

Let $\quad f(t) = \int_L^t \frac{1}{S\sigma(S)^2} dS$

and $\quad g(a,b) = \int_a^b t^{-2} e^{-2rf(t)} dt.$

General Pricing Formula of the One-touch Option

$$V(S) = \frac{S\left(\frac{K_1}{U}\cdot g(L,S) + \frac{K_2}{L}\cdot g(S,U)\right)}{g(L,U)}$$

This is general in a sense that we can deduce the pricing of the one-touch option with only one barrier by eliminating either U or L.

For an option with only upper barrier, let $L \to 0$.

$$\lim_{L\to 0} V(S) = S\frac{K_1}{U}$$

For an option with only lower barrier, let $U \to \infty$.

$$\lim_{U\to\infty} V(S) = \frac{S\frac{K_2}{L}\cdot g(S,\infty)}{g(L,\infty)}$$

For a one-touch option with only a lower barrier, there is a natural question: would the stock even have the chance to strike the lower barrier if the stock gets more volatile as its price increases? We make the following characterization based on $\sigma(S)$.

Theorem

$$\lim_{S\to\infty} V(S) > 0 \iff f(\infty) < \infty$$

Collision Option

Assume $r > 0$, two stocks with stock prices x, y. We use the Black-Scholes PDE:

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 V}{\partial y^2} + rx\frac{\partial V}{\partial x} + ry\frac{\partial V}{\partial y} + \rho\sigma_x\sigma_y xy - rV = 0$$

Assume the option pays K and expires the first time two stocks collide, i.e. the first time $x = y$. If we denote the price of the option as $V(x,y)$, and without loss of generality, assume $x > y$, we have the following conditions:

$$\begin{cases} V(x,y) < K, \forall x > y > 0 \\ V(x,x) = K, \forall x > 0 \end{cases}$$

Since the solution to the above PDE is in the form of $x^a y^b$, for some $a, b \in \mathbb{C}$, and $x^a x^b = K$ for all $x > 0$, we assume $V = \left(\frac{x}{y}\right)^\beta$ for some $\beta \in \mathbb{C}$.

Let $\xi = \frac{x}{y}$, and $\hat{\sigma}^2 = \frac{1}{2}(\sigma_x^2 + \sigma_y^2) - \rho\sigma_x\sigma_y$. We note that $\xi \geq 1$, with change of variables, our PDE becomes an ODE:

$$\hat{\sigma}^2\xi^2 V''(\xi) + (\sigma_x^2 - \rho\sigma_x\sigma_y)V'(\xi) - rV(\xi) = 0$$

Another change of variables, let $z = ln(\xi)$,

$$\hat{\sigma}^2 V''(z) + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)V'(z) - rV(z) = 0$$

Therefore,

$$V(x,y) = V(\xi) = K\left(\frac{x}{y}\right)^\beta$$

$$\text{where } \beta = \frac{-\frac{1}{2}(\sigma_x^2 - \sigma_y^2) - \sqrt{\frac{1}{4}(\sigma_x^2 - \sigma_y^2)^2 + 4\hat{\sigma}^2 r}}{2\hat{\sigma}^2}$$

We notice that $\beta < 0$ as required because the option price is upper-bounded by K. Else if $\beta > 0$, then when $\frac{x}{y}$ approaches positive infinity as $y \to 0$, the option value exceeds K.

Special cases arise when (1) $\sigma_x = \sigma_y$ (i.e. same volatility), (2) $\rho = 0$ (i.e. independent underlying asset price).

Results for Special Cases

Given $\sigma_x = \sigma_y = \sigma$, the price of the collision option that pays K on underlying assets with the same volatility becomes

$$V(x,y) = K\left(\frac{x}{y}\right)^{-\frac{1}{\sigma}\sqrt{\frac{r}{1-\rho}}}$$

Given $\rho = 0$, the price of the collision option that pays K on independent underlying asset becomes,

$$V(x,y) = K\left(\frac{x}{y}\right)^\alpha$$

Where

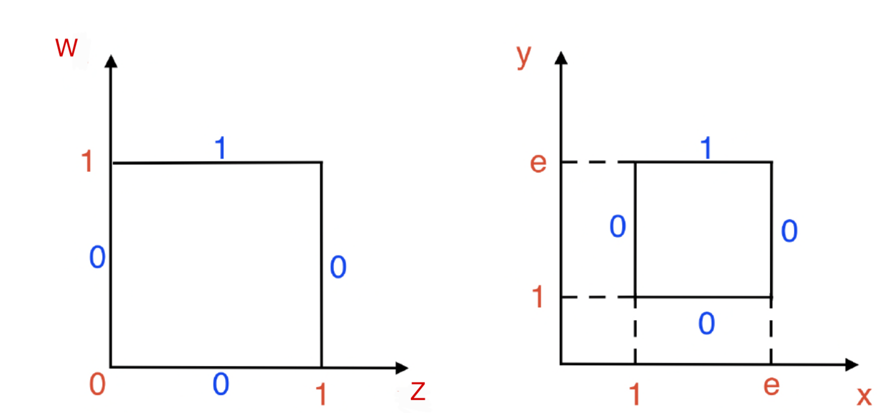
$$\alpha = \frac{-\frac{1}{2}(\sigma_x^2 - \sigma_y^2) - \sqrt{\frac{1}{4}(\sigma_x^2 - \sigma_y^2)^2 + 2(\sigma_x^2 + \sigma_y^2)r}}{\sigma_x^2 + \sigma_y^2}$$

Summary

Essentially, the Boundary Option and the Collision Option are connected by nature. They are both perpetual options with barriers. The Boundary Option has fixed lower barrier or upper barrier or both, while the Collision Option has a dynamic barrier if we consider the higher stock to be the upper barrier or consider the lower stock to be the lower barrier. If we set σ_1 or σ_2 to be zero, then we have a static upper or lower bound in the Collision Option, which corresponds exactly with the Boundary Option.

Further Research: BALSA Option

The BALSA option follows a payoff based on the price of two stocks x, y. It pays \$1 the first time any of the following cases happens: $z = 0$, $z = 1$, $w = 0$, $w = 1$, with the change of variables: $z = \ln x$, $w = \ln y$, respectively corresponds to $x = 1$, $x = e$, $y = 1$, $y = e$.



We've looked into the equal σ and generic r case, the option price follows the fourier series:

$$V(z,w) = \sum_{n=1}^{\infty} C_n \sin(n\pi z)\sinh\left(\sqrt{n^2\pi^2 + 1}w\right) \quad (1)$$

where

$$C_n = \frac{(-1)^{n+1}(e^a + (-1)^{n+1})^2\pi n}{\sinh(Q)(n^2\pi^2 + a^2)} \quad (2)$$

and

$$Q = \sqrt{\frac{2r^2}{\sigma^4} + \frac{1}{2} + n^2\pi^2}$$

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