The Wave Equation, Bessel's Functions, and Vibrations on a Drumhead

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Abstract

Friedrich Wilhelm Bessel (1784 – 1846) was a German Mathematician that discovered the Bessel Functions *. Bessel functions appear frequently in mathematical models of physical phenomena such as: propogation of electromagnetic waves along a wire †, vibrations on a drumhead, the motion of fluids, the diffraction of light, and most vitally, solutions to Laplace's equation for special geometries. In this lightning talk, we derive the Bessel's function from an Ordinary Differential Equation (ODE) and use the method of separation of variables to find a solution the problem of vibrations on a drumhead.

I Pre-requisites

1. Math 20D - Ordinary Differential Equations
2. Math 18 - Linear Algebra
3. Math 20C - Multi-variable Calculus

*Please direct any questions, comments, or concerns to Simon Hu at jhhu@ucsd.edu. Thank you.

*These functions were supposedly actually discovered by Daniel Bernoulli (1700 - 1782), who studied oscillations.

†Look up *Telegrapher's Equation* for more information

II Bessel's Functions

II.1 Bessel's Differential Equation

Bessel's functions are solutions to the second-order, varying coefficient, linear, homogeneous, ordinary differential equation,

$$\frac{d^2 J_n(x)}{dx^2} + \frac{1}{x}\frac{dJ_n(x)}{dx} + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0. \tag{1}$$

There are three types of Bessel's functions. For the purpose of this lightning talk, we will only focus on the first type.

The first are called Bessel's functions of the first kind. In reference to (1), Bessel's functions of the first kind only take positive integer orders of n. These functions do not have a singularity at the origin, $x = 0$.

The second are called Neumann's functions, or Bessel's functions of the second kind. In reference to (1), these functions have singularities at the origin, can take half-integer orders of n, and are generally multi-valued. We will not discuss these functions any further, though they are very important in the field of Mathematical Physics.

The third kind are called Hankel functions, and there are two kinds of Hankel functions: the first and second kinds. They are related to Bessel's functions of the first and second kind, in that Hankel's functions are just linear combinations of Bessel's functions of the first and second kind.

II.2 Solution to Bessel's Differential Equation

We proceed via the method of Frobenius. Assume the solution $J_\alpha(x)$ takes the form

$$J_\alpha(x) = x^\alpha \sum_{k=0}^{\infty} a_k x^k$$

where we assume that $a_0 \neq 0$. Plugging this into (1), we obtain[‡]

$$\sum_{k=0}^{\infty}(\alpha+k)(\alpha+k-1)a_k x^{k-2+\alpha} + \sum_{k=0}^{\infty}(\alpha+k)a_k x^{k-2+\alpha} + \sum_{k=0}^{\infty}a_k x^{\alpha+k} - \sum_{k=0}^{\infty}n^2 a_k x^{k-2+\alpha}$$
$$= \sum_{k=0}^{\infty}(\alpha+k)(\alpha+k-1)a_k x^{k-2+\alpha} + \sum_{k=0}^{\infty}(\alpha+k)a_k x^{k-2+\alpha} + \sum_{k=2}^{\infty}a_{k-2} x^{k-2+\alpha} - \sum_{k=0}^{\infty}n^2 a_k x^{k-2+\alpha} = 0.$$

Equating the like indices of k together, we obtain

$$\sum_{k=0}^{\infty}\left[(\alpha+k)(\alpha+k-1)+(\alpha+k)-n^2\right]a_k x^{k+\alpha-2} + \sum_{k=2}^{\infty}a_{k-2}x^{k+\alpha-2} = 0.$$

There are three cases that we must consider. For obvious reasons, we do not consider the case when the index of the coefficients are negative, that is, $k < 0$.

$k = 0$. For $k = 0$, we have

$$\left[\alpha(\alpha-1)+\alpha-n^2\right]a_0 = \left[\alpha^2-n^2\right]a_0 = 0.$$

This implies that $\alpha = \pm n$, since $a_0 \neq 0$. Both $\alpha = n$ and $\alpha = -n$ produce the same result, but we generally take α to be positive.

$k = 1$. For $k = 1$, we have

$$\left[(n+1)n+n+1-n^2\right]a_1 = \left[n^2+2n+1-n^2\right]a_1 = \left[(n+1)^2-n^2\right]a_1 = 0.$$

This implies that $a_1 = 0$ since $n \neq 0$ and $n \in \mathbb{Z}$. This is due to the fact that the eigenvalue problem for the *Helmholtz equation* only considers positive eigenvalues. This will be discussed later on.

$k \geqslant 2$. For $k \geqslant 2$, we have

$$\left[(n+k)(n+k-1)+n+k-n^2\right]a_k + a_{k-2} = \left[(n+k)^2-n^2\right]a_k + a_{k-2} = 0.$$

[‡] Before someone calls the math police, I know that when you differentiate a power series, you have to increase the index of the power series. However, there is no loss in generality if we start the series index at zero. Why? Plug in $n = 0$ and $n = 1$ into the summation and you will get a zero term in the summation. This properly justifies it.

If $k \geqslant 2$, then we have a recursive relationship between the coefficients. Specifically,

$$a_k = -\frac{a_{k-2}}{[(n+k)^2 - n^2]}. \tag{2}$$

We write out the first three terms.

$$k = 2, \quad a_2 = -\frac{a_0}{[(n+2)^2 - n^2]}$$

$$k = 3, \quad a_3 = -\frac{a_1}{[(n+3)^2 - n^2]}$$

$$k = 4, \quad a_4 = -\frac{a_2}{[(n+4)^2 - n^2]}$$

But we know that $a_1 = 0$. Therefore, all coefficients that involve a_1 must vanish. In the above three terms, the a_3 term vanishes, but the a_0 terms do not. Since the periodicity of the coefficients is 2, we conclude that we will only have non-zero coefficients if k is even. Additionally, due to the periodicity and signage of the coefficients, every other even-coefficient will have the same signage as a_2. That is to say, a_2 will be negative, a_4 will be positive, a_6 is negative, and so on and so forth.

Now that we have an *indicial* formula for the coefficients, we can write out the explicit solution to Bessel's equation. It is convenient here to choose

$$a_0 = \frac{1}{2^n n!}.$$

It may appear that we pulled this number out of thin air, and you would be correct. [§]
Then the solution to (1) is given by

$$\begin{aligned} J_n(x) &= \frac{x^n}{2^n n!}\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2!2^4(n+1)(n+2)} + \cdots\right] \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{(x/2)^{n+2j}}{j!(n+j)!}. \end{aligned}$$

[§]The actual justification requires a bit more analysis than I can cover in this paper. These coefficients are related to the *Gamma function*.

III The Wave Equation in $\mathbb{R}^2$

III.1 Derivation of the Wave Equation

We derive the wave equation by examining the physics of a string under tension. The derivation of this in one dimension is actually quite simple. Recall Newton's famous Force-acceleration equation, $\mathbf{F} = m\mathbf{a}$. Figure

III.2 Conservation of Energy of the Wave Equation

For some ideal wave in a closed system, the energy is conserved! To prove this, we define the

III.3 The Wave Equation in 1-Dimension

Let $u \in C^2$ where $u = u(x,t)$. $u(x,t)$ describes the vertical displacement of the wave at position x and time t. The wave equation equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{3}$$

where c is usually referred to as the speed of propogation of the wave. (26) is considered as a *linear, homogeneous, second-order partial differential equation.* A full solution to this requires we have initial conditions and boundary conditions. For any well-posed partial differential equation problem for the wave equation, we require *two* initial conditions (the initial displacement and the initial velocity of the wave) and a boundary conditions (what happens at the boundary)[¶]. Let $\varphi(x) = u(x,0)$ be the initial displacement of the wave at time $t = 0$. Similarly, let $\psi(x) = u_t(x,0)$ be the initial velocity of the wave at time $t = 0$. To conclude, we want to solve the following problem, for some domain D:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } D \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x). \end{cases} \tag{4}$$

[¶] There are three classes of boundary conditions: Dirichlet, Neumann, and Robin

This is also known as the wave equation with *Cauchy data*.

III.3.1 Form of the Solution

We will first derive the form of the solution. The movitvation here is, that by finding the general form the solution to the wave equation should take, we can make a physical interpretation of the solution. We proceed via the method of *factoring the partial derivatives*. In actuality, this method is called *factorization of the operators*, but we do not assume the reader has had any exposure to operator theory. We factor the partial derivatives:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x,t) = u_{tt} - c^2 u_{xx} = 0.$$

If we first solve

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x,t) = u_t - cu_x = 0 \tag{5}$$

and then, allowing $v(x,t) = u(x,t)$, solve

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v(x,t) = v_t + cv_x = 0, \tag{6}$$

we have a solution to the wave equation. In order to solve (5) and (6), we use the method of characteristics. The solution to (5) is given by $u(x,t) = f(x-ct)$ where $f \in C^2$ (C^2 is the class of all functions with continuous second-order derivatives) is an arbitrary function. The solution to (6) is given by $v(x,t) = g(x+ct)$ where $g \in C^2$ is also an arbitrary function. These two solutions are linearly independent. Therefore, the superposition of these two solutions is also a solution. Therefore, let

$$u(x,t) = f(x-ct) + g(x+ct). \tag{7}$$

What does this imply physically? Well, this means that the solution to the wave equation involves a superposition of a wave traveling to the *left* at speed c and a wave traveling to the *right* at speed c. So for any x, there is a wave traveling to the left and there is a wave traveling to the right, both at the same speed, c.

III.3.2 Solution to the Wave Equation in $\mathbb{R}$

Now that we have a general idea of what form the solution should take, we proceed to derive the solution to the wave equation on the entire line. The idea here is this: we plug in the initial conditions, so now we have a system of equations, which we can easily solve using Gaussian elimination. So we proceed, first by plugging in the initial conditions to the general form of the solution.

$$\begin{aligned} u(x,0) &= f(x) + g(x) = \varphi(x) \\ u_t(x,0) &= -cf(x) - cg(x) = \psi(x). \end{aligned}$$

The solution to this system is:

Then the solution is:

$$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy. \tag{8}$$

(8) is usually referred to as *D'Alembert's Formula*, named after French Mathematician Jean le Rond D'Alembert.

III.3.3 Solution to the Wave Equation in $\mathbb{R}^+$

Now we solve the wave equation on the *half-line*, $(-\infty < x < \infty)$. We will use the *method of odd/even extensions.* The idea is this: we already know the solution to the wave equation on the entire line, so we will extend our functions φ and ψ to the entire line. Based on the boundary conditions given, we will either use an odd or even extension.

(*The case with Dirichlet boundary conditions*). Suppose we wish to solve the problem with homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \text{in } \mathbb{R}^+ \\ u(0,t) = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x). \end{cases} \tag{9}$$

It would be advantageous to use an odd-extension, due to the nature of the boundary condition. The reason for this is beyond the scope of the pre-requisites, but it is related to the idea of symmetric boundary conditions. For the function $\varphi(x)$, define it's odd extension, φ_{odd}, in the following manner:

$$\varphi_{odd}(x) = \begin{cases} \varphi(x) & x > 0 \\ -\varphi(-x) & x < 0. \end{cases}$$

Similarly, for the function $\psi(x)$, define its odd extension, $\psi_{odd}(x)$, in the following manner:

$$\psi_{odd}(x) = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0. \end{cases}$$

Notice that both φ_{odd} and ψ_{odd} are defined on the entire line. So we have now extended (9) into the following problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \\ u(0,t) = 0 \\ u(x,0) = \varphi_{odd}(x) \\ u_t(x,0) = \psi_{odd}(x). \end{cases} \tag{10}$$

We already know the solution to this. It is given by (8). Therefore, the solution is

$$u(x,t) = \frac{1}{2}\left[\varphi_{odd}(x+ct) + \varphi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\, dy. \tag{11}$$

Let's unpack this. There are two regions of interest. $x > c|t|$ and $x < c|t|$.

If $x > c|t|$, then our solution is identically (8).

If $x < c|t|$, then our solution is given by:

$$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) - \varphi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\, dy. \tag{12}$$

(*Case with Neumann boundary conditions*). Suppose we wish to solve the problem with homogeneous Neumann boundary conditions:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R}^+ \\ u_n(0,t) = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x). \end{cases} \tag{13}$$

In this case, it would be advantageous to use an even-extension, due to the nature of the boundary condition. For the function $\varphi(x)$, define it's even extension, $\varphi_{even}(x)$, in the following manner:

$$\varphi_{even}(x) = \begin{cases} \varphi(x) & x > 0 \\ \varphi(-x) & x < 0. \end{cases}$$

Similarly, for the function $\psi(x)$, we define it's even extension, $\psi_{even}(x)$, in the following manner:

$$\psi_{even}(x) = \begin{cases} \psi(x) & x > 0 \\ \psi(-x) & x < 0. \end{cases}$$

We proceed in the same manner for the Dirichlet case and solve the following problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \\ u_n(0,t) = 0 \\ u(x,0) = \varphi_{even}(x) \\ u_t(x,0) = \psi_{even}(x). \end{cases} \tag{14}$$

We already know the solution to this. It is given by (8). Therefore, the solution is

$$u(x,t) = \frac{1}{2}\left[\varphi_{even}(x+ct) + \varphi_{even}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\, dy. \tag{15}$$

Let's unpack this equation. There are two regions of interest. $x > c|t|$ and $x < c|t|$.

If $x > c|t|$, then our solution is identically (8).

If $x < c|t|$, then our solution is given by:

$$u(x,t) = \frac{1}{2}\left[\varphi_{even}(x+ct) + \varphi_{even}(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi_{even}(y)\, dy. \tag{16}$$

III.3.4 Solution on a Finite Interval

On the finite interval, we proceed via the method of *separation of variables.* We want to solve the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} & (0 < x < L, t > 0) \\ u(0,t) = u(L,t) \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases} \tag{17}$$

We assume the solution is of the form $u(x,t) = X(x)T(t)$. The reason we assume this form is because it will reduce solving the PDE to solving an ODE. Since $u(x,t)$ is the product of two functions, if we plug this into the PDE, we obtain the following expression:

$$XT'' = c^2 X'' T.$$

If we divide both sides by $T(t)$ and $X(x)$, we obtain

$$\frac{T''}{cT} = \frac{X''}{X}.$$

However, on each side we have a function in terms of either x or t, but not both. Therefore, we must have that the left-hand side and the right-hand side must be equal to a constant. That is to say, for some constant λ:

$$\frac{T''}{cT} = \frac{X''}{X} = -\lambda.$$

Now we just have to figure out the boundary conditions.

$u(0,t) = X(0)T(t) = 0$ implies that $X(0) = 0$, since we can assume, without loss of generality, that $T(t)$ is not identically zero. That would be silly, and I'm definitely not a silly person. Additionally, $u(L,t) = X(L)T(t) = 0$ implies that $X(0) = 0$, since $T(t) \neq 0 \; \forall \; t$. Now we are done with the boundary conditions and we have a full eigenvalue problem. The problem is:

$$\begin{cases} X'' + \lambda X = 0, & X(0) = X(L) = 0 \\ T'' + c\lambda T = 0. \end{cases} \tag{18}$$

We consider three cases for λ.

$\lambda = 0$. If $\lambda = 0$, then we have $X'' = 0$. The solution to this differential equation is: $X(x) = c_1 x + c_2$. Now we can use the boundary conditions to find the coefficients. From $X(0) = c_2 = 0$, we have that $c_2 = 0$. Therefore, it must be that our solution is actually $X(x) = c_1 x$. Then, the second condition $X(L) = c_1 L = 0$ implies that $c_1 = 0$ since it does not make sense that $L = 0$. Thus, we are left with the trivial solution $X(x) = 0$ and since I'm an interesting man, I am not interested in this trivial solution.

$\lambda < 0$. Let $\beta^2 = \lambda$. The solution to the differential equation must be: $X(x) = c_1 e^{-\beta x} + c_2 e^{\beta x}$. However, we are seeking a bounded solution. Therefore, $c_2 = 0$ – otherwise we'd get an unbounded solution. The condition $X(0) = c_1 = 0$ implies that $c_1 = 0$. Therefore, we have the trivial solution, which we are not interested in.

$\lambda > 0$. Let $\gamma^2 = \lambda$. The solution to the differential equation must be: $X(x) = c_1 \cos(\gamma x) + c_2 \sin(\gamma x)$. The first boundary condition gives us $X(0) = c_1 = 0$, which implies that $c_1 = 0$. Therefore, we are just left with $X(x) = c_2 \sin(\gamma x)$. Then, the second boundary condition gives us $X(L) = \gamma c_1 \sin(\gamma L) = 0$. Since it does not make sense for $\gamma = 0$ (that would be a contradiction), or $c_1 = 0$ since that would give us the trivial solution. Therefore, it must be that $\sin(\gamma L) = 0$. The sine function is zero only if $\gamma L = n\pi$ where $n = 0, 1, 2, \ldots$. This implies that the eigenvalues are:

$$\gamma_n^2 = \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad (n = 0, 1, 2, \ldots). \tag{19}$$

This implies that the *eigenfunctions* must be:

$$X_n(x) = c_n \sin(\gamma_n x). \tag{20}$$

Thus we can now find the solution to $T(t)$. We already know that $\lambda > 0$ gives us a non-trivial solution, so we will assume that $\lambda > 0$. (*Exercise: This is a safe assumption to make. As an exercise, try the case when $\lambda = 0$ and $\lambda < 0$ and convince yourself that this is indeed, a safe assumption*).

Plugging in our λ value into the differential equation for $T(t)$, we find that

$$T_n(t) = c_1 \sin(\gamma_n t) + c_2 \cos(\gamma_n t) \qquad (n = 0, 1, 2, \dots).$$

If our assumption of the form of $u(x, t)$ is correct, then we must have that

$$u_n(x,t) = X_n(x)T_n(t) = c_n \sin(\gamma_n x) \left(\sin(\gamma_n t) + \cos(\gamma_n t)\right)$$

For every n, we have a solution. These solutions are linearly-independent, therefore, we can take an infinite summation of these solutions. We call the summation of these $u_n(x, t)$s and we call this solution $u(x, t)$. That is to say, we have

$$u(x,t) = \sum_{n=0}^{\infty} \sin(\gamma_n x) \left(c_n \sin(\gamma_n t) + b_n \cos(\gamma_n t)\right).$$

But this infinite series solution is absolutely useless! This summation tells us that there are an infinite superposition of solutions! This is where the initial condition will come in. The initial condition allows us to find a specific solution to the PDE, much like the initial conditions helps us find unique solutions to an ODE problem. Using orthogonality relations for the Fourier series, we can solve for the coefficients, c_n and b_n. Let us apply the first initial condition:

$$u(x,0) = \varphi(x) = \sum_{n=0}^{\infty} b_n \sin(\gamma_n x). \tag{21}$$

Then, applying the second initial condition:

$$u_t(x,0) = \psi(x) = \sum_{n=0}^{\infty} c_n \sin(\gamma_n x). \tag{22}$$

To find a formula for the coefficients, we have to use some orthogonality relationships. If we multiply both sides of (21) by $\sin(\gamma_m x)$, then we have

$$\sin(\gamma_m x)\varphi(x) = \sum_{n=0}^{\infty} b_n \sin(\gamma_n x)\sin(\gamma_m x)$$

Then, if we integrate both sides from 0 to L, we are left with:

$$\int_0^L \sin(\gamma_m x)\varphi(x)\ dx = \int_0^L \sum_{n=0}^{\infty} b_n \sin(\gamma_n x)\sin(\gamma_m x)\ dx.$$

Then, we use the fact that:

$$\int_0^L \sin(\gamma_m x)\sin(\gamma_n x)\ dx = \begin{cases} L/2 & m = n \\ 0 & m \neq n \end{cases}$$

in order to deduce that only one term will survive in the integral and the coefficients have been summed. Now we have a formula for the coefficients of the infinite series (these are called the Fourier coefficients, since we are technically dealing with a Fourier series). The coefficients b_n are given by:

$$b_n = \frac{2}{L}\int_0^L \varphi(x)\sin(\gamma_n x)\ dx. \tag{23}$$

Similarly, the coefficients c_n are given by:

$$c_n = \frac{2}{L}\int_0^L \psi(x)\sin(\gamma_n x)\ dx. \tag{24}$$

Therefore, the solution to the PDE is given by:

$$u(x,t) = \sum_{n=0}^{\infty} \sin(\gamma_n x)\left(c_n \sin(\gamma_n t) + b_n \cos(\gamma_n t)\right) \tag{25}$$

where b_n and c_n are given by (23) and (24), respectively.

III.4 Solution to the Wave Equation in n-Dimensions

Let $u \in C^2$ where $u = u(x_1, x_2, x_3, \ldots, x_n, t)$. The wave equation equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u. \tag{26}$$

The formula is the exact same as (**??**) but with the addition of the Laplacian. The general Laplacian in $\mathbb{R}^n$, denoted Δ, is given by

$$\Delta = \nabla \cdot \nabla = |\nabla|^2 = \frac{\partial^2}{dx_1^2} + \frac{\partial^2}{dx_2^2} + \cdots + \frac{\partial^2}{dx_n^2}.$$

The solution to this is given by *Kirchoff's formula* [‖]. Let $B(ct_0, \mathbf{x}_0)$ denote the Ball of radius ct_0, centered at $\mathbf{x}_0$, and equate it to Ω. Then *Kirchoff's formula* is given by

$$u(\mathbf{x}, t_0) = \frac{1}{4\pi c^2 t_0} \int_\Omega \psi(\mathbf{x})\ dS + \frac{d}{dt_0} \left[\frac{1}{4\pi c^2 t_0} \int_\Omega \varphi(\mathbf{x})\ dS \right] \tag{27}$$

The derivation of this formula goes beyond the scope of the prerequisites, but for further discussion, see (insert reference to Strauss's PDEs book here).

IV Vibrations on a Drumhead

IV.1 Physical Description of the Problem

Put the figure here.

The following is a physical description of the problem:

1. Suppose we have a drumhead, which is modeled by a circle of radius a.
2. The drumhead satisfies the wave equation in $\mathbb{R}^2$.
3. Let $\Omega = \left\{(x, y) : x^2 + y^2 < r^2\right\}$ describe the interior of the drumhead.

[‖] This formula is actually due to Poisson (1781 – 1840).

4. Let $\partial\Omega = \{(x, y) : x^2 + y^2 = a^2\}$ describe the boundary of the drumhead.

5. $u = u(x, y, t)$ is the vertical displacement of the drumhead.

6. The drumhead is fixed at the boundary, in other words $u = 0$ for $(x, y) \in \partial\Omega$ (i.e. we have *Dirichlet* boundary conditions).

7. The drumhead has an initial position, given by $u(x, y, t) = \varphi(x, y) \in C$ at $t = 0$.

8. The drumhead has an initial velocity, given by $u_t(x, y, t) = \psi(x, y) \in C$ at $t = 0$.

IV.2 General Solution to the Problem

First, we note that since the drumhead has some sort of radial symmetry, it would be wise to use polar coordinates. The Laplacian in polar coordinates is given by:

$$\Delta_r = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} \tag{28}$$

Thus our PDE problem reduces to

$$\begin{cases} u_{tt} = c^2 \Delta_r u \\ u(0, t) \text{ is finite,} \\ u(\infty, t) = 0. \end{cases} \tag{29}$$

We proceed via the method of separation of variables. Assume that the solution takes the form $u(r, t) = R(r)T(t)\Theta(\theta)$. Plugging this into the PDE, we obtain

$$RT''\Theta = c^2 \left(R''T\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}RT\Theta'' \right).$$

Separating the variables, we obtain

$$\frac{T''}{c^2 T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta}.$$

As with the standard separation of variables argument, we say that the left-hand side of the equal sign is equal to a constant, call it $-\lambda$, and Θ''/Θ is a constant, call it $-\gamma$. Then this gives rise to three ODEs:

$$\Theta'' + \gamma\Theta = 0 \tag{30}$$

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{\gamma}{r^2}\right) R = 0 \tag{31}$$

$$T'' + c^2\lambda T = 0. \tag{32}$$

We will solve each equation in the order they appear. It is appropriate to give (30) 2π-periodic boundary conditions. Convince yourself of this. So, $\Theta(\theta+2\pi) = \Theta(\theta)$. For a circle, 2π periodic boundary conditions implies that $\Theta(0) = \Theta(2\pi)$. We consider three cases for γ.

$\gamma = 0$. If $\gamma = 0$, then the solution to the differential equation is $\Theta(\theta) = c_1\theta + c_2$. The boundary condition $\Theta(\theta) = \Theta(\theta + 2\pi)$ implies that $c_1\theta + c_2 = c_1(\theta + 2\pi) + c_2$. Solving for the coefficients, we see that $c_1\pi = 0$, which means that c_1 must be identically zero. Then we are left with $\Theta(\theta) = c_2$.

$\gamma < 0$. Let $\beta^2 = \gamma$. If $\gamma < 0$, then the solution to the differential equation is $\Theta(\theta) = e^{-i\beta\theta}$. Note that there are actually *two* linearly-independent solutions to the differential equation, but we require that as $r \to \infty, u \to 0$. The solution $\Theta(\theta) = c_2e^{i\beta\theta}$, is not considered, for this reason. The boundary conditions imply $\Theta(\theta) = c_1e^{-i\beta\theta} = c_1e^{-i\beta\theta-i2\pi} = \Theta(\theta + 2\pi)$. Therefore, c_1 must be 0, and we obtain the trivial solution.

$\gamma > 0$. This is the final case. Since we've already shown that the negative eigenvalues produce trivial solutions, it must be that the eigenvalues are zero, and strictly positive. In fact, the eigenvalues are given by

$$\gamma_n = n^2, \quad (n = 1, 2, 3, \dots)$$

and the eigenfunctions are given by

$$\Theta_n(\theta) = c_n \cos(n\theta) + d_n \sin(n\theta).$$

When $\gamma = 0$, the eigenfunction is

$$\Theta(\theta) = \frac{1}{2}c_0.$$

Now we deal with (31). Plugging in the information we have about γ, we now have the following ODE to solve:

$$R_{rr}{}^{**} + \frac{1}{r}R_r + \left(\lambda - \frac{n^2}{r^2}\right) R = 0 \tag{33}$$

for $0 < r < a$ together with the boundary conditions, $R(0)$ is finite and $R(a) = 0$. Here, we only need to consider two cases for λ.

$\lambda = 0$. If $\lambda = 0$, then the solution is given by guessing that $R(r) = r^\alpha$ (this equation is of the *Euler type*). Plugging this into the differential equation, we find that $\alpha = \pm n$. Then $R(r) = c_1 r^n + c_2 r^{-n}$. But $R(r)$ must be finite at $r = 0$ and this eliminates the solution $R(r) = c_2 r^{-n}$. Then, $R(a) = c_1 a^n = 0$. This implies that $c_1 = 0$ and thus we have the trivial solution and we are not interested in this.

$\lambda > 0$. If $\lambda > 0$, then we have to transform this equation into a form that is more familiar. Let $\epsilon = \sqrt{\lambda} r$, so that

$$R_r = R_\epsilon \frac{d\epsilon}{dr} = \sqrt{\lambda} R_\epsilon, \quad R_{rr} = \lambda R_{\epsilon\epsilon}.$$

Then (33) becomes

$$R_{\epsilon\epsilon} + \frac{1}{\epsilon}R_\epsilon + \left(1 - \frac{n^2}{\epsilon^2}\right) R = 0.$$

This is just *Bessel's differential equation of the first kind, of the n-th order* and we know how to solve this. Since $R(0)$ is finite, for some constant c, we have $R(r) = cJ_n(\epsilon)$, where J_n denotes the Bessel function of the first kind, of the n-th order. If we put in the boundary condition, then

$$J_n(\sqrt{\lambda} a) = 0.$$

The Bessel's function has an infinite number of positive roots, which can be seen by graphing the solution. Let's call these roots

$$0 < \lambda_{n1} < \lambda_{n2} < \lambda_{n3} < \cdots.$$

**Note that $R_{rr} = R''$ since R is a function of r

Finally, we deal with (32). There is only one case that we need to consider. The case when $\lambda > 0$ (since $\lambda = 0$ just gives us the trivial solution). If $\lambda > 0$, then the solution to the differential equation is $T(t) = c_1 \cos(\sqrt{\lambda}ct) + c_2 \sin(\sqrt{\lambda}ct)$. We are now ready to comprise a full solution to this problem.

We sum everything up, letting $\beta_{nm} = \sqrt{\lambda_{nm}}$, and the full solution is

$$\begin{aligned} u(r,\theta,t) &= \sum_{m=1}^{\infty} J_0\left(\beta_{0m} r\right)\left(A_{0m}\cos\left(\beta_{0m} ct\right) + C_{0m}\sin\left(\beta_{0m} ct\right)\right) \\ &\quad + \sum_{m,n=1}^{\infty} J_n\left(\beta_{nm} r\right)\Bigg[\left(A_{nm}\cos n\theta + B_{nm}\sin n\theta\right)\cos\left(\beta_{nm} ct\right) \\ &\quad + \left(C_{nm}\cos n\theta + D_{nm}\sin n\theta\right)\sin\left(\beta_{nm} ct\right)\Bigg]. \end{aligned} \tag{34}$$

This equation is...quite formidable to say the least. Now lets put in the initial conditions, φ and ψ. For $u(r,\theta,0) = \varphi(r,\theta)$, we have

$$\varphi(r,\theta) = \sum_{m=1}^{\infty} A_{0m} J_0(\beta_{0m} r) + \sum_{m,n=1}^{\infty} J_n(\beta_{nm} r)(A_{nm}\cos n\theta) + B_{nm}\sin n\theta) \tag{35}$$

and for $u_t(r,\theta,0) = \psi(r,\theta)$, we have

$$\begin{aligned} \psi(r,\theta) &= \sum_{m=1}^{\infty} c\beta_{nm} C_{0m} J_0(\beta_{0m} r) \\ &\quad + \sum_{m,n=1}^{\infty} c\beta_{nm} J_n(\beta_{nm} r)(C_{nm}\cos n\theta + D_{nm}\sin n\theta). \end{aligned} \tag{36}$$

This expansion is much like the one in Fourier series. Except, since the Bessel functions are mutually orthogonal, we can obtain the formula for the coefficients in the same manner.

The formula for the coefficients are given by

$$
\begin{aligned}
A_{0m} &= \frac{1}{2\pi\eta_{0m}} \int_0^a \int_{-\pi}^{\pi} \varphi(r,\theta) J_0(\beta_{0m} r)\; r\; d\theta\; dr \\
A_{nm} &= \frac{1}{\pi\eta_{nm}} \int_0^a \int_{-\pi}^{\pi} \varphi(r,\theta) J_n(\beta_{nm} r) \cos n\theta\; r\; d\theta\; dr \\
B_{0m} &= \frac{1}{\pi\eta_{nm}} \int_0^a \int_{-\pi}^{\pi} \varphi(r,\theta) J_n(\beta_{nm} r) \sin n\theta\; r\; d\theta\; dr \\
C_{0m} &= \frac{1}{2\pi\eta_{0m} c\beta_{nm}} \int_0^a \int_{-\pi}^{\pi} \psi(r,\theta) J_0(\beta_{0m} r)\; r\; d\theta\; dr \\
C_{nm} &= \frac{1}{\pi\eta_{nm} c\beta_{nm}} \int_0^a \int_{-\pi}^{\pi} \psi(r,\theta) J_n(\beta_{nm} r) \cos n\theta\; r\; d\theta\; dr \\
D_{nm} &= \frac{1}{\pi\eta_{nm} c\beta_{nm}} \int_0^a \int_{-\pi}^{\pi} \psi(r,\theta) J_n(\beta_{nm} r) \sin n\theta\; r\; d\theta\; dr.
\end{aligned}
\tag{37}
$$

where η_{nm} is given by the formula

$$
\eta_{nm} = \int_0^a \left[J_n(\beta_{nm} r)\right]^2\; r\; dr = \frac{1}{2} a^2 \left[J_n'(\beta_{nm} a)\right]^2. \tag{38}
$$

The derivation of this formula comes from the orthogonality of the eigenfunctions, but it will not be discussed here.

V Radial Vibrations of a Drumhead

Imagine the following situation: you beat the center of a Taiko drum with a bachi (the stick used to beat it) at time $t = 0$ and you listen for the ensuing vibrations. This means that the initial conditions are

$$
u(x, y, 0) = 0 \text{ and } u_t(x, y, 0) = \psi(r)
$$

and beat is highly concentrated about $r = 0$. Since $\varphi(r, \theta)$ is identically zero, the A_{nm} and B_{nm} in (34) vanish. Furthermore, since $\psi(r)$ is not a function of θ, C_{nm} and D_{nm} vanish as well. So all that remains from (34) is

$$u(r,t) = \sum_{n=1}^{\infty} C_{0m} J_0(\beta_{0m} r) \sin(\beta_{0m} ct) \tag{39}$$

where C_{0m} is given by

$$C_{0m} = \frac{1}{c\beta_{0m}} \cdot \frac{\displaystyle\int_0^a \psi(r) J_0(\beta_{0m} r)\ r\ dr}{\displaystyle\int_0^a \left[J_0(\beta_{0m} r)\right]^2\ r\ dr}. \tag{40}$$

Equations (39) combined with (40) provides a solution to the radial vibrations on a drumhead.