ALGEBRAIC GEOMETRY

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1. Chapter I: Varieties

1.1. Affine Varieties.

Exercise 1.1.

2. Chapter II: Schemes

2.1. Sheaves.

Definition 2.1. Colimit and Filtered colimit

A category $\mathcal{J}$ is **filtered** if

(i) For each $x, y \in \mathcal{J}, \exists z \in \mathcal{J}$, and arrows $x \to y$ and $y \to z$ and

(ii) for every two arrows $u : x \to y, v : x \to y, \exists w : y \to z$ such that $w \circ u =$

Let $\{X_i\}_{i \in I}$ be objects indexed by I. The **colimit** of the system is defined along morphism ι_i : $X_i \to \varinjlim_i X_i, u_i^j : X_i \to X_j$, such that the diagram commutes.

Construct of filtered colimit by using coproduct and coequilizer: note that for convenience, we consider the cases under the category of abelian groups.

Suppose $\{X_i\}_{i \in I}$ is a set of abelian groups, and we define maps $u_i^j : X_i \to X_j$, and $\iota_i : X_i \to \bigoplus_{i \in I} X_i$the injection maps. We require the following diagram to be commutative.

$$\begin{tikzcd} X_i \arrow[r, "u_i^j"] \arrow[dr, "\iota_i"'] & X_j \arrow[d, "\iota_j"] \\ & \bigoplus_{i \in I} X_i \end{tikzcd}$$

Let

$$\varinjlim X_i = \bigoplus_{i \in I} X_i / \{\iota_j u_i^j(x_i) - \iota_i(x_i) | x_i \in X_i\}$$

Consider the new diagram

$$\begin{tikzcd} X_i \arrow[r, "u_j^i"] \arrow[dr, "u^i"'] & X_j \arrow[d, "U^j"] \\ & \varinjlim_i X_i = \bigoplus_{i \in I} X_i / \{\iota_j u_i^j(x_i) - \iota_i(x_i) | x_i \in X_i\} \end{tikzcd}$$

Remark 2.2. In the rest of the document we denote the prime ideal corresponding to a point x in Spec$A$ by $\mathfrak{p}_x$.

Definition 2.3. The **stalk** of a sheaf is defined as:

$$\mathcal{O}_{X,x} = \varinjlim_{x \in U, U \text{is open in} X} \mathcal{O}_X(U)$$

Proposition 2.4. *Suppose* $X = \mathit{Spec}A$, $x \in X$, *then*

$$\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}_x}.$$

Date: May 2025.

Proof. By definition of stalk,

$$\mathcal{O}_{X,x} = \varinjlim_{x \in U, U \text{is open in} X} \mathcal{O}_X(U) = \varinjlim_{x \in D(f)} \mathcal{O}_X(U) = \varinjlim_{f \notin \mathfrak{p}_x} A_f$$

Then it suffices to prove that $\varinjlim_{f \notin \mathfrak{p}_x} A_f = A_{\mathfrak{p}_x}$.

Note that since $f \notin \mathfrak{p}_x$, $f \in (A - \mathfrak{p}_x)$ we can define the image of $\frac{a}{f^n}$ in $A_{\mathfrak{p}_x}$ naturally by identity.

We have a square

$$\begin{tikzcd}
A_f \arrow[r] \arrow[d] \arrow[dr] & \varinjlim_{f \notin \mathfrak{p}_x} A_f \arrow[d] \\
A_{\mathfrak{p}_x} & A_g \arrow[l]
\end{tikzcd}$$

By the universal property of colimits, we have the wanted isomorphism.

$\square$

Definition 2.5. Let $\mathcal{F}$ be a presheaf, let $\phi : \mathcal{F} \to \tilde{\mathcal{F}}$ be a morphism of presheaves If for any presheaf $\mathcal{G}$ and morphism $\psi : \mathcal{F} \to \mathcal{G}$, then there exists a unique morphism $\tilde{\phi} : \tilde{\mathcal{F}} \to \mathcal{G}$.

Exercise 2.6. (Har Ex1.1)

Proof. Only need to prove their stalks are equal.

Let X be a topological space and let A be an abelian group. Given discrete topology on A, for any open set U in A, define the **constant sheaf** $\mathcal{A}(U) := \{f | f : U \to A, f \text{ is continuous}\}$.

The constant presheaf is defined as $\mathcal{A}'(U) := \{f | f(U) = A\}, \forall U \neq \varnothing$.

$\mathcal{A}_x = \varinjlim_{x \in U} \mathcal{A}(U)$. For each U, we denote the connected component of x to be $V \subset U$, then $\mathcal{A}(V) = A$, hence $\mathcal{A}_x = A = \mathcal{A}'_x$

$\square$

Exercise 2.7.

Definition 2.8. Pull-back: The pull-back of a sheaf is defined by:

$$f^+\mathcal{G}(U) = \varinjlim_{f(U) \subset V, V \in \text{Ouv}(Y)} \mathcal{G}(Y).$$

Push-forward: The push-forward of a sheaf is defined by

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)).$$

Let $f^{-1}\mathcal{G}$ denote the sheafication of $f^+\mathcal{G}$.

Theorem 2.9. *(Also refer to Hartshone's Exercise 1.18)*

Prove that f^+ *is left adjoint to* f_*

$$Hom_{PShv}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq Hom_{PShv}(\mathcal{G}, f_*\mathcal{F})$$

Proof. Let $\phi \in \text{Hom}_{\text{PShv}}(f^{-1}\mathcal{G}, \mathcal{F})$, let $V \subset Y$ be a subset of Y. Since $f(f^{-1}(V)) \subset V$, we have a map $\mathcal{G}(V) \to f^+\mathcal{G}(f^{-1}(V))$.

Consider the composition:

$$\mathcal{G}(V) \to f^+\mathcal{G}(f^{-1}(V)) \to f^{-1}\mathcal{G}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V).$$

Let the image of ϕ be the composition of the above maps.

Conversely, let $\psi \in \text{Hom}_{\text{PShv}}(\mathcal{G}, f_*\mathcal{F})$, for some fixed open set $V \subset Y$, we can deduct a map $\mathcal{G}(V) \to f^+(\mathcal{G})(U)$ for open sets $U \subset X, f(U) \subset V$.

Hence, we can define the restriction map $\text{res}^V_U : \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$

Therefore by composition, we have the map

□

Lemma 2.10. *Important lemma in gluing sheaves: Let* X *be a topological space. Let* $X = \bigcup Ui$ *be an open covering. Let F, G be sheaves of sets on X. Given a collection*

$\varphi_i : F|_{Ui} \to G|_{U_i}$ *of maps of sheaves such that for all* $i, j \in I$ *the maps* φ_i, φ_j *restrict to the same map* $F|_{Ui\cap Uj}ßG|_{Ui\cap Uj}$ *then there exists a unique map of sheaves*

$\varphi : A \to B$ *whose restriction to each* U_i *agrees with* φ_i*.*

Definition 2.11. (Definition-Exercise) ***Gluing of sheaves*** Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i, a sheaf $\mathcal{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij} : \mathcal{F}_i \mid_{U_i \cap U_j} \to \mathcal{F}_j \mid_{U_i \cap U_j}$ such that

(1) : For each $\varphi_{ii} = id$, and (2) : for each i, j, k, $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}_i \mid_{U_i} \to \mathcal{F}_\rangle$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$.

Proof. First we show existence:

Let $\mathcal{F}$ be a function on X such that $\mathcal{F} \mid_{U_i} = \mathcal{F}_\rangle$, then we show that $\mathcal{F}$ is a presheaf first.

For any open sets $U \subset V$, there exists

Next we check the sheaf axioms, if □

Definition 2.12. Pushforward and pullback of a sheaf:

2.2. **Schemes.** Important properties:

Proposition 2.13. A *ring* , $(Spec\ A, \mathcal{O})$ *is its spectrum.*

(a) : *For any* $\mathfrak{p} \in SpecA$, *the stalk* $\mathcal{O}_\mathfrak{p}$ *of the sheaf* $\mathcal{O}$ *is isomorphic to the local ring* $A_\mathfrak{p}$.

(b) : *For any element* $f \in A$, *the ring* $\mathcal{O}(D(f))$ *is isomorphic to the localized ring* A_f.

(c) : *In particular,* $\Gamma(Spec, \mathcal{O}) \simeq A$.

Proof. (a) : By definition,

$$\mathcal{O}_\mathfrak{p} = \varinjlim \mathcal{O}(U)$$

and recall that by definition, $\mathcal{O}(U)$ is the set of functions $s : U \mapsto \amalg A_\mathfrak{p}$ such that $s(\mathfrak{p}) \in A_\mathfrak{p}$.

Hence we construct a map $\varphi : \mathcal{O}_\mathfrak{p} \to A_\mathfrak{p}$, $s \mapsto s(\mathfrak{p})$.

Then first we show that φ is surjective;

For any element $\frac{a}{f} \in A_\mathfrak{p}$, $f \notin \mathfrak{p}$, $a \in A$. There exists a neighborhood V of $\mathfrak{p}$, such that for any $\mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{a}{f}$.

□

Theorem 2.14. *The functor* $Ring^{op} \longrightarrow$ *Affsch is fully faithful.*

Theorem 2.15. *Suppose* X *is a locally ringed space,* Y *is an affine scheme, suppose* $Y = SpecB$*. Then there is a natural bijection*

$$Hom[(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)] \longrightarrow Hom(B, \Gamma(X, \mathcal{O}_X))$$

Proof. **(Injectivity):** It suffices to show that for any two morphism of locally ringed space $(f, f^\#), (g, g^\#)$, suppose they have the same image.

Consider the diagram below: Let $x \in X$ and suppose $\mathfrak{q}$ is the corresponding prime ideal of $f(x)$ in Y. Note that $\mathcal{O}_{Y,y}$ is a local ring, its prime ideal is $\mathfrak{q}B$

Let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$, note that since $f^\#_x$ is a local morphism, hence $f^\#_x$ takes $\mathfrak{m}_x$ to the maximal ideal of $\mathcal{O}_{X,x}$, thus the preimage of $\mathfrak{m}_x$ in B is $\mathfrak{q}$, which corresponds to $f(x)$.

Tracing the diagram we can see $f(x) = g(x)$.

$$\begin{tikzcd} \mathcal{O}_X(X) \arrow[r] & \mathcal{O}_{X,x} \\ B = \mathcal{O}_Y(Y) \arrow[u, "\phi"] \arrow[r] & \mathcal{O}_{Y,y} = B_\mathfrak{q} \arrow[u] \end{tikzcd}$$

Next, we want to show $f^\# = g^\#$, or equivalently, $f^\flat = g^\flat$

Consider the diagram below where we take open set $U = D(s)$ for some distinguish open set.

$$\begin{tikzcd}
B = \mathcal{O}_Y(Y) \arrow[r, "\phi"] \arrow[d, "res^Y_U"] & \mathcal{O}_X(X) \arrow[r, "="] \arrow[d, "res^X_{f^{-1}(U)}"] & (f_*\mathcal{O}_X)(Y) \arrow[d, "res^Y_U"] \\
B_s = \mathcal{O}_Y(D(s)) \arrow[r, "f^\flat(U), g^\flat(U)"] & B = \mathcal{O}_X(f^{-1}U) \arrow[r, "="'] & (f_*\mathcal{O}_X)(U)
\end{tikzcd}$$

Note that $\mathcal{O}_X(f^{-1}(U))$ has a $B-$algebra structure, therefore there is at most one map from the $B-$algebra $B_s = B[s^{-1}]$ to $\mathcal{O}_X(f^{-1}(U))$. Hence $f^\flat = g^\flat$.

(Surjectivity)

For any map $\phi : B \longrightarrow \mathcal{O}_X(X)$, need to find a map $(f, f^\#) : (X, \mathcal{O}_X) \longrightarrow (\mathrm{Spec}B, \mathcal{O}_{\mathrm{Spec}B})$.

We define f as following: Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$ and let $\mathfrak{q}$ be the prime ideal of B whose preimage is $\mathfrak{m}_x$. Let $y \in Y$denote the point corresponding to $\mathfrak{q}$. Define $f(x) = y$.

Check continuity: Let $D(s)$ be an open distinguished set in SpecB.

Suppose $x \in f^{-1}(D(s))$,

□

2.2.1. *Gluing of schemes.*

Definition 2.16. Let X_1, X_2 be two schemes, and let U_1, U_2 be open sets in X_1, X_2 respectively, such that there exists an isomorphism $\phi : U_1 \to U_2$. Then, the gluing scheme X of X_1, X_2 is defined as following:

Let $X = X_1 \cup X_2/\sim$ where the equivalence relation is defined to be $x_1 \sim \phi(x_1)$. X is therefore defined by the quotient topology.

Suppose i_1, i_2 are map from X_1, X_2 to X respectively, then for any open set V in X, $i_1^{-1}(V), i_2^{-1}(V)$ must be open in X_1, X_2 respectively.

We define the sheaf on X by:

$\mathcal{O}_X(V) : \{< s_1, s_2 > | s_1 \in \mathcal{O}_{X_1}(i^{-1}(V)), s_2 \in \mathcal{O}_{X_2}(i^{-1}(V)), \phi(s_1|_{i_1^{-1}(V)\cap U_1}) = s_2|_{i_2^{-1}(V)\cap U_2})\}$

Example 2.17. Let $X_1 = X_2 = \mathbb{A}^1_k$, let $U_1 = U_2 = \mathbb{A}^1_k \setminus \{P\}$, where P is the point corresponding to maximal ideal (x). Take ϕ as the identity map.

Then by gluing X_1, X_2 we have a double origin line.

This is a great example where scheme is not affine scheme.

2.2.2. *Fiber product of schemes.*

Definition 2.18.

Lemma 2.19. *Let $X = Spec\ A, Y = Spec B$ be two affine schemes over an affine scheme S. Let $S = Spec R$. Suppose A, B are both $R-$algebras. Then we have the isomorphism:*

$$Spec\ (A \otimes_R B) \simeq Spec A \times_S Spec B$$

Proof. For any scheme Z, to give a morphism from Z to Spec$(A \otimes_R B)$, it suffices to give a map from $A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$.

By the universal property of $A \otimes_R B$, it is equivalent to give a map from $A \times B$ to $\Gamma(Z, \mathcal{O}_Z)$, which is the same as to give maps from A, B to $\Gamma(Z, \mathcal{O}_Z)$ respectively. Hence, we have the maps from Z to X, Y respectively, which implies giving maps from Z to Spec$(A \otimes_R B)$..

□

Theorem 2.20. *For any two schemes X and Y over a scheme S, the fibered product exists and is unique up to isomorphism.*

Definition 2.21. Suppose $f : X \to S, T \to S$ are morphisms of schemes, then we write

$$X_T = \begin{tikzcd}
T \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r] & S
\end{tikzcd}$$

And say X_T is the base change with respect to $T \to S$.

Proposition 2.22. *(Vakil's magical diagram)*

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] & Y \times_S Z \arrow[d, "f \times g"] \\ X \arrow[r, "\Delta_{X/S}"] & X \times_S X \end{tikzcd}$$

This says that $Y \times_X Z$ *is the fiber product* $X \times_{X \times_S X} Y \times_S Z$

2.2.3. *Properties of schemes.*

Definition 2.23. Let $f : X \to Y$ be a morphism of schemes. The diagonal morphism is the unique morphism $\Delta : X \to X \times_Y X$ such that the projection maps $p_1, p_2 : X \times_Y X$ are exactly the identity maps. Say f is separated if the diagonal morphism is a closed immersion. In this case, we say X is separated over Y. Say a scheme X is seperated if X is separated over Spec$\mathbb{Z}$.

Example 2.24. Let k be a field, and let X be the affine line with double origin in 2.17. Then X is not separated over k.

Proposition 2.25. *If* $f : X \to Y$ *is a morphism of affine schemes,* f *is separated.*

Corollary 2.26. *An arbitrary morphism* $f : X \to Y$ *is separated if and only if the image of the diagonal morphism is a closed subset of* $X \times_Y X$

Theorem 2.27. *(Valuation Criterion of Separatedness)*

Let $f : X \to Y$ *be a morphism of schemes, and assume that* X *is noetherian. Then* f *is separated if and only if the following condition holds. For any field* K*, and for any valuation ring* R *with quotient field* K*, let* $T =$ *Spec*R*, let* $U =$ *Spec*K*, and let* $i : U \to T$ *be the morphism induced by the inclusion map* $R \subset K$*. Given a morphism* T *to* Y*, and given a morphism of* U *to* X *which makes a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \\ T \arrow[r] \arrow[ur, dotted] & Y \arrow[u, "f"'] \end{tikzcd}$$

there is at most one morphism from T *to* X *making the diagram commutative.*

Proof. We have to show

□

Exercise 2.28. Let A be a ring, $X = \text{Spec}(A)$, let $f \in A$ and let $D(f) \subset X$ be the distinguished base. Show that the ringed space $(D(f), \mathcal{O}_{X|D(f)})$ is isomorphic to Spec A_f.

Proof. Note the prime ideals of the localized ring A_f has a 1-1 correspondence with prime ideals that do not intersect with $\{f^n | n \in \mathbb{Z}\}$, i.e we have the correspondence

$$\{\text{Prime ideals of } A_f\} \leftrightarrow \{\text{Prime ideals that do not contain } f\}$$

So by definition we notice that

□

Exercise 2.29. Let $(X, \mathcal{O}_X)$

Proof. Since X is a scheme, we can cover X with with affine pieces U_i such that there exists A_i , $g_i : (\text{Spec}A_i, \mathcal{O}_{\text{Spec}A_i}) \simeq (U_i, \mathcal{O}_X|_{U_i})$.

Then consider the covering of U written as $\{U_i \cap U\}$, and since $U_i \cap U$ is open in U_i.

Note that the set $\{D(f)\}_{f \in A_i}$ form an open basis of SpecA_i, therefore $U_i \cap U \simeq \bigcup \text{Spec}(A_i)_{f_{ij}}$.

Then $\mathcal{O}_X|_U|_{\text{Spec}(A_i)_{f_{ij}}} = \mathcal{O}_{\text{Spec}(A_i)}|_{\text{Spec}(A_i)_{f_{ij}}}$. By 2.28, $\mathcal{O}_{\text{Spec}(A_i)}|_{\text{Spec}(A_i)_{f_{ij}}} \simeq \mathcal{O}_{\text{Spec}(A_i)_{f_{ij}}}$. □

Exercise 2.30.

Proof. (a) : If $(X, \mathcal{O}_X)$ is reduced, easy to see that $\mathcal{O}_{X,P}$ has no nilpotent elements. If $s^n = 0$, then $s_P^n = 0$

On the other hand, if for every $P \in X$, $\mathcal{O}_{X,P}$ has no nilpotent elements, by definition, $\mathcal{O}_{X,P} = \lim_{\rightarrow P \in U} \mathcal{O}_X(U)$. By shrinking U we can see that if $\exists$nilpotent elements$_P \in \mathcal{O}_{X,P}$, then, $s_P^n = 0$, Hence we can find a open neighborhood of s_P, such that

Since P is arbitrary, we can therefore obtain a cover of any open subset of X, there is a nilpotent element in every $\mathcal{O}_X(U)$.

(b) : Let $\mathcal{F} : U \mapsto \mathcal{O}_X(U)$ be the presheaf and $\mathcal{F}^+$ be the sheaf associated to it.

$(\mathcal{O}_X(U))_{\text{red}} = \mathcal{O}_X(U)/\text{nil}(\mathcal{O}_X(U))$ is an integral domain. Let $\mathfrak{a} := \text{nil}(\mathcal{O}_X(U)$.

For any point P and its neighborhood $U \subset X$, by the fact that X is a scheme, we can obtain some locally ringed space $(A, \text{Spec}A) \simeq (U, \mathcal{O}_X(U))$. Note that killing the nilradical also induced an isomorphism between Spec(A) and Spec$(A/\mathfrak{a}) := B$.

Hence the image of $D(f)$ in B is also a basis of B, hence we've have the affine pieces by 2.28. □

Exercise 2.31.

Proof. For $f \in \text{Hom}_{\mathfrak{Sch}}(X, \text{Spec}A)$, □

Exercise 2.32.

Proof. Spec $\mathbb{Z} = \{(0), (p)\}$, p is prime. $\mathcal{O}_{\text{Spec}\mathbb{Z}} := \{\}$

For any affine scheme X, suppose A is the corresponding ring of X, then there is only one map from a ring to the generator 1 of $\mathbb{Z}$, i.e., $\mathbb{Z}$ is an final object of the category **Rng.** Thus by 2.31 we can see Spec $(\mathbb{Z})$ is a final object in the category of schemes. □

Exercise 2.33.

Proof. The spectrum of the zero ring is $\varnothing$. Note that a zero ring is the initial object in the category of rings, hence its spectrum is the initial object in the category of schemes. □

Exercise 2.34.

Proof. If we have a morphism from Spec K to X, then the unique point $\{*\}$ maps to some point $x \in X$.

Suppose the maximal ideal in the local ring $\mathcal{O}_x$ is $\mathfrak{m}_x$, then by the bijection we established previously,

$$\text{Hom}[\Gamma(X, \mathcal{O}_X), K] \longrightarrow \text{Hom}[\text{Spec}K, X]$$

The map induces a homomorphism $F \in \text{Hom}[\Gamma(X, \mathcal{O}_X), K]$. Note that by the definition of $\mathcal{O}_{X,x} = \mathcal{O}_x = \varprojlim_{x \in U} \mathcal{O}_X(U)$, there is a map from $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$ to the local ring $\mathcal{O}_x$. Composite it with the natural projection to get the map from $\mathcal{O}_x/\mathfrak{m}_x = k(x)$ to K.

If we have an inclusion map $i : k(x) \rightarrow K$, we can lift i to $\tilde{i} : \mathcal{O}_x \rightarrow K$, with $\tilde{i}(\mathfrak{m}_x) = 0$, and if $t \notin \mathfrak{m}_x, \tilde{i}(t) = i(t)$.

Note that m_x is mapped to 0 in K, which corresponds to the zero ideal in Spec K.

Since X is a scheme, there is an affine open neighborhood U, consider the Zariski topology locally, then we have m_x corresponding to a point $s \in U$.

Hence if we map $\{*\}$ to s, we have a morphism in Hom$(\text{Spec}K, X)$ induced by i. □

Exercise 2.35. Describe Spec $\mathbb{R}[x]$. How does this topological space compare to $\mathbb{R}$.

Proof. The prime ideals in $\mathbb{R}[x]$ are (0); $(x - a)$; $(x^2 + cx + d), c^2 < 4d$. □

Exercise 2.36. Quasi-compactness:

(a): Show that a topological space is Noetherian iff every open subset is quasi-compact.

(b): If X is an affine scheme, show that $sp(X)$ is quasi-compact,but not in general Noetherian. Say a scheme is quasi-compact if $sp(X)$ is.

(c): A is Noetherian ring, then $\mathrm{Spec}(A)$ is Noetherian.

Proof. (a): Note that we prove in general that, if a topological space is Noetherian, then it is quasi-compact Suppose X is a noetherian topological space, then let $\cup U_i|_{i\in I}$ be a open covering of X. Consider the chain $U_1 \subset U_1 \cup U_2 \subset \cdots U_1 \cup U_2 \cup \cdots \cup U_n \subset \cup U_i|_{i\in I}$

Take n as an variable and by noetherian property, the chain stablizes at some N, hence $U_i|_{i\in I} = U_j|_{j=1}^N$. This indicates the finite sub-covering we've found.

Suppose $V \subset X$ is a quasi-compact,open subspace, then suppose $Z_1 \subset Z_2 \subset \cdots$ is an ascending chain of closed subsets in X.

Let $V_N := \bigcup_{1\leq i\leq N} Z_i$.

(b): Suppose X is an affine scheme, let $X = \mathrm{Spec}A$. Note that X has a basis of the form $D(f), f \in A$. □

2.3. Morphism of schemes.

Definition 2.37. A **closed immersion** of schemes is a map $i : Z \to X$, such that the image $i(Z)$ is homeomorphic to a closed subset of Y, and the assoicated map on sheaves $i^\sharp : \mathcal{O}_Y \to i_* O_X$ is surjective.

We have the following equivalent criterion for closed immersions, property $(5), (6)$ will be proven in the next section on quasi-coherent sheaves.

Proposition 2.38. *Let $i : Z \to X$ The following are equivalent:*

(1) : *i is a closed immersion;*

(2) : *For every affine open $U = \mathit{Spec}R \subset X$, there exists an ideal $I \subset R$ such that $i^{-1}(U) = \mathit{Spec}(R/I)$ as schemes over $U = \mathit{Spec}R$.*

(3) : *There exists an affine open covering $X = \bigcap j \in J, U_j = \mathit{Spec}(R_j)$ and for every $j \in J$, there exists an ideal I_j such that $i^{-1}(U_j) = \mathit{Spec}(R_j/I_j)$ as schemes over $U_j = \mathit{Spec}(R_j)$.*

(4) : *The morphism i induces a homeomorphism of Z with a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.*

(5) : *The morphism i induces a homeomorphism of Z with a closed susbet of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\ker(i^\sharp) \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals.*

(6) : *The morphism i induces a homeomorphism of Z with a closed susbet of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\ker(i^\sharp) \subset \mathcal{O}_X$ is a sheaf of ideals locally generated by sections.*

Definition 2.39. The reduced induced closed subscheme:

Let $X = \mathrm{Spec}A$ be an affine scheme, and let Y be a closed subset. Let $\mathfrak{a} \subseteq A$ be the ideal obtained by intersecting all the prime ideals in Y. This is the largest ideal for which $V(\mathfrak{a}) = Y$.

Proposition 2.40. ***Properties of morphisms of finite type***

(a) : *A closed immersion is a morphism of finite type.*

(b) : *A quasi-compact open immersion is of finite type*

(c) : *A composition of two morphisms of finite type is of finite type*

(d) : *Morphisms of finite type are stable under base change.*

(e) : *If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.*

(f) : *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is quasi-compact, and $g \circ f$ is of finite typr, then f is of finite type.*

(g) : *If $f : X \to Y$ is a morphism of finite type, and if Y is Noetherian, then X is Noetherian.*

Proof. (a) : Suppose $i : Y \to Z$ is a closed immersion, then there exists an affine open cover $\{V_i\}$ of Z, $V_i \simeq \mathrm{Spec}(A_i), i^{-1}(V_j) \simeq (\mathrm{Spec}(A_j/I_j))$ for some ideal $I_j \subseteq A_j$.

Note that A_j/I_j is finitely generated as an A_j-algebra, so by definition, the map is finite . Next, we show that a closed immersion is quasi-compact.

For any affine open subset V of Z, say $V = \mathrm{Spec}A$, then since the map is finite, by [HAR]Ex3.4, $i^{-1}(V) \simeq \mathrm{Spec}B$ for some ring B. Hence it is compact by [HAR]Ex3.2.

(b) : Recall that a map $f : X \to Y$ is quasi-compact if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i.

Now in this case, let $i : Y \to Z$ be an quasi-compact open immersion, then $i(Y) \simeq U \subseteq Z$, U is some open subset of Z. Let $\{V_i\}$ be the affine cover of Z satisfying the quasi-compact criterion. Suppose $V_i \simeq \mathrm{Spec}(A_i)$.

Since $i^{-1}(V_i)$ is open and quasi-compact in Y, we can find a finite open covering in Y of $i^{-1}(V_i)$, namely, write $i^{-1}(V_i) = \bigcup_{j=1}^{k} \mathrm{Spec}(B_{ij})$. Using the fact that i is an isomorphism, we can deduce that every B_{ij} is a finitely generated A_i-algebra

Note that since $i^{-1}U = Y \subset \cup i^{-1}(V_i)$, then it can satisfy the criterion of finite-type.

(c) : Let $f : X \to Y, g : Y \to Z$ be two morphisms of finite type, then we consider $g \circ f : X \to Z$.

We can find affine cover $\{V_i = \mathrm{Spec}A_i\}$ of Z and satisfying the finite-type criterion for g. Then we could deduce that for every $g^{-1}(V_i)$, it can be covered by finitely many affine subsets $\mathrm{Spec}A_{ij}, j = 1, \cdots, s$.

Now by [HAR] Ex3.3(b), f is of finite type implies that for every open affine subset $\mathrm{Spec}A_{ij}$ of Y, $f^{-1}(\mathrm{Spec}A_{ij})$ can be covered by finitely many affine open subset of X, denoted as $\mathrm{Spec}(A_{ijk}), k = 1, \cdots, t$. And the finitely generated algebra property inherits

Hence, there exists a affine cover $\{V_i\}$ of Z, such that there exists finite affine cover of finitely generated algebra for every preimage of V_i's.

(d) : Let $f : X \to S$ be morphism of schemes of finite type, and $g : S' \to S$ is any morphism of schemes. Then we consider the base change map $\pi_{S'} : X \times_S S' \to S'$.

Consider the affine case, let $X = \mathrm{Spec}A, S = \mathrm{Spec}R, S' = \mathrm{Spec}T$, then the fiber product $X \times_S S' = \mathrm{Spec}(A \otimes_R T)$. And by the finite type criterion, A is a finitely generated R algebra. Write $A \simeq R[x_1, \cdots, x_n]$.

So $A \otimes_R T \simeq R[x_1, \cdots, x_n] \otimes_R T$, this is a finitely generated T algebra by applying the homomorphism of rings $g^\sharp : R \to T$ associated to the morphism of shemes.

□

2.4. **Proj of a graded ring.** Let S be a graded ring. Write $S = \bigoplus_d S_d$, denote $S_+ = \bigoplus_{d>0} S_d$ called the

Consider the ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$, with the structure sheaf defined by the open sets $D_+(f) : \{\mathfrak{p} | f \notin \mathfrak{p}, f \in S_d, d \geq 1, S_+ \not\subset \mathfrak{p}\}$.

2.5. **Sheaves of Modules.**

2.5.1. *Construction of $\mathcal{O}_X-$modules.*

Definition 2.41. Let $(X, \mathcal{O}_X)$ be a ringed space, then say a sheaf $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$ module if for any open set U of X, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ module. For a subset $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is map compatible with the module structure via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Definition 2.42. Given a ring A and an A-module M, define a presheaf $\tilde{M}$ on X =Spec A by

$$\tilde{M}(D(f)) = M_f$$

Here M_f means the localized module.

Proposition 2.43. *Let $x \in X = \mathrm{Spec}A$, and let $\mathfrak{p} \subset A$ be the corresponding prime ideal in A, then $\tilde{M}_x = M_{\mathfrak{p}}$.*

Proof. By definition,

$$\varinjlim_{x \in D(f)} \tilde{M}(D(f)) = \varinjlim_{f \notin \mathfrak{p}} M_f = M_{\mathfrak{p}}$$

□

Definition 2.44. (Tensor product)

Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X-$modules, the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X} \mathcal{G}(U)$$

Definition 2.45. Direct and inverse image of $\mathcal{O}_X-$modules.

Recall that for a sheaf $\mathcal{G}$ on Y, we consider the presheaf defined by

$$f^+\mathcal{G} : U \mapsto \lim_{f(U)\subset V} \mathcal{G}(V)$$

and let $f^{-1}\mathcal{G}$ be the sheafification of it.

Now let $\mathcal{G}$ be an $\mathcal{O}_Y-$module, giving $f^{-1}\mathcal{G}$ an $\{^{-1}\mathcal{O}_Y-$module structure we can define

$$f^*\mathcal{G} := \mathcal{O}_X \otimes_{\mathcal{O}_{f^{-1}\mathcal{O}_Y}} f^{-1}\mathcal{G}$$

as the inverse image or the pullback of $\mathcal{G}$.

2.5.2. *Sections of sheaves of modules.* Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X-$modules, $s \in \mathcal{F}(X)$ is a global section, there exists a unique map of $\mathcal{O}_X-$modules

$$\mathcal{O}_X \to \mathcal{F}, f \mapsto fs$$

associated to s.

Definition 2.46. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X-$modules, say that $\mathcal{F}$ is generated by global sections if there exists a set I, and global sections $s_i \in \Gamma(X, \mathcal{F}), i \in I$ such that the map

$$\bigoplus_{i\in I} \mathcal{O}_X \to \mathcal{F}$$

which is the map associated to s_i on the summand corresponding to i , is surjective. In this case we say that the $s_i's$ generate $\mathcal{F}$.

Lemma 2.47. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X-$modules, and I be a set. $s_i \in \Gamma(X, \mathcal{F}), i \in I$ are global sections. The sections generate $\mathcal{F}$ iff for all $x \in X$, the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}-$module of $\mathcal{F}_x$.*

Proof. $\Rightarrow$: If the global sections s_i generate $\mathcal{F}$, by definition there is a surjective map $j : \bigoplus_{i\in I} \mathcal{O}_X \to \mathcal{F}$, consider $j_x : \bigoplus_{i\in I} \mathcal{O}_{X,x} \to \mathcal{F}_x$, by sheaf property, this is also surjective. Hence by taking stalks of s_i at x, we can deduce that $s_{i,x}$ generates the $\mathcal{O}_{X,x}-$module $\mathcal{F}_x$.

$\Leftarrow$: Note that by definition, j_x is surjective for every $x \in X$, also by sheaf property, it is surjective on global sections.

$\square$

Lemma 2.48. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}, \mathcal{G}$ be a sheaf of $\mathcal{O}_X-$modules generated by global sections, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is generated by global sections*

Lemma 2.49. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}, \mathcal{G}$ be a sheaf of $\mathcal{O}_X-$modules , and I be a set. $s_i \in \Gamma(X, \mathcal{F}), i \in I$ are collection of local sections, i.e., $s_i \in \mathcal{F}(U_i)$ for some opens $U_i \subset X$. There exists a unique smallest subsheaf of $\mathcal{O}_X-$modules $\mathcal{G}$ such that each s_i corresponds to a local section of $\mathcal{G}$.*

Proof. Consider the subpresheaf of $\mathcal{O}_X-$modules generated by:

$$U \mapsto \{\text{sums} \sum_{j\in J} f_j(s_j|_U), \text{where } J \text{ is finite}, U \subset U_j \text{ for } j \in J, f_i \in \mathcal{O}_X(U)\}$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf.

$\square$

Definition 2.50. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}, \mathcal{G}$ be a sheaf of $\mathcal{O}_X-$modules , say $\mathcal{F}$ is **locally generated by sections** if for every $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is globally generated as a sheaf of $\mathcal{O}_U-$module.

Namely, for every open subset U of X, and for each i a section $s_i \in \mathcal{F}(U)$ such that the associated map

$$\bigoplus_{i \in I} \mathcal{O}_U \to \mathcal{F}|_U$$

is surjective.

Definition 2.51. Let X be some scheme, then a quasi-coherent sheaf on X is a sheaf of $\mathcal{O}_X-$modules $\mathcal{M}$ such that there exists an open affine covering $\{U_i = \text{Spec}A_i\}$, and A_i- modules M_i such that $\mathcal{M}|_{U_i} \simeq \tilde{M}_i$.

Theorem 2.52. *1. The functor from $A-$modules to sheaves of $\mathcal{O}_X-$modules defined by $M \mapsto \tilde{M}$ is fully faithful.*

2. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X-$modules, and assume there exists a cover of X yb

Proof. We prove that for an A-module M and $\mathcal{N}$ a sheaf of $\mathcal{O}_X$ module,

$$\text{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{N}) \to \text{Hom}_A(M, \mathcal{N}(X))$$

is a bijection.

Injectivity: Let $\psi = \phi(U) = \phi(U')$, here U, U' are the principle open set $D(f), D(f')$.

Then the following diagram commutes. Therefore, $M_f = M_{f'}$ Hence $U = U'$.

$$\begin{tikzcd}
M \arrow[r, "res^X_U"] \arrow[d, "\psi"'] & M_f \arrow[d, shift right, "\phi(U)"'] \arrow[d, shift left, "\phi(U')"] \\
\mathcal{N}(X) \arrow[r, "res^X_U"'] & \mathcal{N}(U)
\end{tikzcd}$$

□

Definition 2.53. Say $\mathcal{F}_M$ is the sheaf associated to the module M and the ring map $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ if satisfy the properties above.

If $R = \Gamma(X, \mathcal{O}_X)$ and $\alpha = id_R$, we simply sat $\mathcal{F}_M$ is the sheaf associated to module M.

Proposition 2.54. *Let $(X, \mathcal{O}_X) = (\text{Spec}R, \mathcal{O}_{\text{Spec}R})$ be an affine scheme. Let M be an $R-$module. There exists a canonical isomorphism between the sheaf $\tilde{M}$ associated to the $R-$module and the sheaf $\mathcal{F}_M$ associated to the $R-$module M. The isomorphism is functorial in M. In particular, the sheaf $\tilde{M}$ is quasi-coherent. Moreover, they are characterized by the following mapping property*

$$Hom_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) = Hom_R(M, \Gamma(X, \mathcal{F}))$$

of any sheaf of $\mathcal{O}_X-$modules $\mathcal{F}$. Here a map $\alpha : \tilde{M} \to \mathcal{F}$ corresponds to its effect on the global sections.

Now we come back to what we've temporarily omitted in the last section

Proposition 2.55. *Recall $i : Z \to X$ is a closed immersion;*

(1) : *The morphism i induces a homeomorphism of Z with a closed susbet of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\ker(i^\sharp) \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals.*

(2) : *The morphism i induces a homeomorphism of Z with a closed susbet of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\ker(i^\sharp) \subset \mathcal{O}_X$ is a sheaf of ideals locally generated by sections..*

Definition 2.56. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X-$modules,

(1) : We say $\mathcal{F}$ is locally free if for every point $x \in X$, there is an open neighborhood $x \in U$ such that $\mathcal{F}_U \cong \bigoplus_{i \in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$ module.

(2) :We say $\mathcal{F}$ is finite locally free if we may choose the index sets I to be finite.

(3) : We say $\mathcal{F}$ is finite locally free of rank r if we may choose the index sets I to have cardinality r.

Proposition 2.57. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X-$modules, then if $\mathcal{F}$ is locally free, then it is quasi-coherent.*

Proof. (Sketch:) By definition of locally free, shrink the open neighborhood to some affine open neighborhood. □

Proposition 2.58. *The pullback of a locally free $\mathcal{O}_X-$module is locally free.*

Definition 2.59. An **invertible sheaf** on a ringed space X is locally free $\mathcal{O}_X-$module of rank 1.

Proposition 2.60. *If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves, then so is $\mathcal{L} \otimes \mathcal{M}$. For any invertible sheaf $\mathcal{L}$, there exists an invertible sheaf $\mathcal{L}^{-1}$, such that $\mathcal{L} \otimes \mathcal{L}^{-1} \simeq \mathcal{O}_X$.*

Proof. Check locally, for any $x \in X$, there exists open subsets of $x : U_{\mathcal{L}}, U_{\mathcal{M}}$, such that $\mathcal{L}|_{U_{\mathcal{L}}} \simeq \mathcal{O}_X|_{U_{\mathcal{L}}}, \mathcal{M}|_{U_{\mathcal{M}}} \simeq \mathcal{O}_X|_{U_{\mathcal{M}}}$, therefore $\mathcal{L}$ and $\mathcal{M}$ agree on $U_{\mathcal{L}} \cap U_{\mathcal{M}}$. Note that $\mathcal{O}_X \otimes \mathcal{O}_X \simeq \mathcal{O}_X$. Hence we could prove that $\mathcal{L} \otimes \mathcal{M}$ is also a $\mathcal{O}_X-$module of rank 1.

For the second statement, take $\mathcal{L}^{-1}$ to be the dual sheaf $\mathcal{L}^\vee : \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X)$, then $\mathcal{L} \otimes \mathcal{L}^\vee \simeq \mathrm{Hom}(\mathcal{L}, \mathcal{L}) = \mathcal{O}_X$. □

Definition 2.61. For any ringed space X, the **Picard group** $\mathrm{Pic}(X)$ is the group of isomorphism classes of invertible sheaves on X, under the operation $\otimes$. The above argument 2.60 shows the group property holds.

Lemma 2.62.

2.5.3. *Sheaves of graded modules.* Settings: Let $S = \bigoplus_d S_d$ be a $(\mathbb{Z}-)$graded ring, M is a graded $S-$module.

Proposition 2.63. *There is a natural map*

$$M_0 \to \tilde{M}(D(f))$$

Proof. Take $\frac{m}{1} \in M_{(f)} = \tilde{M}(D_+(f))$, since $\{D_+(f)\}$ is a basis of $\mathrm{Proj}(S)$, and $\tilde{M}$ agree on the intersection of $D_+(f), D_+(g)$ for any f, g. Therefore we can glue them together to a global section. □

Definition 2.64. Let S be a graded ring, $X = \mathrm{Proj}(S)$. Assume that S is generated by S_1 as an S_0-algebra. For any $n \in \mathbb{Z}$, we define the sheaf $\mathcal{O}_X(n) := \widetilde{S(n)} = \widetilde{S_{(n+d)}}$ and call $\mathcal{O}_X(1)$ the Serre twisted module.

For any sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$, define $\mathcal{F}(n) : \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}$.

Proposition 2.65. *Assume that S is generated by S_1 as an S_0-algebra. This is to say that S is generated by the degree 1 terms with coefficients in S_0. An example is $k[x]$ as a $\mathbb{Z}-$graded algebra. Then the following properties hold.*

(a) : *The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf on X.*

(b) : *For any graded $S-$module M, $\widetilde{M}(n) \simeq \widetilde{(M(n))}$. In particular, $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \simeq \mathcal{O}_X(n+m)$.*

(c) : *Let T be another graded ring, generated by T_1 as a T_0-algebra, let $\varphi : S \to T$ be a homomorphism of graded rings, and let $U \subset Y = \mathrm{Proj} T$ and $f : U \to X$ be the morphism determined by φ.*

Then $f^*(\mathcal{O}_X(n)) \simeq \mathcal{O}_Y(n)$ and $f_*(\mathcal{O}_Y(n)|_U) \simeq (f_*\mathcal{O}_U)(n)$.

Proof. (a) : By definition, for $\forall f \in S_1, \mathcal{O}_X(n)(D_+(f)) = \widetilde{S(n)}(D_+(f)) \simeq (S(n)_{(f)})_0 = (S_{(f)})_n$

We can obtain a morphism $\varphi : S_{(f)}$

Since S_1 generated S as an S_0 □

Lemma 2.66. *Let $S = \mathbb{Z}[T_0, T_1, \cdots, T_n]$, the scheme $\mathrm{Proj} = \mathbb{P}^n_{\mathbb{Z}}$ represents the functor which associates to a scheme Y, the pairs $(\mathcal{L}, (s_0, s_1, \cdots, s_n))$ where*

(1) : *$\mathcal{L}$ is an invertible $\mathcal{O}_Y-$module, and*

(2) : *$s_0, \cdots, s_n$ are global sections of $\mathcal{L}$ which generate $\mathcal{L}$.*

up to the following equivalence, $(\mathcal{L}, (s_0, \cdots, s_n)) \sim (\mathcal{N}, (t_0, \cdots, t_n)) \Leftrightarrow$ there exists a morphism $\beta : \mathcal{L} \to \mathcal{N}$ with $\beta(s_i) = t_i$

Proof. □

Definition 2.67. The scheme $\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[T_0, \cdots, T_n])$ is called a projective $n-$space over $\mathbb{Z}$. The base change of $\mathbb{P}^n_{\mathbb{Z}}$ to a scheme S is called a projective $n-$space over S.

Given a scheme Y over S and a pair $(\mathcal{L},(s_0,s_1,\cdots,s_n))$ as the 2.66 shows, and the induced morphism $\varphi_{\mathcal{L},(s_0,s_1,\cdots,s_n)}: Y \to \mathbb{P}^n_S$.

This is the $S-$morphism characterized by

$$\mathcal{L} = \varphi^*_{(\mathcal{L},(s_0,s_1,\cdots,s_n))}\mathcal{O}_{\mathbb{P}^n_S}(1) \text{ and } s_i = \varphi^*_{(\mathcal{L},(s_0,s_1,\cdots,s_n))}T_i$$

T_i are global sections of $\mathcal{O}_{\mathbb{P}^n_S}(1)$ via $S \to \bigoplus_{n\geq 0}\Gamma(X,\mathcal{O}_X(n))$.

Theorem 2.68. *Let k be a field, and let A be a finitely generated $k-$algebra, let X be a projective scheme over A, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X-$module. Then $\Gamma(X,\mathcal{F})$ is a finitely generated $A-$module.*

Corollary 2.69. *Let $f : X \to Y$ be a projective morphism of schemes of finite type over a field k. Let $\mathcal{F}$ be a coherent sheaf on X, then $f_*\mathcal{F}$ is coherent on Y.*

Proof. Only have to check locally on Y. Suppose $Y = \operatorname{Spec} A$, here A is a finite $k-$algebra. Note that $f_*\mathcal{F}$ is always quasi-coherent. By Theorem 2.68 , we can deduce that $\Gamma(X,\mathcal{F})$ is a finitely generated $k-$vector space. Note that $f_*\mathcal{F} = Y\widetilde{\Gamma(Y, f_*(Y))} \simeq \widetilde{\Gamma(X,\mathcal{F})}$. Hence $f_*\mathcal{F}$ is coherent. □

2.5.4. *Finite morphism into $\mathbb{P}^n_k$.*

Proposition 2.70.

2.5.5. *Line bundles.* Let X be a scheme, $\mathcal{L}$ be an invertible $\mathcal{O}_X-$module, suppose $s \in \Gamma(X,\mathcal{L}) = \mathcal{L}(X)$.

Define

$$X_s = \{s_x \in \mathcal{L}_x \text{ generates } \mathcal{L}_x \text{ as } \mathcal{O}_{X,x}-\text{modules.}\}$$

Checking locally, we have the following properties:

Proposition 2.71. *1. X_s is open, and $s|_{X_s} : \mathcal{O}_{X_s} \to \mathcal{L}_{X_s}$ is an isomorphism.*

2. $X_{s\otimes t} = X_s \cap X_t$, when $t \in \Gamma(U,\mathcal{M})$, $s\otimes t \in \Gamma(X,\mathcal{L}\otimes\mathcal{M})$, here $\mathcal{M}$ is the invert of $\mathcal{L}$.

3. $f : Y \to X$, $f^{-1}(X_s) = Y_{f^{-1}(s)}$.

4. For $s_i \in \Gamma(X,\mathcal{L}), i \in I, X = \bigcup X_{s_i} \Leftrightarrow \mathcal{O}_X^{I_i} \to \mathcal{L}$ is surjective.

Definition 2.72. Let X be a scheme, $\mathcal{L}$ be an invertible $\mathcal{O}_X-$module, say $\mathcal{L}$ is ample if

(1) : X is quasi-compact,

(2) : $\forall x \in X, \exists n \geq 0, s \in \Gamma(X,\mathcal{L}^{\otimes n})$ such that X_s is affine and $x \in X_s$.

Example 2.73. Important example: take $X = \mathbb{P}^n_A$, $\mathcal{O}_{\mathbb{P}^n_A}(1)$ is ample.

Lemma 2.74. *Let $f : Y \to X$ be an affine morphism, $\mathcal{L}$ be an ample invertible $\mathcal{O}_X-$mdoule, then $f^*\mathcal{L}$ is ample.*

Proof. Let $X = \bigcup X_{s_i}$ be an affine open cover of X, $s_i \in \Gamma(X,\mathcal{L}^{\otimes n_i}), n_i \geq 0$.

Then since f is affine, $f^{-1}(X_{s_i}) = Y_{f^*s_i}$ is also affine, covering Y. Hence by definition, we've proven the lemma.

□

Lemma 2.75. *Let X be a scheme, and $\mathcal{L}$ an ample invertible $\mathcal{O}_X-$mod, then there exists an affine morphism $f : X \to \mathbb{P}^n = \mathbb{P}^n_{\mathbb{Z}}$ for some $n > 0$. In particular, X is separated.*

Proof. Let $X = \bigcup_{i=0}^n X_{s_i}$ be an open affine covering, $s_i \in \Gamma(X,\mathcal{L}^{\otimes n_i})$, note that from the natural map

Take $N = \prod n_i$, then $t_i = s_i^{N/n_i} \in \Gamma(X,\mathcal{L}^{\otimes N})$ for all i, and $X_{t_i} = X_{s_i}$ is affine for all i.

Then consider for every $t_i, f_i : X_{t_i} \to \mathbb{P}^N$ is induced by $\mathbb{Z} \to \Gamma(X_{t_i},\mathcal{L}^{\otimes N})$. More precisely, t_i is invertible implies that there is a morphism from $X_{t_i} \to \operatorname{Spec}\mathbb{Z}[\frac{x_0}{x_t},\cdots\frac{x_N}{x_t}]$, which is exactly $\mathbb{P}^N$.

Note that on the overlaps, f_i, f_j agree, so we can glue them to a morphism.

□

Theorem 2.76. *Let X be a scheme, $\mathcal{L} \in Pic(X)$ be an ample invertible sheaf, $\mathcal{F} \in QCoh(X)$. Then there exists a surjective morphism*

$$\bigoplus_{i\in I} L^{-n_i} \twoheadrightarrow \mathcal{F}$$

Lemma 2.77. *Let X be a scheme, $\mathcal{L}$ be an invertible $\mathcal{O}_X-$module, $s \in \Gamma(X, \mathcal{L}), \mathcal{F} \in QCoh(X)$, then*

(1) : $t_1, t_2 \in \mathcal{F}(X)$, *such that* $t_1|_{X_s} = t_2|_{X_s}$, *and* X *is quasi-compact, then* $\exists n > 0$ *such that* $s^n t_1 = s^n t_2 \in \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}(X)$.

(2) : *If* X *is quasi-compact and quasi-separated,* $t \in \Gamma(X_s, \mathcal{F})$, *then* $\exists n > 0$, *such that* $s^n t$ *is the restriction to* X_s *of* $t' \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$

Proof. First we consider the affine case, let $X = \mathrm{Spec}(A)$, then $\mathcal{L} \simeq \mathcal{O}_X$. Hence in this case, $s \in \mathcal{O}_X(X)$ is a global section on an affine scheme. Then $\mathcal{F}(X_s) = (\mathcal{F}(X))_s$, therefore if $t_1|_{X_s} = t_2|_{X_s}$, we can clear the denominator by multiplying s to some degree n, such that $s^n t_1 = s^n t_2$.

Now back to the general case:

(1) : We write $X = \bigcup_{i=1}^n U_i$ to be a finite (from quasi-compactness) open affine covering of X, then on each affine piece, consider $t_{1,i} = t_1|_{U_i} \in \mathcal{F}(U_i), i = 1, 2$, $t_{1,1}|_{X_s \cap U_1} = t_{1,2}|_{X_s \cap U_2}$, by what we've shown in the affine case, we can deduce that there exists some $n_i > 0$, $s^{n_i} t_{1,1} = s^{n_i} t_{1,2}$.

Take $N = \prod_i n_i$, then $s^N t_{1,i} = s^N t_{2,i}$, by sheaf property, $s^N t_1 = s^N t_2$.

(2) :

□

Proposition 2.78. *Let S be a graded ring, $X = \mathrm{Proj}\, S$, it is finitely generated by S_1 as a S_0 algebra. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, then there is a natural isomorphism $\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$.*

Here $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$.

Proof.

Lemma 2.79. *Let $X = \mathrm{Spec}\, A$ be an affine scheme. Then for any sheaf of $\mathcal{O}_X$ module $\mathcal{F}$, $A-$module M, show that the following isomorphism holds:*

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$$

Proof of lemma:

Let $f \in \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$, then $f(X)$ gives a natural map in $\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F}))$.

Let $g \in \mathrm{Hom}_A(M, \Gamma(X, \mathcal{F}))$, then we can define $\tilde{g} : \tilde{M} \to \widetilde{\Gamma(X, \mathcal{F})}$.

There is an natural map of A_f-modules $\mathcal{F}(X)_f \to \mathcal{F}(D(f))$, induced from the restriction map. Hence we get a morphism of $\mathcal{O}_X$ modules. from $\widetilde{\Gamma(X, \mathcal{F})}$ to $\mathcal{F}$

One can check that this gives an isomorphism.

Back to the problem, by the lemma it suffices to give the image of a section of $\widetilde{\Gamma_*(\mathcal{F})}$ over $D_+(f)$. Such a section has the form $\frac{m}{f^d}, m \in \Gamma(X, \mathcal{F}(d))$ for some $d \geq 0$. We can regard f^{-d} as a section on $\mathcal{O}_X(-d)$ defined over $D_+(f)$, then we obtain a section of $\mathcal{F}$ over $D_+(f)$ by tensoring: $m \otimes f^{-d}$. This defines β.

Now suppose $\mathcal{F}$ is quasi-coherent. We apply Lemma 2.77, considering F as a global section of the invertible $\mathcal{L} = \mathcal{O}(1)$. Since we have assumed that S is finitely generated by S_1 as an S_0-algebra, we can find finitely many $f_0, f_1, \cdots, f_r \in S_1$, such that X is covered by open affines $D_+(f_i)$. Their intersections are also affine, and $\mathcal{L}_{D_+(f_i)}$ is free. Now we can apply 2.77 , this tells that $\mathcal{F}(D_+(f)) \simeq \Gamma_*(\mathcal{F})_{(f)}$. This implies the isomorphism.

□

2.6. **Divisors.**

2.6.1. *Effective Cartier Divisor.*

Definition 2.80. Let S be a scheme, (1) : A locally principal closed scheme of S is a closed subscheme whose sheaf of ideal is generated by a single element.

(2) : An effective Cartier divisor on S is a closed subscheme $D \subset S$, whose ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_S$ is an invertible $\mathcal{O}_S$ module.

(3) : The invertible sheaf $\mathcal{O}_S(D)$ associated to D is defined by

$$\mathcal{O}_S(D) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{I}_D, \mathcal{O}_S) = \mathcal{I}_D^{\otimes -1}$$

Lemma 2.81. *Let S be a scheme, $D \subset S$ be a closed subscheme. The following are equivalent:*
(1) : *The subscheme D is an effective Cartier divisor on S.*
(2) : *For every $x \in D$, there exists an affine open neighborhood $\mathrm{Spec}(A) = U \subset S$ of x such that $U \cap D = \mathrm{Spec}(A/(f))$ with $f \in A$ a non-zero divisor.*

Proof. (1) $\Rightarrow$ (2) : Let D be an effective Cartier divisor, then the □

Lemma 2.82. *Let X be a scheme, $D, C \subset X$ be effective Cartier divisor with $C \subset D$ and $D' = D + C$. Then there is a short exact sequence*

$$0 \to \mathcal{O}_X(-D)|_C \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$

2.6.2. *Scheme theoretic intersection.*

Definition 2.83. Let $X, Y \subset Z$ be closed

3. Cohomology Theory

3.1. Cohomology of projective schemes.

Definition 3.1. Čech complex

Lemma 3.2. *For any $\mathcal{F} \in \mathrm{Ab}(X)$, the complex $\mathcal{C}(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$, i.e., there is a natrual map $\epsilon : \mathcal{F} \to \mathcal{C}^0$ such that the sequence of sheaves*

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

is exact.

Proof. Define $\epsilon : \mathcal{F} \to \mathcal{C}^0$ by taking the product of the natural maps □

Lemma 3.3. *Assume $H^q(U_{i_0 i_1 \cdots i_p}) = 0$ for all $q > 0, i_0 < i_1 < \cdots < i_p \in I$, then there is the equality*

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(X, \mathcal{F})$$

Proof. We prove by induction on i.

□

Remark 3.4. This lemma can be also intepreted as: If there exists a finite open cover of X, for example if X is quasicompact, then the Čech cohomology coincides with the sheaf cohomology on the topological space X.

Corollary 3.5. *Let X be a separated scheme, $\mathcal{F}$ be a quasi-cohrent O_X module, then, by taking and affine cover $\mathcal{U}$ of X, we have*

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(X, \mathcal{F})$$

Lemma 3.6. *$\mathcal{F} \in \mathrm{Ab}(X), \xi \in H^p(X, \mathcal{F}), p > 0$. Then there exists a open cover of $X = \bigcup U_i$, such that $\xi|_{U_i} = 0$.*

Proof. Take a resolution of $\mathcal{F} : 0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$, and hence $\xi \in \ker(d^p) = \mathrm{Im}(d^{p-1})$.

By definition of an image sheaf, there exists an open cover $X = \bigcup U_i$, and $\xi_i \in \mathcal{I}^{p-1}(U_i)$, such that $d^{p-1}(\xi_i) = \xi|_{U_i}$. Hence $\xi|_{U_i} = 0$ in $H^p(U_i, \mathcal{F})$.

□

Lemma 3.7. *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence in $\mathrm{Ab}(X)$, suppose $\mathcal{F}$ is flasque. Then*
(1) : *$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$ is a short exact sequence for $U \subset X$ open.*
(2) : *If $\mathcal{G}$ is flasque too, then so is $\mathcal{H}$*

Proof. □

Lemma 3.8. *Let $\mathcal{U}$ be a open covering of X, then for each $p \geq 0$, there exists a natrual map, functorial in $\mathcal{F}$,*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$$

Proof. Let $0 \to \mathcal{F} \to \mathcal{J}$ be an injective resolution of $\mathcal{F}$. The Čech complex gives a resolution $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$, hence by a result in algebraic topology, we can find a morphism of two complexes, inducing the map on $\mathcal{F}$, unique up to homotopy.

Applying the functor $\Gamma(X,\cdot)$ and h^p, we get the required map. □

Example 3.9. Let R be a ring and $X = \mathbb{A}^n_R \setminus \{0\} = \operatorname{Spec}(R[T_1,T_2,\cdots,T_n]) \setminus V(T_1,T_2,\cdots,T_n)$. We want to calculate its $i-$th cohomology group.

Note that by 3.5,we only have to calculate the Čech cohomology.

Let $U_i = D(T_i)$, then $D(T_i) \cap D(T_j) = D(T_iT_j)$.

First we consider $n = 1$ case, then $\check{H}^1(\mathcal{U},\mathcal{F}) =$

Definition 3.10.

Theorem 3.11. ***(Theorem B)***

Let $X = \operatorname{Spec}(A)$ be an affine scheme, let $\mathcal{F} \in \operatorname{QCoh}(X)$, then for any $p \geq 1, H^p(X,\mathcal{F}) = 0$

Proposition 3.12. ***The projection formula***

Let $f : X \to Y$ be a morphism of schemes, let $\mathcal{F} \to \operatorname{QCoh}(X), \mathcal{E}$ is a locally free $\mathcal{O}_Y-$module. Then the formula

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} = f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E})$$

holds

Proof. □

Theorem 3.13. ***(Serre's Theorem)***

Let R be a Noetherian ring, X be a projective $R-$scheme. We denote the closed immersion as $i : X \hookrightarrow \mathbb{P}^N_k$.

(1) : *If $\mathcal{F} \in \operatorname{QCoh}(X)$, then $H^i(X,\mathcal{F}) = 0, \forall i > n$.*

(2) : *If $\mathcal{F} \in \operatorname{Coh}(X)$, then $H^i(X,\mathcal{F}(d))$ is a finite generated $R-$mod for $i \geq 0$.*

(3) : *If $\mathcal{F} \in \operatorname{Coh}(X)$, then $H^i(X,\mathcal{F}(d)) = 0$ for $i > 0$, d large enough.*

Definition 3.14. Let X be a projective scheme over k, define its **Euler characteristic** as

$$\chi(\mathcal{F}) = \sum_{i=1}^{n}(-1)^i H^i(X,\mathcal{F})$$

Exercise 3.15. **(Hilbert Polynomial)** c.f [Har] Ex5.2

Let X be a projective $k-$scheme, $\mathcal{O}_X(1)$ is the ample invertible sheaf over X, and suppose $\mathcal{F}$ is a coherent sheaf on X. Show that there exists some polynomial $P(t) \in \mathbb{Q}[t]$, such that $P(n) = \chi(\mathcal{F}(n))$ for all $n \in \mathbb{Z}$. Call P the Hilbert polynomial of $\mathcal{F}$ with respect to $\mathcal{O}_X(1)$.

Proof. Do induction on $\dim \operatorname{Supp} \mathcal{F} := s$.

If $s = 0$, then for any $x \in X, \mathcal{F}_x = 0$. Hence $\mathcal{F} = 0$. The statement is clear.

Claim: There exists $s \in \Gamma(\mathbb{P}^1_k, \mathcal{O}(1))$, such that the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}(1) \longrightarrow \mathcal{G} \longrightarrow 0$$

is exact, with the $\mathcal{F} \to \mathcal{F}(1)$ given by s, and $\dim \operatorname{Supp}(\mathcal{F}) - 1 = \dim \operatorname{Supp}(\mathcal{G})$.

With this claim, we can deduce that

$$\chi(X,\mathcal{F}(d)) + \chi(X,\mathcal{G}(d)) = \chi(X,\mathcal{F}(d+1))$$

Thus by induction $P(d+1) - P(d)$ is a polynomial in $\mathbb{Q}[t]$ of degree $\dim \operatorname{Supp} \mathcal{F} - 1$.

Now we prove the claim.

□

Definition 3.16. Let $X \subset \mathbb{P}^n_k$, say X is a global complete intersection if there exists $D_1,\cdots,D_c$ effective Cartier divisors, such that $X = D_1 \cap \cdots \cap D_c$ scheme theoretically and $\dim X + c = n$.

Lemma 3.17. *Let X be a complete intersection of $D_1,\cdots,D_c$, assume each D_i is defined with a regular section s_i that belongs to some line bundle $\mathcal{O}_{\mathbb{P}^n_k}(d_i)(\mathbb{P}^n_k)$. Then for any $i, 0 \leq i < c, s_{i+1}|_{D_1\cap\cdots\cap D_i} \in H^0(D_1 \cap \cdots \cap D_i, \mathcal{O}_{D_1\cap\cdots\cap D_i}(d_{i+1}))$ is regular.*

Theorem 3.18 (Bezout's Theorem). *Let $X \subset \mathbb{P}^n_k$ be a complete intersection of hypersurfaces $D_1,\cdots D_c$, there the degree of D_i are d_i, the $\deg(X) = d_1 \cdots d_c$.*

3.2. **Serre Duality Theory.**

3.2.1. *Duality for projective space $\mathbb{P}^n_k$..*

Lemma 3.19.

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}()$$

3.2.2. *Serre duality for Cohen-Macaulay schemes.*

Definition 3.20. Let X be a projective variety of dimension d. A sheaf ω_x is said to be a *dualizing sheaf* if

$$\mathrm{Hom}(X, \omega_x) \simeq H^d(X, \mathcal{F})^*$$

Theorem 3.21. *(Serre) Let X be a projective scheme over , and suppose X is Cohen-Macaulay of pure dimension d, then $\forall i \geq 0$, there exists a natural isomorphism*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega_x) \simeq H^{d-i}(X, \mathcal{F})^*$$

for any $\mathcal{F} \in \mathrm{Coh}(X)$.

Corollary 3.22. *Let X be a CM projective variety of pure dimension d, $\mathcal{F}$ a finite locally free $\mathcal{O}_X$−module. Then for $0 < i < d$,*

$$H^i(X, \mathcal{F}(-r)) = 0$$

for $r >> 0$.

Proof. By Serre duality, $H^i(X, \mathcal{F}(-r)) = H^{d-i}(X, \mathcal{F}^\vee \otimes \mathcal{O}_X(r) \otimes \omega_x) = 0$ by Serre vanishing. □

Remark 3.23. Note that when $\mathcal{F}$ is finite locally free,

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega_x) = H^i(X, \mathcal{F}^\vee \otimes \omega_x)$$