ON GALOIS REPRESENTATION

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Abstract. This is a term paper for MATH 254A during the Fall 2025 semester at UC Berkeley. The paper mainly focuses on Galois representations and its properties. We study the from both $\ell-$adic, $p-$adic side, and also use analytic tools such as $L-$function to characterize them.

1. Preliminaries

1.1. **Inverse limits and infinite Galois theory.**

1.1.1. *Profinite topology.* In this subsection, we denote $\mathcal{A}$ to be a category with arbitrary products.

Definition 1.1. Let $\mathcal{A}$ be a category (with arbitrary products) and let I be a directed set.

A family $(A_i)_{i\in I}$ is called an *inverse system* if for each pair $i \leq j$ in I, there exists a morphism $\varphi_{ij} : A_j \to A_i$ such that

(1) $\varphi_{ii} = id$
(2) For any triple $i \leq j \leq k$, $\varphi_{ki} = \varphi_{ji} \circ \varphi_{kj}$.

The inverse limite of an given system $A_\bullet$ is an object in $\mathcal{A}$

$$\varprojlim_{i\in I} A_i := \left\{ (a_i) \in \prod_{i\in I} A_i : \varphi_{ij}(a_j) = a_i \text{ for every pair } i \leq j \right\}$$

such that the natural projection $\varphi_i : A \to A_i$ is a morphism in $\mathcal{A}$ for every i.

Suppose $X_\bullet$ is a inverse system of topological groups. Let $X = \varprojlim_{i\in I} X_i$. If each X_i is a finite set and is endowed with discrete topology, we call X a *profinite set.* In this case, $X = \varprojlim_{i\in I} X_i \subset \prod_{i\in I} X_i$ is closed. Since $\prod_{i\in I} X_i$ is a product of compact spaces, it is compact, and for the closed subgroup X, it is compact too. In fact, it is totally disconnected.

If X_i are finite groups endowed with the discrete topology, the inverse limit is called a *profinite group.* Thus a profinite group is always compact and totally disconnected. As a consequence, all open subgroups if X are closed, and closed subgroup is open if and only if it has finite index.

1.1.2. *Infinite Galois theory.* Let K be a field and L (finite or infinite) Galois extension of K.

Let

$$\mathrm{Gal}(L/K) := \big\{ \sigma \in \mathrm{Aut}(L), \sigma(x) = x, \forall x \in K \big\}$$

Suppose $\mathscr{J}$ is the set of finite Galois extensions of K contained in L and order the set by inclusion. Easy to check that $\mathscr{J}$ is a directed set and $L = \bigcup_{K' \in \mathscr{J}} K'$.

As a consequece, we can study the inverse limit of objects over this directed set.

$$\gamma = (\gamma_{K'}) \in \varprojlim_{E\in\mathscr{J}} \mathrm{Gal}(K'/K) \text{ if and only if } \gamma_{K''}\,|_{K'} = \gamma_{K'} \text{ for } K'' \subset K' \in \mathscr{J}$$

Galois theory implies the isomorphism

$$\mathrm{Gal}(L/K) \simeq \varprojlim_{K'\in\mathscr{J}} \mathrm{Gal}(K'/K), \qquad \sigma \mapsto \sigma|_{K'} \tag{1}$$

is an isomorphism.

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From now on, we identify these groups via the isomorphism, and given discrete topology on each finite group $\mathrm{Gal}(K'/K)$, we get a profinite group $\mathrm{Gal}(L/K)$.

Our main object in this paper is the absolue Galois group.

Definition 1.2. (Absolute Galois group)

Let K be a field and $\overline{K}$ be its separable closure. The *absolute Galois group* of K is the profinite group

$$G_K := \mathrm{Gal}(\overline{K}/K)$$

1.2. **Higher ramification group.** At this point we summarize and recall some facts about higher ramification group with reference in [21] and [17].

A general set up here is that K is a local field with residue field k of characteristic p and the normalized valuation v_K. Let $\mathcal{O}_K$ be its valuation ring, $\mathfrak{m}_K$ be its maximal ideal, and let $U_K^I := 1 + \mathfrak{m}_K^i$.

Definition 1.3. Recall the definition of the $i-$th ramification group G_i. Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$

Let $s \in G, i \geq -1$ an integer, suppose $v_L(s(a) - a) \geq i + 1$ for all $a \in \mathcal{O}_L$, let G_i be the set of s satisfying the inequality. Then we call G_i the $i-$th ramification group.

By convention G_0 is also the inertia subgroup $I(L/K)$ and G_1 is called the *wild inertia subgroup*, denoted $P(L/K)$. Note that the invariant field of $I(L/K)$ is K^{ur} the maximal unramified extension of K in L, and the invariant field of $P(L/K)$ is K^{tame}, the maximal tamely ramified extension of K in L.

We recall two main properties in [17] without proof that will help us discover the profinite structures later on.

Proposition 1.4. *The map*

$$G_i \to U_L^i, s \mapsto s(\pi_L)/\pi_L$$

induces an injective homomorphism

$$\theta_i : G_i/G_{i+1} \hookrightarrow U_L^i/U_L^{i+1}$$

of groups which is independent of the choice of the uniformizer.

And moreover

(1) : *The group G_0/G_1 is cyclic of order prime to p, and is isomorphsic to a subgroup of the group of roots of unity $\mu(k_L)$ in the residue field of L.*

(2) : *G_1 is a $p-$group, and G_0 is the semidirect product of a cyclic group of order prime to p with a normal $p-$subgroup. .*

(3) : *If $L/M/K$ are finite extensions, and let $G = \mathrm{Gal}(L/K), G' = \mathrm{Gal}(M/K), N = e(L/K), N' = e(M/K)$.*

Note that one has commutative diagram:

$$\begin{tikzcd} G_0/G_1 \arrow[r, "\theta_0"] \arrow[d, "res"'] & \mu_N(k_L) \arrow[d, "\epsilon \mapsto \epsilon^{N/N'}"] \\ G_0'/G_1' \arrow[r, "\theta_0"] & \mu_{N'}(k_M) \end{tikzcd}$$

This is a key diagram for us to understand the characters of Galois representations.

1.3. **ℓ-adic Cyclotomic Characters.** Let K be a field of characteristic $\mathrm{char}K \neq \ell$. Let ζ_{ℓ^n} denote (a choice of) the ℓ^n-th root of unity in $\overline{K}$. Consider the one dimensional linear representation

$$\varepsilon_\ell : G_K \to \mathbb{Z}_\ell^\times$$

where $\sigma(\zeta_{\ell^n}) = \zeta_{\ell^n}^{\varepsilon_\ell(\sigma) \mod \ell^n}$.

ϵ_ℓ is called the *$\ell-$adic cyclotomic character*

2. The $p \neq \ell$ case: Grothendieck's Monodromy Theorem

In this section , we require K to be an $\ell-$adic field with finite residue field k of char $k = p$. Suppose $p \neq \ell$. Let $\overline{K}$ be the separable closure of K and $\overline{k}$ its residue ring.

Our goal for this section is to prove the $\ell-$adic Grothendieck's monodromy theorem, and describe the equivalence of categories between the finite dimensional $\ell-$adic Galois representation and finite dimensional Weil-Deligne representation.

2.1. **Structure of absolute Galois group.**

Let G_K be the absolute Galois group of K , and $G_k := \mathrm{Gal}(\overline{k}/k)$.

Write

$$K^{\mathrm{ur}} = \bigcup_{K'/K \text{finite unramified}} K', K^{\mathrm{tame}} = \bigcup_{K'/K \text{fintie tamely unramified}} K'.$$

to be the maximal unramified extension and the maximal tamely unramified extension of K contained in $\overline{K}$.

As usual, I_K is the inertia subgroup and P_K is the the wild inertia subgroup of G_K, both equipped with the profinite topology. By facts from algebraic number theory, $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{ur}}), P_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{tame}})$.

In fact, applying our earlier argument, we can extend proposition 1.4 to the profinite cases.

Lemma 2.1. *P_K is a pro$-p$ group,i.e., an inverse limit of finite $p-$groups.*

Proof. This follows directly from (2) of proposition 1.4 equipping profinite topology. ☐

If N is prime to p, the group $N-$th roots of unity $\mu_N(\overline{k})$ is cyclic of order N. In fact we have a canonical isomorphism

$$I_K/P_K \to \varprojlim_{(p,N)=1} \mu_N(\overline{k})$$

by diagram 1.4.

Let ϖ_K be the uniformizer of K, and we take a compatible sequence (ζ_m). By the above isomorphism, we can define a character (1-dimensional representation) of I_K/P_K :

$$t_\zeta : I_K/P_K \to \prod_{\ell \neq p} \mathbb{Z}_\ell,$$

with formula by

$$\frac{\sigma(\varpi_K)^{\frac{1}{m}}}{\varpi_K^{\frac{1}{m}}} = \zeta_m^{(t_\zeta(\sigma) \mod m)}$$

For $\sigma \in W_K$, $t_\zeta(\sigma\tau\sigma^{-1}) = \varepsilon(\sigma)t_\zeta(\tau), \varepsilon$ here is the cyclotomic character (recall 1.3). Write $t_{\zeta,\ell}$ the projection of t_ζ to $\mathbb{Z}_\ell$.

Remark 2.2. It is a central concern how a Frobenius element act on G_K by conjugation.

Let φ be a lift of geometric Frobenius element in G_k, q is the size of the residue field k. Then for any $x \in G_K$,

$$\varphi\sigma\varphi^{-1}(x) = \sigma^{q^{-1}}(x)$$

Also, applying $\varepsilon(\varphi) = q^{-1}$ to the character formula given above, we get

$$t_\zeta(\varphi\tau\varphi^{-1}) = q^{-1}t_\zeta(\sigma)$$

2.1.1. *Representation of Galois groups.* We define what a ($\ell-$adic) Galois representation is:

Definition 2.3. Let L be a $\ell-$adic field, and let V be a finite dimensional vector space over L. Let $\rho : G_K \to GL(V)$ be a continuous group homomorphism with respect to the $\ell-$adic topology on V. Then we call ρ an $\ell-$adic Galois representation.

We prove a useful lemma for general representation

Lemma 2.4. *Let V be a finite dimensional vector space over L, a finite extension of $\mathbb{Q}_\ell$. For any finite dimensional representation $\rho : G \to GL(V)$, there exists a lattice Λ of V, such that the action of G on Λ is stable, and thus a free $\mathbb{Z}_\ell-$representation of G. In particular, ρ factors though $GL(\Lambda)$.*

Proof. For any lattice Λ_0 of V, $g\Lambda_0 = \{g(x) \mid x \in \Lambda_0\}$, it is still a lattice by definition. Moreover, let $H = \{g(\Lambda_0) = \Lambda_0 \mid g \in G\}$ be the stabilizer of Λ_0. This is a open subset of G, hence G/H is finite.

Let

$$\Lambda = \sum_{g \in G} g(\Lambda_0)$$

This is a finite sum, and easy to check that Λ is a $G-$invariant lattice, and hence it is a $\mathbb{Z}_\ell-$representation of G.

If we denote $\{e_1, \cdots, e_n\}$ as the $\mathbb{Z}_\ell$ basis of Λ, then by definition of a lattice, it also serves as a basis of V over $\mathbb{Q}_\ell$, hence a $\mathbb{Q}_\ell-$representation of V factors thorugh a $\mathbb{Z}_\ell-$lattice. $\square$

2.2. **Construction of Weil group.** In the rest of the section we denote $q = \#k$ be the cardinality of the residue field k. Let $\mathrm{Frob}_K = \mathrm{Frob}_k \in G_k$ be the geometric Frobenius element, which is also a topological generator of $G_k \simeq \widehat{\mathbb{Z}}$.

Definition 2.5. We define the **Weil group** W_K with the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & I_K \arrow[r] & G_K \arrow[r] & G_k \arrow[r] & 0 \\
0 \arrow[r] & I_K \arrow[r] \arrow[u, equal] & W_K \arrow[r] \arrow[u, hook] & \mathrm{Frob}_k^{\mathbb{Z}} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

so that W_K is the subgroup of G_K consisting the elements that maps to an integral power of the Frobenius element in G_k.

Remark 2.6. The group W_K is a topological group, but its topology is NOT the subspace topology of G_K; rather, the topology is determined by assuming I_K is open, and has its usual topology.

In fact, the second row of the exact sequence splits, which implies that $W_K \simeq I_K \rtimes \langle \mathrm{Frob}_K \rangle$, where a geometric Frobenius element acts on I_K by $\mathrm{Frob}_K \, x \, \mathrm{Frob}_K^{-1} = x^q, \forall x \in I_K$.

With definition of Weil groups, the local class field theory could be stated as below:

Theorem 2.7. *Let W_K^{ab} denote the abelianization of W_K. Then there exists unique isomorphisms $\mathrm{Art}_K : K^\times \to W_K^{ab}$ such that*

(1) *If K'/K is finite extension, then $\mathrm{Art}_{K'} = \mathrm{Art}_K \circ N_{K'/K}$*
(2) *We have commutative diagram*

$$\begin{tikzcd}
K^\times \arrow[r, "\mathrm{Art}_K"] \arrow[d, "val_K"'] & W_K^{ab} \arrow[d] \\
\mathbb{Z} \arrow[r, "a \mapsto \mathrm{Frob}_K^a"] & \mathrm{Frob}_K^{\mathbb{Z}}
\end{tikzcd}$$

We turn our attention to studying representations of the Weil groups.

Definition 2.8. Let L be a field of characteristic 0. A representation of W_K over L is a representation on a finite dimensional $L-$vector space which is continuous and if L has the discrete topology, i.e., a representation with open kernel.

2.2.1. *General representations of Weil group.* We discuss semisimplicity of representations of Weil group in this part. First, we give a pure representation theory lemma on semisimplicity: Let $\rho : G \to GL(V)$ be a representation of G.

(1) If V is finite dimensional and $H \triangleleft G$ is a normal subgroup, then ρ is semisimple implies that $\rho|_H$ is semisimple.
(2) If $H < G$ is of finite index r, where r is invertible in the coefficient field k, then ρ is semisimple if and only if $\rho|_H$ is semisimple.

Then, we consider the semisimple properties of representations of Weil groups.

Lemma 2.9. *Let ρ be a representation of the Weil group W_K over a field L and suppose $\rho(I_K)$ is finite. WLOG, suppose that L is algebraically closed, then the following are equivalent:*

(1) *$\rho(\sigma)$ is semisimple for all $\sigma \in W_K$;*
(2) *$\rho(\varphi)$ is semisimple for (any) lift of Frobenius φ;*
(3) *ρ is a semisimple representation.*

Proof. Note that $\operatorname{char} L = 0$. We claim that for any $\sigma \in W_K$, there exists some positive interger n, such that $\rho(\sigma^n)$ likes in the center of $\rho(W_K)$. Since $\rho(I_K)$ is finite, let $\tau_1, \tau_2, \cdots, \tau_m \in I_K$ be the generators, let $\tau_0 := \varphi$ be a lift of Frobenius. Then consider the conjugation $\rho(\tau_i^{k_i}\sigma\tau_i^{-k_i})$. Note that $\tau_i^{k_i}\sigma\tau_i^{-k_i} \in I_K$, and by finiteness there exists some $\rho(\tau_i^{k_i})$ commutes with $\rho(\sigma)$. Let $N = k_0k_1\cdots k_m, \rho(\tau_i^N)$ commutes with $\rho(\sigma)$ for every i. Thus for any $\sigma \in W_K$, by the structure of W_K, there exists some integer n, n' such that $\rho(\sigma^{n'}) = \rho(\varphi^n)$, this shows that (1) and (2) are equivalent.

Now $\rho(\varphi)$ generates a subgroup of finite index in $\rho(W_K)$, by Lemma 2 (2), since $\operatorname{char} L = 0$, ρ is semisimple if and only if $\rho(W_K)|_{\langle\rho(\varphi)\rangle}$ is semisimple, hence if and only if $\rho(\varphi)$ is semisimple, which proved equivalence of (2) and (3). ☐

2.2.2. *Weil-Deligne representations.*

Definition 2.10. A *Weil-Deligne* representation of W_K on a finite dimensional $L-$vector space V is a pair (r, V) consisting of a representation $r : W_K \to GL(V)$, and an endomorphism $N \in \operatorname{End}(V)$, such that for all $\sigma \in W_K$,

$$r(\sigma)Nr(\sigma)^{-1} = (\#k)^{-v_K(\sigma)}N$$

where $v_K : W_K \to \mathbb{Z}$ is determined by $\sigma|_{K^{\mathrm{ur}}} = \mathrm{Frob}^{v_K(\sigma)}$

A Weil-Deligne representation (ρ, N) is $\varphi-$semisimple if ρ is semisimple, or equivalently, $\rho(\varphi)$ is semisimple. We also can define ρ^{ss} the semisimplification of ρ, then (ρ^{ss}, N) is also a Weil-Deligne representation, called the *$\varphi-$semisimplification* of (ρ, N).

Remark 2.11. (1) : Let (ρ, N) be a Weil-Deligne representation, then ρ is irreducible if and only if $N = 0$. This follows from the fact the $\ker N$ is W_K-stable.

(2) : A $\varphi-$semisimple Weil-Deligne representation is **not** a semisimple object in the category $\mathfrak{WD}_L$ of Weil-Deligne representation (over L). A semisimple object in a catgory is the direct sum of simple objects, which restricts $N = 0$ by our last remark.

2.3. **Proof of the Theorem.** Now we come to the heart of our section, proving the Grothendieck's Monodromy Theorem. The proof is given in [22] and also as an exercise in [9] Exercise 2.20, we summarize our own proof below:

Theorem 2.12. *(Grothendieck's Monodromy Theorem)*

Suppose $\ell \neq p, K/\mathbb{Q}_p$ is finite with residue field k of characteristic p. Suppose V is a finite dimensional L vector space, with L an algebraic extension of $\mathbb{Q}_\ell$. Fix $\varphi \in W_K$ as a lift of Frob_K *and a compatible system (ζ_m) of primitive roots of unity.*

If $\rho : G_K \to GL(V)$ is a continuous representation, and $\rho(I_K)$ is finite, then there exists a finite extension K'/K and a uniquely determined nilpotent operator $N \in \operatorname{End}(V)$ such that $\forall\sigma \in I_{K'}$,

$$\rho(\sigma) = \exp(Nt_{\zeta,\ell})(\sigma).$$

For all $\sigma \in W_K$, we have $\rho(\sigma)N\rho(\sigma)^{-1} = \#k^{-v_K(\sigma)}N$.

Proof. First, by 2.4, we see that there exists some G_K stable $\mathcal{O}_L$-lattice Λ, and note that representation $\rho : G_K \to GL_{\mathbb{Q}_\ell}(V)$ is equivalent to $\rho : G_K \to GL_{\mathbb{Z}_\ell}(\Lambda)$, and consider its mod $\ell-$reduction $\tilde{\rho} : G_K \to GL_{\mathbb{Z}_\ell}(\Lambda/\ell\Lambda)$.

Suppose K' is some extension of K such that $\tilde{\rho}(G_{K'}) = \{id\}$. Then $\rho(G_{K'}) = \{\tau(x) \equiv x \mod \ell \mid \tau = \rho(\sigma) \in GL(\Lambda)\}$. Hence $\rho(G_{K'})$ is a pro-ℓ subgroup. Therefore, it acts through a finite quotient, hence $G_{K'}$ is an open subgroup of G_K, which implies K' is a finite extension of K .

Note that taking a finite extension over K is equivalent to a finite indexed subgroup of I_K. Since $I_{K'}$ is a subgroup of $G_{K'}$, $\rho(I_{K'})$ is pro$-\ell$, therefore, the image $\rho(P_{K'})$ is still a pro $\ell-$group. However, by lemma 2.1 it must be trivial since $P_{K'}$ is pro$-p$, and p is prime to ℓ.

Hence $\rho\mid_{I_{K'}}$ factors through the representation $t_{\zeta,\ell} : I_{K'} \to \mathbb{Z}_\ell$. And $t_{\zeta,\ell}(I_{K'})$ is also a pro$-\ell$ group.

Let φ be the geometric Frobenius element $\varphi : x \to x^{\frac{1}{q}}$. Choose $\sigma \in I_{K'}$ such that $t_{\zeta,\ell}(\sigma)$ be the generator of $t_{\zeta,\ell}(I_{K'})$. We see that $\rho(\varphi)\rho(\sigma)\rho(\varphi^{-1}) = \rho(\sigma)^q$ from remark 2.2, so if λ is an eigen-value of $\rho(\sigma)$, so is λ^q.

We now claim that $\lambda^{q^k} = \lambda$ for some λ. If $\lambda \subset \overline{\mathbb{Q}_\ell}$ is of infinite order, then $\lambda, \lambda^q, \lambda^{q^2}, \cdots$ is a infinite sequence of eigen-values of $\rho(\sigma) \subset GL_{\mathbb{Z}_\ell}(\Lambda)$. Note that $\overline{\mathbb{Q}_\ell}$ is equipped with discrete topology, and the sequence must converge somewhere, hence it is of finite order. Note that since λ has finite order and is in a pro $\ell-$group, λ has degree of a power of ℓ.

Now we take some extension K'' of K', such that $\rho(I_{K''})$ is generated by σ^{ℓ^n}. Hence every eigen value of $\rho(\tau), \tau \in I_{K''}$ is 1. This is equivalent to saying that $\rho(I_{K''})$ is unipotent.

For all $\tau \in I_{K''}$, we claim that $N = \frac{\log(\rho(\tau))}{t_{\zeta,p}(\tau)}$ is the wanted unique nilpotent operator. It suffices to prove that $\log(\rho(\tau))$ is well defined which suffices to show that the expansion

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\rho(\tau)-1)^n}{n}$$

converges.

Since $\rho(\tau) \in \rho(I_{K''})$ is unipotent, by definition $(\rho(\tau)-1)^M = 0$ for large enough $M \in \mathbb{Z}$, therefore the expansion converges and it follows that $\log(\rho(\tau))$ is well defined.

Then N is our desired nilpotent operator. From the construction we can see it is unique. □

From the monodromy theorem, notice that every $\ell-$adic Galois representation could be associated with a Weil-Deligne representation. In fact, we have the equivalence of categories below:

Theorem 2.13. *There is an equivalence of categories*

$$\left\{\textit{Finite dimensional } L-\textit{reprsentation of } G_K\right\} \longleftrightarrow \left\{\textit{Weil-Deligne } L-\textit{reprsentation}\right\}$$

taking

$$\rho \mapsto (r, V, N), r(\tau) := \rho(\tau)\exp(-t_{\zeta,\ell}(\varphi^{-v_K(\tau)}\tau)N)$$

Proof. For simplicity, we fix a compatible system and write t instead of $t_{\zeta,\ell}$. Let $\mathrm{Rep}_L(G_K)$ denote the category of finite dimensional $L-$representations and let WD_L denote the category of finite dimensional Weil-Deligne representations.

First we show that (r, V, N) is a Weil-Deligne representation.

By definition, it suffices to compute

$$r(\tau)Nr(\tau)^{-1} = \rho(\tau)\exp(-t(\varphi^{-v_k(\tau)}\tau)N) \cdot N \cdot \exp(t(\varphi^{-v_k(\tau)}\tau)N) \cdot \rho(\tau)^{-1}$$

Since N is nilpotent, we could deduce $\exp(\log N) = N$. Then the formula above reduces to

$$\rho(\tau)\exp(-t(\varphi^{-v_k(\tau)}\tau)N) \cdot \exp(\log N) \cdot \exp(t(\varphi^{-v_k(\tau)}\tau)N) \cdot \rho(\tau)^{-1} = \rho(\tau)N\rho(\tau)^{-1}$$

Hence (r, V, N) is Weil-Deligne.

Fully faithful: Let (σ, U) be a continuous finite dimensional $\mathbb{Q}_\ell-$representation of W_K. Denote the triple (r_σ, U, N_σ) and (r_ρ, V, N_ρ) as the image of (σ, U) and (ρ, V) in Weil-Deligne category respectively.

Let $\phi : U \to V$ be a W_K-equivariant map,i.e.,

$$\phi \circ (\sigma(\tau)\exp(tN_\sigma)) = (\rho(\tau)\exp(tN_\rho)) \circ \phi$$

Hence uniqueness property of the nilpotent operator give that

$$N_\sigma \circ \phi = \phi \circ N_\rho$$

Therefore

$$\mathrm{Hom}_{\mathrm{Rep}(G_K)}((\sigma, U), (\rho, V)) \simeq \mathrm{Hom}_{\mathrm{WD}}((r_\sigma, U, N_\sigma), (r_\rho, V, N_\rho))$$

is bijective, hence it is fully faithful.

Essentially surjective: Let (r', V, N) be a Weil–Deligne representation, so that

$$r'(\sigma)Nr'(\sigma)^{-1} = q^{-v_K(\sigma)}N.$$

Write any $\sigma \in W_K$ uniquely as $\sigma = \varphi^{v_K(\sigma)}u, \qquad u \in I_K$.

Define a representation $\rho : W_K \to GL(V)$ by

$$\rho(u) = r'(u), \qquad u \in I_K,$$

and

$$\rho(\varphi) = r'(\varphi)\exp\big(t(\varphi)N\big).$$

Extend multiplicatively to all $\sigma \in W_K$ by

$$\rho(\sigma) = \rho(\varphi)^{v_K(\sigma)}\rho(u), \qquad \text{where } \sigma = \varphi^{v_K(\sigma)}u.$$

Now define

$$r(\sigma) = \rho(\sigma)\exp\big(-t(\sigma)N\big).$$

Using the identity

$$t(\varphi^{-v_K(\sigma)}\sigma) = t(\sigma) - v_K(\sigma)t(\varphi),$$

We can deduce that

$$r(\sigma) = r'(\sigma), \qquad \forall \sigma \in W_K.$$

Hence we've shown essential surjectivity. And by now we've finished the proof. ☐

We give some conventions of typical $\ell-$adic representations:

Definition 2.14. Let L be an $\ell-$adic field and V a finite dimensional $L-$vector space, $\rho : G_K \to GL(V)$ is an $\ell-$adic Galois representation.

(1) V is **unramified** or **has good reduction** if I_K acts trivially.
(2) V has **potentially good reduction** if $\rho(I_K)$ is finite, in other words, if there exists a finite extension K'/K inside $\overline{K}$, such that V as an $\ell-$adic Galois representation of K' has good reduction
(3) V is **semi-stable** if I_K acts unipotently, in other words, if the semisimplification of V has good reduction.
(4) V is **potentially semi-stable** if there exists a finite extension K' if K contained in $\overline{K}$ such that V is semi-stable as a representation of $G_{K'}$.

Remark 2.15. (4) is equivalent to the condition that there exists an open subgroup of I_K which acts unipotently or that the semisimplification has potentially good reduction.

2.4. **Examples of $\ell-$adic representations.** Now we study some examples of $\ell-$adic representations. We still assume that $\operatorname{char} k = p \neq \ell$ in this part

2.4.1. *Tate module of the multiplicative group $\mathbb{G}_m$.* Consider the exact sequence:

$$1 \longrightarrow \mu_{\ell^n}(\overline{K}) \longrightarrow \overline{K}^{\times} \longrightarrow \overline{K}^{\times} \to 1$$

where $\overline{K}^{\times} \to \overline{K}^{\times}$ is given by $a \mapsto a^{\ell^n}$, and here μ_{ℓ^n} is the group of $\ell - th$ root of unities.

As before, we define the **Tate module** of the multiplicative group $\mathbb{G}_m$:

$$T_\ell(\mathbb{G}_m) = \varprojlim_{n\in\mathbb{N}} \mu_{\ell^n}(\overline{K})$$

Therefore $T_\ell(\mathbb{G}_m)$ is a free $\mathbb{Z}_\ell$ module of rank 1. Fix a compatible sequence

$$t = (\xi_i), \xi_{n+1}^{\ell} = \xi_n, \xi_0 = 1, \xi_i \neq 1, \forall i \geq 1.$$

Then $T_\ell(\mathbb{G}_m) = \mathbb{Z}_\ell t$ with

$$a \cdot t = (\xi_n^{a_n}), an \in \mathbb{Z}, a \equiv a_n \mod \ell^n\mathbb{Z}_\ell$$

We recall the cyclotomic character of G_K

$$\chi : G_K \to \mathbb{Z}_\ell^{\times}.$$

We have $T_\ell(\mathbb{G}_m) = \mathbb{Z}_\ell(\chi)$ is a free $\mathbb{Z}_\ell$ representation of G_K of rank 1. We write

$$T_\ell(\mathbb{G}_m) = \mathbb{Z}_\ell(1), V_\ell(\mathbb{G}_m) = \mathbb{Q}_\ell(1) = \mathbb{Q}_\ell \otimes \mathbb{Z}_\ell(1)$$

We also define the *Tate twists* of these representations to be

$$\mathbb{Z}_\ell(r) = \mathbb{Z}_\ell(1)^{\otimes r} \text{ for } r \geq 0$$

and

$$\mathbb{Z}_\ell(-r) = \mathbb{Z}_\ell(-1)^{\otimes r} \text{ for } r \geq 0, \mathbb{Z}_\ell(-1) = \mathbb{Z}_\ell(1)^{\vee}$$

2.4.2. *The Tate module of an elliptic curve.* Let E be an elliptic curve over a field k, let ℓ be a prime not equal to $\operatorname{char} k$.

The group of $\ell-$torsion points:

$$E[\ell^n] := P \in E(\overline{k}) : \ell^n P = 0$$

Note that we define $f_n : E[\ell^{n+1}] \to E[\ell^n]$ to be $P \mapsto \ell P$

Then as in Example 2.4.1, we define the Tate module of the elliptic curve to be

$$T_\ell(E) = \varprojlim_{n} E[\ell^n]$$

For $\ell \neq \operatorname{char} k, E[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^2$, hence in this case $T_\ell(E) \simeq \mathbb{Z}_\ell^2$ is a free $\mathbb{Z}_\ell$ module of rank 2.

And our final example is a more general case of the previous two.

2.4.3. *The Tate module of an abelian variety.* We start by recalling what an abelian variety is:

Definition 2.16 (Group scheme)**.** Let S be a base scheme. A *group scheme over S* is a scheme G over S together with morphisms of S-schemes

$$m : G \times_S G \to G \quad \text{(multiplication)}, \qquad e : S \to G \quad \text{(identity)}, \qquad i : G \to G \quad \text{(inverse)},$$

satisfying the following group axioms in the category of S-schemes.

Definition 2.17 (Abelian variety)**.** Let k be a field. An *abelian variety over k* is a *group scheme* A over k such that:

(1) A is *proper* over k (the structure morphism $A \to \operatorname{Spec} k$ is proper),
(2) A is *smooth* over k,
(3) A is *geometrically connected* (connected after base change to an algebraic closure of k).

Let $\dim A = g$. Then:

(1) The $\overline{K}-$rational points $A(\overline{K})$ is an abelian group
(2) The ℓ^n-torsion group $A[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$ if $\operatorname{char} k \neq \ell$.

We define the Tate module for abelian varieteies

$$T_\ell(A) = \varprojlim A[\ell^n] \simeq (\mathbb{Z}_\ell)^{2g}$$

2.4.4. *Some historical remarks.* The examples above (especially the geometric ones) serve as an intuition and also introduction to the so called *Neron-Ogg-Shafarevich critierion* [2]. We give some very gentle and vague historical description on it, which could also be seen as a glimpse of (the importance of) $p-$adic Hodge theory.

To clarify this, we first give some definitions.

Definition 2.18. Let K be field, and let $\overline{K}$ be its separable closure of K. Let v be a discrete valuation of K, denote $\mathcal{O}_v$ its valuation ring. Take $\overline{v}$ as an extension of v in $\overline{K}$. We denote the inertia group and decomposition group of $\overline{K}$ with respect to $\overline{v}$ as $I(\overline{v})$ and $D(\overline{v})$.

Say T *is unramified at* v if $I(\overline{v})$ acts trivially on it. In other words, with the isomorphism

$$D(\overline{v})/I(\overline{v}) \simeq \mathrm{Gal}(\overline{k}/k)$$

$D(\overline{v})$ acts on T through its image $\mathrm{Gal}(\overline{k}/k)$

Definition 2.19. Let A be an abelian variety over K, v be a place. Say A has *good reduction at* v if there exists an abelian scheme A_v over $\mathrm{Spec}(\mathcal{O}_v)$, such that $A \simeq A_v \times_{\mathcal{O}_v} K$.

A beautiful theorem characterizing the good reduction in the $\ell \neq p$ case is:

Theorem 2.20. *Let A be an abelian variety. A has good reduction if and only if the $\ell-$adic representation space*

$$V_\ell = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(A)$$

is unramified at v.

In 1967, Ogg[16] proved the case when A is an elliptic curve. Later in 1968, Serre and Tate[22] used the result of Andre Neron [15] and proved the theorem extends for any abelian variety.

This is a striking result since it reveals that a Galois theoretic property can imply a geometry theorem. The Neron-Ogg-Shafarevich critierion is extremely useful in the study of abelian varieties.

However, this only works for the $p \neq \ell$ case. When $p = \ell$, the *unramifiness* breaks down so the proof no longer works. Grothendieck[11] generalized the criterion in Theorem 5.13:

Theorem 2.21. (Grothendieck) *Let A be an abelian variety over K and let ℓ be an arbitrary prime number. Then A has good reduction if and only if $A[\ell^n]$ admits an integral model $\mathcal{G}_n$ for all $n \geq 1$ with $\mathcal{G}_n = \mathcal{G}_{n+1}[\ell^n]$(respecting the $K-$fiber identification $A[\ell^n] = A[\ell^{n+1}][\ell^n]$) for all $n \geq 1$. In such cases, if $\mathcal{A}$ is the abelian scheme over $\mathcal{O}_K$ with $K-$fiber A, then necessarily $\mathcal{G}_n = \mathcal{A}[\ell^n]$ for all $n \geq 1$.*

Now we will step aside from the $\ell-$adic Galois representations and Weil-Deligne representations and put our attention on the case when $p = \ell$.

3. The $p = \ell$ case: $p-$adic Hodge theory

In this section, we want to consider the $p = \ell$ case and give a brief introduction to $p-$adic Hodge theory. In general, p-adic Hodge theory studies finite-dimensional $\mathbb{Q}_p$-representations of the absolute Galois group $G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$ of a p-adic field K by relating them to (semi-)linear-algebraic objects equipped with additional structures such as filtrations, Frobenius, and monodromy operators. Using Fontaine's period rings $B_{\mathrm{HT}}, B_{\mathrm{dR}}, B_{\mathrm{cris}}, B_{\mathrm{st}}$, one attaches to a representation V vector spaces

$$D_{\mathrm{HT}}(V) = (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \quad D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \ldots$$

with filtrations and extra structure. This leads to the classification

$$\text{Crystalline} \subset \text{Semi-stable} \subset \text{de Rham} \subset \text{Hodge-Tate},$$

and provides canonical comparison isomorphisms

$$B \otimes_{\mathbb{Q}_p} V \simeq B \otimes_K D(V),$$

which are compatible with the additional structures. Thus p-adic Hodge theory connects Galois representations with the "filtered (ϕ, N)-modules", giving a powerful tool in arithmetic geometry. It is impossible to construct the entire theory in such a limited space , so we will focus on the construction of period rings (especially the de Rham period ring), and on a typical example classifying $2-$dimensional crystalline representation. We mainly followed the notes of Oliver Brinon and Brian Conrad.[4].

3.1. **Witt rings and perfection.** Let K denote a $p-$adic field and we fix an algebraic closure K^{al}/K. As before, $G_K := \mathrm{Gal}(K^{al}/K)$. Let $\mathbb{C}_K$ be the completion of K^{al} and denote its ring of integers $A := \mathcal{O}_{\mathbb{C}_K}$, and residue field k. We always assume k is perfect in the following.

Before we construct our first period ring B_{dR}, we recall some properties of the Witt ring and also the tilting techniques.

Let $A^\flat := \varprojlim_p A/(p)$, as a result, $A^\flat$ is a strict $p-$ring, hence we could define $A_{\mathrm{inf}} := W(A^\flat)$, define the Fontaine map

$$\theta : A_{\mathrm{inf}} \to A, \theta[x] = x^{(0)}$$

As a standard fact, θ is a ring homomorphism, this was an exercise of Professor Shin's problem sets this semester. Scholze[18] also had them done neatly and concisely.

We now have a G_K-equivariant surjective ring homomorphism

$$\theta_{\mathbb{Q}} : A_{\mathrm{inf}}[\frac{1}{p}] \twoheadrightarrow A[\frac{1}{p}] = \mathbb{C}_K$$

Note that $\ker \theta_{\mathbb{Q}} = \ker \theta[\frac{1}{p}]$. We record some properties of $\ker \theta$.

Proposition 3.1. *Choose $\tilde{p} \in A^\flat$ such that $\tilde{p}^0 = p$,i.e., $\tilde{p} = (p, p^{\frac{1}{p}}, p^{\frac{1}{p^2}}, \cdots) \in A^\flat$, so $v(\tilde{p}) = 1$. Let $\xi = [\tilde{p}] - p = (\tilde{p}, -1, \cdots) \in A_{\mathrm{inf}}$.*

Then (a) : The ideal $\ker \theta \subset A_{\mathrm{inf}}$ is the principal ideal generated by ξ.

(b) : The element $w = (r_0, r_1, \cdots) \in \ker \theta$ is a generator of $\ker \theta$ if and only if $r_1 \in R^\times$.

Corollary 3.2. *For all $j \geq 1$,*

$$A_{\mathrm{inf}} \cap (\ker \theta_{\mathbb{Q}})^j = (\ker \theta)^j$$

Also, $\bigcap (\ker \theta)^j = \bigcap (\ker \theta_{\mathbb{Q}})^j = 0$.

3.2. **Construction of period rings and representations.** Fontaine [7] introduced the constructions of period rings $B_{\mathrm{dR}}, B_{\mathrm{st}}$, and B_{cris} in 1982 and gives the definitive treatment of these period rings in his paper [8].

To give some adequate notations and intuitions, we start with the Hodge-Tate representation/rings:

3.2.1. *Hodge-Tate.*

Definition 3.3. A $\mathbb{C}_K$ representation of G_K is a finite dimensional $\mathbb{C}_K-$vector space W equipped with a continuous G_K-action map $G_K \times W \to W$ which satisfy semilinearality, i.e., for $g \in G_K, c \in \mathbb{C}_K, w \in W, g(cw) = g(c)g(w)$. The category of such objects is denoted $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

For $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ and $q \in \mathbb{Z}$, we define:

$$W\{q\} := W(q)^{G_K} \simeq \{w \in W \mid g(w) = \chi(g)^{-q} w \text{ for all } g \in G_K\}$$

Lemma 3.4. ***(Serre-Tate)*** *For $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$, the natural $\mathbb{C}_K-$linear G_K-equivariant map*

$$\xi_W : \bigoplus_q (\mathbb{C}_K(-q) \otimes_K W\{q\}) \to W$$

is injective. In particular, $\sum_q \dim W\{q\} \leq \dim W$. The equality holds whenever ξ_W is an isomorphism.

With this lemma, we give the definition of a Hodge-Tate representation:

Definition 3.5. If ξ_W in Lemma 3.4 is an isomorphism, then W is Hodge-Tate.

Remark 3.6. We could see from the definition that if W is Hodge-Tate, then it is (non-canonically) isomorphic to $\bigoplus_q \mathbb{C}_K(-q)^{h_q}$, where $h_q = \dim W\{q\}$.

Note that for any object $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ that is Hodge-Tate, we can the *Hodge-Tate weights* of W are the integers $q \in \mathbb{Z}$, such that $W\{q\} := (\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} W)^{G_K}$ is nonzero.

Definition 3.7. The *Hodge Tate* ring of K is the $\mathbb{C}_K-$algebra $B_{\mathrm{HT}} := \bigoplus_{q\in\mathbb{Z}} \mathbb{C}_K(q)$ in which multiplication is defined as $\mathbb{C}_K(q)\otimes_{\mathbb{C}_K} \simeq \mathbb{C}_(q') = \mathbb{C}_K(q+q')$.

The Hodge-Tate representation serves as a quick taste to the $p-$adic representation world, our main focus will be on the constructions of period rings and their corresponding representation.

Remark 3.8. One may ask whether every $W \in \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{C}_K}(G_K)$ always admits a canonical decomposition

$$W \simeq \bigoplus_q \mathbb{C}_K(q) \otimes_{\mathbb{C}_K} W\{q\}.$$

In fact, this is a consequence of the Tate–Sen theorem[19], which we will not prove here, but it is useful to know that such a decomposition exists.

3.2.2. *de Rham.* We keep the notation as before, let $\mathbb{C}_K$ be the completion of the algebraic closure of K, let $A := \mathcal{O}_{\mathbb{C}_K}$ be its valuation ring, and let k be the residue field of $\mathbb{C}_K$. Write $W(k)$ as the Witt ring of k.

From Corollary 3.2, we construct

$$B^+_{\mathrm{dR}} := \varprojlim_j A_{\mathrm{inf}}[1/p]/(\ker\theta_{\mathbb{Q}})^j$$

whose transition maps are G_K-equivariant.

Choose $\varepsilon \in A^\flat$ with $\varepsilon^{(0)} = 1, \varepsilon^{(1)} \neq 1$, so $\theta([\varepsilon]-1) = \varepsilon^{(0)} - 1 = 0$. Hence $\varepsilon^{(0)} \in \ker\theta \subset \ker\theta^+_{\mathrm{dR}}$. We can therefore make use of the logarithm:

$$t := \log([\varepsilon]) = \log(1 + ([\varepsilon]-1)) = \sum_{n=1}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} \in B^+_{\mathrm{dR}}.$$

This lies in the maximal ideal of B^+_{dR}. A key property illustrating B^+_{dR} is as follows:

Proposition 3.9. *The element $t = \log([\varepsilon])$ in B^+_{dR} constructed above is a uniformizer of B^+_{dR}.*

We now give the definition of a *de Rham period ring:*

Definition 3.10. The *field of $p-$adic periods* or the *de Rham period ring* is

$$B_{\mathrm{dR}} := \mathrm{Frac}(B^+_{\mathrm{dR}})$$

equipped with natural G_K action and G_K-invariant filtration via the $\mathbb{Z}-$power of the maximal ideal of B^+_{dR}.

From the definition, we could proposition 3.9 we could see that B^+_{dR} (hence alsoB_{dR}) carries filtration structure. Discussing filtrations on rings and representations will be a key part in our later paragraphs

Filtered representations: Now we turn our attention to de Rham representations. Similar to the definition we gave in3.5, we define a covariant functor (Fontaine functor) $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \to \mathrm{Vect}_K$ by

$$D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

Hence $\dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p}(V)$, and when the equality holds, we say that V is a *de Rham representation.*

Recall that a filtered vector space over a field F is a pair $(D, \{\mathrm{Fil}^\bullet(D)\})$ where D is a finite dimensional $F-$vector space, and has the filtration structure $\mathrm{Fil}^i(D)$.

By our earlier construction and Proposition3.9, we get a natural filtration on B_{dR}, with $\mathrm{Fil}(B_{\mathrm{dR}}) = t^i B^+_{\mathrm{dR}}$. Explicitly, we have

$$\mathrm{Fil}^i(D_{\mathrm{dR}})(V) = (t^i B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

In fact, we could relate to the Hodge-Tate representation we mentioned before:

Proposition 3.11. *If V is a de Rham representation, then V is Hodge-Tate, and* $\mathrm{gr}(D_{\mathrm{dR}}(V)) = D_{\mathrm{HT}}(V)$ *as graded $\mathbb{C}_K-$vector spaces.*

Some nice properties of D_{dR} holds, we give two example as the end of this short introduction on de Rham peroid rings.

Proposition 3.12. *The faithful functor $D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \to \mathrm{Fil}_K$ is an exact functor, and it is compatible with tensors and duals.*

Proof. We give the proof for the first half. Tensor and dual compatibility theorem can be found in chapter 6 of [4].

For any exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

The sequence

$$0 \longrightarrow \mathrm{Fil}^i(D_{\mathrm{dR}}(V')) \longrightarrow \mathrm{Fil}^i(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{Fil}^i(D_{\mathrm{dR}}(V''))$$

is always left exact. In general it might not be right exact. But since now V is de Rham, by Proposition 3.11, all terms in the exact sequence at the top are immediately Hodge-Tate, so the functor D_{HT} applied to the exact sequence yields another exact sequence, By paasing it to the graded parts and calculate their dimensions,

$$\dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V)) = \dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V'')) + \dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V'))$$

and we get that the left exact sequence is also right exact.

□

Corollary 3.13. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $n \in \mathbb{Z}$, V is de Rham if and only if $V(n)$ is de Rham.*

Proof. By 3.12 and applying tensor compatibility, $V(n) \otimes V(-n) \cong V$. □

Proposition 3.14. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$, the G_K-equivariant $B_{\mathrm{dR}}-$linear comparison isomorphism*

$$\alpha : B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \simeq B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$$

respects the filtration and so does its inverse.

Proposition 3.15. *For any complete discretely-valued extension K'/K inside of $\mathbb{C}_K$ and any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, the natural map*

$$K' \otimes_K D_{\mathrm{dR},K}(V) \longrightarrow D_{\mathrm{dR},K'}(V)$$

is an isomorphism in $\mathrm{Fil}_{K'}$. In particular, V is de Rham as a G_K-representation if and only if V is de Rham as a $G_{K'}$-representation.

This property shows that being de Rham is compatible with replacing K with a nice extension K', and also $\widehat{K'^{\mathrm{ur}}}$.

Corollary 3.13 and proposition 3.15 gives an interesting example below:

Example 3.16. In the $1-$dimensional case, de Rham representations are actually equivalent to Hodge-Tate representations. We've shown that de Rham implies Hodge-Tate for any dimension, now for the converse suppose V is a $1-$dimensional Hodge-Tate representation. Therefore, it has some HT-weight i, if we twist V by $V(-i)$, we could reduce to the case when $\psi : G_K \to \mathbb{Z}_p^\times$ with HT wieght 0. Hence $\mathbb{C}_K(\psi)^{G_K} \neq 0$, by the Tate-Sen theorem, $\psi(I_K)$ is finite. Hence by choosing some appropriate ramified extension K'/K, we could have $\psi(I_{K'}) = 1$. Since de Rham representation is compatible with replacing K with $\widehat{K'^{\mathrm{ur}}}$, we could reduce to the case of the trivial character, which is de Rham.

3.2.3. *Crystalline.* All notations inherit from the previous sections.

Consider the G_K-stable $A_{\text{inf}}-$subalgebra

$$A_{\text{inf}}[\frac{\alpha^m}{m!}]_{m\geq 1,\alpha\in\ker\theta} = A_{\text{inf}}[\frac{\xi^m}{m!}]_{m\geq 1}$$

in $A_{\text{inf}}[\frac{1}{p}]$ generated by divided powers of $\ker\theta$. We denote this subalgebra by A^0_{cris}. Since it is a $\mathbb{Z}-$flat domain, if we define

$$A_{\text{cris}} = \varprojlim A^0_{\text{cris}}/p^n A^0_{\text{cris}}$$

to be the $p-$adic completion of A^0_{cris}, then A_{cris} is $p-$adically separated and complete.

We construct the crystallline period ring: Define the G_K-stable $A_{\text{inf}}[\frac{1}{p}]-$subalgebra to be

$$B^+_{\text{cris}} := A_{\text{cris}}[\frac{1}{p}] \subset B^+_{\text{dR}}$$

Then we have the crystalline period ring and the crystalline representations:

Definition 3.17 (crystalline period ring)**.** The crystalline peroid ring B_{cris} for K is the G_K-stable $A_{\text{inf}}[\frac{1}{p}]$ subalgebra $B_{\text{cris}}[\frac{1}{t}] = A_{\text{cris}}[\frac{1}{t}]$ inside $B^+_{\text{dR}}[\frac{1}{t}] = B_{\text{dR}}$.

Definition 3.18 (Crystalline representation)**.** A $p-$adic representation of G_K is *crystalline* if it is $B_{\text{cris}}-$admissible, and the full subcategory is defined as $\text{Rep}^{\text{cris}}_{\mathbb{Q}_p}(G_K.)$

As before, we define the Fontaine functor as $D_{\text{cris}} : \text{Rep}_{\mathbb{Q}_p}(G_K) \to \text{Vec}_{K_0}$.

This allows us to give the filtered structure on B_{cris},i.e., define

$$\text{Fil}^i B_{\text{cris}} = B_{\text{cris}} \cap \text{Fil}^i B_{\text{dR}}$$

We do have to be careful since the filtration is NOT $\phi-$stable! We require a (hard) property on the filtration of B_{cris}

Theorem 3.19. *The space* $(\text{Fil}^0 B_{\text{cris}})^{\phi=1} = \{b \in \text{Fil}^0(B_{\text{cris}}) \mid \phi(b) = b\}$ *of $\phi-$invariant elements in the $0-$th filtered piece of B_{cris} is equal to $\mathbb{Q}_p$.*

Many nice properties of crystalline rings are mentioned in [4] Chapter 9 which we won't summarize in this paper. Instead, we focus on explaining a crucial set up to our later classification of crystalline representations. This is a result given by Colmez and Fontaine[5].

Theorem 3.20 (Colmez-Fontaine)**.** *We define the covariant functor*

$$V_{\text{cris}} : \text{MF}^{\phi}_K \to \mathbb{Q}_p[G_K] - \text{Mod}$$

by $D \longmapsto \text{Fil}^0(B_{\text{cris}} \otimes_{K_0} D)^{\phi=1}$.

The exact tensor-functor $D_{\text{cris}} : \text{Rep}^{\text{cris}}_{\mathbb{Q}_p}(G_K) \to \text{MF}^{\phi}_K$ *is fully faithful. With the inverse essential image given by* V_{cris}.

This marks that every crystalline representation is associated with a weakly admissible filtered $\phi-$module, which somehow transferred our $p-$adic Galois representation problem to a semi-linear algebra problem.

We stop here for the discussion of period ring and turn our attention to some semilinear objects. We work with the field $K_0 := W(k)[\frac{1}{p}]$ in these sections.

Definition 3.21. An *isocrystal* over K_0 is a fintie dimensional K_0-vector space D equipped with a bijective Frobenius semilinear endomorphism $\phi_D : D \to D$.

Definition 3.22. Let K be a $p-$adic field. A *filtered $\phi-$module over K* is a triple $(D, \phi, \text{Fil}^\bullet)$ where (D, ϕ_D) is an isocrystal over K_0 and $(D_K, \text{Fil}^\bullet)$ is an object in Fil_K . We denote the category of filtered $\phi-$modules over K_0 by MF^{ϕ}_K .

3.2.4. *Semi-stable.*

Definition 3.23. As a $B_{\text{cris}}-$algebra with G_K-action,

$$B_{\text{st}}^+ := \text{Sym}_{\mathbb{Q}}(\text{Frac}(A^\flat)) \otimes_{\text{Sym}_{\mathbb{Q}}(R^\times)} B_{\text{cris}}^+$$

and the canonical G_K-equivariant homomorphism $\text{Frac}(A^\flat)^\times \to B_{\text{st}}^+$ via $h \mapsto h \otimes 1$.

Define the *semi-stable period ring* to be $B_{\text{st}} = B_{\text{cris}} \otimes_{B_{\text{cris}}^+} B_{\text{st}}^+ = B_{\text{st}}^+[\frac{1}{t}]$ with the G_K action.

As before, we define the faithful Fontaine functor

$$D_{\text{st}} : \text{Rep}_{\mathbb{Q}_p}^{\text{st}}(G_K) \to \text{MF}_K^{\phi,N}$$

is exact and compatible with duals, tensor products. Likewise, we also have the G_K-equivariant, Frobenius-compatible, and $N-$compatible semi-stable comparison isomorphism

$$\alpha : B_{\text{st}} \otimes_{K_0} D_{\text{st}}(V) \simeq B_{\text{st}} \otimes_{\mathbb{Q}_p} V$$

respecting the filtrations.

As in the crystalline case3.20, we deduce via the comparison isomorphism that:

Proposition 3.24. *The functor* $D_{\text{st}} : \text{Rep}_{\mathbb{Q}_p}^{\text{st}}(G_K) \to \text{MF}_K^{\phi,N}$ *is fully faithful, with quasi-inverse on its essential image given by* V_{st}.

We extend (generalize) our definition of filtered $\phi-$modules:

Definition 3.25. A $(\phi, N)-$*module* over K_0 is an isocrystal (D, ϕ_D) over K_0 equipped with a K_0-linear endomorphism $N_D : D \to D$ such that $N_D\phi_D = p\phi_D N_D$. The category of such modules is denoted $\text{Mod}_{K_0}^{\phi,N}$.

A *filtered* $(\phi, N)-$*module* over K is a $(\phi, N)-$module D over K_0, for which D_K is endowed with a structure of object in Fil_K. The category of such modules is denoted $\text{MF}_K^{\phi,N}$.

In fact, N_D is nilpotent , which follows from the lemma:

Lemma 3.26. *For any* $D \in \text{Mod}_{K_0}^{\phi,N}$, *the monodromy operator* N_D *is nilpotent on* D. *In particular, if* $\dim D = 1, N_D = 0$.

Remark 3.27. In fact, this definition could be somehow interpreted as the $p = \ell$ analogue of our Theorem2.12 in a semi-linear way.

In other words, recall our proof given in 2.12, let φ denote the the geometric Frobenius element in G_k and let N be the nilpotent operator. ε_ℓ is the $\ell-$adic cyclotomic character. Then

$$\varphi N \varphi^{-1} = \varepsilon_\ell(\varphi) N = q^{-1} N$$

This shows that $N\varphi = q\varphi N$, which enables us to think of the Definition3.25 given above. (In our proof, it suffices to take the logarithm on both sides of $\rho(\varphi)\rho(\sigma)\rho(\varphi)^{-1} = \rho(\sigma)^q$.)

To summarize our construction on these period rings and corresponding representations, we have the following relation of $p-$adic Galois representations:

$$\text{Crystalline} \Rightarrow \text{Semi-stable} \Rightarrow \text{de Rham} \Rightarrow \text{Hodge-Tate}$$

3.3. **Classification of lower dimensional semi-stable representation.** According to [4], this section could be considered as a "long linear algebra exercise". We will follow the steps that he gave and summarize a proof.

To be clear, we will focus mainly on the classification of 2−dimensional semistable representations, which is surprisingly equivalent to classifying the weekly admissible $(\phi, N)-$modules. The contribution made by Colmez and Fontaine[5] allows us to turn the problem into pure linear algebra work, and to black-box the precise properties of semi-stable/crystalline representations and the semi-stable/crystalline period rings.

3.3.1. *Some geometry.* Recall the by the end of last topic introducing Grothendieck's approach 2.21 to extend the Neron-Ogg-Shafareviech criterion to the $p = \ell$ case, we could formulate his methods by constructing the $p-$divisible group and the Dieudinné modules.

Definition 3.28. Let k be a perfect field of characteristic $p > 0$, and let $\sigma : W(k) \xrightarrow{\sim} W(k)$ be the Frobenius automorphism lifting the p-power map on k. The *Dieudonné ring* of k is the associative ring

$$\mathscr{D}_k = W(k)[F, V]$$

subject to the relations

$$FV = VF = p, \qquad Fc = \sigma(c)F \text{ for } c \in W(k), \qquad cV = V\sigma(c) \text{ for } c \in W(k).$$

(This is non-commutative when $k \neq \mathbb{F}_p$, and is $\mathbb{Z}_p[x, y]/(xy - p)$ when $k = \mathbb{F}_p$.)

Example 3.29. Let $K_0[\phi] = \mathscr{D}_K[1/p]$ ($\phi = \mathscr{F}$) be the twisted polynomial ring satisfying semilinear operation $\phi(c) = \sigma(c)\phi$ for $c \in K_0$. σ is the Frobenius element on K_0.

Consider a class of isocrystals:

$$D_{r,s} = K_0[\phi]/(K_0[\phi](\phi^r - p^s))$$

In this module, we have $\phi^r = p^s$, so for any eigenvalue λ of ϕ, we have $\lambda^r = p^s$, and hence $\mathrm{ord}_p(\lambda) = \frac{s}{r}$. In fact, the rational number $\frac{s}{r}$ is a *slope* of the isocrystal $D_{r,s}$. We will give a detained definition in the following sections.

Definition 3.30. Let F be a field and let $(D, \{D^i\})$ be a nonzero object in Fil_F. Let $\{i_0 < \cdots < i_n\}$ be the distinct i's such that $\mathrm{gr}^i(D) \neq 0$. The *Hodge polygon* $P_H = P_H(D)$ is the convex polygon in the plane that has leftmost endpoint $(0, 0)$ and has $\dim_F \mathrm{gr}^{i_j}(D)$ consecutive segments with horizontal distance 1 and slope i_j for $0 \leq j \leq r$.

The $y-$coordinate of the rightmost endpoint of $P_H(D)$ is the *Hodge number*

$$t_H(D) = \sum_{i \in \mathbb{Z}} i \cdot \dim_F \mathrm{gr}^i(D).$$

A visualized example of a Hodge polygon can be found in the appendix 2.

Let $\overline{k}$ be the algebraic closure of k, and let $K_0 := W(\overline{k})[1/p]$ and denote K_0^{ur} to be the maximal unramified extension of K_0. Let $\widehat{K_0^{\mathrm{ur}}}$ denote its completion. Then for any non-zero isocrystal D, we denote $\widehat{D} := D \otimes_{K_0} \widehat{K_0^{\mathrm{ur}}}$.

The *Dieudonné-Manin* [13] classification describes the possibilities for $\widehat{D}$.

Theorem 3.31. *For an algebraically closed field k of $\mathrm{char}\, k = p > 0$, the category $\mathrm{Mod}_{K_0}^{\phi}$ of isocrystals over $K_0 = W(k)[1/p]$ is semisimple. Here a category is semisimple means that all objects are finite direct sums of simple objects and all short exact sequences split. Moreover, the simple objects are given up to isomorphism by the isocrytsal $D_{r,s}$ with $\gcd(r, s) = 1$.*

Remark 3.32. The definition of isocrystal $D_{r,s}$ is mentioned in Example 3.29.

It follows from the theorem that if $k = \overline{k}$, then the simple objects of $\mathrm{Mod}_{K_0}^{\phi}$ are in bijection with $\mathbb{Q}$, since by its correspondence with $D_{r,s}$, we can map $(r, s) \mapsto s/r$. Moreover, by Theorem 3.31, ϕ is a semisimple operator. Applying eigenspace decomposition, ϕ can be diagonalized to a diagonal matrix with diagonal entries elements of $p-$adic order $\frac{r}{s}$. We write $\Delta_\alpha := D_{r,s}$, where $\alpha = \frac{r}{s}$. This is called the simple object with *pure slope* α in $\mathrm{Mod}_{K_0}^{\phi}$ when $k = \overline{k}$.

For any perfect field k of characteristic $p > 0$, and any isocrystal; D over K_0, the Dieudenné-Manin classification provides a unique decomposition for $\widehat{D} = \widehat{K_0^{\mathrm{ur}}} \otimes_{K_0} D$:

$$\widehat{D} = \bigoplus_{\alpha \in \mathbb{Q}} \widehat{D}(\alpha)$$

for subobject $\widehat{D}(\alpha) = \Delta_\alpha^{e_\alpha}$ having pure slope α, and $\widehat{D}(\alpha) = 0$ for all but finitely many α. If $\widehat{D}(\alpha) \neq 0$, then α is called a *slope* of D. If $\alpha = \frac{r}{s}$ with $\gcd(r, s) = 1$, then $\dim_{\widehat{K_0^{\mathrm{ur}}}} \widehat{D}(\alpha) = re_\alpha$ is called the *multiplicity* of this slope. Finally, say D is *isoclinic* if $D \neq 0$ and $\widehat{D} = \widehat{D}(\alpha_0)$ for some α_0.

3.3.2. *Hodge and Newton Polygon.*

Definition 3.33. Let D be a non-zero isocrystal over K_0 with slopes $\{\alpha_0 < \cdots < \alpha_n\}$ having multiplicities $\{\mu_0, \cdots, \mu_n\}$. The *Newton polygon* $P_N(D)$ of D is a convex polygon with is the convex polygon in the plane that has leftmost endpoint $(0,0)$ and has μ_i consecutive segments with horizontal distance 1 and slope α_i.

The $y-$coordinate of the rightmost endpoint of $P_N(D)$ is the *Newton number*

$$t_N(D) = \sum \alpha_i \dim \widehat{D}(\alpha_i)$$

With the monodromy operator defined in 3.25, it extends to incorporate

Definition 3.34. A filtered $\phi-$module D over K is *weakly admissible* if $t_N(D') \geq t_H(D')$ for all subobjects $D' \subset D \in \mathrm{MF}_K^{\phi}$, with equality when $D = D'$. Note that $t_H(D) = t_N(D)$ implies $P_H(D)$ and $P_N(D)$ have the same right endpoint.

And Definition 3.34 can extend to a (ϕ, N) module,

Definition 3.35. An object $D \in \mathrm{MF}_K^{\phi,N}$ is *weakly admissible* if $t_N(D') \geq t_H(D')$ for all $D' \subset D \in \mathrm{MF}_K^{\phi,N}$ with equality for $D' = D$. These objects constitute a full subcategory $\mathrm{MF}_K^{\phi,N,w.a.}$ of $\mathrm{MF}_K^{\phi,N}$.

Example 3.36. For $D \in \mathrm{MF}_K^{\phi,N}$, consider the isoclinic decomposition $D = \bigoplus_{\alpha \in \mathbb{Q}} D(\alpha)$ of the underlying isocrystal. By the definition of $D(\alpha)$, its scalar extension $\widehat{D}(\alpha)$ over K_0^{un} is spanned by vectors v such that $\phi_{\widehat{D}}^r(v) = p^s v$ for s/r the reduced form of α. Then

$$\phi_{\widehat{D}}^r(Nv) = p^{-r} N \phi_{\widehat{D}}^r(v) = p^{s-r} Nv.$$

But $(s-r)/r = \alpha - 1$, so $Nv \in D^{\mathrm{b}}(\alpha - 1)$. Hence, by descent from K_0^{un}, we get

$$N(D(\alpha)) \subseteq D(\alpha - 1).$$

From this example, we could deduce that $\oplus_{a \leq \alpha} D(a)$ is $N-$stable and N cuts down the slope by 1.

3.3.3. *Classification Process.* Now as we finishes all the setups, we start our proof on the theorem. Let $K = \mathbb{Q}_p$.

Here is the situation for the special case of $1-$dimensional representations:

Lemma 3.37. *The bijective correspondence $\eta \mapsto D_\eta$ from continuous unramified characters of G_K to isomorphism classes of 1-dimensional weakly admissible filtered (ϕ, N)-modules over K with $t_H = 0$ is the contravariant Fontaine functor*

$$D_{\mathrm{cris}}^* = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(\,\cdot\,, B_{\mathrm{cris}}).$$

That is, $D_{\mathrm{cris}}^*(\mathbf{Q}_p(\eta))$ is in the isomorphism class D_η.

The proof could be found in [4], we'll mainly focus on our $2-$dimensional case below.

For simplicticy, we define an *$i-$fold Tate-twist* of an isocrystal D to be $D\langle i \rangle$ whose underlying K_0 vector space is D, monodromy operator $N_{D\langle i \rangle} = N_D$, Frobenius operator $\phi_{D\langle i \rangle} = p^{-i}\phi_D$, and filtration structure over K is $\mathrm{Fil}^i(D\langle i \rangle) = \mathrm{Fil}^{i+r}(D)$.

Then from this defintion, we could deduce that:

$$t_H(D\langle i \rangle) = t_H(D) - i \dim D$$

and the same equality for Newton polygon.

Taking the $i-$fold Tate twist, we can get to the case where $\mathrm{Fil}^0(D) = D$ and $\mathrm{Fil}^1(D) \neq D$. Let $L := \mathrm{Fil}^0(D)$ be a line in D. Therefore, there exists an invariant r, such that for $1 \leq j \leq r$, we have $\mathrm{Fil}^j(D) = L$, and $\mathrm{Fil}^r(D) = 0$. By applying D_{st}^*, the gap between two Hodge-Tate weights is r. In particular, $t_H(D) = r$.

Now we consider the case when $N = 0$, corresponding to the crystalline case. As above, we could apply the Tate twists to make the smaller Hodge-Tate weight 0.

Theorem 3.38. *The set of isomorphism classes of 2−dimensional crystalline representations V of $G_{\mathbb{Q}_p}$ that have distinct Hodge-Tate weights $\{0, r\}$ with $r > 0$ and are not a direct sum of two characters is naturally parameterized by the set of quadratic polynomials $f(X) = X^2 + aX + b \in \mathbb{Z}_p[X]$, with $\operatorname{ord}_p(b) = r$, where f is the characteristic polynomial of φ on $D = D^*_{\text{cris}}(V)$.*

If f is irreducible then $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with $\operatorname{Fil}^j(D) = \mathbb{Q}_p(e_1 + e_2)$ precisely for $1 \leq j \leq r$ and $[\varphi] = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$. The crystalline Galois representation $V^*_{\text{cris}}(D)$ contravariantly associated to D is irreducible.

If f is reducible with distinct roots then $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with $\operatorname{Fil}^j(D) = \mathbb{Q}_p(e_1 + e_2)$ precisely for precisely for $1 \leq j \leq r$ and each e_i an eigen-vector for φ. If f is reducible with a repeated root λ (so $r \in 2\operatorname{ord}_p(\lambda) \in 2\mathbb{Z}^+$), then the same description holds, except that e_1 spans the λ-eigenspace and φ has the matrix $\varphi = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Before we start our proof, we need a key lemma, which enable us to deduce that the slopes of the polynomial f, which in other words the $p-$adic order of the two roots of f, coincides with the slopes defining the Newton polygon of D.

Lemma 3.39. *Let $T : D \to D$ be a linear automorphism of D of a finite-dimensional $\mathbb{Q}_p-$vector space. Let $K = W(k)[1/p]$ for a perfect field k with $\operatorname{char} k = p > 0$. By extending the Frobenius morphism semilinearly, we get an isocrystal structure on the finite dimensional $K-$vector space $K \otimes_{\mathbb{Q}_p} D$ via $\phi(c \otimes d) = \sigma(c) \otimes T(d)$.*

Then the slopes of ϕ are exactly the $\operatorname{ord}_p(\lambda)'s$, where λ ranges through the eigenvalues of T in $\overline{\mathbb{Q}_p}$, each occurring with multiplicity equal to its eigenvalue multiplicity for T.

Proof of Lemma. Recall that we defined

Let k'/k be any algebraically closed extension of k, not necessarily the algebraic closure of k. Let $K'_0 = W(k')[1/p]$, then the Dieudonné-Manin classification gives the unique decomposition of $\widehat{D'} = \oplus_{a\in\mathbb{Q}} \widehat{D'}(\alpha)$, with $\widehat{D'} := \widehat{K_0'^{\text{ur}}} \otimes_{K'_0} D'$, and subobjects $\widehat{D'}(\alpha) \simeq \Delta'^{e_\alpha}_\alpha$. Therefore, the definition of isocrytals is equivalent to extending the base field to some algebraically closed extension, hence we could reduce the case to $K = \mathbb{Q}_p$. Now $\widehat{D} := \widehat{\mathbb{Q}_p^{\text{ur}}} \otimes_{\mathbb{Q}_p} D$, and by the slope decomposition, we could reduce to the case when $\widehat{D}$ is *isoclinic*, i.e, when $\widehat{D} = \widehat{D}(\alpha) \simeq \Delta^{e_\alpha}_\alpha$ for some pure slope α. If we let $\phi^{-1} = \sigma^{-1} \otimes T^{-1}$ act on the simple objects in 2.4.1 $D_{r,s}$ then $\sigma^{-1}(D_{r,s}) = D_{-r,s}$, and T^{-1} do not change the $\pm-$signs, hence the sign of the slope α is inverted. So it suffices to consider the case when $\alpha \geq 0$. Write $\alpha = s/r$ in the reduced form with $r \geq 1, s \geq 0$. Consider the extension $\mathbb{Q}_p(p^{1/r})$ and the base change $\mathbb{Q}_p(p^{1/r}) \otimes_K D$, scaling the Frobenius by $(p^{1/r})^{-s} \otimes T$ we can change the slope of the original isocrystal to $\alpha - s/r = 0$. Therefore, we've reduced to the case when $\alpha = 0$.

Now we make a technical definition: define an isocrystal over a $p-$adic field to be *power-bounded* if there exists a $W(k)-$lattice $\Lambda \subset \Delta$ such that the sequence of $W(k)-$lattices $\{\phi^n(\Lambda)\}$ for $n \geq 0$ are contained in a common $W(k)-$lattice of Δ. We claim that the definition is well defined, i.e., if such a property holds for one $\phi-$stable lattice, then it holds for every one. In fact, by Dieudonné-Manin classification we first reduce the isocrystal to isoclinic, and suppose L is a $\phi-$stable power-bounded lattice. Any two lattices are isomorphic in an isoclinic isocrystal. This follows from that if we have L, L' being lattices of some simple isocrystal $D_{r,s}, r, s \in \mathbb{Z}$, then there exists some $n, p^n L' = L$. And since p is invertible in $D_{r,s}$, the two lattices are isomorphic. Hence for any isoclinic isocrystal $\Delta_\alpha = D_{r,s}^{\oplus m}$, any two lattices are also isomorphic. So in an isoclinic isocrystal, if one lattice is power-bounded, applying the isomorphism, any lattice is power-bounded.

When $\alpha = 0$, for any $\mathbb{Z}_p-$lattice $L \subset D$, family of lattices $\{T^n L\}$ is bounded for any $n \geq 0$. Hence all eigen-values of T are integral units in $\overline{\mathbb{Q}_p}$ by suitable scalar extensions (making T diagonalizable), and the order of the slopes are exactly 0, and if we reverse our reduction process, the slope of α are exactly $r/s = \operatorname{ord}_p(\alpha)$. And for general λ being the eigen-values of T, we've also finished the proof. □

Now, we prove our classification result for the crystalline case.

Proof. If D has Hodge-Tate weights of $\{0, r\}$, then by definition,

$$D \cong \mathbb{C}_K(-r) \oplus \mathbb{C}_K.$$

Recall that $L = \mathrm{Fil}^1(D)$, then the filtration on D is :

$$\mathrm{Fil}^i(D) = \begin{cases} 0, & i \geq r+1 \\ L = \mathbb{C}_K(r), & 1 \leq i \leq r \\ D, & i \leq 0. \end{cases}$$

Let $f(X) = X^2 + aX + b \in \mathbb{Q}_p[x]$ be the characteristic polynomial of φ acting on D, so $b \neq 0$. The condition $r = t_H(D) = t_N(D) = \mathrm{ord}_p(b)$ forces $b \in p^r\mathbb{Z}_p^\times$.

Case 1: If f is irreducible over $\mathbb{Q}_p$, then their roots in $\overline{\mathbb{Q}_p}$ has the same valuation and hence its valuation is strictly greater then 0 since $r \geq 1$. In fact, $a \in p^{[r/2]}\mathbb{Z}_p$. In this case there is no nontrivial subobjects of D, and in particular, $\varphi(L)$ is not contained in L. Thus if we choose a basis vector e_1 of L, $e_2 = \varphi(e_1)$ is linearly independent with e_1, and they together form a basis of D.

Note that the matrix of φ under the basis e_1, e_2 is $\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$ by simple computation, and $L = \mathbb{Q}_p e_1$ by our construction. According to our filtration formula above, the theorem holds for the irreducible case.

Case 2: If f is reducible, with roots λ_1, λ_2. Then $f(X) = (X - \lambda_1)(X - \lambda_2)$, and without loss of generality, suppose $\mathrm{ord}_p(\lambda_1) \geq \mathrm{ord}_p(\lambda_2)$. By Lemma 3.39, $\mathrm{ord}_p(\lambda_1) + \mathrm{ord}_p(\lambda_2) = t_N(D)$, hence by weekly admissibility,

$$\mathrm{ord}_p(\lambda_1) + \mathrm{ord}_p(\lambda_2) = 0 + r = r.$$

(1) First, assume the roots are distinct, let e_i be an eigen vector of λ_i, then $D \simeq \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$. Hence the only nontrival subobjects of D are $\mathbb{Q}_p e_1, \mathbb{Q}_p e_2$ the two lines. By weakly admissible assumption, we have $t_H(e_i) \leq t_N(e_i) = \mathrm{ord}_p(\lambda_i)$.

The filtration structure on $\mathbb{Q}_p e_i$ is either a line that is equal to $L = \mathrm{Fil}^1(D)$, or is just 0. As we assumed that $\mathrm{ord}_p(\lambda_1) \leq \mathrm{ord}_p(\lambda_2)$, by our equality $\mathrm{ord}(\lambda_1) + \mathrm{ord}\,(\lambda_2) = r > 0, \mathrm{ord}_p(\lambda_1) < r$. If the filtration structure on $\mathbb{Q}_p e_1$ is equal to L, $t_H(\mathbb{Q}_p e_1) = r$. Contradiction!. Hence $t_H(\mathbb{Q}_p e_1) = 0$. Now it remains to check whether $t_H(\mathbb{Q}_p e_2) = L$ or not.

If $L \neq \mathbb{Q}_p e_2$, then since L is a $1-$dimensional filtered subobject of D, by rescaling e_1, e_2 we get $L = \mathbb{Q}_p(e_1 + e_2)$. Then it remains to check that this is in fact the only crystalline representation.

Note that by weakly admissibility of D, we have $t_H(\mathbb{Q}_p e_i) \leq t_N(\mathbb{Q}_p e_i)$. And since $\lambda_1, \lambda_2 \in \mathbb{Q}_p^\times$ are distinct eigenvalue of the operator $\varphi, \mathrm{ord}_p(\lambda_1) \leq \mathrm{ord}_p(\lambda_2)$ and $\mathrm{ord}_p(\lambda_1) + \mathrm{ord}_p(\lambda_2) = r$.Hence $\mathrm{ord}_p(e_2) > 0$. Note that $t_H(\mathbb{Q}_p e_2) = 0 < t_N(\mathbb{Q}_p e_2)$, so $\mathbb{Q}_p e_2$ is never weekly admissible. Thus the $L = \mathbb{Q}_p e_2$ case will not happen. Hence the only possibility for a weakly admissible suboject to exists is $\mathbb{Q}_p e_1$, which is equivalent to $\mathrm{ord}_p(\lambda_1) = 0$.

Therefore, we've obtained all crystalline representation of $G_{\mathbb{Q}_p}$ with distinct weights.

(2) If $\lambda_1 = \lambda_2 = \lambda$, then $2\,\mathrm{ord}_p(\lambda) = r$. If φ is a scalar, i.e., $\lambda-$eigenspace is 2-dimensional, then $L = \mathrm{Fil}^1(D)$ would be a subobject and $t_H(L) = r = 2\,\mathrm{ord}_p(\lambda)$, whereas $t_N(L) = \mathrm{ord}_p(\lambda) < t_H(L)$, which contradicts with the weekly admissibility. Hence the $\lambda-$eigenspace must be $1-$dimensional. We choose an eigenvector e_1 of λ, and let e_2 be an eigenvector such that $L = \mathbb{Q}_p(e_1 + e_2)$ as the argument before.

Therefore by linear algebra, φ must be strictly upper triangular, thus by Jordan decomposition, it has the matrix

$$\varphi = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We have uniquely determined the filtration on D.

□

Finally, we consider the $2-$dimensional semi-stable $G_{\mathbb{Q}_p}$ representations that are not crystalline, i.e., $N_D \neq 0$.

Proposition 3.40. *The non-crystalline semistable* 2*-dimensional representations* V *of* $G_{\mathbb{Q}_p}$ *with smallest Hodge–Tate weight* 0 *are parameterized as follows: there is a Hodge–Tate weight* $r > 0$ *of* *the form $r = 2m+1$ with $m > 0$, and V is parameterized (up to isomorphism) by a pair (λ, c) with $\lambda \in p^m\mathbb{Z}_p^\times$ and $c \in \mathbb{Q}_p$.*

For a given (λ, c), the contravariant filtered (φ, N)-module $D = D^*_{\mathrm{st}}(V)$ is $D = \mathbb{Q}_pe_1 \oplus \mathbb{Q}_pe_2$ with N and φ as in (8.3.3), and $\mathrm{Fil}^j(D) = \mathbb{Q}_p(ce_1 + e_2)$ for $1 \leq j \leq 2m+1$.

There is a unique nontrivial subobject $D' = \mathbb{Q}_pe_1$, with $t_H(D') = 0$ and $t_N(D') = m$.

In particular, D' is weakly admissible iff $m = 0$, i.e. iff $\lambda \in \mathbb{Z}_p^\times$.

Proof. Since 0 is the smallest Hodge-Tate weight, $\mathrm{Fil}^0(D) = D$, and $\mathrm{Fil}^1(D) \neq D$. We claim that $\mathrm{Fil}^1(D) \neq 0$.

If not, then the Hodge polygon of D will be a straight line, and by the week admissibility, so will the Newton polygon. Recall the action of the nilpotent operator N on an simple object $N(D(\alpha)) \subset D(\alpha - 1)$ by Example 3.36, so it reduces the slope by 1. But the slope is larger than 0 as we've claimed that 0 is the smallest Hodge-Tate weight. So the monodromy operator must vanish, $N = 0$. But this is no longer our "non-crystalline" case. Hence, $\mathrm{Fil}^i(D)$ is equal to a line L on D.

Let r be the unique integer such that $\mathrm{Fil}^j(D) = L$ for $1 \leq j \leq r$ and $\mathrm{Fil}^j(D) = 0$ for $j > r$. Let $m = \mathrm{ord}_p(\lambda_1) = \mathrm{ord}_p(\lambda_2) - 1$. By weakly admissibility, we have $r = t_H(D) = t_N(D) = \mathrm{ord}_p(\lambda_1) + \mathrm{ord}_p(\lambda_2) = 2m + 1$.

By Example 3.36 again we see that N carries the λ_2 line to the λ_1 line, and therefore we can choose some eigenvector e_2 such that $\varphi(e_2) = \lambda_2e_2$ and define $e_1 = N(e_2)$. Note that by definition of the nilpotent operator

$$p\lambda_1e_1 = p\varphi(e_1) = N\varphi(e_2) = p\varphi N(e_2) = \lambda_2N(e_2) = \lambda_2e_1.$$

This forces $\lambda_2 = p\lambda_1$.

To conclude, there exists some $\lambda \in p^m\mathbb{Z}_p^\times$, and under the basis $\{e_1, e_2\}, D = \mathbb{Q}_pe_1 \oplus \mathbb{Q}_pe_2$, the nilpotent operator $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \varphi = \begin{pmatrix} \lambda & 0 \\ 0 & p\lambda \end{pmatrix}$.

It remains to discuss the possibilities of the line L. By the $(\phi, N)-$stability, the only nontrivial subobject in the line L is $\mathbb{Q}_pe_1$. If $L = \mathbb{Q}_pe_1$, we know that $t_N(\mathbb{Q}_pe_1) = \mathrm{ord}_p \lambda_1 = m < r$ always holds, and in this case $t_H(\mathbb{Q}_pe_1) = 2m + 1 = r$, $\mathbb{Q}_pe_1$ is never weakly admissible.

Thus $L = \mathbb{Q}_p(se_1 + e_2), s \in \mathbb{Q}_p$, s is unique. In this case, since $L \neq \mathbb{Q}_pe_1, t_H(\mathbb{Q}_pe_1) = 0 \leq m = t_N(\mathbb{Q}_pe_1)$. The weak admissibility implies that there are no restrictions on s and this is all the possibilities.

□

Remark 3.41. In our proof 3.3.3, we claimed that we could "rescale" $ae_1 + e_2$ to $e_1 + e_2$. This is because if we let $e_1' = ae_1, \varphi(e_1') = \phi(a)\lambda_1e_1 = \frac{\phi(a)}{a}\lambda_1(ae_1)$ is still an eigenvector. Hence we can always regulate the coefficient to $e_1 + e_2$. But for the (ϕ, N) case, the nilpotent operator is not preserved under random scaling:

$$\phi(a)\lambda_2e_2 = N\varphi(e_1') = p\varphi N(e_1') = p\varphi N(ae_1) = p\phi(a)\lambda_2e_2$$

However, if we replace the initial choice of e_2 with a $\mathbb{Q}_p^\times$ multiple, then $e_1 = N(e_2)$ is scaled the same way so the fixed s does not change.

Now we've finished the classification of $2-$dimensional crystalline representations, a natural question to ask is what does this classification indicate and the reason why we shed light on this interesting example.

In fact, this relates to the crystalline modularity lifting theorem:

Theorem 3.42. *Let $p \geq 5$ be a prime. Let*

$$\rho : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{Q}_p)$$

be a continuous representation, unramified outside finitely many primes, and let

$$\bar{\rho} : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{F}_p)$$

be its semisimplified reduction.

Assume:

(1) (Oddness) $\det(\rho(c)) = -1$ *for any complex conjugation* $c \in G_{\mathbf{Q}}$.

(2) (Residual modularity) $\bar{\rho}$ *is absolutely irreducible and modular, i.e. there exists a cuspidal eigenform* f *such that*
$$\bar{\rho} \cong \bar{\rho}_f.$$

(3) (Crystalline at p) *The restriction* $\rho|_{G_{\mathbf{Q}_p}}$ *is crystalline with Hodge–Tate weights*
$$\mathrm{HT}(\rho|_{G_{\mathbf{Q}_p}}) = \{0, k-1\}$$
for some integer $k \geq 2$.

(4) (Regularity) *The Hodge–Tate weights are distinct.*

(5) (Big image) *The image of* $\bar{\rho}$ *contains* $\mathrm{SL}_2(\mathbf{F}_p)$.

Then there exists a cuspidal modular eigenform f *of weight* k *such that*
$$\rho \cong \rho_f.$$

I want to thank CJ Dowd for introducing me to this theorem in a discussion. This finishes our discussion on $p-$adic Hodge theory.

4. Artin $L-$functions, Brauer's Theorem and Galois representations

4.1. **Adeles and Ideles.**

4.1.1. *Topologies.* The main reference here is [10].

Definition 4.1. A **global field** F is a field which is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$ for some prime p. Global fields are called **number fields** when they are extensions over $\mathbb{Q}$ and **function fields** when they are extensions over $\mathbb{F}_p(t)$.

Definition 4.2. A **place** of a global field F is an equivalence class of absolute values. A place is (non-)Archimedean if it consists of (non-)Archimedean absolute values.

Definition 4.3. (Adeles)
Let F be a global field. The ring of **adeles** of F, denoted $\mathbb{A}_F$, is the restricted direct product of the completions of F_v with respect to the valuation ring $\mathcal{O}_{F_v}$.

More explicitly,
$$\mathbb{A}_F = \left\{ (x_v) \subset \prod_v F_v : x_v \in \mathcal{O}_{F_v} \text{ for all but finite } v \right\}.$$

If S is a non-empty finite set of places, write
$$\mathbb{A}_F^S = \prod_{v \notin S} F_v := \left\{ (x_v) \subset \prod_v F_v : x_v \in \mathcal{O}_{F_v} \text{ for all but finite } v \notin S \right\}.$$

And
$$F_S : \mathbb{A}_{F,S} = \prod_{v \in S} F_v$$

Thus we may identify $F_S \times \mathbb{A}_F^S = \mathbb{A}_F$. For finite set of places $S' \subset S$, we set
$$F_S^{S'} := \prod_{v \in S - S'} F_v.$$

We build the topology as follows: The open sets of $\mathbb{A}_F$ should be of the form
$$U_S \times \prod_{v \in S} U_v$$
where S in a finite set of places of F including the infinite places and $U_S \subset F_S$ is an open set. For simplicity, we use the notation
$$\widehat{\mathcal{O}_F} := \prod v \notin S \mathcal{O}_{F_v}$$

Proposition 4.4. *The adele ring $\mathbb{A}_F$ is locally compact.*

Proposition 4.5. *The image of F in $\mathbb{A}_F$ under the diagonal embedding is discrete. The quotient $\mathbb{A}_F/F$ is compact.*

Now we consider the multiplicative group of $\mathbb{A}_F$. Denoted $\mathbb{A}_F^{\times}$.

$$\mathbb{A}_F^{\times} = \left\{ (x_v) \subset \prod_v F_v : x_v \in \mathcal{O}_{F_v}^{\times} \text{ for all but finite } v \right\}$$

This group is called **idele** of the global field F.

4.2. **Aside from Number Theory: Brauer's Theorem.** In this section, we prove a purely representation theorem inspired by Brauer's paper [3] in 1947. The theorem pays off when we try to argue the meomorphic continuity of Artin $L-$functions in the later section. We mainly follow the proof given in [20].

Definition 4.6. Let G be a finite group, $x \in G$. Take p as a prime number. Say x is $p-$**unipotent** if x has order of a power of p, and say x is p**-regular** if its order is prime to p.

A group H is said to be $p-$elementary if it is a direct product of a cyclic group C of order prime to p with a $p-$group P.

Recall that if G is a finite group, and $\chi_1, \chi_2, \cdots, \chi_h$ be its distinct irreducible characters. Let $R(G)$ be the finite abelian group generated by these distinct characters. In other words,

$$R(G) = \mathbb{Z}\chi_1 \bigoplus \mathbb{Z}\chi_2 \bigoplus \cdots \bigoplus \mathbb{Z}\chi_h$$

If H is a subgroup of G, then restriction of representations defines a ring homomorphism $R(G) \to R(H)$, denoted by Res_H^G.

Similarly, the induction of representations defines a homomorphism $R(H) \to R(G)$, denoted by Ind_H^G.

Note that there is a natural linear extension of Ind and the Res maps for a commutative ring A:

$$A \otimes \mathrm{Res}_{\mathrm{H}}^{\mathrm{G}} : A \otimes R(G) \to A \otimes R(H)$$
$$A \otimes \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} : A \otimes R(H) \to A \otimes R(G)$$

Let $X(p)$ be the family of $p-$elementary subgroups of G. Let V_p be the image of the homomorphism

$$\mathrm{Ind} : \bigoplus_{H \subset X(p)} \to R(G)$$

Serre gives a road map in [20], we follow the steps there but omit some details.

Let $A = \mathbb{Z}[\zeta_n]$ be the subring of $\mathbb{C}$ generated by $n-$th root of unity,here ζ_n is a primitive $n-$th root of unity.

Lemma 4.7. *The image of $A \otimes \mathrm{Ind}$ is $A \otimes V_p$, and moreover, $(A \otimes V_p) \cap R(G) = V_p$.*

Lemma 4.8. *Each class function on G with integer values divisible by n is an $A-$linear combination of characters induced from characters of cyclic subgroups of G.*

Proof. Let f be such a function, then one can write $f = n \cdot \chi$, here χ is a class function with integer values. Let C be a cyclic subgroup of G, we define a function

$$\theta_C = \begin{cases} |C|, & x \text{ generate } A, \\ 0, & \text{otherwise} . \end{cases}$$

Then, by Prop.27 in[20]

$$n = \sum_C \mathrm{Ind}_C^G(\theta_C)$$

Hence

$$f = n\chi = \sum_C \mathrm{Ind}_C^G(\theta_C)\chi = \sum_C \mathrm{Ind}_C^G(\theta_C \cdot \mathrm{Res}_C^G \chi).$$

By definition, the value of $\chi_C = \theta_C \cdot \mathrm{Res}_C^G \chi$ is divisible by $|C|$, then for any character ϕ of C

$$\langle \chi_C, \psi \rangle = \sum_{g \in C} |C| \cdot \psi(g).$$

□

Lemma 4.9. *Let χ be an element of $A \otimes R(G)$ with integer values, let $x \in G$, and let x_r be the $p'-$component of x. Then:*

$$\chi(x) \equiv \chi(x_r) \mod p$$

Lemma 4.10. *Let x be a $p'-$element of G, and let H be a $p-$elementary subgroup of G associated with x. Then there exists a function $\psi \in A \otimes R(H)$, with integer values, such that the induced function $\psi = \mathrm{Ind}_H^G \psi$ has the following properties:*

(1) $\psi'(x) \not\equiv 0 \mod p$
(2) $\psi'(s) = 0$ *for each $p'-$element of G which is not conjugate to x.*

Lemma 4.11. *There exists an element ψ of $A \otimes V_p$ with integer values, such that $\psi(x) \not\equiv 0 \mod p$ for each $x \in G$.*

Theorem 4.12. *Let G be a finite group and V_p be the subgroup of $R(G)$ generated by characters induced from those of $p-$elementary subgroups of G. Then the index of V_p in $R(G)$ is finite and prime to p.*

Proof. Let $g = p^n l$ be the order of $G, (p, l) = 1$. It suffices to show $l \in A \otimes V_p$. Let ψ be an element satisfying Lemma 4.11. The values of $\psi \not\equiv 0 \mod p$. We choose $N = \varphi(p^n)$ to be the Euler function value at p^n, then $\psi(x)^N \equiv 1 \mod p$ for all $x \in G$. The value of the function $l(\psi^N - 1)$ is divisible by $lp^n = g$. By Lemma 4.8, the function is an $A-$linear combination of characters induced from the characters of the cyclic subgroup of G. Since every cyclic subgroup is $p-$elementary, we have $l(\psi^N - 1) \in A \otimes V_p$. $A \otimes V_p$ is an ideal of $A \otimes R(G)$, hence $l\psi^N \in A \otimes V_p$. By subtracting, we see that $l \in A \otimes V_p$, and we've finished the proof.

□

Finally, we state the Brauer's Theorem and give a proof to it:

Theorem 4.13. ***Brauer's Theorem***

Each character of G is a linear combination with integer coeffiecients of characters induced from characters of elementary subgroups

Proof. Let V_p be the subgroup of $R(G)$ defined in 4.12. It suffices to show that

$$\bigoplus_{p \text{ prime}} V_p = R(G).$$

Now since V contains V_p, the index of V in $R(G)$ divides the index of V_p in $R(G)$, hence is prime to p by Theorem 4.12.

Since p is arbitrary, the index is equal to 1. And then we've finished the proof.

□

Theorem 4.14. *Every character of G is a linear combination with integer coefficients of monomial characters.*

(Recall that a character is **monomial** if it is induced from a character of a degree 1 subgroup.)

Proof. Recall a fact from [20] Theorem 8.5.16, if G is a supersolvable group. Then there exists a normal abelian group of G which is not contained in the center of G.

Then, applying Theorem 4.13, and the fact that each character of an elementary group is monomial. □

4.3. **Artin $L-$functions.** In this section, after introducing the concept of the Artin $L-$function, our main theme is to discover the nature of Artin $L-$functions, such as convergence, analytic continuity, etc.

Remark 4.15. Many analytic properties of the functions will be used without proof, as I want to focus on a more algebraic side. These analytic tools or strategies can be found in Tate's Thesis [23].

We will mention necessary properties Fourier analysis on number fields without proof. More explicit detail are shown in [23] and [14].

4.3.1. *Abelian $L-$functions.*

Definition 4.16. Let G be a (topological) group, F be a global field. A quasi-character $\chi : G \to F^\times$ is a (continuous) group homomorphsim.

We introduce a useful lemma frequently mentioned in the following text:

Lemma 4.17. *Let G be a finite abelian group, and denote the group of characters of G to be $\widehat{G}$. Then we have a (non-canonical) isomorphism of groups: $G \simeq \widehat{G}$.*

In 1920, Hecke [12] introduced the so called Hecke character. Later in 1946, Margret Matchett [6] showed that Hecke's original definition is equivalent to the character of the idele class group in her doctoral thesis . We will use the definition below:

Definition 4.18. Let F be a number field, a *Hecke character* is the character of the group $\mathbb{A}_F^\times/F^\times$.

Remark 4.19. Let χ be a character of $\mathbb{A}_F^\times$, then if we require $\chi(F^\times) = 1$, it is (naturally induces) a Hecke character.

We introduce a result that would allow us to reduce to study the characters of local fields.

Proposition 4.20. *Let $\chi : \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a character, then we can always write χ in local factors. In other words,*

$$\chi = \prod_v \chi_v$$

Here χ_v is a character of $F_v^\times$, and for almost all nonarchimeadean places v, χ_v is trivial on $\mathcal{O}_{F_v}^\times$.

This allows us to define the Hecke $L-$function:

Definition 4.21. Let F be a number field, and $\chi = \prod_v \chi_v$ is a Hecke character. Let S be a finite set of places, such that if $v \notin S$, then v is a non-archemidean place and χ_v is unramified. We define the *Hecke $L-$function* (associated to χ):

$$L^S(s, \chi) = \prod_{v \in S} (1 - \chi(\mathfrak{p}_v) N_{\mathfrak{p}_v}^{-s})^{-1}$$

For simplicity, we could also simply write $L(s, \chi)$.

Hecke showed in his paper [12] in 1920 (although not in the exact same form) that the above product converges when $\mathrm{Re}(s) > 1$, and it could analytically continued to a meromorphic function on $\mathbb{C}$. However, his proof was very complicated, which involved computing transformation of θ function on the number fields.

In 1950, Tate gave a concise proof using Fourier analysis on adele rings. This marks the start of modern studies on automorphic forms.

We formally state this as a theorem:

Theorem 4.22. *The Hecke $L-$function converges when* $\mathrm{Re}(s) > 1$, *and extends to a meromorphic function on $\mathbb{C}$. For nontrivial characters, the extension is entire; for the trivial character, there is exactly one simple pole at $s = 1$.*

4.3.2. *The Artin $L-$functions.*

Definition 4.23. (Artin $L-$functions)
Let $\rho : \mathrm{Gal}(K/k) \to GL_n(V_\rho)$ be a $n-$dimensional representation of the Galois group, the *Artin $L-$function* is defined to be the Euler product:

$$L(s,\rho) = L(s,\rho,K/k) = \prod_{\mathfrak{p}} \det(I_n - N(\mathfrak{p})^{-s}\rho(\mathrm{Frob}_{\mathfrak{p}})|_{V_\rho^{I_{\mathfrak{P}}}})^{-1}$$

Here $\mathfrak{p}$ is any prime, unramified or ramified, and $V_\rho^{I_{\mathfrak{P}}}$ is the subspace fixed by $I_{\mathfrak{P}}$. $\mathrm{Frob}_{\mathfrak{p}}$ denotes the Frobenius element

In particular, when $\mathfrak{p}$ is unramified, $I_{\mathfrak{P}}$ is trivial, then the local Euler factor of $\mathfrak{p}$ is

$$L_{\mathfrak{p}}(s,\rho) = \det(I_n - N(\mathfrak{p}^{-s})\rho(\mathrm{Frob}_{\mathfrak{p}}))^{-1}$$

In 1923, Artin proved in his paper [1] the following properties of his L function:

Theorem 4.24. (a) : *Let ρ be a $n-$dimensional representation of* $\mathrm{Gal}(K/k)$, *then Artin $L-$function $L(s,\chi_\rho,K/k)$ converges for* $\mathrm{Re}(s) > 1$.

(b) : *The collection of $L(s,\chi_\rho,K/k)$ as L and ρ vary, are additive, satisfy inflation, and are inductive.*

Remark 4.25. We give the explicit formulas of additivity, inflation, and inductivity.

Let ρ: $\mathrm{Gal}(K/k) \to GL_n(V)$ be a $n-$dimensional representation, and let ρ_1, ρ_2 be $n-$dimensional representation corresponding to V_1, V_2.

(1) :**Additivity**

$$L(s,\chi_{\rho_1} \oplus \chi_{\rho_2}, K/k) = L(s,\chi_{\rho_1},K/k)L(s,\chi_{\rho_2},K/k)$$

(2) :**Inflation**

$$L(s,\chi_{\mathbf{1}},K/k) = \zeta_K(s) = L(s,\chi_{\mathbf{1}},k/k)$$

(3) :**Inductivity**

Let $k \subset M \subset K$ be an extension of fields, and let $\rho_0 : \mathrm{Gal}(K/M) \to GL(V)$ be the restriction of ρ to $\mathrm{Gal}(K/M)$. Then,

$$L(s,\rho,K/k) = L(s,\mathrm{Ind}_{\mathrm{Gal}(K/M)}^{\mathrm{Gal}(K/k)},K/k) = L(s,\rho_0,K/M)$$

Proof. **(1) Additivity:**

The Artin L-function is defined as

$$L(s,\rho,K/k) = \prod_{\mathfrak{p}} \det\left(1 - \rho(\mathrm{Frob}_{\mathfrak{p}})N\mathfrak{p}^{-s} \mid V^{I_{\mathfrak{p}}}\right)^{-1}.$$

If $\rho = \rho_1 \oplus \rho_2$, then the action of Frobenius on $V = V_1 \oplus V_2$ is block-diagonal, so

$$\det\left(1 - \rho(\mathrm{Frob}_{\mathfrak{p}})N\mathfrak{p}^{-s}\right) = \det\left(1 - \rho_1(\mathrm{Frob}_{\mathfrak{p}})N\mathfrak{p}^{-s}\right)\det\left(1 - \rho_2(\mathrm{Frob}_{\mathfrak{p}})N\mathfrak{p}^{-s}\right).$$

Multiplying over all primes gives the desired equality.

(2) Inflation:

If $\rho = \mathbf{1}$ is the trivial representation, then $\rho(\mathrm{Frob}_{\mathfrak{p}}) = 1$ for all primes $\mathfrak{p}$, and $V^{I_{\mathfrak{p}}} = V$. Hence the Euler factor at $\mathfrak{p}$ is

$$\det(1 - \rho(\mathrm{Frob}_{\mathfrak{p}})N\mathfrak{p}^{-s})^{-1} = (1 - N\mathfrak{p}^{-s})^{-1},$$

so the Artin L-function is precisely

$$L(s,\mathbf{1},K/k) = \prod_{\mathfrak{p}}(1 - N\mathfrak{p}^{-s})^{-1} = \zeta_K(s).$$

(3) Inductivity:

Let ρ_0 be a representation of $G_M = \mathrm{Gal}(K/M)$, and $\rho = \mathrm{Ind}_{G_M}^{G_k}\rho_0$. Then by Frobenius reciprocity and properties of determinants, the Euler factors satisfy

$$\det\left(1 - \rho(\mathrm{Frob}_{\mathfrak{p}})N\mathfrak{p}^{-s}\right) = \prod_{\mathfrak{P}|\mathfrak{p}} \det\left(1 - \rho_0(\mathrm{Frob}_{\mathfrak{P}})N\mathfrak{P}^{-s}\right),$$

where $\mathfrak{P}$ runs over primes of M over $\mathfrak{p}$. Multiplying over all $\mathfrak{p}$ gives

$$L(s,\rho,K/k) = L(s,\rho_0,K/M).$$

□

Later on we would like to follow Artin's interest in discovering the continuation and establish functional equations for his $L-$functions.

We recall Brauer's Theorem in this context:

Theorem 4.26. *Let $G = \mathrm{Gal}(K/k)$ be the Galois group of a finite extension of number fields, and $\rho : G \to GL_n(\mathbb{C})$ is a $n-$dimensional representation. Then there exists cyclic subgroups $H_i \subset G$, Hecke characters ψ_i of H, and **integer** n_i, such that*

$$L(s,\chi_\rho,K/k) = \prod_i L(s,\psi_i,K/K^{H_i})^{n_i}$$

Proof. By Brauer's Theorem, there exists elementary subgroups $H_i \subset G$, such that

$$\rho = \sum_i n_i \operatorname{Ind}_{H_i}^G \psi_i$$

where $n_i \in \mathbb{Z}$ and ψ_i are characters of the elementary subgroups H_i.

Let K^{H_i} be the fixed field corresponding to H_i by Galois theory, then by Property 4.24, the additivity and inductivity of Artin $L-$function ensures

$$L(s,\chi_\rho,K/k) = \prod_i L(s,\psi_i,K/K^{H_i})^{n_i}.$$

□

With the help of Brauer' theorem, Artin proved the following theorem

Theorem 4.27. *The Artin $L-$function $L(s,\chi_\rho,K/k)$ extends to a meromorphic function of s.*

Remark 4.28. Artin also found a functional equation for his $L-$function:

$$\Lambda(s,\rho,K/k) = \varepsilon(s,\rho,K/k)\Lambda(1-s,\rho^\vee,K/k)$$

Proof. By Theorem 4.26, we get the factorization

$$L(s,\chi_\rho,K/k) = \prod_i L(s,\psi_i,K/K^{H_i})^{n_i}$$

Every factor $L(s,\psi_i,K/K^{H_i})$ is associated to a Hecke character ψ_i and by Theorem 4.22, $L(s,\psi_i,K/K^{H_i})$ extends to a meromorphic function on $\mathbb{C}$.

Since an integral power and product of meromorphic function, the Artin $L-$function $L(s,\chi_\rho,K/k)$ extends meromorphically to $\mathbb{C}$.

□

Remark 4.29. **Warning:** We can not assume n_i to be positive integers.

Remark 4.30. Artin then made the famous conjecture on the analytic behavior of his $L-$functions:

Conjecture 4.31. *If ρ is irreducible, and $\rho \neq \mathbf{1}$, then the Artin $L-$function $L(s,\rho,K/k)$ is entire.*

For simplicity, we assume that F is a number field, and $F/\mathbb{Q}$ is an abelian extension (or else we could take the maximal abelian subfield of F over $\mathbb{Q}$).

When $\dim\rho = 1$, which by convention we could call ρ an irreducible character of G. By the isomorphism of global class field theory:

$$r_{F/\mathbb{Q}} : A_{\mathbb{Q}}^\times/\mathbb{Q}^\times N_{F/\mathbb{Q}}\mathbb{A}_F^\times \to \mathrm{Gal}(F/\mathbb{Q})$$

We get the Hecke character $\chi = \rho \circ r_{F/\mathbb{Q}}$.

By calculation,

$$L(s,\rho) = L(s,\chi)$$

Where the left hand side is the Artin $L-$function and the right hand side the the Hecke $L-$function (in this case it is also the Dirichlet $L-$function). Since the Hecke $L-$function is entire, so is the Artin $L-$function.

when $\dim \rho = 2$, let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ be an irreducible 2-dimensional Galois representation, and let

$$L(s, \rho) = \prod_p \det\bigl(1 - \rho(\mathrm{Frob}_p)p^{-s}\bigr)^{-1}$$

denote its associated Artin L-function. This conjecture is known to hold for *odd* 2-dimensional representations, i.e., those for which $\det(\rho(c)) = -1$ for complex conjugation c, by the Langlands–Tunnell theorem: such representations are *modular*, arising from weight 1 cuspidal modular forms, and thus their L-functions are entire. This result constitutes the first nontrivial verification of the Artin conjecture beyond the one-dimensional (abelian) case, and it illustrates the power of modularity in establishing analytic continuation.

However, the cases of even 2-dimensional representations and all higher-dimensional representations remain largely open, highlighting the central role of the Langlands program in the general problem. □

5. Appendix

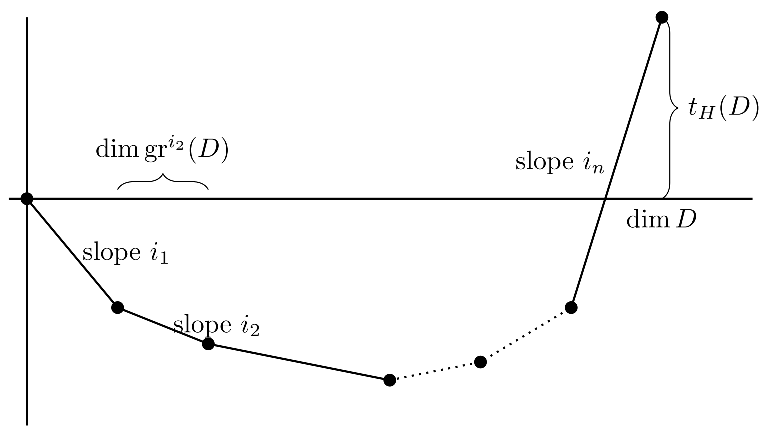

Figure 1. The Hodge polygon of D illustrating slopes and graded pieces.

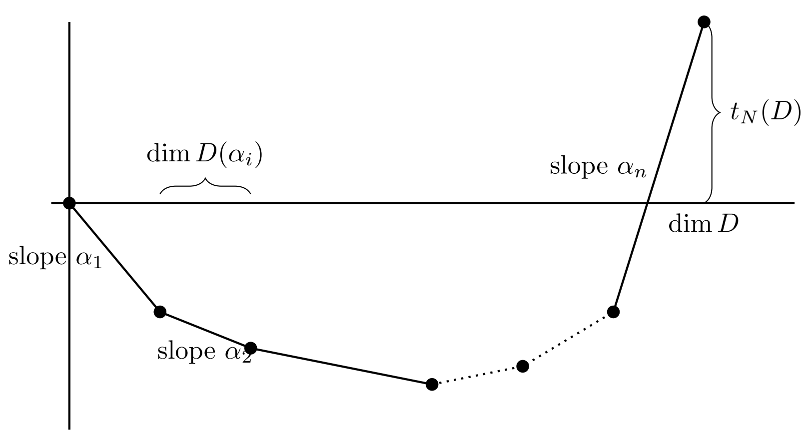

Figure 2. The Newton polygon of D illustrating slopes and isoclinic pieces.

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