CONSTRUCTING IRREDUCIBLE REPRESENTATIONS OF SYMMETRIC GROUPS VIA YOUNG TABLEAUX

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ABSTRACT. In this paper, we try to find the relations between irreducible representations of symmetric group S_n and a seemingly less related combinatorial object Young Tableaux. We introduce the basic ideas in representation theory and the connection with Combinatorics.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Young Tableau is an interseting and rich topic ever since it was introduced by Alfred Young at Cambridge University in 1900. In 1903, they were applied in studying the symmetric groups by George Frobenius and the theory was further developed later. In this paper we mainly focus constructing the correspondence between irreducible representations of symmetric groups with modules generated by certain Young tableaux. We would also want to show the applications of the theory to more combinatorial works.

The paper will be consisted of 4 main parts. For this section we introduce our work and some basic definitions of Young's Theory.

In the second part we review fundamental knowledge of representation theory and Young tableau. Using the basic insights we will see the natural correspondence between representations and tableaux.

For the third part we construct the Specht modules (which are the irreducible modules of S_n) and find the algebraic properties for it. The third section would be the most important part of the paper since we finally see how Young Tableaux illustrates irreducible representations.

For the final part we introduce the RSK algorithm and reprove some theorems mentioned in the third section .

Some basic definition that would be frequently used without emphasizing:

Definition 1.1. A **Young diagram** (also called a **Ferrers diagram**, particularly when represented using dots) is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.

Definition 1.2. A **Young tableau** is obtained by filling in the boxes of the Young diagram with symbols taken from some alphabet, which is usually required to be a totally ordered set.

Definition 1.3. A tableau is called **standard** if the entries in each row and each column are increasing.

Definition 1.4. A tableau is called **semistandard**, or column strict, if the entries weakly increase along each row and strictly increase down each column.

Key words and phrases. Young tableaux, Combinatorics, Representation theory, RSK Algorithm.

Remark 1.5. We often write **SYT** for standard young tableau and **SSYT** for semistandard young tableau.

2. Basics of Representation Theory and Young Tableau

Definition 2.1. A **representation** ρ of a group G is a homomorphism $\rho : G \to GL(V)$ for some (finite-dimensional) vector space V. The dimension of V is called the **degree** of ρ.
We write ρ_g for $\rho(g)$ and $\rho_g v$ for the action of ρ_g on $v \in V$

Definition 2.2. Two representations $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ are **equivalent** if there exists an isomorphism $T : V \to W$ such that $\rho_g = T\varphi_g T^{-1}$ for all $g \in G$,i.e. $\rho_g T = T\varphi_g$ for all $g \in G$.

Definition 2.3. The **Group Algebra**, denoted $\mathbb{C}[G]$, is the set of all linear combinations of the elements in group G. For example, let $G = \{g_1, g_2, ..., g_n\}$, then $\mathbb{C}[G] = \{c_1 g_1 + c_2 g_2 + ... + c_n g_n \mid c_1, c_2, ..., c_n \in \mathbb{C}\}$

Definition 2.4. A G**-module** is an vector space M with a group action $\rho : G \times M \to M, (g, m) \mapsto gm$ on M.

Definition 2.5. The character χ of representation $\rho : G \to GL(V)$ is a function $G \to \mathbb{C}$, such that $\chi(g) = Tr(\rho_g)$. Here Tr is the trace of ρ_g.

Definition 2.6. Let $\rho : G \to GL(V)$ be a representation. A subspace $W \subseteq V$ is **G-invariant** if , for all $g \in G$ and $w \in W$, one has $\rho_g w \in W$

Definition 2.7. A representation $\rho : G \to GL(V)$ is **irreducible** if the only G -invariant subspaces of V are 0 and V.

Definition 2.8. A representation $\rho : G \to GL(V)$ is **completely reducible** if for every G-invariant subspace $W \subseteq V$, there exists a G-invariant subspace $W^{\perp}$, such that $W \oplus W^{\perp} = V$.

Proposition 2.9. *A representation of a finite group G can be seen as a $\mathbb{C}G$ module*

We come to our first important theorem of representation theory :

Theorem 2.10. *(Maschke's Theorem)*
Every representation of a finite group G is completely reducible.

The proof could be found in any textbook of representation theory , see [**Ste**] for further details.
By Mashcke's Theorem, we can decompose group algebra $\mathbb{C}[G]$ of G into:

$$\mathbb{C}[G] = \bigoplus m_i V^{(i)}$$

where $V^{(i)}$ runs through all (inequivalent) irreducible submodule of V. If we take χ as the character of V, and $\chi^{(i)}$ as the characters of V, then $m_i = \langle \chi^{(i)}, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi^{(i)}(g^{-1})$.

In order to clarify the irreducible representations, which are extremely improtant when we later construct the Specht modules, we need the following proposition:

Proposition 2.11. *Let G be a finite group and suppose $\mathbb{C}[G] = \bigoplus m_i V^{(i)}$,where V^i runs through all (inequivalent) irreducible submodule of V. Then*
1.$m_i = dimV^{(i)}$
2. The number of $V^{(i)}$ equals the number of conjugacy classes of G.

The proof could be found in [**Sag**].
Next we introduce some concepts relating to partitions of n:

Definition 2.12. A **partition** of an integer n is a tuple of $k(k \in \mathbb{N}_{\geq 1})$ integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ in non-increasing order and satisfies $\lambda_1 + \lambda_2 + ... + \lambda_k = n$ Sometimes we call the collection of Young Tableaux of the same partition **Young Tabloid of partition** λ.

Definition 2.13. Suppose $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_m)$ are two partitions of n. Then λ **dominates** μ, denoted $\lambda \trianglerighteq \mu$, if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \text{ for all } i \geq 1.$$

Lemma 2.14. ***Dominance Lemma for Partitions***
Let t^λ and s^μ be tableaux of shape λ and μ, respectively. If for each index i , the elements of row i of s^μ are all in different columns in t^λ, then $\lambda \trianglerighteq \mu$.

Proof. Sketch: Let $S_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i$, $T_i = \mu_1 + \mu_2 + \cdots + \mu_i$, then if for each index i, the elements of row i of s^μ are all in different columns in t^λ, it 'contributes' 1 to the sum S_i. Therefor the sum T_i is less or equal to the sum S_i. □

After introducing the basic definitions we need to state out the relations between a Young tableau with the irreducible modules on S_n. Therefore, we give a statement that illustrates the structure of equivalence relations in S_n in the following proposition.

Proposition 2.15. *Let permutations $\sigma_1, \sigma_2 \in S_n$, σ_1 is the conjugate of σ_2 if and only if σ_1 and σ_2 are of the same type.*
Here having the ***same type*** *means the cycles composing the two permutations have the same length and cardinals.*

Proof. First we prove the claim below

Claim 2.16. *Let σ be a permutation in S_n, $1 \leq k \leq n$, $\sigma(a_1a_2\cdots a_k)\sigma^{-1} = (\sigma(a_1)\sigma(a_2)\cdots\sigma(a_k))$*

This is easy to check since $\sigma(a_1a_2\cdots a_k)\sigma^{-1}[\sigma(a_i)] = \sigma(a_1a_2\cdots a_k)a_i = \sigma(a_{i+1})$ for all $1 \leq i \leq k$, $a_{k+1} = a_1$. So every $\sigma(a_i)$ permutes to $\sigma(a_{i+1})$ on the right hand side of the identity.

On one hand, if σ_1 and σ_2 are in the same conjugacy class, there exists $\tau \in S_n$, such that $\tau\sigma_1\tau^{-1} = \sigma_2$. We then (uniquely) decompose σ_1 to composition of disjoint cycles $C_1C_2\cdots C_m$. Noticing that $\tau\sigma_1\tau^{-1} = (\tau C_1\tau^{-1})(\tau C_2\tau^{-1})\cdots(\tau C_m\tau^{-1}) = \sigma_2$, we now have a (unique) disjoint cycle decomposition of σ_2. Therefore σ_2 is the same type of σ_1.

On the other hand, if σ_1 and σ_2 have the same shape, let $\sigma_1 = C_1C_2\cdots C_k$, $\sigma_2 = L_1L_2\cdots L_k$, where C_i and L_i are disjoint cycles that have the same length for every $1 \leq i \leq k$

By Claim 2.10, for each cycle $C_i = (a_{i_1}a_{i_2}\cdots a_{i_t})$, $L_i = (b_{i_1}b_{i_2}\cdots b_{i_t})$, there exists $\sigma_i \in S_n$, such that $\sigma_i(a_{i_k}) = b_{i_k}$, $\sigma_iC_i\sigma_i^{-1} = L_i$,therefore C_i and L_i are in the same conjugacy class for $1 \leq i \leq k$. Thus it's not hard to find that σ_1 and σ_2 are in the same conjugacy class. □

Remark 2.17. When proving Claim 2.10, we assume the fact that every permutation has a unique disjoint cycle decomposition, the proof can be found in [**Hun**].

Remark 2.18. Proposition 2.15 is true for every finite group G.

After explaining Proposition 2.15, we can see the relations between Young tableau and the symmetric group S_n. In order to set up the relation of irreducible representations of S_n with Young tableau, rather than S_n, we need to add structures to the vector space of Young tableaux. As we stated before, a representation of a group G corresponds with a G-module, we should consider the action of S_n on the vector space generated by λ-tabloids.

Therefore we introduce the permutation module

Definition 2.19. The **permutation module** of a partition λ is the set $M^\lambda = \mathbb{C}\{\{t_1\}, \{t_2\}, \cdots, \{t_k\}\}$ where $\{t_1\}, \cdots, \{t_k\}$ is the complete list of λ-tabloids.

We give some examples of the permutation module:

Example 2.20. If $\lambda = (n)$, then $M^{(n)} = \mathbb{C}\ \{\ \boxed{1\ 2\ \cdots\ n}\ \}$

Example 2.21. If $\lambda = (n-1, 1)$, then $M^{(n-1,1)} \simeq \mathbb{C}\{1, 2, \cdots, n\}$. Which is because each tabloid is only determined by the element on the second row.

We introduce basic algebraic properties of M:

Definition 2.22. Any G-module M is **cyclic** if there is a $\mathbf{v} \in M$ such that $M = \mathbb{C}G\mathbf{v}$,where $G\mathbf{v} = \{g\mathbf{v} \mid g \in G\}$. In this case we say that M is generated by $\mathbf{v}$.

Proposition 2.23. *If λ is a partition of n, then M^λ is a cyclic module generated by any λ-tabloid. In addition, if $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ $dimM^\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_k!}$*

Proof. For any tabloid $\mathbf{v} \in M^\lambda$, consider the action of S_n on M^λ as following: for any $\sigma \in S_n$, define $\sigma \circ \mathbf{v} := \sigma(\mathbf{v})$ i.e, σ acts on every row of the Young tabloid respectively. Let σ run through all elements, by Proposition 2.15, $\sigma \circ \mathbf{v}$ runs through all tabloids of partition λ. Therefore $\mathbf{v}$ generates the spanning set of the permutation module. By definition, $\mathbf{v}$ generates M^λ.

$dim M^\lambda = \#\lambda$ -tabloids$= \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_k!}$.

□

3. Construction of Specht Modules

Borrowing the definitions from [1], we have :

Definition 3.1. Given tableau T, suppose it has rows $R_1, R_2, ...R_l$ and columns $C_1, C_2, ..., C_t$. Let $R_t = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l}$, and $C_t = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$ be the **row-stabilizer** and **column stabilizer** respectively

For example if we take $T = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 5 & 2 \\ \hline 3 & 6 & 8 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}$

$R_t = S_{\{1,4,2,5\}} \times S_{\{3,6,8\}} \times S_{\{7,9\}}$ and $C_t = S_{\{1,3,7\}} \times S_{\{4,6,9\}} \times S_{\{5,8\}} \times S_{\{2\}}$

Under this notation our equivalence class can be described as $\{t\} = R_t t$.
We construct a map from the set of tableaux T to the permutation module M^λ

Definition 3.2. Let T be a tableau, then the **associated polytabloid** is $e_T = \sum_{\pi \in C_t} sgn(\pi)\pi$.
Denote $\sum_{\pi \in C_t} sgn(\pi)\pi := \kappa_T$

Then we have some basic properties for the polytabloids

Proposition 3.3. *1.* $R_{\pi T} = \pi R_T \pi^{-1}$
2.$C_{\pi T} = \pi C_T \pi^{-1}$
3.$R_{\pi T} = \pi R_T \pi^{-1}$
4.$e_{\pi T} = \pi e_T$

The proof can be found in [**Sag**]
Finally, we can define the irreducible modules of S_n

Definition 3.4. For partition λ, the corresponding Specht module S^λ is the submodule of M^λ spanned by the polytabloids e_T

Remark 3.5. By Lemma 2.2 we immediately have S^λ are cyclic groups generated by any given polytabloid.

Example 3.6. If $\lambda = (n-1, 1)$, write the $(n-1,1)$-tabloids as $\{T\} = \begin{array}{|l|}\hline i\ i+1 \cdots\ j \\ \hline k \\ \hline\end{array} := \mathbf{k}$

Then by definition we can write the polytabloid of T as $e_T = \mathbf{k} - \mathbf{i}$ and the polytabloids spans the Specht module $S^{(n-1,1)}$ $=\{c_1\mathbf{1} + c_2\mathbf{2} + ... + c_n\mathbf{n} : c_1 + c_2 + ... + c_n = 0\}$

Theorem 3.7. *The Specht modules S^λ for λ form the complete list of irreducible modules on S_n over $\mathbb{C}$*

We follow the proof given in [**Sag**], but skip some of the details.

Notation 3.8. 1.Given a subgroup H of S_n, define $H^- = \sum_{\pi \in H} sgn(\pi)\pi$. If $H = \{\pi\}$, then write π^- for H^-.

2.Define the unique inner product on M^λ as $\langle \{\mathbf{t}\}, \{\mathbf{s}\} \rangle = \delta_{\{t\},\{s\}}$. Here $\delta_{j,k}$ is the Kronecker symbol.

3. Let $S^{\mu\perp}$ denote the orthogonal complement of S^μ.

Lemma 3.9. *Let $H \leq S_n$ be a subgroup*

1. *If $\pi \in H$, then $\pi H^- = H^- \pi = (sgn\pi)H^-$.*
2. *For any $\mathbf{u}, \mathbf{v} \in M^\lambda$, $\langle H^- \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, H^- \mathbf{v} \rangle$, i.e, H^- is a self-adjoint operator.*
3. *If the transposition $(b, c) \in H$, then we can factor $H^- = k(\epsilon - (b, c))$, where $k \in \mathbb{C}[S_n]$, and ϵ denotes the identity permutation.*
4. *If t is a tableau with b, c in the same row of t and $(b, c) \in H$, then $H^-\{\mathbf{t}\} = \mathbf{0}$.*

Corollary 3.10. *Let $t = t^\lambda$ be a λ-tableau and $s = s^\mu$ be a μ-tableau, where λ, μ are partitions of n. If $\kappa_t\{s\} \neq 0$ then $\lambda \trianglerighteq \mu$. And if $\lambda = \mu$, then $\kappa_t\{s\} = \pm\{e_t\}$.*

Corollary 3.11. *If $\mathbf{u} \in M^\mu$ and the t has shape μ, then $\kappa_t \mathbf{u}$ is a multiple of $\mathbf{e}_t$.*

The proof of the statements above are directly from [**Sag**]. However we do want to emphasize proof of the **Submodule Theorem** found by James[**Jam**]. Note that the submodule theorem applies over arbitrary fields, only having to change the formula for the inner product on M^λ.

First we consider the case when $char(F) = 0$ ($F = \mathbb{C}$).

Theorem 3.12. *Let U be a submodule of M^μ. Then $S^\mu \subseteq U$ or $U \subseteq S^{\mu\perp}$. In particular, when the field is $\mathbb{C}$, the S^μ are irreducible.*

Proof. Let $\mathbf{u} \in U$, then since $\mathbf{u} \in M^\lambda$, by Corollary 3.10, for any $t \in S^\lambda$,$\kappa_t \mathbf{u} = x \cdot e_t$. Here x denote the scaler in $\mathbb{C}$ and e_t denotes the polytabloid of t. $\mathbf{u} = \sum c_i\{t_i\}$.

If there exists t, such that $x \neq 0$. By definition, $\kappa_t = \sum_{\pi \in C_t} sgn(\pi)\pi$, therefore $\kappa_t \mathbf{u} = x \cdot e_t \in U$. Thus, $e_t \in U$. As remark 3.5 shows, S^λ is cyclic and can be generated by e_t, so we have $S^\lambda \in U$.

If for all $t \in S^\lambda$, $\kappa_t \mathbf{u} = 0$, then for all λ-tableaux $\mathbf{v}$, $\langle \kappa_t \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \kappa_t \mathbf{v} \rangle = \langle \mathbf{u}, e_\mathbf{v} \rangle = 0$. Here by definition the polytabloid of $\mathbf{v}$ is $e_\mathbf{v} = \sum \kappa_\mathbf{v}\{\mathbf{v}\}$. Therefore, since $e_\mathbf{v}$ spans S^λ, $\langle U, S^\lambda \rangle = 0$, i.e., $U \subseteq S^{\lambda\perp}$ and we are done. □

Proposition 3.13. *Suppose the field of scalars is $\mathbb{C}$ and $\theta \in Hom(S^\lambda, M^\mu)$ is nonzero. Thus $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$, then θ is a multiplication by scalar.*

And now we come to the point of proving **Theorem 3.7**:

Proof. First we prove S^λ are irreducible from Theorem 3.12

If $\exists \lambda$ such that S^λ is reducible then let W be a (non-trivial) submodule of S^λ . By Theorem 3.12 we know that either $W \supseteq S^\lambda$ or $W \subseteq S^{\lambda\perp}$.

In field $\mathbb{C}$, $S^\lambda \cap S^{\lambda\perp} = \{0\}$. And we have the contradiction.

By Proposition 2.11 we see that the number of irreducible S_n modules equals to the conjugacy classes of S_n, which also equals to the number of partitions of n. Therefore we only have to show that S^λ is pairwise inequivalent.

If $\exists \lambda \neq \mu, S^\lambda \cong S^\mu$. Let $\sigma : S^\lambda \to S^\mu$ be an moudule isomorphism. Let $\theta \in Hom(S^\mu, M^\mu)$, then by proposition 3.13 we know that θ is a multiplication by scalar. Since the embedding $i : S^\mu \hookrightarrow M^\mu$ is a multiplication of scalar 1, there exists nonzero $\theta_0 \in Hom(S^\mu, M^\mu)$. Let $\tau = \theta_0 \circ \sigma$, then $\tau \neq 0$ and $\tau \in Hom(S^\lambda, M^\mu)$. (We leave this for the reader to check the well-definess)

Therefore by Proposition 3.13, $\lambda \trianglerighteq \mu$. Conversely, if we consider $\phi : S^\mu \to S^\lambda$ and we have $\mu \trianglerighteq \lambda$.

Thus, $\lambda = \mu$ and we've proven that S^λ is pairwise inequivalent. □

Remark 3.14. For field $\mathbb{C}$ or any field F of character 0, the theorem 3.7 holds. However when we discuss the circumstances when $charF = p$, S^λ is no longer irreducible. This will be a further topic in modular representation theory. See [Jam] for reference.

Corollary 3.15. *The permutation modules has unique decomposition :*

$$M^\mu = \bigoplus_{\lambda \geq \mu} m_{\lambda\mu} S^\lambda$$

with the diagonal multiplicity $m_{\mu\mu} = 1$.

Now we want to point out more information of the structures of the Specht modules by exploring the basis and dimension of it.

Definition 3.16. A tableau T is **standard** if the rows and columns of T are increasing sequences. A polytabloid and tabloid is **standard** if the corresponding tableau is **standard**.

We can construct the basis of Specht modules using Definition 3.16 :

Theorem 3.17. *The set B_T : $\{e_T$: T is a standard λ-tableau$\}$ is a basis for the corresponding Specht module S^λ.*

The proof can be found in [**Sag**].

Notation 3.18. $f^\lambda = \#\ \{\text{standard } \lambda \text{ tableaux}\}$

We conclude this section with the theorems below:

Theorem 3.19. *For any partition λ and any base field F, we have:*
1. $B_t = \{e_t : t$ is a standard tableau$\}$ is a basis for S^λ ,
2. $dim S^\lambda = f^\lambda$, and
3. $\Sigma_\lambda (f^\lambda)^2 = n!$.

4. The RSK Algorithm

In this section we give a description of the RSK Algorithm and some of its interesting applications. The first intuition of the RSK algorithm could be found in developing a combinatorial proof for formula :

$$\textstyle\sum_\lambda (f^\lambda)^2 = n!.$$

We start with introducing the **Robinson-Schensted Correspondence** originally discovered by Robinson in1938 [**Rob**] and found in an another way by Schensted [**Sch**] in 1961:

Definition 4.1. The **Robinson-Schensted Correspondence** is a map $\tau : S_n \to SYT(\lambda)$, $\pi \mapsto (P, Q)$. (Here $SYT(\lambda)$ denotes the set of all standard Young tableaux.)

We first describe in details the map "$\pi \mapsto (P, Q)$" Suppose π is written in the two-line notation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$$

We construct a sequence of tableaux pairs:

$$(P, Q) = (\varnothing(P_0), \varnothing(Q_0)), (P_1, Q_1), (P_2, Q_2), \cdots, (P_n, Q_n),$$

Generally speaking, R-S begins with two empty SYTs, denoted P_0 and Q_0. The algorithm inserts the bottom row into P_0 left-to-right, and the final output is P. As $\pi(i)$ is inserted into P_{i-1} to get P_i , the number i is inserted into Q_{i-1} to get Q_i. These methods of insertion are different between P and Q.

Definition 4.2. A tableau P be a **partial tableau** if it's an array with distinct entries whose rows and columns increase. Which means a partial tableau is standard when it consists of all elements in $\{1, 2, \cdots, n\}$.

Let $x \notin P$, to row insert x in P, we describe the procedures (Here "$\rightleftharpoons$" means replacement)
Step 1: Set $R \rightleftharpoons$ the first row of P
Step 2: If x is less than some element of row R,
  step 2-a: Let y be the least element in R larger than x and replace y by x in R
  step 2-b: Set $x \rightleftharpoons y$ and $R \rightleftharpoons$ the next row down.
Step 3: Now x is greater than every element of R, so place x at the end of row R and stop the algorithm.

The result of row inserting x into P yields to tableau P', then write $r_x(P) = P'$. Then we describe the construction of (P_k, Q_k) by induction:
$P_k = r_{xk}(P_{k-1})$,
Q_k = place k into Q_{k-1} at the cell (i, j) where the insertion terminates.
Then we are ready to prove the correspondence between S_n and SYT.

Theorem 4.3. *The map $\pi \to (P, Q)$ is a bijection between elements of S_n and pairs of standard tableaux of the same shape λ.*

Next we introduce the **(RS)-Knuth algorithm**, which stands for K in the RSK-algorithm

Definition 4.4. Let $A = (a_{ij})_{n\times m}$ be an $\mathbb{N}$-matrix with finite supports, i.e. there is only finite number of non-zero entries in A. The **generalized permutation** of A is a $2 \times n$ matrix

$$w_A = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

such that 1. $i_1 \geq i_2 \geq \cdots \geq i_n$

2.If $i_r = i_s$then $j_r = j_s$

3.for each pair (i, j), there are exactly a_{ij} occurences of the column $\begin{bmatrix} i \\ j \end{bmatrix}$.

Theorem 4.5. ***The main RSK algorithm***

1. RSK is a bijection between $\mathbb{N}$-matrices with finite support and pairs of SSYT of the same shape.

2. $j$occurs in P exactly $\sum_i a_{ij}$ times, and i occurs in Q exactly $\sum_j a_{ij}$times.

We leave the proof to the readers.

Now we finish our paper with the intuition we mentioned at the start of the section and prove the identity using the RSK-Algorithm:

Theorem 4.6. $\sum_\lambda (f^\lambda)^2 = n!$

Here λ runs through all the partitions of n and f^λ denotes the number of standard λ tableaux.

Proof. Let σ be a map between an element $\pi \in S_n$ and a pair of SYTs (P, Q) following the RSK-algorithm.

By Theorem 4.3 we know that σ is a bijection.

We calculate the cardinality of S_n and the SYT pairs (P_i, Q_i) (which are eventually equal).

Therefore we have $card(S_n) = n! = \#$ SYT pairs $(P_i, Q_i) = \sum_\lambda f^\lambda \cdot f^\lambda = \sum_\lambda (f^\lambda)^2$. (For every partition λ, P_i and Q_i has f^λ choices respectively.) □

5. Acknowledgement

This paper is completed under the supervision of Professor Dan Ciubotaru at the University of Oxford. He gave amazing lectures and provided excellent topics for participants to do research on. The process of writing this paper was hard but enjoyable.

I would also want to thank Professor Yingfang Fan from Dalian Maritime University and Professor Zhiqiang Hu from Dalian University of Technology for the first lectures on scientific research and academic writing. Their outstanding performances in their relevant research field also stimulates me to pursue higher goals.

They are also excellent parents and I would not have grown to who I am today if it weren't for their passion and persistence for science and life.

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