Development of Continuum Damage Mechanics for Elastic Solids Sustaining Anisotropic and Unilateral Damage

J.-L. Chaboche

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Development of Continuum Damage Mechanics for Elastic Solids Sustaining Anisotropic and Unilateral Damage

J.-L. CHABOCHE
Onera
29, Avenue de la Division Leclerc
92320 Chatillon (Hauts-de-Seine)
France

ABSTRACT: This article deals with the phenomenological approach of Continuum Damage Mechanics and the consistent modeling of damaging processes incorporating two specificities:

- the anisotropic nature of the damage, which is related to the directionality of the past loading history
- the unilateral character of damage, corresponding to the possible closure of defects under compressive loading conditions

The discussion is limited to the simple case of the elastic behaviour and the rate independent damaging processes in brittle materials like concrete, that can be considered as initially isotropic. This case is important in order to focus on the true difficulties and to solve them.

A recently proposed unilateral damage condition is recalled, that preserves symmetry of the elastic operators and eliminates the possibility of discontinuous stress-strain responses, present in the previous theories. Moreover, a simple but general form of the damage rate equation is proposed, which has the potentiality to describe fairly well the development of the anisotropic damage, using a fourth-order damage tensor. Various specialized forms are discussed that are based on simple elastic solutions in solids containing distributed microcracks.

1. INTRODUCTION

THE CONTINUUM DAMAGE Mechanics (CDM) has been widely developed and used during the past twenty years. In this macroscopic phenomenological theory, the material is still considered as a continuum, even during the damaging process where voids, cavities, microcracks . . . nucleate and grow at the micro-scale. To some extent, CDM is a concept lying somewhere between classical Continuum Mechanics and Fracture Mechanics (FM): at the micro-scale, the FM criteria can still be used, but are averaged, through CDM methods, leading

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to simplified constitutive and damage equations at the macro-level, equations that can be really used at the level of the component analysis.

Reviews of CDM can be found [5,6,16,21,25,27] which summarize the present state of the phenomenological approach of CDM. These use damage effect variables in the sense of Kachanov [18,19] and Rabotnov [28] and the effective stress/effective strain concepts.

Several generalizations to anisotropic damage situations have been proposed incorporating damage variables like vectors [20], second-order tensors [26,31], and fourth-order tensors [4,17,30]. One of the difficulties in the CDM development concerns the nonlinearities associated to the unilateral nature of the damage. Roughly speaking the damage is active only if microcracks are open. In the other case, called "passive damage" by Krajcinovic and Fonseka [20], damage is still present but affects differently the mechanical properties of the material.

Until recently, the Continuum Damage theories were unable to describe simultaneously the following facts:

- The damage is anisotropic in nature, due to the directionality of the defects.
- The damage can be active or inactive, when the microcracks are respectively open or closed, which leads to a unilateral behaviour.

The review of the possibilities of various theories using scalars, vectors, second-order and fourth-order tensors has been made in the particular case of the elastic-damaged materials, plasticity being neglected. This case is of particular importance for elastic brittle materials like concrete but allows also a simple conceptual case in order to check the validity of damage theories. In fact, with the actually proposed unilateral conditions, the theories considered were showing one of the following properties [7,8]:

- no anisotropy of damage effect
- nonsymmetry of the elastic compliance of the material
- response discontinuities (stress or strain) when the unilateral condition takes place

The author proposed recently a new way of formulating the unilateral condition for active/passive damage [9]. It considers the so-called "principal directions" of the actual damage state and the normal strains associated to these directions. This method was shown as the only way to eliminate the above inconsistencies. This new formulation is discussed in Section 3.1.

In the present article, we discuss with more details the various possibilities offered to define the special "principal directions of damage state" and show the application of the new concept for both second-order and fourth-order damage tensors.

Moreover, in Section 4, a simple general form is proposed for the anisotropic damage rate equation. Combined with the new unilateral condition, it will have the potentiality to describe the damage induced anisotropy of initially isotropic materials, submitted to any kind of complex nonproportional loadings. No particular applications are considered in the present article, where only the general concepts and properties are demonstrated.

2. GENERALITIES

2.1 The Thermodynamic Framework

The damage theories are considered in their rate-independent form, i.e., we limit ourselves to the case of the elastic damaged material under small strains and isothermal conditions. From a thermodynamic point of view, the damage equations can be presented through the definition [14]:

The thermodynamic potential defines the present (damage) state of the materials. Under isothermal conditions, the specific free energy for instance can be written:

$$\Psi = \Psi(\underset{\sim}{\epsilon}, d_a) = \frac{1}{2}\underset{\sim}{\epsilon} : \underset{\approx}{\tilde{\Lambda}}(d_a) : \underset{\sim}{\epsilon} \tag{1}$$

where $\underset{\sim}{\epsilon}$ is the elastic strain tensor and d_a denotes a family of damage variables ($a = 1,2. . .$). $\underset{\approx}{\tilde{\Lambda}}(d_a)$ is a fourth-order symmetric tensor, the secant stiffness, function of damage. It corresponds here to the case of active damage (open microcracks). From Equation (1) we obtain:

$$\underset{\sim}{\sigma} = \frac{\partial \Psi}{\partial \underset{\sim}{\epsilon}} = \underset{\approx}{\tilde{\Lambda}}(d_a) : \underset{\sim}{\epsilon} \tag{2}$$

$$y_a = -\frac{\partial \Psi}{\partial d_a} = -\frac{1}{2}\underset{\sim}{\epsilon} : \frac{\partial \underset{\approx}{\tilde{\Lambda}}}{\partial d_a} : \underset{\sim}{\epsilon} \tag{3}$$

where y_a denotes the thermodynamic affinities to the damage variables d_a, that was introduced [3] as the damage energy release rate. Note that the thermodynamic potential contains all the information about the damage effect on the material stress-strain behaviour.

The dissipative potential describes the damage evolution and the corresponding irreversible processes. It is written in terms of the thermodynamic affinities associated to damage. In the present case, within the simplifying assumption of a time-independent process, we may replace the notion of dissipative potential and corresponding normality hypotheses by the following damage criterion:

$$g_a(y_a; d_a) = y_a - K_a(d_a) \leq 0 \tag{4}$$

Such a criterion states that the damage in the material is increased when the energy release y_a attains the damage threshold K_a and increases further. In fact, K_a evolves with damage, with the condition $y_a = K_a(d_a)$, and stores the maximum value attained by y_a. With respect to the loading/unloading condition, evolution of damage is defined with the Kuhn-Tucker relationship [17]:

$$\dot{d}_a \geq 0 \qquad g_a(y_a\,; d_a) \leq 0 \qquad \dot{d}_a\, g_a(y_a\,; d_c) = 0 \tag{5}$$

Within this framework, the second thermodynamic principle reduces to the following inequality:

$$\dot{\Phi}_d = y_a \dot{d}_a \geq 0 \tag{6}$$

Provided the damage energy release rate y_a is always positive, this principle is automatically checked.

2.2 The Active/Passive Unilateral Problem

The mechanical behaviour that can be described within the present framework is illustrated schematically by Figure 1 for a tension-compression loading:

- The states a, c, f, e, d, i, j correspond to the elastic behaviour before or after damage.
- The states b, g, h, k correspond to the damage evolution, during loading, with $g_a = 0$.

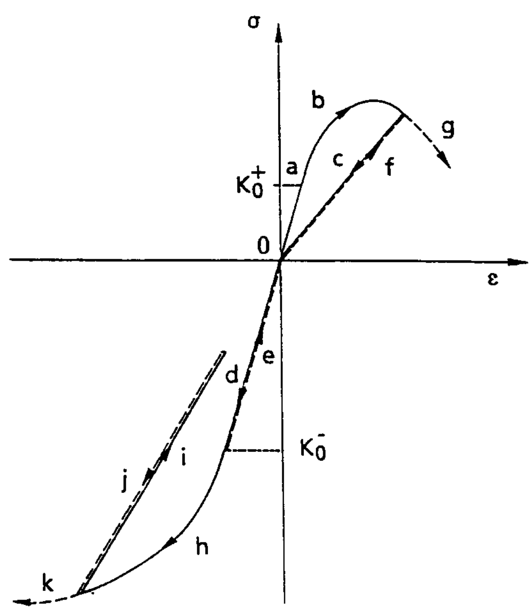

Figure 1. Schematics of the tension-compression elastic behaviour with damage and a unilateral condition.

- Unloading after the damaging process is linear elastic (c), with a reduced stiffness.
- The unilateral condition takes place during unloading when $\epsilon = 0$ (or $\sigma = 0$). In the compression regime the initial stiffness is recovered, corresponding to an inactive damage. The path c-d-e-f describes a bilinear elastic behaviour.
- When reloading (f), the damage grows further when the strain is increased above the previous maximum strain (g).

We have shown [7,8] the limitations of the previously proposed continuum damage theories in what concerns the simultaneous description of an anisotropic damage state together with an active/passive damage condition. It was shown that the proposed unilateral condition does affect both the "diagonal" terms* of the compliance (or stiffness) tensors and the "non-diagonal" terms. In the theories developed by Ju [17] with a fourth-order "damage tensor" or by Krajcinovic and Fonseka [20] with "damage vectors," there results stress-strain response discontinuities for complex multiaxial strain paths. In Table 1, for example, we have indicated for each theory the components $\tilde{\Lambda}_{1111}$, $\tilde{\Lambda}_{1122}$, $\tilde{\Lambda}_{2222}$, of the damaged elastic stiffness (Voigt notations) for two situations in the principal strains. When the principal strain ϵ_1 changes sign, and ϵ_2 is non-zero valued, a stress discontinuity clearly takes place with the two above-mentioned theories: here it is due to the change of the non-diagonal term $\tilde{\Lambda}_{1122}$. According to the Ramtani Theory [29], with second-order "damage tensors," the stiffness tensor is non-symmetric, which is not acceptable (see Table 1).

3. A CONSISTENT UNILATERAL DAMAGE CONDITION

The proposed formulation for the unilateral condition takes its origin in a study made by Andrieux, Bamberger and Marigo [1] at the level of a micro-mechanical analysis. As in the theory by Krajcinovic and Fonseka [20], they used a vector representation for the microcrack systems, but with a different condition to describe the closure of each crack system.

3.1 The Principle of the Proposed Formulation

In the present framework of a macroscopic phenomenological approach of anisotropic damage, we consider its description through second-order or fourth-order tensors. The damage effect on the elastic compliance (or on the elastic stiffness) is not specified for the moment, so that any solution or any anisotropic damage rate equation can be used associated to the proposed formulation.

Assume the present damaged elastic behaviour for fully active conditions is known. We use, for instance, the elastic stiffness $\tilde{\Lambda}(D)$ where the damage variable is now called D. We also consider a given unit vector $\vec{n}$, assumed to be a principal direction of damage (or a direction of maximum damage). The damage

*The terms "diagonal" or "non-diagonal" are used for a matrix representation of the fourth-order tensors, using the standard Voigt notations.

Table 1. *The elastic compliance matrix for biaxial principal strains with four anisotropic and unilateral damage theories. λ and μ are the Lame's coefficients.*

	$\epsilon_1 > 0$ $\epsilon_2 > 0$		$\epsilon_1 < 0$ $\epsilon_2 > 0$	
Vectors	$\lambda + 2\mu + 2(C_1 + C_2)\omega_1^2$	$\lambda + C_1(\omega_1^2 + \omega_2^2)$	$\lambda + 2\mu$	$\lambda + C_1\omega_2^2$
(Krajcinovic-Fonseka)	$\lambda + C_1(\omega_1^2 + \omega_2^2)$	$\lambda + 2\mu + 2(C_1 + C_2)\omega_2^2$	$\lambda + C_1\omega_2^2$	$\lambda + 2\mu + 2(C_1 + C_2)\omega_2^2$
2nd Order Tensors	$h_{11}^{+2} + \lambda(1 - \delta)$	$h_{12}^{+2} + \lambda(1 - \delta)$	$h_{11}^{-2} + \lambda(1 - \delta)$	$h_{12}^{+2} + \lambda(1 - \delta)$
[29]	$h_{12}^{+2} + \lambda(1 - \delta)$	$h_{22}^{+2} + \lambda(1 - \delta)$	$h_{12}^{-2} + \lambda(1 - \delta)$	$h_{12}^{+2} + \lambda(1 - \delta)$
4th Order Tensors	$(\lambda + 2\mu)(1 - d)$	$\lambda(1 - d)$	$\lambda + 2\mu$	λ
[17]	$\lambda(1 - d)$	$(\lambda + 2\mu)(1 - d)$	λ	$(\lambda + 2\mu)(1 - d)$
New Formulation:				
2nd Order Tensor	$(\lambda + 2\mu)(1 - d_1)^2$	$\lambda(1 - d_1)(1 - d_2)$	$\lambda + 2\mu$	$\lambda(1 - d_1)(1 - d_2)$
[12] (Section 4.3)	$\lambda(1 - d_1)(1 - d_2)$	$(\lambda + 2\mu)(1 - d_2)^2$	$\lambda(1 - d_1)(1 - d_2)$	$(\lambda + 2\mu)(1 - d_2)^2$

will be considered as fully active if the normal strain $\epsilon_n = \vec{n} \cdot \underset{\sim}{\epsilon} \cdot \vec{n}$ associated to that direction is positive.

As in the formulation by Marigo [1], if the normal strain ϵ_n is negative the term $\tilde{\Lambda}_{1111} - \Lambda_{1111}$ in the stiffness change must disappear [10]. This can be obtained with:

$$\underset{\approx}{C}_{\text{eff}} = \underset{\approx}{\tilde{\Lambda}}(D) + H(-\epsilon_n)\,\underset{\approx}{P} : (\underset{\approx}{\Lambda} - \underset{\approx}{\tilde{\Lambda}}(D)) : \underset{\approx}{P} \tag{7}$$

where $\underset{\approx}{P}$ is the fourth-order projection tensor on the direction $\vec{n}$: $\underset{\approx}{P} = \vec{n} \otimes \vec{n} \otimes \vec{n} \otimes \vec{n}$. The normal strain can be written $\epsilon_n = T_r(\underset{\approx}{P} : \underset{\sim}{\epsilon})$, where T_r is the Trace operator on a second-order tensor.

Now, considering three orthogonal directions $\vec{n}_i$, corresponding to three orthogonal "principal directions" for damage, which will be defined more precisely in the next section, we may easily generalize the above equation as:

$$C_{\text{eff}} = \underset{\approx}{\tilde{\Lambda}}(D) + \eta \sum_{i=1}^{3} H(-T_r(\underset{\approx}{P}_i : \underset{\sim}{\epsilon}))\,\underset{\approx}{P}_i : (\underset{\approx}{\Lambda} - \underset{\approx}{\tilde{\Lambda}}_{(D)}) : \underset{\approx}{P}_i \tag{8}$$

where $\underset{\approx}{P}_i = \vec{n}_i \otimes \vec{n}_i \otimes \vec{n}_i \otimes \vec{n}_i$ and η is a phenomenological material dependent coefficient.

With $\eta = 1$, it is easy to check that Equation (8), written in the coordinate system $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$, gives the matrix C^{eff}_{ijkl} identical to $\tilde{\Lambda}_{ijkl}$ in the same system, except the diagonal terms C^{eff}_{iiii} (i not summed) corresponding to a negative normal strain ϵ_{ii} (i not summed), for which $\tilde{\Lambda}_{iiii}$ is replaced by its initial value Λ_{iiii} (i not summed) in the same system. It shows one important property of the proposed unilateral condition : only the diagonal term corresponding to a negative normal strain is replaced by its value corresponding to the initial undamaged state. The fact that nondiagonal terms are not modified by the active/passive unilateral condition is the key for preserving continuity of stress and strain responses under any loading path. The material coefficient η, with $0 \le \eta \le 1$, allows to describe the eventuality of a residual damage effect for compressive loading conditions.

The dual formulation can easily be written, where the unilateral condition is based on the normal stress. In that case, we have to decompose the elastic compliance:

$$\underset{\approx}{C}^{-1}_{\text{eff}} = \underset{\approx}{\tilde{\Lambda}}^{-1}_{(D)} + \eta \sum_{i=1}^{3} H(-Tr(\underset{\approx}{P}_i : \underset{\sim}{\sigma}))\,\underset{\approx}{P}_i : (\underset{\approx}{\Lambda}^{-1} - \underset{\approx}{\tilde{\Lambda}}^{-1}_{(D)}) : \underset{\approx}{P}_i \tag{9}$$

The two formulations are not equivalent.

Returning back to the thermodynamic framework, it is easy to incorporate the two above formulations as nonlinear (bilinear) elastic laws. In the first case, we use the free energy:

$$\Psi = \frac{1}{2}\underset{\sim}{\epsilon} : \underset{\approx}{C}_{\text{eff}} : \underset{\sim}{\epsilon} \tag{10}$$

where $\underset{\approx}{C}_{\text{eff}}$ is given by Equation (8). In the second case, we use the free enthalpy:

$$\Psi^* = \frac{1}{2}\underset{\sim}{\sigma} : \underset{\approx}{C}_{\text{eff}}^{-1} : \underset{\sim}{\sigma} \tag{11}$$

where the active compliance is given by Equation (9).

3.2 Possible Definitions of the "Principal Directions" of Damage

An advantage of the above unilateral condition is that the definition of the so-called "principal directions of damage" is completely free. We propose below two kinds of choices, related either to the present damage state or to the present strain state.

3.2.1 DIRECTIONS BASED ON THE DAMAGE STATE

In principle, the opening-closure condition of microcracks, has to be related to the directions of cracks, then transposed to a macroscopic description, to the principal directions of the damage tensors:

- *In the case of a second-order damage tensor,* we may choose naturally for $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$, the principal directions of the damage tensor. For instance, if we use the description proposed by Cordebois and Sidoroff [12], with an elastic energy equivalence to define the effective stresses and effective strains, the second-order damage tensor $\underset{\sim}{d}$ is represented by a diagonal matrix of components d_1, d_2, d_3 in its principal axes. In this theory, also used by Lee et al. [23] and Chow and Wang [11], the stiffness tensor expresses, in the same system (using Voigt notations):

$$\frac{\tilde{\Lambda}_{(d)}}{E^*} = \begin{bmatrix} (1-\nu)(1-d_1)^2 & \nu(1-d_1)(1-d_2) & \nu(1-d_1)(1-d_3) \\ \nu(1-d_1)(1-d_2) & (1-\nu)(1-d_2)^2 & \nu(1-d_2)(1-d_3) \\ \nu(1-d_1)(1-d_3) & \nu(1-d_2)(1-d_3) & (1-\nu)(1-d_3)^2 \\ & & \\ & & \\ & & \end{bmatrix}$$

$$\left.\begin{matrix} (1-2\nu)(1-d_2)(1-d_3) & & \\ & (1-2\nu)(1-d_3)(1-d_1) & \\ & & (1-2\nu)(1-d_1)(1-d_2) \end{matrix}\right] \qquad (12)$$

where $E^* = E/(1+\nu)(1-2\nu)$, E and ν being the elastic parameters of the initially isotropic material. In that case, let us note that the direction of the maximum principal damage is also the direction in which the effective Young's modulus (in a uniaxial tensile state) is minimum. The symmetry and continuity with the present unilateral condition is easy to check in the example given for the comparison in Table 1.

- *In the case of a fourth-order damage tensor,* we could also use for $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$ the principal directions of orthotropy of the damage tensor $\underset{\approx}{D}$ or of the associated stiffness tensor $\underset{\approx}{\Lambda}(D)$, if one of them can be assumed as always orthotropic. Unfortunately, this is not the case in general under complex nonproportional loading histories, where superposition of several orthotropic damage increments (of different principal directions), leads to a nonorthotropic resulting damage tensor. In that case, we propose to introduce the "principal directions" as the ones that render minimum the effective Young's modulus in a uniaxial tensile state. More precisely the "maximum principal damage direction" is taken with this definition; the second principal direction is searched in the orthogonal plane, still as the direction corresponding to the minimum effective modulus.

  In some particular cases the principal directions will be undetermined for these two definitions based on the principal directions of damage. This is true for isotropic elastic behaviour or, more frequently, of a transversely isotropic behaviour. In that case, the unilateral active/passive condition has to be taken for the principal strain directions. Physically, among the randomly distributed microcracks that correspond to this isotropic case, the ones that close first are perpendicular to the principal strain that changes from a positive to a negative value.

3.2.2 DIRECTIONS BASED ON THE STRAIN STATE (OR STRESS STATE)

The previous macroscopic theories with unilateral damage, discussed in Section 2, were using a closure condition based on the principal strain (or stress) directions. Even if we write the closure consequences in the way proposed in Section 3.1, using basically a decomposition of the actual elastic stiffness tensor, we still have the opportunity to choose the "principal directions for closure" as the principal strain directions. In addition to its simplicity, an advantage of the present choice is to eliminate automatically the above-mentioned difficulty associated with isotropic situations.

4. A DAMAGE EVOLUTION LAW

In the previous sections, we have defined the present state of a damaged elastic material, incorporating in a consistent way both the anisotropy of damage and its unilateral character. Now, we develop complementary equations in order to define the damage kinetics. This part of the theory is presently limited to the case of the rate independent material.

The damage variable is considered as a fourth-order tensor. The main reason for this choice is the larger descriptive possibilities offered in comparison to the case of a second-order damage tensor. For example, the actual stiffness of a damaged material, even orthotropic, cannot be correctly described by Equation (12).

4.1 Description of the Active Elastic Behaviour

The symmetric fourth-order damage tensor $\underset{\sim}{D}$ describes directly the elastic behaviour of the damaged material submitted to completely "active" loading conditions. Then we adopt the formulation of Section 3.2, with a decomposition of the stiffness given by Equation (8) and a unilateral damage condition based on a normal strain criterion. For additional simplification, we choose the principal strain directions to define the "principal directions for closure."

In the particular case of positive principal strains, that is when all microcracks are open, then active, we use the damage tensor as a linear transformation between the initial and actually damaged elastic stiffness of the material. We assume the following symmetric form:

$$\tilde{\underset{\approx}{\Lambda}}_{(D)} = \frac{1}{2}\left[(\underset{\approx}{1} - \underset{\approx}{D}) : \underset{\approx}{\Lambda} + \underset{\approx}{\Lambda} : (\underset{\approx}{1} - \underset{\approx}{D}) \right] \tag{13}$$

$\underset{\approx}{\Lambda}$ being the initial elastic stiffness and $\underset{\approx}{1}$ the fourth-order identity operator (which components u_{ijkl} are 0, except when $i = k = j = l$, where $u_{iiii} = 1$).

Note two differences with the theory proposed by Chaboche [4]. The damage tensor is forced to be symmetric (in the previous theory it was deduced from the knowledge of $\tilde{\underset{\approx}{\Lambda}}$ by $\underset{\approx}{D} = \underset{\approx}{1} - \tilde{\underset{\approx}{\Lambda}} : \underset{\approx}{\Lambda}^{-1}$ which is not consistent here). Moreover, the previous form $\tilde{\underset{\approx}{\Lambda}} = (\underset{\approx}{1} - \underset{\approx}{D}) : \underset{\approx}{\Lambda}$ was symmetric only for an initial isotropic behaviour and, for varying temperature conditions, with a constant Poisson's ratio. Here, the symmetry is preserved for any mechanical and thermal history.

The notion of effective stress [3,4] is still applicable. The free energy is of the form:

$$\Psi = \frac{1}{2} \underset{\sim}{\epsilon} : \underset{\approx}{C}_{\text{eff}} : \underset{\sim}{\epsilon} \tag{14}$$

so that the actual stress becomes:

$$\underset{\sim}{\sigma} = \underset{\approx}{C}_{\text{eff}} : \underset{\sim}{\epsilon} \tag{15}$$

where $\underset{\approx}{C}_{\text{eff}}$ is given by Equation (8). The effective stress is defined in terms of a strain equivalence – the effective stress $\tilde{\underset{\sim}{\sigma}}$ is the stress tensor to be applied to the initial undamaged material in order to restitute the same strain tensor as the one produced on the damaged material by the presently applied stress tensor $\underset{\sim}{\sigma}$. With this definition at hand, we easily find:

$$\tilde{\underset{\sim}{\sigma}} = \underset{\approx}{M}^{-1} : \underset{\approx}{\sigma} \tag{16}$$

where the fourth-order operator $\underset{\approx}{M}$ is given by $\underset{\approx}{C}_{\text{eff}} : \underset{\approx}{\Lambda}^{-1}$. In the case of active damage, using Equation (13), $\underset{\approx}{M}$ expresses as

$$\frac{1}{2}[(\underset{\approx}{1} - \underset{\approx}{D}) + \underset{\approx}{\Lambda} : (\underset{\approx}{1} - \underset{\approx}{D}) : \underset{\approx}{\Lambda}^{-1}]$$

Such a form is different from the one used earlier [4]: $\underset{\approx}{M} = \underset{\approx}{1} - \underset{\approx}{D}$.

We now have to express the affinities associated to the damage variable $\underset{\approx}{D}$. They depend on the actual strain state through:

$$\underset{\approx}{Y} = -\frac{\partial \Psi}{\partial \underset{\approx}{D}} = -\frac{1}{2}\underset{\sim}{\epsilon} : \frac{\partial \underset{\approx}{C}_{\text{eff}}}{\partial \underset{\approx}{D}} : \underset{\sim}{\epsilon} \tag{17}$$

For the case of a purely active damage, we recover a relation similar to the one obtained classically [22,24]:

$$\underset{\approx}{Y} = \frac{1}{2}[\underset{\sim}{\epsilon} \otimes (\underset{\approx}{\Lambda} : \underset{\sim}{\epsilon})]_s \tag{18}$$

where $\otimes$ stands for the tensorial product ($\underset{\approx}{Y}$ is a symmetric fourth-order tensor). In the reverse case, we have additional terms, which can be expressed easily in the principal strain system. To Y_{iiii} and Y_{iijj} (i and j not summed). We subtract respectively:

$$\frac{1}{2}\eta \Lambda_{iiii} (\epsilon_i^-)^2 \text{ and } \frac{1}{4}\eta \Lambda_{iijj} [(\epsilon_i^-)^2 + (\epsilon_j^-)^2]$$

where ϵ_i^- are the negative principal strains and are taken as 0 for positive principal strains.

4.2 The Choice of a Damage Loading Surface

As in Section 2, for the rate independent damaging process, we use a damage criterion in the form:

$$g(\underset{\approx}{Y}\,;\,\underset{\sim}{\epsilon},\,\delta) = y - r(\delta) = Tr(\underset{\approx}{Q} : \underset{\approx}{Y}) - r(\delta) \leq 0 \tag{19}$$

$\underset{\approx}{Q}$ is a fourth-order operator to be defined below; it can depend on the present strain. $Tr(\underset{\approx}{A})$ means the Trace operator on the fourth-order $\underset{\approx}{A}$, which can also be written $\underset{\approx}{A} : : \underset{\approx}{1}$ or, with indices A_{ijij}. δ represents a scalar damage measure and $r(\delta) \geq 0$ is the associated present damage threshold.

Assuming the normality rule for the damage rate equations, we write:

$$\dot{\underset{\approx}{D}} = \dot{\mu}\frac{\partial g}{\partial \underset{\approx}{Y}} = \dot{\mu}\underset{\approx}{Q} \qquad \dot{\delta} = -\dot{\mu}\frac{\partial g}{\partial r} = \dot{\mu} \tag{20}$$

which leads to:

$$\dot{\underset{\approx}{D}} = \underset{\approx}{Q}\dot{\delta} \tag{21}$$

Note that the Kuhn-Tucker relations [Equation (5)] are taken into account by the fact that r is the maximum value attained in the past by $y = Tr(\underset{\approx}{Q} : \underset{\approx}{Y})$, that is $\dot{\delta} = 0$ if $g < 0$. Otherwise, we have $\dot{\delta} > 0$.

Equation (21) describes the anisotropic damage rate. It allows separation, in the simplest way of the two following aspects:

- The direction of the damage rate, which is given by the tensor $\underset{\approx}{Q}$. During proportional loading conditions, this tensor will be considered as constant.
- The non-linearity of the damage evolution, which will be contained in the scalar damage rate $\dot{\delta}$ or, consistently for the rate independent material, in the knowledge of the scalar function $y = r(\delta)$.

The expression of the Second Principle of Thermodynamics in the present case is:

$$\dot{\Phi}_d = -\frac{\partial \psi}{\partial \underset{\approx}{D}} : : \dot{\underset{\approx}{D}} = Tr(\underset{\approx}{Y} : \dot{\underset{\approx}{D}}) = Tr(\underset{\approx}{Q} : \underset{\approx}{Y})\dot{\delta} = y\dot{\delta} \geq 0 \tag{22}$$

4.3 The Choice of Damage Rate Directionality

The direction of the damage rate is given in Equation (21) by the fourth-order symmetric tensor $\underset{\approx}{Q}$. Many forms can be proposed for it, depending both on the material and on the loading conditions. One possibility, similar to the one used by Chaboche in 1979 [4] is to consider $\underset{\approx}{Q}$ as defined, in the principal directions of the effective strain, by a constant (material dependent) tensor:

$$\underset{\approx}{Q}^* = (1 - \xi)\,\underset{\approx}{1} + \xi\,\underset{\approx}{\Gamma}^* \tag{23}$$

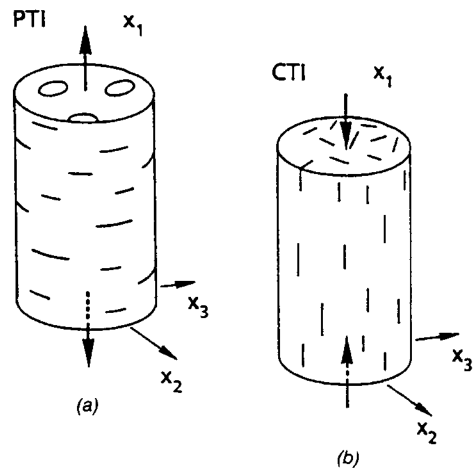

Figure 2. *Two specific cracking arrangements. (a) Planar transverse isotropy, produced by $\sigma_1 > 0$, $\sigma_2 = \sigma_3 = 0$; (b) cylindrical transverse isotropy, produced by $\sigma_1 < 0$, $\sigma_2 = \sigma_3 = 0$.*

where ξ is a material parameter giving the degree of anisotropy of the damage growth rate. $\underset{\approx}{\Gamma}^*$ is a fourth-order tensor, fixed from a simple completely orthotropic case, where the microcracks grow perpendicular to the maximum principal effective stress. From the elastic analysis of this particular case, with randomly distributed parallel microcracks (Figure 2), one can find the macroscopic elastic compliance, with a planar transverse isotropy (PTI), as given by Hoenig [15] for an initially isotropic material (Voigt notations in the order 11, 22, 33, 23, 31, 12):

$$\underset{\approx}{\tilde{\Lambda}}^{-1} = \frac{1}{E}\begin{bmatrix} \frac{1}{H_1} & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & -1 & & & \\ & & & 2(1+\nu) & & \\ & & & & \frac{2(1+\nu)}{H_4} & \\ & & & & & \frac{2(1+\nu)}{H_4} \end{bmatrix} \tag{24}$$

H_1 and H_4 are the damage parameters. The corresponding stiffness has the following form:

$$\frac{\tilde{\Lambda}}{E^*} = \begin{bmatrix} (1-\nu)(1-D_1) & \nu(1-D_1) & \nu(1-D_1) & & & \\ \nu(1-D_1) & (1-\nu)(1-D_2) & \nu(1-D_{23}) & & & \\ \nu(1-D_1) & \nu(1-D_{23}) & (1-\nu)(1-D_2) & & & \\ & & & 1-2\nu & & \\ & & & & (1-2\nu)(1-D_4) & \\ & & & & & (1-2\nu)(1-D_4) \end{bmatrix} \tag{25}$$

with $D_2 = (\nu/(1-\nu))^2 D_1$ and $D_{23} = (\nu/(1-\nu))D_1$; D_4 is proportional to D_1. The relationship between $\tilde{\Lambda}$ and the fourth-order damage tensor being assumed in the form of Equation (25), the tensor Γ^* in Equation (23) is obtained for $\xi = 1$:

$$\Gamma^* = a \begin{bmatrix} 2-3\nu-\nu^2 & \nu(1-\nu) & \nu(1-\nu) & & & \\ \nu(1-\nu) & \nu^2 & \nu^2 & & & \\ \nu(1-\nu) & \nu^2 & \nu^2 & & & \\ & & & 0 & & \\ & & & & b/a & \\ & & & & & b/a \end{bmatrix} \tag{26}$$

with two material dependent coefficients a and b, corresponding respectively to tensile and shear terms.

However, the above choice suffers of a deficiency. The unilateral condition is not taken into account for a uniaxial tensile loading in the direction 1. The normal strain condition formulated in Section 3.1, needs to replace D_2 by 0 in the active damaged stiffness [Equation (25)].

For this reason, we do propose an alternative choice, obeying to the following conditions:

1. The direction of the damage rate, representative of the direction of the microcrack propagation, is given by a linear combination like Equation (23), the simplest one, in order to restitute the possibility for a limiting isotropic damage theory (with $\xi = 0$).
2. In order to restitute closure effects and their influence on the damage growth, we have to consider a separation between positive and negative principal strains.
3. In the particular case where the three principal strains are negative, the damage rate can correspond to the isotropic case.
4. $\underset{\approx}{\Gamma}$ is dependent on the principal strains. Its form is given in order to restitute approximately the planar transverse isotropy (PTI) considered above, for a proportional uniaxial tensile loading.
5. The formulation must restitute also approximately, for a uniaxial compressive loading, the microcrack situation stated by Hoenig [15] as: cylindrical transverse isotropy (CTI). The cracks are randomly distributed [(Figure 2(b)]; their normals are perpendicular to the compressive direction (they are produced by the positive transverse principal strains $\epsilon_2 = \epsilon_3 > 0$). In that case, the elastic compliance takes the Hoenig form:

$$\underset{\approx}{\tilde{\Lambda}}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & \dfrac{1}{H_2} & -\tilde{\nu} & & & \\ -\nu & -\tilde{\nu} & \dfrac{1}{H_2} & & & \\ & & & \dfrac{1+\nu}{H_4} & & \\ & & & & \dfrac{1+\nu}{H_5} & \\ & & & & & \dfrac{1+\nu}{H_5} \end{bmatrix} \tag{27}$$

where $\tilde{\nu}$ is taken so as to check the transverse isotropy condition:

$$\tilde{\nu} = \frac{1+\nu}{H_4} - \frac{1}{H_2}$$

There are probably many forms of the damage rate operator $\underset{\approx}{Q}$ that follow the five conditions. Among them, we propose the following candidate formula. $\underset{\approx}{Q}$ depends on the actual strain and is expressed in the principal strain space:

$$\underset{\approx}{Q}(\underset{\sim}{\epsilon}) = (1 - \xi)\underset{\approx}{1} + \frac{\xi}{\epsilon_{eq}^2}\underset{\approx}{\Gamma}(\underset{\sim}{\epsilon}) \tag{28}$$

where ϵ_{eq} is taken as a norm of the strain tensor, for instance its second invariant $\epsilon_{eq} = (\epsilon_{ij}\,\epsilon_{ij})^{1/2}$. The fourth-order symmetric tensor $\underset{\approx}{\Gamma}$ is written as:

$$\underset{\approx}{\Gamma}(\epsilon) = \sum_{i=1}^{3}\sum_{j=1}^{3}\{[(\underset{\approx}{\Gamma}^* : \underset{\sim}{N}^{ij})\otimes\underset{\sim}{N}^{ij}]_s H(\epsilon_j)$$

$$- c\underset{\sim}{N}^{ij}\otimes\underset{\sim}{N}^{ij}H(-\epsilon_j)\}H(\epsilon_i)\epsilon_i\epsilon_j \tag{29}$$

$\underset{\sim}{N}^{ij}$ is the second-order symmetric tensor constructed with the unit directions $\vec{n}_i$ and $\vec{n}_j$ associated to the principal strain ϵ_i and ϵ_j:

$$\underset{\sim}{N}^{ij} = (\vec{n}_i\otimes\vec{n}_j)_s \tag{30}$$

H is the Heaviside function. Γ^* is the material dependent fourth-order tensor:

$$\underset{\approx}{\Gamma}^* = \begin{bmatrix} a & b & b & & & \\ b & a & b & & & \\ b & b & a & & & \\ & & & 2(a-b) & & \\ & & & & 2(a-b) & \\ & & & & & 2(a-b) \end{bmatrix} \tag{31}$$

In the present formulation, we have only five free coefficients to define the directionality of the damage rate: η for the active/passive condition, ξ for the linear combination of the pure isotropic case and of the "pure anisotropic" one; a and b in the tensor $\underset{\approx}{\Gamma}^*$, associated to positive principal strains, and c associated to the combinations of a positive and a negative principal strain.

The role of the term $2(a - b)$ in $\underset{\approx}{\Gamma}^*$ produces the evolution of shear moduli in $\underset{\approx}{\Lambda}$ associated with two positive principal strains. The value is taken as $a - b$ in order to restitute the CTI, cylindrical transverse isotropy, when these two principal strains are positive and equal.

Note that b must be taken as zero if we want to recover exactly the transverse isotropy after damage under pure uniaxial tension. In that case, the compliance change does not correspond exactly to the PTI solution of Hoenig [15] as given by Equation (24). This solution is only approximately recovered, when damage is small.

The second term in Equation (29), with the material coefficient c gives the evolution of the shear moduli associated with mixed principal strains (one positive, one negative). It will play a role for instance for uniaxial tension ($\epsilon_1 > 0$, $\epsilon_2 = \epsilon_3 < 0$), and describe the evolution of $\tilde{\Lambda}_{1212}$ and $\tilde{\Lambda}_{3131}$.

5. CONCLUSION

The description of the progressive deterioration of materials is one of the most difficult tasks, even when using simplified phenomenological approaches of Continuum Damage Mechanics. One of the open problems was associated with the simultaneous description of:

- the directionality of damage and the corresponding induced anisotropy for the (elastic) behaviour of the material
- the unilateral nature of damage, which is associated to the possibility for the existing microcracks to be open (active damage) or closed (passive damage) by the external loads

In previous papers we have shown that none cf the presently existing damage theories with anisotropic *and* unilateral damage can be accepted. Either inconsistent nonsymmetries of the elastic stiffness or nonphysical discontinuities of the stress-strain response can be observed. This is true for every kind of anisotropic damage description: vectors, second-order or fourth-order tensors.

Based on an initial work by Andrieux et al. [1] a new general formulation has been proposed recently [9] and is briefly discussed. The unilateral condition does modify only the "diagonal" principal terms of the stiffness or compliance operators; these being evaluated in a system consistent with the present "principal directions of damage." Two versions of the new condition are formulated using:

- a compliance decomposition and a stress-based closure condition
- a stiffness decomposition and a strain-based closure condition

Three different possibilities have been proposed and discussed for the definition of the "principal directions of damage":

- the principal directions of a second-order damage tensor
- the directions that render minimum the effective Young's modulus (in a tensile test) for fourth-order damage tensors
- the principal strain themselves, as in other theories with unilateral and anisotropic damage

A consistent form for the anisotropic damage growth equation has been proposed, based on the concept of a damage loading surface for a rate independent elastic material. The rate form of this equation combines, in a very simple way:

- the directionality of damage rate, which is considered as constant for a given loading direction
- the non-linearity of the evolution, which is taken as independent of the direction
- a linear combination of a pure isotropic case and a "pure anisotropic" one to define the directionality of the damage rate

- an evaluation of the anisotropic term on the basis of simple solutions for elastic solids containing specific microcrack arrangements

The proposed constitutive and growth equations have been formulated in their general form. Further investigations are necessary to evaluate all the practical possibilities of the theory. The various proposed criteria need to be combined systematically in order to obtain the most consistent and powerful formulation, i.e., second-order or fourth-order tensors, unilateral condition based on stress or strain, choice for the "principal directions," form of the damage rate equation. Simple procedures need to be established for the determination of the material constants from experimental data.

The present article has dealt with the elastic, but damaging material. Additional research is needed to consistently formulate the inelastic constitutive equations and the corresponding damage evolutions. The notion of effective stresses/effective strains could offer interesting possibilities.

REFERENCES

1. Andrieux, S., Y. Bamberger and J. J. Marigo. 1986. "Un Modèle de Matériau Microfissuré pour les Bétons et les Roches," *J. Mécanique Théorique et Appliquée*, 5(3):471–513.
2. Budiansky, B., R. J. O'Connell. 1976, "Elastic Moduli of a Cracked Solid," *Int. J. Solids Structures*, 12:81–97.
3. Chaboche, J.-L. 1977. "Sur l'Utilisation des Variables d'état Interne pour la Description du Comportement Viscoplastique et de la Rupture par Endommagement," *Symposium Franco-Polonais de Rhéologie et Mécanique, Cracovie.*
4. Chaboche, J.-L. 1979. "Le Concept de Contrainte Effective, Appliqué à l'Élasticité et à la Viscoplasticité en présence d'un Endommagement Anisotrope," Euromech. 115, Grenoble, *Edition du CNRS*, no. 295, 1982, pp. 737–760, English translation by R. A. E. Farnborough, Trans 2039, 1980.
5. Chaboche, J.-L. 1981. "Continuous Damage Mechanics. A Tool to Describe Phenomena before Crack Initiation," *Nuclear Engineering and Design*, 64:233–247.
6. Chaboche, J.-L. 1988. "Continuous Damage Mechanics. Parts I and II," *J. of Applied Mechanics*, 55:59–72.
7. Chaboche, J.-L. 1990. "On the Description of Damage Induced Anisotropy and Active/Passive Effect," *ASME 1990 Winter Annual Meeting*, Dallas.
8. Chaboche, J.-L. 1992. "Damage Induced Anisotropy: On the Difficulties Associated with the Active/Passive Unilateral Condition," *Int. J. of Damage Mechanics*, 1(2):148–171.
9. Chaboche, J.-L. 1992. "A New Unilateral Condition for the Description of Material Behaviour with Anisotropic Damage," Paris: C. R. Acad. Sci., t. 314, Série II, pp. 1395–1401.
10. Chaboche, J.-L. 1993. "A Continuum Damage Theory with Anisotropic and Unilateral Damage," proposed to *ASME J. of Applied Mechs.*
11. Chow, C. L. and Wang June. 1987. "An Anisotropic Theory of Continuum Damage Mechanics for Ductile Fracture," *Engineering Fracture Mechanics*, 27(5):547–558.
12. Cordebois, J. P. and F. Sidoroff. 1979. "Anistropie Élastique Induite par Endommagement," Euromech. 115, Grenoble, *Editions du CNRS*, no. 295, 1982.
13. Delameter, W. R. and G. Hermann. 1974. "Weakening of Elastic Solids by Doubly-Periodic Arrays of Cracks," *Topics in Applied Continuum Mechanics*. Springer Verlag.
14. Germain, P., Q. S. Nguyen and P. Suquet. 1983. "Continuum Thermodynamics," *J. of Applied Mechanics*, ASME, 50:228–232.

15. Hoenig, A. 1979. "Elastic Moduli of a Non Randomly Cracked Body," *Int. J. Solids Structures*, 15:137–154.
16. Hult, J. 1979. "Continuum Damage Mechanics—Capabilities, Limitations and Promises," *Mechanisms of Deformation and Fracture*. Oxford: Pergamon, pp.233–247.
17. Ju, J. W. 1989. "On Energy-Based Coupled Elastoplastic Damage Theories: Constitutive Modeling and Computational Aspects," *Int. J. Solids and Structures*, 25(7):803–833.
18. Kachanov, L. M. 1958. "Time of the Rupture Process under Creep Conditions," *Isv. Akad. Nauk. SSR*, Otd Tekh. Nauk, (8):26–31.
19. Kachanov, M. 1980. "Continuum Model of Medium with Cracks," *J. of the Engineering Mechanics Division*, ASCE, 106(EM5):1039–1051.
20. Krajcinovic, D. and G. U. Fonseka. 1981. "The Continuous Damage Theory of Brittle Materials, Parts I and II," *J. of Applied Mechanics*, ASME, 48:809–824.
21. Krajcinovic, D. 1984. "Continuum Damage Mechanics," *Applied Mechanics Reviews*, 37(1):1–6.
22. Ladeveze, P. and J. Lemaitre. 1984. "Damage Effective Stress in Quasi-Unilateral Material Conditions," *IUTAM Congress, Lyngby*, Denmark.
23. Lee, H., K. Peng and J. Wang. 1985. "An Anisotropic Damage Criterion for Deformation Instability and Its Application to Forming Limit Analysis of Metal Plates," *Engng. Fract. Mech.*, 21(5):1031–1054.
24. Lemaitre, J. and J.-L. Chaboche. 1985, "Mécanique des Matériaux Solides," *Dunod*. Paris: Cambridge Univ. Press, English edition, 1990.
25. Mazars, J., G. Pijaudier-Cabot. 1989. "Continuum Damage Theory—Application to Concrete," *J. of Engineering Mechanics*, 115(2):345–365.
26. Murakami, S. and N. Ohno. 1980. "A Continuum Theory of Creep and Creep Damage," *3rd IUTAM Symposium on Creep in Structures, Leicester.*
27. Ortiz, M. 1985. "A Constitutive Theory for the Inelastic Behavior of Concrete," *Mech. Mater.*, 4:67–93.
28. Rabotnov, Y. N. 1969. "Creep Problems in Structural Members," *North-Holland.*
29. Ramtani, S. 1990. "Contribution à la Modélisation du Comportement Multiaxial du Béton Endommagé avec Description du Caractère Unilatéral," Thèse, Univ. Paris VI.
30. Simo, J. C. and J. W. Ju. 1987. "Strain- and Stress-Based Continuum Damage Models—I-Formulation," *Int. J. Solids Structures*, 23(7):821–840.
31. Vakulenko, A. A. and M. L. Kachanov. 1971. "Continuum Theory of Medium with Cracks," *Mekhanika Tverdogo Tela*, (4):159–166 (in Russian).