Pergamon

Int. J. Mech. Sci. Vol. 39, No. 4, pp. 473–486, 1997

0020-7403/97 $17.00 + 0.00

PII: S0020-7403(96)000034-3

CONSTITUTIVE AND DAMAGE EVOLUTION EQUATIONS OF ELASTIC-BRITTLE MATERIALS BASED ON IRREVERSIBLE THERMODYNAMICS

S. MURAKAMI and K. KAMIYA

Department of Mechanical Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-01, Japan

(Received 5 April 1995; and in revised form 19 February 1996)

Abstract—Unified descriptions of the constitutive and evolution equations of elastic-brittle damage materials are developed on the basis of irreversible thermodynamic theory for constitutive equations. The Helmholtz free energy is assumed to be a function of elastic strain tensor and second rank symmetric damage tensor. In order to take account of the effects of unilateral condition of damage due to the opening and closure of microcracks, modified elastic strain tensor is introduced into the Helmholtz free energy. A damage dissipation potential related to the entropy production rate is expressed in terms of damage conjugate force. The constitutive and the damage evolution equations derived by these potentials were applied to an elastic-brittle damage material. The anisotropic elastic-brittle damage behavior of high-strength concrete under uniaxial, proportional and non-proportional combined loading was analysed to elucidate the utility and the limitations of the present theory. Finally, the initial damage surfaces in the axial-shear and biaxial stress spaces are calculated. Copyright © 1996 Elsevier Science Ltd.

Keywords: elastic-brittle material, damage, constitutive equation, evolution equation, irreversible thermodynamics, Helmholtz free energy, dissipation potential, unilateral damage, combined loading, damage surface.

NOTATION

A_k	thermodynamic conjugate force corresponding to V_k
B	thermodynamic conjugate force of β
$\boldsymbol{D}$ and D_{ij}	second rank symmetric damage tensor and its components
D_i	principal value of the tensor $\boldsymbol{D}$
E_0	initial Young's modulus
F	dissipation potential
$\boldsymbol{g}$	temperature gradient; $\boldsymbol{g} \equiv -\operatorname{grad} T$
$\boldsymbol{J}$	mechanical flux vector
K_d	material constant
$\boldsymbol{L}$	material constant tensor of fourth rank
$\boldsymbol{n}_i$	unit vector of principal direction of the tensor $\boldsymbol{D}$
$\boldsymbol{q}$	heat flux
s	specific entropy
T	absolute temperature
$\boldsymbol{V}_k$	internal state variable tensor of even rank
$\boldsymbol{X}$	thermodynamic conjugate force corresponding to $\boldsymbol{J}$
$\boldsymbol{Y}$ and Y_{ij}	damage conjugate force tensor and its components
Y_{eq}	equivalent damage conjugate force
β	scalar variable prescribing further development of damage
$\boldsymbol{\varepsilon}$	strain tensor
$\boldsymbol{\varepsilon}^e$ and ε^e_{ij}	elastic strain tensor and its components
$\tilde{\boldsymbol{\varepsilon}}^e$	modified elastic strain tensor
$\boldsymbol{\varepsilon}^p$	plastic strain tensor
$\eta_1, \eta_2, \eta_3, \eta_4$	material constants
λ and μ	Lamé constants
ν_0	initial Poisson's ratio
ρ	mass density
$\boldsymbol{\sigma}$ and σ_{ij}	Cauchy stress tensor and its components
ψ	specific Helmholtz free energy
$\boldsymbol{\Phi}$	specific entropy production rate
ζ	material constant describing the crack closure effect

1. INTRODUCTION

Continuum Damage Mechanics (CDM) has made significant development in the past two decades, and has proved to be a systematic and promising approach to the analysis of damage and fracture process, or the process of the damage and deformation interaction in wide variety of materials [1–4]. The most crucial problems for the further development of CDM, however, will be the proper and accurate modeling of material damage, besides the numerical schemes for the solution of initial and boundary value problems in computational CDM. For this purpose, a number of irreversible thermodynamics theories of CDM [2–19] have been developed so far as a systematic framework to the unified description of the inelastic deformation and the related internal structural change due to material damage.

Lemaitre, Chaboche and their co-workers [2–6, 19, 20] in particular, developed a series of irreversible thermodynamic theory for wide variety of problems ranging from elastic-brittle to elastic-plastic-ductile, creep, and fatigue damage of metals, geological materials and composites.

Krajcinovic and others [4, 7–10] on the other hand, developed thermodynamic theories of anisotropic damage mainly by use of vector damage variables, and discussed in detail the damage process of brittle and creep materials. Chow and his co-workers [11–14] proposed an energy-based elastic-plastic damage model in order to describe the difference in the observed failure modes of geological materials under compression and tension, by use of a damage tensor identified by the fourth rank elasticity tensor and a fourth rank projection tensor. Simo and Ju [15, 16] and Voyiadjis and Kattan [17, 18] furthermore, discussed the coupling between elastic-plastic deformation and anisotropic damage by use of a second rank symmetric damage tensor.

In the present paper, we will develop the description of anisotropic damage behavior of elastic-brittle materials under multiaxial state of stress by employing a second rank symmetric damage tensor [21, 22] in the irreversible thermodynamic frameworks. The Helmholtz free energy is first expressed as a function of simultaneous invariants of the elastic strain tensor and the damage tensor by taking account of the unilateral behavior of elastic-brittle material. A dissipation potential is represented as a quadratic function of the conjugate force of the damage tensor to ensure the necessary condition for a thermodynamic potential. Constitutive equations are applied to describe the anisotropic elastic damage behavior of high-strength concrete under various loading histories.

2. DAMAGE VARIABLES

Damage in materials is usually induced by nucleation, growth and coalescence of certain microscopic cavities. Since the development of these cavities is governed by the action of applied stress and strain, material damage is essentially anisotropic; this feature is especially important in brittle materials damaged by the development of distributed and oriented microscopic cracks [23, 24]. Thus, a scalar damage variable often has a serious limitation to the description of actual material damage, and a number of theories have been developed to model the anisotropic damage state by means of damage variables ranging from a vector to higher rank tensors [21, 28].

Vector damage variables [7–10, 25], above all, would be straightforward to be applied to describe the distributed plane cracks; i.e. to describe the normal direction of each plane crack and the crack density of the specific plane. The vector damage variables, however, cannot represent the effect of a set of plane cracks of different orientations by means of a simple addition of the corresponding vectors. Moreover, odd rank tensors in general can be excluded from the set of internal variables, since they do not conform to the invariance with respect to the rotation of the frame [26].

Thus, in spite of the complexity of their mathematical structures, more elaborate theories have been proposed for the description of the state and the effect of the distributed cavities by use of higher and even rank tensors. Among them, second rank symmetric damage tensors [21, 22, 26–28] are most commonly employed because they are mathematically simpler than the higher rank of tensors, and yet can describe most essential features of anisotropic damage.

In order to develop a feasible damage theory in this paper, we will postulate that the anisotropic damage state can be described by a damage tensor

$$\boldsymbol{D} = \sum_{i=1}^{3} D_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{1}$$

where D_i and $\boldsymbol{n}_i$ are the principal value and the unit vector of principal direction of the tensor $\boldsymbol{D}$. The damage tensor $\boldsymbol{D}$ of Eqn (1) was derived by Murakami and Ohno [21, 22] by postulating that the principle effect of the material damage consists in the net area reduction due to three-dimensional (3D) distribution of microscopic cavities in the material. Then, D_i in Eqn (1) can be interpreted as the ratio of area reduction in the plane perpendicular to $\boldsymbol{n}_i$ caused by the development of cavities. Though the second rank symmetric damage tensor $\boldsymbol{D}$ cannot describe more complicated damage state than orthotropy, it has been often employed in the development of anisotropic damage theories [11–24].

3. IRREVERSIBLE THERMODYNAMIC THEORY FOR MATERIAL DAMAGE

Irreversible thermodynamics furnishes rational frameworks for the unified formulation of constitutive and evolution equations of damaged material caused by internal change of the material [2–20, 29, 30]. For the sake of the subsequent development of this paper, we will first make a brief review of the constitutive theory of thermodynamics with internal variables [3, 4, 8, 28, 29].

In the case of infinitesimal deformation, the Clausius-Duhem inequality has the following form:

$$\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \rho(\dot{\psi} + s\dot{T}) - \frac{\boldsymbol{q}}{T}\cdot \mathrm{grad}\ T \geqslant 0 \tag{2}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are stress and strain tensor, ρ, ψ, s, T and $\boldsymbol{q}$ are mass density, specific Helmholtz free energy, specific entropy, absolute temperature and heat flux, respectively.

We will first assume that the total strain tensor $\boldsymbol{\varepsilon}$ can be decomposed into elastic and inelastic strain tensor $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and $\boldsymbol{\varepsilon}^{\mathrm{p}}$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}. \tag{3}$$

The microstructural change in materials in the process of inelastic deformation and damage may be induced by the changes in density and configuration of dislocations, by the development of microscopic cavities and other imperfections. We then postulate that internal states of materials characterized by these changes can be properly described by a set of tensor variables of even ranks:

$$\{\boldsymbol{V}_k; k = 1, 2, \cdots\}. \tag{4}$$

In view of Eqn (3), we have the Helmholtz free energy as follows:

$$\psi = \psi(\boldsymbol{\varepsilon}^{\mathrm{e}}, \boldsymbol{V}_k, T) \tag{5}$$

Substitution of Eqns (3) and (5) into Eqn (2) furnishes the following constitutive equations and the equation of specific entropy production rate:

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}}, \quad s = -\rho \frac{\partial \psi}{\partial T} \tag{6}$$

$$\rho \boldsymbol{\Phi} = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{A}_k \cdot \dot{\boldsymbol{V}}_k + \frac{\boldsymbol{g}}{T}\cdot \boldsymbol{q} \geqslant 0 \tag{7}$$

where $\boldsymbol{A}_k$ and $\boldsymbol{g}$ are the thermodynamic conjugate forces corresponding to $\boldsymbol{V}_k$ and temperature gradient:

$$\boldsymbol{A}_k \equiv -\rho \frac{\partial \psi}{\partial \boldsymbol{V}_k} \quad (k = 1, 2, \cdots), \quad \boldsymbol{g} \equiv -\mathrm{grad}\ T. \tag{8}$$

Now, if we define the mechanical flux vector $\boldsymbol{J}$ and their thermodynamic conjugate force vector $\boldsymbol{X}$ as follows;

$$\boldsymbol{J} = \rho\{\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{V}}_k, \boldsymbol{q}\}^{\mathrm{T}} \tag{9}$$

$$\boldsymbol{X} = \left\{\boldsymbol{\sigma}, \boldsymbol{A}_k, \frac{\boldsymbol{g}}{T}\right\} \tag{10}$$

then Eqn (7) of entropy production has the alternative form:

$$\rho \boldsymbol{\Phi} = \boldsymbol{X}\cdot \boldsymbol{J} \geqslant 0. \tag{11}$$

Rice [30] supposed that the rate at which any structural rearrangement occurs is entirely determined by the forces associated with the rearrangement. In this case, if each component J_k of mechanical flux vector $\boldsymbol{J}$ can be assumed to be a function of the corresponding components X_k of the thermodynamic force vector $\boldsymbol{X}$ and the current state:

$$J_k = \tilde{J}_k(X_k, T, \boldsymbol{J}), \tag{12}$$

then there exists a dissipation potential $F(\boldsymbol{X})$ which is a homogeneous, convex function of the corresponding conjugate forces as follows:

$$J_k = \frac{\partial F}{\partial X_k}. \tag{13}$$

By use of Eqns (9), (10) and (13), each component of $\boldsymbol{J}$ can be given as follows:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\partial F}{\partial \boldsymbol{\sigma}}, \quad \dot{\boldsymbol{V}}_k = \frac{\partial F}{\partial \boldsymbol{A}_k}, \quad \boldsymbol{q} = \frac{\partial F}{\partial(\boldsymbol{g}/T)}. \tag{14}$$

The advantage of the proposed procedure is quite obvious. Instead of separate formulation of constitutive and damage evolution equations in the CDM theory, a unified description of CDM is possible only by establishing a single potential. Furthermore, the damage theory established in this way is formally similar to the common theory of plasticity. Extension of the present theory to the case of finite deformation will not induce any essential difficulties.

4. APPLICATION TO ELASTIC-BRITTLE MATERIALS

4.1. *Formulation of Helmholtz free energy for elastic-brittle materials*

We will now apply the thermodynamic theory discussed in the previous section to the damage process of unilateral elastic-brittle materials. By the elastic-brittle materials we imply a material which is damaged by the development of distributed microscopic cracks and leads to the final fracture by their coalescence without significant inelastic deformation. The orientation of the distributed cracks saliently depends on the direction of the external load. Geological materials under confined uniaxial compression, for example, will be damaged mainly by nucleation and growth of microcracks almost parallel to the loading direction.

In order to represent the state of anisotropic damage characterized by these cracks, we will employ a second rank symmetric damage tensor $\boldsymbol{D}$ discussed in Section 2. Besides the damage effects represented by $\boldsymbol{D}$, an additional effect may be also needed to identify the damage state which is responsible mainly to the further development of damage, and thus we will introduce another scalar damage variable β for this purpose, which corresponds to the isotropic hardening variable r in plastic deformation theory [19].

By restricting the present discussion to a purely mechanical process and by disregarding inelastic strain, the internal state variables of Eqn (4) will be given by

$$\boldsymbol{V}_k = \{\boldsymbol{D}, \beta\}. \tag{15}$$

In view of this relation, Eqn (8) furnishes the thermodynamic conjugate force $\boldsymbol{A}_k = \{\boldsymbol{Y}, -B\}$ corresponding to $\boldsymbol{V}_k$ as follows:

$$\boldsymbol{Y} \equiv -\rho\left(\frac{\partial \psi}{\partial \boldsymbol{D}}\right), \quad -B \equiv -\rho\left(\frac{\partial \psi}{\partial \beta}\right) \tag{16}$$

where the minus sign of the conjugate force B is taken for the convenience of further formulation. The second rank symmetric tensor $\boldsymbol{Y}$ given above, in particular, represents the energy release of material due to the development of damage, and is equivalent to the energy release rate G in fracture mechanics [3, 14, 23].

We are now in a position to determine a proper form of Helmholtz free energy. It will be assumed first that the free energy may consist of two parts; one is the free energy mainly related to elastic deformation but affected by damage, and the other is the energy exclusively related to the damage development. Namely, we will postulate the following expression for the Helmholtz free energy:

$$\rho\psi(\boldsymbol{\varepsilon}^{\mathrm{e}},\boldsymbol{D},\beta) = \rho\psi^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}},\boldsymbol{D}) + \rho\psi^{\mathrm{d}}(\beta). \tag{17}$$

According to the representation theory of non-linear algebra [31, 32], the most general form of the tensor valued scalar function $\rho\psi^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}},\boldsymbol{D})$ can be expressed by the combination of ten basic invariants of two symmetric tensors $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and $\boldsymbol{D}$. At the initial undamaged state, the elastic-brittle material is assumed to be isotopic and linear elastic, and the elastic strain energy $\rho\psi^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}},\boldsymbol{D})$ may decrease with the process of damage $\boldsymbol{D}$. Thus, the function $\rho\psi^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}},\boldsymbol{D})$ is quadratic in $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and linear in $\boldsymbol{D}$, and can be given as the linear combination of the following terms:

$$(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})^2,\quad \mathrm{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}})^2,\quad (\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})^2\ \mathrm{tr}\boldsymbol{D},\quad \mathrm{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}})^2\ \mathrm{tr}\boldsymbol{D}$$

$$\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}}\ \mathrm{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}}\boldsymbol{D}),\quad \mathrm{tr}[(\boldsymbol{\varepsilon}^{\mathrm{e}})^2\boldsymbol{D}]. \tag{18}$$

One of the difficulties in the formulation of CDM theory for the elastic-brittle material is concerned with the unilateral nature of damage. Thus, the anisotropic nature of damage and its unilateral character have been often discussed [15, 16, 20, 33–36]. Ramtani [36] for example, employed a second rank symmetric damage tensor and strain decomposition in order to describe the unilateral nature of damage. Ju and his co-workers [15, 16] used the fourth rank damage tensor and fourth rank positive projection tensor to discuss this problem. Chaboche [34], on the other hand, reviewed the possibilities of the stress-strain discontinuity for multiaxial strain paths in the preceding unilateral damage theories, and showed that the discontinuous stress-strain response may occur when unilateral condition affects both the diagonal and the non-diagonal terms of the compliance (or stiffness) tensors. Recently, he proposed a more general theory which eliminates the possibility of discontinuous stress-strain response [35].

In the present paper, we will employ the following modified elastic strain tensor $\bar{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ to represent unilateral response of the damaged materials as already employed by Mazars *et al.* [20], Ladveze and Lemaitre [33] and Ramtani [36]:

$$\bar{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \langle\boldsymbol{\varepsilon}^{\mathrm{e}}\rangle - \zeta\langle -\boldsymbol{\varepsilon}^{\mathrm{e}}\rangle \tag{19}$$

$$[\langle\boldsymbol{\varepsilon}^{\mathrm{e}}\rangle] = \begin{bmatrix} \langle\varepsilon_1^{\mathrm{e}}\rangle & 0 & 0 \\ 0 & \langle\varepsilon_2^{\mathrm{e}}\rangle & 0 \\ 0 & 0 & \langle\varepsilon_3^{\mathrm{e}}\rangle \end{bmatrix},\quad [\langle -\boldsymbol{\varepsilon}^{\mathrm{e}}\rangle] = \begin{bmatrix} \langle -\varepsilon_1^{\mathrm{e}}\rangle & 0 & 0 \\ 0 & \langle -\varepsilon_2^{\mathrm{e}}\rangle & 0 \\ 0 & 0 & \langle -\varepsilon_3^{\mathrm{e}}\rangle \end{bmatrix} \tag{20}$$

where $\langle\ \rangle$ is Macauley bracket, and $\varepsilon_i^{\mathrm{e}}$ $(i = 1, 2, 3)$ and ζ are the principal values of $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and a material constant describing the crack closure effect.

Since the unilateral effect is observed only for non-vanishing damage $\boldsymbol{D}$, the strain tensor in the invariants of Eqn (18) except for the first and second ones are assumed to be replaced by the modified strain tensor of Eqns (19) and (20). By use of Eqns (18)–(20), and by taking account of the avoidance of the stress-strain discontinuity [34, 35], the final form of the damaged strain energy $\rho\psi^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}},\boldsymbol{D})$ can be given as follows:

$$\rho\psi^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}},\boldsymbol{D}) = \tfrac{1}{2}\lambda(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})^2 + \mu\mathrm{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}})^2 + \eta_1\mathrm{tr}\boldsymbol{D}(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})^2 + \eta_2\mathrm{tr}\boldsymbol{D}\mathrm{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}})^2 + \eta_3\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}}\mathrm{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}}\boldsymbol{D}) + \eta_4\mathrm{tr}[(\bar{\boldsymbol{\varepsilon}}^{\mathrm{e}})^2\boldsymbol{D}] \tag{21}$$

where symbols λ and μ are Lamé constants, and η_1–η_4 are newly introduced material constants.

The Helmholtz free energy $\rho\psi^{\mathrm{d}}(\beta)$ has been introduced here to represent the effect of the damage state on the further development of damage, and thus we will assume the following quadratic form as a simplest expression in β as

$$\rho\psi^{\mathrm{d}}(\beta) = \tfrac{1}{2}K_{\mathrm{d}}\beta^2 \tag{22}$$

where K_{d} is a material constant.

4.2. *Constitutive and damage evolution equations of elastic-brittle materials*

Substitution of the Helmholtz free energy (21) into Eqns (6) and (16) yields the following constitutive equations together with the expression for the thermodynamic conjugate forces:

$$\boldsymbol{\sigma} = \frac{\partial(\rho\psi)}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = \frac{\partial(\rho\psi^{\mathrm{e}})}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}}$$

$$= [\lambda(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}}) + 2\eta_1(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})(\mathrm{tr}\boldsymbol{D}) + \eta_3\mathrm{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}}\boldsymbol{D})]\boldsymbol{I}$$

$$+ 2[\mu + \eta_2(\mathrm{tr}\boldsymbol{D})]\boldsymbol{\varepsilon}^{\mathrm{e}} + \eta_3(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})\boldsymbol{D} + 2\eta_4\bar{\boldsymbol{\varepsilon}}^{\mathrm{e}}:\left(\frac{\partial\bar{\boldsymbol{\varepsilon}}^{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{e}}}\boldsymbol{D} + \boldsymbol{D}\frac{\partial\bar{\boldsymbol{\varepsilon}}^{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{e}}}\right) \tag{23}$$

$$\boldsymbol{Y} = -\frac{\partial(\rho\psi)}{\partial\boldsymbol{D}} = -\frac{\partial(\rho\psi^{\mathrm{e}})}{\partial\boldsymbol{D}}$$

$$= -[\eta_1(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})^2 + \eta_2\mathrm{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}})^2]\boldsymbol{I} - \eta_3(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}}) - \eta_4\bar{\boldsymbol{\varepsilon}}^{\mathrm{e}}\bar{\boldsymbol{\varepsilon}}^{\mathrm{e}} \tag{24}$$

$$B = \frac{\partial(\rho\psi)}{\partial\beta} = \frac{\partial(\rho\psi^{\mathrm{d}})}{\partial\beta} = K_{\mathrm{d}}\beta. \tag{25}$$

The damage evolution equation will be furnished, provided a proper expression of the dissipation potential of Eqn (13) can be established. Furthermore, according to the experiments of acoustic emission on geological materials [37], a finite region was observed in strain (stress) space within which no brittle damage occurs for any loading path. By considering this experimental fact, we can assume the existence of a damage criterion in the space of the thermodynamic conjugate forces $\{\boldsymbol{Y}, -B\}$. This criterion will be associated with the dissipation potential $F(\boldsymbol{X})$, which should be a convex and homogeneous function in the space of thermodynamic conjugate forces. Similarly to Chow *et al.* [11, 14], we will assume the following homogeneous function of degree one for the surface of dissipation potential (damage surface):

$$F(\boldsymbol{Y}, B) = Y_{\mathrm{eq}} - (B_0 + B) = 0 \tag{26}$$

$$Y_{\mathrm{eq}} = \sqrt{\tfrac{1}{2}\boldsymbol{Y}:\boldsymbol{L}:\boldsymbol{Y}} \tag{27}$$

where B_0 is a material constant and represents the initial threshold of the damage evolution. The symbol $\boldsymbol{L}$ in Eqn (27) is a fourth rank tensor which may depend on the state of damage $\boldsymbol{D}$ in general. However, according to the assumption of Eqn (27), $\boldsymbol{L}$ may be given as the following isotropic tensor:

$$L_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{28}$$

In view of Eqn (14), the evolution equations for $\dot{\boldsymbol{D}}$ and $\dot{\beta}$ are given by the dissipation potential of Eqn (26) as follows:

$$\dot{\boldsymbol{D}} = \dot{\lambda}_{\mathrm{d}}\frac{\partial F}{\partial\boldsymbol{Y}} \tag{29}$$

$$\dot{\beta} = \dot{\lambda}_{\mathrm{d}}\frac{\partial F}{\partial(-B)} = \dot{\lambda}_{\mathrm{d}} \tag{30}$$

where $\dot{\lambda}_{\mathrm{d}}$ is a multiplier to be determined by the consistency condition on the damage surface. Namely, in order that the damage may further develop, the conjugate forces should remain on the surface, and thus we have

$$\dot{\lambda}_{\mathrm{d}} = \alpha\frac{\partial F}{\partial\boldsymbol{Y}}:\dot{\boldsymbol{Y}}\bigg/\left(\frac{\partial B}{\partial\beta}\right) = \alpha\frac{\boldsymbol{L}:\boldsymbol{Y}}{2K_{\mathrm{d}}Y_{\mathrm{eq}}}:\dot{\boldsymbol{Y}} \tag{31}$$

$$\begin{cases} \alpha = 1, & \text{if } F = 0 \text{ and } \dfrac{\partial F}{\partial\boldsymbol{Y}}:\dot{\boldsymbol{Y}} > 0 \\ \alpha = 0, & \text{if } F < 0 \text{ or } \dfrac{\partial F}{\partial\boldsymbol{Y}}:\dot{\boldsymbol{Y}} \leqslant 0. \end{cases} \tag{32}$$

5. ELASTIC-BRITTLE DAMAGE ANALYSIS FOR HIGH STRENGTH CONCRETE

5.1. *Uniaxial compression*

To examine the utility of the constitutive and damage evolution equations formulated in the preceding section, the process of elastic deformation and brittle fracture of a high strength concrete will be analysed.

In order to determine the material constants of the preceding equations, we will first analyse uniaxial compression of the column specimen of a high strength concrete, and then compare the results with those of the experiments [38].

Let us employ a rectangular coordinate system (x_1, x_2, x_3), where the axis x_1 is taken in the direction of column axis, while x_2 and x_3 are perpendicular to x_1. Furthermore, in case of uniaxial loading, there exist components of the elastic strain tensor ε^e_{11}, ε^e_{22} and ε^e_{33}. Because of the fact that $\varepsilon^e_{11} < 0$ and $\varepsilon^e_{22}, \varepsilon^e_{33} > 0$, the components of modified elastic strain specified by Eqns (19) and (20) are given as follows:

$$[\bar{\varepsilon}^e] = \begin{bmatrix} \zeta\varepsilon^e_{11} & 0 & 0 \\ 0 & \varepsilon^e_{22} & 0 \\ 0 & 0 & \varepsilon^e_{33} \end{bmatrix}. \tag{33}$$

The state of stress, on the other hand, has the following components:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\sigma_{11} < 0). \tag{34}$$

In view of Eqns (33) and (34), the constitutive equation (23) can be expressed as follows:

$$\begin{aligned} \sigma_{11} &= [\lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}\boldsymbol{D} + 2(\eta_3 + \eta_4\zeta^2)D_{11}]\varepsilon^e_{11} \\ &\quad + [\lambda + 2\eta_1\mathrm{tr}\boldsymbol{D} + \eta_3(D_{11} + D_{22})]\varepsilon^e_{22} + [\lambda + 2\eta_1\mathrm{tr}\boldsymbol{D} + \eta_3(D_{11} + D_{33})]\varepsilon^e_{33} \end{aligned} \tag{35}$$

$$\begin{aligned} \sigma_{22} = 0 &= [\lambda + 2\eta_1\mathrm{tr}\boldsymbol{D} + \eta_3(D_{11} + D_{22})]\varepsilon^e_{11} \\ &\quad + [\lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}\boldsymbol{D} + 2(\eta_3 + \eta_4)D_{22}]\varepsilon^e_{22} \\ &\quad + [\lambda + 2\eta_1\mathrm{tr}\boldsymbol{D} + \eta_3(D_{22} + D_{33})]\varepsilon^e_{33} \end{aligned} \tag{36}$$

$$\begin{aligned} \sigma_{33} = 0 &= [\lambda + 2\eta_1\mathrm{tr}\boldsymbol{D} + \eta_3(D_{11} + D_{33})]\varepsilon^e_{11} \\ &\quad + [\lambda + 2\eta_1\mathrm{tr}\boldsymbol{D} + \eta_3(D_{22} + D_{33})]\varepsilon^e_{22} \\ &\quad + [\lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}\boldsymbol{D} + 2(\eta_3 + \eta_4)D_{33}]\varepsilon^e_{33}. \end{aligned} \tag{37}$$

By use of Eqns (21) and (24), each component of the conjugate force $\boldsymbol{Y}$ of the damage tensor, on the other hand, are given as follows:

$$Y_{11} = -\eta_1(\mathrm{tr}\boldsymbol{\varepsilon}^e)^2 - \eta_2\mathrm{tr}(\boldsymbol{\varepsilon}^e)^2 - \eta_3(\mathrm{tr}\boldsymbol{\varepsilon}^e)\varepsilon^e_{11} - \eta_4\zeta^2(\varepsilon^e_{11})^2 \tag{38}$$

$$Y_{22} = -\eta_1(\mathrm{tr}\boldsymbol{\varepsilon}^e)^2 - \eta_2\mathrm{tr}(\boldsymbol{\varepsilon}^e)^2 - \eta_3(\mathrm{tr}\boldsymbol{\varepsilon}^e)\varepsilon^e_{22} - \eta_4(\varepsilon^e_{22})^2 \tag{39}$$

$$Y_{33} = -\eta_1(\mathrm{tr}\boldsymbol{\varepsilon}^e)^2 - \eta_2\mathrm{tr}(\boldsymbol{\varepsilon}^e)^2 - \eta_3(\mathrm{tr}\boldsymbol{\varepsilon}^e)\varepsilon^e_{33} - \eta_4(\varepsilon^e_{33})^2. \tag{40}$$

Thus, Eqns (29), (31), (32) and (38)–(40) furnish the following damage evolution equations:

$$\dot{D}_{11} = \alpha\frac{Y_{11}(Y_{11}\dot{Y}_{11} + Y_{22}\dot{Y}_{22} + Y_{33}\dot{Y}_{33})}{2K_d(Y_{11}^2 + Y_{22}^2 + Y_{33}^2)} \tag{41}$$

$$\dot{D}_{22} = \alpha\frac{Y_{22}(Y_{11}\dot{Y}_{11} + Y_{22}\dot{Y}_{22} + Y_{33}\dot{Y}_{33})}{2K_d(Y_{11}^2 + Y_{22}^2 + Y_{33}^2)} \tag{42}$$

$$\dot{D}_{33} = \alpha\frac{Y_{33}(Y_{11}\dot{Y}_{11} + Y_{22}\dot{Y}_{22} + Y_{33}\dot{Y}_{33})}{2K_d(Y_{11}^2 + Y_{22}^2 + Y_{33}^2)} \tag{43}$$

where α is a constant defined in Eqn (31).

The value of undamaged initial Young's modulus E_0 and Poisson's ratio ν_0 are determined from literature [7]:

$$E_0 = 21.4\,\text{GPa}, \quad \nu_0 = 0.2. \tag{44}$$

The material constants in Eqns (19)–(22) and (26), on the other hand, are determined so that the present theory can describe the elastic and the damage behavior of the high strength concrete under uniaxial compression (shown by symbols ○ in Fig. 1):

$$\begin{aligned} &\eta_1 = -400\,\text{MPa}, \quad \eta_2 = -900\,\text{MPa}, \\ &\eta_3 = 100\,\text{MPa}, \quad \eta_4 = -23500\,\text{MPa}, \\ &\zeta = 0.1, \quad K_d = 0.04, \quad B_0 = 2.6\times 10^{-3}\,\text{MPa}. \end{aligned} \tag{45}$$

The symbols and the solid lines in Fig. 1, respectively, show the experimental and the predicted results by means of Eqns (34)–(45) for the stress-strain relation of high strength concrete under uniaxial compression. As observed in this figure, the material constants of Eqns (44), (45) and the constitutive and evolution equations (35)–(43) describe the experimental results with sufficient accuracy. The dashed line, on the other hand, shows the relation between stress and volumetric strain. This result predicts the dilatancy of this material as observed in a usual geological materials. This phenomena may be caused by the anisotropic development of damage in this material.

Figures 2 and 3 show the process of material damage predicted by the present analysis by use of Eqns (35)–(45). Figure 2 is the evolution of the components of damage variable in the above high-strength concrete under uniaxial compression. Though the difference between D_{11} and $D_{22} = D_{33}$ observed in Fig. 2 is relatively small, this results is consistent with the commonly observed experiments; i.e. both of internal microcracks perpendicular and parallel to the loading direction develop dominantly under compression in the case of elastic-brittle damage material.

In actual process of damage, since final fracture usually occurs before the magnitude of the damage variable $\|\boldsymbol{D}\|$ attains to unity, a critical value of damage variable D_{cr} is often employed to define an additional criterion of the fracture [3, 4]. However, the relevant experimental data on D_{cr} are not available in this case, and hence no fracture criterion has been incorporated in the present formulation. However, it will be observed that the damage state $D_{22} = D_{33} \approx 0.4$ may be quite practical in the critical stage of this material.

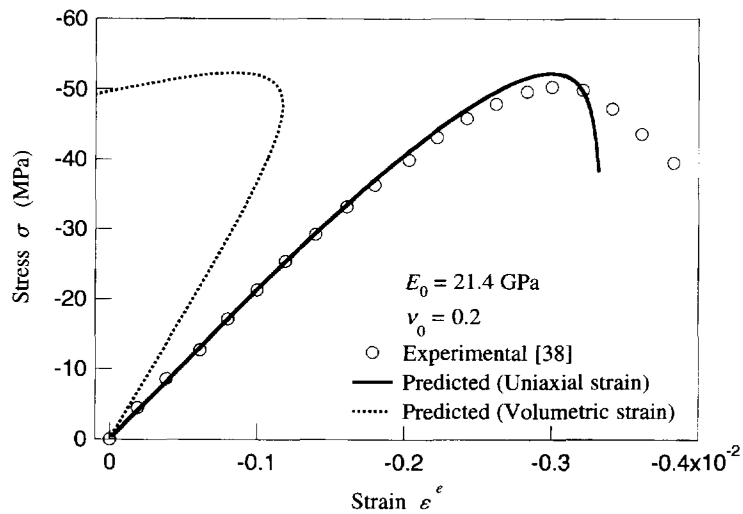

Fig. 1. Uniaxial compression of high-strength concrete; experimental and predicted results.

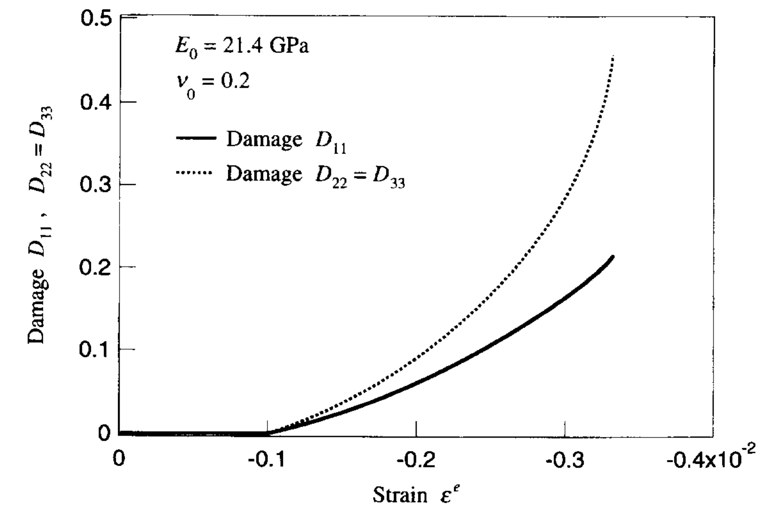

Fig. 2. Damage-strain relations of high-strength concrete under uniaxial compression.

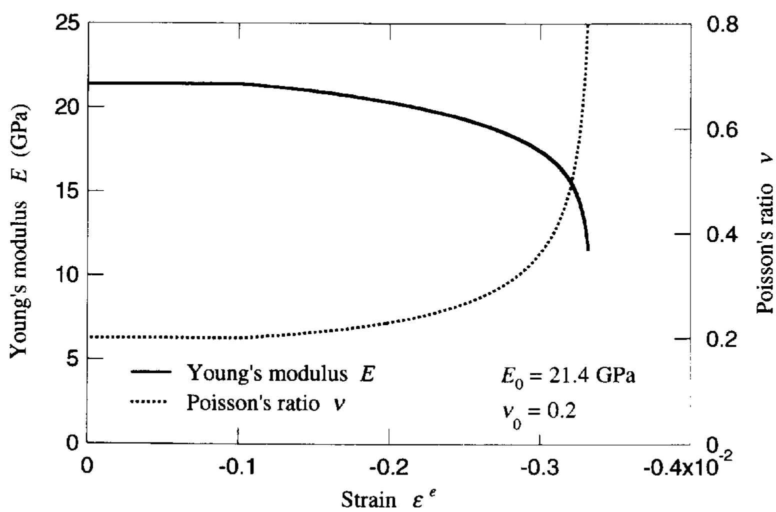

Fig. 3. Predicted results of Young's modulus, E, and Poisson's ratio, ν, under uniaxial compression.

Figure 3 is the prediction of Young's modulus-strain relation and the apparent Poisson's ratio-strain relation. These results also represent the characteristic aspects of the elastic-brittle damage materials. The decrease of Young's modulus due to the development of damage is clearly shown in Eqn (35). The increase of apparent Poisson's ratio, on the other hand, will have close relation with the dilatancy of elastic-brittle damage material as observed in common experimental results of the materials.

5.2. *Uniaxial tension*

In case of uniaxial tension, ε^e_{22} and ε^e_{33} perpendicular to the loading direction x_1 take negative values. Hence, the modified elastic tensor of Eqns (19) and (20), the constitutive equation (23) and the expression of the thermodynamic force, respectively, have the following forms:

$$[\bar{\varepsilon}^e] = \begin{bmatrix} \varepsilon^e_{11} & 0 & 0 \\ 0 & \zeta\varepsilon^e_{22} & 0 \\ 0 & 0 & \zeta\varepsilon^e_{33} \end{bmatrix}. \tag{46}$$

$$\sigma_{11} = [\lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}\boldsymbol{D} + 2(\eta_3 + \eta_4)D_{11}]\varepsilon^e_{11}$$
$$+ [\lambda + 2\eta_1\mathrm{tr}\boldsymbol{D} + \eta_3(D_{11} + D_{22})]\varepsilon^e_{22} + [\lambda + 2\eta_1\mathrm{tr}\boldsymbol{D} + \eta_3(D_{11} + D_{33})]\varepsilon^e_{33} \tag{47}$$

$$\sigma_{22} = 0 = [\lambda + 2\eta_1 \mathrm{tr}\boldsymbol{D} + \eta_3(D_{11} + D_{22})]\varepsilon^e_{11} + [\lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}\boldsymbol{D} + 2(\eta_3 + \eta_4\zeta^2)D_{22}]\varepsilon^e_{22} + [\lambda + 2\eta_1 \mathrm{tr}\boldsymbol{D} + \eta_3(D_{22} + D_{33})]\varepsilon^e_{33} \tag{48}$$

$$\sigma_{33} = 0 = [\lambda + 2\eta_1 \mathrm{tr}\boldsymbol{D} + \eta_3(D_{11} + D_{33})]\varepsilon^e_{11} + [\lambda + 2\eta_1 \mathrm{tr}\boldsymbol{D} + \eta_3(D_{22} + D_{33})]\varepsilon^e_{22} + [\lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}\boldsymbol{D} + 2(\eta_3 + \eta_4\zeta^2)D_{33}]\varepsilon^e_{33} \tag{49}$$

$$Y_{11} = -\eta_1(\mathrm{tr}\boldsymbol{\varepsilon}^e)^2 - \eta_2 \mathrm{tr}(\boldsymbol{\varepsilon}^e)^2 - \eta_3(\mathrm{tr}\boldsymbol{\varepsilon}^e)\varepsilon^e_{11} - \eta_4(\varepsilon^e_{11})^2 \tag{50}$$

$$Y_{22} = -\eta_1(\mathrm{tr}\boldsymbol{\varepsilon}^e)^2 - \eta_2 \mathrm{tr}(\boldsymbol{\varepsilon}^e)^2 - \eta_3(\mathrm{tr}\boldsymbol{\varepsilon}^e)\varepsilon^e_{22} - \eta_4\zeta^2(\varepsilon^e_{22})^2 \tag{51}$$

$$Y_{33} = -\eta_1(\mathrm{tr}\boldsymbol{\varepsilon}^e)^2 - \eta_2 \mathrm{tr}(\boldsymbol{\varepsilon}^e)^2 - \eta_3(\mathrm{tr}\boldsymbol{\varepsilon}^e)\varepsilon^e_{33} - \eta_4\zeta^2(\varepsilon^e_{33})^2. \tag{52}$$

The expressions of damage evolution equations under this condition are same as those under compression, Eqns (41)–(43). However, the components of the damage conjugate force $\boldsymbol{Y}$ under uniaxial tension (50)–(52) are different from those under uniaxial compression (38)–(40), and thus we have different damage development for these cases.

Figures 4 and 5 show the stress-strain and the damage evolution under uniaxial tension predicted by Eqns (46)–(52) by use of the material constants of Eqns (44) and (45) identified by the uniaxial compression test. Unstable fracture in brittle material under uniaxial tension will occur no later than the ultimate stress. This critical state in Figs 4 and 5 corresponds to $(\sigma)_u = 12.3$ MPa, $(\varepsilon^e)_u = 7.8 \times 10^{-4}$, $(D_{11})_u = 0.13$ and $(D_{22} = D_{33})_u = 0.01$, and hence the curves in Figs 4–6 beyond this stage have been entered by thin lines.

Comparison between Figs 1 and 4 shows that the stress at the final fracture under uniaxial tension is about four times smaller than that under uniaxial compression. This feature may be in contrast to the usual observations in concrete in which the fracture stress in tension is usually smaller by five times or more than than that of compression [20]. Thus, though the introduction of modified elastic strain tensor $\bar{\boldsymbol{\varepsilon}}^e$ gives considerably smaller strength for tension than that for compression and describes the unilateral nature of elastic-brittle damage materials qualitatively; some quantitative limitation is observed in these results. An improvement of this discrepancy will be facilitated by employing the critical damage D_{cr} in the present theory, as already pointed out in Section 5.1.

As regards the unilateral behavior of different elastic-brittle materials, preceding theories have assumed the different formulation, or different material constants for under a tensile and compressive condition [7]. In the present paper, however, we can describe the different behavior in these conditions by use of only single set of equations and material constants, without any additional assumptions for each condition.

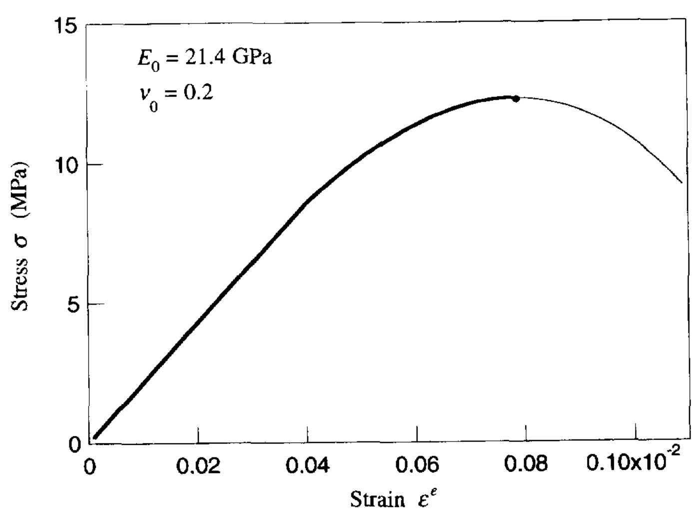

Fig. 4. Predicted results of stress-strain relation under uniaxial tension.

As observed in Fig. 5, damage component D_{11} which represents the damage of loading direction is much larger than the other components D_{22} and D_{33}. This result is different from that under compression observed in Fig. 2. These results show that the anisotropic behavior of damage under uniaxial tension is more significant than under uniaxial compression. The different aspects of damage progress under each condition are consistent with the actual behavior of elastic-brittle damage materials.

The change in Young's modulus and the apparent Poisson's ratio of the high strength concrete under uniaxial compression are shown in Fig. 6. By comparing Fig. 6 with Fig. 3, we can observe much more rapid decrease of Young's modulus in Fig. 6 than in Fig. 3. On the other hand, the value of Poisson's ratio under uniaxial compression increases as the strain grows, while that under uniaxial tension decreases. These different aspects are accounted for by the difference between constitutive Eqns (35)–(37) and (47)–(49). Though the first bracketed term on the right-hand side of Eqn (47) and that of Eqn (35) play a significant role on the decrease in the value of Young's modulus, a decrease in the value of the term of Eqn (47) due to damage development is much larger than that of Eqn (35). This difference results from the introduction of the modified elastic strain tensor (19) into the Helmholtz free energy (21).

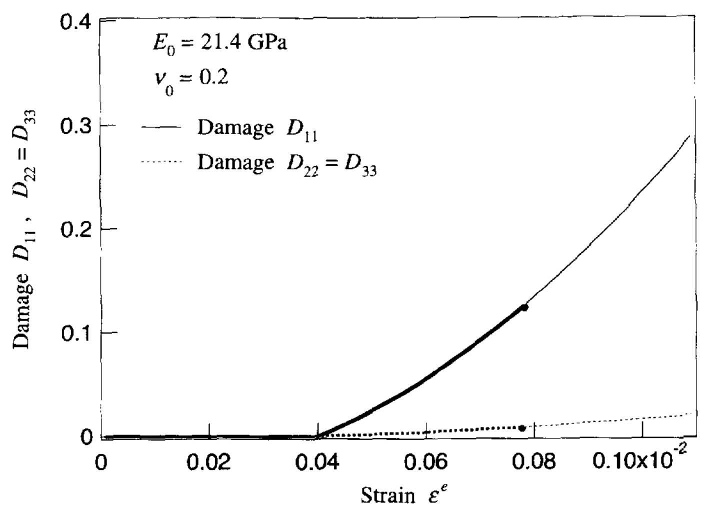

Fig. 5. Predicted results of damage-strain relation under uniaxial tension.

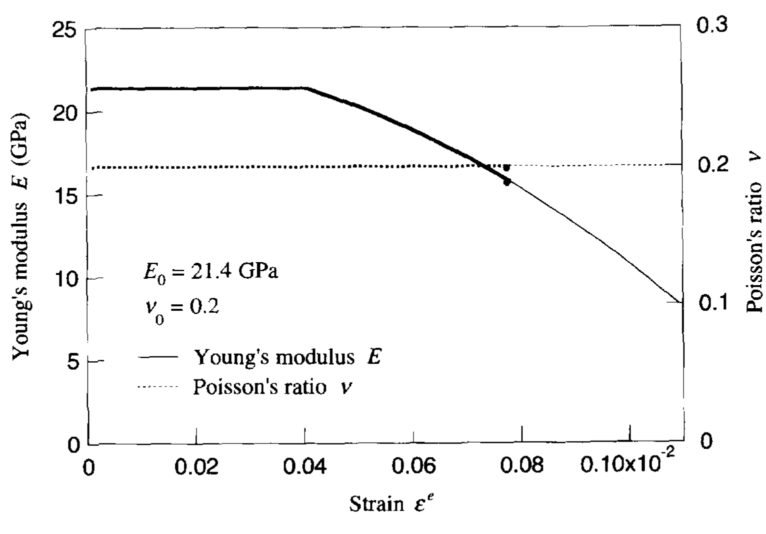

Fig. 6. Predicted results of Young's modulus, E, and Poisson's ratio, ν, under uniaxial tension.

5.3. *Strain path dependence on elastic-brittle materials*

Let us now calculate the elastic-brittle damage behavior under non-proportional loading paths 1, 2 and 3 shown in Fig. 7.

Figure 8 shows the three strain paths corresponding to stress paths in Fig. 7. As observed in Fig. 8, despite that each stress path in Fig. 7 leads to the same stress state P, the corresponding strain paths strain to the slightly different strain state P_1, P_2 and P_3. Furthermore, each strain path shows the significant non-linearity due to the different way of development of anisotropic damage. In particular, as observed in the strain path 3 in Fig. 8, the axial strain is developed even though shear stress only is applied (see, around point B). However, such a phenomenon can not be observed in strain path 1. These features are attributable to the path dependence of the damage development [22].

5.4. *Initial damage surface*

The proposed damage surface of Eqn (26) is described in the space of thermodynamic conjugate forces $\boldsymbol{Y}$ of damage tensor. The proposed constitutive equation (23) and expression (24) of $\boldsymbol{Y}$ enable us to describe the shape of damage surface (26) in stress (or strain) space. In general, the damage surface will depend on the stress (or strain) path which the elastic-brittle damage materials has experienced. However, as far as the initial damage surface is concerned, it is independent from any stress (or strain) path, because the proposed equation show a usual elastic material behavior without damage.

Figures 9(a) and (b) show the initial damage surface in the axial stress-shear stress space and that in biaxial stress space. These surfaces are obtained as the stress loci where the damage starts to

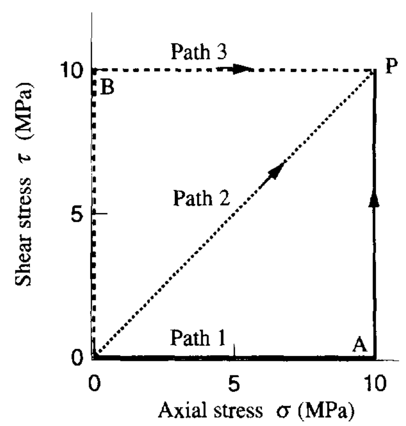

Fig. 7. Shear stress τ-axial stress σ paths.

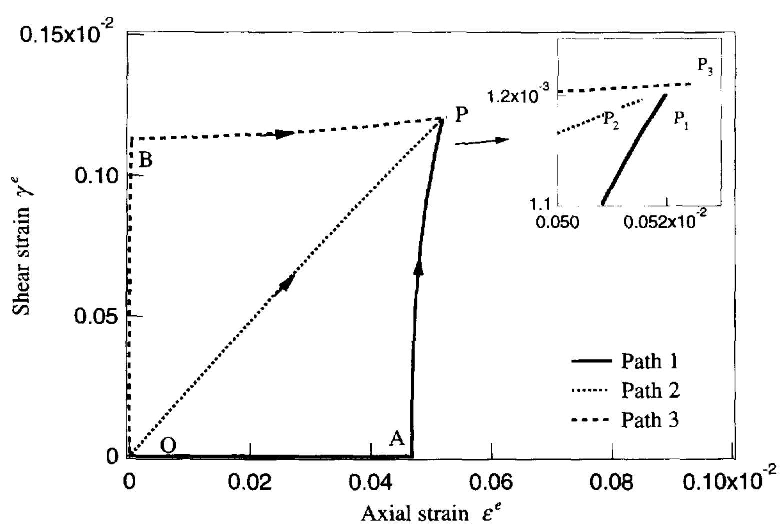

Fig. 8. Predicted results of shear strain γ^e-axial strain ε^e paths.

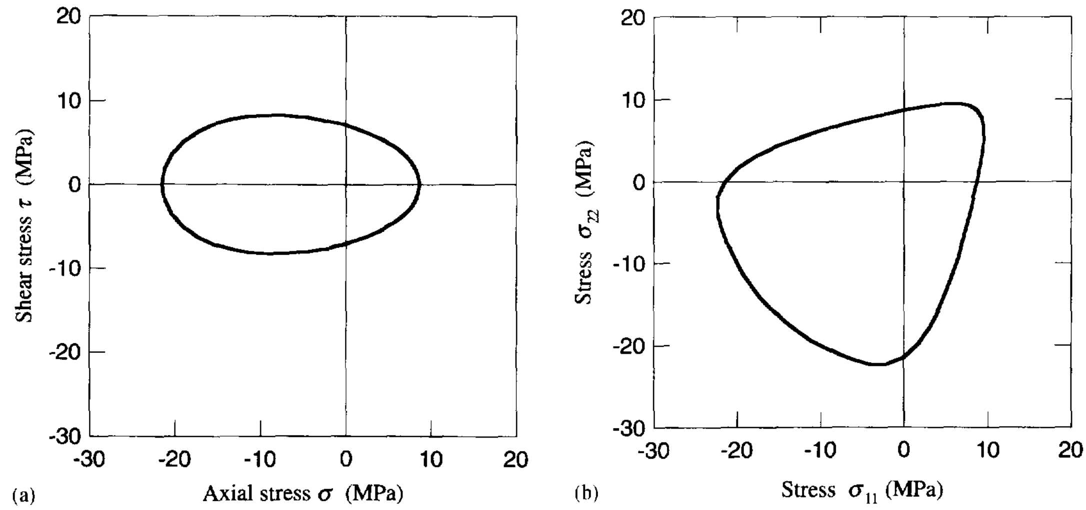

Fig. 9. Predicted initial damage surface: (a) shear stress τ-axial stress σ space; (b) biaxial stress space.

develop by various proportional loading paths in the corresponding stress spaces. Though the unilateral nature of damage initiation is well observed in these figures, the quantitative discussion on the validity of these predictions should have recourse to the experiments.

Further quantitative improvement of the present theory may be possible by introducing the terms which depends on the sign of the volumetric strain besides the modified elastic strain tensor $\tilde{\varepsilon}^e$ of Eqns (19) and (20) into the Helmholtz free energy of Eqn (21), if the influence of the stress discontinuity stated in Section 4.1 [34, 35] is not so significant.

6. CONCLUSIONS

Irreversible thermodynamic theory of the constitutive and the damage evolution equations for elastic-brittle damage materials was developed by employing a second rank symmetric damage tensor in order to describe the state of anisotropic damage. The Helmholtz free energy was represented in terms of the modified elastic strain tensor in order to represent the anisotropic and the unilateral effects. The resulting theory was applied to describe the anisotropic and unilateral aspects in elastic damage behavior, the change in elastic moduli in a high-strength concrete under uniaxial monotonic tension and compression as well as under non-proportional combined axial-shear loading. The damage surfaces in the axial-shear and biaxial stress space were also discussed.

Material constants for the resulting equations were identified by applying them to describe the results of the uniaxial compression tests on the high-strength concrete, and then the response under tensile test was predicted.

The prediction of uniaxial loading show that, the present formulation can describe the unilateral nature of the elastic-brittle damage materials without assuming the specific independent equations, or assuming different material constants for different loading conditions. However, further improvement of the accuracy of the prediction of the fracture under tensile loading may be facilitated by incorporating the critical damage value D_{cr} in the present modeling.

The application to the different loading paths reveals the path dependence of the damage progress as well as the interaction of shear strain to axial strain under the shear loading path. The path dependence, however, is not so significant in the present result.

Though the initial damage surface calculated in the axial stress-shear stress and biaxial stress space can describe qualitatively the behavior of the high-strength concrete, further research will be necessary from theoretical and experimental point of view to verify its validity.

The discussion of the behavior of the damage surface under different loading conditions will be needed for the further development of the present formulation.

Acknowledgement—The authors are grateful for the support in part for the present work by the Ministry of Education, Science, Sports and Culture of Japan under the Grant-in-Aid for Scientific Research (B) [No. 05452125] for the fiscal years of 1994 and 1995, and the Grant-in-Aid for Developmental Scientific Research (B) [No. 7555346] for the fiscal year of 1995.

REFERENCES

1. L. M. Kachanov, *Introduction to Continuum Damage Mechanics*. Martinus-Nijhoff, Dordrecht (1986).
2. J. Lemaitre, *A Course on Damage Mechanics*. Springer-Verlag, Berlin (1992).
3. J. Lemaitre and J. -L. Chaboche, *Mechanics of Solid Materials*. Cambridge University Press, Cambridge, U.K. (1990).
4. D. Krajcinovic and J. Lemaitre (eds.), *Continuum Damage Mechanics—Theory and Applications*. Springer-Verlag, Berlin (1986).
5. J. Lemaitre, Damage modeling for prediction of plastic or creep fatigue failure in structures, *Trans. Struct. Mech. React. Technol.* **L** (L 5/1*b) (1979).
6. J. -L. Chaboche, Sur l'utilisation des variables d'etat interne pour la description due comportment viscoplastique et de la rupture par endommagement. In *Problems Non-Lineaires de Mécanique*, p. 137. PWN, Warsaw (1980).
7. D. Krajcinovic and G. U. Fonseka, The continuous damage theory of brittle materials—I and II. *J. Appl. Mech., Trans. ASME* **48**, 809 (1981).
8. D. Krajcinovic, Constitutive equation for damaging materials. *J. Appl. Mech., Trans. ASME* **50**, 355 (1983).
9. D. Krajcinovic and S. Selvaraj, Creep rupture of metals—An analytical model. *J. Engng Mater. Technol., Trans. ASME* **106**, 405 (1984).
10. R. Ilankamban and D. Krajcinovic, A constitutive theory for progressively deteriorating brittle solids. *Int. J. Solids Structures* **23**, 1521 (1987).
11. C. L. Chow and J. Wang, An anisotropic theory of continuum damage mechanics for ductile fracture. *Engng Fract. Mech.* **27**, 547 (1987).
12. C. L. Chow and J. Wang, A finite element analysis of continuum damage mechanics for ductile fracture. *Int. J. Fract.* **38**, 83 (1988).
13. C. L. Chow and T. J. Lu, On evolution laws of anisotropic damage. *Engng Fract. Mech.* **34**, 679 (1989).
14. T. J. Lu and C. L. Chow, On constitutive equations of inelastic solids with anisotropic damage. *Theor. Appl. Fract. Mech.* **14**, 187 (1990).
15. J. C. Simo and J. W. Ju, Strain- and stress-based constitutive damage models—I and II. *Int. J. Solids Structures* **23**, 821–841 (1987).
16. J. W. Ju, On energy-based coupled elastoplastic damage theories: constitutive modeling and computational aspects. *Int. J. Solids Structures* **25**, 803 (1989).
17. G. Z. Voyiadjis and P. I. Kattan, A coupled theory of damage mechanics and finite strain elasto-plasticity—I. Damage and elastic deformations, *Int. J. Engng Sci.* **28**, 421 (1990).
18. P. I. Kattan and G. Z. Voyiadjis, A coupled theory of damage mechanics and finite strain elasto-plasticity—II. Damage and finite strain plasticity, *Int. J. Engng Sci.* **28**, 505 (1990).
19. J. P. Cordebois and F. Sidoroff, Endommagement Anisotrope en Élasticité et Plasticité. *J. Méc. Théor. Appl. Numéro Spécial*, 45 (1982).
20. J. Mazars and G. Pijaudier-Cabot, Continuum damage theory—Application to concrete. *J. Engng Mech.* **115**, 345 (1989).
21. S. Murakami and N. Ohno, Continuum theory of creep and creep damage. In *Creep in Structures* (Edited by A. R. S. Ponter and D. R. Hayhurst), p. 422. Springer-Verlag, Berlin (1981).
22. S. Murakami, Mechanical modeling of material damage. *J. Appl. Mech., Trans. ASME* **55**, 280 (1988).
23. J. -L. Chaboche, Continuum damage mechanics—I and II. *J. Appl. Mech., Trans. ASME* **55**, 59, 65 (1988).
24. S. Murakami, Anisotropic damage in metals. In *Failure Criteria of Structured Media* (Edited J. P. Boehler), p. 99. A. A. Balkema, Rotterdam (1993).
25. L. M. Kachanov, *Foundation of Fracture Mechanics*. Nauka, Moscow (1974).
26. E. T. Onat and F. A. Leckie, Representation of mechanical behavior in the presence of changing internal structures. *J. Appl. Mech., Trans. ASME* **55**, 1 (1988).
27. A. A. Vakulenko and L. M. Kachanov, Continuum theory of media with cracks. *Mekh. Tverdogo Tiela* **4**, 159 (1971) (in Russian).
28. J. Betten, Damage tensor in continuum mechanics. *J. Méc. Théor. Appl.* **2**, 13 (1983).
29. B. D. Coleman and M. E. Gurtin, Thermodynamics with internal state variables. *J. Chem. Phys.* **47**, 597 (1967).
30. J. R. Rice, Inelastic constitutive relation for solids: an internal variable theory and its application to metal plasticity. *J. Mech. Phys. Solids* **19**, 433 (1971).
31. C. Truesdell and W. Noll, The non-linear field theories of mechanics. In *Encyclopedia of Physics* (Edited by S. Flügge), Volume 3. Springer-Verlag, Berlin (1965).
32. A. J. M. Spencer, Theory of invariants. In *Continuum Physics*, Vol. I. (Edited by A. C. Eringen), p. 240. Academic Press, New York (1981).
33. P. Ladeveze and J. Lemaitre, Damage effective stress in quasi-unilateral material conditions, In *IUTAM Congress*, Lyngby, Denmark (1984).
34. J. -L. Chaboche, Damage induced anisotropy: on the difficulties associated with the active/passive unilateral condition. *Int. J. Damage Mech.* **1**, 148 (1922).
35. J. -L. Chaboche, Development of continuum damage mechanics for elastic solids sustaining anisotropic and unilateral damage. *Int. J. Damage Mech.* **2**, 311 (1993).
36. S. Ramtani, Contribution à la modélisation du comportement multiaxial du béton endommagé avec description du caractère unilatéral. Thèse, Université de Paris VI (1990).
37. D. J. Holcomb, Discrete memory in rock: a review. *J. Rheology* **28**, 725 (1984).
38. P. T. Wang, Complete stress–strain curves of concrete and its effect on ductility of reinforced concrete members. Ph.D. Thesis, University of Illinois at Chicago Circle, IL (1977).